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Product topology

In topology and related areas of mathematics, a **product space** is the Cartesian product of a family of topological spaces equipped with a natural topology called the **product topology**. This topology differs from another, perhaps more natural-seeming, topology called the box topology, which can also be given to a product space and which agrees with the product topology when the product is over only finitely many spaces. However, the product topology is "correct" in that it makes the product space a categorical product of its factors, whereas the box topology is too fine; in that sense the product topology is the natural topology on the Cartesian product.

Definition

Throughout, I will be some non-empty index set and for every index $i \in I$, let X_i be a topological space. Denote the Cartesian product of the sets X_i by

$$X := \prod X_\bullet := \prod_{i \in I} X_i$$

and for every index $i \in I$, denote the i-th ***canonical projection*** by

$$\begin{aligned} p_i : \ \prod_{j \in I} X_j &\to X_i, \\ (x_j)_{j \in I} &\mapsto x_i. \end{aligned}$$

The ***product topology***, sometimes called the ***Tychonoff topology***, on $\prod_{i \in I} X_i$ is defined to be the coarsest topology (that is, the topology with the fewest open sets) for which all the projections $p_i : \prod X_\bullet \to X_i$ are continuous. The Cartesian product $X := \prod_{i \in I} X_i$ endowed with the product topology is called the ***product space***. The open sets in the product topology are arbitrary unions (finite or infinite) of sets of the form $\prod_{i \in I} U_i$, where each U_i is open in X_i and $U_i \neq X_i$ for only finitely many i. In particular, for a finite product (in particular, for the product of two topological spaces), the set of all Cartesian products between one basis element from each X_i gives a basis for the product topology of $\prod_{i \in I} X_i$. That is, for a finite product, the set of all $\prod_{i \in I} U_i$, where U_i is an element of the (chosen) basis of X_i, is a basis for the product topology of $\prod_{i \in I} X_i$.

The product topology on $\prod_{i \in I} X_i$ is the topology generated by sets of the form $p_i^{-1}(U_i)$, where $i \in I$ and U_i is an open subset of X_i. In other words, the sets

$$\left\{ p_i^{-1}(U_i) \,\middle|\, i \in I \text{ and } U_i \subseteq X_i \text{ is open in } X_i \right\}$$

form a subbase for the topology on X. A subset of X is open if and only if it is a (possibly infinite) union of intersections of finitely many sets of the form $p_i^{-1}(U_i)$. The $p_i^{-1}(U_i)$ are sometimes called open cylinders, and their intersections are cylinder sets.

The product topology is also called the *topology of pointwise convergence* because a sequence (or more generally, a net) in $\prod_{i\in I} X_i$ converges if and only if all its projections to the spaces X_i converge. Explicitly, a sequence $s_\bullet = (s_n)_{n=1}^{\infty}$ (respectively, a net $s_\bullet = (s_a)_{a\in A}$) converges to a given point $x \in \prod_{i\in I} X_i$ if and only if $p_i(s_\bullet) \to p_i(x)$ in X_i for every index $i \in I$, where $p_i(s_\bullet) := p_i \circ s_\bullet$ denotes $(p_i(s_n))_{n=1}^{\infty}$ (respectively, denotes $(p_i(s_a))_{a\in A}$). In particular, if $X_i = \mathbb{R}$ is used for all i then the Cartesian product is the space $\prod_{i\in I} \mathbb{R} = \mathbb{R}^I$ of all real-valued functions on I, and convergence in the product topology is the same as pointwise convergence of functions.

Examples

If the real line $\mathbb{R}$ is endowed with its standard topology then the product topology on the product of n copies of $\mathbb{R}$ is equal to the ordinary Euclidean topology on $\mathbb{R}^n$. (Because n is finite, this is also equivalent to the box topology on $\mathbb{R}^n$.)

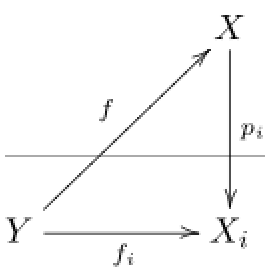
The Cantor set is homeomorphic to the product of countably many copies of the discrete space $\{0, 1\}$ and the space of irrational numbers is homeomorphic to the product of countably many copies of the natural numbers, where again each copy carries the discrete topology.

Several additional examples are given in the article on the initial topology.

Properties

The set of Cartesian products between the open sets of the topologies of each X_i forms a basis for what is called the box topology on X. In general, the box topology is finer than the product topology, but for finite products they coincide.

The product space X, together with the canonical projections, can be characterized by the following universal property: if Y is a topological space, and for every $i \in I$, $f_i : Y \to X_i$ is a continuous map, then there exists *precisely one* continuous map $f : Y \to X$ such that for each $i \in I$ the following diagram commutes:



This shows that the product space is a product in the category of topological spaces. It follows from the above universal property that a map $f : Y \to X$ is continuous if and only if $f_i = p_i \circ f$ is continuous for all $i \in I$. In many cases it is easier to check that the component functions f_i are continuous. Checking whether a map $X \to Y$ is continuous is usually more difficult; one tries to use the fact that the p_i are continuous in some way.

In addition to being continuous, the canonical projections $p_i : X \to X_i$ are open maps. This means that any open subset of the product space remains open when projected down to the X_i. The converse is not true: if W is a subspace of the product space whose projections down to all the X_i are open, then W need not be open in X (consider for instance $W = \mathbb{R}^2 \setminus (0,1)^2$.) The canonical projections are not generally closed maps (consider for example the closed set $\left\{(x,y) \in \mathbb{R}^2 : xy = 1\right\}$, whose projections onto both axes are $\mathbb{R} \setminus \{0\}$).

Suppose $\prod_{i \in I} S_i$ is a product of arbitrary subsets, where $S_i \subseteq X_i$ for every $i \in I$. If all S_i are *non-empty* then $\prod_{i \in I} S_i$ is a closed subset of the product space X if and only if every S_i is a closed subset of X_i. More generally, the closure of the product $\prod_{i \in I} S_i$ of arbitrary subsets in the product space X is equal to the product of the closures:[1]

$$\operatorname{Cl}_X\left(\prod_{i \in I} S_i\right) = \prod_{i \in I} \left(\operatorname{Cl}_{X_i} S_i\right).$$

Any product of Hausdorff spaces is again a Hausdorff space.

Tychonoff's theorem, which is equivalent to the axiom of choice, states that any product of compact spaces is a compact space. A specialization of Tychonoff's theorem that requires only the ultrafilter lemma (and not the full strength of the axiom of choice) states that any product of compact Hausdorff spaces is a compact space.

If $z = (z_i)_{i \in I} \in X$ is fixed then the set

$$\left\{x = (x_i)_{i \in I} \in X \,\middle|\, x_i = z_i \text{ for all but finitely many } i\right\}$$

is a dense subset of the product space X.[1]

Relation to other topological notions

Separation

- Every product of T_0 spaces is T_0.
- Every product of T_1 spaces is T_1.
- Every product of Hausdorff spaces is Hausdorff.
- Every product of regular spaces is regular.
- Every product of Tychonoff spaces is Tychonoff.
- A product of normal spaces *need not* be normal.

Compactness

- Every product of compact spaces is compact (Tychonoff's theorem).
- A product of locally compact spaces *need not* be locally compact. However, an arbitrary product of locally compact spaces where all but finitely many are compact *is* locally compact (This condition is sufficient and necessary).

Connectedness

- Every product of connected (resp. path-connected) spaces is connected (resp. path-connected).
- Every product of hereditarily disconnected spaces is hereditarily disconnected.

Metric spaces

- Countable products of metric spaces are metrizable spaces.

Axiom of choice

One of many ways to express the axiom of choice is to say that it is equivalent to the statement that the Cartesian product of a collection of non-empty sets is non-empty.[2] The proof that this is equivalent to the statement of the axiom in terms of choice functions is immediate: one needs only to pick an element from each set to find a representative in the product. Conversely, a representative of the product is a set which contains exactly one element from each component.

The axiom of choice occurs again in the study of (topological) product spaces; for example, Tychonoff's theorem on compact sets is a more complex and subtle example of a statement that requires the axiom of choice and is equivalent to it in its most general formulation,[3] and shows why the product topology may be considered the more useful topology to put on a Cartesian product.

See also

- Disjoint union (topology) – space formed by equipping the disjoint union of the underlying sets with a natural topology called the disjoint union topology
- Final topology – Finest topology making some functions continuous
- Initial topology – Coarsest topology making certain functions continuous - Sometimes called the projective limit topology
- Inverse limit – Construction in category theory
- Pointwise convergence – A notion of convergence in mathematics
- Quotient space (topology) – Topological space construction
- Subspace (topology) – Inherited topology
- Weak topology – Mathematical term

Notes

1. Bourbaki 1989, pp. 43–50.
2. Pervin, William J. (1964), *Foundations of General Topology*, Academic Press, p. 33
3. Hocking, John G.; Young, Gail S. (1988) [1961], *Topology* (https://archive.org/details/topology00hock_0/page/28), Dover, p. 28 (https://archive.org/details/topology00hock_0/page/28), ISBN 978-0-486-65676-2

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