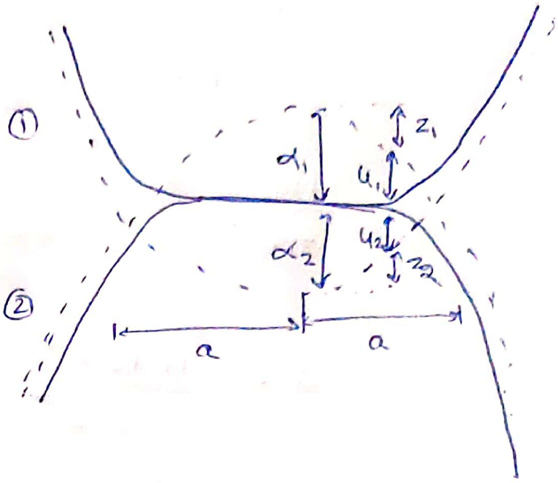
Normal Interaction

$$P(r) = q_0\left[1-\left(\frac{r}{a}\right)^2\right]^{1/2} \quad \text{Hertz theory (eq. 3.39)} \quad -(1)$$



$$u_z(r) = \frac{1-\nu^2}{E}\,\frac{\pi q_0}{4a}\,(2a^2 - r^2) \quad \text{(eq. 3.41a)} \quad -(2)$$

$$\because \alpha = \alpha_1 + \alpha_2,$$

$$u_1(r) + u_2(r) = \alpha - \left(\frac{r^2}{2R^*}\right) \quad \text{(eq. 4.17)} \quad -(3)$$

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2} \quad -(4)$$

Substituting eq. (2) into eq. (3)

$$\frac{\pi q_0}{4a}(2a^2 + r^2)\left[\left(\frac{1-\nu_1^2}{E_1}\right) + \left(\frac{1-\nu_2^2}{E_2}\right)\right] = \alpha - \frac{r^2}{2R^*}$$

$$\frac{1}{E^*} = \left(\frac{1-\nu_1^2}{E_1}\right) + \left(\frac{1-\nu_2^2}{E_2}\right)$$

$$\frac{\pi q_0}{4aE^*}(2a^2 - r^2) = \alpha - \left(\frac{r^2}{2R^*}\right) \quad -(5)$$

Substituting $r = 0$ into eq. (5)

$$\alpha = \frac{\pi q_0 a}{2E^*} \quad -(7)$$

substituting $r = a$ in eqn (5)

$$\frac{\pi q_0}{4aE^*}(2a^2 - a^2) = \frac{\pi q_0 a}{2E^*} - \frac{a^2}{2R^*}$$

$$-\frac{\pi p_0 a}{4E^*} + \frac{\pi p_0 a}{2E^*} = \frac{a^2}{2R^*}$$ ②

$$\frac{-\pi p_0 a + 2\pi p_0 a}{4E^*} = \frac{a^2}{2R^*}$$

$$\frac{\pi p_0 a}{4E^*} = \frac{a^2}{2R^*}$$

$$a = \frac{\pi p_0 R^*}{2E^*} \quad \text{— ⑧}$$

Total normal force, F_n

$$F_n = \int_0^a P(r)\, 2\pi r\, dr$$

$$= \int_0^a p_0\left(1-\left(\frac{r}{a}\right)^2\right)^{1/2} \cdot 2\pi r\, dr$$

$$= \frac{2\pi p_0}{a}\int_0^a r \cdot \sqrt{a^2-r^2}\, dr$$

$$= \frac{2\pi p_0}{a}\left[\frac{-(a^2-r^2)^{3/2}}{3}\right]_0^a$$

$$= \frac{2\pi p_0}{a}\left[0 + \frac{a^3}{3}\right]$$

$$F_n = \frac{2}{3}\pi p_0 a^2 \quad \text{— ⑨}$$

writing eqn. 8 in terms of F_n

$$\boxed{F_n = \frac{2}{3} P_0 \pi a^2}$$

(3)

$$a^2 \times \frac{2}{3} \times a = \frac{\pi P_0 R^*}{2E^*} \times \frac{2}{3} \times a^2$$

$$\frac{2}{3} \times a^3 = \frac{F_n R^*}{2E^*}$$

$$a^3 = \frac{3 F_n R^*}{4 E^*} \quad \text{— (10)}$$

writing eqn. 7 in terms of F_n

$$\alpha = \frac{\pi P_0 a}{2E^*}$$

$$\alpha^3 = \frac{\pi^3 P_0^3 a^3}{8 E^{*3}}$$

$$a \times \frac{2}{3} \times \frac{2}{3} \times \alpha^3 = \left(\frac{2}{3} \pi P_0 a^2\right) \left(\frac{2}{3} \pi P_0 a^2\right) \left(\frac{\pi P_0}{8 E^{*3}}\right)$$

$$\because F_n = \frac{2}{3} \pi P_0 a^2$$

$$\alpha^3 = \frac{\pi F_n^2 P_0}{8 E^{*3}} \times \frac{9}{4a}$$

$$\because a = \frac{\pi P_0 R^*}{2E^*}$$

$$\alpha^3 = \frac{9 F_n^2 \times \cancel{\pi P_0} \times \cancel{2} \times \cancel{E^*}}{\cancel{32}_{16} \times E^{*3} \times \cancel{\pi P_0} \times R^*}$$

$$\alpha^3 = \frac{9 F_n^2}{16 E^{*2} R^*} \quad \text{— (11)}$$

$$a^2 = R^* \alpha \quad \text{--- (12)}$$

Rearranging eq. (11).

$$\cancel{F_n = \frac{4}{3}} \quad \alpha^3 = \frac{9 F_n^2}{16 R^* E^{*2}}$$

$$F_n = \sqrt{\frac{16 R^* E^{*2} \alpha^3}{9}} = \frac{4}{3} E^* (R^* \alpha^3)^{1/2} \quad \text{--- (13)}$$

Contact stiffness.

$$K_n = \frac{dF_n}{d\alpha} = \frac{d}{d\alpha}\left(\frac{4}{3} E^* R^{*1/2} \alpha^{3/2}\right)$$

$$= \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times E^* \times R^{*1/2} \times \alpha^{1/2}$$

$$= 2 E^* (R^* \alpha)^{1/2} \qquad \because a^2 = R^* \alpha$$

$$K_n = 2 E^* a \quad \text{--- (14)}$$

Tangential Interaction -

For limiting condition, $F_t = \mu F_n$ - Distribution of tangential traction ($q(r)$):

$$q(r) = \mu\, p(r)$$

from hertz theory

$$p(r) = p_0\left(1 - \left(\frac{r}{a}\right)^2\right)^{1/2}$$

from eqn. (9)

$$p_0 = \frac{3 F_n}{2\pi a^2}$$

$$p(r) = \frac{3 F_n}{2\pi a^3}\left(a^2 - r^2\right)^{1/2}$$

$$q(r) = \frac{3\mu F_n}{2\pi a^3}\left(a^2 - r^2\right)^{1/2} \quad \text{for} \quad 0 \le r \le a \qquad (17)$$

for stick region superimposing -ve traction from $0 \le r \le b$

$$q(r) = -\left(\frac{3\mu F_n}{2\pi a^3}\right)\left(b^2 - r^2\right)^{1/2} \qquad 0 \le r \le b \qquad (18)$$

∴ Distribution of tangential traction over the area

$$q(r) = \frac{3\mu F_n}{2\pi a^3}\left[\left(a^2 - r^2\right)^{1/2} - \left(b^2 - r^2\right)^{1/2}\right] \qquad 0 \le r \le b$$

$$q(r) = \frac{3\mu F_n}{2\pi a^3}\left(a^2 - r^2\right)^{1/2} \qquad b \le r \le a \qquad - \text{ Eq. } 19$$

Relative tangential displacement

$$\delta = \left(\frac{3\mu F_n}{16 G^* a}\right)\left(1 - \frac{b^2}{a^2}\right) \qquad - \text{ mindlin (1949)} \qquad - (20)$$

(eq. 7.42)

$$\frac{1}{G^*} = \frac{(2-\nu_1)}{G_1} + \frac{(2-\nu_2)}{G_2} \quad — (21)$$

Magnitude of tangential force, Integrating eq. (19)

$$F_t = 2\pi \int_0^a q\, r\, dr$$

$$= 2\pi \int_0^b \frac{3\mu F_n}{2\pi a^3}\left[(a^2-r^2)^{1/2} - (b^2-r^2)^{1/2}\right] r\, dr + \int_b^a \frac{3\mu F_n}{2\pi a^3}\left[(a^2-r^2)^{1/2}\right] r\, dr$$

$$= \frac{2\pi \times 3\mu F_n}{2\pi a^3}\left[\int_0^b \left(\sqrt{a^2-r^2}\, r - \sqrt{b^2-r^2}\, r\right) dr + \int_b^a \sqrt{a^2-r^2}\, r\, dr\right]$$

$$= \frac{3\mu F_n}{a^3}\left\{\left[-\frac{(a^2-r^2)^{3/2}}{3}\right]_0^b - \left[-\frac{(b^2-r^2)^{3/2}}{3}\right]_0^b + \left[-\frac{(a^2-r^2)^{3/2}}{3}\right]_b^a\right\}$$

$$= \frac{3\mu F_n}{a^3}\left\{\left[-\frac{(a^2-b^2)^{3/2}}{3} + \frac{(a^2)^{3/2}}{3}\right] - \left[0 + \frac{(b^2)^{3/2}}{3}\right] + \left[0 + \frac{(a^2-b^2)^{3/2}}{3}\right]\right\}$$

$$= \frac{\mu F_n}{a^3}\left[-(a^2-b^2)^{3/2} + a^3 - b^3 + (a^2-b^2)^{3/2}\right]$$

$$= \frac{\mu F_n}{a^3}\left[a^3 - b^3\right]$$

$$F_t = \mu F_n\left(1 - \frac{b^3}{a^3}\right) \quad — (22)$$

Rearranging eq. 22 and substituting in eq 20

$$\frac{b^3}{a^3} = 1 - \frac{F_t}{\mu F_n}$$

$$\left(\frac{b}{a}\right)^2 = \left(1 - \frac{F_t}{\mu F_n}\right)^{2/3}$$

$$\delta = \frac{3\mu F_n}{16G^* a}\left[1 - \left(1 - \frac{F_t}{\mu F_n}\right)^{2/3}\right] \quad \text{—— (23)}$$

differentiating eq. 23 w.r.t. F_t and inverting.

$$\frac{d\delta}{dF_t} = \frac{\cancel{3\mu F_n}}{\cancel{16}G^* a} \times \frac{\cancel{2}}{\cancel{3}}\left(1 - \frac{F_t}{\mu F_n}\right)^{-1/3} \times \frac{1}{\mu F_n}$$

$$K_t = \frac{dF_t}{d\delta} = 8G^* a\left(1 - \frac{F_t}{\mu F_n}\right)^{1/3} \quad \text{—— (24)}$$

Similarly for unloading.

$$q(r) = -2\left(\frac{3\mu F_n}{2\pi a^3}\right)\left[\left(a^2 - r^2\right)^{1/2} - \left(c^2 - r^2\right)^{1/2}\right] \qquad 0 \leq r \leq c$$

$$\text{—— (25)}$$

$$q(r) = -2\left[\frac{3\mu F_n}{2\pi a^3}\right]\left(a^2 - r^2\right)^{1/2} \qquad c \leq r \leq a$$

Resultant traction distribution by adding eq. 19 and 25

$$q(r) = -\frac{3\mu F_n}{2\pi a^3}\left[\left(a^2 - r^2\right)^{1/2} - 2\left(c^2 - r^2\right)^{1/2} + \left(b^2 - r^2\right)^{1/2}\right] \qquad 0 \leq r$$

$$q(r) = \frac{-3\mu F_n}{2\pi a^3}\left[\left(a^2 - r^2\right)^{1/2} - 2\left(c^2 - r^2\right)^{1/2}\right] \qquad b \leq r \leq c$$

$$q(r) = -\frac{3\mu f_n}{2\pi a^3}\left(a^2 - r^2\right)^{1/2} \qquad c \le r \le a$$

Integrating eq. 26 over the contact area

$$F_t = 2\pi \int_0^a q(r)\, r\, dr = 2\pi \left[\underbrace{\int_0^b q(r)\, r\, dr}_{1} + \underbrace{\int_b^c q(r)\, r\, dr}_{2} + \int_c^a q(r)\, r\, dr \right]$$

1

$$= \frac{-3\mu f_n \times \cancel{2\pi}}{\cancel{2\pi} a^3} \int_0^b \left(r\sqrt{a^2 - r^2} - 2r\sqrt{c^2 - r^2} + r\sqrt{b^2 - r^2} \right) dr.$$

$$= \frac{-3\mu f_n}{a^3} \left[\frac{-(a^2 - r^2)^{3/2}}{3} - 2\left(\frac{-(c^2 - b^2)^{3/2}}{3} \right) + \left(\frac{-(b^2 - r^2)^{3/2}}{3} \right) \right]_0^b$$

$$= \frac{-3\mu f_n}{a^3} \left[\frac{-(a^2 - b^2)^{3/2}}{3} - 2\left(\frac{-(c^2 - b^2)^{3/2}}{3} \right) + 0 + \frac{a^3}{3} - \frac{2c^3}{3} + \frac{b^3}{3} \right]$$

2

$$= \frac{-3\mu f_n}{a^3} \int_b^c \left(\cancel{a} r\sqrt{a^2 - r^2} - 2r\sqrt{c^2 - r^2} \right) dr$$

$$= \frac{-3\mu f_n}{a^3} \left[\frac{-(a^2 - r^2)^{3/2}}{3} - 2\left(\frac{-(c^2 - r^2)^{3/2}}{3} \right) \right]_b^c$$

$$= \frac{-3\mu f_n}{a^3} \left[\frac{-(a^2 - c^2)^{3/2}}{3} + 0 + \frac{(a^2 - b^2)^{3/2}}{3} - \frac{2(c^2 - b^2)^{3/2}}{3} \right]$$

3

$$= -\frac{3\mu F_n}{a^3} \int_c^a r\sqrt{a^2-r^2}\, dr$$

$$= \frac{-3\mu F_n}{a^3}\left[\cancel{\ell} - \frac{(a^2-r^2)^{3/2}}{3}\right]_c^a$$

$$= -\frac{3\mu F_n}{a^3}\left[\cancel{0} + \frac{(a^2-c^2)^{3/2}}{3}\right]$$

Adding 1 + 2 + 3

$$F_t = -\frac{3\mu F_n}{a^3}\left[-\cancel{\frac{(a^2-b^2)^{3/2}}{3}} + \cancel{\frac{2(c^2-b^2)^{3/2}}{3}} + \frac{a^3}{3} - \frac{2c^3}{3} + \frac{b^3}{3} - \cancel{\frac{(a^2-c^2)^{3/2}}{3}} + \cancel{\frac{(a^2-b^2)^{3/2}}{3}} - \cancel{\frac{2(c^2-b^2)^{3/2}}{3}} + \cancel{\frac{(a^2-c^2)^{3/2}}{3}}\right]$$

$$= -\frac{\mu F_n}{a^3}\left[a^3 - 2c^3 + b^3\right]$$

$$= \mu F_n\left(1-\frac{b^3}{a^3}\right) - 2\mu F_n\left(1-\frac{c^3}{a^3}\right) \quad -(27)$$

$$F_t = F_t^* - 2\mu F_n\left(1-\frac{c^3}{a^3}\right)$$

tangential force from which unloading starts

$$\because F_t^* = \mu F_n\left(1-\frac{b^3}{a^3}\right)$$

$$c = a\left[1-\left\{\frac{(F_t^*-F_t)}{2\mu F_n}\right\}\right]^{1/3} \quad -(28)$$

$$\delta = \frac{3\mu F_n}{16G^* a}\left(2\frac{c^2}{a^2} - \frac{b^2}{a^2} - 1\right)$$

$$= \frac{3\mu F_n}{16G^* a}\left[2\left(1 - \frac{(F_t^* - F_t)}{2\mu F_n}\right)^{2/3} - \left(1 - \frac{F_t^*}{\mu F_n}\right)^{2/3} - 1\right] \quad -(29)$$

$$\frac{d\delta}{dF_t} = \frac{3\cancel{\mu F_n}}{16G^* a}\left[2\times\frac{2}{3}\left(1 - \frac{(F_t^* - F_t)}{2\mu F_n}\right)^{-1/3} \times \left(\frac{1}{\cancel{2\mu F_n}}\right)\right]$$

$$K_t = \frac{dF_t}{d\delta} = \frac{\cancel{16}\, G^* a}{\cancel{3}}\left[\frac{\cancel{3}}{\cancel{2}}\left(1 - \frac{(F_t^* - F_t)}{2\mu F_n}\right)^{1/3}\right]$$

$$K_t = 8G^* a\left[1 - \frac{(F_t^* - F_t)}{2\mu F_n}\right]^{1/3} \quad -(30)$$

Similarly for reloading,

$$F_t = \mu F_n\left(1 - \frac{b^3}{a^3}\right) - 2\mu F_n\left(1 - \frac{c^3}{a^3}\right) + 2\mu F_n\left(1 - \frac{d^3}{a^3}\right) \quad -(32)$$

$$d = a\left[1 - \frac{(F_t - F_t^{**})}{2\mu F_n}\right] \quad -(33)$$

$$\delta = \frac{3\mu F_n}{16G^* a}\left(1 - \frac{b^2}{a^2} + 2\frac{c^2}{a^2} - 2\frac{d^2}{a^2}\right) \quad -(34)$$

$$K_t = \frac{dF_t}{d\delta} = 8G^* a\left[1 - \frac{(F_t - F_t^{**})}{2\mu F_n}\right]^{1/3} \quad -(35)$$