Capacity Upper Bounds for the Symmetric Diamond Relay Network

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***Abstract*—This report considers the discrete memoryless symmetric diamond relay network, where a source X wants to transmit information to a destination Z via two relays Y_1 and Y_2, with there being no direct link between X and Z. We also impose the condition that Y_1 and Y_2 are conditionally independent and identically distributed given X. We derive upper bounds to the capacity of this channel that are tighter than the celebrated cut-set bound. Our approach builds on Wu et al' s approach [1] of using measure concentration inequalities to obtain tighter information-theoretic inequalities.**

I. Introduction

Relay Channels are typically used to model wireless communication networks where power is of importance. For example, consider a scenario wherein a transmitter on earth wishes to communicate with a satellite in deep space. In such a scenario, due to power constraints, a direct link between the transmitter and the receiver in deep space is not possible and relays are used to forward the message to the destination.

In the information-theoretic framework, relay channels were first studied by van der Muelen [2] in 1971. Through the seminal work of Cover and El Gamal [3] in 1979, an upper bound on the capacity of the general relay channel was obtained, which is now popularly known as the *cut-set bound*. This work also proposed two achievability schemes, now popularly known as decode-and-forward and compress-and-forward. The parallel relay network also called the diamond relay channel ([4]) was introduced by Schein and Gallager in [5] and [6] and the cut-set bound for this channel was obtained. [7] extended compress-and-forward and decode-and-forward for the case of multile relays. Since then, several achievability schemes have been proposed, namely amplify-and-forward, quantize-map-and-forward, compute-and-forward, noisy network coding, hybrid coding etc for the single relay channel. Majority of the work on relay channels has been targeted towards designing achievability schemes and obtaining achievable rates that are close to the cut-set bound.

However, the cut-set bound has been shown to be loose in general. [8], [9], [1] derive tighter upper bounds for some special cases of the single relay channel. To the best of our knowledge, the cut-set bound is the best known bound for the general diamond relay network. [1] uses measure concentration inequalities, specifically, the generalized blowing-up lemma to bound information-theoretic expressions, in turn leading to strictly tighter upper bounds for the discrete memoryless symmetric primitive relay channel. The blowing-up lemma has found multiple uses in the context of information theory, especially in strong converses for the degraded broadcast channel.(see [10] for details).

In this work, we first review Wu et al [1]'s upper bound for the capacity of the discrete memoryless symmetric primitive relay channel in Section II. We then present a tighter upper bound using [1]'s approach for the symmetric diamond relay network in Section III.

II. Symmetric Primitive Single Relay Channel

In this section, we consider the symmetric primitive single relay channel and provide an overview of the upper bound given by Theorem 4.1 in [1]. This section focuses mainly on the idea behind the upper bound and on the application of the generalized blowing-up lemma [10] in obtaining the bound. The details can be found in [1].

A. Channel Model

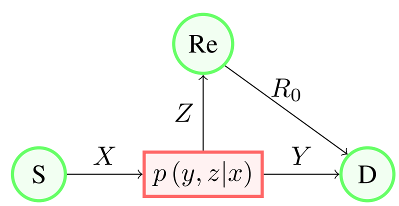

Fig. 1. Primitive single relay channel

The symmetric primitive single relay channel as shown in Figure 1 consists of a source whose input is X, a relay whose output is Z and a destination, whose input is Y. The relay to destination link is noiseless with capacity R_0. The channel through which the source sends X is modelled as

$$(\Omega_X, p(y,z|x), \Omega_Y \times \Omega_Z)$$

where $\Omega_X, \Omega_Y, \Omega_Z$ are finite sets. The symmetric condition is imposed in the form of the following two constraints a) $p(y,z|x) = p(y|x)\,p(z|x)$ b) $\Omega_Y = \Omega_z = \Omega$, and $\Pr(Y=\omega|X=x) = \Pr(Z=\omega|X=x), \forall w \in \Omega, x \in \Omega_X$. For n channel uses, define an encoding function at the relay, $f_n : \Omega^n \to \{1, 2, \dots 2^{nR_0}\}$. Denote the relay's transmission as $I_n = f_n(Z^n)$.

B. Cut-set bound

We present the cut-set bound and it's derivation to illustrate the approach towards obtaining a tighter upper bound.

Proposition 2.1. *(Cut-Set bound) : For the general primitive relay channel, if a rate R is achievable, then there exists some $p(x)$ such that*

$$\begin{cases} R \leq I(X;Y,Z) \\ R \leq I(X;Y) + R_0 \end{cases}$$

Proof. From Fano's inequality, we get

$$nR \leq I(X^n;Y^n,I_n) + n\epsilon$$

The two bounds given in Proposition 2.1 are obtained by bounding the above expression in two different ways. First,

$$\begin{aligned} nR &\leq I(X^n;Y^n,I_n) + n\epsilon \\ &\leq I(X^n;Y^n,Z^n) + n\epsilon \\ &\leq nI(X;Y,Z) + n\epsilon \end{aligned}$$

Following is another way of bounding the same expression

$$\begin{aligned} nR &\leq I(X^n;Y^n,I_n) + n\epsilon \\ &\leq I(X^n;Y^n) + H(I_n|Y^n) - H(I_n|X^n) + n\epsilon \qquad (1) \\ &\leq nI(X;Y) + nR_0 + n\epsilon \end{aligned}$$

□

Note that, after (1), we use $H(I_n|Y^n) \leq H(I_n)$ and $H(I_n|X^n) \geq 0$ to get the final expression. The crux of [1]'s approach is to come up with a tighter upper bound on $H(I_n|Y^n) - H(I_n|X^n)$ by letting $H(I_n|X^n) = na_n$ and bounding $H(I_n|Y^n)$ using typicality arguments combined with the generalized blowing-up lemma. The single-letter characterizations are obtained from standard time-sharing arguments.

C. Wu et al's upper bound

The upper bound for the symmetric primitive relay channel in [1] is given by the following proposition.

Proposition 2.2. *For the symmetric primitive relay channel, if a rate R is achievable, then there exists some $p(x)$ and*

$$a \in \left[0, \min\left(R_0, H(Z|X), \frac{2}{\ln 2}\left(\frac{|\Omega|-1}{|\Omega|}\right)^2\right)\right]$$

such that

$$\begin{aligned} &R \leq I(X;Y,Z) \\ &R \leq I(X;Y) + R_0 - a \\ &R \leq I(X;Y) + H_2\left(\sqrt{\frac{a\ln 2}{2}}\right) + \sqrt{\frac{a\ln 2}{2}}\log(|\Omega|-1) - a \end{aligned}$$

where $H_2(.)$ is the binary entropy function.

The proof requires Lemma 2.3, the proof of which in turn requires Lemma 2.4. We list out the lemmas and then proceed by briefly outlining the proofs given in [1]

Lemma 2.3. *For a fixed n,*

$$H(I_n|Y^n) \leq nV\left(\sqrt{\frac{a_n \ln 2}{2}}\right),$$

$$V(r) := \begin{cases} \log|\Omega|, & \text{if } r > \frac{|\Omega|-1}{|\Omega|} \\ H_2(r) + r\log(|\Omega|-1), & \text{if } r \leq \frac{|\Omega|-1}{|\Omega|} \end{cases}$$

Before presenting Lemma 2.4, we list out some notation. Consider the B-letter i.i.d extensions of X^n, Y^n, Z^n, I_n,

$$\{(X^n(b), Y^n(b), Z^n(b), I_n(b))\}_{b=1}^{B}$$

where for any $b \in [1:B]$, $(X^n(b), Y^n(b), Z^n(b), I_n(b))$ has the same distribution as (X^n, Y^n, Z^n, I_n). We denote $[X^n(1)\ldots X^n(B)]$ by $\mathbf{X}$ and similarly for $\mathbf{Y}, \mathbf{Z}, \mathbf{I}$. Note that $\mathbf{I} := f(\mathbf{Z})$

Lemma 2.4. *Let $f^{-1}(\boldsymbol{i}) := \{\underline{\omega} \in \Omega^{nB} : f(\underline{\omega}) = \boldsymbol{i}\}$ and*

$$\Gamma_\lambda\left(f^{-1}(\boldsymbol{i})\right) := \left\{\underline{\omega} \in \Omega^{nB} : \exists\, \underline{\omega}' \in f^{-1}(\boldsymbol{i})\ s.t.\ d(\underline{\omega}, \underline{\omega}') \leq \lambda\right\}$$

where $\lambda = nB\left(\sqrt{\frac{a_n \ln 2}{2}} + \delta\right)$ and $d(\underline{\omega}, \underline{\omega}')$ represents the Hamming distance between $\underline{\omega}$ and $\underline{\omega}'$. Then for any $\delta > 0$ and B sufficiently large,

$$Pr\left(\boldsymbol{Y} \in \Gamma_\lambda\left(f^{-1}(\boldsymbol{I})\right)\right) \geq 1 - \delta$$

Proof of Lemma 2.3 : Using time-sharing arguments, it is easy to see that

$$H(\boldsymbol{I}|\boldsymbol{Y}) = BH(I_n|Y^n) \qquad (2)$$

So, we upper bound bound $H(\boldsymbol{I}|\boldsymbol{Y})$ which leads to an upper bound on $H(I_n|Y^n)$. Let

$$E = \mathbb{1}\left(\boldsymbol{Y} \in \Gamma_\lambda\left(f^{-1}(\boldsymbol{I})\right)\right)$$

$H(\boldsymbol{I}|\boldsymbol{Y})$ can be bounded as follows

$$\begin{aligned} H(\boldsymbol{I}|\boldsymbol{Y}) &\leq H(\boldsymbol{I}, E|\boldsymbol{Y}) \\ &= H(E|\boldsymbol{Y}) + H(\boldsymbol{I}|\boldsymbol{Y}, E) \\ &\leq 1 + H(\boldsymbol{I}|\boldsymbol{Y}, E) \\ &= \Pr(E=0)\, H(\boldsymbol{I}|\boldsymbol{Y}, E=0) + \\ &\quad \Pr(E=1)\, H(\boldsymbol{I}|\boldsymbol{Y}, E=1) + 1 \\ &\leq H(\boldsymbol{I}|\boldsymbol{Y}, E=1) + \delta nBR_0 + 1 \qquad (3) \end{aligned}$$

where (3) follows from Lemma 2.4. We now make the following key observation : given $E = 1$, i.e. $\boldsymbol{Y}$ is in the blown-up set of $f^{-1}(\boldsymbol{I})$, for any such possible $\boldsymbol{i}$, there exists atleast one $\underline{\omega} \in f^{-1}(\boldsymbol{i})$ such that $d(\underline{\omega}, \boldsymbol{Y}) \leq \lambda$. Since $f^{-1}(\boldsymbol{i})$ are disjoint for distinct $\boldsymbol{i}$, the number of possible $\boldsymbol{i}$'s is given by the number of sequences in the Hamming ball of radius λ, centered around $\boldsymbol{Y}$, $\text{Ball}(\boldsymbol{Y}, \lambda)$. Therefore,

$$\begin{aligned} H(\boldsymbol{I}|\boldsymbol{Y}, E=1) &\leq \log|\text{Ball}(\boldsymbol{Y}, \lambda)| \\ &= nBV\left(\sqrt{\frac{a_n \ln 2}{2}} + \delta\right) \qquad (4) \end{aligned}$$

where (4) follows from the characterization of the volume of the Hamming ball given in Appendix H of [1]. From (2), choosing δ arbitrarily small and B sufficiently large, we get

$$H\left(I_n|Y^n\right) \leq nV\left(\sqrt{\frac{a_n \ln 2}{2}}\right)$$

$\square$

The proof of Lemma 2.4 makes use of a sharper version of the generalized blowing-up lemma stated in [11]. The sharper version is stated in [10].

Proof of Lemma 2.4 : Let $(\boldsymbol{x}, \boldsymbol{i}) \in \mathcal{T}_\epsilon^{(B)}\left(X^n, I_n\right)$, where $\mathcal{T}_\epsilon^{(B)}\left(X^n, I_n\right)$ denotes the ϵ jointly typical sets of $\left(X^n, I_n\right)$ From [12],

$$p\left(\boldsymbol{i}|\boldsymbol{x}\right) \geq 2^{-B\left(H\left(I_n|X^n\right)+\epsilon\right)} \geq 2^{-B\left(na_n+\epsilon\right)} \tag{5}$$

Therefore,

$$\Pr\left(\boldsymbol{Z} \in f^{-1}\left(\boldsymbol{i}\right)|\boldsymbol{x}\right) \geq 2^{-B\left(na_n+\epsilon\right)}$$

By applying the generalized blowing-up lemma,

$$\begin{aligned}
&\Pr\left(\boldsymbol{Z} \in \Gamma_{nB\left(\sqrt{\frac{a_n \ln 2}{2}}+2\sqrt{\epsilon}\right)}\left(f^{-1}\left(\boldsymbol{i}\right)\right)|\boldsymbol{x}\right) \\
&\geq \Pr\left(\boldsymbol{Z} \in \Gamma_{nB\left(\sqrt{\frac{(a_n+\epsilon) \ln 2}{2}}+\sqrt{\epsilon}\right)}\left(f^{-1}\left(\boldsymbol{i}\right)\right)|\boldsymbol{x}\right) \\
&\geq 1-e^{-2nB\epsilon} \\
&\geq 1-\sqrt{\epsilon}
\end{aligned} \tag{6}$$

for a sufficiently large B. Since $\boldsymbol{Y}, \boldsymbol{Z}$ are i.i.d conditioned on $\boldsymbol{X}$, (6) holds when $\boldsymbol{Z}$ is replaced by $\boldsymbol{Y}$.

$$\begin{aligned}
&\Pr\left(\boldsymbol{Y} \in \Gamma_{nB\left(\sqrt{\frac{a_n \ln 2}{2}}+2\sqrt{\epsilon}\right)}\left(f^{-1}\left(\boldsymbol{I}\right)\right)\right) \\
&= \sum_{(\boldsymbol{x},\boldsymbol{i})} \Pr\left(\boldsymbol{Y} \in \Gamma_{nB\left(\sqrt{\frac{a_n \ln 2}{2}}+2\sqrt{\epsilon}\right)}\left(f^{-1}\left(\boldsymbol{i}\right)\right)|\boldsymbol{x},\boldsymbol{i}\right) p\left(\boldsymbol{x},\boldsymbol{i}\right)
\end{aligned} \tag{7}$$

Restricting the sum to pairs $(\boldsymbol{x}, \boldsymbol{i}) \in \mathcal{T}_\epsilon^B\left(X^n, I_n\right)$ and applying (6) concludes the proof by choosing δ to be $2\sqrt{\epsilon}$. $\square$

The proof of Proposition 2.2 follows from Lemma 2.3 by standard time-sharing arguments.

III. Symmetric Diamond Relay Network

In this section we extend results from Section II for the case of the symmetric diamond relay network. We first provide an overview of existing upper bounds for the capacity of the general diamond relay network with noiseless links from relays to destination, introduced by [6] and then present our main results.

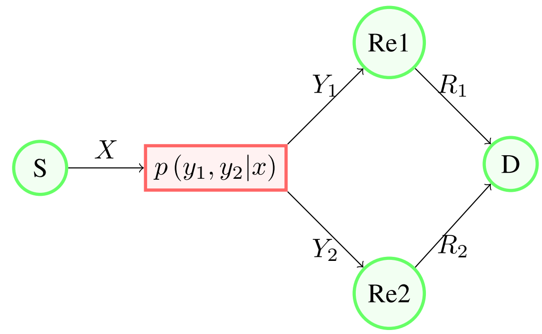

Fig. 2. Diamond relay network

A. Channel Model

The symmetric diamond relay network as shown in Figure 2 consists of a source whose input is X, two relays whose outputs are Y_1, Y_2 respectively. The relay to destination links are noiseless with capacity R_1, R_2 respectively. The channel through which the source sends X, is modeled as

$$\left(\Omega_X, p\left(y_1, y_2|x\right), \Omega_{Y_1} \times \Omega_{Y_2}\right)$$

The symmetric condition is imposed in the form of the following constraints a) $p\left(y_1, y_2|x\right) = p\left(y_1|x\right) p\left(y_2|x\right)$ b) $\Omega_{Y_1} = \Omega_{Y_2} = \Omega$, $\Pr\left(Y_1 = \omega|X = x\right) = \Pr\left(Y_2 = \omega|X = x\right)$. For n channel uses, we define encoding functions f_{n1}, f_{n2} for the relays

$$\begin{aligned}
f_{n1} &: \Omega^n \to \left\{1, 2, \ldots 2^{nR_1}\right\} \\
f_{n2} &: \Omega^n \to \left\{1, 2, \ldots 2^{nR_2}\right\}
\end{aligned}$$

Denote the relay transmissions as $T_{1n} = f_{n1}\left(Y_1^n\right)$, $T_{2n} = f_{n2}\left(Y_2^n\right)$.

B. Cut-Set bound

We review the cut-set bounds for the capacity of the discrete memoryless diamond relay network, C_{net} given in [6]. From [6], $C_{net} = \sup \frac{1}{n} I\left(X^n; T_{1n}, T_{2n}\right)$. Deriving upper bounds for $I\left(X^n; T_{1n}, T_{2n}\right)$ in different ways leads to the following set of bounds. We elaborate on only those bounds that will be used to obtain tighter upper bounds on C_{net} later.

1) Broadcast cut-set bound:

$$C_{net} \leq \max_{p(x)} I\left(X; Y_1, Y_2\right)$$

2) Multiaccess cut-set bound:

$$\begin{aligned}
\frac{1}{n} I\left(X^n; T_{1n}, T_{2n}\right) &\leq \frac{1}{n} H\left(T_{1n}, T_{2n}\right) - \frac{1}{n} H\left(T_{1n}, T_{2n}|X^n\right) \\
&\leq \frac{1}{n} H\left(T_{1n}, T_{2n}\right) \\
&\leq R_1 + R_2
\end{aligned}$$

Therefore, $C_{net} \leq R_1 + R_2$

3) Cross cut-set bounds: Define the capacities of the point-to-point link between source and each of the relays as C_1, C_2 respectively. Specifically,

$$C_1 = \max_{p(x)} I(X; Y_1)$$
$$C_2 = \max_{p(x)} I(X; Y_2)$$

$$\begin{aligned}\frac{I(X^n; T_{1n}, T_{2n})}{n} &= \frac{1}{n}\left(I(X^n; T_{2n}) + I(X^n; T_{1n}|T_{2n})\right) \\ &\le \frac{1}{n}\left(I(X^n; Y_2^n) + I(X^n; T_{1n}|T_{2n})\right) \\ &\le C_2 + \frac{(H(T_{1n}|T_{2n}) - H(T_{1n}|T_{2n}, X^n)}{n} \\ &\le C_2 + \frac{1}{n}H(T_{1n}) \\ &\le C_2 + R_1\end{aligned}$$

Thus, $C_{net} \le C_2 + R_1$. Similarly, $C_{net} \le C_1 + R_2$

C. New upper bound

We now derive strictly tighter versions of the multiaccess cut-set bounds and the cross cut-set bounds for the symmetric diamond relay network using methods similar to the those discussed in Section II.

Theorem 3.1. *For the symmetric diamond relay network, if a rate R is achievable, then there exists $p(x)$ and b such that*

$$b \in \left[0, \min\left(R_1 + R_2, H(Y_1, Y_2|X), \frac{1}{2\ln 2}\left(\frac{|\Omega| - 1}{\Omega}\right)^2\right)\right]$$

$$R \le I(X; Y_1, Y_2)$$
$$R \le R_1 + R_2 - b$$
$$R \le \min(R_1, R_2) + H_2\left(\sqrt{2b\ln 2}\right) + \sqrt{2b\ln 2}\log(|\Omega| - 1)$$
$$R \le \min(C_1, C_2) + H_2\left(\sqrt{2b\ln 2}\right) + \sqrt{2b\ln 2}\log(|\Omega| - 1)$$

As before, the proof of Theorem 3.1 requires Lemma 3.2, whose proof in turn requires Lemma 3.3. We present the lemmas with their proofs, followed by the proof of Theorem 3.1.

We use the definition of $V(.)$ from Lemma 2.3

Lemma 3.2. *For a fixed n,*

$$H(T_{1n}|T_{2n}) \le nV\left(\sqrt{2b_n \ln 2}\right)$$

We again consider the B-letter i.i.d extensions of the random variables $(X^n, Y_1^n, Y_2^n, T_{1n}, T_{2n})$ and denote these by $(\boldsymbol{X}, \boldsymbol{Y_1}, \boldsymbol{Y_2}, \boldsymbol{T_1}, \boldsymbol{T_2})$. Using the same definitions for $f^{-1}(.), \Gamma_\lambda(.)$ as in Lemma 2.4, we derive the following lemma

Lemma 3.3. *For any $\delta > 0$ and sufficiently large B,*

$$Pr\left(\boldsymbol{Y_1} \in \Gamma_\lambda\left(f_1^{-1}(\boldsymbol{T_1})\right) \cap \Gamma_\lambda\left(f_2^{-1}(\boldsymbol{T_2})\right)\right) \ge 1 - \delta$$

where $\lambda = nB\sqrt{\frac{b_n \ln 2}{2}}$

We now present the proofs of Lemma 3.2 and Lemma 3.3.

Proof of Lemma 3.2 : Using time-sharing arguments, we get $H(\boldsymbol{T_1}|\boldsymbol{T_2}) = BH(T_{1n}|T_{2n})$. Therefore, we upper bound $H(\boldsymbol{T_1}|\boldsymbol{T_2})$. Define

$$E = \mathbb{1}\left(\boldsymbol{Y_1} \in \Gamma_\lambda\left(f_1^{-1}(\boldsymbol{T_1})\right) \cap \Gamma_\lambda\left(f_2^{-1}(\boldsymbol{T_2})\right)\right)$$

where λ is as defined in Lemma 3.3. Using similar arguments from the proof of Lemma 2.4,

$$H(\boldsymbol{T_1}|\boldsymbol{T_2}) \le H(\boldsymbol{T_1}|\boldsymbol{T_2}, E = 1) + \delta nBR_1 + 1$$

Since $E = 1$, i.e. $\boldsymbol{Y_1}$ is in the intersection of the blown-up sets of $f_1^{-1}(\boldsymbol{T_1})$ and $f_2^{-1}(\boldsymbol{T_2})$, $\exists\, \omega_1, \omega_2$, such that $d(\omega_1, \boldsymbol{Y_1}) \le \lambda$ and $d(\omega_2, \boldsymbol{Y_1}) \le \lambda$. Hence, given $\boldsymbol{T_2}$, the maximum number of possible $\boldsymbol{T_1}$ sequences will lie in a Hamming ball centered at $\boldsymbol{T_2}$ with radius 2λ. Therefore,

$$\begin{aligned}H(\boldsymbol{T_1}|\boldsymbol{T_2}, E = 1) &\le \log|\text{Ball}(\boldsymbol{T_2}, 2\lambda)| \\ &= nBV\left(\sqrt{2b_n \ln 2}\right)\end{aligned}$$

Choosing δ arbitrarily small and B sufficiently large concludes the proof. □

Proof of Lemma 3.3 :
Assume $(\boldsymbol{x}, \boldsymbol{t_1}, \boldsymbol{t_2}) \in \mathcal{T}_\epsilon^{(B)}(X^n, T_{1n}, T_{2n})$. By [12],

$$\begin{aligned}p(\boldsymbol{t_1}, \boldsymbol{t_2}|\boldsymbol{x}) &\ge 2^{-B(H(T_{1n}, T_{2n}|X^n) + \epsilon)} \\ &= 2^{-nB(b_n + \epsilon)}\end{aligned}$$

Therefore,

$$\Pr\left(\boldsymbol{Y_1} \in f_1^{-1}(\boldsymbol{t_1}), \boldsymbol{Y_2} \in f_2^{-1}(\boldsymbol{t_2})\,|\boldsymbol{x}\right) \ge 2^{-nB(b_n + \epsilon)}$$

Because of the symmetry of the diamond relay network considered,

$$\Pr\left(\boldsymbol{Y_1} \in \left(f_1^{-1}(\boldsymbol{t_1}) \cap f_2^{-1}(\boldsymbol{t_2})\right)|\boldsymbol{x}\right) \ge 2^{-nB(b_n + \epsilon)}$$

For ease of notation, we denote $f_1^{-1}(\boldsymbol{t_1}) \cap f_2^{-1}(\boldsymbol{t_2})$ by $f_{12}^{-1}(\boldsymbol{t_1}, \boldsymbol{t_2})$. By applying the generalized blowing-up lemma [10] and for a sufficiently large B,

$$\Pr\left(\boldsymbol{Y_1} \in \Gamma_{nB\left(\sqrt{\frac{b_n \ln 2}{2}} + 2\sqrt{\epsilon}\right)}\left(f_{12}^{-1}(\boldsymbol{t_1}, \boldsymbol{t_2})\right)|\boldsymbol{x}\right) \ge 1 - \sqrt{\epsilon}$$

As in the proof of Lemma 2.4, averaging the above expression over all possible $(\boldsymbol{x}, \boldsymbol{t_1}, \boldsymbol{t_2})$ gives

$$\Pr\left(\boldsymbol{Y_1} \in \Gamma_{nB\left(\sqrt{\frac{b_n \ln 2}{2}} + 2\sqrt{\epsilon}\right)}\left(f_{12}^{-1}(\boldsymbol{T_1}, \boldsymbol{T_2})\right)\right)$$

which can be lower bounded by restricting the average over $\mathcal{T}_\epsilon^B(X^n, T_{1n}, T_{2n}), B \to \infty$ which leads to a lower bound of $1 - 2\sqrt{\epsilon}$. The proof finally follows from the following observation

$$\Gamma_\lambda\left(f_1^{-1}(\boldsymbol{T_1})\right) \cap \Gamma_\lambda\left(f_2^{-1}(\boldsymbol{T_2})\right) \supseteq \Gamma_\lambda\left(f_{12}^{-1}(\boldsymbol{T_1}, \boldsymbol{T_2})\right)$$

□

Proof of Theorem 3.1 : Let $H(T_{1n}, T_{2n}|X^n) = nb_n$, where $b_n \in [0, \min(R_1 + R_2, H(Y_1, Y_2|X))]$. Substituting this in the multiaccess cut-set bound (III-B2), we get

$$R \leq R_1 + R_2 - b$$

Since Lemma 3.2 is symmetric in T_1, T_2, we apply it to the cross-cut-set bounds III-B3 to get

$$R \leq \min(C_1, C_2) + H_2\left(\sqrt{2b\ln 2}\right) + \sqrt{2b\ln 2}\log(|\Omega| - 1)$$

Finally,

$$H(T_{1n}, T_{2n}) = H(T_{1n}|T_{2n}) + H(T_{2n})$$

Applying Lemma 3.2 to the multiaccess cut-set bound we have

$$R \leq \min(R_1, R_2) + H_2\left(\sqrt{2b\ln 2}\right) + \sqrt{2b\ln 2}\log(|\Omega| - 1)$$

□

IV. Conclusions

We presented a summary of Wu et al's approach to obtain tighter upper bounds for the symmetric primitive single relay channel. Further, we derived new upper bounds that are strictly tighter than the cut-set bound for the symmetric diamond relay network using Wu et al's approach of deriving information-theoretic inequalities using measure concentration inequalities.

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