EEE 243 Signals and Systems

2022

Lecture 14: Practice Problems: Application of Laplace Transform

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// Application of Laplace Transform to electrical circuits (1) → (RLC)

to (2) Differential equations

Inverse Problems Laplace Transform

considerable information about a signal and its transform that can be useful either in characterizing the signal or in checking a calculation. In Sections 9.7 and 9.8 and in some of the problems at the end of this chapter, we give several other examples of the uses of these properties.

9.6 SOME LAPLACE TRANSFORM PAIRS

As we indicated in Section 9.3, the inverse Laplace transform can often be easily evaluated by decomposing $X(s)$ into a linear combination of simpler terms, the inverse transform of each of which can be recognized. Listed in Table 9.2 are a number of useful Laplace

Lecture 12 (Ver) Laplace Transform

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$

Example 9.16

The initial- and final-value theorems can be useful in checking the correctness of the Laplace transform calculations for a signal. For example, consider the signal $x(t)$ in Example 9.4. From eq. (9.24), we see that $x(0^+) = 2$. Also, using eq. (9.29), we find that

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s^3 + 5s^2 + 12s}{s^3 + 4s^2 + 14s + 20} = 2,$$

which is consistent with the initial-value theorem in eq. (9.110).

9.5.11 Table of Properties

In Table 9.1, we summarize the properties developed in this section. In Section 9.7, many of these properties are used in applying the Laplace transform to the analysis and characterization of linear time-invariant systems. As we have illustrated in several examples, the various properties of Laplace transforms and their ROCs can provide us with

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to\infty} sX(s)$$

S. No.	Type of proper rational function	Partial fraction
(i)	$\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
(ii)	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$, $a \neq b \neq c$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
(iii)	$\dfrac{px+q}{(x-a)^3}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-a)^3}$
(iv)	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-b)}$
(v)	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$
(vi)	$\dfrac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where (x^2+ax+b) and (x^2+cx+d) can not be factorised.	$\dfrac{Ax+B}{x^2+ax+b}+\dfrac{Cx+D}{x^2+cx+d}$

Impedances in the Laplace domain

①

Impedances in the Laplace domain

$Z_L = sL$ [Inductor]

$Z_C = 1/sC$ [Capacitor]

$Z_R = R$ [Resistor]

For Inductor

$$V(t) = L \cdot \frac{di}{dt}$$

$$\mathcal{L}\{V(t)\} = L \cdot sI(s)$$

$$V(s) = \underbrace{Ls}_{\text{impedance in the Laplace-domain}} I(s)$$

so $\boxed{Z_L(s) = sL}$ Ω $\rightarrow \left[\frac{H}{s} = \frac{Wb}{As} = \frac{Vs}{As} = \frac{V}{A} = \Omega\right]$

(H: Henry, Wb: Weber, V: Volt, s: second, A: Ampere, s: second)

so Ohm's law applies

For Capacitor.

$$V(t) = \frac{1}{C}\int i(t)\,dt$$

$$\mathcal{L}(v(t)) = \frac{1}{C}\frac{I(s)}{s}$$

$$V(s) = \underbrace{\frac{1}{sC}}_{\text{Impedance of Capacitor in the s-domain}} \cdot I(s) \rightarrow \left[\frac{s}{F} = \frac{sV}{C} = \frac{V}{(C/s)} = \frac{V}{A} = \Omega\right]$$

(s: second, F: Farad, V: volt, C: coulomb, C/s → I)

$$\boxed{Z_C(s) = \frac{1}{sC}}\ \Omega$$

for ~~Regi~~ Resistor

$$V(t) = R\,i(t)$$

$$\boxed{V(s) = R\,I(s)}$$

$$Z_R(s) = R$$

(2)

* Application of Laplace Transforms to Electrical circuits

→ Redrawing Circuits in the S-domain

Resistor

Fig-1

$$\mathcal{L}[V] = \mathcal{L}(iR)$$

$$V(s) = R\,I(s)$$

Fig 1.1

Inductor

Fig-2

Approach-1 (voltage source to represent initial condition)

$$\mathcal{L}[V] = \mathcal{L}\left[L \cdot \frac{di}{dt}\right]$$

$$V(s) = L\left[s\,I(s) - i(0^-)\right]$$

$$\boxed{V(s) = sL\,I(s) - L\,i(0^-)} \quad \text{—(1)}$$

Fig. 2.1

Approach-2 (Current Source to represent Initial Condition)

→ From (1) $\quad I(s) = \frac{1}{sL}V(s) + \frac{\cancel{L}\,i(0^-)}{s\cancel{L}}$

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$

Fig. 2-2

Here, $V(s) = \underbrace{sL}_{\text{impedance in s-domain}}\,I(s)$

$$I(s) = \frac{V(s)}{sL}$$

Capacitor

$i = C\frac{dV}{dt}$ Approach-1: [current source to represent Initial Condition]

$\xrightarrow{\text{taking Laplace Transform on both sides}}$ $\mathcal{L}[i] = \mathcal{L}\left[C \cdot \frac{dV}{dt}\right)$

Fig-3

$$I(s) = C\left[s\,V(s) - V(0^-)\right]$$

$$I(s) = C\,s\,V(s) - C\,V(0^-)$$

$$\boxed{I(s) = \frac{V(s)}{(1/sC)} - C\,V(0^-)} \quad \text{—(II)}$$

← impedance of Capacitor in s-domain

Fig. 3.1

Approach-2

Voltage source to represent Initial Condition

from (II) $\quad V(s) = \frac{1}{sC}\cdot I(s) + C\,V(0^-)\cdot\frac{1}{sC}$

$$\boxed{V(s) = \frac{I(s)}{sC} + \frac{V(0^-)}{s}}$$

Fig.3.2

5.2.6

(3)

given $i_L(0^-) = -2$

$v_c(0^-) = 2$

$x(t) = u(t)$

Initial Conditions

Find $y(t)$ using Laplace transform. in fig. 1

Sol^n

at node ① $[\mathcal{L}[u(t)] = 1/s]$

$$\frac{Y(s)}{1} - 2 + \frac{Y(s)}{(1/s)} - I_1 = 0$$

$$\Rightarrow \quad I_1 = Y(s) - 2 + sY(s) \quad \text{—(1)}$$

Fig. 2. equivalent S-domain circuit

in loop-1

$$X(s) - 2I_1(s) - sI_1(s) + 2 - Y(s) = 0$$

$$\Rightarrow \frac{1}{s} - I_1(s)(2+s) - Y(s) + 2 = 0$$

$$I_1(s) = \frac{-Y(s) + 1/s + 2}{(2+s)}$$

from eq ①

so, $$\frac{-Y(s) + 1/s + 2}{2+s} = Y(s) - 2 + sY(s)$$

$$\Rightarrow 2 - \left[\frac{Y(s) - 1/s - 2}{2+s}\right] - Y(s) - sY(s) = 0$$

$$\Rightarrow 2(2+s) - Y(s)(2+s) - sY(s)(2+s) = 0$$

$$\Rightarrow 4 + 2s - 2Y(s) - sY(s) - 2sY(s) - s^2Y(s) = 0$$

$$\Rightarrow 4 + 2s - 2Y(s) - 3sY(s) - s^2Y(s) = 0$$

$$\Rightarrow 4 + 2s - Y(s)(2 + 3s + s^2) = 0$$

$$\Rightarrow 2(2+s) - Y(s) + 1/s + 2 - Y(s)(2+s) - sY(s)(2+s) = 0$$

$$\Rightarrow 4 + 2s - Y(s) + 1/s + 2 - 2Y(s) - sY(s) - 2sY(s) - s^2Y($$

$$\Rightarrow 4s + 2s^2 - sY(s) + 1 + 2s - 2sY(s) - s^2Y(s) - 2s^2Y(s) - s^3Y(s) = 0$$

$$\Rightarrow -s^3Y(s) - 3s^2Y(s) - 3sY(s) + 2s^2 + 6s + 1 = 0$$

$$\Rightarrow (-s^3 - 3s^2 - 3s)\,Y(s) = -(2s^2 + 6s + 1)$$

$$\Rightarrow Y(s) = \frac{2s^2+6s+1}{s^3+3s^2+3s}$$

$$\boxed{Y(s) = \frac{2s^2+6s+1}{s(s^2+3s+3)}} \quad *$$

Using Partial ~~Differen~~ fraction ~~expansion~~

$$\frac{2s^2+6s+1}{s(s^2+3s+3)} = \frac{A}{s} + \frac{(Bs+C)}{s^2+3s+3}$$

Since it cannot be factored

multiply by $s(s^2+3s+3)$ both sides

$$2s^2+6s+1 = A(s^2+3s+3) + (Bs + C)\,s. \quad \text{—— (1)}$$

$\Rightarrow$ putting $s=0$

$1 = A \cdot 3 \Rightarrow A = 1/3$

for s from (1)

$2s^2+6s+1 = \frac{1}{3}(s^2+3s+3)$

(4)

$$2s^2 + 6s + 1 = As^2 + 3sA + 3A + Bs^2 + Cs$$

equating coefficients of order s^2, s, and constant

$$2 = A + B$$

$$6 = 3A + C$$

$$1 = 3A \Rightarrow \boxed{A = 1/3}$$

So, $2 = 1/3 + B \Rightarrow \boxed{B = 2 - 1/3 = 5/3.}$

Also, $6 = 3 * 1/3 + C \Rightarrow \boxed{C = 6 - 1 = 5.}$

Hence, $$Y(s) = \frac{2s^2 + 6s + 1}{s(s^2 + 3s + 3)} = \frac{1/3}{s} + \frac{(5/3)s + (5)}{s^2 + 3s + 3}$$

$$Y(s) = \frac{1}{3s} + \frac{5s + 15}{3}$$

$$= \frac{1}{3s} + \frac{(5/3)s + 5}{\left(s^2 + 2\cdot\frac{3}{2}\cdot s + \frac{9}{4}\right) + \frac{3}{4}}$$

$$= \frac{1}{3s} + \frac{5/3\, s + 5}{\left(s + \sqrt{3}/2\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Note: $\frac{3}{2} * 5/3 + 5/2 = 5/2 + 5/2 = 5$

$$= \frac{1}{3s} + \frac{\left[5/3\, s + 3/2 * 5/3\right] + 5/2}{\left(s + 3/2\right)^2 + \left(\sqrt{3}/2\right)^2}$$

$$Y(s) = \frac{1}{3s} + 5/3\left[\frac{s + 3/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2}\right] + \frac{5}{\sqrt{3}}\,\frac{\sqrt{3}/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2}$$

$\left[\frac{5}{\sqrt{3}} * \frac{\sqrt{3}}{2} = 5/2\right]$

Taking ILT of both sides results in

$$y(t) = \frac{1}{3}u(t) + 5/3\left[e^{-3/2\,t}\cos \sqrt{3}/2\, t\right]u(t) + 5/\sqrt{3}\left[e^{-3/2\,t}\sin \frac{\sqrt{3}}{2}t\right.$$

Here, $\alpha = 3/2$, $\omega_0 = \sqrt{3}/2$

Re{s}

We know that

$$\mathcal{L}\left[\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}\right] = \left[e^{-\alpha t}\cos\right.$$

$$\mathcal{L}\left[\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}\right] = \left[e^{-\alpha t}\sin\omega_0 t\right]$$

Re{s}>

$\mathcal{L}^{-1}[1/s] = u(t)$, $Re\{s\}>0$

ROC: Pole-zero plot

So, 'ROC' must be greater than zero

ROC>0;

$$Y(s) = \frac{2s^2+6s+1}{s(s^2+3s+3)}$$

poles at $s=0$

at $s = -3/2 \pm j\sqrt{3}/2$

s^2+3s+3

$$\Rightarrow s = \frac{-3 \pm \sqrt{3^2-4\cdot1\cdot3}}{2\cdot1}$$

$$= -3/2 \pm \frac{\sqrt{9-12}}{2}$$

$$= -3/2 \pm \frac{\sqrt{-3}}{2}$$

$$= -3/2 \pm \frac{\sqrt{j^2\cdot3}}{2}$$

$$= -3/2 \pm \frac{j\sqrt{3}}{2}$$

Zeros:

$2s^2+6s+1$

$$= s = \frac{-6 \pm \sqrt{6^2-4\cdot2\cdot1}}{2\cdot2}$$

$$= \frac{-6}{4} \pm \frac{\sqrt{36-8}}{4}$$

$$= -3/2 \pm \frac{\sqrt{28}}{4}$$

$$= -3/2 \pm \frac{2\sqrt{7}}{2\cdot2}$$

$$= -3/2 \pm \sqrt{7}/2$$

Zeros $\rightarrow \{-3/2 + \sqrt{7}/2,\ -3/2 - \sqrt{7}/2\}$

Exp. 02

Switch has been closed for a very long time. Without the circuit is in steady state

(a) Find the initial condition ~~after~~ before the switch is opened,

(b) Find $i_L(t)$ ~~to~~ when switch is opened

(a) at $t = 0^-$, i.e. when the switch was closed

So, $I_{L0} = \frac{12}{4+4} = 1.5\,A.$

$V_{C0} = 12 * \frac{4}{4+4} = 6V.$

So, two initial conditions: $I_{L0} = 1.5A$ and $V_{C0} = +6V.$

(b) when the switch is opened,

$1/Cs = 1/0.1s = 10/s$

$\hookrightarrow \frac{V_{C0}}{s} = 6/s$

$L*I_{L0}$
$= 2*1.5$
$= 3A$

So, the Right.

let's Draw the circuit for the Right hand side ⑥

⑥ switch is opened

$$I_L(s) = \frac{6/s+3}{8+2s+10/s} = \frac{3s+6}{2s^2+8s+10}$$ [see below]

So ILT,

$$i_L(t) = 1.5\, e^{-2t} \sin\left(t+\pi/2\right)$$

Ans

see the other side

$$\mathcal{L}\left[\frac{d\,f(t)}{dt}\right] = s\,F(s)$$

$$\mathcal{L}\left[t\cdot f(t)\right] = -\frac{d}{ds}F(s)$$

in loop-1 → KCL →

$$-3 - 2sI - 4I - 4I - \frac{10}{s}I - \frac{6}{s} = 0$$ [∵ loop current $I = -I_L$]

$$-3 + 2sI_L + 4I_L + 4I_L + \frac{10}{s}I_L + 6/s = 0$$

$$I_L\left[2s + 8 + 10/s\right) = \left(6/s + 3\right)$$

$$I_L = \frac{6/s+3}{8+2s+10/s} = \frac{6+3s}{8s+2s^2+10} = \frac{3s+6}{2s^2+8s+10}$$

$$\boxed{I_L = \frac{3s+6}{2s^2+8s+10}}$$

ILT

$$\mathcal{L}^{-1}\left[\frac{3s+6}{2s^2+8s+10}\right]$$

$$= \frac{3}{2}\left[\frac{s+2}{s^2+4s+5}\right]$$

$$= \frac{3}{2}\left[\frac{s+2}{(s^2+4s+4)+1}\right]$$

$$= 3/2\left[\frac{s+2}{(s+2)^2+1^2}\right]$$

$\frac{1}{6} \times 9 = 3/2$

$$= 1.5\, e^{-2t} \cdot \cos t$$

$$\boxed{i(t) = 1.5\, e^{-2t} \sin\left(\pi/2\, t\right)}$$

[where $\omega_0 = 1$, $\alpha = 2$]

$$\left(e^{-\alpha t}\cos\omega_0 t\right)u(t) \leftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \quad Re\{s\} > -\alpha$$

$$\left[1.5\, e^{-2t} \sin\left(t+\pi/2\right)\right.$$

$3/2$ cost

$$\begin{cases} \omega_0 = 1 \\ \alpha = 2 \end{cases}$$

Exp. 02 ended

Part of Pole-Zero

(c) Initial Value Theorem (I.V.T) and Final Value Theorem (FVT)

$h(t) \longleftrightarrow H(s)$

IVT

IVT: $\boxed{h(0) = \lim_{s\to\infty} sH(s)}$ [No condition]

$t=0$

FVT: If the eigen values in the ~~Theorem~~ System are on the left side of the 's' complex plane [i.e. $\boxed{Re\{s\}}$ are Negative]

Real Part of the eigen values are on the Negative Part of 's'.

$t\to\infty$

FVT $\rightarrow$ $\boxed{h(\infty) = \lim_{s\to 0} sH(s)}$

Example-06

$I_L(s) = \dfrac{3s+6}{2s^2+8s+10}$, Find Initial Value and Final Value of the current in the inductor, $I_L(s)$

IVT

$i_L(0) = \lim_{s\to\infty} s\, I_L(s)$

$= \lim_{s\to\infty} \dfrac{s(3s+6)}{2s^2+8s+10} = \dfrac{\cancel{s^2}[3+6/s]}{\cancel{s^2}[2+\frac{8}{s}+\frac{10}{s^2}]}$

$= \dfrac{3+0}{2+0+0} = 3/2$

$= \lim_{s\to\infty} \dfrac{3s^2}{2s^2}$ [when $s\to\infty$, only consider the higher order]

$= 3/2 = 1.5$ A. [Which is consistent with $i_L(t) = 1.5e^{-2t}$ Cont]

Next, We check that ~~E~~ Eigen Values are on the LHS of the Complex plane.

The characteristic equation is given by the denominator,

So, $2s^2 + 8s + 10 = 0 \Rightarrow s_1 = -2 + j$

$s_2 = -2 - j$

Since The Real Parts of s_1 and s_2 are Negative, the eigen values are ~~for~~ on the LHS. So we can use F.V.T. to find Final Value.

F.V.T., $i_L(\infty) = \lim_{s \to 0} \dfrac{s(3s+6)}{2s^2+8s+10}$

Eigen Values are on the LHS.

$= 0$

Soln of Differential Equations using LT.

①

Exp. 5.8.1

// Consider the second-order, linear, constant-coefficient differential equation.

$$y''(t) + 5y'(t) + 6y(t) = \exp[-t]\, u(t),$$

$$y'(0^-) = 1 \text{ and } y(0^-) = 2. \quad //$$

Solution of Differential eqn

Note Differentiation property in the Time-domain: (extended)

$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)}$$

Extended to yield

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - s^{(n-1)} x(0^-) - , \ldots , - s^1 x^{(n-2)}(0^-) - s^{0} x^{(n-1)}(0^-)$$

Generalized

$$\boxed{\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - s^{(n-1)} x(0^-) - , \ldots , - s\, x^{(n-2)}(0^-) - x^{(n-1)}(0^-)}$$

Exple:

$$\mathcal{L}\left\{\frac{dY(t)}{dt}\right\} = sY(s) - s^{(1-1)} \cdot y^{1}(0^{-1}) = sY(s) - Y(0^-)$$

$$\mathcal{L}\left\{\frac{d^2 Y(t)}{dt^2}\right\} = s^2 Y(s) - s^{(2-1)} \cdot y^{1}(0^-) - s^{(2-2)} \cdot y'(0^-)$$

$$= s^2 Y(s) - sY(0^-) - Y'(0^-)$$

→ s decreases in order

$Y(0^-)$ increases in order of its derivative

s^{n-1},	s^{n-2},	s^{n-3},	...
$Y(0^-)$	$Y'(0^-)$	$Y''(0^-)$	....
1,	2,	3,	...
no derivative	1st derivative	2nd derivative	...

(2)

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Exp. 5.8.1

Consider the second-order, linear, constant-coefficient differential equation

$$y''(t) + 5y'(t) + 6y(t) = \exp[-t]\,u(t),$$

where, $y'(0^-) = 1$ and $y(0^-) = 2$.

Soln

Taking Laplace Transform on both sides, results in

$$\left[s^2Y(s) - sY(0^-) - y'(0^-)\right] + 5\left[sY(s) - y(0^-)\right] + 6Y(s) = \frac{1}{s+1}$$

$$\Rightarrow \left[s^2Y(s) - s\cdot 2 - 1\right] + 5\left[sY(s) - 2\right] + 6Y(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2Y(s) + 5sY(s) + 6Y(s) - 2s - 11 = \frac{1}{s+1}$$

$$\Rightarrow Y(s)(s^2 + 5s + 6) - (2s + 11) = \frac{1}{s+1}$$

$$\Rightarrow Y(s)(s^2 + 5s + 6)(s+1) - (2s+11)(s+1) = 1$$

$$\Rightarrow Y(s)(s^2 + 5s + 6)(s+1) = 1 + \left[2s^2 + 2s + 11s + 11\right]$$

$$\Rightarrow Y(s)(s^2 + 5s + 6)(s+1) = (2s^2 + 13s + 12)$$

$$\Rightarrow Y(s) = \frac{2s^2 + 13s + 12}{(s+1)\underbrace{(s^2 + 5s + 6)}_{(s+2)(s+3)}}$$

Taking Partial expansion ~~on both sides~~ –

$$Y(s) = \frac{2s^2 + 13s + 12}{(s+1)\underbrace{(s^2 + 5s + 6)}_{(s+2)(s+3)}} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \quad \text{—(1)}$$

(2)

Multiply by $(s+1)(s+2)(s+3)$ on both sides of (1)

$$\Rightarrow \quad 2s^2 + 13s + 12 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \quad \text{--- (11)}$$

putting $s = -1, -2,$ and -3 to (11)

$\Rightarrow$ $s=-1$; $\quad 2(-1)^2 + 13\cdot(-1) + 12 = A(-1+2)(-1+3) + 0 + 0$

$$\Rightarrow 2 - 13 + 12 = 2A \Rightarrow \boxed{A = 1/2}$$

$s=-2$; $\quad 2\cdot(-2)^2 + 13(-2) + 12 = 0 + B(-2+1)(-2+3) + 0$

$$\Rightarrow 8 - 26 + 12 = -B \Rightarrow -B = -6 \Rightarrow \boxed{B = 6}$$

$s=-3$; $\quad 2\cdot(-3)^2 + 13\cdot(-3) + 12 = 0 + 0 + C\cdot(-3+1)(-3+2)$

$$\Rightarrow 18 - 39 + 12 = C(-2)(-1) \Rightarrow \boxed{C = \frac{-9}{2}}$$

putting these values of A, B, and C to (1)

$$Y(s) = \frac{2s^2+13s+12}{(s+1)(s^2+5s+6)} = \frac{1}{2(s+1)} + \frac{6}{s+2} - \frac{9}{2(s+3)}$$

Now taking ILT on both sides.

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left[\frac{1}{2(s+1)} + \frac{6}{s+2} - \frac{9}{2(s+3)}\right]$$

$$\boxed{y(t) = \left[\frac{1}{2}e^{-t} + 6e^{-2t} - \frac{9}{2}e^{-3t}\right]u(t)}$$

Ans

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