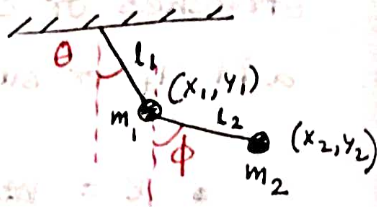
Double Pendulum confined in a plane:



$$\begin{cases} x_1 = l_1 \sin\theta \\ \dot{x}_1 = l_1 \cos\theta \, \dot{\theta} \\ y_1 = l_1 \cos\theta \\ \dot{y}_1 = -l_1 \sin\theta \, \dot{\theta} \end{cases}$$

$$\begin{cases} x_2 = l_1 \sin\theta + l_2 \sin\phi \\ \quad = l_1 \cos\theta \, \dot{\theta} + l_2 \cos\phi \, \dot{\phi} \\ y_2 = l_1 \cos\theta + l_2 \cos\phi \\ \quad = -\left[l_1 \sin\theta \, \dot{\theta} + l_2 \sin\phi \, \dot{\phi} \right] \end{cases}$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi} (\cos\theta \cos\phi + \sin\theta \sin\phi \right.$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \cos(\theta - \phi) \dot{\theta} \dot{\phi} \right)$$

$$\cancel{V = -m_1 g l_1 \cos\theta - m_2 l_1 \sin\theta - m_2 l_2 \cos\phi}$$

$$V = -m_1 g l_1 \cos\theta - m_2 g (l_1 \cos\theta + l_2 \cos\phi)$$

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \cos(\theta - \phi) \dot{\theta} \dot{\phi} \right)$$
$$+ m g l_1 \cos\theta + m_2 g (l_1 \cos\theta + l_2 \cos\phi)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\Rightarrow \frac{d}{dt} \left(m_1 l_1^2 \dot{\theta} + m_2 l_1^2 \dot{\theta} + m_2 l_1 l_2 \cos(\theta - \phi) \dot{\phi} \right) = -\left(m_2 l_1 l_2 \sin(\theta - \phi) \dot{\theta} \dot{\phi} + m_1 g l_1 \sin\theta + m_2 g l_1 \sin\theta \right)$$

$$\Rightarrow (m_1 l_1^2) \ddot{\theta} + m_2 l_1 l_2 \cos(\theta - \phi) \ddot{\phi} - m_2 \cancel{l_1} l_2 \sin(\theta - \phi) \dot{\theta} \dot{\phi}$$
$$+ m_2 l_1 l_2 \sin(\theta - \phi) \dot{\phi}^2 = -\left(m_2 \cancel{l_1} l_2 \sin(\theta - \phi) \dot{\theta} \dot{\phi} \right.$$
$$\left. + (m_1 + m_2) g l_1 \sin\theta \right)$$

$$(m_1+m_2)l_1^2\ddot{\theta} + m_2 l_1 l_2 \cos(\theta-\phi)\ddot{\phi} + m_2 l_1 l_2 \sin(\theta-\phi)\dot{\phi}^2 + (m_1+m_2) g l_1 \sin\theta = 0$$

$$\boxed{(m_1+m_2)l_1\ddot{\theta} + m_2 l_2 \cos(\theta-\phi)\ddot{\phi} + m_2 l_2 \sin(\theta-\phi)\dot{\phi}^2 + (m_1+m_2) g \sin\theta = 0} \quad -(1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\Rightarrow \frac{d}{dt}\left[m_2 l_2^2\dot{\phi} + m_2 l_1 l_2 \cos(\theta-\phi)\dot{\theta}\right] = + m_2 l_1 l_2 \sin(\theta-\phi)\dot{\theta}\dot{\phi} - m_2 g l_2 \sin\phi$$

$$\Rightarrow m_2 l_2^2\ddot{\phi} + m_2 l_1 l_2 \cos(\theta-\phi)\ddot{\theta} - m_2 l_1 l_2 \sin(\theta-\phi)\dot{\theta}^2 + \cancel{m_2 l_1 l_2 \sin(\theta-\phi)\dot{\theta}\dot{\phi}}$$
$$= \cancel{m_2 l_1 l_2 \sin(\theta-\phi)\dot{\theta}\dot{\phi}} - m_2 g l_2 \sin\phi$$

$$\Rightarrow m_2 l_2^2\ddot{\phi} + m_2 l_1 l_2 \cos(\theta-\phi)\ddot{\theta} - m_2 l_1 l_2 \sin(\theta-\phi)\dot{\theta}^2 + m_2 g l_2 \sin\phi = 0$$

$$\boxed{m_2 l_2\ddot{\phi} + m_2 l_1 \cos(\theta-\phi)\ddot{\theta} - m_2 l_1 \sin(\theta-\phi)\dot{\theta}^2 + m_2 g \sin\phi = 0} \quad -(2)$$

$$\dot{\theta} \to \omega$$
$$\dot{\phi} \to 0$$

eq(1):
$$(m_1+m_2)l_1\dot{\omega} + m_2 l_2 \cos(\theta-\phi)\dot{0} + m_2 l_2 \sin(\theta-\phi) 0^2 + (m_1+m_2) g \sin\theta = 0$$

eq(2):
$$m_2 l_2 \dot{0} + m_2 l_1 \cos(\theta-\phi)\dot{\omega} - m_2 l_2 \sin(\theta-\phi)\omega^2 + m_2 g \sin\phi = 0$$

Let $\theta = \omega_1$; $\dot{\theta} = \omega_2$; $\phi = \omega_3$; $\dot{\phi} = \omega_4$

$$(m_1+m_2)l_1\dot{\omega}_2 + m_2 l_2 \cos(\theta-\phi)\dot{\omega}_4 + m_2 l_2 \sin(\theta-\phi)\omega_4^2 + (m_1+m_2) g \sin\theta = 0$$

$$\Rightarrow (m_1+m_2)l_1\dot{\omega}_2 + m_2 l_1 \sin(\theta-\phi)\cos(\theta-\phi)\omega_2^2 - m_2 l_1 \cos^2(\theta-\phi)\dot{\omega}_2$$
$$- m_2 g \sin\phi \cos(\theta-\phi) + m_2 l_2 \sin(\theta-\phi)\omega_4^2$$
$$+ (m_1+m_2) g \sin\theta = 0.$$

$$\Rightarrow \boxed{\left[(m_1+m_2)l_1 - m_2 l_1 \cos^2(\theta-\phi)\right]\dot{\omega}_2 = m_2 g \sin\phi \cos(\theta-\phi) - m_2 l_1 \sin(\theta-\phi)\cos(\theta-\phi) - (m_1+m_2) g \sin\theta.}$$

$$m_2 l_2 \dot{\omega}_4 + m_2 l_1 \cos(\theta-\phi)\,\dot{\omega}_2 - m_2 l_1 \sin(\theta-\phi)\,\omega_2^2 + m_2 g \sin\phi = 0$$

$$\Rightarrow \boxed{m_2 l_2 \dot{\omega}_4 = m_2 l_1 \sin(\theta-\phi)\,\omega_2^2 - m_2 l_1 \cos(\theta-\phi)\,\dot{\omega}_2 - m_2 g \sin\phi}$$

$$= m_2 l_1 \sin(\theta-\phi)\,\omega_2^2 - m_2 g\sin\phi - m_2 \cos(\theta-\phi)(l_1\dot{\omega}_2).$$

$$= m_2 l_1 \sin(\theta-\phi)\,\omega_2^2 - m_2 g \sin\phi + \frac{m_2\cos(\theta-\phi)}{m_1+m_2}\Big[m_2 l_2 \cos(\theta-\phi)\dot{\omega}_4 + m_2 l_2 \sin(\theta-\phi)\,\omega_4^2 + (m_1+m_2) g \sin\theta\Big]$$

$$\Rightarrow \left(\cancel{m_2} l_2 - \frac{m_2^{\cancel{2}} l_2 \cos^2(\theta-\phi)}{m_1+m_2}\right)\dot{\omega}_4 = \cancel{m_2} l_1 \sin(\theta-\phi)\,\omega_2^2 - \cancel{m_2} g \sin\phi + \cancel{m_2}\, g\sin\theta\cos(\theta-\phi) + \frac{m_2^{\cancel{2}}}{m_1+m_2}\, l_2 \cos(\theta-\phi)\sin(\theta-\phi)\,\omega_4^2$$

~~$\Rightarrow \dot{\omega}_4 =$~~ *Therefore:*

$$\dot{\omega}_2 = \left[(m_1+m_2)l_1 - m_2 l_1 \cos^2(\omega_1-\omega_2)\right]^{-1}\Big[m_2 g \sin(\omega_3)\cos(\omega_1-\omega_3) - m_2 l_1 \sin(\omega_1-\omega_3)\cos(\omega_1-\omega_3)\,\omega_2^2 - (m_1+m_2) g \sin\omega_1\Big]$$

$$\cancel{\dot{\omega}_4 = \left(l_2 - \frac{m_2 l_2 \cos^2(\theta-\phi)}{m_1+m_2}\right)^{-1}\left[l_1 \sin(\theta-\phi)\,\omega_2^2 - g\sin\phi + g\sin\theta\right.}$$

$$\dot{\omega}_4 = \left(l_2 - \frac{m_2 l_2 \cos^2(\omega_1-\omega_3)}{m_1+m_2}\right)^{-1}\Big[l_1 \sin(\omega_1-\omega_3)\,\omega_2^2 - g\sin\omega_3 + g\sin\omega_1\cos(\omega_1-\omega_3) + \frac{m_2}{m_1+m_2}\, l_2 \cos(\omega_1-\omega_3)\sin(\omega_1-\omega_3)\,\omega_4^2\Big]$$