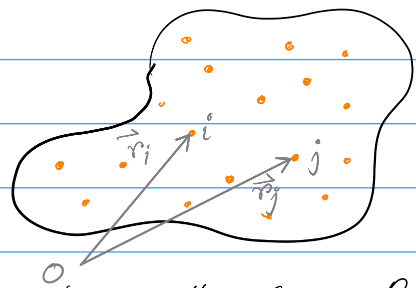
Classical Mechanics :

A Inbuilt defination: Any restriction of a motion is called constraint of a motion, and the forces responsible for it called constraint forces.

- Constraints that can be expressed as an algebraic equation involving the coordinate is called holonomic constraints. And which are not expressed in that way is called non holonomic constraints.

Let us take some example to understand the problem,

Ex.1: The Rigid body

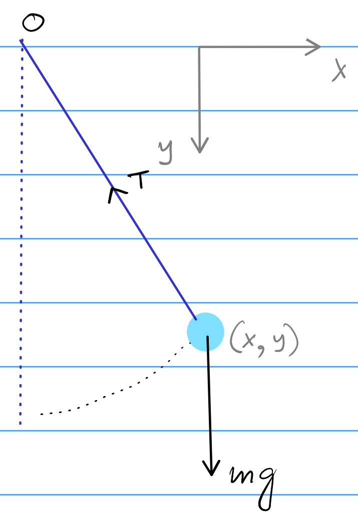


In a rigid body, the distance between any two particle is constant. The constancy is maintained by the internal forces between two particle. Yes, we don't know what is the magnitude of the forces. But we do know their effect viz., to maintain the particle separation const.

Regidity of the constraints here.

Constraint eq^n: $|\vec{r}_i - \vec{r}_j| = $ const. can also be write $(\vec{r}_i - \vec{r}_j)^2 = $ const.

Ex.2: The Simple harmonic motion

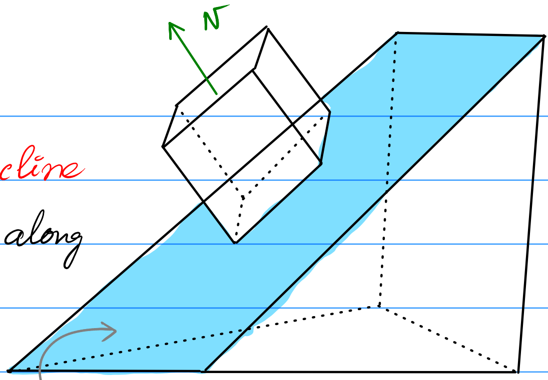


How we go about this problem? We take a tension T and the gravitional force mg. After doing some simple calculation we end up with eq of motion.

But say we don't know T rather we know the effect of T i.e, it maintain the bob in a circular arc.

Constraint eq^n: $x^2 + y^2 = $ const.

Ex 3: The Incline Plane

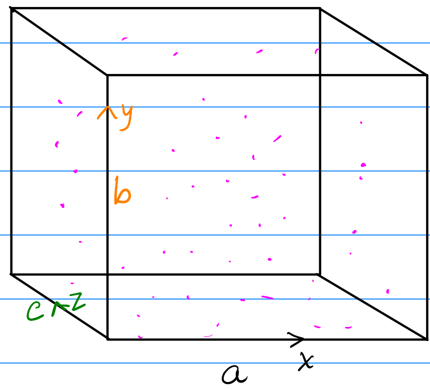


Here the constrain is the incline plane, because the block have to move along the plane. And the force of constraint is normal force.

eq^n of constraint:

$$lx + my + nz = \beta$$

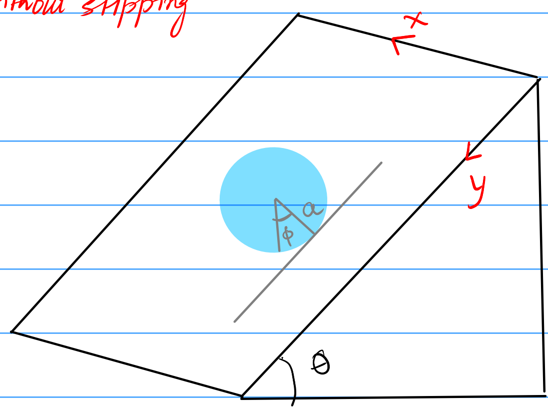
Ex. 4: Gases inside a box



Here, constraint is finite volume of the encloser,

eq:
$0 \leq x \leq a$
$0 \leq y \leq b$
$0 \leq z \leq c$

Ex 5: Rolling disk in inclined plane rolling without slipping



***Note:**

All the above cases you will find that workdone by the constraint forces is zero.

- In case of Simple harmonic motion T is perpendicular to arc. So workdone by T is zero.

- Reaction force or normal force is perpendicular to the incline plane, so workdone by it also zero.

- Let take a look at Rigid body motion,

Consider the particles in the rigid body - if $\vec{F}_{ij}$ is force on i^{th} particle due to j^{th} particle, we have the work done in a displacement of the work done in a displacement of i^{th} particle,

$$W_i = \sum_j \vec{F}_{ij} \cdot d\vec{r}_i$$

where $\vec{F}_{ii} = 0$

Self-force

Now considering all particle, total work done

$$W = \sum_i W_i = \sum_i \sum_j \vec{F}_{ij} \cdot d\vec{r}_i \quad \text{— (1)}$$

changing the indices,

$$W = \sum_j \sum_i \vec{F}_{ji} \cdot d\vec{r}_j = \sum_i \sum_j \vec{F}_{ij} \cdot (-d\vec{r}_j) \quad \text{— (2)}$$

By Newton's third law, $\vec{F}_{ij} = -\vec{F}_{ji}$

Using (1) & (2)

$$W = \frac{1}{2} \sum_i \sum_j \vec{F}_{ij} \cdot (d\vec{r}_i - d\vec{r}_j)$$

Remember the constraint eq^n;

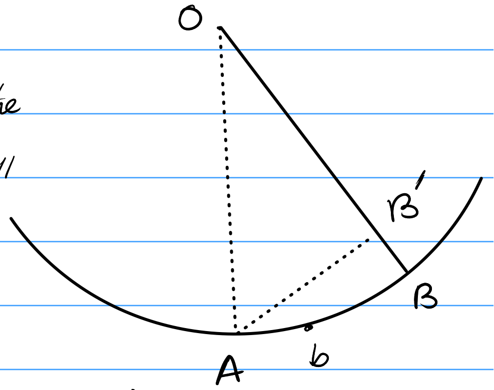
$$(\vec{r}_i - \vec{r}_j)^2 = \text{cont}$$

$$\Rightarrow (\vec{r}_i - \vec{r}_j)(d\vec{r}_i - d\vec{r}_j) = 0 \Rightarrow \boxed{d\vec{r}_i - d\vec{r}_j = 0}$$

$d\vec{r}_i - d\vec{r}_j$ is perpendicular to $\vec{r}_i - \vec{r}_j$.

So, $W = 0$

Ex 6: Elastic string in Simple harmonic motion



String may not be in-extensible. Centre of the bob moves along AB' instead of AB so work done will not vanish at any more.

A way out: Virtual displacement ($\delta \vec{r}$ instead of $d\vec{r}$)

- Virtual or hypothetical in the scence that there is no passege of time during the displacement concerned.

Since there is no passage of time, the infinitesimal displacement will be along Ab and the work done or virtual work done is vanish.

D'Alembert's Principal:

In any virtual displacement, the total work done by the forces of constraint vanish, unless of course the constraint is associated with frictional force.

Let us consider a system of particles, the force on the i^{th} particle being split into two parts

$$\vec{F}_i = \vec{F}_i^a + \vec{F}_i^c$$

constraint force.

$$\Rightarrow m_i \ddot{\vec{r}}_i = \vec{F}_i^a + \vec{F}_i^c$$

work done in a virtual displacement is,

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_i \vec{F}_i^a \cdot \delta \vec{r}_i + \underbrace{\sum_i \vec{F}_i^c \cdot \delta \vec{r}_i}_{}^{\;=0}$$

$$\Rightarrow \boxed{\sum_i \left(\vec{F}_i^a - m_i \ddot{\vec{r}}_i\right) \cdot \delta \vec{r}_i = 0}$$

□ Note: Constraint forces in the examples are elastic in nature. modulus of elasticity tending to infinity. eg.,

- Inextensible string Young modulus $\to \infty$
- Rigid body modulus is arbitrarily large.
- Incline plane also have large modulus.

$$Y = \frac{\overbrace{F/A}^{\text{stress}}}{\underbrace{\Delta l / l}_{\text{strain}}}$$

$$\boxed{Y = \frac{Fl}{A\Delta l}}$$

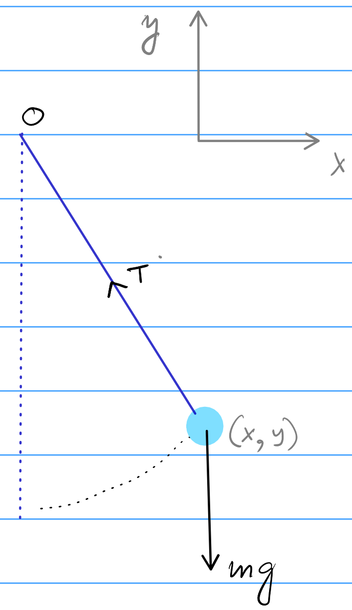
Elastic Potential Energy: stress × strain × volume

$$= \text{Modulus of elasticity} \times (\text{strain})^2 \times \text{volume}$$

As modulus of elasticity will increase (upto ∞), the strain also decreases ($\to 0$) and since there is square term in energy so, potential energy will vanish.

Aplications:-

Ex.1: Simple Harmonic Motion



eqⁿ of constraints:

$$x^2 + y^2 = \text{const.}$$

$$\Rightarrow x\,\delta x + y\,\delta y = 0 \qquad \text{— ③}$$

D'Alembert's principal:-

$$(m\ddot{x} - X)\,\delta x + (m\ddot{y} + Y)\,\delta y = 0$$

X & Y are forces along x and Y axis excluding contraint forces.

$$Y = mg \qquad X = 0$$

$$\ddot{x}\,\delta x + (\ddot{y} + g)\,\delta y = 0$$

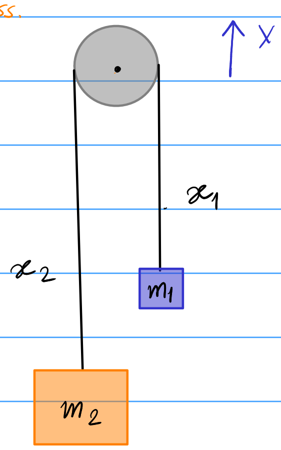
$$\Rightarrow \ddot{x}\left(-\frac{y}{x}\right)\delta y + (\ddot{y} + g)\,\delta y = 0$$

$$\Rightarrow -\ddot{x}y + \ddot{y}x + gx = 0$$

For small angle approximation, $\ddot{y} \approx 0$ & $y = -l$

$$\text{So,} \quad \ddot{x} + \frac{g}{l} x = 0$$

Problem 1: Solve the Atwood's machine by D'Alembert's principal. Assuming string is vertical, inextensible and pulley is frictionless.



String is inextensible, so $x_1 + x_2 = \text{const.}$

$$\Rightarrow \delta x_1 + \delta x_2 = 0$$

$$\ddot{x}_1 + \ddot{x}_2 = 0$$

D'Alembert's principal gives,

$$(m_1\ddot{x}_1 - m_1 g)\,\delta x_1 + (m_2\ddot{x}_2 - m_2 g)\,\delta x_2 = 0$$

or, $m_1\ddot{x}_1 - m_1 g - m_2\ddot{x}_2 + m_2 g = 0$

or, $m_1\ddot{x}_1 - m_2\ddot{x}_2 = (m_1 - m_2)\,g$

or, $(m_1 + m_2)\,\ddot{x}_1 = (m_1 - m_2)\,g$

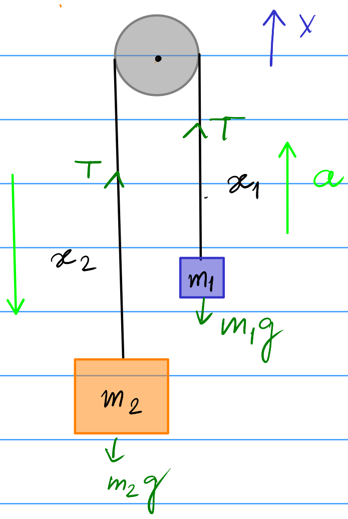
or, $(m_1 + m_2)\,\ddot{x}_2 = (m_2 - m_1)\,g$

If we consider T:

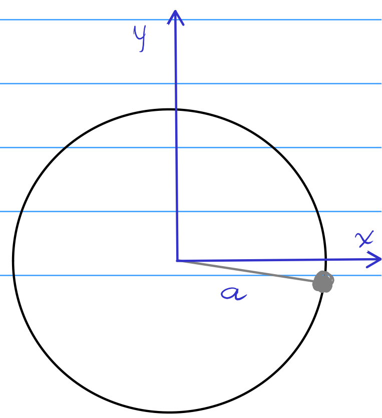
$$m_1 a = T - m_1 g$$

$$m_2 a = m_2 g - T$$

$$\overline{(m_1 + m_2)\,a = (m_2 - m_1)\,g}$$



Problem 2: A particle is constrained to move on the circumfarence of a circle. If no external force is acting on the partical, show show by D'Alembert's principle that the particle moves with uniform angular velocity.



Eqn of constrained: $x^2 + y^2 = a^2$

$$\Rightarrow x\,\delta x + y\,\delta y = 0$$

D'Alembert's principal,

$$\sum\left(\vec{F}_i^{\,a} - m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i = 0$$

So, $m\ddot{x}\,\delta x + m\ddot{y}\,\delta y = 0$

$$\Rightarrow \ddot{x}\left(-\frac{y}{x}\right)\delta y + \ddot{y}\,\delta y = 0$$

$$\Rightarrow -y\ddot{x} + \ddot{y}x = 0$$

$$\Rightarrow \frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = k \text{ (say}$$

$\ddot{x} = kx$ • Trial solⁿ: $x(t) = e^{\lambda t}$

$$\Rightarrow \lambda^2 - k = 0$$

$$\Rightarrow \lambda = \pm\sqrt{k}$$

So, $x(t) = A e^{\sqrt{k}t} + B e^{-\sqrt{k}t}$

Problem 3: A rigid rod moves within a spherical bowl so that its ends are always in contact with the inner surface of the bowl

Write down the equation of the constriant and find the number of degree of freedom.

Problem 4: *Suppose the motion of a pendulum is not restricted to a plane. Such a pendulum is called a spherical pendulum. Write the equation of constraint and set up the equation of constraint and set up the equation of D'Alembert's principal.*