Exercise 1

Suppose we have n available instructors that we can hire and m proseminar groups, each with a particular time and place. Every instructor should teach exactly 2 proseminar groups, but there are some proseminar groups whose time slots overlap and thus cannot be taught by the same instructor. We want to find an assignment such that every proseminar is taught by one instructor (it is okay if some instructors teach no groups).

Use your knowledge of graph theory to model this problem and explain how to find a solution if there is one (and how to determine whether one exists in the first place). Hint: Matchings!

Can you adapt your method to also work if instructors can also teach just 1 group?

Solution: If $2n < m$ there is obviously no solution. Otherwise we model the problem as an undirected graph with no loops whose nodes are the m proseminars and two nodes are connected if their time slots do not overlap (i.e. an instructor could, in principle, teach both of them).

Now a (partial) assignment of instructors to pairs of proseminars is simply a matching in this graph, where an edge $\{u, v\}$ being in the matching means that one instructor teaches both u and v. The problem has a solution if and only if there exists a perfect matching. If this does not exist, the best we can do is to teach $2k$ proseminars, where k is the cardinality of a maximum matching.

If we also want to allow instructors to teach only one group, we can still compute the maximum matching as before. If it has cardinality k, there are $m - 2k$ unassigned proseminars, and these need to be assigned to one of the remaining $n - k$ instructors each, so the problem is solvable iff $n - k \geq m - 2k$, i.e. $n \geq m - k$.

Note: For 3 groups per instructor this problem would already be NP-complete and thus (as far as we know) *much* harder.

Exercise 2

a) Consider the set of nodes $\{1, \dots, n\}$. How many ways are there to draw undirected edges between them?

b) How many undirected graphs are there with nodes $\{1, \dots, n\}$? How many with exactly k edges? What if we do not allow loops?

c) What changes in b) if we want to count multigraphs instead of graphs? (two edges are still considered the same if they connect the same two nodes, but there can now be several instances of the same edge)

Solution:

a) Edges in undirected graphs are unordered pairs of nodes, i.e. in our case subsets of $\{1, \dots, n\}$ with size 1 or 2. There are n loop edges of the form $\{i\}$ and $\binom{n}{2} = \frac{1}{2}n(n-1)$ non-loop edges of the form $\{i, j\}$. This gives us a total of $\frac{1}{2}n(n+1)$ edges.

   Alternatively, one can count the non-loop edges by understanding that there are $n \cdot (n-1)$ ordered pairs of two different nodes and every unordered pair $\{x, y\}$ with $x \neq y$ corresponds to two ordered pairs (x, y) and (y, x), so we need to divide that number by 2.

b) The nodes are fixed, so we can only choose what the edges are. The set of edges is simply an arbitrary subset of the possible edges. In a) we determined that there are $\frac{1}{2}n(n+1)$ possible edges, so there are $2^{n(n+1)/2}$ graphs. If we do not allow loops, this changes to $2^{n(n-1)/2}$.

   If we want to count the graphs with *exactly* k edges, we need to use the Binomial coefficients instead and get $\binom{\frac{1}{2}n(n+1)}{k}$ and $\binom{\frac{1}{2}n(n-1)}{k}$.

c) The number of multigraphs with nodes $\{1, \ldots, n\}$ is infinite if $n \geq 1$ since we can add the loop edge from 1 to itself as often as we want. Similarly, the number of multigraphs without loops is also infinite as soon as $n \geq 2$. (We do not yet have a way of classifying the size of infinite sets beyond 'they are infinite', so we will be content with 'infinite' here.)

   The number of multigraphs with 0 nodes and the number of multigraphs with 0 or one node and no loops is, of course, 1 (since they cannot contain any edges).

   As for the number of multigraphs with exactly k edges, the set of edges can essentially be seen as a multiset of size k consisting of the edges we determined in a). The formula from the lecture thus gives us $\binom{\frac{1}{2}n(n+1)+k-1}{k}$ if loops are allowed and $\binom{\frac{1}{2}n(n-1)+k-1}{k}$ if they are not.

Exercise 3

The n guests at a party are lining up in a big row for some photos. Of these guests, k are men and l are women and $n-k-l$ identify as neither man nor woman. ($k+l \leq n$)

Among the guests, there are r couples ($2r \leq n$), i.e. $2r$ of the guests are in attendence with their partner and $n-2r$ are by themselves.

Determine how many possible ways there are for them to stand for each photo.

a) For the first photo, the guests may stand in the row in whichever order they want.

b) For the second photo, the men put on a white mask that renders them indistinguishable from one another and all the women put on a black mask that likewise renders them indistinguishable from one another. The remaining $n-k-l$ people remain distinguishable from one another and from the men and women.

c) For the third photo, the guests take off the masks again and must arrange themselves in such a way that everyone who is in a couple stands next to their partner.

d) For the last picture, we ignore the couples again, but all the men must stand to the left of all the women (i.e. if there is a man in position i and a woman in position j, then $i < j$). The remaining $n-k-l$ people may stand wherever they wish.

e) The photographer charges a fee of s Euros to the host and the guests all chip in to pay it. How many ways are there for them split the fee amongst themselves (assuming everyone pays in multiples of €1)?

f) How many ways are there for them to split the fee if the host makes sure that no guest can get away with contributing no money?

Solution:

a) This is simply a permutation of n elements, of which there are $n!$ different ones.

b) We can model such a configuration as a word over the alphabet $\{M, W, N_1, \ldots, N_{n-k-l}\}$ where M stands for a man, W for a woman, and N_i for one of the $n-k-l$ remaining people. Such a word must contain the letter M exactly k times, the letter W exactly l times, and each of the letters N_i exactly once. The Mississippi formula then tells us that there are $\binom{n}{k,l,1,\ldots,1} = \frac{n!}{k!l!}$ such words.

c) One possible way to think of this is to model each of the r couples as a single element and every person not in a couple (of which there are $n-2r$) also as a single element. Then there are $n-r$ elements and $(n-r)!$ ways to arrange them in a line. Since every couple can independently stand in two different ways (depending on who stands left and who right), we get $2^r(n-r)!$ possible configurations.

d) We first choose which of the n spots to reserve for the $n-k-l$ people who are neither man nor woman. There are $\binom{n}{n-k-l} = \binom{n}{k+l}$ ways to do this. We then have $(n-k-l)!$ ways to distribute them to these spots. Of the remaining $k+l$ spots, the leftmost k must be filled with men (with $k!$ ways to do this) and the other l with women (with $l!$ ways to do this).

   This gives us the overall number of combinations as $\binom{n}{k+l}(n-k-l)!k!l! = \frac{n!k!l!}{(k+l)!}$

e) We essentially distribute s indistinguishable units of money onto n distinguishable guests with no restrictions. This corresponds to the leftmost column of the second row in the Twelvefold Way, i.e. there are $\binom{s+n-1}{s}$ ways for them to pay.

   Equivalently, we can think of this as picking a multiset with s entries where each element is one of the guests. The number of times that a guest is in the multiset is the amount of Euros they pay.

f) For this we simply move to the rightmost column of the Twelvefold Way and get $\binom{s-1}{n-1}$ (multiset of size s with elements from a set of size n where each element occurs at least once).

Bonus exercise

a) Rephrase the first row of the Twelvefold Way from the lecture slides in terms of functions: describe each entry as 'How many functions are there from ... to ... that are ...'

b) Give an intuitive explanation for the formula $k!\left\{ {n \atop k} \right\}$ in the top right entry of the Twelvefold Way. Where does the $k!$ come from? Where does the $\left\{ {n \atop k} \right\}$ come from?

c) If we apply the same reasoning from b) to the other two columns of the table we would expect the third row to contain the entries $k^n/k!$ and $k^{\underline{n}}/k! = \binom{n}{k}$, but instead it contains the entries '$\sum_{i \leq k} \left\{ {n \atop i} \right\}$' and '0 or 1', which is not the same. In fact, $k^n/k!$ is usually not even an integer. How can that be?

d) For the Twelvefold Way, we saw that

   - The first row counts the number of functions from a set of size n to a set of size k (with various conditions).
   - The second row counts the number of multisets of size n with elements from a set of size k (with various conditions).
   - The third row counts the number of partitions of a set of size n into (exactly/at most) k equivalence classes.

   Can you find a similar interpretation for the last row?

Solution:

a) From left to right: If $|A| = n$ and $|B| = k$, how many functions are there from A to B

   1. without any restrictions
   2. that are injective
   3. that are surjective

b) The Stirling numbers of the 2nd kind are defined as the number of ways to distribute n labelled balls into k unlabelled boxes. The top-right entry on the other hand talks about n labelled balls into k *labelled* boxes. One way to do this is to first distribute the balls into unlabelled boxes ($\left\{ {n \atop k} \right\}$ possibilities) and then put labels from 1 to k on the boxes, and there are $k!$ ways to do this.

The important thing here is that the boxes were already distinguishable before being labelled due to the fact that they all contain different numbers, so each of the $k!$ ways to put labels on the boxes indeed gives us a different result. We will see that this is important in sub-exercise c).

Another way to look at the formula is that there is a one-to-one correspondence between

- surjective functions from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$
- pairs consisting of a partitioning of $\{1, \ldots, n\}$ into k non-empty distinct sets and permutations of $\{1, \ldots, k\}$

In other words, we can (uniquely) construct a surjective function from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$ by first deciding which of the n input values will be mapped to the same output value (this is a partitioning of $\{1, \ldots, n\}$ into k non-empty groups) and then distributing the k possible output values to the groups (which is to pick a bijective function from the k groups to the k values).

c) The point where the argument from b) breaks down is that if we drop the requirement that every box contain at least one ball, it is now possible for two different labellings of the boxes to lead to the same result.

For example, if we want to distribute 2 labelled balls onto 4 unlabelled boxes, we can e.g. put Ball 1 into one box and Ball 2 into another box and leave the remaining two boxes empty. There are now in principle 4! ways to label the boxes, but some of them lead to the same outcome: for example, if we label the box containing Ball 1 with 1 and the one containing Ball 2 with 2, then it does not matter if we label the two empty ones with 3 and 4 or with 4 and 3. In both cases we will get the overall result 'Box 1: $\{1\}$, Box 2: $\{2\}$, Box 3: $\emptyset$, Box 4: $\emptyset$.'

d) From left to right: The last row counts the number of ways to write the number n as a sum of

1. exactly k natural numbers (or equivalently: up to k positive integers)
2. k zeros and ones
3. exactly k positive integers

where the order of the summands does not matter (i.e. $1+2$ is considered the same combination as $2+1$).

Again, it is obvious that the second column is trivial: if $n > k$ then there is clearly no way to write n as a sum of k zeros and ones, and if $n \leq k$ there is exactly one way (namely $\underbrace{1 + \ldots + 1}_{n \text{ times}} + \underbrace{0 + \ldots + 0}_{k-n \text{ times}}$)