Lagrangian Formulation

Recap of Generalised coordinate : We take new coordinate such that they are independent from each-other. And degrees of freedom is minimum number of independent coordinate by which we can describe the system.

By independent we mean that for the n variables given by $u_1, u_2, \dots, u_n$ (n being the DoF) if we have n constants $c_1, c_2, \dots, c_n$ satisfying

$$\sum_{i=1}^{n} c_i \, du_i = 0$$

at any point, then it is necessary follows that

$$c_1 = c_2 = c_3 = \dots = c_n = 0$$

Generalised coordinate transformation,

$$x_1 = x_1(q_1, q_2, \dots q_n, t)$$

$$\vdots \qquad \qquad \vdots$$

$$x_r = x_r(q_1, q_2, \dots, q_n, t)$$

N being the no. of particles.

$$\vdots \qquad \qquad \vdots$$

$$x_{3N} = x_{3N}(q_1, q_2, \dots, q_n, t)$$

Lagrange's Equation of 2nd Kind :

The D'Alembert's principal was,

$$(m_i \frac{d^2 x_i}{dt^2} - F_i^a)\delta x_i = 0$$

I've used Einstein notation.

- First term, $m_i \frac{d^2 x_i}{dt^2} \delta x_i$, before compute it we have to do some other such that we just plug that here!

Cancelation of dots;

By generalised coordinate,

$$x_r = x_r(q_1, q_2, q_3, \dots, q_n, t)$$

$$\Rightarrow \underset{\dot{x}_r}{\overset{\|}{\frac{dx_r}{dt}}} = \frac{\partial x_r}{\partial t} + \sum_{r=1}^{n} \frac{\partial x_r}{\partial q_r} \dot{q}_r \quad (1)$$

$\Rightarrow$ For virtual displace.

$$\delta x_i = dx_i\big|_{dt=0}$$

$$\Rightarrow \boxed{\delta x_i = \frac{\partial x_i}{\partial q_j} \delta q_j}$$

running index

or,

$$\boxed{\frac{\partial \dot{x}_r}{\partial \dot{q}_r} = \frac{\partial x_r}{\partial q_r}}$$

Property 2:

Consider another term,

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial q_j}\right) + \frac{\partial}{\partial q_l}\left(\frac{\partial x_i}{\partial q_j}\right)\dot{q}_l$$

running index

partial diff.

order does not matter

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial x_i}{\partial t}\right) + \frac{\partial}{\partial q_j}\left(\frac{\partial x_i}{\partial q_l}\dot{q}_l\right)$$

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial q_l}\dot{q}_l\right)$$

From (i)

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial \dot{x}_i}{\partial q_j}$$

Now we ready to compute the D'Alembert's first term,

$$\text{ie,} \quad m_i \frac{d^2 x_i}{dt^2} \delta x_i = m_i \left(\frac{d^2 x_i}{dt^2}\right)\left(\frac{\partial x_i}{\partial q_j}\right)\delta q_j$$

Here i & j are both running index.

Crucial Step,

$$= \frac{d}{dt}\left(m_i \frac{dx_i}{dt} \underset{\frac{\partial \dot{x}_i}{\partial \dot{q}_j}}{\underset{\|}{\frac{\partial x_i}{\partial q_j}}}\right)\delta q_j - m_i \frac{dx_i}{dt} \underset{\frac{\partial \dot{x}_i}{\partial q_j}}{\underset{\|}{\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)}}\delta q_j$$

$$= \frac{d}{dt}\left(m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right)\delta q_j - \left(m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}\right)\delta q_j$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m \dot{x}_i^2\right)\right)\delta q_j - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{x}_i^2\right)\delta q_j$$

Non-relativistic kinetic energy: $T = \frac{1}{2} m \dot{x}_i^2$

$$= \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j$$

- Second term of D'Alembert's eq^n:

$$-F_i^{(a)}\delta x_i = -F_i^{(a)}\left(\frac{\partial x_i}{\partial q_j}\right)\delta q_j$$

$$= - Q_j \delta q_j$$

So, $\left(m_i \frac{d^2 x_i}{dt^2} - F_i^{(a)}\right)\delta x_i = 0$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

where $\boxed{Q_j = F_i^{(a)}\left(\frac{\partial x_i}{\partial q_j}\right)}$

Generalised force

- But if force is conservative then there exist a scaler (may not unique) (V):

Case 1: $V(q_1, q_2, \dots, q_n, t)$ ← ordinary potential energy

$$\Rightarrow Q_j = -\frac{\partial V}{\partial q_j}.$$

Lagrange's 2nd kind become,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

We define, $L = T - V$; $L = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$ ← Euler-Lagrange eq^n

Cyclic Coordinate:

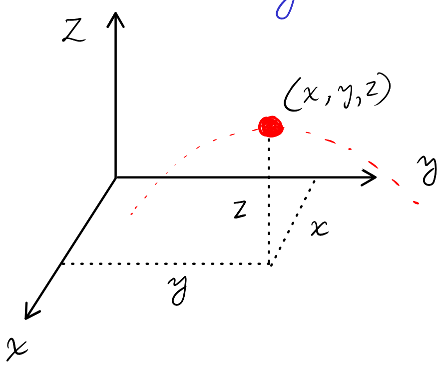
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

Say, L does not depend upon q_j, i.e, $\frac{\partial L}{\partial q_j} = 0$.

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_j} = \text{const.} \Rightarrow \text{independent of time.}$$

We say q_j is "cyclic" coordinate and $\frac{\partial L}{\partial \dot{q}_j}$ is a conserved quantity (i.e, it is independent of time). Most of the problem cyclic quantity have some-kind of symmetry. We call $\frac{\partial L}{\partial \dot{q}_j}$ as generalised momentum. Because in cartesian co-ordinate it gives momentum. One thing to note that generalised coordinate need not have the unit of momentum.

Ex1: Let say you throw a ball in the air. Find the cyclic coordinate.



$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgh.$$

Here you see $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial y}$ is zero. So x and y are cyclic coordinate. So there must exist conserved momentum. In this case linear momentums along x (p_x) and y (p_y) axis must conserved, ie they do not change with time.

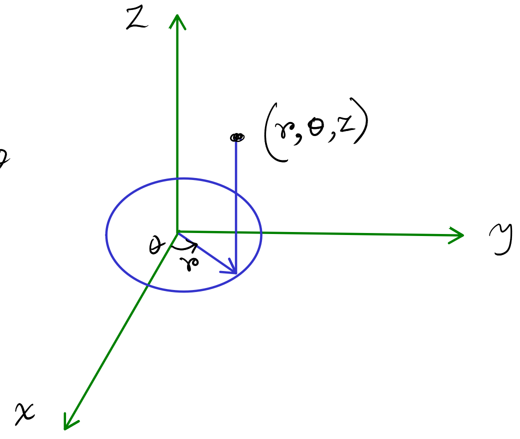
If you like terminology we call conservation of momentum is arises because of spatial traslation invarience.

- Angular and linear momentum is conserved in cylindrical symmetry

$$x = r\cos\theta \Rightarrow \dot{x} = -r\sin\theta\,\dot{\theta} + \dot{r}\cos\theta$$

$$y = r\sin\theta \Rightarrow \dot{y} = r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta$$

$$z = z$$



$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$= r^2\dot{\theta}^2 + \dot{r}^2 + \dot{z}^2$$

$$L = \frac{1}{2}m\left(r^2\dot{\theta} + \dot{r}^2 + \dot{z}^2\right) - V(r)$$

$\because$ Cylindrical symmetry $V = V(r)$

So here $\frac{\partial L}{\partial \theta}$ & $\frac{\partial L}{\partial z}$ is zero. θ and z is cyclic co-ordinate and momentum along z-axis $(m\dot{z})$ and angular momentum $(mr^2\dot{\theta})$ must conserved.

Problem: Find the cyclic coordinate for spherical symmetric case.

- Now what I am going to do write down a quantity and justify that is indeed this.

$$\boxed{E \equiv \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L} \quad \text{— (2)}$$

Let consider a particle in N-dimensions.

$$\mathcal{L} = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dots + \dot{x}_N^2) - V(x_1, x_2, \dots, x_N)$$

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}_j} \dot{x}_j = m\dot{x}_j^2 \equiv 2T$$
- $$\mathcal{L} = T - V$$

$$\Rightarrow E = 2T + T - V = T + V = \text{Total energy of the sys.}$$

② does not always be total energy of a system. But if it satisfies a certain condition it is indeed total energy of the system.

$$E \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j - \mathcal{L}$$

** If Lagrangian is independent of time then energy of the system must conserved.

Proof:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j - \frac{d\mathcal{L}}{dt}$$

Now, $$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\mathcal{L}(q_j, \dot{q}_j, t)\right)$$

$$= \frac{\partial \mathcal{L}}{\partial q_j} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial \mathcal{L}}{\partial t}$$

$$\therefore \frac{dE}{dt} = \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j\right) - \left(\frac{\partial \mathcal{L}}{\partial q_j} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial \mathcal{L}}{\partial t}\right)$$

$$\boxed{\frac{dE}{dt} = -\frac{\partial \mathcal{L}}{\partial t}}$$

if $\mathcal{L}$ is independent of time i.e, time invarience then energy must conserved.

Now I am going to state a theorem which is most beautiful result for ever. She is a mathematician,

$\boxed{\text{Noether's Theorem}}$: For each symmetry of a Lagrangian, there is conserved quantity.

Symmetry : If we change little bit of some coordinate then Lagrangian has no first-order change.

Conserved : The quantity is independent of time.

Proof : Let the Lagrangian be invarient, in the small number ϵ under the change of coordinates,

$$q_i \longrightarrow q_i + \epsilon K_i(q)$$

q is shorthand notation for all q_i's.

The fact that Lagrangian does not change in the first order in ϵ.

$$\frac{dL}{\partial \epsilon} = 0 = \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon}$$

$$= \frac{\partial L}{\partial q_i} K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} K_i\right)$$

E-L eq^n

$$\left[\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} K_i\right) = 0$$

Therefore, the quantity $P(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}_i} K_i$ is conserved quantity.

* * Cyclic coordinate is special case of Noether's theorem, just put $K_i = 1$.

i.e, $q_i \rightarrow q_i + \epsilon$ for symmetric coordinate.