Let $x_1, ..., x_n \sim \ N(\mu, \frac{1}{\tau})$. For the prior, we have

$$
\begin{aligned}
p(\mu, \tau) &= p(\mu|\tau)p(\tau) \\
\mu|\tau &\sim N(\mu_0, \frac{1}{\lambda_0 \tau}) \\
\tau &\sim \text{Gamma}(a_0, b_0) \\
p(\tau) &= \frac{b_0^{a_0}}{\Gamma(a_0)} \tau^{a_0 - 1} e^{-b_0 \tau}
\end{aligned}
$$

Suppose $q(\mu, \tau) = q_\mu(\mu) q_\tau(\tau)$. Then we have

$$
\begin{aligned}
\log q_\mu^*(\mu) &= E_\tau \Big[\log p(\mu, \tau, x_1, ..., x_n) \Big] + \text{const} \\
&= E_\tau \Big[\log \prod_{i=1}^n p(x_i|\mu, \tau) + \log p(\mu|\tau) + \log p(\tau) \Big] + \text{const} \\
&= E_\tau \Big[\frac{n}{2} \log \Big(\frac{\tau}{2\pi} \Big) - \frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2 \Big] \\
&\quad + E_\tau \Big[\frac{1}{2} \log \Big(\frac{\lambda_0 \tau}{2\pi} \Big) - \frac{\lambda_0 \tau}{2} (\mu - \mu_0)^2 \Big] + E_\tau \Big[\log p(\tau) \Big] + \text{const} \\
&= -\frac{E_\tau(\tau)}{2} \Big[\sum_{i=1}^n (x_i - \mu)^2 + \lambda_0 (\mu - \mu_0)^2 \Big] + \text{const} \\
&= -\frac{E_\tau(\tau)}{2} \Big[(\lambda_0 + n)\mu^2 - 2\Big(\lambda_0 \mu_0 + \sum_{i=1}^n x_i \Big) \mu \Big] + \text{const}
\end{aligned}
$$

We see that $q_\mu^*(\mu)$ is the pdf of $N(\mu^*, \sigma^{2*})$, where

$$
\mu^* = \frac{\lambda_0 \mu_0 + \sum_{i=1}^n x_i}{\lambda_0 + n} \text{ and } \sigma^{2*} = \frac{1}{(\lambda_0 + n) E_\tau(\tau)}
$$

$$
\begin{aligned}
\log q_\tau^*(\tau) &= E_\mu \Big[\log p(\mu, \tau, x_1, ..., x_n) \Big] + \text{const} \\
&= E_\mu \Big[\log \prod_{i=1}^n p(x_i|\mu, \tau) + \log p(\mu|\tau) + \log p(\tau) \Big] + \text{const} \\
&= E_\mu \Big[\frac{n}{2} \log \Big(\frac{\tau}{2\pi} \Big) - \frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2 \Big] \\
&\quad + E_\mu \Big[\frac{1}{2} \log \Big(\frac{\lambda_0 \tau}{2\pi} \Big) - \frac{\lambda_0 \tau}{2} (\mu - \mu_0)^2 \Big] + \log p(\tau) + \text{const} \\
&= \frac{n}{2} \log \tau - \frac{\tau}{2} \sum_{i=1}^n E_\mu[(\mu - x_i)^2] + \frac{1}{2} \log \tau - \frac{\lambda_0 \tau}{2} E_\mu[(\mu - \mu_0)^2] \\
&\quad + (a_0 - 1) \log \tau - b_0 \tau + \text{const} \\
&= \Big(\frac{n+1}{2} + a_0 - 1 \Big) \log \tau - \Big[b_0 + \frac{1}{2} \sum_{i=1}^n E_\mu[(\mu - x_i)^2] + \frac{\lambda_0}{2} E_\mu[(\mu - \mu_0)^2] \Big] \tau + \text{const.}
\end{aligned}
$$

We see that $q_\tau^*(\tau)$ be the pdf of Gamma(a^*, b^*), where

$$
a^* = \frac{n+1}{2} + a_0 \text{ and } b^* = b_0 + \frac{1}{2} \sum_{i=1}^n E_\mu[(\mu - x_i)^2] + \frac{\lambda_0}{2} E_\mu[(\mu - \mu_0)^2]
$$

.

We have

$$\begin{aligned}
E_\tau(\tau) &= \frac{a^*}{b^*}, \\
E_\mu[(\mu - x_i)^2] &= E_\mu(\mu^2) - 2x_i E_\mu(\mu) + x_i^2 \\
&= \sigma^{2*} + \mu^{*2} - 2x_i\mu^* + x_i^2, \\
E_\mu[(\mu - \mu_0)^2] &= \sigma^{2*} + \mu^{*2} - 2\mu_0\mu^* + \mu_0^2.
\end{aligned}$$

$$\begin{aligned}
ELBO(q_\mu^*(\mu), q_\tau^*(\tau)) &= ELBO(\mu^*, \sigma^{2*}, a^*, b^*) \\
&= E_{\mu,\tau}(\log p(x_1, ...x_n, \mu, \tau) - \log(q_\mu^*(\mu) q_\tau^*(\tau))) \\
&= E_{\mu,\tau}[\log \prod_{i=1}^n p(x_i|\mu,\tau)p(\mu|\tau)p(\tau)] - E_{\mu,\tau}[\log q_\mu^*(\mu)] - E_{\mu,\tau}[\log q_\tau^*(\tau)],
\end{aligned}$$

in which

$$\begin{aligned}
E_{\mu,\tau}[\log q_\mu^*(\mu)] &= E_{\mu,\tau}\Big[-\frac{1}{2}\log\sigma^{2*} - \frac{1}{2}\log 2\pi - \frac{(\mu-\mu^*)^2}{2\sigma^{2*}}\Big] \\
&= -\frac{1}{2}\log\sigma^{2*} - \frac{E_\mu[(\mu-\mu^*)^2]}{2\sigma^{2*}} + \text{const} \\
&= -\frac{1}{2}\log\sigma^{2*} - \frac{\sigma^{2*}}{2\sigma^{2*}} + \text{const} \\
&= -\frac{1}{2}\log\sigma^{2*} + \text{const}
\end{aligned}$$

and

$$\begin{aligned}
E_{\mu,\tau}[\log q_\tau^*(\tau)] &= E_{\mu,\tau}\Big[a^*\log b^* - \log\Gamma(a^*) + (a^*-1)\log\tau - b^*\tau\Big] \\
&= a^*\log b^* - \log\Gamma(a^*) + (a^*-1)E_\tau[\log\tau] - b^* E_\tau(\tau) \\
&= a^*\log b^* - \log\Gamma(a^*) + (a^*-1)(-\log b^* + \Psi(a^*)) - b^*\frac{a^*}{b^*} \\
&= \log b^* - \log\Gamma(a^*) + (a^*-1)\Psi(a^*) - a^*,
\end{aligned}$$

where $\Psi(x) = \frac{\partial}{\partial x}\log\Gamma(x)$ and

$$\begin{aligned}
&E_{\mu,\tau}[\log \prod_{i=1}^n p(x_i|\mu,\tau)p(\mu|\tau)p(\tau)] \\
&= E_{\mu,\tau}\Big[\frac{n}{2}\log\tau - \frac{n}{2}\log(2\pi) - \frac{\tau}{2}\sum_{i=1}^n (x_i-\mu)^2 \\
&\qquad + \frac{1}{2}\log\lambda_0 + \frac{1}{2}\log\tau - \frac{1}{2}\log(2\pi) - \frac{\lambda_0\tau}{2}(\mu-\mu_0)^2\Big] + E_\tau[\log p(\tau)],
\end{aligned}$$

where

$$\begin{aligned}
&E_{\mu,\tau}\Big[\frac{n}{2}\log\tau - \frac{n}{2}\log(2\pi) - \frac{\tau}{2}\sum_{i=1}^n (x_i-\mu)^2 \\
&\qquad + \frac{1}{2}\log\lambda_0 + \frac{1}{2}\log\tau - \frac{1}{2}\log(2\pi) - \frac{\lambda_0\tau}{2}(\mu-\mu_0)^2\Big] \\
&= E_{\mu,\tau}\Big[\frac{n+1}{2}\log\tau - \frac{\tau}{2}\Big(\sum_{i=1}^n (x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big)\Big] + \text{const} \\
&= \frac{n+1}{2}E_\tau[\log\tau] - \frac{1}{2}E_\tau(\tau)\Big[\sum_{i=1}^n E_\mu[(x_i-\mu)^2] + \lambda_0 E_\mu[(\mu-\mu_0)^2] + \text{const} \\
&= \frac{n+1}{2}(-\log b^* + \Psi(a^*)) \\
&\qquad - \frac{1}{2}\frac{a^*}{b^*}\Big[\sum_{i=1}^n (\sigma^{2*} + \mu^{*2} - 2x_i\mu^* + x_i^2) + \lambda_0(\sigma^{2*} + \mu^{*2} - 2\mu_0\mu^* + \mu_0^2)\Big] + \text{const},
\end{aligned}$$

and

$$\begin{aligned}
E_\tau[\log p(\tau)] &= E_\tau[a_0 \log b_0 - \log \Gamma(a_0) + (a_0 - 1)\log \tau - b_0 \tau] \\
&= (a_0 - 1)E_\tau[\log \tau] - b_0 E_\tau(\tau) + \text{const} \\
&= (a_0 - 1)[-\log b^* + \Psi(a^*)] - b_0 \frac{a^*}{b^*} + \text{const.}
\end{aligned}$$