The Wave Equation

Waves on a stretched string can have different wave shapes, or wave forms, but they all have the same wave speed, for a string with a set tension - but what are some other features?

To find these features we can form a differential equation. If we assume that the amplitude of the wave is small, and take a small section of the string l, we can apply Newton's second law

$$F = ma$$

and we know the linear density of the string μ and it's length l so

$$m = \mu l$$

and, knowing the vertical acceleration a_y we can find F_y

$$F_y = \mu l a_y$$

and we also know that

$$a_y = \frac{d^2 y}{dt^2}$$

The string has two tensions on it, one pulling it left, $\vec{F_1}$, and one pulling it right, $\vec{F_2}$.

Using this we can find the resultant vertical force on l

$$F_y = F_{2_y} - F_{1_y}$$

So, to find F_{i_y}

$$\frac{F_{i_y}}{F_{i_x}} = S_i$$

$$F_{i_y} = F_{i_x} S_i \approx \tau S_2$$

So the total force

$$F_y = \tau S_2 - \tau S_1$$

Which we can then use to form the differential equation

$$\frac{\tau S_2 - \tau S_1}{dx} = \frac{\mu}{\tau}\frac{d^2y}{dt^2}$$

Which we can then manipulate to give

$$\frac{\delta^2 y}{\delta x^2} = \frac{\mu}{\tau}\frac{\delta^2 y}{\delta dt^2}$$

and, because we know that $v = \sqrt{\tau/\mu}$

$$\frac{\delta^2 y}{\delta x^2} = \frac{1}{v^2}\frac{\delta^2 y}{\delta dt^2}$$

Example: Does the following equation for a periodic wave satisfy the wave equation?

$$y(x,t) = y_m \sin(kx - \omega t)$$

So, substituting

$$\frac{d^2}{dx^2}\big(y_m \sin(kx - \omega t)\big) = \frac{1}{v^2}\frac{d^2}{dt^2}\big(y_m \sin(kx - \omega t)\big)$$

Then differentiating

$$-k^2 y_m \sin(kx - \omega t) = -\frac{1}{v^2} y_m \omega^2 \sin(kx - \omega t)$$

Removing like terms

$$k^2 = \frac{\omega^2}{v^2}$$

$$k = \frac{\omega}{v}$$

Which we know is true from previous lectures, therefore the wave satisfies the wave equation.

A general wave has the following form

$$y(x,t) = h(kx \pm \omega t)$$

It is therefore possible to prove that any wave fits the wave formula.