EEE 243 Signals and Systems

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Lecture 05: Problems on Classifications of Signals and Operations on Signals

Department of Electrical and Electronic Engineering
School of Engineering
BRAC University

Inspiring Excellence

5.1. Problems (Classifications of Signals)

Periodic vs. Aperiodic

Note 1. The sum of two periodic signals is periodic only if the ratio of their respective periods can be expressed as a rational number.

Proof: Let $x(t)$ and $y(t)$ are two periodic signals with fundamental periods T_1 and T_2, respectively, such that their sum is given by,

$$z(t)=ax(t)+by(t)$$

Since $x(t)$ and $y(t)$ are two periodic with periods T_1 and T_2, respectively, we can write the following.

$$x(t)=x(t+kT_1)$$
$$y(t)=y(t+lT_2)$$

where k, and l are integers. Hence,

$$z(t)=ax(t+kT_1)+by(t+lT_2)$$

If z(t) is periodic with period T, then the following holds.

$$z(t)=ax(t+T)+by(t+T)=ax(t+kT_1)+by(t+lT_2)$$

Hence,

$$T=kT_1=lT_2$$

$$\frac{T_1}{T_2}=\frac{l}{k}, \qquad (\text{i.e., a rational number})$$ ■

Note 2. Linear operations (i.e., in case of addition) do not affect the periodicity of the resulting signal.

Proof. If $x(t)$ and $y(t)$ have the same period T, then,

$z(t)=x(t)+y(t)$ is periodic with period T.

Hence, addition does not change the period of the resulting signal. ■

Note 3. Nonlinear operations on periodic signal, e.g., multiplication, produce periodic signals with different periods.

Proof. Let $x(t)=\cos\omega_1 t$ and $y(t)=\cos\omega_2 t$.

Let $z(t)=x(t)y(t)$

$$z(t)=\cos\omega_1 t\cos\omega_2 t$$

$$z(t)=\frac{1}{2}\left[\cos(\omega_2-\omega_1)t+\cos(\omega_2+\omega_1)t\right]$$

If $\omega_2=\omega_1=\omega$, then

$$z(t)=\frac{1}{2}+\frac{1}{2}\cos 2\omega t$$

So, $z(t)$ has a constant term $\frac{1}{2}$ and a second-harmonic term $\left(\frac{1}{2}\cos 2\omega t\right)$.

Hence, nonlinear operations on periodic signals can produce higher order harmonics. ■

Note 4. Since a periodic signal is a signal of infinite duration, which start at $t=-\infty$ and go on to $t=\infty$, it implies that all practical signals are aperiodic.

Problem 01. Determine which of the following signals are periodic?

(a) $x_1(t)=\sin\left(\frac{2\Pi}{3}\right)t$

(b) $x_2(t)=\sin\left(\frac{2\Pi}{5}\right)t\cos\left(\frac{4\Pi}{3}\right)t$

(c) $x_3(t)=\sin 3t$

(d) $x_4(t)=x_1(t)-2x_3(t)$

Solution.

(a) $x_1(t)$ is periodic with period $T_1=3$.

(b)

$$x_2(t)=\frac{1}{2}\left(\sin\left(\frac{2\Pi}{5}+\frac{4\Pi}{3}\right)t+\sin\left(\frac{2\Pi}{5}-\frac{4\Pi}{3}\right)t\right)$$

$$x_2(t)=\frac{1}{2}\left(\sin\left(\frac{26\Pi}{15}\right)t-\sin\left(\frac{14\Pi}{15}\right)t\right)$$

$$x_2(t)=\frac{1}{2}\left(\sin\left(2\frac{\Pi}{15/13}\right)t-\sin\left(2\frac{\Pi}{15/7}\right)t\right)$$

$x_2(t)$ is the sum of two sinusoids with periods $T_{21}=15/13=T_2/13$ and $T_{22}=15/7=T_2/7$. Since $T_{21}/T_{22}=7/13$ is a rational number, $x_2(t)$ is periodic with period $T_2=15$.

(c)

$$x_3(t) = \sin 13t$$

$$x_3(t) = \sin 13\frac{2\Pi}{2\Pi}t$$

$$x_3(t) = \sin\frac{2\Pi}{(2\Pi/13)}t$$

Hence, $x_3(t)$ is periodic with period $T_3 = 2\Pi/13$.

(d)

$$x_4(t) = x_1(t) - 2x_3(t)$$

$$x_4(t) = \sin\left(\frac{2\Pi}{3}\right)t - 2\sin 13t$$

Here $T_1 = 3$ and $T_3 = 2\Pi/13$ such that $\frac{T_1}{T_3} = \frac{3}{2\Pi/13} = \frac{39}{2\Pi}$

$$2\Pi T_1 = 39T_3$$

Hence, we cannot find $\frac{T_1}{T_3} = \frac{l}{k}$, $\quad$ (i.e., a rational number) since Π is irrational. So, $x_4(t)$ is not periodic.

Power vs. energy signals

Problem 02. Consider the signals in Figure 1. Determine whether these signals are energy or power signals.

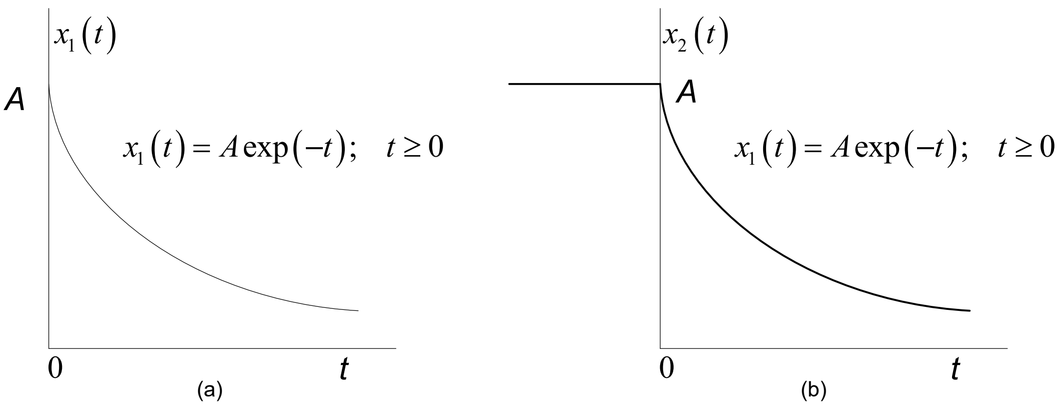

Figure 1.

Solution.

(a) The signal in Figure 1(a), $x_1(t)$, is aperiodic with total energy

$$E = \int_0^{\infty} \left| A \exp(-t) \right|^2 dt$$
$$E = \int_0^{\infty} A^2 \exp(-2t)\, dt$$
$$E = A^2 \frac{\left[\exp(-\infty) - \exp(-0)\right]}{-2}$$
$$E = \frac{A^2}{2}$$

which is finite. So, this signal is an energy signal with energy $A^2/2$.

The <u>average power</u> is given by

$$P = \lim_{L \to \infty} \left(\frac{1}{2L} \int_{-L}^{L} A^2 \exp(-2t)\, dt \right)$$
$$P = \lim_{L \to \infty} \left(\frac{1}{4L} A^2 \right)$$
$$P = 0$$

(b) The <u>energy</u> in the signal in Figure 1(b), $x_2(t)$, is given by,

$$E_2 = \lim_{L \to \infty} \left[\int_{-L}^{0} A^2 dt + \int_0^{L} A^2 \exp(-2t)\, dt \right]$$
$$E_2 = \lim_{L \to \infty} A^2 \left[L + \frac{1}{2}\left(1 - \exp(-2L)\right) \right]$$

As $L \to \infty$, $E \to \infty$ (unbounded). Hence, $x_2(t)$ is not an energy signal.

Now the <u>average power</u> of $x_2(t)$ is given by,

$$P_2 = \lim_{L \to \infty} \frac{1}{2L} \left[\int_{-L}^{0} A^2 dt + \int_0^{L} A^2 \exp(-2t)\, dt \right]$$
$$P_2 = \lim_{L \to \infty} \frac{A^2}{2L} \left[L + \frac{1}{2}\left(1 - \exp(-2L)\right) \right]$$
$$P_2 = \frac{A^2}{2}$$

Since P_2 is finite, $x_2(t)$ is a power signal.

Practice problem 01: Consider the signals in Figure 2. Determine whether these signals are energy or power signals.

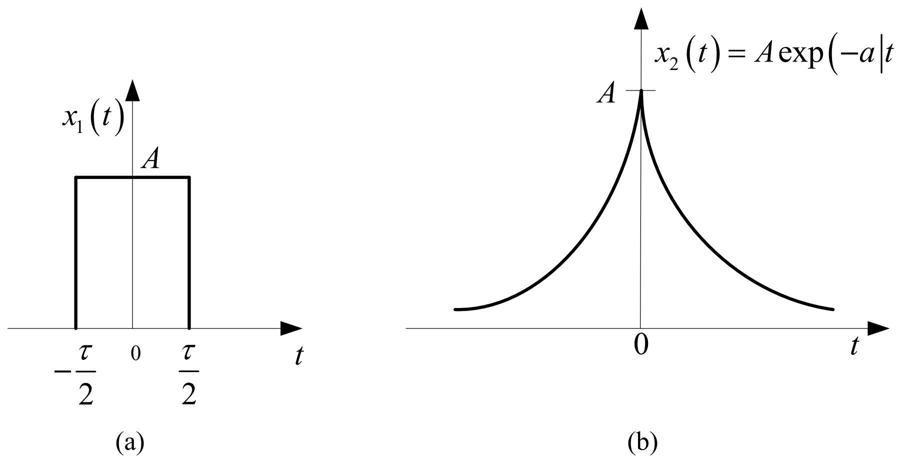

Figure 2.

Practice problem 02: All time limited signals in practice are energy signals. Justify the statement.

Even vs. odd signals

An arbitrary signal $x(t)$ can be represented as a sum of even (i.e., even symmetric) and odd (i.e., odd symmetric) signals as follows.

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ is called the even part of $x(t)$ and is given by

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

And $x_o(t)$ is called the odd part of $x(t)$ and is given by

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

Note 5. A signal $x(t)$ is called Even signal whenever $x(t)$ coincides with its reflection $x(-t)$. Such a signal is symmetric with respect to the time origin.

Likewise, a signal $x(t)$ is called odd signal whenever $x(t)$ coincides with $-x(-t)$, i.e., the negative of its reflection. Such a signal is asymmetric with respect to the time origin.

Problem 03. Consider the signal $x(t)$ defined by

$$x(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$$

Determine if it is even or odd signal. Also determine the even part and odd part of the signal.

Solution.

Even part of $x(t)$ is given by

$$x_e(t)=\begin{cases}\frac{1}{2}\left[x(t)+x(-t)\right] & t>0 \quad [\text{replace } t \text{ by} +t \text{ since } t>0 \,\text{in}\, x_e]\\ \frac{1}{2}\left[x(-t)+x(-(-t))\right] & t<0 \quad [\text{replace } t \text{ by} -t \text{ since } t<0 \,\text{in}\, x_e]\end{cases}$$

$$x_e(t)=\begin{cases}\frac{1}{2}[1+0] & t>0\\ \frac{1}{2}[0+1] & t<0\end{cases}$$

$$x_e(t)=\begin{cases}\frac{1}{2} & t>0\\ \frac{1}{2} & t<0\end{cases}$$

Odd part of $x(t)$ is given by

$$x_o(t)=\begin{cases}\frac{1}{2}\left[x(t)-x(-t)\right] & t>0 \quad [\text{replace } t \text{ by} +t \text{ since } t>0 \,\text{in}\, x_0]\\ \frac{1}{2}\left[x(-t)-x(-(-t))\right] & t<0 \quad [\text{replace } t \text{ by } -t \text{ since } t<0 \,\text{in}\, x_o]\end{cases}$$

$$x_o(t)=\begin{cases}\frac{1}{2}[1-0] & t>0\\ \frac{1}{2}[0-1] & t<0\end{cases}$$

$$x_o(t)=\begin{cases}\frac{1}{2} & t>0\\ -\frac{1}{2} & t<0\end{cases}$$

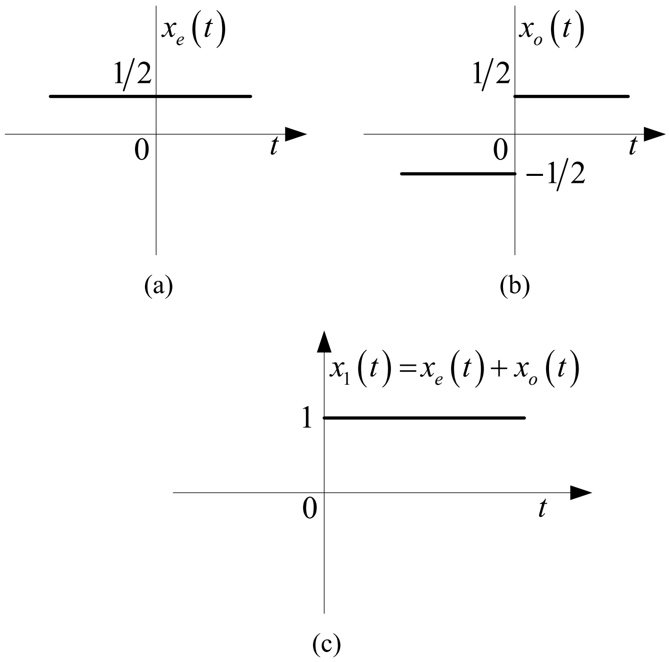

Figure 3. (a) $x_e(t)$; (b) $x_o(t)$; (c) $x(t)$.

From Figure 3(c), the signal $x(t)$ is neither even nor odd since its values for t<0 are zero.

Problem 04. Consider the signal given as follows.

$$x(t) = \begin{cases} 2\cos(4t), & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine the signal whether it is even signal or odd signal. Find its even and odd decomposition. What would happen if $x(0) = 2$ instead of 0, i.e., when we define the sinusoid at t = 0? Explain.

Solution.

Even part of $x(t)$ is given by

$$x_e(t)=\begin{cases}\frac{1}{2}\left[x(t)+x(-t)\right] & t>0\\ \frac{1}{2}\left[x(-t)+x\left(-(-t)\right)\right] & t<0\\ 0 & t=0\end{cases}$$

$$x_e(t)=\begin{cases}\frac{1}{2}\left[2\cos(4t)+0\right] & t>0\\ \frac{1}{2}\left[0+2\cos(4t)\right] & t<0\\ 0 & t=0\end{cases}$$

$$x_e(t)=\begin{cases}\cos(4t) & t>0\\ \cos(4t) & t<0\\ 0 & t=0\end{cases}$$

Odd part of $x(t)$ is given by

$$x_o(t)=\begin{cases}\frac{1}{2}\left[x(t)-x(-t)\right] & t>0\\ \frac{1}{2}\left[x(-t)-x\left(-(-t)\right)\right] & t<0\\ 0 & t=0\end{cases}$$

$$x_o(t)=\begin{cases}\frac{1}{2}\left[2\cos(4t)-0\right] & t>0\\ \frac{1}{2}\left[0-2\cos(4t)\right] & t<0\\ 0 & t=0\end{cases}$$

$$x_o(t)=\begin{cases}\cos(4t) & t>0\\ -\cos(4t) & t<0\\ 0 & t=0\end{cases}$$

If $x(0)=2$, then the even part will be changed as follows while the odd part is the same as before.

$$x_e(t)=\begin{cases}\cos(4t) & t>0\\ \cos(4t) & t<0\\ 2 & t=0\end{cases}$$

This means that the even part has a discontinuity at t=0.

Also, the signal $x(t)$ is neither even nor odd given that its values for $t \le 0$ are zero.

Practice Problem 03. Consider the signal given as follows.

$$x(t) = \begin{cases} A\exp(-\alpha t), & t > 0 \\ 0, & t < 0 \end{cases}$$

Determine the signal whether it is even signal or odd signal. Find its even and odd decomposition.

Problem 05: Characterize the sinusoidal signal

$$x(t) = \sqrt{2}\cos\left(\Pi t/2 + \Pi/4\right) \qquad -\propto < t < \propto .$$

Solution

- The signal $x(t)$ is <u>Deterministic</u>, as the value of the signal can be obtained for any possible value of *t*.
- The signal $x(t)$ <u>is Analog</u>, as there is a continuous variation of the time variable *t,* i.e*,* $-\propto < t < \propto$ and the amplitude of the signal varies between $-\sqrt{2}$ and $\sqrt{2}$.
- The signal $x(t)$ is of <u>infinite support</u>, as the signal does not become zero outside any finite interval. Because of the infinite support, this signal cannot exist in practice.

Operations on signals

Problem 06 (Time shifting, scaling, and reversal). Consider an analog pulse

$$x(t)=\begin{cases}1, & 0\le t>1\\ 0, & \text{otherwise}\end{cases}$$

Find mathematical expressions for $x(t)$ scaled by 2, delayed by 2, advanced by 2, and the reflected signal $x(-t)$.

Solution.

The **delayed** signal $x(t-2)$ can be found mathematically by replacing the variable t by $(t-2)$ so that

$$x(t-2)=\begin{cases}1, & 0\le t-2\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$x(t-2)=\begin{cases}1, & 2\le t\le 3\\ 0, & \text{otherwise}\end{cases}$$

The value $x(0)$ (which in $x(t)$ occurs at $t = 0$) in $x(t-2)$ now occurs when $t = 2$, so that the signal $x(t)$ has been shifted to the right two units of time, and since the values are occurring later, the signal $x(t-2)$ is said to be "delayed" by 2 with respect to $x(t)$.

Similarly, the **advanced** signal $x(t+2)$ can be found mathematically by replacing the variable t by $(t+2)$ so that

$$x(t+2)=\begin{cases}1, & 0\le t+2\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$x(t+2)=\begin{cases}1, & -2\le t\le -1\\ 0, & \text{otherwise}\end{cases}$$

The signal $x(t+2)$ can be seen to be the advanced version of $x(t)$, as it is this signal shifted to the left by two units of time. The value $x(0)$ for $x(t+2)$ now occurs at $t=-2$, which is ahead of $t = 0$.

Furthermore, the signal $x(-t)$ **(i.e., time reversed)** is given by

$$x(-t)=\begin{cases}1, & 0\le -t\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$x(-t)=\begin{cases}1, & -1\le t\le 0\\ 0, & \text{otherwise}\end{cases}$$

This signal is a mirror image of the original: the value $x(0)$ still occurs at the same time, but $x(1)$ occurs when *t*=-1.

The **scaled** signal $x(2t)$ (i.e., compressed) and $x(t/2)$ (i.e., expanded) can be found mathematically by

$$x(2t)=\begin{cases}1, & 0\le 2t\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$x(2t)=\begin{cases}1, & 0\le t\le \dfrac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$x(t/2)=\begin{cases}1, & 0\le t/2\le 1\\ 0, & \text{otherwise}\end{cases}\quad \text{replace } t \text{ by } t/2$$

$$x(t/2)=\begin{cases}1, & 0\le t\le 2\\ 0, & \text{otherwise}\end{cases}\quad \text{multiply } t/2 \text{ by } 2$$

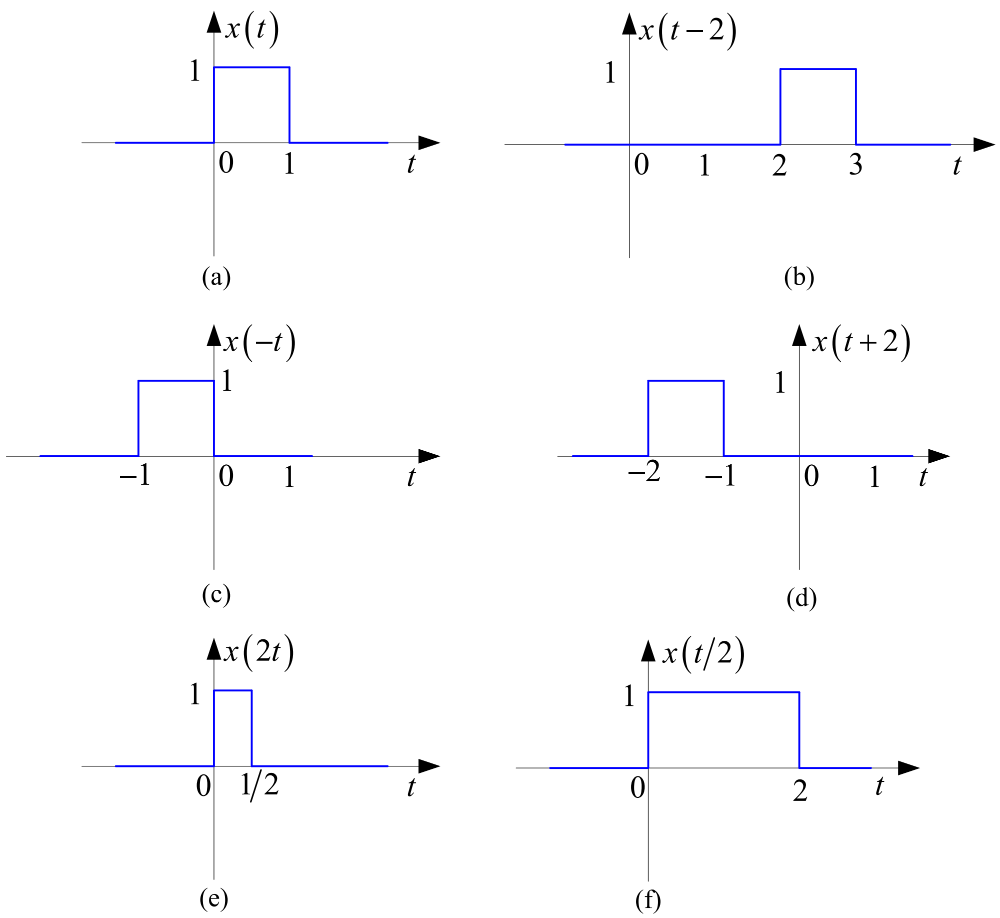

Figure 4. (a) $x(t)$; (b) $x(t-2)$; (c) $x(-t)$; (d) $x(t+2)$; (e) $2x(t)$.

References

[1] Alan V. Oppenheim, Alan S. Willsky, With S. Hamid, Syed Hamid Nawab, “Signals and Systems” Pearson, second edition.
[2] Samir S. Soliman, Mandyam D. Srinath “Continuous and discrete signals and systems,” Prentice Hall, second edition.
[3] P. Lathi, “Linear Systems and Signals”, Oxford University Press, Inc, second edition.
[4] J. Calic, “Communication Systems”. Online: < http://cnx.org/content/col10631/1.3/ >