Coupled Oscillators

A coupled oscillator is where twi SHM systems are attached, for example, connecting two mass-sprint systems with another spring.

Using Hooke's Law, $F = -kx$, and Newton II, $F = ma = m\frac{d^2x}{dt^2}$:

Dynamics of m_1:

$$\begin{aligned} m\frac{\delta^2 x_1}{\delta t^2} &= -kx_1 - k(x_1 - x_2) = \\ &= -2kx_1 + kx_2 \end{aligned}$$

Dynamics of m_2:

$$\begin{aligned} m\frac{\delta^2 x_2}{\delta t^2} &= -kx_2 - k(x_2 - x_1) = \\ &= -2kx_2 + kx_1 \end{aligned}$$

From which you can get:

$$\frac{\delta^2(x_1 + x_2)}{\delta t^2} = -\frac{k}{m}(x_1 + x_2)$$

So, further:

$$x_1 + x_2 = x_{m1}\cos{(\omega t + \phi_1)}$$

$$x_1 - x_2 = x_{m2}\cos{(\omega t + \phi_2)}$$

Even further:

$$x_1(t) = \frac{1}{2}x_{m1}\cos(\omega_1 t + \phi_1) + \frac{1}{2}x_{m1}\cos(\omega_2 t + \phi_2)$$

$$x_2(t) = \frac{1}{2}x_{m1}\cos(\omega_1 t + \phi_1) - \frac{1}{2}x_{m1}\cos(\omega_2 t + \phi_2)$$

Forced Coupled Oscillators

Just as in normal forced SHM we apply a harmonic driving force, only to one of the two masses:

$$F_e(t) = F_0 \cos(\omega_d t)$$

Where ω_d can be the same or different as ω_1 or ω_2.

Using Newton II ($F = ma$) we can derive a differential equation for the two masses:

$$x_1 : \quad m\frac{d^2x_1}{dt^2} = -kx_1 - k(x_1 - x_2) + F_0 cos(\omega_d t)$$

$$x_2 : \quad m\frac{d^2x_2}{dt^2} = -kx_1 - k(x_1 - x_2)$$

So, adding the two equations gives:

$$m\frac{d^2(x_1 + x_2)}{dt^2} = -k(x_1 + x_2) + F_0 \cos(\omega_d t)$$

and subtracting them

$$m\frac{d^2(x_1 - x_2)}{dt^2} = F_0 \cos(\omega_d t) - 3k(x_1 - x_2)$$

and we know that

$$x_m = \frac{\frac{F_0}{m}}{\omega^2 - \omega_d^2}$$

so

$$x_1 + x_2 = \frac{\frac{F_0}{m}}{\omega_1^2 - \omega_d^2} \cos(\omega_d t)$$