Elastic Spheres in Contact Under Varying Oblique Forces[1]

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An investigation is made of the phenomena occurring at the contact surfaces of like elastic spheres subjected to a variety of applied forces. Owing to the presence of slip, with its associated energy dissipation and permanent set, the changes in tractions and displacements depend not only upon the initial state of loading, but upon the entire past history of loading and the instantaneous relative rates of change of the normal and tangential forces. On the basis of three of the simplest cases of varying oblique forces, a set of rules of procedure is assembled and then applied to two types of problems. In the first, the initial tangential compliances are calculated for a variety of past histories and instantaneous rates of loading; in the other, a detailed history of a varying oblique force is investigated.

1 Introduction

The mathematical study of the effects produced by mutual compression of elastic bodies was initiated by Hertz[4] who considered the case in which the forces between bodies are normal to the contact surfaces. Recent studies[5, 6, 7] of the effects of forces tangential to the contact surfaces have revealed the necessity for taking slip into account. This phenomenon, with its accompanying energy dissipation and permanent set, introduces nonlinearities of a different character than the Hertz nonlinearity. Not only do the changes in stresses and displacements depend upon the initial state of loading, but also upon the entire past history of loading and the instantaneous relative rates of change of the normal and tangential forces. For example, different phenomena are involved and different results obtained depending upon whether the normal or tangential force is held constant, while the other varies; whether they both vary, and whether the sense of the variation is such that one increases while the other decreases, both increase, or both decrease; whether the relative rate of change is greater or less than the coefficient of friction; whether the immediate past history of loading was in the same or opposite sense as the current loading. This situation makes it necessary to study a variety of special cases. The ones selected are those which seem likely to arise in applications and, at the same time, reveal new phenomena or require new techniques. In every case, the elastic bodies considered are identical spheres.

After a brief statement, in Section 2, of the results of the Hertz theory, three of the simplest cases of varying oblique forces are considered in Sections 3 to 5. In these, the normal force is held constant while the tangential force increases, decreases, and oscillates. On the basis of the considerations employed and the results obtained in the preceding cases, a set of rules for further procedure is stated in Section 6. The rules are then applied to two types of problems. In the first (Sections 7 to 13), the initial tangential compliances are calculated for a variety of past histories and instantaneous rates of loading. In the second (Sections 14 to 19), a detailed history of a varying oblique force is investigated. An integral is encountered which necessitates a definite choice of loading history in order to proceed further. The one selected is the case of an initial normal load, followed by an oblique force whose inclination remains constant while its magnitude varies. The force first increases, then decreases, and, finally, oscillates.

2 Normal Force Varying, Tangential Force Zero

This is the case to which the Hertz theory[4] is applicable. The theory predicts a plane, circular contact surface of radius

$$a = (KNR)^{1/3} \qquad [1]$$

where N is the normal force, R is the radius of the spheres, and $K = 3(1-\nu^2)/4E$, in which ν and E are Poisson's ratio and Young's modulus, respectively, of the material.

The distribution of normal traction on the contact surface is given by

$$\sigma = \frac{3N}{2\pi a^3}(a^2 - \rho^2)^{1/2} \qquad [2]$$

where ρ is the radial distance from the center of the contact surface.

The theory also gives the relative approach of the spheres

$$\alpha = 2(KN/R^{1/2})^{2/3} \qquad [3]$$

Hence the normal compliance of one sphere is

$$c_n = \frac{1}{2}\frac{d\alpha}{dN} = \frac{2}{3}\left(\frac{K^2}{RN}\right)^{1/3} = \frac{1-\nu}{4\mu a} \qquad [4]$$

where μ is the shear modulus of the material.

3 Normal Force Constant, Tangential Force Increasing

This is the case investigated by Cattaneo[5] and Mindlin.[6] The normal force is applied first, following which the tangential force

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[4] "Theory of Elasticity," by S. Timoshenko and J. N. Goodier, McGraw-Hill Book Company, Inc., New York, N. Y., 1951, p. 372.

[5] "Sul contatto di due Corpi Elastici," by C. Cattaneo, Accademia dei Lincei, Rendiconti, Series 6, vol. 27, 1938, pp. 342–348, 434–436, and 474–478.

[6] "Compliance of Elastic Bodies in Contact," by R. D. Mindlin, Journal of Applied Mechanics, Trans. ASME, vol. 71, 1949, pp. A-259–268.

[7] "Effects of an Oscillating Tangential Force on the Contact Surfaces of Elastic Spheres," by R. D. Mindlin, W. P. Mason, T. F. Osmer, and H. Deresiewicz, Proceedings of the First U. S. National Congress of Applied Mechanics, 1951, pp. 203–208.

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T increases monotonically from zero. Considerations of symmetry show that the distribution of normal traction is not altered during application of T, as long as the two spheres are alike.[8]

If it is assumed that no slip occurs, symmetry considerations lead to the conclusion that the tangential component of displacement at the contact surface is that of a rigid body. The situation is thus reduced to a mixed boundary-value problem in elasticity. On the contact surface, the normal component of traction (zero) and the tangential component of displacement (constant) are given. On the remainder of the surface of the sphere, approximated as plane, the three components of traction are given (all zero). Solution of this problem gives the tangential component of traction τ on the contact surface. This is everywhere parallel to the displacement (and the force T), is axially symmetric in magnitude and increases to infinity at the edge of the contact surface. In addition to yielding the initial tangential compliance, this solution reveals that slip must accompany tangential forces, as infinite tractions are required in the absence of slip.

Slip may be expected to start at the edge of the contact surface, where infinite tangential tractions would occur without slip, and progress radially inward. Since τ, in the absence of slip, is radially symmetric, it is reasonable to suppose that slip occurs on an annulus. Further, as a first approximation, it is assumed that the tangential traction on the annulus of slip is in the direction of the tangential force and has the magnitude $\tau = f\sigma$, where f is a constant coefficient of friction. On account of symmetry, the tangential component of displacement is constant on the portion of the contact surface on which no slip occurs (the "adhered portion"). Thus the situation is reduced to another mixed boundary-value problem in elasticity. The tangential displacement (constant) and the normal traction (zero) are given on the adhered portion and the traction is given over the remainder of the boundary (tangential traction $= f\sigma$, normal traction zero on the annulus of slip and all three components of traction zero outside). Solution of this problem[5,6] yields the inner radius of the annulus of slip

$$c = a\,(1 - T/fN)^{1/3} \qquad [5]$$

the distribution of tangential traction on the contact surface

$$\left.\begin{aligned} \tau &= \frac{3fN}{2\pi a^3}\,(a^2 - \rho^2)^{1/2}, \quad c \leqslant \rho \leqslant a \\ \tau &= \frac{3fN}{2\pi a^3}\,[(a^2 - \rho^2)^{1/2} - (c^2 - \rho^2)^{1/2}], \quad \rho \leqslant c \end{aligned}\right\} \qquad [6]$$

and the displacement of distant points with respect to the uniform displacement of the adhered portion

$$\left.\begin{aligned} \delta &= \frac{3(2 - \nu)fN}{16\mu a}\left(1 - \frac{c^2}{a^2}\right) \\ &= \frac{3(2 - \nu)fN}{16\mu a}\left[1 - \left(1 - \frac{T}{fN}\right)^{2/3}\right] \end{aligned}\right\} \qquad [7]$$

The distribution of tangential traction on the contact surface and the relation between tangential load and displacement are illustrated in Figs. 1 and 2, respectively. As T approaches fN, point c in Fig. 1 approaches the center of the contact surface (point O) in accordance with Equation [5]. For any $T < fN$ there corresponds a point on the curve in Fig. 2, in accordance with Equation [7]. When $T = fN$, the adhered portion of the contact surface has shrunk to zero and the displacement has reached the abscissa of point F in Fig. 2, at which point the displacement becomes indeterminate, as rigid-body sliding takes place over the whole contact surface. A distinction is thus made between slip and slide. The term slip is used here when there is relative tangential displacement over only a portion of contiguous surfaces. When T reaches the value for which the inner boundary of the slip surface shrinks to zero, there is relative displacement over the whole contact surface; this situation is termed slide.

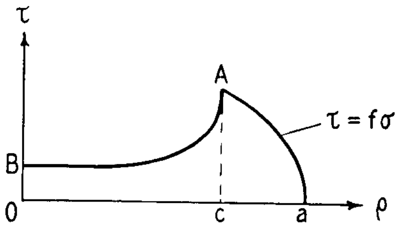

FIG. 1

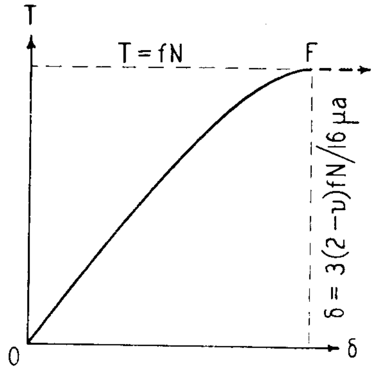

FIG. 2

The tangential compliance for the case of constant normal force and monotonically increasing tangential force is the reciprocal of the slope of the curve in Fig. 2 and is given by the formula

$$c_t = \frac{d\delta}{dT} = \frac{2 - \nu}{8\mu a}\left(1 - \frac{T}{fN}\right)^{-1/3} \qquad [8]$$

Thus the tangential compliance is $(2 - \nu)/8\mu a$ at $T = 0$ and is infinite at $T = fN$; i.e., free sliding takes place at the latter load.

The ratio of the initial tangential compliance, $(2 - \nu)/8\mu a$, to the normal compliance, Equation [4], is $(2 - \nu)/2(1 - \nu)$. Hence the ratio ranges from unity, for $\nu = 0$, to $3/2$, for $\nu = 1/2$.

4 NORMAL FORCE CONSTANT, TANGENTIAL FORCE DECREASING

Suppose, now, that after having reached a value T^*, where $0 < T^* < fN$, the tangential force T is reduced. If slip were prevented, the tangential traction τ would tend to negative infinity on the boundary $\rho = a$. This conclusion is reached from the solution of the appropriate boundary-value problem in elasticity, as described in the preceding section. Hence slip, opposite in sense to the initial slip, is presumed to start at $\rho = a$ and penetrate to a radius b, assumed, temporarily, to lie in the interval $c \leqslant b \leqslant a$. As in the preceding section, the tangential traction on the annulus $b \leqslant \rho \leqslant a$ is equal to $f\sigma$ but its sense is now opposite to that of the initial traction. Hence the change of traction over the annulus $b \leqslant \rho \leqslant a$ is $-2f\sigma$. Since no additional slip occurs on the surface $\rho \leqslant b$, the change of tangential displacement in that region must, from symmetry, be that of a rigid body. Thus the change due to reduction of T presents a boundary-value problem in elasticity identical in form with that

[8] A case in which the normal traction is changed as a result of application of a tangential force is considered by H. Poritzky, JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 72, 1950, pp. 191–201.

encountered in the preceding section. Hence the change in tangential traction, by analogy with Equations [6], is

$$\left.\begin{aligned} \tau_c &= -\frac{3fN}{\pi a^3}(a^2-\rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a \\ \tau_c &= -\frac{3fN}{\pi a^3}[(a^2-\rho^2)^{1/2}-(b^2-\rho^2)^{1/2}], \quad \rho \leqslant b \end{aligned}\right\} \quad ..[9]$$

This distribution is illustrated by curve a-A'-B' in Fig. 3.

The resultant traction accompanying a reduction in T is obtained by adding the initial traction, Equations [6], and the change, Equations [9], with the result

$$\left.\begin{aligned} \tau &= -\frac{3fN}{2\pi a^3}(a^2-\rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a \\ \tau &= -\frac{3fN}{2\pi a^3}[(a^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2}], \quad c \leqslant \rho \leqslant b \\ \tau &= -\frac{3fN}{2\pi a^3}[(a^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2} \\ &\qquad + (c^2-\rho^2)^{1/2}], \quad \rho \leqslant c \end{aligned}\right\} \quad ..[10]$$

This is illustrated, in Fig. 3, by curve a-D-E-F, which is the sum of a-A-B and a-A'-B'.

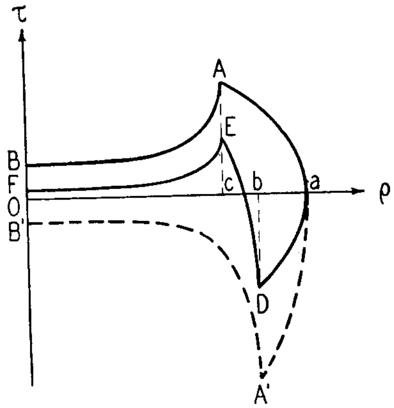

FIG. 3

further decrease of T to $-fN$ the phenomenon is again the same as the one described in the preceding section for a monotonic T.

The associated displacement of the adhered portion is found by a similar process of superposition. The change in displacement is obtained by multiplying the first of Equations [7] by -2 and replacing c by b as given in Equation [13]. The initial displacement is given by the second of Equations [7] with T replaced by T^*. The sum of the two is

$$\begin{aligned} \delta_d &= \frac{3(2-\nu)fN}{16\mu a}\left(2\frac{b^2}{a^2}-\frac{c^2}{a^2}-1\right) \\ &= \frac{3(2-\nu)fN}{16\mu a}\left[2\left(1-\frac{T^*-T}{2fN}\right)^{2/3}-\left(1-\frac{T^*}{fN}\right)^{2/3}-1\right] \end{aligned} \quad ....[14]$$

The tangential load-displacement curve is illustrated by the full lines in Fig. 4.

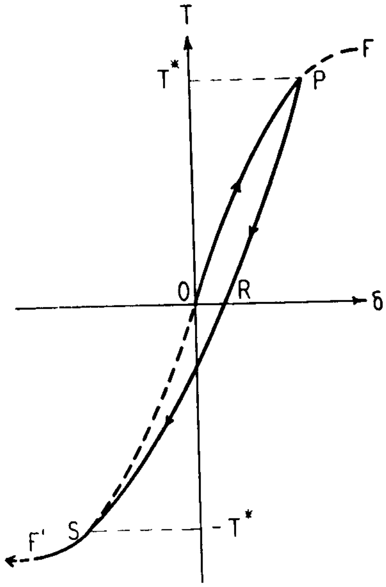

FIG. 4

The equilibrium condition

$$T = \int_0^{2\pi}\int_0^a \tau\, \rho d\rho d\theta \quad ....[11]$$

yields, upon insertion of τ from Equations [10]

$$T = \frac{fN}{a^3}(a^3-c^3) - \frac{2fN}{a^3}(a^3-b^3) \quad ....[12]$$

Noting, from Equation [5], that the first term on the right of Equation [12] is the initial tangential force T^*, the depth of penetration of counterslip is

$$b = a\left(1-\frac{T^*-T}{2fN}\right)^{1/3} \quad ....[13]$$

Thus the assumption $c \leqslant b \leqslant a$ is valid as long as $-T^* \leqslant T \leqslant T^*$. When $T = -T^*$, i.e., when the tangential load is fully reversed, $b = c$, i.e., counterslip has penetrated to the depth of the initial slip. At the same time, Equations [10] reduce to Equations [6] with signs reversed. The traction is then distributed just as the initial traction at $T = T^*$ was, but with opposite sense. The situation is the same as if no positive T had ever been applied, but only a negative T of magnitude $-T^*$. Upon

The compliance for unloading is

$$\frac{d\delta_d}{dT} = \frac{2-\nu}{8\mu a}\left(1-\frac{T^*-T}{2fN}\right)^{-1/3} \quad ....[15]$$

It is interesting to note that the initial compliance on unloading ($T = T^*$ in Equation [15]) is the same as the initial compliance on loading ($T = 0$ in Equation [8]); that is, in Fig. 4, the slope of PR at P is the same as the slope of OP at O.

When T has been reduced from T^* to zero, there is a permanent set given by OR, in Fig. 4, the magnitude of which is obtained by setting $T = 0$ in Equation [14]. The accompanying traction is not zero, but is a self-equilibrating distribution obtained by setting $b = a(1-T^*/2fN)^{1/3}$ in Equations [10].

When T has been reduced to $-T^*$, the displacement has reached the negative of the displacement at $T = T^*$, i.e., the abscissa of S in Fig. 4 is the negative of the abscissa of P, and the compliance is identical with that of curve OP at $T = T^*$. Hence the unloading curve P-R-S is tangent, at S, to the central perversion OS, of the loading curve OP. Upon further decrease of T^* to $-fN$, the unloading curve follows the reflected loading curve SF', in accordance with the results for monotonic T given in Section 3.

5 Normal Force Constant, Tangential Force Oscillating

In the preceding section the entire situation at $T = -T^*$ is identical with that at $T = T^*$, except for reversal of sign. Hence a subsequent increase of T from $-T^*$ to T^* will be accompanied by the same events as occurred during the decrease from T^* to $-T^*$, except for reversal of sign. Thus, in starting along SU, Fig. 5, the compliance at S is the same as the compliance at P of PR. Slip again starts at $\rho = a$ in the same sense as occurred along path OP. At an intermediate point of SU, the traction will be like a-D-E-F, Fig. 3, with sign reversed. When the tangential force once more reaches T^*, b will again have penetrated to c and the traction will be exactly a-A-B.

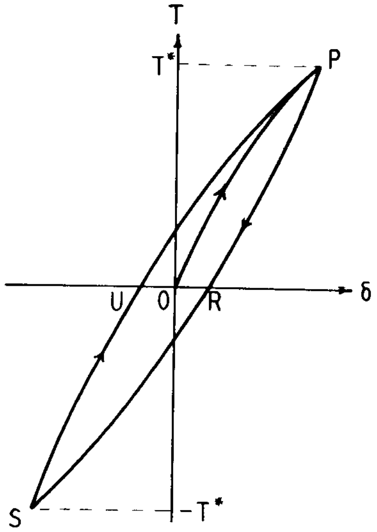

Fig. 5

The displacement along path S-U-P is

$$\delta_i = -\delta_d(-T) = -\frac{3(2-\nu)fN}{16\mu a}\left[2\left(1-\frac{T^*+T}{2fN}\right)^{2/3} - \left(1-\frac{T^*}{fN}\right)^{2/3} - 1\right] \qquad [16]$$

Hence, when $T = T^*$, the terminal points P of S-U-P and P of OP, Fig. 5, are identical, as may be seen by comparing Equation [16] with Equation [7] for that value of T.

Subsequent diminution of T will then produce a repetition of the events accompanying the first diminution.

It is now evident that during oscillation of T between $\pm T^*$, with N maintained constant, the load-displacement curve traverses the loop P-R-S-U-P, Fig. 5.

The area enclosed in the loop gives the frictional energy loss per cycle

$$F = \int_{-T^*}^{T^*} (\delta_d - \delta_i)dT$$
$$= \frac{9(2-\nu)f^2N^2}{10\mu a}\left\{1 - \left(1-\frac{T^*}{fN}\right)^{5/3} - \frac{5T^*}{6fN}\left[1 + \left(1-\frac{T^*}{fN}\right)^{2/3}\right]\right\} \qquad [17]$$

For small T^*/fN, Equation [17] reduces to

$$F = \frac{(2-\nu)T^{*3}}{36\mu a f N} \qquad [18]$$

i.e., the energy loss per cycle varies as the cube of the maximum tangential force.

Some of the conclusions reached in this section have been subjected to experimental test.[7] The occurrence of the annulus of slip has been verified, as well as the constancy of the coefficient of friction and the relation between the size of the inner radius of the annulus and the magnitude of T^*. In the tests, the energy dissipation, for magnitudes of T^*/fN close to unity, agreed with the values predicted by the theory; for T^*/fN in the vicinity of zero, however, it appeared to vary more nearly as the square than the cube of the maximum tangential force.

6 Rules of Procedure

The results of the two preceding sections have been obtained by means of extensions of Cattaneo's and Mindlin's solutions for the case of constant N and monotonically increasing T. Before proceeding to cases of greater complexity, it may be helpful to set down, explicitly, the rules upon which the two preceding and the subsequent developments are based:

Rule 1. The radius of the contact surface and the normal component of traction on it are given by the Hertz formulas, Equations [1] and [2].

Rule 2. With every application or change of tangential force, slip will be initiated whenever, in the absence of slip, the solution of the appropriate boundary-value problem in the theory of elasticity yields a tangential traction, at any point, in excess of the product of a constant coefficient of friction and the normal component of traction at that point.

Rule 3. Slip, in the direction of the force causing it,[9] progresses concentrically, radially inward from the boundary of the contact surface, forming an "annulus of slip."

Rule 4. At any point on a contact surface, the magnitude of the tangential component of traction is at most equal to the product of a constant coefficient of friction and the normal component of traction at that point. The equality necessarily holds at a point at which slip has just occurred, in which case the traction has the same sense as the slip.[9]

Rule 5. The adhered portion of the contact surface, i.e., the portion encircled by an annulus on which slip occurs, is subjected to a change of tangential traction and undergoes a rigid-body tangential displacement. The radius of the adhered portion, the distribution of the traction and the magnitude of the displacement are obtained from Cattaneo's and Mindlin's formulas, Equations [5], [6], [7].

Rule 6. Beginning with an equilibrium position, for which the displacement and the distribution of traction have been established in accordance with the preceding rules, the effects of a change in the state of loading are obtained by advancing to the desired state through a sequence of equilibrium positions, each of which is obtained from its predecessor by applying Rules 1 to 5. In particular, the condition of equilibrium, Equation [11], is useful.

In the following sections these rules are applied to the calculation of tangential compliances for a number of past histories, initial states, and variations of both N and T.

7 N Increasing, T Increasing

Suppose that, after an initial state has been reached by first applying a normal force N and then a tangential force T, both N and T are to increase at an arbitrary relative rate.

The initial state is given by Equations [1], [2], [5], [6], and [7]. The distribution of tangential traction is illustrated by

[9] There is a small lateral component of relative tangential displacement which accompanies the major slip in the direction of the applied force. The lateral tangential traction, which accompanies the lateral slip, is neglected.

curve a-A-B in Fig. 6, the tangential load-displacement history by curve OA in Fig. 7.

Now, increase N by an increment ΔN. The radius of the contact surface increases to a value a_1 given by Equation [1] with N replaced by $N + \Delta N$

$$a_1 = [K(N + \Delta N)R]^{1/3} \qquad [19]$$

The tangential traction remains unchanged.

With the normal load held constant at $N + \Delta N$, increase the tangential force by an increment $\Delta T < f\Delta N$. In accordance with the rules of Section 6, slip begins at $\rho = a_1$ and progresses inward to a radius c_1 given by Equation [5] with c, a, T, and N replaced by c_1, a_1, ΔT_1, and $N + \Delta N$, respectively

$$c_1 = a_1\,[1 - \Delta T/f(N + \Delta N)]^{1/3} \qquad [20]$$

From Equations [1] and [19]

$$a^3(N + \Delta N) = a_1^3 N \qquad [21]$$

Hence

$$c_1 = a(fN)^{-1/3}\,(fN + f\Delta N - \Delta T)^{1/3} \qquad [22]$$

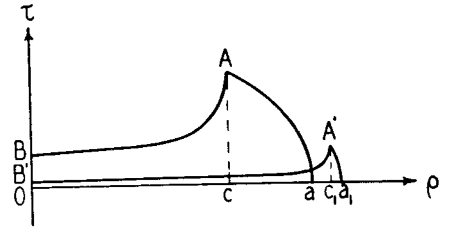

Fig. 6

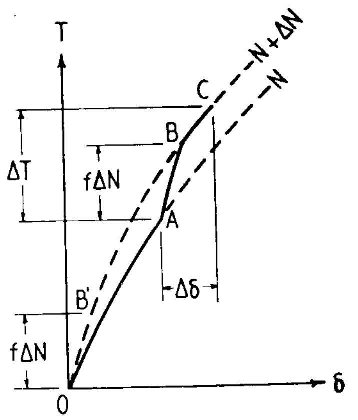

Fig. 7

The same replacements in Equations [6] and [7] give the increments in traction and displacement, the former of which is illustrated by curve a_1-A'-B' in Fig. 6. The load-displacement relation, during the increment, is illustrated by curve AB, in Fig. 7. The slope of AB, at A, is $8\mu a_1/(2 - \nu)$. These conclusions may be ascertained from Equation [7] and its derivative with respect to ΔT, after the appropriate replacements have been made.

Now increase ΔT to $f\Delta N$. From Equation [22], c_1 progresses in to a; i.e., the inner radius of the incremental annulus of slip shrinks to the radius of the initial contact surface. The increment in tangential traction is now given by

$$\left.\begin{aligned} \Delta\tau &= \frac{3f(N + \Delta N)}{2\pi a_1^3}\,(a_1^2 - \rho^2)^{1/2}, \quad a \leqslant \rho \leqslant a_1 \\ \Delta\tau &= \frac{3f(N + \Delta N)}{2\pi a_1^3}\,[(a_1^2 - \rho^2)^{1/2} - (a^2 - \rho^2)^{1/2}], \; \rho \leqslant a \end{aligned}\right\} \qquad [23]$$

and is illustrated by curve a_1-A'-B' in Fig. 8. The total tangential traction is the sum of Equations [6] and [23]. Using Equation [21], this becomes

$$\left.\begin{aligned} \tau_R &= \tau + \Delta\tau = \frac{3f(N + \Delta N)}{2\pi a_1^3}\,(a_1^2 - \rho^2)^{1/2}, \; c \leqslant \rho \leqslant a_1 \\ \tau_R &= \frac{3f(N + \Delta N)}{2\pi a_1^3}\,[(a_1^2 - \rho^2)^{1/2} - (c^2 - \rho^2)^{1/2}], \; \rho \leqslant c \end{aligned}\right\} \qquad [24]$$

But this is the distribution that would have been obtained if the entire history had been an initial application of normal force $N + \Delta N$ followed by a single application of tangential force $T + f\Delta N$. Thus the sum of curves a-A-B and a_1-A'-B', in Fig. 8,

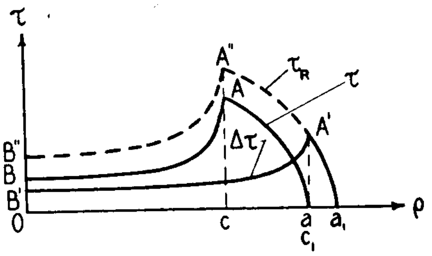

Fig. 8

is a_1-A''-B'', which is the curve corresponding to Equations [24]. Further increase in ΔT will then produce tractions in accordance with the findings in Section 3 for monotonic increase in T.

Further, when $\Delta T = f\Delta N$, the total displacement is the same as would have been obtained if an initial normal force $N + \Delta N$ had been followed by a tangential force $T + f\Delta N$; i.e., point B of AB, in Fig. 7, coincides with point B of the load-displacement curve, O-B'-B, for initial normal force $N + \Delta N$. Upon further increase in T, the load-displacement curve follows path BC.

The compliance at A of curve AB, in Fig. 7, is

$$\frac{2 - \nu}{8\mu a_1}$$

The compliance at B of curve O-B-C is

$$\frac{2 - \nu}{8\mu a_1}\left(1 - \frac{T}{f(N + \Delta N)}\right)^{-1/3}$$

Hence, for small $\Delta T \geqslant f\Delta N$, the increment in displacement is

$$\Delta\delta = \frac{2 - \nu}{8\mu a}\left[f\Delta N + \left(1 - \frac{T}{f(N + \Delta N)}\right)^{-1/3}(\Delta T - f\Delta N)\right]$$

The compliance at $(N,\, T)$, for N and T increasing, is then

$$c_t = \lim_{\substack{\Delta N \to 0 \\ \Delta T \to 0}} \frac{\Delta\delta}{\Delta T} = \frac{2 - \nu}{8\mu a}\left[f\frac{dN}{dT} + \left(1 - f\frac{dN}{dT}\right)\left(1 - \frac{T}{fN}\right)^{-1/3}\right], \; 0 \leqslant \frac{dN}{dT} \leqslant \frac{1}{f} \qquad [25]$$

If $\Delta T < f\Delta N$, the load-displacement curve does not extend beyond B in Fig. 7. Hence

$$\Delta\delta = \frac{2 - \nu}{8\mu a}\,\Delta T$$

and the compliance is

$$c_t = \frac{2 - \nu}{8\mu a}, \qquad \frac{dN}{dT} \geqslant \frac{1}{f} \qquad [26]$$

8 N Decreasing, T Increasing

Starting with the same initial state as in Section 7, it is desired first to decrease N by an amount ΔN, with the tangential load held constant. Such a decrease would reduce the contact radius to a value a_1, given by Equation [1] with N replaced by $N - \Delta N$. However, the portions of the spheres in annulus $a_1 \leqslant \rho \leqslant a$ would no longer be in contact, and would be unable to sustain tangential traction. Consequently, before a reduction of normal load may be effected, it is necessary to remove the existing tangential traction from this region. This removal may be accomplished by means of the following procedure:

(a) The contact area $\rho \leqslant a_1$ is "frozen"; i.e., no slip is permitted to take place.

(b) A distribution of traction is added, such that it will cause the annulus $a_1 \leqslant \rho \leqslant a$ to become free of traction, and, at the same time, obey the injunction stated in (a).

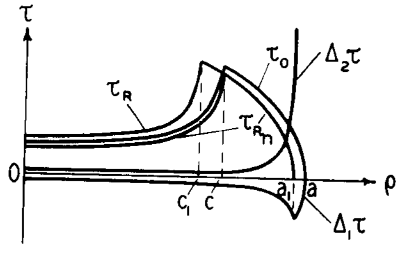

Fig. 9

Such a distribution (see Fig. 9) is given by

$$\left.\begin{aligned} \Delta_1\tau &= -\frac{3fN}{2\pi a^3}(a^2-\rho^2)^{1/2}, \quad a_1 \leqslant \rho \leqslant a \\ \Delta_1\tau &= -\frac{3fN}{2\pi a^3}[(a^2-\rho^2)^{1/2}-(a_1^2-\rho^2)^{1/2}], \ \rho \leqslant a_1 \end{aligned}\right\} \quad \ldots [27]$$

The resultant traction (which, it should be noted, corresponds to a condition of equilibrium with the force $T - f\Delta N$, but not with T) will be

$$\left.\begin{aligned} \tau_{R_n} &= 0, \quad a_1 \leqslant \rho \leqslant a \\ \tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad c \leqslant \rho \leqslant a_1 \\ \tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-(c^2-\rho^2)^{1/2}], \rho \leqslant c \end{aligned}\right\} \quad \ldots [28]$$

Since annulus $a_1 \leqslant \rho \leqslant a$ is now free of traction, it is permissible to decrease N by ΔN. The distribution of traction, Equations [28], is the same as if the order of loading had been: (1) $N - \Delta N$, (2) $T_1 = f(N - \Delta N)\ (1 - c^3/a_1^3) = T - f\Delta N$, where the value of T_1 is obtained by applying the integral condition, Equation [11]. The displacement may be thought to follow path AB, Fig. 10.

A portion of the tangential force, represented by the volume under the traction distribution, Equations [27], was removed in the course of the reduction of the normal load. But the spheres must be in equilibrium with T; hence the force $f\Delta N$ is transferred to the remaining contact surface (i.e., $\rho \leqslant a_1$). In consequence of the assumption stated in (a) of this section, the additional tangential traction[10] resulting from this transfer is given by (see Fig. 9)

$$\Delta_2\tau = \frac{f\Delta N}{2\pi a_1}(a_1^2-\rho^2)^{-1/2}, \quad \rho \leqslant a_1 \ldots\ldots\ldots [29]$$

[10] See Equation [76], reference (6).

This is the distribution, referred to in Section 3, which increases without limit as ρ approaches a_1.

Now unfreeze the contact surface. Slip will progress in the direction of the applied tangential force in accordance with the fundamental rules. The resultant tangential traction will be distributed as given by Equations [28] with c replaced by c_1, where

$$c_1 = a_1\left(1 - \frac{T}{f(N-\Delta N)}\right)^{1/3}$$

The displacement may be thought to take path BC in Fig. 10.

Now, holding the normal force constant (at $N - \Delta N$), increase

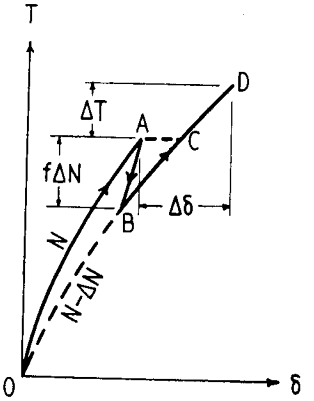

Fig. 10

the tangential force by an amount ΔT. At this point the tangential traction will be given by Equations [28] with c replaced by c_2, where

$$c_2 = a_1\left(1 - \frac{T+\Delta T}{f(N-\Delta N)}\right)^{1/3} < c_1$$

It is interesting that the resultant traction has, qualitatively, the same form as the initial traction, Equations [6].

The displacement will follow path CD on curve O-B-D, Fig. 10. Hence the compliance will be

$$c_t = \lim_{\substack{\Delta N\to 0\\ \Delta T\to 0}}\left\{\frac{1}{\Delta T}\frac{2-\nu}{8\mu a}\left[-f\Delta N + \left(1-\frac{T}{fN}\right)^{-1/3}(f\Delta N+\Delta T)\right]\right\}$$
$$= \frac{2-\nu}{8\mu a}\left[-f\frac{dN}{dT} + \left(1+f\frac{dN}{dT}\right)\left(1-\frac{T}{fN}\right)^{-1/3}\right] \ldots [30]$$

Since, inherently, $dN/dT < 0$, its absolute value must be used in this formula.

9 N Increasing, T Decreasing

1 Apply N, increase the tangential force monotonically from zero to T^*, then reduce it to T. The tangential traction is given by Equations [10]; its distribution is shown by τ_0 in Fig. 11.

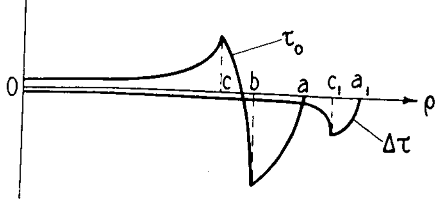

Fig. 11

2 Keeping the tangential load constant, increase the normal load to $N + \Delta N$. The radius a_1 of the new contact area is given by Equation [19]. The tangential traction remains unchanged.

3 Keeping the normal force constant (at $N + \Delta N$), reduce the tangential force by ΔT. The additional tangential traction (see $\Delta\tau$, Fig. 11) will be given by Equations [23] with sign reversed.

Again, if $\Delta T = f\Delta N$, so that $c_1 = a$ (Fig. 11), the resultant traction will be

$$\left.\begin{aligned}
\tau_R &= -\frac{3f(N+\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a_1 \\
\tau_R &= -\frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2}], \\
&\qquad\qquad c \leqslant \rho \leqslant b \\
\tau_R &= -\frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2} \\
&\qquad\qquad + (c^2-\rho^2)^{1/2}], \ \rho \leqslant c
\end{aligned}\right\} \quad \ldots [31]$$

Hence the situation is the same as that obtained by (1) imposing a normal load $N + \Delta N$, (2) applying a tangential load T_1^*, (3) reducing the tangential load to $T_1 = T - f\Delta N$. T_1^* may be found by noting that $T_1^* = f(N + \Delta N)(1 - c^3/a_1^3) = T^* + f\Delta N$.

4 The displacement (see Equation [14]) at the conclusion of step 1 will have traversed the path O-A-B (Fig. 12); following step 3 it will have reached point C, at which stage path O-A-B-C meets path O-A'-C.

If $\Delta T > f\Delta N$, splitting the decrement into $f\Delta N$ and $\Delta T - f\Delta N$ and applying these parts consecutively will yield a result entirely similar to that found in part 3 of this section. The additional displacement will follow path CD along curve $A'D$ (Fig. 12).

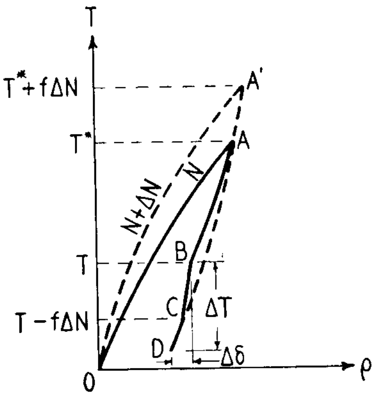

FIG. 12

Remembering that, again, inherently, $dN/dT < 0$, so that its absolute value must be employed, the compliance is

$$\left.\begin{aligned}
c_t &= \lim_{\substack{\Delta N\to 0\\ \Delta T\to 0}}\left\{\frac{1}{\Delta T}\frac{2-\nu}{8\mu a}\left[f\Delta N + \left(1-\frac{T^*-T}{2fN}\right)^{-1/3}(\Delta T - f\Delta N)\right]\right\} \\
&= \frac{2-\nu}{8\mu a}\left[f\frac{dN}{dT} + \left(1-f\frac{dN}{dT}\right)\left(1-\frac{T^*-T}{2fN}\right)^{-1/3}\right], \quad 0 > \frac{dN}{dT} \geqslant -\frac{1}{f} \\
c_t &= \frac{2-\nu}{8\mu a}, \quad \frac{dN}{dT} \leqslant -\frac{1}{f}
\end{aligned}\right\} \quad \ldots [32]$$

10 *N* DECREASING, *T* DECREASING

1 Repeat step 1 of Section 9.

2 A decrease of normal load by an amount ΔN must be preceded by the expedients discussed in (*a*) and (*b*) of Section 8, i.e., a freezing of the contact area, and freeing of the annulus $a_1 \leqslant \rho \leqslant a$ of tangential traction by adding, with sign reversed, $\Delta\tau$ of Equations [27]. The resultant traction, which is not in equilibrium with the force T, will be (Fig. 13)

$$\left.\begin{aligned}
\tau_{R_n} &= 0, \quad a_1 \leqslant \rho \leqslant a \\
\tau_{R_n} &= -\frac{3f(N-\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \ b \leqslant \rho \leqslant a_1 \\
\tau_{R_n} &= -\frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2}], \\
&\qquad\qquad c \leqslant \rho \leqslant b \\
\tau_{R_n} &= -\frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b^2-\rho^2)^{1/2} \\
&\qquad\qquad + (c^2-\rho^2)^{1/2}], \ \rho \leqslant c
\end{aligned}\right\} \quad \ldots [33]$$

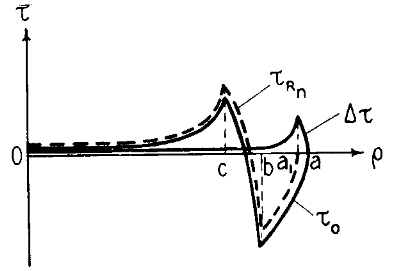

FIG. 13

The decrease in normal load is effected concurrently with the removal of the traction. This distribution may be reached by: (1) applying $N - \Delta N$, (2) applying $T_1^* = T^* - f\Delta N$, (3) reducing from T_1^* to $T + f\Delta N$. The transferred load (see Section 8) gives rise to a distribution of traction given by Equation [29] with sign reversed.

3 Unfreeze: slip will progress in the direction opposite to that of the initial tangential force. The displacement may be thought to take path B-B'-C (Fig. 14). The traction will be given by Equations [33] with b replaced by b_1, where

$$b_1 = a_1\left(1 - \frac{T^*-T}{2f(N-\Delta N)} + \frac{\Delta N}{2(N-\Delta N)}\right)^{1/3}$$

4 Further displacement, due to a decrease of tangential load at constant normal load ($N - \Delta N$), will proceed along curve A'-B'-C to point D. Thus the compliance will be

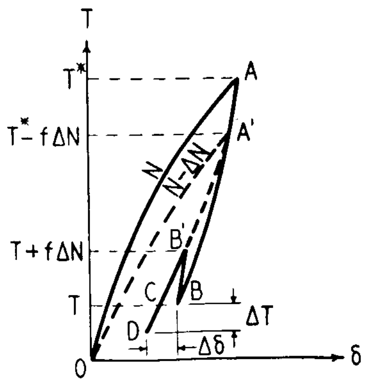

FIG. 14

$$c_t = \lim_{\substack{\Delta N \to 0 \\ \Delta T \to 0}} \left\{ \frac{1}{\Delta T} \frac{2-\nu}{8\mu a} \left[-f\Delta N + \left(1 - \frac{T^* - T}{2fN}\right)^{-1/3} (f\Delta N + \Delta T) \right] \right\}$$

$$= \frac{2-\nu}{8\mu a} \left[-f\frac{dN}{dT} + \left(1 + f\frac{dN}{dT}\right)\left(1 - \frac{T^* - T}{2fN}\right)^{-1/3} \right], \quad \frac{dN}{dT} \geqslant 0 \quad \ldots [34]$$

In the preceding four sections the discussion concerned tangential compliances due to simultaneously varying normal and tangential forces applied at nonzero levels of increasing or decreasing tangential load, in particular, at levels having a given previous history in the direction of the additional tangential force. In each case, the magnitude of the initial tangential force at these levels was reached at constant initial normal load.

In addition, it is of interest to find the values of compliances resulting from oblique loading at levels of tangential load which have no previous loading history in the direction of the tangential component of the system of oblique forces. The cases

(*a*) N increasing, T increasing, at $T^* = 0$
(*b*) N decreasing, T increasing, at $T^* = 0$
(*c*) N increasing, T decreasing, at $T = T^*$

are no more than special instances of loading considered in Sections 7, 8, and 9, respectively. The compliances are obtained directly from the general expressions given in the corresponding sections. On the other hand, there are several cases of interest for which the results cannot be obtained from previously considered cases. These are considered in Sections 11 to 13.

11 N Decreasing, T Decreasing, at $T = T^*$

1 Repeat step 1 of Section 9, with the proviso $T^* - T = \Delta T$ is an infinitesimal.

2 The reduction of the normal load by ΔN again must be preceded by freezing the contact surface and removal of traction, as described in Sections 8 and 10. Here, however, it is necessary to treat separately the cases $\Delta T \geqslant 2f\Delta N$ and $\Delta T < 2f\Delta N$.

(*a*) $\Delta T \geqslant 2f\Delta N$. Reference to Equations [1] and [13] reveals that

$$\left(\frac{b}{a}\right)^3 = 1 - \frac{\Delta T}{2fN} \leqslant 1 - \frac{\Delta N}{N} = \left(\frac{a_1}{a}\right)^3$$

i.e., $a_1 \geqslant b$. Hence the situation is the same as in the general case (Section 10); the compliance, a special case of Equation [34], is given by

$$c_t = \frac{2-\nu}{8\mu a}$$

The connection between this loading and the one discussed in Section 10 may be established by noting that, in Section 10, the load is decreased from T^* with $\Delta N = 0$, i.e., $dT/dN = \infty > 2f$.

(*b*) $\Delta T < 2f\Delta N$. In this case

$$\left(\frac{b}{a}\right)^3 = 1 - \frac{\Delta T}{2fN} > 1 - \frac{\Delta N}{N} = \left(\frac{a_1}{a}\right)^3$$

i.e., $a_1 < b$.

Now freeze the contact area. In order to free annulus $a_1 \leqslant \rho \leqslant a$ of tangential traction, add the following distributions of traction (Fig. 15) neither of which causes slip over circle $\rho \leqslant a_1$

$$\left.\begin{aligned} \Delta_1\tau &= \frac{3fN}{\pi a^3}(a^2 - \rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a \\ \Delta_1\tau &= \frac{3fN}{\pi a^3}[(a^2-\rho^2)^{1/2} - (b^2-\rho^2)^{1/2}], \quad \rho \leqslant b \end{aligned}\right\} \ldots [35]$$

$$\left.\begin{aligned} \Delta_2\tau &= -\frac{3fN}{2\pi a^3}(a^2 - \rho^2)^{1/2}, \quad a_1 \leqslant \rho \leqslant a \\ \Delta_2\tau &= -\frac{3fN}{2\pi a^3}[(a^2-\rho^2)^{1/2} - (a_1^2-\rho^2)^{1/2}], \quad \rho \leqslant a_1 \end{aligned}\right\} \ldots [36]$$

The resultant traction (not in equilibrium with T), found by adding tractions of Equations [10], [35], and [36], is

$$\left.\begin{aligned} \tau_{R_n} &= 0, \quad a_1 \leqslant \rho \leqslant a \\ \tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}(a_1^2 - \rho^2)^{1/2}, \quad c \leqslant \rho \leqslant a_1 \\ \tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2} - (c^2-\rho^2)^{1/2}], \quad \rho \leqslant c \end{aligned}\right\} \ldots [37]$$

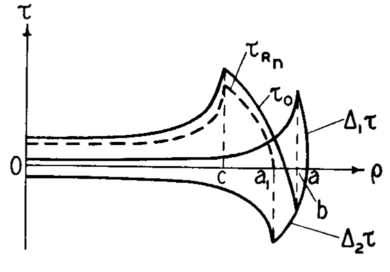

Fig. 15

Again, the normal load may be reduced by ΔN, provided that, concurrently, a traction distribution, similar to Equation [29], is applied over the area $\rho < a_1$. Its magnitude will be such as to provide the tangential force contained in the difference of volumes under the traction surfaces given by Equations [35] and [36]. Hence

$$\Delta\tau = -\frac{\Delta T - f\Delta N}{2\pi a_1}(a_1^2 - \rho^2)^{-1/2}, \quad \rho < a_1 \ldots\ldots [38]$$

The traction of Equations [37] is the same as if the order of loading had been: (1) $N - \Delta N$, (2) $T_1^* = T^* - f\Delta N$.

Now unfreeze; slip will progress in the direction indicated by the sign of the traction of Equation [38], i.e., for $\Delta T < f\Delta N$, in the direction of initial tangential force, for $f\Delta N < \Delta T < 2f\Delta N$, in the opposite direction. In the former case, the path of the displacement may be thought to be *O-A-B-A-C-D* (Fig. 16); the expression for the compliance is

$$c_t = \lim_{\substack{\Delta N \to 0 \\ \Delta T \to 0}} \left\{ \frac{1}{\Delta T} \left[\frac{2-\nu}{8\mu a} f\Delta N - \frac{2-\nu}{8\mu a_1}\left(1 - \frac{T_1^*}{f(N-\Delta N)}\right)^{-1/3} (f\Delta N - \Delta T) \right] \right\}$$

$$= \frac{2-\nu}{8\mu a}\left[f\frac{dN}{dT} + \left(1 - f\frac{dN}{dT}\right)\left(1 - \frac{T^*}{fN}\right)^{-1/3} \right], \quad \frac{dN}{dT} > \frac{1}{f} \ldots\ldots [39]$$

In the latter case, the displacement may be thought to traverse path O-A-B-A-C-D in Fig. 17, where CD is part of an unloading path such as $A'B'$ in Fig. 14. The compliance is $c_t = (2-\nu)/8\mu a$; thus this expression is valid for all $0 \leqslant dN/dT \leqslant 1/f$.

The traction of Equation [38] vanishes for $\Delta T = f\Delta N$, indicating that, in this instance, Equations [37] correspond to a condition of equilibrium with the force T; hence no slip accompanies unfreezing.

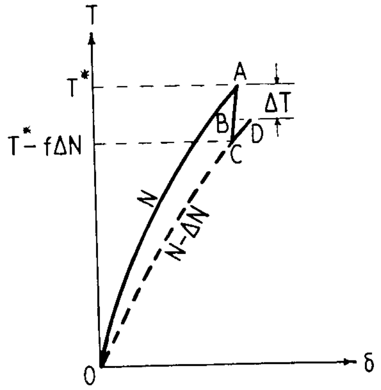

FIG. 16

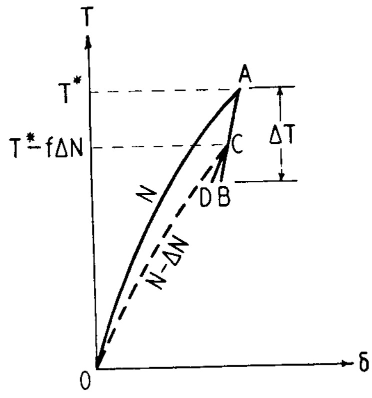

FIG. 17

12 N INCREASING, T INCREASING, AT POINT ON UNLOADING CURVE

1 Repeat steps 1 and 2 of Section 9.

2 Keeping the normal force constant (at $N + \Delta N$), increase the tangential force by ΔT.

Three cases arise, depending on whether $\Delta T \gtreqless f\Delta N$.

(a) $\Delta T = f\Delta N$. The additional traction will be such as to cause slip, in the direction of the newly applied force, in an annulus $c_1 \leqslant \rho \leqslant a_1$ and no further slip on circle $\rho \leqslant c_1$. Hence

$$\left.\begin{aligned} \Delta\tau &= \frac{3f(N+\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad c_1 \leqslant \rho \leqslant a_1 \\ \Delta\tau &= \frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-(c_1^2-\rho^2)^{1/2}], \\ &\qquad \rho \leqslant c_1 \end{aligned}\right\} \quad ..[40]$$

where

$$c_1 = a_1\left(1-\frac{\Delta T}{f(N+\Delta N)}\right)^{1/3}$$

Since $\Delta T = f\Delta N$, $c_1 = a$; hence the resultant traction (the sum of Equations [10] and [40]) becomes (see Fig. 18)

$$\left.\begin{aligned} \tau_R &= \frac{3f(N+\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad a \leqslant \rho \leqslant a_1 \\ \tau_R &= \frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(a^2-\rho^2)^{1/2}], \\ &\qquad b \leqslant \rho \leqslant a \\ \tau_R &= \frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(a^2-\rho^2)^{1/2} \\ &\qquad + 2(b^2-\rho^2)^{1/2}], \ c \leqslant \rho \leqslant b \\ \tau_R &= \frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(a^2-\rho^2)^{1/2} \\ &\qquad + 2(b^2-\rho^2)^{1/2}-(c^2-\rho^2)^{1/2}], \ \rho \leqslant c \end{aligned}\right\} \quad ..[41]$$

Hence the situation is the same as if the order of loading had been: (1) $N + \Delta N$, (2) $\underline{T_1}^*$, (3) reduction of tangential load to T_1, (4) its increase by $\overline{\Delta T}$. It may be noted that

$$A — \qquad T_1^* = f(N+\Delta N)\left(1-\frac{c^3}{a_1^3}\right)$$
$$= T^* + f\Delta N$$

$$B — \qquad T_1^* - T_1 = 2f(N+\Delta N)\left(1-\frac{b^3}{a_1^3}\right)$$
$$= T^* - T + 2f\Delta N$$
$$\therefore T_1 = T - f\Delta N$$

$$C — \qquad \overline{\Delta T} = 2f(N+\Delta N)\left(1-\frac{a^3}{a_1^3}\right)$$
$$= 2f\Delta N$$

Thus, on the load-displacement curve, Fig. 19, path O-A-B-C-C' may be replaced by path O-A'-B'-C'. The compliance is given by

$$c_t = \frac{2-\nu}{8\mu a}$$

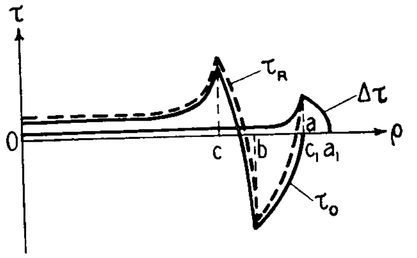

FIG. 18

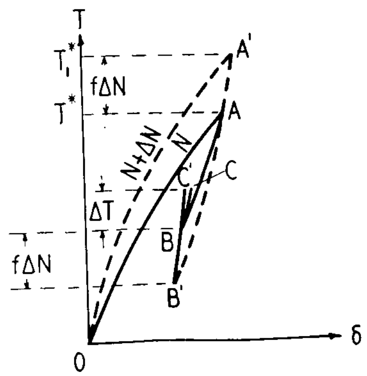

FIG. 19

(b) $\Delta T > f\Delta N$ ($c_1 < a$). If the increment ΔT is applied in two steps: $\Delta_1 T = f\Delta N$, $\Delta_2 T = \Delta T - f\Delta N$, in the usual manner, the course of events will be found the same as in (a) of this section, with the exception that $\overline{\Delta T} = \Delta T + f\Delta N$. Hence, again

$$c_t = \frac{2-\nu}{8\mu a}$$

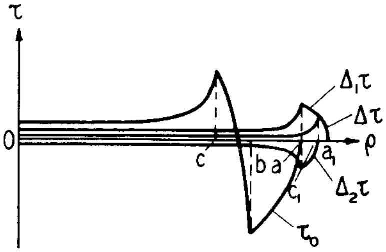

FIG. 20

(c) $\Delta T < f\Delta N$ ($c_1 > a$). The additional traction $\Delta\tau$, Equations [40], may be replaced by an equivalent distribution in the form (see Fig. 20)

$$\left.\begin{aligned}
\Delta_1\tau &= \frac{3f(N+\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad a \leqslant \rho \leqslant a_1\\
\Delta_1\tau &= \frac{3f(N+\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-(a^2-\rho^2)^{1/2}],\\
&\qquad\qquad \rho \leqslant a
\end{aligned}\right\} \quad \ldots [42]$$

and

$$\left.\begin{aligned}
\Delta_2\tau &= 0, \quad c_1 \leqslant \rho \leqslant a_1\\
\Delta_2\tau &= \frac{3fN_{c_1}}{2\pi c_1^3}(c_1^2-\rho^2)^{1/2}, \quad a \leqslant \rho \leqslant c_1\\
\Delta_2\tau &= \frac{3fN_{c_1}}{2\pi c_1^3}[(c_1^2-\rho^2)^{1/2}-(a^2-\rho^2)^{1/2}], \quad \rho \leqslant a
\end{aligned}\right\} \quad \ldots [43]$$

where $N_{c_1}/c_1^3 = N/a^3$.

It is seen that $\tau_0 + \Delta_1\tau$ is the resultant traction τ_R, Equations [41], obtained in case (a). Hence the load-displacement curve will be as shown in Fig. 21. There, path O-A-B-C-D may be replaced by path O-A'-B'-C'-D. The compliance becomes

$$c_t = \lim_{\substack{\Delta N\to 0\\ \Delta T\to 0}}\left\{\frac{1}{\Delta T}\left[\frac{2-\nu}{8\mu a_1}f\Delta N - \frac{2-\nu}{8\mu c_1}(f\Delta N - \Delta T)\right]\right\}$$

$$= \frac{2-\nu}{8\mu a} \quad \ldots\ldots\ldots [44]$$

Thus, Equation [44] is valid for all values of $dN/dT > 0$.

13 N Decreasing, T Increasing, at Point on Unloading Curve

1 Repeat step 1 of Section 9.

2 Keeping the normal force constant, increase the tangential load by an amount ΔT. In view of the discussion leading to Equations [10], the increment in traction will be (Fig. 22)

$$\left.\begin{aligned}
\Delta_1\tau &= \frac{3fN}{\pi a^3}(a^2-\rho^2)^{1/2}, \quad b_1 \leqslant \rho \leqslant a\\
\Delta_1\tau &= \frac{3fN}{\pi a^3}[(a^2-\rho^2)^{1/2}-(b_1^2-\rho^2)^{1/2}], \quad \rho \leqslant b_1
\end{aligned}\right\} \quad \ldots [45]$$

where

$$b_1 = a\left(1 - \frac{\Delta T}{2fN}\right)^{1/3}$$

Hence the resultant traction τ_{R0} is given by Equations [41] in which $N + \Delta N$, a_1 and a are replaced by N, a and b_1, respectively.

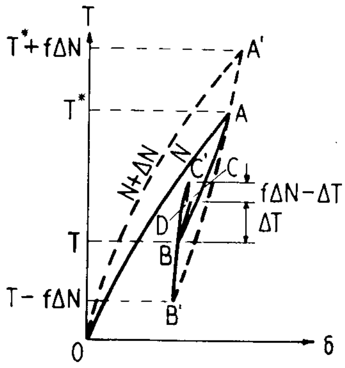

FIG. 21

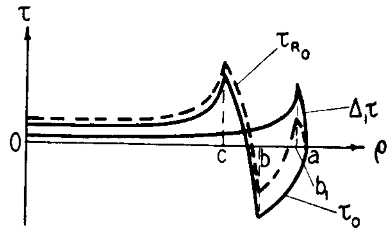

FIG. 22

3 Now, keeping the tangential force constant (at $T + \Delta T$), decrease the normal force by ΔN. The new contact radius is a_1. Several cases arise depending on the relative magnitudes of ΔT and $f\Delta N$. The solution follows the pattern established in Section 8; i.e., (1) freeze the contact area, (2) remove the traction in annulus $a_1 \leqslant \rho \leqslant a$ in such a way that surface $\rho \leqslant a_1$ remains frozen, (3) decrease N by ΔN, (4) unfreeze and permit slip to occur until equilibrium with force $T + \Delta T$ is re-established.

(a) $\Delta T \geqslant 2f\Delta N$ ($a_1 \geqslant b_1$). The additional traction necessary to free annulus $a_1 \leqslant \rho \leqslant a$ is $\Delta_2\tau$ (see Fig. 23) given by Equations [36]. Hence the resultant traction (which is not in equilibrium with $T + \Delta T$) becomes

$$\left.\begin{aligned}
\tau_{R_n} &= 0, \quad a_1 \leqslant \rho \leqslant a\\
\tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad b_1 \leqslant \rho \leqslant a_1\\
\tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b_1^2-\rho^2)^{1/2}],\\
&\qquad\qquad b \leqslant \rho \leqslant b_1\\
\tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b_1^2-\rho^2)^{1/2}\\
&\qquad + 2(b^2-\rho^2)^{1/2}], \quad c \leqslant \rho \leqslant b\\
\tau_{R_n} &= \frac{3f(N-\Delta N)}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-2(b_1^2-\rho^2)^{1/2}\\
&\qquad + 2(b^2-\rho^2)^{1/2}-(c^2-\rho^2)^{1/2}], \quad \rho \leqslant c
\end{aligned}\right\} \quad \ldots [46]$$

This distribution of traction may be attained by applying (1) $N - \Delta N$, (2) $T_1{}^*$, then (3) reducing tangential force to T_1, and

(4) again increasing it by $\overline{\Delta T}$, where the values of these load levels are

$$A \text{ — } \quad T_1^* = f(N - \Delta N)\left(1 - \frac{c^3}{a_1^3}\right) = T^* - f\Delta N$$

$$B \text{ — } \quad T_1^* - T_1 = 2f(N - \Delta N)\left(1 - \frac{b^3}{a_1^3}\right) = T^* - T - 2f\Delta N$$

hence

$$T_1 = T + f\Delta N$$

$$C \text{ — } \quad \overline{\Delta T} = 2f(N - \Delta N)\left(1 - \frac{b^3}{a_1^3}\right) = \Delta T - 2f\Delta N$$

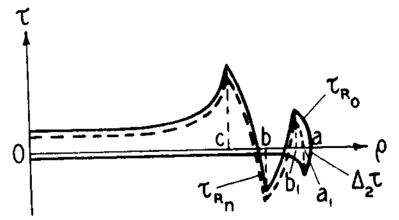

FIG. 23

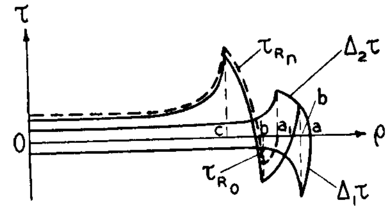

FIG. 25

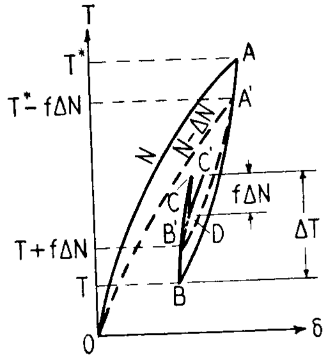

FIG. 24

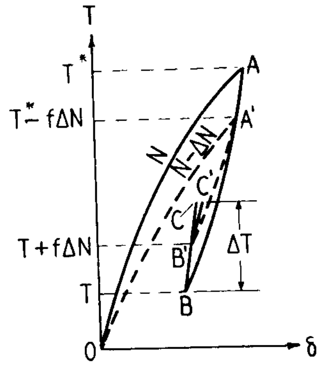

FIG. 26

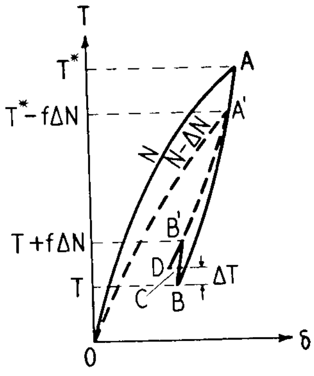

FIG. 27

Thus, in the load-displacement plane, Fig. 24, path O-A-B-C-D may be thought of as replaced by path O-A'-B'-D. Unfreezing will be followed by displacement along DC'.

The resulting compliance is

$$c_t = \lim_{\substack{\Delta N \to 0 \\ \Delta T \to 0}} \left\{ \frac{1}{\Delta T}\left[\frac{2-\nu}{8\mu a}\left(1 - \frac{\Delta T}{2fN}\right)^{-1/3} \Delta T - \frac{2-\nu}{8\mu a}\left(1 - \frac{\Delta T - f\Delta N}{2fN}\right)^{-1/3} f\Delta N + \frac{2-\nu}{8\mu a_1}\left(1 - \frac{f\Delta N}{2f(N - \Delta N)}\right)^{-1/3} f\Delta N \right]\right\} = \frac{2-\nu}{8\mu a}$$

(*b*) $\Delta T < 2f\Delta N$ $(a_1 < b_1)$. The addition to the traction in step 1 of $\Delta_1\tau$, Equations [35], and $\Delta_2\tau$, Equations [36], both with signs reversed, and with b_1 replacing b in the expression for $\Delta_1\tau$, will serve to remove the traction in annulus $a_1 \leqslant \rho \leqslant a$. Hence the resultant traction (not in equilibrium with $T + \Delta T$) will be (see Fig. 25)

$$\left.\begin{aligned}
\tau_{R_n} &= 0, \quad a_1 \leqslant \rho \leqslant a \\
\tau_{R_n} &= -\frac{3f(N - \Delta N)}{2\pi a_1^3}(a_1^2 - \rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a_1 \\
\tau_{R_n} &= -\frac{3f(N - \Delta N)}{2\pi a_1^3}[(a_1^2 - \rho^2)^{1/2} - 2(b^2 - \rho^2)^{1/2}], \\
&\qquad c \leqslant \rho \leqslant b \\
\tau_{R_n} &= -\frac{3f(N - \Delta N)}{2\pi a_1^3}[(a_1^2 - \rho^2)^{1/2} - 2(b^2 - \rho^2)^{1/2} \\
&\qquad + (c^2 - \rho^2)^{1/2}], \quad \rho \leqslant c
\end{aligned}\right\} \quad \text{. . [47]}$$

Again, the situation is the same as if the order of loading had been: (1) $N - \Delta N$, (2) $T_1^* = T^* - f\Delta N$, then (3) decrease of tangential load to $T_1 = T + f\Delta N$.

If $f\Delta N \leqslant \Delta T < 2f\Delta N$, path O-A-B-C-B-B' in load-displacement plane, Fig. 26, may be replaced by path O-A'-B'; unfreezing follows path $B'C'$. The compliance may be seen to be $c_t = (2-\nu)/8\mu a$.

If $\Delta T < f\Delta N$, unfreezing follows path $B'D$, Fig. 27. The compliance is given by

$$c_t = \lim_{\substack{\Delta N \to 0 \\ \Delta T \to 0}} \left\{ \frac{1}{\Delta T}\left[\frac{2-\nu}{8\mu a} f\Delta N - \frac{2-\nu}{8\mu a_1}\left(1 - \frac{T_1^* - T_1}{2f(N - \Delta N)}\right)^{-1/3}(f\Delta N - \Delta T)\right]\right\}$$
$$= \frac{2-\nu}{8\mu a}\left[f\frac{dN}{dT} + \left(1 - f\frac{dN}{dT}\right)\left(1 - \frac{T^* - T}{2fN}\right)^{-1/3}\right], \quad \frac{dN}{dT} \leqslant -\frac{1}{f}$$

and

$$c_t = \frac{2-\nu}{8\mu a}, \quad 0 > \frac{dN}{dT} \geqslant -\frac{1}{f} \quad \text{. . . . . [48]}$$

Here, once again, the absolute value of dN/dT is to be used in the formula.

In examining the cases of Sections 11 and 13, the reader will discover an apparent paradox to the effect that the expressions for the tangential compliances depend on the order of application of ΔN and ΔT. However, an imposition of a large number of infinitesimal increments, followed by a limiting process, leads to the results obtained regardless of the initial order of application.

14 Tangential Tractions Due to Oblique Loading

In this section a calculation is made of the distribution of tangential traction on the contact area of two spheres, pressed together by an initial force N_0 normal to their common tangent plane, and acted upon by a monotonically increasing oblique force. It is assumed that the resultant force remains planar. A study is made of the traction distribution because, by means of it, it will be possible to calculate the relative displacement of the two spheres.

Let a_0 denote the radius of the initial contact circle. An increase, by an amount $\Delta_i N$, of the normal force increases the contact radius to a value a_i; an application, at constant normal force, of an incremental tangential force $\Delta_i T$ gives rise to a tangential traction $\Delta_i \tau$, expressed by Equations [6] in which the appropriate values of N and T have been inserted. The three cases $\Delta_i T \lesseqgtr f\Delta_i N$ are considered separately.

(*a*) $\Delta_i T < f\Delta_i N$ ($c_i > a_{i-1}$). After n increments, by amounts $\Delta_i N$, $\Delta_i T$ ($i = 1, 2, \ldots, n$) of the normal and tangential forces, respectively, the resultant traction on circle $\rho < a_0$ will be, from Equations [6] and [1]

$$\tau_n = \frac{3f}{2\pi KR}\sum_{i=1}^{n}[(a_i^2 - \rho^2)^{1/2} - (c_i^2 - \rho^2)^{1/2}]$$

$$= \frac{3f}{2\pi KR}\sum_{i=1}^{n} a_i(1 - \rho^2/a_i^2)^{1/2}\left[1 - \left(1 - \frac{1 - c_i^2/a_i^2}{1 - \rho^2/a_i^2}\right)^{1/2}\right] \quad \ldots\ldots [49]$$

At the ith stage the value of the normal load is

$$N_i = N_0 + \sum_{\mu=1}^{i}\Delta_\mu N$$

Now, let

$$\sum_{\mu=1}^{i}\Delta_\mu N = \psi_i(T) \quad \ldots\ldots [50]$$

so that

$$N_i = N_0 + \psi_i(T) \quad \ldots\ldots [51]$$

Hence

$$a_i = [KR(N_0 + \psi_i)]^{1/3} \quad \ldots\ldots [52]$$

and

$$c_i = a_i\left[1 - \frac{\Delta_i T}{f(N_0 + \psi_i)}\right]^{1/3} \quad \ldots\ldots [53]$$

Now, substitution of Equations [52] and [53] in Equation [49], and passage to the limit as $\Delta_i N$ and $\Delta_i T$ approach zero and n becomes infinite, yield

$$\tau = \frac{S^2}{2\pi\rho^2}\int_0^T\left[1 - \frac{S^2}{(N_0 + \psi)^{2/3}}\right]^{-1/2}(N_0 + \psi)^{-2/3}dT \quad \ldots [54]$$

where

$$S = \rho(KR)^{-1/3}$$

To proceed further it is necessary to specify $\psi(T)$, i.e., the relation between N and T. The simplest case is

$$\frac{dT}{dN} = \text{const, say, } \beta \quad \ldots\ldots [55]$$

Using Equations [50] and [55], the integration in Equation [54] can be performed, with the result

$$\tau = \frac{3\beta(N_0 + N^*)}{2\pi a_*^3}[(a_*^2 - \rho^2)^{1/2} - (a_0^2 - \rho^2)^{1/2}], \quad \rho < a_0 \ldots [56]$$

where $N_0 + N^*$ is the final normal load and a_* the corresponding contact radius.

The resultant traction in annulus $a_{r-1} \leqslant \rho \leqslant c_r$ at the ith stage of loading will be

$$\tau_n = \frac{3f}{2\pi KR}\sum_{i=r}^{n}[(a_i^2 - \rho^2)^{1/2} - (c_i^2 - \rho^2)^{1/2}]$$

$$= \frac{3f}{2\pi KR}\sum_{i=r}^{n} a_i(1 - \rho^2/a_i^2)^{1/2}\left[1 - \left(1 - \frac{1 - c_i^2/a_i^2}{1 - \rho^2/a_i^2}\right)^{1/2}\right]$$

which may be rewritten

$$\tau_n = \frac{3f}{2\pi KR}a_r(1 - \rho^2/a_r^2)^{1/2}\left[1 - \left(1 - \frac{1 - c_r^2/a_r^2}{1 - \rho^2/a_r^2}\right)^{1/2}\right]$$

$$+ \frac{3f}{2\pi KR}\sum_{i=r+1}^{n} a_i(1 - \rho^2/a_i^2)^{1/2}\left[1 - \left(1 - \frac{1 - c_i^2/a_i^2}{1 - \rho^2/a_i^2}\right)^{1/2}\right] \quad \ldots\ldots [57]$$

Again, employment of Equations [52] and [53] and passage to the limit as vanishing increments of force are applied infinitely many times, followed by imposition of the condition in Equation [55] in conjunction with Equation [50], yield for the value of the resultant traction in Equation [57]

$$\tau = \frac{3\beta(N_0 + N^*)}{2\pi a_*^3}(a_*^2 - \rho^2)^{1/2}, \quad a_0 \leqslant \rho \leqslant a_* \ldots [58]$$

Equations [56] and [58] constitute the expressions for the resultant traction. It may be noted they are exactly like Equations [6], with β playing the role of the coefficient of friction. Since $\tau < f\sigma$, no slip, in the form heretofore understood, has taken place. Instead, there appears to be an overlapping action, proceeding outward from a_0 to a_*, in the direction of the tangential force.

(*b*) $\Delta_i T = f\Delta_i N$ ($c_i = a_{i-1}$), i.e., $\beta = f$. The resultant traction in annulus $a_{r-1} \leqslant \rho \leqslant a_r$ is given by

$$\tau_n = \frac{3f}{2\pi KR}\left\{(a_r^2 - \rho^2)^{1/2} + \sum_{i=r+1}^{n}[(a_i^2 - \rho^2)^{1/2} - (a_{i-1}^2 - \rho^2)^{1/2}]\right\}$$

$$= \frac{3f}{2\pi KR}(a_n^2 - \rho^2)^{1/2}$$

$$= \frac{3fN_n}{2\pi a_n^3}(a_n^2 - \rho^2)^{1/2}$$

Thus

$$\tau = \lim_{\substack{\Delta N, \Delta T \to 0 \\ n \to \infty}}\{\tau_n\} = \frac{3f(N_0 + N^*)}{2\pi a_*^3}(a_*^2 - \rho^2)^{1/2},$$

$$a_0 \leqslant \rho \leqslant a_* \ldots\ldots [59]$$

On circle $\rho \leqslant a_0$

$$\tau_n = \frac{3f}{2\pi KR}\sum_{i=1}^{n}[(a_i^2-\rho^2)^{1/2}-(a_{i-1}^2-\rho^2)^{1/2}]$$

$$= \frac{3fN_n}{2\pi a_n^3}[(a_n^2-\rho^2)^{1/2}-(a_0^2-\rho^2)^{1/2}]$$

so that

$$\tau = \lim_{\substack{\Delta N,\Delta T\to 0\\ n\to\infty}}\{\tau_n\} = \frac{3f(N_0+N^*)}{2\pi a_*^3}[(a_*^2-\rho^2)^{1/2} - (a_0^2-\rho^2)^{1/2}], \quad \rho \leqslant a_0 \ldots\ldots [60]$$

Equations [59] and [60], which are essentially the same as Equations [6], completely specify the distribution of traction in this case.

(c) $\Delta_i T > f\Delta_i N$, i.e., $\beta > f$. The usual division of $\Delta_i T$ into $f\Delta_i N$ and $\Delta_i T - f\Delta_i N$ is made and the two parts applied consecutively. After n applications the expression for the resultant traction will be

$$\tau_n = \frac{3f}{2\pi KR}(a_n^2-\rho^2)^{1/2}, \quad c_n \leqslant \rho \leqslant a_n$$

$$\tau_n = \frac{3f}{2\pi KR}[(a_n^2-\rho^2)^{1/2}-(c_n^2-\rho^2)^{1/2}], \quad \rho \leqslant c_n$$

where

$$c_n = a_n\left(1-\frac{\sum_{i=1}^{n}\Delta_i T}{fN_n}\right)^{1/2} < a_0$$

Passage to the limit results in

$$\left.\begin{aligned}\tau &= \frac{3f(N_0+N^*)}{2\pi a_*^3}(a_*^2-\rho^2)^{1/2}, \quad c_* \leqslant \rho \leqslant a_*\\ \tau &= \frac{3f(N_0+N^*)}{2\pi a_*^3}[(a_*^2-\rho^2)^{1/2}-(c_*^2-\rho^2)^{1/2}],\\ &\qquad \rho \leqslant c_*\end{aligned}\right\}\ldots[61]$$

where

$$c_* = a_*\left(1-\frac{T^*}{f(N_0+N^*)}\right)^{1/2}$$

15 Tangential Tractions Due to Oblique Unloading

Let it now be required to find the tractions when the loading described in the preceding section is followed by a monotonically decreasing oblique force of constant inclination.

(a) $\beta \leqslant f$. The traction at the start of unloading is given by Equations [56] and [58]. Keeping the normal load constant, decrease the tangential force by $\Delta_1 T$. Slip will take place in the direction opposite to that of the initial tangential force. By the fundamental rules, the traction over the region on which new slip has occurred will be in the direction of slip and of magnitude $f\sigma$. Let b_1 denote the inner radius of the new slip annulus. Then, in order that no new slip occur in circle $\rho \leqslant b_1$, the additional traction must be the sum, with sign reversed, of the tractions of Equations [56] and [58], in which a_0 is replaced by b_1, and of Equations [6], in which b_1 replaces c. Hence the resultant traction will be

$$\tau_{R0} = -\frac{3f(N_0+N^*)}{2\pi a_*^3}(a_*^2-\rho^2)^{1/2}, \quad b_1 \leqslant \rho \leqslant a_*$$

$$\tau_{R0} = -\frac{3f(N_0+N^*)}{2\pi a_*^3}\left[(a_*^2-\rho^2)^{1/2} - \left(1+\frac{\beta}{f}\right)(b_1^2-\rho^2)^{1/2}\right], \quad a_0 \leqslant \rho \leqslant b_1$$

$$\tau_{R0} = -\frac{3f(N_0+N^*)}{2\pi a_*^3}\left[(a_*^2-\rho^2)^{1/2} - \left(1+\frac{\beta}{f}\right)(b_1^2-\rho^2)^{1/2} + \frac{\beta}{f}(a_0^2-\rho^2)^{1/2}\right], \quad \rho \leqslant a_0$$

The equilibrium condition, Equation [11], yields

$$b_1 = a_*\left(1-\frac{\Delta T}{\left(1+\frac{\beta}{f}\right)f(N_0+N^*)}\right)^{1/3}$$

Reduction of the normal load by $\Delta_1 N$ to N_1, resulting in a contact surface of radius a_1 ($a_1 < b_1$, since $\Delta_1 N > \Delta_1 T/f$), must be preceded by a removal of the traction from annulus $a_1 \leqslant \rho \leqslant a$ in such a manner that no slip occur on $\rho \leqslant a_1$ (see, for example, Section 11, 2b). This may be accomplished by: (1) subtracting from the foregoing τ_{R0} the same tractions which, when added to the traction of Equations [56] and [58], yielded τ_{R0}, (2) adding, with sign reversed, tractions of Equations [56] and [58], in which a_0 is replaced by a_1. This will result in a distribution of traction

$$\tau_{R1} = \frac{3\beta N_1}{2\pi a_1^3}(a_1^2-\rho^2)^{1/2}, \quad a_0 \leqslant \rho \leqslant a_1$$

$$\tau_{R1} = \frac{3\beta N_1}{2\pi a_1^3}[(a_1^2-\rho^2)^{1/2}-(a_0^2-\rho^2)^{1/2}], \quad \rho \leqslant a_0$$

where

$$N_1 = N_0 + N^* - \Delta_1 N$$

and

$$\frac{a_0}{a_1} = \left(1-\frac{T^*-\Delta_1 T}{\beta N_1}\right)^{1/3}$$

which, because it corresponds to a condition of equilibrium (with force $T^* - \Delta_1 T$), remains unaltered by the reduction of normal load and subsequent unfreezing. But this distribution of traction is, qualitatively, the same as at the start of unloading; hence the unloading process may be repeated with similar results.

After the nth reduction the traction will be

$$\tau_{Rn} = \frac{3\beta N_n}{2\pi a_n^3}(a_n^2-\rho^2)^{1/2}, \quad a_0 \leqslant \rho \leqslant a_n$$

$$\tau_{Rn} = \frac{3\beta N_n}{2\pi a_n^3}[(a_n^2-\rho^2)^{1/2}-(a_0^2-\rho^2)^{1/2}], \quad \rho \leqslant a_0$$

where

$$N_n = N_0 + N^* - \sum_{i=1}^{n}\Delta_i N$$

a_n is the contact radius at normal load N_n and

$$\frac{a_0}{a_n} = \left(1-\frac{T^*-\sum_{i=1}^{n}\Delta_i T}{\beta N_n}\right)^{1/3}$$

Finally

$$\left.\begin{aligned}\tau &= \lim_{\substack{\Delta N, \Delta T \to 0 \\ n \to \infty}} \{\tau_{R_n}\} \\ &= \frac{3\beta(N_0 + N)}{2\pi a^3} (a^2 - \rho^2)^{1/2}, \quad a_0 \leqslant \rho \leqslant a \\ \tau &= \frac{3\beta(N_0 + N)}{2\pi a^3} [(a^2 - \rho^2)^{1/2} - (a_0^2 - \rho^2)^{1/2}], \\ &\qquad \rho \leqslant a_0\end{aligned}\right\} \quad [62]$$

where

$$\frac{a_0}{a} = \left(1 - \frac{T}{\beta(N_0 + N)}\right)^{1/3} = \left(\frac{N_0}{N_0 + N}\right)^{1/3}$$

and $N_0 + N$, a, $T(> 0)$ denote final normal load, contact radius, and tangential load, respectively.

It may be observed that Equations [62] have the same form as their counterpart in loading, Equations [56] and [58].

(*b*) $\beta \geqslant f$. The traction at the start of unloading is given by Equations [61]. Reduction, at constant normal load, of the tangential load by $\Delta_1 T$ changes the traction to that given by Equations [10], in which N, a and $T^* - T$ are replaced by $N_0 + N^*$, a_* and $\Delta_1 T$, respectively.

In the process of reducing the normal force it is necessary to distinguish between the two cases $f \leqslant \beta < 2f$ and $\beta \geqslant 2f$; however, both follow closely the pattern of events discussed in Section 11. Thus the resultant equilibrium traction after one cycle of unloading will be

$$\tau_{R_1} = -\frac{3fN_1}{2\pi a_1^3} (a_1^2 - \rho^2)^{1/2}, \quad b_1 \leqslant \rho \leqslant a_1$$

$$\tau_{R_1} = -\frac{3fN_1}{2\pi a_1^3} [(a_1^2 - \rho^2)^{1/2} - 2(b_1^2 - \rho^2)^{1/2}], \quad c \leqslant \rho \leqslant b_1$$

$$\tau_{R_1} = -\frac{3fN_1}{2\pi a_1^3} [(a_1^2 - \rho^2)^{1/2} - 2(b_1^2 - \rho^2)^{1/2} + (c^2 - \rho^2)^{1/2}], \quad \rho \leqslant c$$

where the expressions

$$c = a_1 \left(1 - \frac{T^*}{f(N_0 + N^*)}\right)^{1/3} \left(\frac{N_0 + N^*}{N_1}\right)^{1/3}$$

and

$$b_1 = a_1 \left(1 + \frac{\Delta_1 N}{2N_1} - \frac{\Delta_1 T}{2fN_1}\right)^{1/3}$$

are obtained by using Equation [11].

After the nth reduction the traction will be given by the foregoing equations in which the subscript 1 is replaced by n and the symbol $\Delta_n \equiv \sum_{i=1}^{n} \Delta_i$. Hence

$$\left.\begin{aligned}\tau_R &= \lim_{\substack{\Delta N, \Delta T \to 0 \\ n \to \infty}} \{\tau_{R_n}\} \\ &= -\frac{3f(N_0 + N)}{2\pi a^3} (a^2 - \rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a \\ \tau_R &= -\frac{3f(N_0 + N)}{2\pi a^3} [(a^2 - \rho^2)^{1/2} - 2(b^2 - \rho^2)^{1/2}], \\ &\qquad c_* \leqslant \rho \leqslant b \\ \tau_R &= -\frac{3f(N_0 + N)}{2\pi a^3} [(a^2 - \rho^2)^{1/2} - 2(b^2 - \rho^2)^{1/2} \\ &\qquad + (c_*^2 - \rho^2)^{1/2}], \quad \rho \leqslant c_*\end{aligned}\right\} \quad [63]$$

where $N_0 + N$, a, $T > 0$ are defined in Equations [62]

$$\frac{c_*}{a} = \left(1 - \frac{T^*}{f(N_0 + N^*)}\right)^{1/3} \left(\frac{N_0 + N^*}{N_0 + N}\right)^{1/3}$$

and

$$\frac{b}{a} = \left(1 - \frac{T^* - T}{2f(N_0 + N)} + \frac{N^* - N}{2(N_0 + N)}\right)^{1/3}$$

16 Tangential Tractions Due to Oscillating Oblique Force, $dT/dN \geqslant f$

It is now possible to calculate the distribution of traction on the contact surface of a pair of elastic spheres when the spheres are subjected to an initial normal compression, followed by an oscillating oblique force. This is the same problem as that of Section 5 except that the restriction to constant normal force is now removed. As will be seen, the additional variable introduces the complication that a steady state is not reached until after an additional half cycle.

Again, it is necessary to distinguish between cases $dT/dN \geqslant f$ and $dT/dN \leqslant f$. In this section the discussion is limited to the former.

1 Starting with an initial normal load N_0, apply a monotonically increasing oblique load until the contact surface of the two spheres is acted upon by normal and tangential forces $N_0 + N^*$ and T^*, respectively. It should be noted that $N^* = T^*/\beta$. The distribution of tangential traction is given by Equations [61] in which the expression for c_* may now be rewritten

$$\frac{c_*}{a_*} = \left(1 - \frac{T^*/fN_0}{1 + T^*/\beta N_0}\right)^{1/3} \quad [64]$$

Since the expression for a_0/a_* may be put in the form

$$\frac{a_0}{a_*} = \left(\frac{N_0}{N_0 + N^*}\right)^{1/3} = \left(\frac{1}{1 + T^*/\beta N_0}\right)^{1/3} = \left(1 - \frac{T^*/\beta N_0}{1 + T^*/\beta N_0}\right)^{1/3} \quad [65]$$

it is evident from a comparison of Equations [64] and [65] that $c_* \leqslant a_0$.

2 Now unload from $\{N_0 + N^*, T^*\}$. The traction at level $\{N_0 + N, T\}$ will be given by Equations [63] in which the expressions for b/a and c_*/a may be written

$$\frac{b}{a} = \left[1 - \frac{(T^* - T)/2fN_0}{1 + T/\beta N_0} + \frac{(T^* - T)/2\beta N_0}{1 + T/\beta N_0}\right]^{1/3} \quad [66]$$

and

$$\frac{c_*}{a} = \left[1 - \frac{(T^* - T)/2fN_0}{1 + T/\beta N_0} - \frac{(T^* + T)/2fN_0}{1 + T/\beta N_0} + \frac{(T^* - T)/\beta N_0}{1 + T/\beta N_0}\right]^{1/3} \quad [67]$$

It is seen from Equations [66] and [67] that $b = c_*$ for that value of T which satisfies the relation

$$\frac{T}{T^*} = \frac{1/\beta - 1/f}{1/\beta + 1/f} = \lambda \quad [68]$$

where $-1 \leqslant \lambda \leqslant 0$.

At $T = \lambda T^*$ the traction of Equations [63] degenerates to

$$\tau = -\frac{3f(N_0 + N_\lambda)}{2\pi a_\lambda^3} (a_\lambda^2 - \rho^2)^{1/2}, \quad c_* \leqslant \rho \leqslant a_\lambda$$

$$\tau = -\frac{3f(N_0 + N_\lambda)}{2\pi a_\lambda^3}[(a_\lambda^2 - \rho^2)^{1/2} - (c_*^2 - \rho^2)^{1/2}], \quad \rho \leqslant c_* \qquad \text{[69]}$$

where

$$N_\lambda = \frac{T_\lambda}{\beta} = \frac{\lambda T^*}{\beta}$$

and

$$\frac{a_\lambda}{a_0} = \left(1 + \frac{\lambda T^*}{\beta N_0}\right)^{1/3} \leqslant 1$$

Further unloading to $\{N_0 - N^*, -T^*\}$ causes the traction to become

$$\left.\begin{aligned}\tau &= -\frac{3f(N_0 - N^*)}{2\pi a_{-*}^3}(a_{-*}^2 - \rho^2)^{1/2}, \quad c_{-*} \leqslant \rho \leqslant a_{-*}\\ \tau &= -\frac{3f(N_0 - N^*)}{2\pi a_{-*}^3}[(a_{-*}^2 - \rho^2)^{1/2} - (c_{-*}^2 - \rho^2)^{1/2}],\\ &\qquad \rho \leqslant c_{-*}\end{aligned}\right\} \qquad \text{[70]}$$

where a_{-*}, the contact radius corresponding to a normal force $N_0 - N^*$, satisfies the relation

$$\frac{a_{-*}}{a_0} = \left(\frac{N_0 - N^*}{N_0}\right)^{1/3} = \left(1 - \frac{T^*}{\beta N_0}\right)^{1/3} \qquad \text{[71]}$$

and c_{-*} is given by

$$\frac{c_{-*}}{a_{-*}} = \left(1 - \frac{T^*}{f(N_0 - N^*)}\right)^{1/3} = \left(1 - \frac{T^*/fN_0}{1 - T^*/\beta N_0}\right)^{1/3} \qquad \text{[72]}$$

Comparison of Equation [67] at $T = 0$ and Equation [71] reveals that $c_* \gtreqless a_{-*}$, according as $\beta \lesseqgtr 2f$.

3 Reload; the traction at stage $\{N_0 + N, T\}$ will be

$$\left.\begin{aligned}\tau &= \frac{3f(N_0 + N)}{2\pi a^3}(a^2 - \rho^2)^{1/2}, \quad b_r \leqslant \rho \leqslant a\\ \tau &= \frac{3f(N_0 + N)}{2\pi a^3}[(a^2 - \rho^2)^{1/2} - 2(b_r^2 - \rho^2)^{1/2}],\\ &\qquad c_{-*} \leqslant \rho \leqslant b_r\\ \tau &= \frac{3f(N_0 + N)}{2\pi a^3}[(a^2 - \rho^2)^{1/2} - 2(b_r^2 - \rho^2)^{1/2}\\ &\qquad + (c_{-*}^2 - \rho^2)^{1/2}], \quad \rho \leqslant c_{-*}\end{aligned}\right\} \qquad \text{[73]}$$

where

$$|N| \leqslant N^*, \qquad |T| \leqslant T^*$$

Using the relation

$$\frac{c_{-*}}{a} = \frac{c_{-*}}{a_{-*}}\frac{a_{-*}}{a}$$

and the equilibrium condition, Equation [11], one obtains for the value of b_r/a

$$\frac{b_r}{a} = \left[\frac{1}{2}\left(1 + \frac{N_0 - N^*}{N_0 + N}\right) - \frac{T^* + T}{2f(N_0 + N)}\right]^{1/3} \qquad \text{[74]}$$

At $\{N_0 + N^*, T^*\}$

$$a = a_*$$

$$\frac{c_{-*}}{a_*} = \left(1 - \frac{T^*}{f(N_0 - N^*)}\right)^{1/3}\left(\frac{N_0 - N^*}{N_0 + N^*}\right)^{1/3} = \left[1 - \frac{(1 + 2f/\beta)T^*/fN_0}{1 + T^*/\beta N_0}\right]^{1/3} \qquad \text{[75]}$$

and

$$\frac{b_*}{a_*} = \left(\frac{N_0}{N_0 + N^*} - \frac{T^*}{f(N_0 + N^*)}\right)^{1/3} = \left[1 - \frac{(1 + f/\beta)T^*/fN_0}{1 + T^*/\beta N_0}\right]^{1/3} \qquad \text{[76]}$$

Hence, comparison of Equations [64], [75], and [76] shows that

$$c_* > b_* > c_{-*}$$

4 Unload once again; the traction at load level $\{N_0 + N, T\}$ will be

$$\left.\begin{aligned}\tau &= -\frac{3f(N_0 + N)}{2\pi a^3}(a^2 - \rho^2)^{1/2}, \quad b_u \leqslant \rho \leqslant a\\ \tau &= -\frac{3f(N_0 + N)}{2\pi a^3}[(a^2 - \rho^2)^{1/2} - 2(b_u^2 - \rho^2)^{1/2}],\\ &\qquad b_* \leqslant \rho \leqslant b_u\\ \tau &= -\frac{3f(N_0 + N)}{2\pi a^3}[(a^2 - \rho^2)^{1/2} - 2(b_u^2 - \rho^2)^{1/2}\\ &\qquad + 2(b_*^2 - \rho^2)^{1/2}], \quad c_{-*} \leqslant \rho \leqslant b_*\\ \tau &= -\frac{3f(N_0 + N)}{2\pi a^3}[(a^2 - \rho^2)^{1/2} - 2(b_u^2 - \rho^2)^{1/2}\\ &\qquad + 2(b_*^2 - \rho^2)^{1/2} - (c_{-*}^2 - \rho^2)^{1/2}], \quad \rho \leqslant c_{-*}\end{aligned}\right\} \qquad \text{[77]}$$

In order for equilibrium with the force T to obtain, the relation

$$\frac{b_u}{a} = \left[\frac{1}{2}\left(1 + \frac{N_0 + N^*}{N_0 + N}\right) - \frac{T^* - T}{2f(N_0 + N)}\right]^{1/3} \qquad \text{[78]}$$

must hold (see Equation [11]).

When $T = -T^*$, the contact radius is once again a_{-*}, and $b_u = b_{-*}$. Furthermore

$$\frac{b_{-*}}{a_{-*}} = \left[\frac{N_0}{N_0 - N^*} - \frac{T^*}{f(N_0 - N^*)}\right]^{1/3} = \frac{b_*}{a_{-*}}$$

so that $b_{-*} = b_*$. Hence the distribution of traction, Equations [77], is the same as that obtained on first unloading to $\{N_0 - N^*, -T^*\}$ (see Equations [70]). Henceforth the cycle may be repeated with identical results, i.e., a steady state is reached after one and three-quarter cycles.

17 Load-Displacement Relation for Oscillating Oblique Force, $dT/dN \geqslant f$

The load-displacement relations corresponding to the tractions of the preceding section may be obtained by superposition of equations in the generic form of Equation [7]. By the aid of Equation [1] and the relation $E = 2(1 + \nu)\mu$, the first factor of the right-hand side of Equation [7] may be transformed to

$$\frac{3f(2 - \nu)N}{16\mu a} = \frac{f(2 - \nu)}{2(1 - \nu)}\frac{a^2}{R} \qquad \text{[79]}$$

Hence, during the first loading from $\{N_0, 0\}$ to $\{N_0 + N^*, T^*\}$ the displacement will be given by (see curve OP, Fig. 28)

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta_l}{a_0^2} = \frac{a^2}{a_0^2}\left(1-\frac{c^2}{a^2}\right)$$
$$= (1+\theta L)^{2/3} - [1-(1-\theta)L]^{2/3}, \quad 0 \leqslant L \leqslant L^* \ldots\ldots [80]$$

where

$$L = \frac{T}{fN_0}, \qquad L^* = \frac{T^*}{fN_0}, \qquad \theta = \frac{f}{\beta}$$

and the subscript on δ_l stands for "loading." The range of θ is $0 \leqslant \theta \leqslant 1$.

From Equation [80], and the fact that

$$a = a_0\,(1+\theta L)^{1/3} \ldots\ldots\ldots\ldots\ldots\ldots [81]$$

the expression for the tangential compliance in this load interval is obtained. Thus

$$c_l = \frac{d\delta_l}{dT} = \frac{2-\nu}{8\mu a}\left[\theta + (1-\theta)\left(1-\frac{L}{1+\theta L}\right)^{-1/3}\right] \ldots [82]$$

If, now, Equation [55] and the definition of θ (see Equation [80]) are substituted into Equation [25] for the compliance due to the corresponding incremental process, the resulting relation becomes identical with Equation [82]. This result is to be expected, since the tractions in both cases are identical.

During the unloading process following initial loading, the load-displacement relation takes two different forms, according as $L \gtrless \lambda L^*$. With the aid of the expressions defined in Equations [66] and [67] one obtains (curves P-R-Y and YS, Fig. 28)

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta_{u11}}{a_0^2} = \frac{a^2}{a_0^2}\left(2\frac{b^2}{a^2} - \frac{c_*^2}{a^2} - 1\right)$$
$$= 2\left[1+\frac{\theta}{2}(L^*+L) - \frac{1}{2}(L^*-L)\right]^{2/3}$$
$$- [1-(1-\theta)L^*]^{2/3} - (1+\theta L)^{2/3},$$
$$L^* \geqslant L \geqslant \lambda L^* \ldots [83]$$

where λ is defined in Equation [68] and

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta_{u12}}{a_0^2} = -(1+\theta L)^{2/3}$$
$$+ [1+(1+\theta)L]^{2/3}, \quad \lambda L^* \geqslant L \geqslant -L^* \ldots\ldots [84]$$

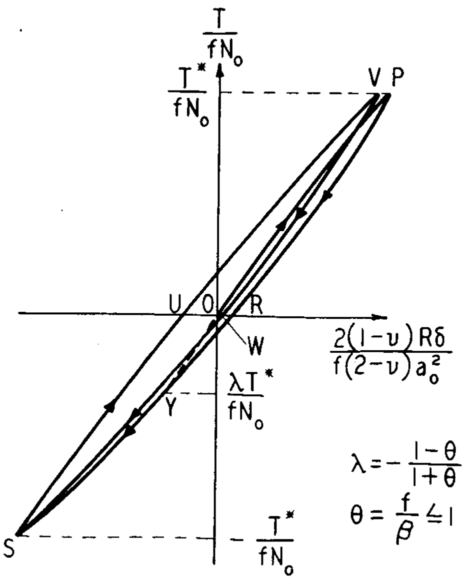

Fig. 28

The subscripts on δ_{u11} and δ_{u12} stand for "first portion of first unloading" and "second portion of first unloading," respectively. At point Y (Fig. 28), curves P-R-Y and YS have a common tangent.

In the load interval from $\{N_0 + N^*, T^*\}$ to $\{N_0 + N_\lambda, \lambda T^*\}$ the compliance is

$$c_{u11} = \frac{d\delta_{u11}}{dT}$$
$$= \frac{2-\nu}{8\mu a}\left\{-\theta + (1+\theta)\left[1-(1-\theta)\frac{L^*-L}{2(1+\theta L)}\right]^{-1/3}\right\}$$
$$\ldots\ldots [85]$$

It will be recalled that the compliance obtained by the corresponding incremental process is given by Equation [34] which, however, does not yield Equation [85] when fdN/dT is replaced by θ. This is because the respective traction distributions, although qualitatively alike, differ quantitatively; that is, in both cases

$$\left.\begin{aligned}
\tau &= -\frac{3f\mathfrak{N}}{2\pi a^3}(a^2-\rho^2)^{1/2}, \quad b \leqslant \rho \leqslant a \\
\tau &= -\frac{3f\mathfrak{N}}{2\pi a^3}[(a^2-\rho^2)^{1/2} - 2(b^2-\rho^2)^{1/2}], \\
&\qquad\qquad c_* \leqslant \rho \leqslant b \\
\tau &= -\frac{3f\mathfrak{N}}{2\pi a^3}[(a^2-\rho^2)^{1/2} - 2(b^2-\rho^2)^{1/2} \\
&\qquad + (c_*^2-\rho^2)^{1/2}], \quad \rho \leqslant c_*
\end{aligned}\right\} \ldots [86]$$

where $\mathfrak{N}$ denotes the total normal load. However, while in the "incremental" case

$$c_*/a = (1 - T^*/f\mathfrak{N})^{1/3} \text{ and } b/a = [1-(T^*-T)/2f\mathfrak{N}]^{1/3}$$

in the "continuous" case

$$\frac{c_*}{a} = \left(1-\frac{T^*}{f(N_0+N^*)}\right)^{1/3}\left(\frac{N_0+N^*}{N_0+N}\right)^{1/3}$$
$$= \left(1-\frac{T^*}{f\mathfrak{N}} + \frac{N^*-N}{\mathfrak{N}}\right)^{1/3}$$

and

$$\frac{b}{a} = \left(1-\frac{T^*-T}{2f\mathfrak{N}} + \frac{N^*-N}{\mathfrak{N}}\right)^{1/3}$$

where $|N| \leqslant N^*$.

In the load interval from $\{N_0 + N_\lambda, \lambda T^*\}$ to $\{N_0 - N^*, -T^*\}$ the compliance is

$$c_{u12} = \frac{d\delta_{u12}}{dT} = \frac{2-\nu}{8\mu a}\left[-\theta + (1+\theta)\left(1+\frac{1}{1+\theta L}\right)^{-1/3}\right]$$
$$\ldots\ldots [87]$$

The constant λ, incidentally, expressed in terms of θ, becomes

$$\lambda = -\frac{1-\theta}{1+\theta} \leqslant 0 \ldots\ldots\ldots\ldots\ldots [88]$$

The displacement during reloading from $\{N_0 - N^*, -T^*\}$ to $\{N_0 + N^*, T^*\}$, obtained by aid of Equations [72] and [74], is (curve S-U-V, Fig. 28)

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta_r}{a_0^2} = \frac{a^2}{a_0^2}\left(1+\frac{c_{-*}^2}{a^2} - 2\frac{b_r^2}{a^2}\right)$$
$$= (1+\theta L)^{2/3} + [1-(1+\theta)L^*]^{2/3}$$
$$-2\left[1-\frac{\theta}{2}(L^*-L) - \frac{1}{2}(L^*+L)\right]^{2/3}, \quad -L^* \leqslant L \leqslant L^*$$
$$\ldots [89]$$

while the displacement during the second unloading process becomes, with reference to Equations [75], [76], and [78] (curve *V-W-S*, Fig. 28)

$$\frac{2(1-\nu)\,R\delta_{u2}}{f(2-\nu)\,a_0^2} = \frac{a^2}{a_0^2}\left(2\,\frac{b_u^2}{a^2} - 2\,\frac{b_*^2}{a^2} + \frac{c_{-*}^2}{a^2} - 1\right)$$
$$= 2\left[1 + \frac{\theta}{2}(L^* + L) - \frac{1}{2}(L^* - L)\right]^{2/3} - 2(1 - L^*)^{2/3}$$
$$+ \{1 - (1+\theta)L^*\}^{2/3} - (1+\theta L)^{2/3}, \quad L^* \geqslant L \geqslant -L^* \quad \ldots\ldots [90]$$

The subscripts on δ_r and δ_{u2} stand for "reloading" and "second unloading," respectively.

Further repetition of the loading cycle will, of course, yield the displacements of Equations [89] and [90]. These displacements have the same meaning for the case of an oblique oscillating force as displacements of Equations [14] and [16] have for the case of pure tangential oscillating force; the two pairs of equations become identical when $\theta = 0$, so that the loop *S-U-V-W-S*, Fig. 28, degenerates to *S-U-P-R-S*, Fig. 5.

It should be noted that the character of the load-displacement curves, Fig. 28, varies with θ. Thus, the curvature (with respect to the abscissa) of OP at O is negative, as in Fig. 30, for values of θ in the interval $1/2 < \theta \leqslant 1$ and becomes positive, as shown in Fig. 28, for $\theta < 1/2$. The curvature of *S-U-V* at S is positive, as shown in Fig. 28, for values of $\theta < \sqrt{2} - 1$.

For the sake of completeness, the tangential compliances are given for the load intervals corresponding to which the displacements are δ_r and δ_{u2}, respectively. Thus

$$c_r = \frac{d\delta_r}{dT} = \frac{2-\nu}{8\mu a}\left\{\theta + (1-\theta)\left[1 - (1+\theta)\,\frac{L^* + L}{2(1+\theta L)}\right]^{-1/3}\right\} \quad \ldots\ldots [91]$$

and

$$c_{u2} = \frac{d\delta_{u2}}{dT} = \frac{2-\nu}{8\mu a}\left\{-\theta + (1+\theta)\left[1 - (1-\theta)\,\frac{L^* - L}{2(1+\theta L)}\right]^{-1/3}\right\} \quad \ldots\ldots [92]$$

In all equations of this section there exists a limitation on the value of T^*, namely

$$0 \geqslant \frac{-T^*}{f(N_0 - N^*)} \geqslant -1$$

which may be put in the form

$$0 \leqslant L^* \leqslant \frac{1}{1+\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots [93]$$

18 Oscillating Oblique Force, $dT/dN \leqslant f$

1 Repeat step 1 of Section 16; the traction is given by Equations [56] and [58]. It will be recalled (see Section 14) that no slip occurs during loading.

2 During unloading from $\{N_0 + N^*, T^*\}$ to $\{N_0, 0\}$ the traction at any level of loading is given by Equations [62], and vanishes at $T = 0$. Again, no slip has taken place.

Further unloading constitutes, in effect, the situation treated in Section 8, except for change in sign of the tangential force and, hence, of the tangential traction. Consequently, the present process yields a distribution of traction expressed by Equations [6] with sign reversed, in which, of course, N stands for the total normal load. At $\{N_0 - N^*, -T^*\}$, $a = a_{-*}$ and $c = c_{-*}$. The expression for the resultant traction is given by Equations [70]; for c_{-*}/a_{-*}, by Equation [72].

It may be noted that, since, in the load interval $0 > T \geqslant -T^*$, the absolute value of the traction in annulus $c_{-*} \leqslant \rho \leqslant a_{-*}$ is $f\sigma$, slip does take place during this portion of the loading cycle.

3 During reloading to $\{N_0 + N^*, T^*\}$, the additional traction to be superposed on the traction of Equations [70] is obtained by the procedure of Section 14, leading, at the terminal load, to Equations [56] and [58] with a_0 replaced by a_{-*}. The distribution of the initial and the additional tractions is illustrated by curves τ_0 and τ_a, respectively, in Fig. 29. Their algebraic sum yields the following resultant traction

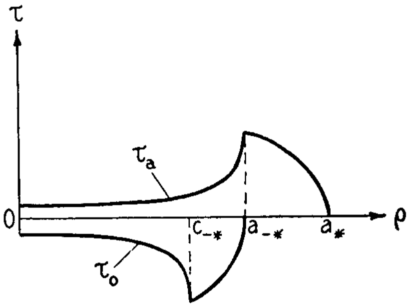

Fig. 29

$$\left.\begin{aligned}
\tau &= \frac{3\beta(N_0 + N^*)}{2\pi a_*^3}(a_*^2 - \rho^2)^{1/2}, \quad a_{-*} \leqslant \rho \leqslant a_* \\
\tau &= \frac{3\beta(N_0 + N^*)}{2\pi a_*^3}\left[(a_*^2 - \rho^2)^{1/2} - \left(1 + \frac{f}{\beta}\right)(a_{-*}^2 - \rho^2)^{1/2}\right], \quad c_{-*} \leqslant \rho \leqslant a_{-*} \\
\tau &= \frac{3\beta(N_0 + N^*)}{2\pi a_*^3}\left[(a_*^2 - \rho^2)^{1/2} - \left(1 + \frac{f}{\beta}\right)(a_{-*}^2 - \rho^2)^{1/2} + \frac{f}{\beta}(c_{-*}^2 - \rho^2)^{1/2}\right], \quad \rho \leqslant c_{-*}
\end{aligned}\right\} \quad \ldots [94]$$

Here

$$\frac{a_{-*}}{a_*} = \left(\frac{N_0 - N^*}{N_0 + N^*}\right)^{1/3} = \left(1 - \frac{2T^*/\beta N_0}{1 + T^*/\beta N_0}\right)^{1/3} \quad \ldots [95]$$

and

$$\frac{c_{-*}}{a_*} = \frac{c_{-*}}{a_{-*}}\,\frac{a_{-*}}{a_*} = \left(1 - \frac{(2 + \beta/f)T^*/\beta N_0}{1 + T^*/\beta N_0}\right)^{1/3} \quad \ldots [96]$$

4 In part 1 of Section 15 it was shown that, for the loading under consideration, the distribution of traction during unloading is the same as that during loading. Hence, during unloading from $\{N_0 + N^*, T^*\}$ to $\{N_0 - N^*, -T^*\}$ the additional traction (τ_a in Fig. 29), discussed in part 3 of this section, is gradually removed, until, at $\{N_0 - N^*, -T^*\}$, the remaining traction is, once again, the same as at the end of the first unloading (see part 2 of this section and τ_0 in Fig. 29). A stable cycle is thus established, during which no slip takes place (for further discussion see Section 19).

The load-displacement relations are obtained in a manner similar to those in Section 17. Thus, during initial loading and unloading

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta}{a_0^2} = \frac{1}{\theta}\left[(1+\theta L)^{2/3} - 1\right], \qquad L^* \geqslant L \geqslant 0 \quad [97]$$

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta}{a_0^2} = -(1+\theta L)^{2/3} + [1+(1+\theta)L]^{2/3}, \qquad 0 \geqslant L \geqslant -L^* \quad [98]$$

The relations of Equation [97] are illustrated by curves OP and PO in Fig. 30; of Equation [98], by curve OS in the same figure.

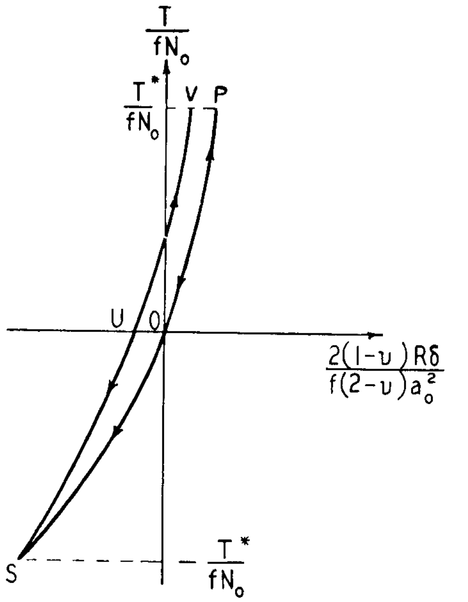

FIG. 30

The tangential compliance corresponding to the cycle in Equation [97] is

$$c_t = \frac{2-\nu}{8\mu a} \quad [99]$$

that corresponding to the load interval of Equation [98]

$$c_t = \frac{2-\nu}{8\mu a}\left[-\theta + (1+\theta)\left(1 + \frac{L}{1+\theta L}\right)^{-1/3}\right] \quad [100]$$

During each subsequent loading and unloading the displacement will be given by (curves S-U-V and V-U-S, Fig. 30)

$$\frac{2(1-\nu)}{f(2-\nu)}\frac{R\delta}{a_0^2} = \frac{1}{\theta}(1+\theta L)^{2/3} - \left(1+\frac{1}{\theta}\right)(1-\theta L^*)^{2/3} + [1-(1+\theta)L^*]^{2/3}, \qquad |L| \leqslant L^* \quad [101]$$

The compliance during the stable load cycle $|L| \leqslant L^*$ is the same as in Equation [99].

In Equations [97], [98] and [101] the range of θ is $1 \leqslant \theta \leqslant \infty$.

19 Frictional Energy Loss per Cycle

As in Section 5, the frictional energy loss per cycle is given by the area of the load-displacement loop (S-U-V-W-S, Fig. 28)

$$F = \int_{-T^*}^{T^*} (\delta_{u2} - \delta_r)\, dT = \frac{9(2-\nu)(fN_0)^2}{10\mu a_0}\left\{\frac{1}{4\theta}\left[\frac{1+\theta}{1-\theta}(1-\theta L^*)^{5/3} - \frac{1-\theta}{1+\theta}(1+\theta L^*)^{5/3}\right] - \frac{1}{1-\theta^2}\left(1 - \frac{1+5\theta^2}{6}L^*\right)(1-L^*)^{2/3}\right\} \quad [102]$$

in which $0 \leqslant \theta \leqslant 1$, i.e., $\beta \geqslant f$. The expressions for δ_{u2} and δ_r were obtained from Equations [90] and [89], respectively.

For $\theta = 0$, i.e., N = const, Equation [102] reduces to Equation [17].

For small values of L^*, series expansion of Equation [102] yields

$$F = \frac{(2-\nu)T^{*3}}{36\,\mu a_0 f N_0}(1-\theta^2) \quad [103]$$

which expression should be compared with Equation [18].

It should be noted that, in the case for which $1 \leqslant \theta \leqslant \infty$, i.e., $\beta \leqslant f$, the displacement retraces its path in loading and unloading (see Equation [101], and curves S-U-V and V-U-S, Fig. 30), so that no loop is formed; i.e., there is no frictional energy loss involved. This result is to be expected since, by one of the fundamental rules, slip does not occur if $\tau < f\sigma$ (see discussion following Equation [58]). It is interesting that, for $\beta = f$, i.e., $\theta = 1$, the energy, Equation [102], vanishes, indicating that dT/dN must exceed the coefficient of friction if energy is to be dissipated.