LVA 703070
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Exercise 1

Recall that the 'Big Sigma' notation for indexed sums $\sum_{i=k}^{l} x_i$ means $x_k + x_{k+1} + \ldots + x_l$.

Consider the sequence of numbers defined by the recurrence

$$a_0 = 1 \qquad \text{and} \qquad a_n = 1 + \sum_{i=0}^{n-1} a_i \quad \text{for } n \geq 1$$

a) Prove the auxiliary fact that $\left(\sum_{i=0}^{k} 2^i\right) = 2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

b) Prove that $a_n = 2^n$ by strong induction on n.

c) Can you find an alternative proof of b) that only requires 'normal' mathematical induction? **Hint:** Try to manipulate the recurrence for a_n into another, simpler recurrence.

Solution:

a) This is easily proven by mathematical induction on k. The base case is obvious. In the induction step, we assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$. Then:

$$\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i \overset{\text{IH}}{=} 2^{k+1} + 2^{k+1} - 1 = 2^{k+2} - 1$$

□

b) We show the statement $P(n) = (a_n = 2^n)$ by strong induction over n.

Induction step: Let $n \in \mathbb{N}$ be arbitrary. Assume as the IH: $a_k = 2^k$ for any $k < n$. We need to show that $a_n = 2^n$.

The case $n = 0$ is obvious. If $n > 0$, we have:

$$a_n = 1 + \sum_{i=0}^{n-1} a_i \overset{\text{IH}}{=} 1 + \sum_{i=0}^{n-1} 2^i \overset{\text{a)}}{=} 1 + 2^n - 1 = 2^n$$

□

c) The recurrence from the exercise statement is a so-called *full-history recurrence* because the n-th value of the sequence depends on all previous values. Proving the correctness of a solution is usually a straightforward strong induction (as we have seen in b)), but finding a solution in the first place can be tricky. In this case, simply guessing a solution from the sequence $1, 2, 4, 8, 16, \ldots$ is not too hard, but the usual approach with full-history recurrences is to try to cancel most of the terms – typically by adding a multiple of a shifted version of the recurrence to itself. This is also what we shall do here.

We have

$$a_n = 1 + a_0 + \ldots + a_{n-1} \qquad \text{for all } n \geq 1 \tag{1}$$

If we substitute $n + 1$ for n in (1) we get

$$a_{n+1} = 1 + a_0 + \ldots + a_{n-1} + a_n \;. \tag{2}$$

Subtracting (1) from (2), we find that all terms on the right-hand side except for a_n cancel, so we have

$$a_{n+1} - a_n = a_n \tag{3}$$

Rearranging yields the recurrence

$$a_{n+1} = 2a_n \qquad\qquad \text{for all } n \geq 1 \tag{4}$$

or, equivalently

$$a_n = 2a_{n-1} \qquad\qquad \text{for all } n \geq 2\ . \tag{5}$$

Substituting $n = 1$ in (1), we get $a_1 = 1 + a_0 = 2$, so we can actually extend the validity of (5) to all $n \geq 1$. To summarise, we have the recurrence

$$a_0 = 1 \qquad a_n = 2a_{n-1} \qquad\qquad \text{for all } n \geq 1 \tag{6}$$

from which it is immediately obvious that $a_n = 2^n$ for all n. This can be now be shown by a mathematical induction on n:
Base case: $a_0 = 1 = 2^0$.
Induction step: Let $n \in \mathbb{N}$ be arbitrary and assume the IH $a_n = 2^n$.
Then $a^{n+1} = 2a_n \overset{IH}{=} 2 \cdot 2^n = 2^{n+1}$.

Exercise 2
Consider the following relations:

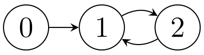
a) Draw the graph of each of the following relations. In case of an infinite relation, sketch a part of the graph similarly to the example on the lecture slides.
   i) $R_1 \subseteq \{0,1,2\}^2$, $R_1 = \{(0,1),(1,2),(2,1)\}$
   ii) $R_2 \subseteq (\{0,1\}^3)^2$, $R_2 = \{(x,y) \mid x \text{ and } y \text{ contain the same number of 1s}\}$
   Example: $((0,1,1),(1,0,1)) \in R_2$
   iii) $R_3 \subseteq \mathbb{N}^2$, $R_3 = \{(m,n) \mid n = m+1 \text{ or } n = 2m+1\}$

b) Describe in your own words: Given graphs of relations R and S, how do you determine
   - the graphs of R^{-1}, RS, R^n, $R^=$, R^+, R^*?
   - the image and preimage of some set?
   - the domain and range?
   - whether R is (ir-)reflexive, (anti-)symmetric, or transitive?
   - whether R is left-/right-total or left-/right-unique?

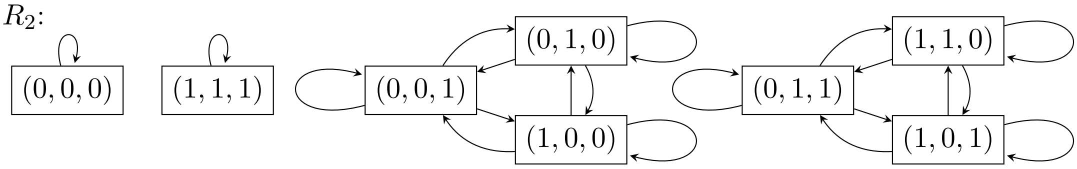
   You need not write all of this down, but you should be able to explain it.

c) Determine the domain and range of the three relations from a) and check which of the above properties they have.

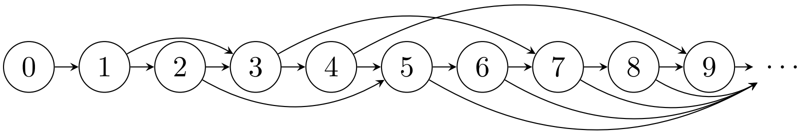
Solution:

a) i) R_1: 

   ii) R_2:



iii) R_3:



b)
- R^{-1} Flip the direction of all the arrows.
- RS: Suppose $R \subseteq A \times B$ and $S \subseteq B \times C$. Draw the graphs of R and S next to each other so that they share the elements in B in the middle. Then, for each arrow $a \to b$ in the left graph and each arrow $b \to c$ in the right graph that share the same middle node b, draw an arrow $a \to c$. The result graph consists of all the arrows going from set A to set C (i.e. we now delete the sets B in the middle).
- R^n: draw all the elements of A and connect precisely those elements $a \to b$ where b is reachable from a in the graph of R in precisely n steps.
- $R^=$: for each element $x \in A$, add a loop from x to itself.
- R^+: draw all the elements of A and connect precisely those elements $a \to b$ where b is reachable from a in the graph of R in any number of steps. However, do not add loops unless a lies on a cycle.
- R^*: same as R^+, but additionally add a loop to each element.
- $R(X)$: find all elements that have an arrow that comes from an element of X.
- $R^{-1}(Y)$: find all elements that have an arrow that goes to an element of Y.
- $\text{dom}(R)$: find all elements that have outgoing arrows.
- $\text{ran}(R)$: find all elements that have incoming arrows.
- Reflexive: each element has an arrow to itself (a 'loop').
- Irreflexive: no element has a loop
- Symmetric: for each arrow $x \to y$, there is also an arrow going in the other direction $y \to x$.
- Anti-symmetric: if there is an arrow $x \to y$ between two different elements x, y, there is no arrow $y \to x$ in the other direction.
- Transitive: If there is a sequence of two arrows $x \to y \to z$, there is also an arrow $x \to z$. (if there is an indirect connection, there is also a direct one)
- Left-total: each element of A has at least one outgoing arrow
- Right-total: each element of B has at least one incoming arrow
- Left-unique: no element has more than one incoming arrow
- Right-unique: no element has more than one outgoing arrow

Note: More precise definitions of 'reachable' and 'cycle' will be given later in the lecture.

c)
- R_1: domain: $\{0, 1, 2\}$, range: $\{1, 2\}$, properties: irreflexive, left-total, right-unique
- R_2: domain and range: $\{0, 1\}^3$, properties: reflexive, symmetric, transitive, left- and right-total
- R_3: domain: $\mathbb{N}$, range: $\mathbb{N} \setminus \{0\}$, properties: irreflexive, antisymmetric, left-total

Exercise 3
Consider relations S and T on $\mathbb{Z}$ defined by $S = \{(a, a+1) \mid a \in \mathbb{Z}\}$ and $T = \{(a, 2a) \mid a \in \mathbb{Z}\}$.

a) In the lecture, we introduced the concept of a *closure* of a relation with respect to some property. In particular, we looked at the reflexive and transitive closures.

   Now let R^s denote the *symmetric closure* of a relation R on A. Can you give a simple expression for R^s? (similar to $R^= = R \cup \mathrm{Id}_A$ from the slides)

b) Find simple intensional descriptions of the following relations and sketch the graphs of the first 6:

i) $S^=$	ii) S^{-1}	iii) S^s	iv) S^n	v) S^+
vi) S^*	vii) ST	viii) TS	ix) T^sS	x) $((S^2)^* \cap (S^3)^*)$

c) True or false? $(R^=)^n = (R^n)^=$ for any relation R. Explain your answer!

d) True or false? $(R^m)^n = (R^n)^m$ for any relation R. Explain your answer!

Solution:

a) The desired expression is $R^s = R \cup R^{-1}$. It is obvious that no matter what R is, $R \cup R^{-1}$ is symmetric, since

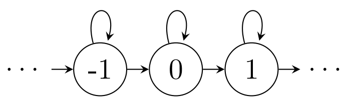
$$\begin{aligned}(x, y) \in R \cup R^{-1} &\longleftrightarrow (x, y) \in R \vee (y, x) \in R \\ &\longleftrightarrow (y, x) \in R \vee (x, y) \in R \longleftrightarrow (y, x) \in R \cup R^{-1} .\end{aligned}$$

   This shows that $R \cup R^{-1}$ is a symmetric superset of R.

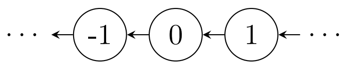
   It remains to show that $R \cup R^{-1}$ is the smallest symmetric superset of R (with respect to inclusion). That is, we have to show that, if R' is a symmetric relation with $R \subseteq R' \subseteq A^2$, $R \cup R^s \subseteq R'$. To that end, note that it suffices to show that $R^s \subseteq R'$. To show this, suppose $(x, y) \in R^s$ for some x, y. Then $(y, x) \in R$ and thus $(y, x) \in R'$ since $R \subseteq R'$. But then also $(x, y) \in R'$ because R' is symmetric. We thus have $R^s \subseteq R'$.

   Note: This solution is extremely detailed. Your solution need not be as detailed.

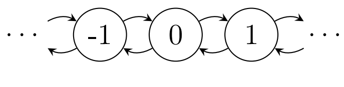
b) i) $\{(a, b) \mid b = a \text{ or } b = a + 1\}$



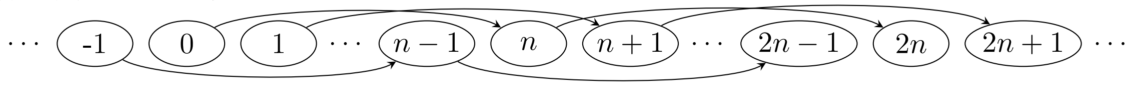
   ii) $\{(a, b) \mid b = a - 1\}$



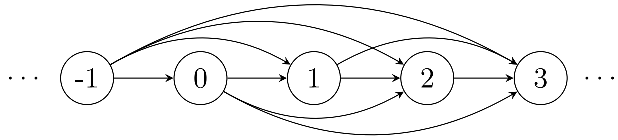
   iii) $\{(a, b) \mid |a - b| = 1\}$



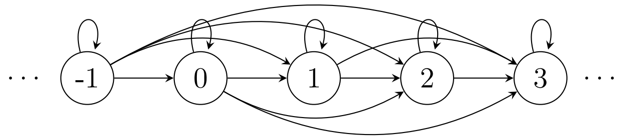
   iv) $\{(a, b) \mid b = a + n\}$



   v) $\{(a, b) \mid a < b\}$



vi) $\{(a, b) \mid a \leq b\}$



vii) $\{(a, b) \mid b = 2a + 2\}$

viii) $\{(a, b) \mid b = 2a + 1\}$

ix) $\{(a, b) \mid b = 2a + 1 \text{ or } a = 2b - 2\}$

x) $(S^6)^* = \{(a, b) \mid a \leq b \text{ and } b - a \text{ is a multiple of } 6\}$

c) False. If we let $R = S$ we have $(R^=)^n = \{(a, b) \mid a \leq b \leq a + n\}$ whereas we have $(R^n)^= = \{(a, b) \mid b = a \text{ or } b = a + n\}$.

Intuitively speaking, $(R^=)^n$ is the relation of taking between 0 and n steps in R. In the graph representation, it connects any two nodes that are reachable from one another in the graph of R in at most n steps.

d) True. We first show the auxiliary fact $(R^m)^n = R^{mn}$ for any $m, n \in \mathbb{N}$ by induction on n.
Base case: $(R^m)^0 = \mathrm{Id} = R^{m \cdot 0}$.
Induction step: Let $n \in \mathbb{N}$ be arbitrary and assume the IH $(R^m)^n = R^{mn}$. Then:

$$(R^m)^{n+1} = (R^m)^n R^m \overset{\text{IH}}{=} R^{mn} R^m = R^{mn+m} = R^{m(n+1)} \ .$$

With the commutativity of multiplication, it follows that $(R^m)^n = R^{mn} = R^{nm} = (R^n)^m$.

Bonus Exercise

a) For each of the following list of properties, give a non-empty set A and a relation on A that has these properties – or prove that there is no such relation.

i) reflexive, antisymmetric, not transitive

ii) reflexive, symmetric, not transitive

iii) irreflexive, symmetric, not transitive

iv) reflexive and irreflexive

v) irreflexive, symmetric, transitive, left-total

b) Show that if A is a set and R is a relation on A, then R is symmetric and antisymmetric if and only if $R \subseteq \mathrm{Id}_A$, i.e. if $(x, y) \in R$ implies $x = y$.

Solution:

a) i) $\{(a, b) \mid b = a \text{ or } b = a + 1\} = S^=$ on the set $\mathbb{Z}^2$
(where S is the relation from Exercise 3)

ii) $\{(a, b) \mid |a - b| \leq 1\} = (S^s)^=$ on the set $\mathbb{Z}^2$

iii) $\{(a, b) \mid |a - b| = 1\} = S^s$ on the set $\mathbb{Z}^2$

iv) Not possible. Let $x \in A$. Then reflexivity demands $(x, x) \in A$ but irreflexivity demands $(x, x) \notin A$. This is a contradiction, so there cannot be such a relation.

Note however that if $A = \emptyset$ *were* allowed, the relation $\emptyset$ on A is indeed both reflexive and irreflexive.

v) Not possible. Let $x \in A$. By left-totality, there is a y with $(x, y) \in A$. By symmetry, we have $(y, x) \in A$. With transitivity, we get $(x, x) \in A$. But this contradicts irreflexivity.

Again, if A were allowed to be empty, the empty relation would have all these properties.

b) For the first direction, suppose $R \subseteq \mathrm{Id}_A$. Then R trivially satisfies antisymmetry, because $(x, y) \in R$ with $x \neq y$ is not possible. Symmetry is also satisfied because if $(x, y) \in R$, then $x = y$ and thus also $(y, x) \in R$.

   For the other direction, suppose R is both symmetric and antisymmetric. We shall show that $S \subseteq \mathrm{Id}_A$. To that end, consider arbitrary x, y with $(x, y) \in R$. By symmetry, $(y, x) \in R$ and then by antisymmetry $x = y$ and thus $(x, y) \in \mathrm{Id}_A$. Thus we can conclude $R \subseteq \mathrm{Id}_A$.