Lagrange's Equation of the first kind

New Notation: Instead of x, y, z we are going to use x_1, x_2, x_3. So if we have n number of particle we have $3n$ number coordinate.

Einstein summation notation: Instead $\sum_i a_i b_i$ we only write $a_i b_i$

- A holonomic constraint can be written as

$$\phi(x_1, x_2, x_3; x_4, x_5, x_6; \ldots; t) = 0$$

by Einstein notation,

$$\Rightarrow \phi(x_i, t) = 0$$

- If there are r constraints we can write,

$$\sum_{i=1}^{3n} \sum_{\alpha=1}^{r} \frac{\partial \phi_\alpha}{\partial x_i} dx_i + \sum_{\alpha=1}^{r} \frac{\partial \phi_\alpha}{\partial t} dt = 0 \quad \text{--- (1)}$$

Now if we consider real displacement to virtual displacement

i.e, $dx_i \rightarrow \delta x_i$

As in the concept of virtual displacement in no time the particle is displace by δx.

So eq (1) can be written,

$$\boxed{\frac{\partial \phi_\alpha}{\partial x_i} \delta x_i = 0} \quad \text{--- (2)}$$

Remember D'Alembert's principal,

$$(m_i \ddot{x}_i - X_i) \delta x_i = 0 \quad \text{--- (3)}$$

Now multiply each of the eqs (2) by multiplier λ_α add them together,

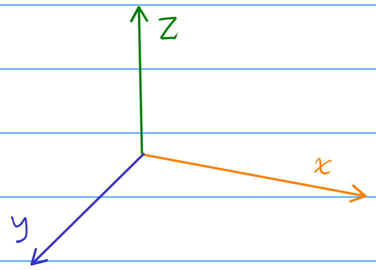
$$\left(m_i \ddot{x}_i - X_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i}\right) = 0$$

Remembering δx_i's are not arbitrary. But choose the λ_α's in such a way that n of the following eqⁿ satisfied:

$$\boxed{m_i \ddot{x}_i - X_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i} = 0} \rightarrow$$

Problem 1: *Using Lagrange's equation of 1st kind find the equation of motion of spherical pendulum.*

In spherical pendulum eqⁿ of constraint is



$$\phi = x^2 + y^2 + z^2 - l^2 = 0$$

- $m\ddot{x} - F_x + \lambda \frac{\partial \phi}{\partial x} = 0$
- $m\ddot{y} - F_y + \lambda \frac{\partial \phi}{\partial y} = 0$
- $m\ddot{z} - F_z + \lambda \frac{\partial \phi}{\partial z} = 0$

But force is only along negative z direction, $F_z = -mg$

$F_x = F_y = 0$

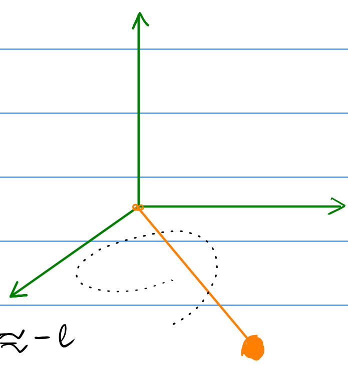
$$\left\{ \begin{array}{ll} m\ddot{x} + 2\lambda x = 0 & \text{— (a)} \\ m\ddot{y} + 2\lambda y = 0 & \text{— (b)} \\ m\ddot{z} + mg + 2\lambda z = 0 & \text{— (c)} \end{array} \right.$$

Now again, $x^2 + y^2 + z^2 = l^2$

$$\Rightarrow z^2 = l^2 - x^2 - y^2$$

$$\Rightarrow z^2 = l^2\left(1 - (x/l)^2 - (y/l)^2\right)$$

$$\therefore z = \pm l \left[1 - (x/l)^2 - (y/l)^2\right]^{1/2}$$



Note:

① We hung the Pendulum from origin. so $z \approx -l$

② We choose a unit for which $x << l$. $\nearrow \Rightarrow \dot{z} = \ddot{z} = 0$

eq(c) becomes,

$$mg - 2\lambda l = 0 \Rightarrow \boxed{\lambda = \frac{mg}{2l}}$$

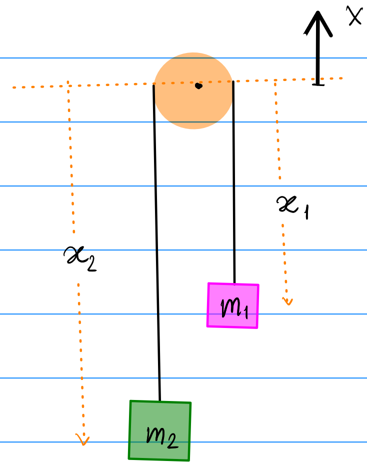
$$\begin{cases} m\ddot{x} + 2\lambda x = 0 & \text{— (a)} \\ m\ddot{y} + 2\lambda y = 0 & \text{— (b)} \\ m\ddot{z} + mg + 2\lambda z = 0 & \text{— (c)} \end{cases}$$

Other two, $m\ddot{x} + \frac{mg}{l}x = 0 \Rightarrow \boxed{\ddot{x} + \frac{g}{l}x = 0}$

$$\ddot{y} + \frac{g}{l}y = 0$$

They have same frequency

Problem 2: *Solve the Atwood's machine problem by Lagrange's equation of the first kind. What is the tension in the string?*



Eqn of constrain : $x_1 + x_2 = \text{cont.}$

$$\Rightarrow \phi = x_1 + x_2 - c = 0.$$

$$m_1\ddot{x}_1 - F_1 + \lambda\frac{\partial\phi}{\partial x_1} = 0$$

$$m_2\ddot{x}_2 - F_2 + \lambda\frac{\partial\phi}{\partial x_2} = 0$$

Now, here only gravitional force is present so, $F_1 = -m_1 g$ & $F_2 = -m_2 g$

So, $$\begin{cases} m_1\ddot{x}_1 + m_1 g + \lambda = 0 \\ m_2\ddot{x}_2 + m_2 g + \lambda = 0 \end{cases}$$

Again $\ddot{x}_1 + \ddot{x}_2 = 0$

$$m_1\ddot{x}_1 + m_1 g - m_2\ddot{x}_2 - m_2 g = 0$$

$$\Rightarrow m_1\ddot{x}_1 + m_2\ddot{x}_1 + (m_1 - m_2)g = 0$$

$$\Rightarrow \boxed{\ddot{x}_1 = \frac{m_2 - m_1}{m_1 + m_2}g}$$

$$m_1\left(\frac{m_2 - m_1}{m_1 + m_2}\right)g + m_1 g + \lambda = 0$$

$$\Rightarrow \lambda = \left(m_1\left(\frac{m_1 - m_2}{m_1 + m_2}\right) - m_1\right)g$$

$$= \left(\frac{\cancel{m_1} - m_2 - \cancel{m_1} - m_2}{m_1 + m_2}\right)m_1 g$$

$$\boxed{\lambda = -\frac{2m_1 m_2}{m_1 + m_2}g}$$

Note: If you calculate tension of the string you will end up with

$$T = \frac{2m_1 m_2}{m_1 + m_2}g$$

Conclusion: Lagrange multiplier have the information about constraint force.

So far we built a unseen principal D'Alembert's principal i.e, no work done by constraint forces. We built up virtual concept namely "virtual displacement" where with no time we can displace. Then we had done a job for holonomic constant where we meet Lagrange multiplier. Information of constraint force is inside the Lagrange multiplier. Thus we built Lagrange eqn of first kind.

But we can't get rid of Cartesian-coordinate. We want to get rid of Cartesian-Coordinate because they are not independent from each-other in many problem.

So our present intest to get generalised coordinate for which coordinates are independent from one-another.

They have properties:

① general in the sense that they need not be lengths of angles.

② in number just equal to the number of degrees of freedom.

DoF: Smallest number of coordinates required to specify completely the position or configuration of the system.

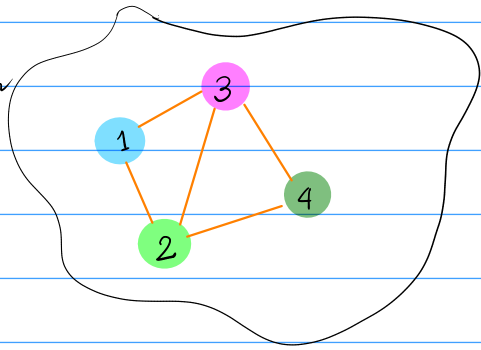
Usually, $\text{DoF} = 3n - r$

no. of particles (n)   no. of constraints (r)

** Except Rigid body: As they have infinite number of particle of DOF is something else!

$3+2+1$

Consider a particle 1 it have DOF 3. Now take particle 2. Fix 1 and try to move 2. Then they have 2 DOF. Again fix 1 & 2 and try to move 3 then it have only 1 DOF. Now if you take 4th particle it can't move. So degrees of freedom is 6, generally



3 rotational + 3 translation.