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Coefficient of tangential restitution for viscoelastic spheres

T. Schwager[1], V. Becker[1], and T. Pöschel[2,a]

[1] Charité, Augustenburger Platz 1, 13353 Berlin, Germany
[2] Physikalisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany

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Abstract. The collision of frictional granular particles may be described by an interaction force whose normal component is that of viscoelastic spheres while the tangential part is described by the model by Cundall and Strack (Géotechnique **29**, 47 (1979)) being the most popular tangential collision model in Molecular Dynamics simulations. Albeit being a rather complicated model, governed by 5 phenomenological parameters and 2 independent initial conditions, we find that it is described by 3 independent parameters only. Surprisingly, in a wide range of parameters the corresponding coefficient of tangential restitution, ε_t, is well described by the simple Coulomb law with a cut-off at $\varepsilon_t = 0$. A more complex behavior of the coefficient of restitution as a function on the normal and tangential components of the impact velocity, g_n and g_t, including negative values of ε_t, is found only for very small ratio g_t/g_n. For the analysis presented here we neglect dissipation of the interaction in normal direction.

PACS. 45.70.-n Granular systems – 45.50.Tn Collisions – 45.50.-j Dynamics and kinematics of a particle and a system of particles

1 Introduction

The collision of granular particles may be characterized by two complementary descriptions: first, the collision may be described by the time-dependent interaction forces in normal and tangential direction with respect to the contact plane, F_n and F_t, being the foundation of Molecular Dynamics simulations. The components of the relative velocity in normal and tangential direction after the collision are obtained by integrating Newton's equation of motion. Second, the collision may be described by the coefficients of restitution, ε_n and ε_t, quantifying the ratios of the incoming and outgoing relative velocities. The latter description is the basis of event-driven Molecular Dynamics.

As both models describe the same physical phenomenon, the coefficients of restitution should be derivable from the forces, including material parameters, and the impact velocity.

In this paper we consider the collision of frictional viscoelastic spheres. The interaction force in normal direction with respect to the contact plane, that is, in the direction of the unit vector between the centers of the particles, $\mathbf{e}_n \equiv (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|$ is described by the Hertz contact force [1] extended to viscous deformations [2],

$$F_n = \max\left[0, \frac{2Y\sqrt{R_{\mathrm{eff}}}}{3(1-\nu^2)}\left(\xi^{3/2} + A\sqrt{\xi}\dot{\xi}\right)\right], \qquad (1)$$

where $R_{\mathrm{eff}} \equiv R_iR_j/(R_i + R_j)$, with R_i and R_j being the radii of the particles. The dynamical argument of this force is the time-dependent mutual deformation $\xi \equiv \max(0, R_i + R_j - |\mathbf{r}_i - \mathbf{r}_j|)$ of the particles and the max[...] takes into account that there are no attracting forces between granular particles. The material is described by the Young modulus Y and the Poisson ratio ν and by the dissipative constant A being a function of the material viscosity, for details see [2]. The compression obeys, thus, Newton's equation

$$m_{\mathrm{eff}}\,\ddot{\xi} + F_n\left(\dot{\xi}, \xi\right) = 0; \qquad \dot{\xi}(0) = g_n\,; \qquad \xi(0) = 0, \quad (2)$$

with the normal component of the impact velocity $g_n = -(\mathbf{v}_i - \mathbf{v}_j)\cdot\mathbf{e}_n$ and $m_{\mathrm{eff}} \equiv m_im_j(m_i + m_j)$. Note that the normal and tangential force components F_n and F_t are signed quantities. Their combination forms the vector $\mathbf{F} = F_n\mathbf{e}_n + F_t\mathbf{e}_t$ of the total force.

The coefficient of normal restitution, defined as the ratio of the post-collisional and the pre-collisional values of the normal component of the relative velocity follows from the solution of equation (2) through

$$\varepsilon_n \equiv -\frac{g_n'}{g_n} \equiv \frac{\dot{\xi}(t_c)}{\dot{\xi}(0)}\,, \qquad (3)$$

where t_c is the duration of the collision. From its definition, obviously, $0 \le \varepsilon_n \le 1$, where $\varepsilon_n = 1$ describes an

[a] e-mail: thorsten.poeschel@uni-bayreuth.de

elastic collision. The analytical solution of equation (2) and, thus, equation (3) is technically complicated [3,4] and the computation of t_c needed in equation (3) is even more complicated [5].

While the normal interaction force originates from the viscoelastic properties of the particle material and can be, therefore, mathematically derived from considering the deformation of the bulk of the material [2], the tangential force is mainly determined by the surface properties. In the literature, there are several models for the description of the tangential force between rough spheres, for a review see [6]. In this paper we refer to the model by Cundall and Strack [7]. This model was used in many Molecular Dynamics simulations of granular systems and yields very realistic results [8–16]. Therefore, it is the most popular tangential collision model for the simulation of rough spheres. In the next section we will discuss the tangential force in more detail.

2 Tangential forces and the coefficient of tangential restitution

The collision of frictional particles changes not only the normal component of the relative velocity but also its tangential component as well as the particles' rotational velocity.

The relative velocity of the surfaces of the particles at the point of contact is

$$\mathbf{g} \equiv \mathbf{v}_i - \mathbf{v}_j - (R_i \boldsymbol{\Omega}_i + R_j \boldsymbol{\Omega}_j) \times \mathbf{e}_n \,, \tag{4}$$

with $\boldsymbol{\Omega}_{i/j}$ and $R_{i/j}$ being the angular velocities and radii of the particles. Its components in normal and tangential direction read

$$\mathbf{g}_n = (\mathbf{e}_n \cdot \mathbf{g})\mathbf{e}_n \,, \tag{5}$$

$$\mathbf{g}_t = (\mathbf{e}_n \times \mathbf{g}) \times \mathbf{e}_n \,. \tag{6}$$

The direction of the initial tangential velocity defines the unit vector $\mathbf{e}_t$ in tangential direction. The tangential component g_t of the impact velocity is defined as $g_t(t) = \mathbf{g}(t) \cdot \mathbf{e}_t$, similar to the normal component g_n. By definition, the initial value of $g_t = g_t(0)$ is non-negative, $g_t(0) \geq 0$. Similar to the coefficient of normal restitution for the change of the relative velocity in normal direction, equation (3), the coefficient of tangential restitution describes the change of the tangential component[1],

$$\varepsilon_t \equiv \frac{g_t'}{g_t} = \frac{g_t(t_c)}{g_t(0)} \,. \tag{7}$$

The coefficient of tangential restitution has two elastic limits, $\varepsilon_t = \pm 1$. In the case $\varepsilon_t = 1$ the tangential relative velocity keeps conserved, that is, this limit corresponds to perfectly smooth particles. The other case, $\varepsilon_t = -1$, corresponds to reversal of the relative tangential velocity as it would occur for the collision of very elastic gear wheels. Consequently, $-1 \leq \varepsilon_t \leq 1$. The value $\varepsilon_t = 0$ describes the total loss of relative tangential velocity after a collision. Let us elaborate now how the coefficient of tangential restitution relates to the corresponding component of the interaction force.

From a theoretical point of view, the description of the tangential force is problematic: If the particles slide on each other (dynamic friction), the tangential force adopts the value

$$F_t = -\operatorname{sgn}[g_t(t)]\mu F_N \,, \tag{8}$$

called the *Coulomb friction law* with the friction coefficient μ. The function $\operatorname{sgn}(\cdot)$ is the sign function. If the particles do *not* slide, obviously, the tangential force must be $|F_t| < \mu F_n$. So, which value is adopted then? In case of no sliding, the tangential force adopts the value which keeps the particles from sliding. This tautological statement reflects the fact that the tangential force is not well defined unless we make assumptions on the microscopic details of the deformation of the asperities at the contact surface, see *e.g.* [17] for such a model.

For practical applications, that is, Molecular Dynamics simulations, however, we need the tangential component of the interaction force as a function of the particles' relative position and velocity and possibly of the history of the contact. Therefore, for the application in particle simulations, interaction force models were elaborated to mimic static friction between contacting particles, see [6].

One of the most realistic models used in many Molecular Dynamics simulations is the model by Cundall and Strack [7] where static friction is described by means of a spring acting in the contact plane. The spring is initialized at the time of first contact, $t = 0$, and exists until the surfaces of the particles separate from one another after the collision. The elongation

$$\zeta(t) = \int_0^t g_t(t')\mathrm{d}t' \tag{9}$$

quantifies the restoring tangential force, limited by Coulomb's friction law, equation (8). The tangential interaction force reads then

$$F_t = -\operatorname{sgn}(\zeta) \min(\mu F_n, k_t|\zeta|), \tag{10}$$

where k_t is a phenomenological constant which characterizes the elastic resistance of the surface asperities against deformation in tangential direction. Note that the tangential force at some time t^*, $F_t(t^*)$, does not only depend on the relative positions and velocities of the particles at the same time, $\xi(t^*)$ and $\dot{\xi}(t^*)$, but on the full history of the contact, $0 \leq t \leq t^*$.

From equation (10) there follows the elongation of the spring $|\zeta| = \mu F_n/k_t$ in the regime of sliding, according to the Coulomb criterion. The spring acts as a reservoir of energy which may be released in a later stage of the collision. As shown below, this reservoir is the reason why the model by Cundall and Strack is able to model also negative values of the coefficient of tangential restitution.

[1] Note that the definition of ε_t differs from the corresponding definition, equation (3), of ε_n by the sign. In the literature both definitions are used.

Albeit rather successful in Molecular Dynamics simulations, we would like to mention that the addressed model is a phenomenological one which cannot be *directly* related to microscopic properties of the material. Insofar, the constant k_t of the above-introduced tangential spring cannot be computed from material and surface properties. Nevertheless, using a rather crude estimate we can obtain the order of magnitude of the phenomenological parameter k_t if we relate the imaginary tangential spring to the spheres' surface roughness[2]. A more realistic two-dimensional model for the interaction forces in normal and tangential direction, which is based on the microscopic details of the surfaces can be found in [18]. Here, a particle (disk) is modeled by 800 mass points which are connected to their neighbors via non-linear springs. Because of its great complexity, this model is, however, not suited for the simulation of a many-particle system.

3 Limiting case of pure Coulomb sliding

Let us assume that during the entire collision the particles slide on each other, that is, the friction force is not sufficient to stop the tangential relative motion. The force is therefore described by equation (8), *i.e.* $F_t = -\operatorname{sgn}(g_t)\mu F_n$.

We assume nearly instantaneous collisions, that is, the unit vector $\mathbf{e}_n$ does not change during the collision [19] and obtain from equations (5) and (6) Newton's law for the components of the relative velocities,

$$\frac{\mathrm{d}g_n(t)}{\mathrm{d}t} = -\frac{1}{m_{\text{eff}}}F_n\,; \qquad \frac{\mathrm{d}g_t(t)}{\mathrm{d}t} = \frac{1}{\alpha m_{\text{eff}}}F_t\,, \tag{11}$$

with the effective mass, $m_{\text{eff}} \equiv m_i m_j/(m_i + m_j)$, and

$$\alpha \equiv \left[1 + \frac{m_{\text{eff}} R_i^2}{J_i} + \frac{m_{\text{eff}} R_j^2}{J_j}\right]^{-1}, \tag{12}$$

where J_i and J_j are the moments of inertia of the particles. The parameter α characterizes the mass distribution within the particles. For homogeneous spheres it adopts the value $\alpha = 2/7$.

During the collision the normal component of the velocity changes by

$$g_n' - g_n = -\frac{1}{m_{\text{eff}}}\int_0^{t_c} F_n(t)\mathrm{d}t = -(1+\varepsilon_n)g_n \tag{13}$$

and the tangential component after the collision is

$$\begin{aligned} g_t' - g_t &= \frac{1}{\alpha m_{\text{eff}}}\int_0^{t_c} F_t \mathrm{d}t = -\frac{\mu}{\alpha m_{\text{eff}}}\int_0^{t_c} F_n \mathrm{d}t \\ &= -\frac{\mu(1+\varepsilon_n)}{\alpha} g_n. \end{aligned} \tag{14}$$

Using the definition of the coefficient of tangential restitution, we obtain

$$\varepsilon_t^C = 1 - \frac{\mu(1+\varepsilon_n)}{\alpha}\,\frac{g_n}{g_t}\,, \tag{15}$$

where the superscript C stands for pure Coulomb regime. Note that this result was derived independently from the functional form of the normal and tangential force laws (see [20,21]). Moreover, ε_t^C depends *always* significantly on both the normal and the tangential components of the relative velocity.

According to our basic assumption of pure Coulomb friction, the particles slide on each other during the entire contact, that is, $g_t' \geq 0$ and $\varepsilon_t^C \geq 0$. Consequently, equation (15) must be cut off:

$$\varepsilon_t^C = \max\left(0, 1 - \frac{\mu(1+\varepsilon_n)}{\alpha}\,\frac{g_n}{g_t}\right). \tag{16}$$

Figure 1 shows the coefficient of tangential restitution, equation (16), as it follows from the simple Coulomb law.

4 Numerical integration of the equations of motion

4.1 Example

Newton's equations of motion, equations (11), can be solved numerically to obtain the coefficient of tangential restitution as a function of the components of the impact velocity and the particle material constants. Before discussing the more general scaling behavior of the problem, here we want to present an example of this integration in its phenomenological variables. Figure 2 shows $\varepsilon_t(g_n, g_t)$ for material parameters of steel. The values of the corresponding system parameters are given in the caption of the figure.

[2] The order of magnitude of k_t can be estimated from the following argument: The maximal elastic force (neglecting the dissipative part) during an impact of spheres,

$$F_n^{\max} = \frac{\mu}{3}\left[\frac{250\,Y^2\pi^3\rho^3}{(1-\nu^2)}\right]^{1/5} R^2 g_n^{6/5}\,,$$

can be obtained by equating the elastic energy of the compressed spheres (Hertz force) and the initial kinetic energy according to the normal component of the relative velocity g_n. We assume that at least in the moment of maximum compression, the particles do not slide, due to the Coulomb criterion, equation (8). The maximum tangential force, $F_t = \mu F_n^{\max}$, corresponds to the deformation $\zeta = F_t/k_t$ of the spring in tangential direction. With the realistic material constants used later (see caption of Fig. 2), $\mu = 0.5$, $Y = 210\,\text{MPa}$, $\rho = 7850\,\text{kg/m}^3$, $R = 0.02\,\text{m}$, $\nu = 0.3$ and assuming the initial normal velocity $g_n = 20\,\text{m/s}$, we obtain $\zeta = 1.109 \cdot 10^5\,\text{m}\,\text{s}^2/\text{kg}\,k_t^{-1}$. Thus, the value $k_t = 10^{12}\,\text{kg/s}^2$ used in the simulations below corresponds to $\zeta = 1.109 \cdot 10^{-7}\,\text{m} = 0.1109\,\mu\text{m}$. In a microscopic interpretation, the tangential force originates from the deformation ζ of microscopic asperities of the surface (in tangential direction). Obviously, the deformation of the asperities, ζ, should not exceed the size of the asperities themselves. The numerical value for ζ given above corresponds to the surface roughness of non-polished steel, therefore, the assumed value $k_t = 10^{12}\,\text{kg/s}^2$ seems to be the right order of magnitude.

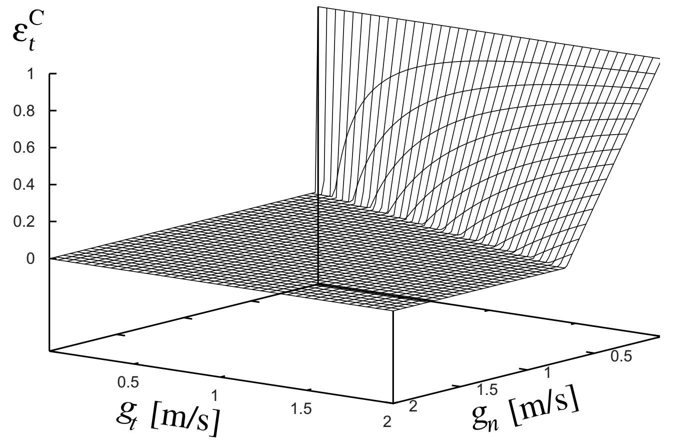

Fig. 1. The coefficient of tangential restitution as a function of the components of the impact velocity, g_n and g_t, in the case of pure Coulomb friction. The relevant parameters are $\mu = 0.4$ and $\alpha = 2/7$.

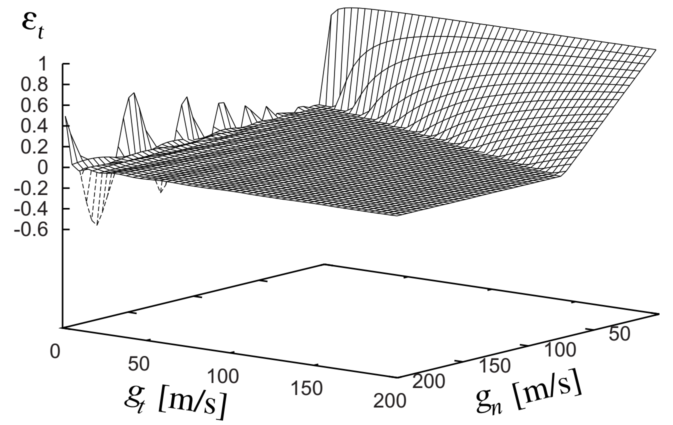

Fig. 2. Coefficient of tangential restitution for the collision of identical steel spheres of $R = 2\,\mathrm{cm}$ ($Y = 210\,\mathrm{GPa}$, $\rho = 7850\,\mathrm{kg/m^3}$, and $\nu = 0.3$, $\alpha = 2/7$) as a function of the impact velocities, $\varepsilon_t(g_n, g_t)$. Dissipation in normal direction was neglected, the tangential spring constant was $k_t = 10^{12}\,\mathrm{kg/s^2}$, the friction constant was $\mu = 0.4$. For small tangential impact velocities the coefficient of restitution may become negaitve.

It turns out that $\varepsilon_t(g_n, g_t)$ resembles the coefficient $\varepsilon_t^C(g_n, g_t)$ of the simplified model discussed in Section 3 in a wide range of impact velocities. Significant deviations are found only in the region of small g_t where also negative values of ε_t are found. Differently from the model of pure Coulomb friction, here ε_t is not cut off at some value, but the plateau comes from the inherent dynamics of the Cundall-Strack model and the corresponding equation of motion. The plateau at $\varepsilon_t \approx 0$ as well as the oscillations for small g_t can be understood from the analysis of the scaled equations of motion given below.

4.2 Scaling properties

In this paper we focus on the coefficient of tangential restitution, therefore, to derive scaling properties we neglect the damping of the motion in normal direction and assume $A = 0$ for the dissipative constant in equation (1). At first glance, this assumption seems to be an important simplification, however, the complex behavior of the tangential motion discussed here is caused exclusively by the tangential interaction force. Thus, the choice of the interaction force in normal direction is less important. The main difference between $A = 0$ and $A > 0$ is the duration of the contact of the colliding spheres. Except for these quantitative changes, the system behavior will be the same for $A > 0$ since (naturally) the normal and tangential forces are perpendicular to one another and, thus, do not interfere.

We write the equations of motion in a more convenient way,

$$\begin{aligned}
&\ddot{\xi} + \beta\xi^{3/2} = 0,\\
&\ddot{\zeta} + \min\left(\frac{\mu\beta}{\alpha}\xi^{3/2}, \frac{k_t}{\alpha m_{\mathrm{eff}}}\zeta\right) = 0, \qquad (17)\\
&\xi(0) = 0; \qquad \dot{\xi}(0) = g_n\,; \qquad \zeta(0) = 0; \qquad \dot{\zeta}(0) = g_t\,,
\end{aligned}$$

with

$$\beta \equiv \frac{2Y}{3(1-\nu^2)}\frac{R_{\mathrm{eff}}^{1/2}}{m_{\mathrm{eff}}}\,. \qquad (18)$$

In the Coulomb regime, *i.e.* if $F_t = -\operatorname{sgn}(\zeta)\mu m_{\mathrm{eff}}\beta\xi^{3/2}$, the elongation of the tangential spring is $\zeta = \operatorname{sgn}(\zeta)\mu m_{\mathrm{eff}}\beta\xi^{3/2}/k_t$ with the sign, $\operatorname{sgn}(\zeta)$, taken from the value of ζ before applying the limiting rule.

The above problem contains 5 phenomenological parameters: β, k_t, μ, m_{eff}, α and 2 initial velocities g_n and g_t. Using suitable units of length and time, we can reduce the number of independent parameters to only 3. The proper length scale is the maximal compression,

$$\xi_{\max} = \left(\frac{5}{4}\right)^{2/5}\frac{g_n^{4/5}}{\beta^{2/5}}\,. \qquad (19)$$

The proper time scale is the duration of the collision,

$$t_c = \frac{C}{\beta^{2/5}g_n^{1/5}}\,, \qquad (20)$$

with C being a pure number. We drop the numerical prefactors and obtain the time and length scales

$$\begin{aligned}
t_{\mathrm{scale}} &= \beta^{-2/5}g_n^{-1/5}\,,\\
\xi_{\mathrm{scale}} &= g_n^{4/5}\beta^{-2/5}\,. \qquad (21)
\end{aligned}$$

We define the scaled compression and elongation of the tangential spring, $x \equiv \xi/\xi_{\mathrm{scale}}$, $z \equiv \zeta/\xi_{\mathrm{scale}}$ and write time derivatives with respect to the scaled time t/t_{scale}. The initial conditions of the collision in scaled variables are

$$\begin{aligned}
\dot{x}(0) &= g_n\frac{t_{\mathrm{scale}}}{\xi_{\mathrm{scale}}} = 1,\\
\dot{z}(0) &= g_t\frac{t_{\mathrm{scale}}}{\xi_{\mathrm{scale}}} = \frac{g_t}{g_n}\,, \qquad (22)
\end{aligned}$$

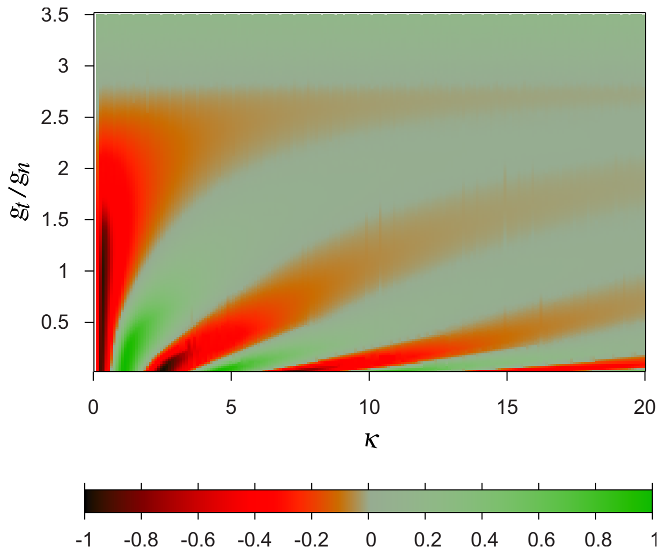

Fig. 3. The coefficient of tangential restitution as obtained from the numerical solution of the scaled equation of motion, equation (23). For explanation see text.

and the equations of motion are

$$\ddot{x} + x^{3/2} = 0,$$
$$\ddot{z} + \min\left(\frac{\mu}{\alpha} x^{3/2}, \kappa z\right) = 0, \qquad (23)$$
$$x(0) = 0; \qquad \dot{x}(0) = 1; \qquad z(0) = 0; \qquad \dot{z}(0) = \frac{g_t}{g_n},$$

with

$$\kappa \equiv \frac{k_t}{\alpha m_{\text{eff}} \beta^{4/5} g_n^{2/5}}. \qquad (24)$$

If the Coulomb limit is active, the tangential elongation is determined by $z = \mu x^{3/2}/(\alpha\kappa)$. Hence, the scaled problem contains only three parameters: μ/α, κ, and the ratio g_t/g_n. Figure 3 shows the solution of Newton's equation of motion in the reduced variables. As shown in the example, Figure 2, for sufficiently large values of g_t or g_n we obtain a very small coefficient of tangential restitution, $\varepsilon_t \approx 0$. For small g_t/g_n or large κ corresponding to a stiff tangential spring, the plot reveals the typical oscillations. The limit of large κ is discussed in Section 4.4.

4.3 Switching between the modes of the tangential force

Depending on the parameters of the motion κ, μ and g_t/g_n the motion of the particles during contact can be complicated. For sufficiently high scaled elasticity κ during the collision, the tangential force, equation (10), may switch multiple times between the Coulomb regime, $F_t = \pm\mu F_n$, and the Cundall-Strack regime, $F_t = -k_t\zeta$. This is shown in Figure 4.

For large values of $g_t/g_n \gtrsim 3$ we see that the collision takes place entirely in the Coulomb regime, except for very small values of κ. This corresponds to the upper region in Figure 3 where the resulting coefficient of restitution is almost zero.

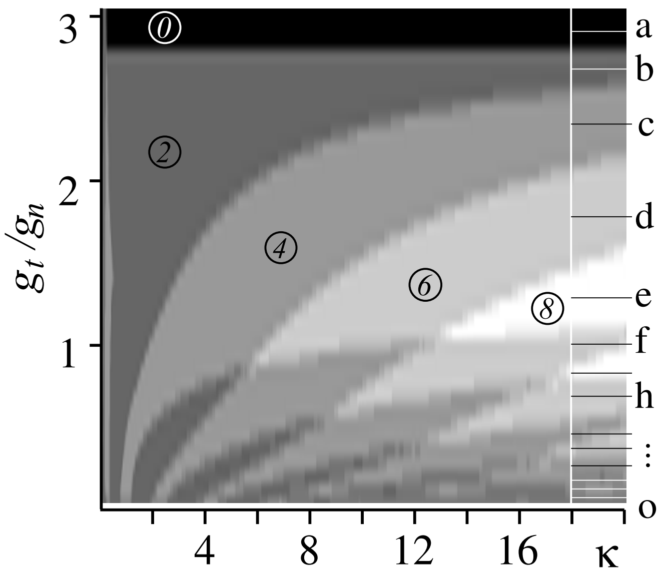

Fig. 4. During the collision the tangential force according to equation (10) may switch multiple times between the Coulomb regime, $F_t = \pm\mu F_n$ and the Cundall-Strack regime, $F_t = -k_t\zeta$. The figure shows the number of switching events during the collision in dependence on the system variables κ and g_t/g_n for identical homogeneous spheres ($\alpha = 2/7$) and $\mu = 0.4$. The numbers in circles indicate the number of switching events. For $\kappa = 18$ and the indicated values of g_t/g_n ("a" ... "o" (not all letters are shown)) the time-dependent forces are drawn in Figure 5.

For larger values of κ, depending on g_t/g_n the force switches repeatedly between the regimes. For the example $\kappa = 18$, the time-dependent forces are drawn in Figure 5, where the letters on the right-hand side of Figure 4 indicate the ratio of the impact velocity components. These letters correspond to the same letters in Figure 5.

First one notes that the motion always starts in the Coulomb regime (Fig. 5a-o). This is due to the fact that at the beginning of the collision the normal force increases roughly as $t^{3/2}$ while the tangential force increases as t. Thus, in the beginning of the contact, $t \gtrsim 0$, the tangential force is always larger than μF_n for a certain period of time. The same is true for the final part of the collision: with the same line of reasoning, one can convince oneself that the collision must terminate in the Coulomb regime —except for the trivial case $g_t \equiv 0$. Therefore, the number of switches between Cundall-Strack regime and Coulomb regime is always even.

Furthermore, for very small values of κ the system cannot remain in the Coulomb regime. It can only remain in the regime of pure Coulomb friction if $\mu/\alpha x^{3/2}$ is smaller than κz (see Eq. (23)). Hence for sufficiently small values of κ we will alway observe a switch to the Cundall-Strack regime (and back to the Coulomb regime at the end of the collision). This explains the vertical line on the left-hand side of Figure 4 which indicates this region of small κ with a non-zero number of switches.

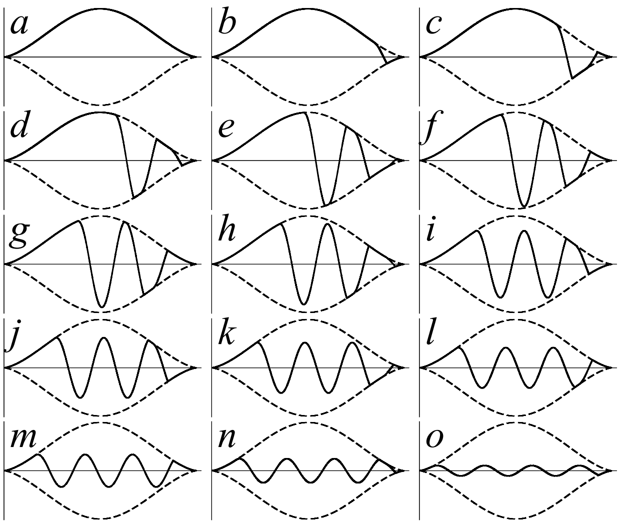

Fig. 5. The normal and tangential components of the interaction force as a function of time for $\kappa = 18$ and the values g_t/g_n indicated in Figure 4. The full lines show the tangential component, F_t/μ, due to equation (10) and the dashed lines show the normal component, $\pm F_n$, due to equation (1) (with $A = 0$).

For a large ratio of the velocity components, g_t/g_n (Fig. 5a) we observe the regime of pure Coulomb friction, that is, the number of switches is zero, except for very small values of κ.

For a velocity slightly smaller than the limit of pure Coulomb friction (Fig. 5b) the friction force is sufficient to stop the particle (in tangential direction) in the course of the collision. Hence, the force switches to the Cundall-Strack regime. The direction of the relative motion of the particles at the contact is reversed and the force switches back to the Coulomb regime eventually. Since we have one motion reversal, the coefficient of tangential restitution is negative.

For even smaller initial tangential velocity (Fig. 5c) the motion is stopped a second time after the switch back to the Coulomb regime (second switch). Hence, a second motion reversal takes place and subsequent switch to the Coulomb regime. As we have two motion reversals (and four switches) the coefficient of tangential restitution is positive.

Note that the elongation of the Cundall-Strack spring at the instant of the first switch from the initial Coulomb regime increases with decreasing initial tangential velocity, as all switches take place in the final decompression phase. Therefore, the tangential oscillation cannot complete even half a period —the switch to the Coulomb regime after only one motion reversal is inevitable. The number of motion reversals is half the number of switches.

For further decreasing initial tangential velocity the number of switches increases to its maximum of eight (for the chosen elasticity κ), see Figure 5e. In this case the coefficient of tangential restitution is positive (due to four motion reversals). The initial switch from the Coulomb regime takes place just before the moment of maximum compression. Now, for further decreasing initial tangential velocity (Fig. 5f) the tangential oscillation may complete half a cycle once —the number of switches decreases to six. However, we have four motion reversals, the coefficient of tangential restitution is positive again.

For an even smaller velocity (Fig. 5g) we have again eight switches as there is an additional switch from the Coulomb regime just before the end of collision, accompanied by a motion reversal and a final switch back to the Coulomb regime. The coefficient of tangential restitution is negative.

For further decreasing tangential velocity the initial elongation of the tangential spring decreases. The tangential oscillation performs more and more cycles before switching back to the Coulomb regime. This switch takes place later and later, reducing the number of switches between the regimes that may take place. So, for decreasing tangential velocities we have a decreasing number of switches. This decrease is not monotonous, there are tangential velocities where the number of switches briefly rises by two due to short episodes of motion in the Cundall-Strack regime near the end of the collision (Fig. 5j).

Finally, for sufficiently small initial tangential velocity (Fig. 5o) the motion switches from the Cundall-Strack regime once, the tangential spring performs a number of undamped oscillations and finally switches back to the Coulomb regime. The number of switches is two.

Note that, due to the undamped nature of the tangential spring there can only be energy dissipation during the Coulomb regime.

4.4 Limit of stiff tangential springs

For very stiff tangential springs, that is, large k_t or large κ in scaled variables, and a certain range of g_t/g_n we can estimate the absolute value of ε_t. We will show that in this case $|\varepsilon_t|$ is small with the limit $\lim_{\kappa\to\infty} |\varepsilon_t| = 0$.

Assume the ratio of the velocity components, g_t/g_n, sufficiently small such that the relative tangential motion of the particles stops at least once during the course of the collision (see Sect. 4.3). The largest elongation of the scaled tangential spring is $z_{\max} = \mu/(\alpha\kappa)$. Figure 6 (top panel) shows a typical situation. Therefore, when the tangential motion of the particle ceases, the maximum energy stored in the tangential spring is

$$E_{\tan}^{\max} = \frac{1}{2}\kappa z_{\max}^2 = \frac{\mu^2}{2\alpha^2\kappa}\,. \qquad (25)$$

This is also the maximum amount of energy of the tangential motion that can be recovered after the collision, provided there are no further switches to the Coulomb regime[3]. Therefore, equation (25) is an upper limit of

[3] As discussed above, close to the end of the collision the force turns back to the Coulomb regime where further energy is dissipated. Without damping, this switch back to the Coulomb regime is inevitable, except for the trivial case of vanishing initial tangential velocity. For non-zero normal damping there exists a range of initial conditions where the collision ends in the Cundall-Strack regime. Nonetheless, the above estimate, equation (25), being an upper limit, stays correct.

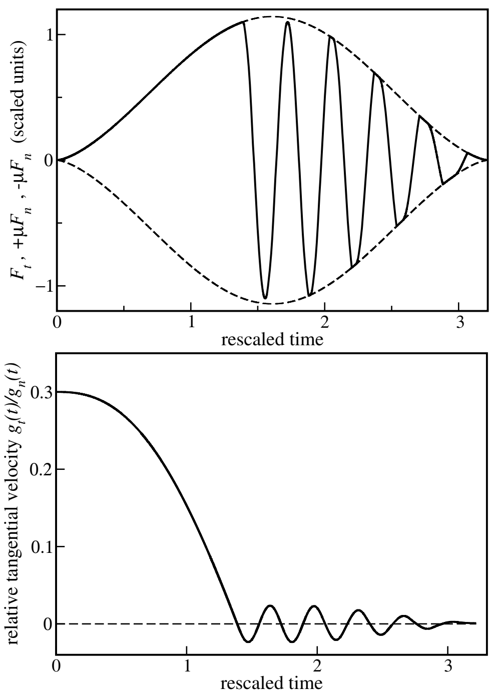

Fig. 6. Top panel: the normal (dashed lines) and tangential (full line) component of the interaction force for the case of very stiff tangential springs ($\kappa = 100$). The initial value $z(0) = g_t/g_n = 0.3$. As discussed in the text, the tangential motion is stopped near the point of maximal normal compression. For better presentation, the tangential force was multiplied by $1/\mu$. After this stop of tangential motion there are many subsequent episodes of Coulomb sliding. Bottom panel: the tangential velocity over time. For better orientation, the value of $g_t = 0$ is marked (dashed line). As discussed in the text, the velocity estimated by equation (26) (first minimum shortly after $t = 1$) grossly overestimates the final velocity.

the final energy of the tangential motion and with $\dot{z} = g_t(t)/g_n$, defined in equation (22), we obtain

$$|g_t'| \le g_n \frac{\mu}{\alpha\sqrt{\kappa}} \,. \tag{26}$$

Hence, for very stiff tangential springs we have

$$\varepsilon_t \le \frac{g_n}{g_t} \frac{\mu}{\alpha\sqrt{\kappa}} \,. \tag{27}$$

The above considerations are valid if the initial tangential energy is larger than $E_{\tan}^{\max}$, *i.e.*

$$g_t \ge g_n \frac{\mu}{\alpha\sqrt{\kappa}} \,. \tag{28}$$

Otherwise the initial kinetic energy of the tangential motion would not be sufficient to fully load the tangential spring. Note that there is no contradiction between equations (26) and (28), the former concerning g_t' and the latter concerning g_t. Combining both equations leads to the trivial but correct $|g_t'| \le g_t$. Furthermore, the result equation (27) is only valid if the tangential motion ceases at least once. As a rough estimate, there will be no stopping if

$$g_t \ge \frac{2\mu}{\alpha} g_n \,, \tag{29}$$

see equation (15) for $\varepsilon_n = 1$ and $\varepsilon_t \ge 0$. The actual limiting tangential velocity of pure Coulomb friction is even smaller. However, the limit equation (27) still holds, as, according to equation (15), the coefficient of tangential friction is zero for $g_t = 2\mu g_n/\alpha$ in the case of pure Coulomb friction. Note that in the whole discussion we assumed that there is no further energy loss after the first switch from the Coulomb regime to the Cundall-Strack regime. However, this assumption is not realistic as for high tangential stiffness there will be *many* episodes of Coulomb sliding with the accompanying energy loss (see Fig. 6, top panel). The resulting final tangential velocity will actually be much smaller than the estimate equation (27). Figure 6, bottom panel, illustrates the mechanism. Hence, for large tangential stiffness κ there is a range

$$g_n \frac{\mu}{\sqrt{\alpha\kappa}} \ll g_t \le \frac{2\mu}{\alpha} g_n \,, \tag{30}$$

where we can approximate the coefficient of tangential restitution in good accuracy by zero! Therefore, the pure Coulomb model, equation (15), as discussed before, is a good approximation for the velocity dependence of the tangential coefficient of restitution. This vindicates the pragmatic model $\varepsilon_t = \max[\varepsilon_t^{\text{const}}, 1 - \mu(1+\varepsilon_n)/\alpha \cdot g_n/g_t]$ (with $\varepsilon_t^{\text{const}}$ being a constant from the interval $[-1, 1]$) as used in the literature [20–23]. However, as we have seen in the course of the discussion, the only sensible choice of $\varepsilon_t^{\text{const}}$ for Cundall-Strack–like tangential interaction models is $\varepsilon_t^{\text{const}} = 0$.

5 Conclusions

We investigated the coefficient of tangential restitution, ε_t, for the dissipative collision of frictional viscoelastic spheres. The tangential force was described by the model by Cundall and Strack [7] which was shown to deliver rather realistic results in many Molecular Dynamics simulations of granular systems, *e.g.* [8–16]. Here the tangential force is modeled by means of a linear spring in the direction of the contact plane where the force is cut off due to the Coulomb friction law. It was shown that the dynamics of the system depends on 3 independent variables, including material constants and initial velocities of the collision. The detailed dynamics of the collision may be rather complex, revealing multiple transitions between (dissipative) Coulomb friction and (conservative loading and unloading) of the spring. Nevertheless, in a rather wide range of parameters, the collision is well described by simple Coulomb friction in tangential direction with a cut-off at $\varepsilon_t = 0$, thus, $0 \le \varepsilon_t \le 1$. Only for small ratio, g_t/g_n, of the tangential and normal components of the impact velocity the coefficient of tangential restitution shows a more complicated behavior, that is, it oscillates between

$\varepsilon_t = -1$ and $\varepsilon_t = 1$ in dependence on g_t/g_n and on κ being mainly a function of the particle properties, in particular the strength of the tangential spring. Only in this region we observe negative values of the coefficient of tangential restitution. For stiff tangential springs, asymptotically we obtain $\varepsilon_t \to 0$.

As a consequence, we obtain that the coefficient of tangential restitution as obtained from pure Coulomb friction, cut off at $\varepsilon \geq 0$, equation (16), is a rather good approximate description for the colliding spheres. A similar approximation where, instead of the limit of zero in equation (16) a constant from the interval $[-1, 1]$ was assumed, was already used in event-driven Molecular Dynamics simulations, *e.g.* [20–23], where the justification of this approximation was not considered.

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