(Almost) Fair Allocation of Indivisible Goods and Chores

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Abstract

Fair allocation of goods and chores falls under the broad category of multi-agent systems and economics and has been extensively studied due to its practical importance in industrial and social decision-making. Although the academic research on the allocation of divisible goods/chores started in 1948, the study of indivisible items posed new challenges for algorithm designers. Here, we consider the problem of dividing a set of indivisible items among agents fairly and efficiently. While the vast majority of literature on this topic assumes a subset of this problem where all items contain a non-negative utility (i.e., goods), we want to focus on a general setting where an item can be either a good or a chore for different agents. In this review article, we first introduce different *relaxed* versions of fairness notions and highlight the recent advances in the fair allocation of indivisible goods and chores. We also highlight open problems that persist for goods-only, chore-only, and mixed-manna setting that remains to be solved before further advancement in this field.

1 Introduction

The problem of fairly dividing a set of resources among agents is of central importance in various fields, including economics, computer science, and political science. Although the divide-and-choose algorithm can be traced back to the Bible (5), the modern research of fair allocation started with the work of Steinhaus in 1947 (20). His work motivated a large body of research on algorithmic fair allocation of objects among individuals. However, the vast majority of the earlier work dealt with divisible items, on the classic 'cake-cutting' problem. Aziz and Mackenzie solved that problem in their 2016 article (6) by showing that an *envy-free*(15) and *proportional*(20) allocation of a divisible cake always exists and can be computed in bounded steps.

Indivisible items pose additional challenges because it is not always possible to allocate indivisible items in *envy-free* and *proportional* manner, even in simple cases. For example, let us assume there is one (indivisible) candy that we have to allocate between two children. No matter how much we try, it is impossible to allocate that candy fairly. This impossibility of absolute fairness sparked much interest in the last decade where computer scientists, economists, and mathematicians tried to come up with reasonable relaxations of *envy-freeness* (e.g., EF1, EFX, EFL...) and *proportinality* (e.g., PROP1, PROPX, MMS...) and design algorithms that can allocate indivisible items obeying those *relaxed* notions of fairness. However, the majority of the work in the last decade focused on only a sub-area of the indivisible item allocation problem where all the items are considered *goods*. Refer to these excellent review papers (5; 2) for a broad overview of different algorithmic achievements and challenges in indivisible *good* allocation problem.

In this review article, we will focus on a general version of the indivisible item allocation problem where an item can be a *good* or a *chore* depending upon the agent (mixed manna setting). Although the number of open problems in indivisible *good* allocation is not small, adding *chores* along with it poses extra complications which forced researchers to restrict several parts of the mixed manna problem (e.g. *separable* setting, *restricted mixed goods* setting...) for providing provably fair allocations. The flow of the article will be following: We will first formalize the problem statement and define the fairness notions and then present some of the most recent advancements and open problems. However, unlike most traditional review articles, we will also discuss brief sketches of some of the algorithms and their correctness proves, rather than just mentioning their existence in the literature.

2 The mixed manna setting

We can formalize our problem using a *fair division instance* (N, M, U) where we allocate a set of m indivisible items M among n agents N. Here, $U = \{u_1, u_2, ..., u_n | u_i : 2^M \to \mathbb{R}\ \forall i \in [n]\}$ is the set of utility functions, one for each agent. Each utility function u_i is specified by m numbers $u_{ij} \in \mathbb{R}\ \forall j \in M$. Intuitively, u_{ij} denotes the value of item j by agent i. An item j is considered good (respectively, chore) by agent i if $u_{ij} > 0$ (respectively, $u_{ij} < 0$). An item with $u_{ij} = 0$ is generally considered a dummy item which often is included among goods or chores depending upon the formulation of the individual authors.

An allocation is represented by an n-tuple $X = (X_1, ..., X_n)$ of M, where $X_i \subseteq M$ is the bundle allocated to agent i under the constraint $X_i \cap X_j = \emptyset\ \forall i \neq j$ and $\cup_{i \in N} X_i = M$. To keep our problems reasonably simple, we consider *additive utilities*, namely $u_i(X) = \sum_{o \in X_i} u_i(o)$.

In the mixed manna setting, a useful way to partition the set of items M is into M^+ (set of *mixed goods*), M^0 (set of *dummy chores*), and M^- (set of *pure chores*). These terms are defined as following:

$$
\begin{aligned}
&M^+ := \{j \in M |\ \exists\, i \in N, u_{ij} > 0\} \\
&M^0 := \{j \in M |\ \forall\, i \in N, u_{ij} \leq 0\ \&\ \exists\, i \in N u_{ij} = 0\} \\
&M^- := \{j \in M |\ \forall\, i \in N, u_{ij} < 0\}
\end{aligned}
$$

To make the mixed manna setting more manageable, the fair division instance (N, M, U) is subclassified into different types such as *separable*, *restricted mixed goods*, *binary mixed goods*, and *identical ordering* instances.

- A fair division instance is called *separable* if the items in M can be partitioned into $M^{\geq 0} := \{j \in M |\ \forall\, i \in N, u_{ij} \geq 0\}$, the set of items that are not chore for any agent, and set of pure chores M^-.
- An instance is called *restricted mixed goods (RMG)* if $\forall\, j \in M^+$, there exists a value $v_j > 0$ such that $\forall\, i \in N$, if $u_{ij} > 0$, then $u_{ij} = v_j$. An important point to notice here is that RMG instance does not impose any restriction on u_{ij} values if they are ≤ 0.
- A *binary mixed goods* instance is a special case of RMG instance where $\forall\, j, j' \in M^+, v_j = v_{j'}$.
- An instance is called *identical ordering* (IDO) if all the agents have the same ordinal preference for all the items in M. A special case of IDO is called *identical* setting where for every $j \in M, u_{ij} = u_{i'j}\ \forall\, i, i' \in N$.

3 Pareto-optimality

In addition to fairness, efficiency criterion is also considered in many algorithms. The most common efficiency concept used in fair allocation literature is *Pareto-optimality*

Definition 3.1. *(**PO**) Given an allocation X, another allocation X' is a Pareto-improvement of X if $u_i(X'_i) \geq u_i(X_i)$ for all $i \in N$ and $u_j(X'_j) > u_j(X_j)$ for some $j \in N$. We say that an allocation is Pareto-optimal (PO) if there is no allocation that is a Pareto-improvement of X.*

4 Welfare functions

Two of the most common social welfare function we are going to use in this review are *(utilitarian) social welfare* and *Nash welfare*, which we will define next.

Definition 4.1. ***(SW)*** *Given an allocation X, the (utilitarian) social welfare $[SW(X)]$ of X is the sum of agent's utilities under X.*

$$SW(X) \coloneqq \sum_{i \in N} u_i(X_i) = \sum_{i \in N} \sum_{j \in X_i} u_{ij}$$

Definition 4.2. ***(NW)*** *Given an allocation X, the Nash welfare $[NW(X)]$ of X is the geometric mean of the agent's utilities under X.*

$$NW(X) \coloneqq \left(\prod_{i \in N} u_i(X_i) \right)^{1/n}$$

Lemma 4.3. *An allocation in which every item is allocated to the agent that values it the most or hates it the least is PO and maximizes the social welfare. (19)*

Proof. Since every item is given to the agent that values it the most or hates it the least in our allocation. Re-allocating any item from their already allocated agent to a new one will not be a Pareto-improvement of the proposed allocation. Hence, the proposed allocation is PO.

In our proposed allocation X,

$$\begin{aligned} SW(X) &= \sum_{i \in N} \sum_{j \in X_i} \max_{i \in N} (u_{ij}) \\ &= \sum_{j \in M} \max_{i \in N} (u_{ij}) \\ &\geq \sum_{i \in N} \sum_{j \in Y_i} \max_{i \in N} (u_{ij}) \\ &\geq SW(Y) \quad \forall \text{ allocation } Y \end{aligned}$$

Hence, the proposed allocation also maximized the social welfare function. □

5 Envy-freeness

Definition 5.1. ***(EF)*** *An allocation X is envy-free (EF) if for any two agents $i, j \in N$, we have $v_i(X_i) \geq v_i(X_j)$.*

But, since the problem of checking if a given instance is an EF allocation is NP-complete even for restricted settings (4; 10), researchers have turned their attention to several relaxations of EF which we will discuss next.

5.1 EF1

Envy-freeness upto one item (EF1) for good only setting, although was first introduced by Lipton et al. in 2004 (18), was formally defined by Budish (2011) (11). Here, we present a general version of EF1 for the mixed manna setting.

Definition 5.2. ***(EF1)*** *Given allocation X, we say that agent i envies agent j by more than one item if i envies j and $u_i(X_i - o) < u_i(X_j - o)$[1] for each item $o \in X_i \cup X_j$. An allocation X is envy-free upto one item (EF1) if $\forall i, j \in N$, i does not envy j by more than one item.*

Aziz et al.(3) proposed a *double round-robin algorithm* (Algorithm 1) and showed that it can construct an EF1 allocation. The sketch of the *double round-robin algorithm* is as followed: (1) Partition the objects into M^+ and M^-. (2) Let the agents choose the items from M^- in $(1, 2, ..., n)^*$ round-robin sequence and then items from M^+ in the reverse sequence.

Theorem 5.3. *Algorithm 1, in $O(\max\{m \log m, mn\})$ time, returns an EF1 allocation.*

[1] For notational ease, I am using $u(X - o)$ and $u(X + o)$ instead of $u(X \backslash \{o\})$ and $u(X \cup \{o\})$ respectively in the whole review article.

Algorithm 1 EF1 allocation in mixed-manna setting

1: **function** ALLOCATE_EF1((N, M, U))
2: $X \leftarrow (\emptyset, \emptyset, ..., \emptyset)$
3: Partition M into M^+ and $M^\dagger := (M^- \cup M^0)$
4: If $|M^\dagger| = an - k$ for some positive integer a and $k \in [n-1]$, create k dummy null items for which each agent has utility 0. Then, add those items to $M^\dagger$
5: Allocate items in $M^\dagger$ in round-robin sequence $(1, 2, ..., n)^*$ such that each agent chooses his most preferred item.
6: Allocate items in M^+ in round-robin sequence $(n, n-1, ..., 2, 1)^*$ such that each agent chooses his most preferred item. If an agent has no available item that gives him strictly positive utility, he doesn't chooses any item and pretends to pick a dummy item with 0 utility.
7: $X' \leftarrow X \setminus \{\text{dummy items}\}$
8: Return X'

Proof. (Sketch)

Pick two agents i, j such that $i < j$ in the round-robin sequence. Let, c_r^t be the item allocated to agent r in t-th round $\forall t \in [a]$ of line 5 of the algorithm. Similarly g_r^t be the item allocated to agent r in t-th round $\forall t \in [b]$ of line 6 of the algorithm.

Since i gets to select items from $M^\dagger$ before j, i does not envy j w.r.t $M^\dagger$. Similarly, j does not envy i w.r.t M^+. Now, i will prefer his own allocation w.r.t M^+ if we remove g_j^1 from j's allocation. Similarly, j will prefer his own allocation w.r.t $M^\dagger$ if we remove c_j^a from j's allocation. Hence, just by removing one good or chore we can make the allocation given by Algorithm 1 envy-free which implies EF1.

Line 3 of the algorithm takes $O(mn)$ time since each item needs to be checked for each agent separately. Line 5 and 6 requires $O(m \log m)$ time as there can be at max m iterations and in each iteration, the most preferred item for each agent can be found by sorting the items according to the preference of every agent at the beginning. Thus, the total running time of the algorithm is $O(\max\{m \log m, mn\})$. □

The presence of PO+EF1 allocation in mixed manna setting has been proved for **two player** setting by Aziz et al.(3) in the Algorithm 2.

Algorithm 2 EF1+PO allocation in mixed-manna setting in two agent game

function ALLOCATE_EF1_PO((N, M, U) where $N = \{w, l\}$)
$X(w) \leftarrow (\emptyset)$ and $X(l) \leftarrow (\emptyset)$
Let $O_w^* = \{o \in M \mid u_w(o) \geq 0 \,\&\, u_l(o) \leq 0\}$ and $O_l^* = \{o \in M \mid u_l(o) \geq 0 \,\&\, u_w(o) < 0\}$
Let $O^+ = \{o \in O \mid u_i(o) > 0 \; \forall \; i \in N\}$ and $O^- = \{o \in O \mid u_i(o) < 0 \; \forall \; i \in N\}$
For each item $o \in O^+ \cup O_w^*$, allocate o to agent w. For each item $o \in O^- \cup O_l^*$, allocate o to agent l.
Sort items in $O^+ \cup O^- = \{o_1, o_2, ..., o_r\}$ where $|u_l(o_1)|/|u_w(o_1)| \geq ... \geq |u_l(o_r)|/|u_w(o_r)|$
Set $t = 1$
while agent l envies w by more than one item **do**
  if $o_t \in O^+$ **then**
    $X(w) \leftarrow X(w) - o_t$
    $X(l) \leftarrow X(l) + o_t$
  else if $o_t \in O^-$ **then**
    $X(w) \leftarrow X(w) + o_t$
    $X(l) \leftarrow X(l) - o_t$
  $t \leftarrow t + 1$
Return $X = (X(w), X(l))$

Theorem 5.4. *For two agents, a PO + EF1 allocation always exists and can be computed in $O(m^2)$ time using Algorithm 2.*

Proof. *(Sketch)* We can easily show that, in the Algorithm 2, line 5 onward the allocation satisfies PO. For showing this PO invariance, we can argue that items in O_w^* will always remain with the winner w and items in O_l^* will remain with the looser l. Because, reallocating any item in those set will harm both l and w. Next, we can show that existence of any Pareto-improvement of X including the other items will lead to a contradiction.

Once, the PO invariance is proved, we can argue that if the loser envies the winner at allocation X, the winner does not envy the loser. Because if that is not true, then there must exists a Pareto-improvement of X which is not possible. Let's consider that at allocation X, the loser does not envy the winner, but the winner envies the loser. This mean, just one iteration ago in the while loop, when the allocation is X' (let's say), the loser envies the winner by more than one item. Now, we can exclude the item being transferred in the last iteration (the iteration that converts $X' \to X$), the loser envies the winner by more than one agent (by construction). Now, if we consider that the winner envies the loser by more than one item at X, contradictions arise, resulting in the fact that winner doesn't envy the loser at X by more than one item. Hence, EF1.

The sorting of the items can be done in $O(m \log m)$ time. Each single iteration of the while loop requires us to make sure the allocation is EF1 from loser's perspective. This requires $O(m^2)$ time. Hence, the time complexity of Algorithm 2 is $O(m^2)$. □

A natural question is whether PO+EF1 allocations exist for ≥ 3 agents and whether it can be computed in polynomial time, which is still an **open problem**.

5.2 EFX

Although, EF1 is one of the most studied relaxation of EF, it is quite weak in itself. EF1 insists that envy disappears after the removal of the *most valuable good* according to the envying agent from the envied agent's bundle or removal of the *costliest chore* from his own bundle. But, in many cases the most valuable good / costliest chore might be primary reason for very large envy to exist in the first place (13). This led to the search for stronger notions of fairness which are often more desirable in practical scenario and envy-free up to any item (EFX) is one such fairness notion.

EFX relaxation was proposed originally for the allocation of goods by Caragiannis et al. (12). In the mixed manna setting, EFX requires that envy can be eliminated by removing any one good from envied agent's bundle or any one chore from his own bundle. Formally we can define it the following way

Definition 5.5. *(**EFX**) An allocation X is envy-free upto any item (EFX) if, for every pair of agents $i, j \in N$, it holds that $(u_i(X_i) - o) \geq (u_i(X_j) - o)\ \forall\, o \in X_i \cup X_j$ such that $|o| > 0$.*

There exists another slightly stronger version of EFX named EFX_0 (17) which requires that envy must disappear after removal of any non-negative valued item. It is easy to see that $\text{EFX}_0 \Rightarrow$ EFX, but the other way round (19). It is also worth nothing that EFX $\Rightarrow$ EF1, which is obvious from their definitions.

But, unlike EF1 allocation, existence of EFX allocations in a general mixed manna setting is an **open problem** and remains unknown till date.

In the goods only scenario, Chaudhury et al. (13) proved the existence of EFX allocation for three agent and there are several work done on existence and polynomial time computation of EFX allocation in restricted settings (2; 5). But, the existence of an EFX allocation for general goods only setting for $n \geq 4$ agents remains an **open problem**. The importance of this problem has also led to multiple articles showing the existence of a more relaxed version of EFX, α-EFX allocation for different values of α (2). But, the finding the best value of $alpha$ for which α-EFX allocation exists remains an **open problem**.

Livanos et al. proved that EFX+PO allocation exists for a fair division instance with restricted mixed goods (RMG) and identical chores setting, which also maximizes the social welfare. They proved their claim by providing a polynomial time algorithm, which we shall discuss in the following paragraphs.

But before explaining the algorithm we need to discuss an important solution concepts named *envy graph* (19) and a standard algorithmic technique known as *envy-cycle elimination* (it was first proposed by Lipton et al. (18)). An *envy graph* is a graph representation of an allocation where each agent is a node and there exits a directed edge from agent i to agent h iff i envies h. Now, notice that if there exists a directed cycle in the envy-graph for some allocation, then we can reallocate the bundles among the agents in that cycle in reverse order and all those agents will receive a bundle that they prefer over their current bundle, without changing the utility of the agents outside that cycle. This procedure is called *envy-cycle elimination* that results in Pareto-improvement. A slightly different version of *envy-graph* is called *top envy-graph* where a directed edge is presented from agent i to agent h if i envies h the **most** (among all other agents).

The polynomial time algorithm proposed by Livanos et al. (19) for EFX+PO allocation first divides the items into M^+, M^0 and M^-. Then it allocates those sets among the agents in such a way that after allocating one of those three sets, the allocation satisfies EFX+PO. The pseudocode of the algorithm is presented in Algorithm 3. An interesting thing to notice here is that this algorithm allocates an item to one of the agents who value it the most, hence it also maximizes social welfare.

Algorithm 3 EFX+PO for RMG + identical chores setting

function ALLOCATE((N, M, U))
    $X \leftarrow (\emptyset, \emptyset, ..., \emptyset)$ ▷ Empty Allocation
    Partition M into M^+, M^-, and M^0
    $X \leftarrow$ GoodAllocate($X, (N, M^+, U)$) ▷ Phase 1: Allocating M^+ items
    $X \leftarrow$ ZeroAllocate($X, (N, M^0, U)$) ▷ Phase 2: Allocating M^0 items
    $X \leftarrow$ ChoreAllocate($X, (N, M^-, U)$) ▷ Phase 3: Allocating M^- items
    Return X ▷ This allocation is proved to be EFX+PO

function GOODALLOCATE($X, (N, M^+, U)$)
    Order and relabel the goods such that $v_1 \geq v_2 \geq v_3 \geq ... > 0$
    while $M^+ \neq \emptyset$ **do**
        Pick $j \in M$
        $N_j \leftarrow \{i \in N : u_{ij} = v_j\}$
        Let G_X be the envy-graph defined by X
        Let $G_j = G_X[N_j]$ be the sub-graph of G_X induced by N_j
        Let $i \in N_j$ be a source in G_j
        $X_i \leftarrow X_i + j$
        $M^+ \leftarrow M^+ - j$
    Return X

function ZEROALLOCATE($X, (N, M^0, U)$)
    while $M^0 \neq \emptyset$ **do**
        Pick $j \in M^0$
        Let $i \in N$ be such that $u_{ij} = 0$
        $X_i \leftarrow X_i + j$
        $M^0 \leftarrow M^0 - j$
    Return X

function CHOREALLOCATE($X, (N, M^-, U)$)
    Order chores in M^- according to $\prec$ such that $j \prec j'$ *iff* $-v_j \geq -v_{j'}$
    while $M^- \neq \emptyset$ **do**
        Pick smallest $j \in M^-$ according to $\prec$
        Let G_X be the envy-graph defined by x
        Let $i \in N$ be a *sink* in G_X
        $X_i \leftarrow X_i + j$
        $M^- \leftarrow M^- - j$
    Return X

Livanos et al. (19) also proposed a similar algorithm for EFX$_0$+PO allocation for *binary mixed goods* and *identical chores* setting. The primary different between Algorithm 3 and this one is the part where we allocate the M^+ and M^0 items. This algorithm uses 'ALG-BINARY' algorithm of Amantidis et al. (1) for allocating M^+, which is directly inspired by articles (14; 9). The pseudocode of the M^0 allocation algorithm is presented in Algorithm 4.

Algorithm 4 M^0 allocation for EFX$_0$+PO for binary mixed goods + indentical chores setting

function ZEROALLOCATE-2($X, (N, M^0, U)$)
  while $M^0 \neq \emptyset$ **do**
    Pick $j \in M^0$
    $N_j \coloneqq \{i \in N | u_{ij} = 0\}$
    Let G_X be the envy-graph defined by X
    Let $G_j = G_X[N_j]$ be the sub-graph defined by X
    Let $i \in N$ be a source in G_j
    $X_i \leftarrow X_i + j$
    $M^0 \leftarrow M^0 - j$
  Return X

The third contribution of Livanos et al. (19) was providing another polynomial time algorithm for EFX+PO allocation computation, this time for RMG + IDO chores setting. This algorithm is also quire similar to that of Algorithm 3, with a change in the M^- allocation which is described in Algorithm 5.

Algorithm 5 M^- allocation for EFX$_0$+PO for RMG + IDO chores setting

function CHOREALLOCATE-3($X, (N, M^0, U)$)
  Order chores in M^- in IDO order $u_{i1} \leq u_{i2} \leq ... \leq u_{i|M^-|}$
  while $M^0 \neq \emptyset$ **do**
    Pick smallest $j \in M^-$ according to $\prec$
    Let G^*_X be the top-envy-graph defined by X
    while any sink in G^*_X **do**
      Let C be a cycle in G^*_X
      Reallocate bundles according to C
    Let $i \in N$ be a sink in G^*_X
    $X_i \leftarrow X_i + j$
    $M^- \leftarrow M^- - j$
  Return X

6 Proportionality

Proportionality (PROP), first proposed by Steinhaus (20), is the most widely studied threshold-based solution concept in fair allocation literature (5). Formally it can be defined as:

Definition 6.1. *(**PROP**) An allocation X is proportional (PROP) if for every agent $i \in N$, we have $u_i(X_i) \geq$PROP$_i$, where PROP$_i \coloneqq (1/n) \cdot u_i(M)$.*

Aziz et al. (3) showed that for additive utilities and general mixed manna setting EF $\Rightarrow$ PROP but since existence of PROP is not possible for indivisible objects we look into different relaxations of this concept. One such relaxation novel relaxation is PROP1.

6.1 PROP1

Definition 6.2. *(**PROP1**) An allocation X satisfies proportionality up to one item (PROP1) if for each agent $i \in N$,*

- $u_i(X_i) \geq u_i(M)/n$*; or*
- $u_i(X_i) + u_i(o) \geq u_i(M)/n$ *for some* $o \in M \backslash X_i$*; or*
- $u_i(X_i) - u_i(o) \geq u_i(M)/n$ *for some* $o \in X_i$.

Aziz et al. (7) proposed a polynomial time algorithm for computing PO+PROP1 allocation in mixed manna setting. The proposed algorithm starts (unusually) with a partial allocation (which assumes that items are not indivisible) that satisfies PROP. Then it builds an acyclic *'consumption graph'* which rounds the fractional allocation of the items into an integral allocation (which at the end will satisfy the indivisibility of the items) satisfying PO and PROP1. This fractional to integral allocation rounding happens in the following way. In the trivial case, when one agent has fraction 1.0 of the objects and others have 0.0 fraction, the items is allocated to that agent. The general case is explained using the Algorithm 6.

Algorithm 6 fractional to integral allocation

$Q \leftarrow$ empty FIFO queue
while $\exists$ agent i sharing ≥ 1 item o with others **do**
    add i to Q
    while $Q \neq \{\emptyset\}$ **do**
        Take first agent j out of Q
        Add all neighbors of j to Q ▷ neighbors in the consumption graph
        for each o shared by j **do**
            if $u_j(o) > 0$ **then**
                give o to j fully
            else if $u_j(o) < 0$ **then**
                give o to a neighbor with whom o is shared
            Update the allocation

The authors proved that this overall algorithm will converge in a PROP1+PO allocation of the indivisible items in (strongly) polynomial time.

For general mixed manna setting and additive utilities an EF1 allocation implies PROP1 allocation(3). There exists another, slightly stringent version of PROP1 named PROP1$_{outer}$(3) which also enforces connectedness along with proportionality. An allocation X is called *connected* if for each agent $i \in N$, X_i is connected in the path $(o_1, o_2, ..., o_m)$. The formal definition of PROP1$_{outer}$ is as follows:

Definition 6.3. *(**PROP1$_{outer}$**) A connected allocation X is PROP1$_{outer}$ if for agent $i \in N$,*

- $u_i(X_i) \geq u_i(M)/n$*, or*
- $u_i(X_i) + u_i(o) \geq u_i(M)/n$ *for some item $o \in M$ such that $X_i + o$ is connected; or*
- $u_i(X_i) - u_i(o) \geq u_i(M)/n$ *for some item $o \in M$ such that $X_i - o$ is connected.*

Aziz et al. (3) provided an algorithm and proved that *"for additive utilities, a connected PROP1$_{outer}$ allocation of a path always exists and can be computed in polynomial time"*. Their *generalized moving-knife algorithm* is a recursive algorithm that treats the set M as a cake where different agents have different uniform preferences / dislike for some parts of the cake. The algorithm moves a knife from left to right (or right to left depending upon there is any agent that finds the chunk of the cake under consideration to be positive or not) and the agent who shouts first (assuming his proportionality is being achieved by that point) is given the chunk. The algorithm then recurses the same procedure over the leftover chunk and the rest of the agents.

7 Maximin Share Fairness

One of the most widely studied fairness notion in discrete fair division literature is *maximin share fairness* (MMS) introduced by Budish (2011) (11). In this fairness notion, the goal is to make sure agent i at least receives as much as her *maximim share* $d_i^n(M)$. The *maximin share* is the maximum value agent i can guarantee for herself by allocating the goods into n disjoint bundles and keeping the worst of them. It is worth noting that PROP $\Rightarrow$ MMS.

Definition 7.1. *(**MMS**) Let $A_n(M)$ be the collection of possible allocations of the goods in M for n agents. An allocation A is said to be maximin share fair (MMS) if $\forall i \in N$, $u_i(A_i) \geq d_i^n(M)$ holds* *true. Here,*

$$d_i^n := \max_{B \in A_n(M)} \min_{S \in B} u_i(S)$$

But, computing MMS allocations is NP-hard (2). Hence, we should look into other relaxations/versions of MMS.

Definition 7.2. ***(PropMX)*** *A mixed manna instance is called proportional up to the maximin good or any bad (PropMX) if $\forall i \in N$ either:*

- $u_i(X_i) + d_i(X) \geq Prop_i$, *where* $d_i(X) = \max_{\{i' \neq i\}} \min_{\{j \in X_{i'}, u_{ij} > 0\}} \{u_{ij}\}$, *or*
- $\forall c \in X_i$ *such that* $u_{ic} < 0, u_i(X_i - c) \geq PROP_i$.

Livanos et al. (19) proposed an polynomial time algorithm for computing a PropMX allocation in a separable instance with IDO chores of a mixed manna setting. The pseudocode of the algorithm is presented in Algorithm 7.

Algorithm 7 PropMX allocation in separable instances with IDO chores

function ALLOCATE4((N, M, U))
    $X \leftarrow (\emptyset, \emptyset, ..., \emptyset)$ ▷ Empty Allocation
    Partition M into $M^{\geq 0}$ and M^-
    Order the chores in M^- such that $u_{i1} \leq u_{i2} \leq ... \leq u_{i|M^-|}$
    $M^{\geq 0} \leftarrow \{j \in M : \forall i \in N, u_{ij} \geq 0\}$
    $M^- \leftarrow M \backslash M^{\geq 0}$
    $X^{\geq 0} \leftarrow$ GOODS($M^{\geq 0}$) ▷ PropM$_0$ algorithm of (8)
    $X \leftarrow X^{\geq 0}$
    for $j \leftarrow 1$ to $|M^-|$ **do**
        while There is no any sinks in G_x^* **do**
            Let C be a cycle in G_x^*
            Reallocate bundles according to C
        Let i be a sink in G_x^*
        $X_i \leftarrow X_i + j$
    Return X
=0

8 Conclusion

We have so far discussed some of the recent algorithmic achievements in the field of fair allocation of indivisible items as well as the open problems. For all the algorithms and theorems in this article we considered the problem of allocating a set of items among a set of agents. But, this problem statement assumes that every item is distinct in nature, which need not be the case in a general setting. A very recent pre-print by Gafni et al. (16) tried to tackle the general case of mixed manna setting where items have copies as well. That article proposed different version of the EF and PROP relaxations, this time with copies in mind and showed how those fairness properties stand in the mathematical framework along with the more widely used ones. The authors also proposed a *duality theorem* for goods/chore allocation, which is a promising approach for proving more stronger version of different fairness notions in mixed manna with copies setting.

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