Complex notes

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A path or a curve is a continuous function, $\gamma : [a, b] \to \mathbb{C}$ ($Rng(\gamma) \subset \mathbb{C}$)

$\gamma(a)$: initial point of path; $\gamma(b)$: endpoint of path

$[a, b]$: parameter interval

γ is said to be:

1. closed if $\gamma(a) = \gamma(b)$
2. smooth or C^1 if γ is differentiable and $\gamma^{'}$ is continuous
3. simple if γ is one-one
4. simple closed if $\gamma(a) = \gamma(b)$ and γ is one-one on (a, b)
5. piecewise smooth if there are finitely many points $s_0, s_1 \ldots s_n \in [a.b]$ with
   $a = s_0 < s_1 < s_2 \cdots < s_n = b$ such that the restriction of γ to each (s_i, s_{i+1}) is smooth.

$-\gamma$ or γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(a + b - t)$

$\phi : [0, 1] \to [a, b]$ defined as: $\phi(t) = a + (b - a)t$ (one-one and differentiable)

Line integral: $f : [a, b] \to \mathbb{C}$: continuous

$f = u + iv$, where $u, v : [a, b] \to \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$

Properties:

1. $\int_a^b c.f(t)dt = c. \int_a^b f(t)dt$
2. $| \int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Length of a smooth curve: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve.

$L(\gamma) = \int_a^b |\gamma^{'}(t)|dt = \int_a^b \sqrt{\gamma_1^{'}(t)^2 + \gamma_2^{'}(t)^2}dt$ $(\gamma(t) = \gamma_1(t) + i.\gamma_2(t))$

If $\gamma : [a, b] \to C$ is piecewise smooth then $L(\gamma)$ is the sum of the length of its smooth parts.

Defn(orientation):A curve γ is positively oriented if traversed in anti-clockwise direction else is negatively oriented.

Examples:

1. $\gamma(t) = r.e^{it}$, $(t \in [0, 2\pi])$ $(r > 0$: simple, smooth curve);

   $L(\gamma) = \int_0^{2\pi} |i.r.e^{it}|dt = r.(2\pi)$

2. $\gamma(t) = e^{it}$, $(t \in [0, 4\pi])$: closed, smooth, traverses the unit circle twice in the positive direction

Integration over paths: $\gamma[a, b] \to \mathbb{C}$ is a smooth curve and $f : \gamma \to \mathbb{C}$: continuous

Defn: $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma^{'}(t)dt = \int_a^b g(t)dt$

$(g(t) = f(\gamma(t))\gamma^{'}(t)$ where $g : [a, b] \to \mathbb{C})$

Let $[a_1, b_1]$ be any closed interval. Then $\exists \phi : [a_1, b_1] \to [a, b]$ (one-one,differentiable and $\phi(a_1) = a; \phi(a_2) = b)$

$\phi[a_1, b_1] \to \mathbb{C}$: smooth

$\int_{a_1}^{b_1} f(\gamma_1(t)).\gamma_1^{'}(t)dt$ $(= \int_{\gamma_1} f(z)dz)$

$= \int_{a_1}^{b_1} f(\gamma(\phi(t))).\phi^{'}(t)dt = \int_\gamma f(\gamma(s))\gamma^{'}(s)ds = \int_\gamma f(z)dz$ $(\phi(t) = s)$

If γ is piecewise smooth, the integral can be split into the sum of its smooth components: if $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$, then $\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f$.

Note that $\gamma_i's$ are smooth.

Proposition: If f and g are continuous on a smooth curve γ, then

1. $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$

2. $\int_{\gamma^-} f = -\int_\gamma f$

3. $|\int_\gamma f(z)dz| \leq \|f\|_{\infty,\gamma} L(\gamma)$ $(\|f\|_{\infty,\gamma} = sup_{z\in\{\gamma\}}|f(z)|)$

$|\int_\gamma f| = |\int_a^b f(\gamma(t)).\gamma^{'}(t)dt| \leq \int_\gamma |f(\gamma(t)).\gamma^{'}(t)|dt \leq \|f\|_{\infty,\gamma} \int_a^b |\gamma^{'}(t)|dt$ $(L(\gamma) = \int_a^b |\gamma^{'}(t)|dt)$

Examples:

(i)Let γ be the arc of a circle of radius 3 $(|z| = 3)$ from 3 to $3i$.

Show that:

$$|\int_\gamma \frac{z+4}{z^3-1} dz| \leq \frac{21\pi}{52}$$

(ii) $\gamma : |z| = 2$ (traverse curve in positive direction)

Prove:

$$|\int_\gamma \frac{e^z dz}{z^2+1}| \leq \frac{4\pi e^2}{3}$$

Fundamental theorem of calculus:

If $f:[a,b]\to\mathbb{R}$ has a primitive F, then $\int_a^b f(x)dx = F(b)-F(a)$ $(F^{'}(x)=f(x), \forall x\in[a,b])$

Definition: Suppose $G\in\mathbb{C}$ be a domain. If a continuous function $f:G\to\mathbb{C}$ has a primitive F on G and if γ is a smooth curve in G with initial and terminal points ω_1 and ω_2 respectively, then:

$\int_\gamma f = F(\omega_1)-F(\omega_2)$

Proof: Let $[a,b]\in\mathbb{R}$ be a parameter interval for γ and $\gamma(a)=\omega_1$; $\gamma(b)=\omega_2$

Given $F^{'}(z)=f(z)$ $(\forall z\in G)$

$$\int_\gamma f = \int_a^b f(\gamma(t)).\gamma^{'}(t)dt = \int_a^b F^{'}(\gamma(t))\gamma^{'}(t)dt$$

$$= \int_a^b (F\circ\gamma)^{'}(t)dt = F\circ\gamma(b) - F\circ\gamma(a) = F(\omega_2)-F(\omega_1)$$

Corollary-1: If γ is a closed curve (smooth), then

$\int_\gamma f = 0$ (Proof follows from FTC)

Corollary-2: If $f\in H(\Omega)$ for a region $\Omega\in\mathbb{C}$ and if $f^{'}=0$ on Ω, then f is a constant function.

Proof: Fix a point $\omega_0\in\Omega$. It suffices to show that $f(\omega)=f(\omega_0), \forall\omega\in\Omega$

Simple Closed Curve:

Jordan-curve theorem: Every simple closed curve in $\mathbb{C}$ divides the complex plane into two regions. One of these regions is bounded and the other is unbounded. The bounded region is called the interior of the curve.

Example: $G=\mathbb{C}\setminus\{0\}$

$f(z)=\frac{1}{z}$ on G, $\gamma:|z|=1$, $\gamma(t)=e^{it}$, $(t\in[0,2\pi])$

$$\int_\gamma f = \int_0^{2\pi} f(\gamma(t)).\gamma^{'}(t)dt = \int_0^{2\pi}\frac{i.e^{it}}{e^{it}}dt = 2\pi i \neq 0$$

Winding number or index of a closed curve: Letγ be a closed curve on $\mathbb{C}$ and let $\alpha\in\mathbb{C}\setminus\{\gamma\}$. The winding number of γ about α or the index of γ with respect to α is denoted by: $\eta(\gamma;\alpha)/Ind_\gamma(\alpha)$ defined by:

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha}$$

Example: $\gamma : [0, 6\pi] \to \mathbb{C}$

$\gamma(t) = a + re^{it}$

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{\gamma^{'}(t)}{\gamma(t)-a} = \frac{1}{2\pi i}\int_\gamma \frac{1}{a+re^{it}-a}.ire^{it}dt = 1$$

Theorem: Let γ be a smooth, closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then $\eta(\gamma;\alpha) \in \mathbb{Z}$.

Proof: To be done