Sinusoidal Sources

You can introduce a sinusoidal source into a circuit, with $V = V_0 \sin \omega t$. The Fourier series says that you can decompose any signal into the sum of sine waves. The response to single sine waves is called AC theory.

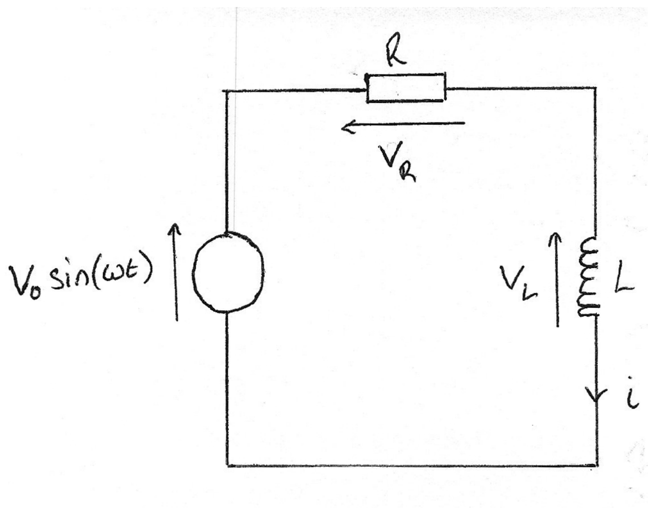

Figure 1: An LC Circuit with a Sinusoidal Source.

KVL: $V_0 \sin(\omega t) - V_R - V_L = 0$
KCL: One loop therefore one current i

A combination with the ideal relations gives

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V_0}{L} \sin(\omega t)$$

To which we can guess a solution

Guess

$$i(t) = I_0 \sin(\omega t + \phi)$$

Differentiate

$$\frac{di}{dt} = I_0 \omega \cos(\omega t + \phi)$$

Substitute

$$I_0\omega\cos(\omega t+\phi)+\frac{R}{L}I_0\sin(\omega t+\phi)=\frac{V_0}{L}\sin(\omega t)$$

To verify that this is correct we need to find values of (I_0, ϕ) that work for $t \geq 0$. So, for $t = 0$

$$0 = I_0\omega\cos(\phi)+\frac{R}{L}\sin(\phi)$$

Factorising I_0

$$\omega\cos(\phi)+\frac{R}{L}\sin(\phi)=0$$

Then, dividing though by $\cos(\phi)$

$$\omega+\frac{R}{L}\tan(\phi)=0$$

So, for our guess to be correct

$$\phi=\tan^{-1}\left(-\frac{\omega L}{R}\right)$$

Then, you can do the same for $t = \frac{\pi}{2\omega}$. You use this value because it shifts a sine function into the cosine function, and the cosine function to the negative of the sine function.

$$I_0\omega\cos(\omega t+\phi)+\frac{R}{L}I_0\sin(\omega t+\phi)=\frac{V_0}{L}\sin(\omega t)$$

$$\frac{V_0}{L}\sin\left(\frac{\pi}{2}\right)=I_0\cos\left(\frac{\pi}{2}+\phi\right)+\frac{R}{L}I_0\sin\left(\frac{\pi}{2}+\phi\right)$$

$$\frac{V_0}{L}=-I_0\sin(\phi)+\frac{R}{L}I_0\cos(\phi)$$

Now, using trigonometric identities you can find a nicer expression for substitution, using our know value for ϕ.

$$\sin\phi=\frac{-\omega L}{\sqrt{R^2+\omega^2L^2}}$$

$$\cos\phi=\frac{R}{\sqrt{R^2+\omega^2L^2}}$$

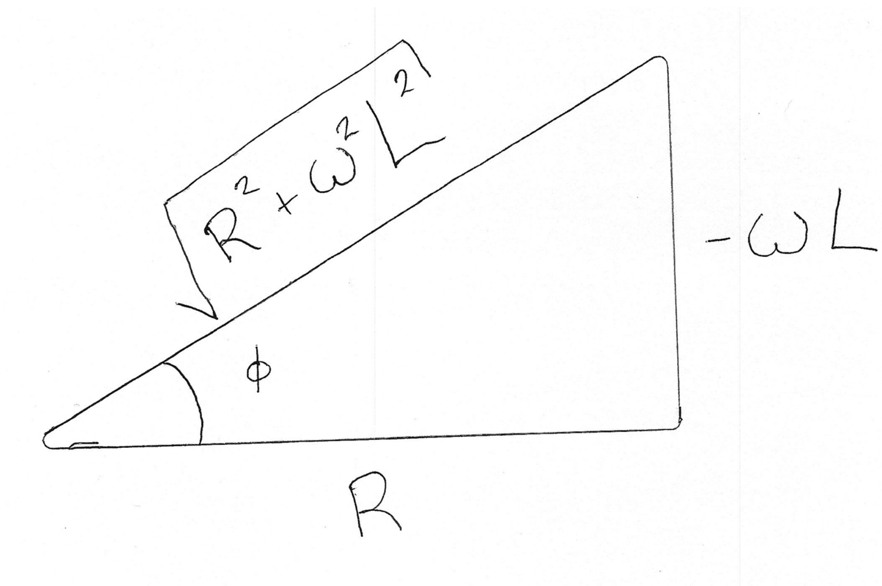

Figure 2: Diagram reprisenting $\tan\phi$ as a triangle.

Then, by substitution

$$\frac{V_0}{L} = -I_0\omega\left(\frac{-\omega L}{\sqrt{R^2+\omega^2L^2}}\right) + I_0\frac{R}{L}\left(\frac{R}{\sqrt{R^2+\omega^2L^2}}\right)$$

$$\frac{V_0}{L} = \frac{I_0}{L}\frac{R^2+\omega^2L^2}{\sqrt{R^2+\omega^2L^2}}$$

$$\frac{V_0}{L} = \frac{I_0}{L}\sqrt{R^2+\omega^2L^2}$$

$$I_0 = \frac{V_0}{\sqrt{R^2+\omega^2L^2}}$$

If the guess is correct then our equations for I_0 and ϕ are correct. So finally we have to verify our solution by a final substitution back into $i(t)$

$$i(t) = \frac{V_0}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$