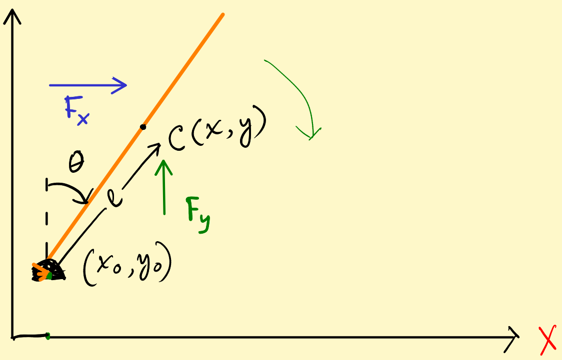
Inverted Pendulum

A rod of mass m whose centre of mass is located at (x, y) and a end is situated at (x_0, y_0) which is $f(t)$. The rod is free to move the pivot point.



If moment of inertia about the com is I_c. And force acting on the pivot point is F_x & F_y along x and y direction respectively. Then net torque on the system about c,

Newtonian approach:

$$I_c \ddot{\theta} = F_y l \sin\theta - F_x l \cos\theta \quad — (1)$$

Now, $F_x = m\ddot{x}$; $F_y = m\ddot{y} - mg \quad — (2)$

$x = x_0 + l\sin\theta$	$y = y_0 + l\cos\theta$
$\Rightarrow \dot{x} = \dot{x}_0 + l\cos\theta\,\dot{\theta}$	$\Rightarrow \dot{y} = \dot{y}_0 - l\sin\theta\,\dot{\theta}$
$\Rightarrow \ddot{x} = \ddot{x}_0 + l\cos\theta\,\ddot{\theta} - l\dot{\theta}^2\sin\theta \quad \ldots (3.a)$	$\Rightarrow \ddot{y} = \ddot{y}_0 - l\sin\theta\,\ddot{\theta} - l\dot{\theta}^2\cos\theta \quad \ldots (3.b)$

Now putting the values in eq (1) from (2) & (3).

$$I_c\ddot{\theta} = ml\sin\theta\left(\ddot{y}_0 - l\sin\theta\,\ddot{\theta} - l\dot{\theta}^2\cos\theta - g\right) - ml\cos\theta\left(\ddot{x}_0 + l\cos\theta\,\ddot{\theta} - l\dot{\theta}^2\sin\theta\right)$$

cancel each-other

$$\Rightarrow I_c\ddot{\theta} = ml\left(\sin\theta\,\ddot{y}_0 - \cos\theta\,\ddot{x}_0\right) - ml^2\ddot{\theta}\left(\sin^2\theta + \cos^2\theta\right) - mlg\sin\theta$$

$$\Rightarrow \left(I_c + ml^2\right)\ddot{\theta} = ml\sin\theta(\ddot{y}_0 + g) - ml\cos\theta\,\ddot{x}_0$$

Moment of inertia of the rod about the pivot point

$$\boxed{\ddot{\theta} = \left(\frac{ml}{I}\right)\left((\ddot{y}_0 + g)\sin\theta - \ddot{x}_0\cos\theta\right)} \quad \left[I = I_c + ml^2\right]$$

- **Lagrangian approach :-**

$x = x_0 + l\sin\theta \qquad \dot{x} = \dot{x}_0 + l\cos\theta\,\dot\theta$

$y = y_0 + l\cos\theta \qquad \dot{y} = \dot{y}_0 - l\sin\theta\,\dot\theta$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

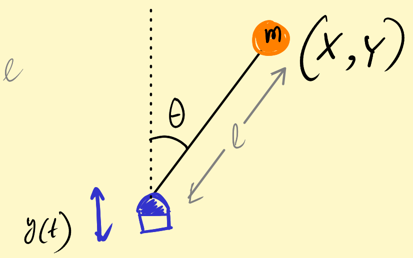
$$= \frac{1}{2}m\left(\dot{x}_0^2 + 2x_0 l\cos\theta\,\dot\theta + l^2\dot\theta + \dot{y}_0^2 + 2y_0 l\sin\theta\right)$$

$$V = mgy = mg(y_0 + l\cos\theta)$$

$\mathcal{L} = T - V$ and use EL rel^n for all generalised coordinate.

- **Case 1:** Vertically driven $\overbrace{\text{simple}}^{\text{rod have mass = 0}}$ pendulum:

Given, $y(t) = A\sin(\omega_d t)$ $\quad A \ll l$



$X = l\sin\theta$ $\quad$ | $\quad Y = l\cos\theta + y(t)$

$\Rightarrow \dot{x} = l\cos\theta\,\dot\theta$ $\quad$ | $\quad \Rightarrow \dot{Y} = -l\sin\theta\,\dot\theta + A\omega\cos(\omega_d t)$

$$T = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) = \frac{1}{2}m\left(l^2\cos^2\theta\,\dot\theta^2 + l^2\sin^2\theta\,\dot\theta^2 - 2A\omega_d l\sin\theta\cos(\omega_d t)\dot\theta + A^2\omega^2\cos^2(\omega_d t)\right)$$

$$\Rightarrow T = \frac{1}{2}m\left(l^2\dot\theta^2 - 2A\omega_d l\sin\theta\cos(\omega_d t)\dot\theta + A^2\omega_d^2\cos^2(\omega_d t)\right)$$

$$V = mgY = mg\left(l\cos\theta + A\sin(\omega_d t)\right)$$

$$\mathcal{L} = T - V \quad \& \quad \boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta}}$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = m\left(l^2\dot\theta - A\omega l\sin\theta\cos(\omega_d t)\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = m\left(l^2\ddot\theta - A\omega_d l\cos\theta\cos(\omega_d t)\dot\theta + A\omega_d^2 l\sin\theta\sin(\omega_d t)\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mA\omega l \cos\theta \cos(\omega_d t)\dot{\theta} + mgl\sin\theta$$

$$\therefore \quad ml^2\ddot{\theta} = mgl\sin\theta - Am\omega_d^2 l\sin\theta\sin(\omega_d t)$$

$$\ddot{\theta} = \left(g - A\omega^2\sin(\omega_d t)\right)\left(\frac{\sin\theta}{l}\right)$$

This eq^n can be derived also from Newtonian appraoch,

$$I\ddot{\theta} = ml\left(\sin\theta\,(\ddot{y}_0 + g) - \cos\theta\,\ddot{x}_0\right)$$

Here $I = ml^2$ & $\ddot{y}_0 = -A\omega^2\sin(\omega_d t)$ & $\ddot{x} = 0$

$$\ddot{\theta}ml^2 = ml\sin\theta\left(g - A\omega_d^2\sin(\omega_d t)\right)$$

$$\Rightarrow \ddot{\theta} = \left(\frac{\sin\theta}{l}\right)\left(g - A\omega_d^2\sin(\omega_d t)\right)$$

- Non linear eq^n so we have solve it using numerically.

- Simultaneous eq^n: $\dot{\theta} = \omega$ $\quad\quad$ $\dot{\omega} = \left[g - A\omega_d^2\sin(\omega_d t)\right]\left(\frac{\sin\theta}{l}\right)$

We can get some idea using small angle approximation,
$\sin\theta \simeq \theta$

So, $$\ddot{\theta} = \frac{\theta}{l}\left(g - A\omega_d^2\sin(\omega_d t)\right)$$

$$\theta = e^{\lambda t}$$

$$\Rightarrow \lambda^2 = \frac{1}{l}\left(g - A\omega_d^2\sin(\omega_d t)\right)$$

$$\therefore \lambda = \pm\sqrt{\frac{1}{l}\left(g - A\omega_d^2\sin(\omega_d t)\right)}$$

$$\theta(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - Am\omega_d^2\,\ell\sin\theta\sin(\omega_d t)$$

$$= -\left[A\omega_d^2\sin(\omega_d t) - g\right] ml\sin\theta$$

If $A\omega_d^2 > g$ → System will try to stabilize itself

Case 2: only horizontal driving force:-

$y_0 = 0;\ \dot{y}_0 = 0$ & $x_0 = A\sin(\omega_d t) \Rightarrow \ddot{x}_0 = -A\omega_d^2\sin(\omega_d t)$

$$\boxed{\ddot{\theta} = \left(\frac{m\ell}{I}\right)\left((\ddot{y}_0 + g)\sin\theta - \ddot{x}_0\cos\theta\right)}$$

So, eq^n of motion,

$$\ddot{\theta} = \frac{m\ell}{m\ell^2}\left(g\sin\theta + A\omega_d^2\sin(\omega_d t)\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell}\left(g\sin\theta + A\omega_d^2\sin(\omega_d t)\cos\theta\right)$$