LVA 703070
Department of Computer Science
University of Innsbruck

Exercise 1
Show that each of the following sets A is countably infinite by giving either an explicit bijective enumeration $\mathbb{N} \to A$ or an explicit bijective numbering $A \to \mathbb{N}$.

a) $\{n \in \mathbb{N} \mid n > 42\}$
b) $\{n \in \mathbb{Z} \mid n < 0, n \text{ odd}\}$
c) $\mathbb{N}^3$
d) $\mathbb{N}^k$ for any $k \geq 1$
e) $\mathbb{N}^*$ (the set of finite-length lists of natural numbers)

You may use Cantor's pairing function $\pi : \mathbb{N}^2 \to \mathbb{N}$ from Slide 6.

Solution:

a) $n \mapsto n + 43$ is an enumeration.
b) $n \mapsto -2n - 1$ is an enumeration.
c) $(x, y, z) \mapsto \pi(x, \pi(y, z))$ is a numbering
d) We define a numbering by simply applying Cantor's pairing function $n - 1$ times:

$$(x_1, \dots, x_k) \mapsto \pi(x_1, \pi(x_2, \dots, \pi(x_{n-1}, x_k) \dots))$$

e) We first encode any non-empty list of length k as the a pair consisting of $k-1$ and the encoding of the tuple according to the scheme from the previous sub-exercise. Then we encode that pair as a natural number using Cantor's pairing function. That gives us a bijection between all the non-empty list of naturals and $\mathbb{N}$. To take care of the empty list, we add 1 to the number we have so far and let 0 encode the empty list. Formally:

$$\nu(\varepsilon) = 0 \qquad \nu(x_1, \dots, x_k) = \pi(k - 1, \pi(x_1, \pi(x_2, \dots, \pi(x_{k-1}, x_k) \dots))) + 1 \quad \text{if } k \geq 1$$

Exercise 2
Classify the following sets into the categories 'finite', 'countably infinite', 'uncountable' with a brief explanation for each.

a) $\mathbb{R}_{\geq 0}$, the set of non-negative real numbers.
b) $[0; 1]$, the set of all real numbers x with $0 \leq x \leq 1$
c) The set of all functions $f : \mathbb{N} \to \mathbb{N}$ with finite support, i.e. for which there exists a number N such that $f(n) = 0$ for all $n \geq N$.
d) The set of all sequences $(a_n)_{n \in \mathbb{N}}$ of natural numbers where $a_n = 0$ for any odd n.
e) $\mathcal{P}_n(\mathbb{N})$, i.e. the set of all subsets of $\mathbb{N}$ with exactly n elements

Solution:

a) Uncountable. If it were countable, then $\{-x \mid x \in \mathbb{R}_{\geq 0}\}$ would also be countable. But this is just the set $\mathbb{R}_{\leq 0}$ of non-positive real numbers, and then the union of the two (which is $\mathbb{R}$) would also be countable.

b) Uncountable. If it were countable, then we could just shift this interval by some number y and find that $\{x + y \mid x \in [0; 1]\}$, i.e. the interval $[y; y + 1]$, is also countable. But $\mathbb{R}$ can be partitioned into countably many intervals of length 1:

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n; n+1] = \ldots \cup [-2; -1] \cup [-1; 0] \cup [0; 1] \cup [1; 2] \cup \ldots$$

Since a union of countably many countable sets is countable, this would mean that $\mathbb{R}$ is countable.

c) Countably infinite. We can map any list $(a_0, \ldots, a_{n-1})$ of natural numbers to a function $f : \mathbb{N} \to \mathbb{N}$ with $f(i) = a_n$ if $i < n$ and $f(i) = 0$ otherwise. This is a surjective function from $\mathbb{N}^*$ to our set of interest, and since $\mathbb{N}^*$ is countable, so is our set of interest. That it is infinite is also clear, since for every natural number c, the function $f_c : \mathbb{N} \to \mathbb{N}$ defined by $f_c(0) = c$ and $f_c(n) = 0$ is a different function with finite support.

d) Uncountable. We can take any sequence $a_0, a_1, a_2, \ldots$ in $\mathbb{N}^\mathbb{N}$ and map it to the sequence $a_0, 0, a_1, 0, a_2, 0, \ldots$. This map is an injective function from $\mathbb{N}^\mathbb{N}$ to our set of interest, so since $\mathbb{N}^\mathbb{N}$ is uncountable, our set of interest is as well.

e) If $n = 0$ it is clearly finite because $\mathcal{P}_0(\mathbb{N}) = \{\emptyset\}$. So let us assume that $n > 0$. Then it is countably infinite.

It is countable because mapping $(a_1, \ldots, a_n)$ to $\{a_1, \ldots, a_n\}$ gives us a surjection from $\mathbb{N}^n$ to $\mathcal{P}_n(\mathbb{N})$ and $\mathbb{N}^n$ is countable. It is infinite because mapping k to the set $[n - 1] \cup \{n + k\}$ is an injection from $\mathbb{N}$ to $\mathcal{P}_n(\mathbb{N})$. E.g. for $n = 3$ this gives us the infinite sequence of sets $\{1, 2, 3\},\ \{1, 2, 4\}, \{1, 2, 5\}, \ldots$

Another way to prove that it is countably infinite is to use the bijection $P_n(\mathbb{N}) \to \mathbb{N}$ where we map a set $A \subseteq \mathbb{N}$ to the number $\sum_{k\in A} 2^k$, with the inverse function that maps a natural number n to the set of all i such that the i-th digit in the binary representation of n is 1.

Exercise 3

Given a countable infinite set A with a bijective enumeration $e : \mathbb{N} \to A$, we can write out that enumeration as an infinite sequence $e(0), e(1), e(2), \ldots$.

Example: We showed in the lecture on Slide 4 that if A and B are countably infinite disjoint sets with the (bijective) enumerations $a_0, a_1, a_2, \ldots$ for A and $b_0, b_1, b_2, \ldots$, then $A \cup B$ is also countably infinite. The enumeration we used to show this is $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$. Similarly, if $f : A \to C$ is injective, then $f(A)$ is countably infinite with the enumeration $f(a_0), f(a_1), f(a_2), \ldots$.

The following sets were all shown to be countably infinite in the lecture (directly or indirectly). Write out the first 10 elements of the enumeration we (implicitly) constructed in the proof.

a) $\mathbb{N}$ (direct proof on Slide 3)

b) $\mathbb{Z}$ (by writing $\mathbb{Z} = \mathbb{N} \cup f(\mathbb{N})$ with $f(n) = -n - 1$, on Slide 4)

c) the even integers (because they are an infinite subset of $\mathbb{Z}$, which is countable; see Slide 7)

d) $\mathbb{N} \times \mathbb{N}$ (see Slide 5)

e) $\mathbb{Z} \times \mathbb{Z}$ (see Slide 5)

f) $\mathbb{Q}$ (see Slides 11 and 10; feel free to list fewer than 10 elements here)

Hint: You can also try implementing enumerations as infinite lists in Haskell to save yourself some work.

Solution:

a) The bijective enumeration we used here was simply the identity function, i.e. we have $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$.

b) Since f is injective and the union here is disjoint, we can simply apply f to each natural number to get the enumeration $-1, -2, -3, \ldots$ for $f(\mathbb{N})$. To get an enumeration of the disjoint union of $\mathbb{N}$ and $f(\mathbb{N})$ we simply alternated between the enumerations for each, i.e. we get $0, -1, 1, -2, 2, -3, 3, -4, 4, -5, \ldots$.

c) We simply skip all the numbers in the enumeration of $\mathbb{Z}$ that are not even: $0, -2, 2, -4, 4, -6, 6, -8, 8, -10, \ldots$

d) The enumeration of this can be read directly from the proof on the slides as:

$$(0,0),\ (0,1),\ (1,0),\ (0,2),\ (1,1),\ (2,0),\ (0,3),\ (1,2),\ (2,1),\ (3,0),\ \ldots$$

e) We simply substitute the enumeration for $\mathbb{Z}$ into both components of the enumeration of $\mathbb{N} \times \mathbb{N}$ – for example, we replace 0 with 0, 1 with -1, 2 with 1, etc. and get:

$$(0,0),\ (0,-1),\ (-1,0),\ (0,1),\ (-1,-1),\ (1,0),\ (0,-2),\ (-1,1),\ (1,-1),\ (-2,0),\ \ldots$$

f) First we drop all the elements that contain a 0 in the second component from the enumeration of $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0\}$. That gives us:

$$(0,-1),\ (0,1),\ (-1,-1),\ (0,-2),\ (-1,1),\ (1,-1),\ (0,2),\ (-1,-2),\ (1,1),\ (-2,-1),\ \ldots$$

Now $\mathbb{Q}$ is the image of this set under the function $f(a, b) = \frac{a}{b}$. We cannot simply apply f to every element of the enumeration to get a bijective enumeration of $\mathbb{Q}$ (as we have done above for the enumeration of $\mathbb{Z}$) because f is not injective (e.g. $f(1, 2) = f(2, 4)$). The way we defined the enumeration in the proof in the lecture is that we always pick the *smallest* number for every element in the image – in other words, we enumerate as $f(0), f(1), \ldots$, but whenever we see a number we have already seen, we drop it.

Applying this to the sequence above, we get:

$$0,\ 1,\ -1,\ \tfrac{1}{2},\ 2,\ -\tfrac{1}{2},\ -2,\ \tfrac{1}{3},\ 3,\ -\tfrac{1}{3},\ \ldots$$

A (not particularly efficient) Haskell implementation with which one can easily compute all of these results is provided in OLAT in the file `Enumerations.hs`.

Bonus exercise

Let A and B be (not necessarily disjoint) countable sets. We have seen in the lecture that there are then injective functions $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

a) Show that $A \times B$ is countable by constructing an injective function $A \times B \to \mathbb{N}$.
Hint: Remember that Cantor's pairing function $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective.

b) Show that $A \cup B$ is countable by constructing an injective function $A \cup B \to \mathbb{N}$.

c) Show that A^* is countable in whichever way you want. You may use results from all previous exercises on this sheet (in particular Exercise 1e) might be useful).

Solution:

a) We define $h : A \times B \to \mathbb{N} \times \mathbb{N},\ h(x, y) = (f(x), g(y))$. It is easy to see that h is injective. Then $\pi \circ h$ is the desired injective function from $A \times B$ to $\mathbb{N}$.

b) The obvious solution would be to map every element x of A to $f(x)$ and every element y of B to $g(y)$. There are, however, two complications: first of all, A and B might not be disjoint (so what do we do if x is in both?), and second the ranges of f and g might not be disjoint, so we might end up mapping an element of A to e.g. 0 and an element of B also to 0.

   The first issue can be fixed easily by simply mapping every element x of A to $f(x)$ and only if that is not the case (i.e. $x \in B \setminus A$) do we map it to $g(x)$. As for the second issue, we simply change $f(x)$ to $2f(x)$ and $g(x)$ to $2g(x) + 1$ so that they are still injective but one maps to even integers and the other to odd integers.

   Putting it all together: Define the function $h : A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) + 1 & \text{otherwise} \end{cases}$$

   Then h is injective: all elements of A map to even numbers and all elements of $B \setminus A$ map to odd numbers, so if $h(x) = h(y)$ then either x and y are both in A or both in $B \setminus A$. If $x, y \in A$ we have $2f(x) = 2f(y)$ and thus $f(x) = f(y)$ and thus $x = y$. If $x, y \in B \setminus A$ we have $2g(x) + 1 = 2g(y) + 1$ and thus $g(x) = g(y)$ and thus also $x = y$.

c) If we want an explicit injection $A^* \to \mathbb{N}$, we can simply take a list $(x_1, \ldots, x_n) \in A^*$ and then feed the list $(f(x_1), \ldots, f(x_n))$ to the bijection we found in Exercise 1e).

   A less explicit proof would be that $A^* = \bigcup_{n \geq 0} A^n$, which is a union of countably many countable sets and therefore countable.