If anything about an exercise is unclear or you believe you have found a mistake, ask in the OLAT forum. If doing so would reveal significant information about your solution, email Manuel Eberl instead.

Exercise 1

a) We define: $A = \{0, 1, 3\}$, $B = \{0, 2, 3, 5\}$, $C = \{0, 2, 3\}$, $D = \{0, 3, 4, 5\}$, $M = \{A, B, C, D\}$
Give full extensional expressions for the following sets (i.e. write out all their elements).

i) $[0]$	ii) $[1]$	iii) $\{\{n\} \mid n \in [4]\}$	iv) $\mathcal{P}(A)$
v) $\bigcup M$	vi) $\bigcap M$	vii) $B \setminus (A \cap C)$	viii) $(A \times C) \setminus (B \times D)$
ix) $\{n^2 \mid n \in \mathbb{N}, n \leq 10, n \text{ even}\}$		x) $\{X \in \mathcal{P}(\mathbb{Z}_6) \mid X \subseteq Y \text{ for all } Y \in M\}$	

Solution:

i) $\emptyset$

ii) $\{1\}$

iii) $\{\{1\}, \{2\}, \{3\}, \{4\}\}$

iv) $\{\emptyset, \{0\}, \{1\}, \{3\}, \{0, 1\}, \{0, 3\}, \{1, 3\}, \{0, 1, 3\}\}$

v) $\{0, 1, 2, 3, 4, 5\}$

vi) $\{0, 3\}$

vii) $\{2, 5\}$

viii) $\{(0, 2),\ (1, 0),\ (1, 2),\ (1, 3),\ (3, 2)\}$

ix) $\{0, 4, 16, 36, 64, 100\}$

x) The sets X that we are looking for are those that are a subset of *all* the sets in M. Equivalently, X must be a subset of $\bigcap M$, i.e. $\{0, 3\}$. Thus, the result is $\mathcal{P}(\{0, 3\}) = \{\emptyset, \{0\}, \{3\}, \{0, 3\}\}$

Exercise 2

a) What is the cardinality of the following sets?

i) $\{\emptyset\}$	ii) $\{\emptyset, \{\emptyset\}\}$	iii) $\{\{1, 2\}, \{2, 1\}\}$	iv) $[n]$	v) $\{n^2 \mid n \in \mathbb{Z}, -5 \leq n \leq 5\}$
vi) $\mathbb{Z}_n$	vii) $\{(1, 2), (2, 1)\}$	viii) $\{\mathbb{N}\}$	ix) $\mathcal{P}([10])$	x) $\{\mathbb{Z}_{10} \setminus \{0\}, [9]\}$

b) Consider the n-th von Neumann numeral V_n as defined in the lecture. What is the cardinality of V_n? Prove it using the recursive definition of V_n!

Solution:

a)

i) 1	ii) 2	iii) 1	iv) n	v) 6 (since e.g. $(-2)^2 = 2^2$)
vi) n	vii) 2	viii) 1	ix) $2^{10} = 1024$	x) 1 (since $\mathbb{Z}_{10} \setminus \{0\} = [9]$)

b) From the slides it seems that $V_n = \{V_0, \ldots, V_{n-1}\}$, which suggests that $|V_n| = n$. Let $P(n)$ denote the statement '$|V_n| = n$'. We prove $P(n)$ for all n by induction on n.

Base case: The cardinality of V_0, which is $\emptyset$, is 0 by definition. Thus $P(0)$ holds.

Induction step: Let n be an arbitrary natural number. Assume as induction hypothesis that $P(n)$ holds, i.e. $|V_n| = n$. We now need to show that $P(n + 1)$ holds, i.e. $|V_{n+1}| = n + 1$.

To do this, we expand the recursive definition of V_{n+1}, namely $V_{n+1} = V_n \cup \{V_n\}$. Note that $V_n \notin V_n$ (no set contains itself). Therefore by the definition of cardinality for insertion of an element we have:

$$|V_{n+1}| = |V_n \cup \{V_n\}| = |V_n| + 1 \overset{\text{IH}}{=} n + 1$$

where in the last step we used the induction hypothesis. This completes the proof.

Note: Your proof does not have to be as explicit as the one above. E.g. you do not have to say explicitly what the property $P(n)$ is. We simply strive, in our sample solutions, to present proofs in the greatest possible detail and clarity.

Exercise 3

Similary to what we have done in the lecture, we will now look at all the sets that can be constructed using only $\emptyset$ and extensional definition. Let us call those 'DS sets'.

We define the *rank* of a DS set as its 'maximal nesting depth'. That is, $\emptyset$ has rank 0 and the rank of a non-empty set A is the maximum of the ranks of its elements, plus 1.

Example: $\text{rank}(\emptyset) = 0$ $\quad$ $\text{rank}(\{\emptyset\}) = 1$ $\quad$ $\text{rank}(\{\{\emptyset\}\}) = \text{rank}(\{\emptyset, \{\emptyset\}\}) = 2$

a) List all DS sets of ranks 0, 1, 2, 3.
Hint: To avoid excessive writing, consider using some abbreviations (e.g. the V_n notation for the von-Neumann numerals from the slides).

b) How many DS sets are there with rank i for $i = 0, 1, 2, 3$?

c) How many DS sets are there with rank $\leq i$ for $i = 0, 1, 2, 3$?

Solution:

a) We use the 'von Neumann' numerals from the lecture as abbreviations.

Rank 0: $\emptyset = V_0$

Rank 1: A rank 1 set can only contain sets of rank 0 and must contain at least one set of rank 0. Since there only is one set of rank 0, the only possibility is: $\{\emptyset\} = \{V_0\} = V_1$

Rank 2: To obtain a set of rank 2, we need an element of rank 1 (there is only one choice) and optionally elements of rank 0 (again, there is only one choice). This gives us: $\{\{\emptyset\}\} = \{V_1\}$ and $\{\emptyset, \{\emptyset\}\} = \{V_0, V_1\} = V_2$

Rank 3: A set of rank 3 must contain one of the two sets of rank 2 (namely $\{V_1\}$ and V_2) or both of them. In addition to that, it may or may not contain each of the sets of rank 0 and 1 (namely V_0 and V_1). We thus construct the following table:

	Contains $\{V_1\}$	Contains V_2	Contains $\{V_1\}$ and V_2
No sets of rank < 2	$\{\{V_1\}\}$	$\{V_2\}$	$\{\{V_1\}, V_2\}$
Contains V_0	$\{\{V_1\}, V_0\}$	$\{V_2, V_0\}$	$\{\{V_1\}, V_2, V_0\}$
Contains V_1	$\{\{V_1\}, V_1\}$	$\{V_2, V_1\}$	$\{\{V_1\}, V_2, V_1\}$
Contains V_0 and V_1	$\{\{V_1\}, V_0, V_1\}$	$\{V_2, V_0, V_1\} = V_3$	$\{\{V_1\}, V_2, V_0, V_1\}$

b) As we can see from a), we have 1, 1, 2, 12 DS sets of rank 0, 1, 2, 3, respectively.

c) Summing up the numbers from b), we have 1, 2, 4, 16 DS sets of rank at most 0, 1, 2, 3, respectively.

Bonus Exercise

In this exercise, we go back to the setting from Exercise 3. Let A_n denote the set of all DS sets of rank at most n.

a) Find $|A_4|$. **Warning:** Do *not* try to manually list all the DS sets of rank 4.

b) Find a general formula to compute $|A_n|$.

c) What is the rank of A_n?

d) What is the biggest DS set of rank at most n? What about exactly rank n?

Solution:

a) A DS set of rank at most 4 must clearly consist entirely of sets of rank 3. Conversely, any set consisting entirely of sets of rank 3 does have rank at most 4.

Consequently, A_4 is simply the set of all sets whose elements are from A_3. This is precisely the power set of A_3, i.e. $A_4 = \mathcal{P}(A_3)$. Thus we have $|A_4| = 2^{16} = 65{,}536$.

b) The argument from a) generalises to $A_n = \mathcal{P}(A_{n-1})$. Therefore, the number of DS sets with rank at most n is

$$|A_n| = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^1}}}}}}_{n \text{ times}}$$

This 'power tower' is also sometimes called *tetration* and written as $2 \uparrow\uparrow n$. Needless to say, these numbers grows *very* fast. ($|A_5|$ already has almost 20,000 decimal digits!)

c) Since A_n consists entirely of sets of rank $\leq n$, its rank cannot be higher than $n+1$. On the other hand, there obviously exists an element of rank n for any n (e.g. V_n), so A_n contains an element of rank n and must by definition have rank at least $n+1$. Thus $\text{rank}(A_n) = n+1$.

d) For $n = 0$ there is only one set, namely $\emptyset$. Otherwise, a DS set of rank at most n must consist entirely of sets of rank at most $n-1$. The biggest such set is the one that consists of *all* sets of rank at most $n-1$ – namely A_{n-1}.

Since A_{n-1} is the biggest DS set of rank at most n and $\text{rank}(A_{n-1}) = n$, it is also the biggest DS set of rank exactly n.