Exercise 1

Let $f : A \to B$ be a function. Recall that for any sets $X \subseteq A$ and $Y \subseteq B$, we defined the *image* $f(X) = \{f(x) \mid x \in X\}$ and preimage $f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}$.

a) Prove: $f(X \cup X') = f(X) \cup f(X')$

b) If you attempt to prove $f(X \cap X') = f(X) \cap f(X')$ (which does not hold in general) with the same approach you took in a), where does your proof attempt fail?

c) Show that $f^{-1}(f(X)) \subseteq X$ does not hold in general by giving a counterexample.

d) Prove: $f(X \cap X') = f(X) \cap f(X')$ holds for all $X, X' \subseteq A$ if and only if f is injective.

Solution:

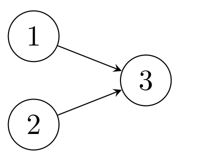
a) **Direction** $\subseteq$: Let y be arbitrary with $y \in f(X \cup X')$. Then by definition of the image there exists $x \in X \cup X'$ with $f(x) = y$. If $x \in X$ we then have $y \in f(X)$ and thus $y \in f(X) \cup f(X')$. If on the other hand $x \in X'$ we have $y \in f(X')$ and thus also $y \in f(X) \cup f(X')$.

Direction $\supseteq$: Let y be arbitrary with $y \in f(X) \cup f(X')$. If $y \in f(X)$ then there exists $x \in X$ with $y = f(x)$ and hence also $x \in X \cup X'$ and thus $y \in f(X \cup X')$. If on the other hand $y \in f(X')$ there exists $x \in X'$ with $y = f(x)$ and hence also $x \in X \cup X'$ and thus $y \in f(X \cup X')$.

b) The direction $\subseteq$ works just the same as before, but the direction $\supseteq$ does not: the problem is that the assumption $y \in f(X) \cap f(X')$ gives us $y \in f(X)$ and $y \in f(X')$ and from that $x \in X$ and $x' \in X'$ with $f(x) = f(x') = y$. But x and x' are not necessarily the same, so we cannot conclude $x \in X \cap X'$, and thus cannot move forward.

c) What we need to achieve is the existence of an $x' \in A \setminus X$ such that $x' \in f^{-1}(f(X))$, i.e. $f(x') \in f(X)$. In other words: if we draw the graph of f as a relation, then we need to find an $x' \in A \setminus X$ that can be reached from some $x \in X$ by taking an arrow to the right and then taking an arrow in reverse direction. That means there has to be some $y \in B$ such that $f(x) = y = f(x')$.

We can easily construct such a situation by choosing e.g. $A = \{1, 2\}$, $B = \{3\}$, $X = \{1\}$, and $f : \{1, 2\} \to \{3\},\ f(x) = 3$. The graph representation of f as a relation then looks like this:



Now $f(X) = \{f(1)\} = \{3\}$ and $f^{-1}(f(X)) = \{1, 2\} \neq \{1\} = X$.

d) **Direction** $\to$: Assume that $f(X \cap X') = f(X) \cap f(X')$ holds for all $X, X' \subseteq A$. We have to prove that f is then injective. We proceed by contradiction.

Suppose f were not injective. Then there exist $x, x' \in A$ such that $x \neq x'$ and $f(x) = f(x')$. By our assumption, it holds that $f(\{x\} \cap \{x'\}) = f(\{x\}) \cap f(\{x'\})$. But because $x \neq x'$ we have $f(\{x\} \cap \{x'\}) = f(\emptyset) = \emptyset$. Moreover, because $f(x) = f(x')$ we have $f(\{x\}) \cap f(\{x'\}) = \{f(x)\} \cap \{f(x')\} = \{f(x)\} \neq \emptyset$. This is a contradiction.

Direction $\leftarrow$: Assume that f is injective. Let $X, X' \subseteq A$ be arbitrary. We must prove that $f(X \cap X') = f(X) \cap f(X')$. We proceed by proving that either side is a subset of the other.

Direction $\subseteq$: Let $y \in f(X \cap X')$. Then there exists an $x \in X \cap X'$ with $f(x) = y$. Then also $y \in f(X)$ and $y \in f(X')$ and thus $y \in f(X) \cap f(X')$ follows.

Direction $\supseteq$: Let $y \in f(X) \cap f(X')$. Then there exist $x \in X$ and $x' \in X'$ with $f(x) = f(x') = y$. By injectivity of f we then also have $x = x'$ and thus $x \in X \cap X'$ and from this $y \in f(X \cap X')$.

Note: This is essentially the proof you tried to do in b), but now we *can* conclude that $x = x'$ because of the injectivity assumption.

Exercise 2

Let Σ be a finite non-empty set and Σ^* the set of words over Σ. We will look at the relation $\sqsubseteq$ on Σ^* where $w_1 \sqsubseteq w_2 \longleftrightarrow$ (there exist $u, v \in \Sigma^*$ such that $w_2 = uw_1v$) .

That is: $w_1 \sqsubseteq w_2$ holds if w_1 is a *subword* of w_2. E.g. ε, abc, bc, e are all $\sqsubseteq abcde$.

a) Show that $(\Sigma^*, \sqsubseteq)$ is a poset. Is it also totally ordered?

b) Draw Hasse diagrams of the following posets:

   i) $([5], \leq_{\mathbb{N}})$

   ii) $(\{a, b\}^*, \sqsubseteq)$ with $\sqsubseteq$ defined as above (only draw it for words up to length 3)

   iii) $([3] \times [3], \leq_{\text{comp}})$, where $\leq_{\text{comp}}$ is the product of the natural order $\leq_{\mathbb{N}}$ on $[3]$ with itself.

   iv) $(A, \subseteq)$ where $A = \{X \subseteq [4] \mid 1 \in X \text{ or } 2 \in X\}$.

c) How can you read off the minimal, maximal, least, greatest elements from a Hasse diagram? Mark them in the diagrams you drew!

d) Which of these orders are total? Can you tell from the Hasse diagrams? For non-total orders, indicate two incomparable elements.

Solution:

a) **Reflexivity:** For any $w \in \Sigma^*$, we have $w = \varepsilon w \varepsilon$ and thus $w \sqsubseteq w$ (by choosing $u = v = \varepsilon$).

Transitivity: Suppose $w_1 \sqsubseteq w_2 \sqsubseteq w_3$. Then there exist u_1, u_2, v_1, v_2 such that $w_2 = u_1w_1v_1$ and $w_3 = u_2w_2v_2$. Let $u = u_2u_1$ and $v = v_1v_2$. Then $w_3 = u_2u_1w_1v_1v_2 = uw_1v$ and thereby $w_1 \sqsubseteq w_3$ as desired.

Antisymmetry: Suppose $w_1 \sqsubseteq w_2$ and $w_2 \sqsubseteq w_1$. We must now show that $w_1 = w_2$.

There are two observations that can make this proof a bit easier:

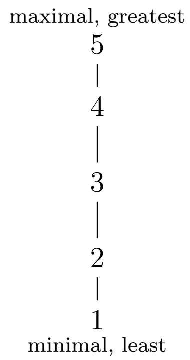
- First, if $x \sqsubseteq y$ for two words, then $|x| \leq |y|$. This is obvious since if $y = uxv$ then $|y| = |u| + |x| + |v| \geq |x|$.
- Second, if $x \subseteq y$ and $|x| = |y|$ then $x = y$. This is also not too hard to see: if $y = uxv$ and $u \neq \varepsilon$ or $v \neq \varepsilon$, we would have $|y| > |x|$, which contradicts $|x| = |y|$. Thus $u = v = \varepsilon$ and then $x = y$.

With the first of these observations, $w_1 \sqsubseteq w_2$ and $w_2 \sqsubseteq w_1$ directly impliy $|w_1| = |w_2|$ and then with the second observation $w_1 = w_2$.

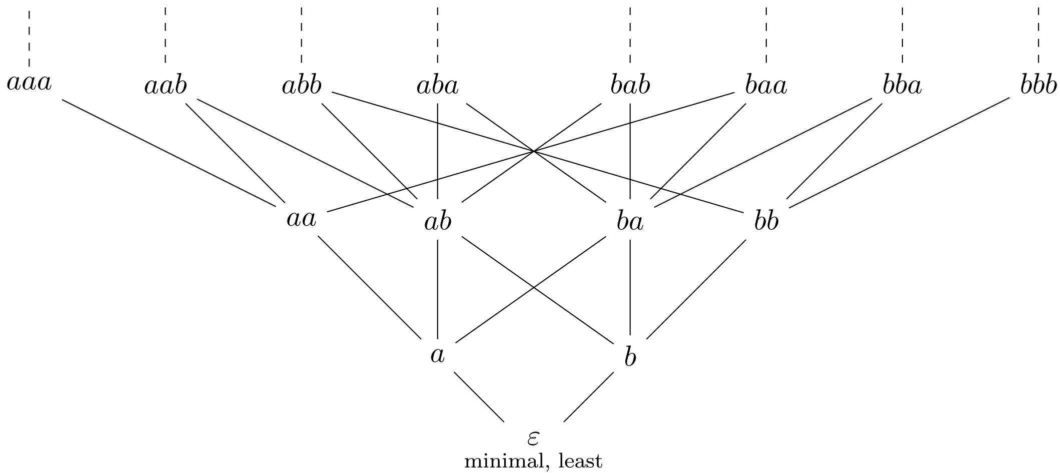
Totality: If Σ is a singleton set (say $\Sigma = \{a\}$) it is a total order, since we can then write any word in the form a^n for some $n \in \mathbb{N}$ and $w_1 \sqsubseteq w_2$ holds if and only if $|w_1| \leq |w_2|$, which can easily be seen to be total.

If on the other hand Σ is not a singleton set (and, as we assumed, nonempty), then there exist $a, b \in \Sigma$ with $a \neq b$ and e.g. the words a and b are incomparable. That these two are incomparable can be seen from Observation 2 from our antisymmetry proof: since $|a| = |b| = 1$, if $a \sqsubseteq b$ or $b \sqsubseteq a$ then $a = b$, which is a contradiction.

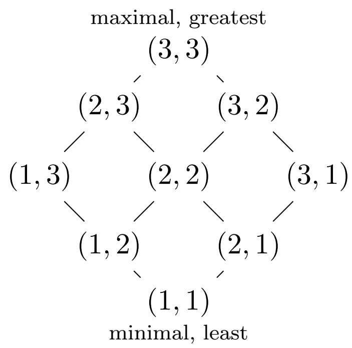
b) i) There are no incomparable elements since this is a total order.



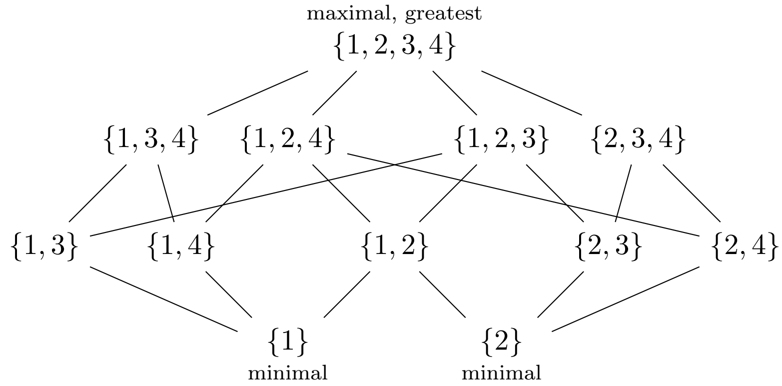
ii) E.g. a and b or aa and bab are incomparable.



iii) E.g. $(1,2)$ and $(2,1)$ or $(1,2)$ and $(3,1)$ are incomparable.



iv) E.g. $\{1,3\}$ and $\{1,2\}$ or $\{1,3\}$ and $\{2,3,4\}$ are incomparable.



c) The minimal elements are those that have no edges going down. A least element x additionally has to be connected to every other element, i.e. for every other element y there must be a path from x to y that only ever goes up.

For maximal/greatest element, the same holds switching all occurrences of 'up' and 'down'.

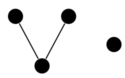
d) Only i) is total. There, the elements are arranged in one long line so that for every pair of elements x and y, we can reach one from the other by only going up or only going down. This is also why total orders are sometimes called *linear orders*.

In the others we have pairs of elements that cannot be reached from one another just by going only up or only down. Such pairs of elements are incomparable.

Exercise 3

The following are two different posets, but they are also 'the same' in some sense because one can obtain one from the other by simply renaming elements ($1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 2$):

While the specific elements are different, the orders themselves have 'the same shape'. One says that these two posets are *isomorphic*. If we only care about different 'isomorphism types', we can draw the above partial order e.g. like this:



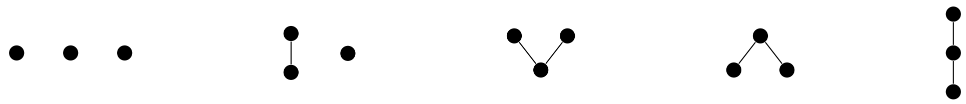
Draw diagrams like these for all posets of size n for $n = 0, 1, 2, 3, 4$. Be careful not to draw the same isomorphism type twice by accident!

Hint: Just so that you can be sure not to have missed one: the number of posets up to isomorphism for $n =$ 0, 1, 2, 3, 4 is 1, 1, 2, 5, 16. (Sequence A000112 on OEIS)

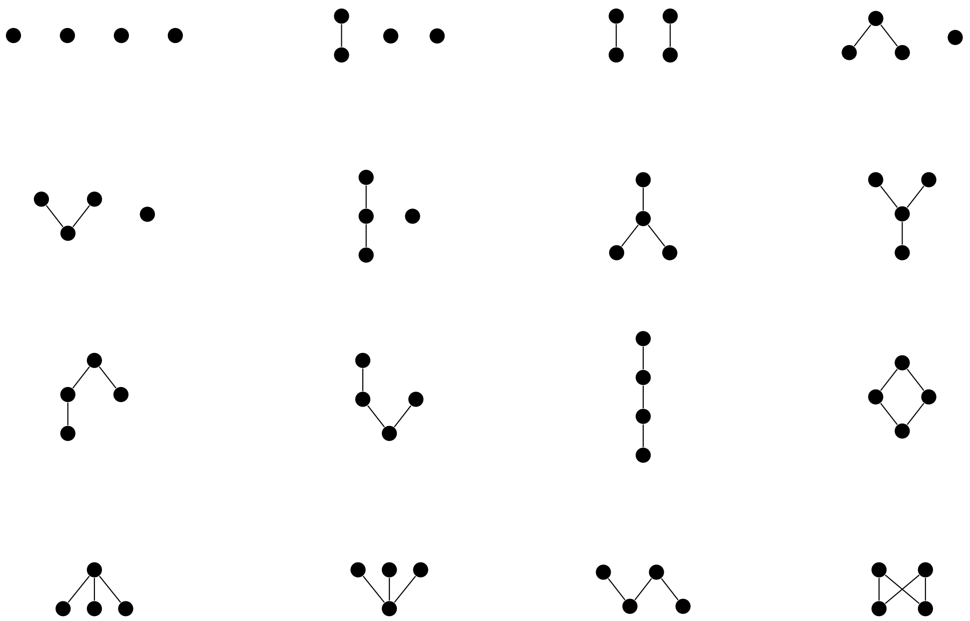
Solution: The only partial order on $\emptyset$ is the empty relation (whose diagram is empty). The only partial order on a singleton set $\{x\}$ is $\mathrm{Id}_{\{x\}}$, whose diagram is just a single $\bullet$. As for the rest:

- $n = 2$:

- $n = 3$:



- $n = 4$:



One good strategy to come up with all of these for some size n is to first list all the ones that are simply disjoint unions of smaller partial orders (see the Bonus Exercise for a precise definition of what a disjoint union of two orders is).

E.g. for $n = 4$, we can decompose $4 = 1 + 1 + 1 + 1$ or $4 = 2 + 1 + 1$ or $4 = 2 + 2$ or $4 = 3 + 1$. This accounts for the first 6 diagrams.

Next, we look at all diagrams of size 3 that are not decomposable into disjoint unions of smaller orders (there are 3 of those) and go through all the different places where we can add a new element. However, one has to be very careful about eliminating duplicates (it may be difficult to tell at first glance if two diagrams are actually the same).

Bonus exercise

a) Show that if $(A, \preceq_1)$ and $(B, \preceq_2)$ are posets that have a least element, then their product order $(A \times B, \preceq_{\text{comp}})$ also has a least element.

b) Let $\preceq_1$ and $\preceq_2$ be preorders on A and B with $A \cap B = \emptyset$. We define the preorder $\preceq_{1\uplus 2}$ as $\preceq_{1\uplus 2} = \preceq_1 \cup \preceq_2$. That is:

$$x \preceq_{1\uplus 2} y \quad \longleftrightarrow \quad (x, y \in A \text{ and } x \preceq_1 y) \text{ or } (x, y \in B \text{ and } x \preceq_2 y)$$

We call $\preceq_{1\uplus 2}$ the *disjoint union* of $\preceq_1$ and $\preceq_2$.

Is $\preceq_{1\uplus 2}$ a partial order if $\preceq_1$ and $\preceq_2$ are partial orders? What about total? What are the minimal, maximal, least, greatest elements of $\preceq_{1\uplus 2}$? What does its Hasse diagram look like if $\preceq_1$ and $\preceq_2$ are partial orders?

c) In the lecture we showed that if a partial order $\preceq$ on a finite set has exactly one minimal element x, then x is also least. Show that this is not true if we do not assume finiteness.

Solution:

a) Let x_1 and x_2 be least elements of $(A, \preceq_1)$ and $(B, \preceq_2)$, respectively. Let $(y_1, y_2) \in A \times B$ be arbitrary. Then we have $x_1 \preceq_1 y_1$ and $x_2 \preceq_2 y_2$ because x_1 and x_2 are least in their respective orders. Then, by definition of the product order, $(x_1, x_2) \preceq_{\text{comp}} (y_1, y_2)$.

b) If A is empty, then $\preceq_{1\uplus 2}$ is just $\preceq_2$ and thus inherits all its properties (including whether it is a partial/total order), and analogously if B is empty.

   Otherwise, if A and B are both nonempty, the disjoint union order can never be total since any element from A is incomparable to any element from B by definition. For the same reason, there also cannot be any least or greatest elements (least/greatest elements must be comparable to every element). It is however a partial order (as you can check easily).

   As for minimal/maximal elements, it is easy to see that any minimal element of $\preceq_1$ is also minimal in $\preceq_{1\uplus 2}$: Let $x \in A$ be minimal in $\preceq_1$. To show that it is also minimal in $\preceq_{1\uplus 2}$, let $y \in A \cup B$ with $y \prec_{1\uplus 2} x$. Then by definition of $\preceq_{1\uplus 2}$ (and the fact that $x \in A$ and $A \cap B = \emptyset$) we have $y \in A$ and $y \prec_1 x$. But that is not possible because x is minimal in $\prec_1$.

   By duality, it follows that maximal elements of $\preceq_1$ are also preserved. Lastly, due to the commutativity of $\cup$ and the definition of the disjoint union order, the minima and maxima of $\preceq_2$ are then also preserved.

   If $\preceq_1$ and $\preceq_2$ are preorders, the Hasse diagram of $\preceq_{1\uplus 2}$ can be obtained simply by writing those of $\preceq_1$ and $\preceq_2$ next to one another independently.

c) Let $\diamond$ be some object that is $\notin \mathbb{Z}$.
   Consider the poset $(\mathbb{Z}_{\leq 0} \cup \{\diamond\}, \preceq)$ where $\mathbb{Z}_{\leq 0} = \{0, -1, -2, \ldots\}$ and $\preceq$ as follows:

$$\begin{array}{ccc} 0 & & \\ | & & \\ -1 & \qquad & \diamond \\ | & & \\ -2 & & \\ \vdots & & \end{array}$$

   It is easy to see from this picture that $\diamond$ is the only minimal element, but it is clearly not least (since it is incomparable to e.g. 0).

   $\preceq$ can be formally defined as:

$$x \preceq y \quad \longleftrightarrow \quad x = y = \diamond \ \text{ or } \ (x, y \in \mathbb{Z} \text{ and } x \leq y)$$

   That is: all the integers are ordered by their natural ordering in $\mathbb{Z}$ and $\diamond$ is incomparable to the integers.

   An alternative view of this argument is that our $\preceq$ is simply the disjoint union of the natural order $(\mathbb{Z}_{\leq 0}, \leq)$ and the trivial order $(\{\diamond\}, \mathrm{id}_{\{\diamond\}})$. And as we have seen in c), such an order inherits all the minimal/maximal elements from its constituent orders (in this case the minimal element $\diamond$ and the maximal element 0) but has no least/greatest elements.