EEE 243 Signals and Systems

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Lecture 11: The Continuous-Time Fourier Transform

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INTRODUCTION

In the previous lecture, we developed a representation of periodic signals as linear combinations of complex exponentials. In this lecture, we extend these concepts to apply to signals that are **not periodic**.

For **periodic signals**, the complex exponential building blocks are *harmonically* related, for **aperiodic signals** they are *infinitesimally close in frequency*, and the **representation** in terms of a linear combination takes the form of *an integral rather than a sum*. The resulting spectrum of coefficients in this representation is called the **Fourier transform**, and the synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the **inverse Fourier transform**.

Fourier reasoned that *an aperiodic signal can be viewed as a periodic signal with an infinite period*. More precisely, in the Fourier series representation of a periodic signal, as the *period increases* the *fundamental frequency decreases* and the *harmonically related components become closer in frequency*. As the **period becomes infinite**, the frequency components form a continuum and the **Fourier series sum becomes an integral**.

$T \to \infty;\ \omega \to 0$ such that $\sum \ \to \int$

4.1 REPRESENTATION OF APERIODIC SIGNALS: THE CONTINUOUS-TIME FOURIER TRANSFORM

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Consider Fourier series representation for the continuous-time **periodic** square wave examined in Example 3.5. Specifically, over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T, as shown in Figure 4.1.

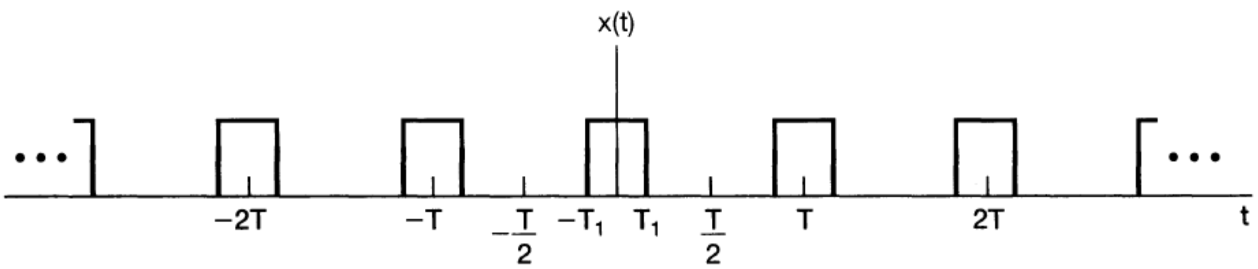

Figure 4. 1 A continuous-time periodic square wave.

As determined in Example 3.5, the Fourier series coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \quad (4.1)$$

where $\omega_0 = 2\Pi/T$.

An alternative way of interpreting eq. (4.1) is as **samples of an envelope function**, specifically,

$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega=k\omega_0} \quad (4.2)$$

That is, with ω thought of as a continuous variable, the function $\frac{(2\sin\omega T_1)}{\omega}$ represents the envelope of Ta_k and the coefficients a_k are simply equally spaced samples of this envelope. Also, for fixed T_1, the envelope of Ta_k is independent *of* T.

In Figure 4.2, we again show the Fourier series coefficients for the periodic square wave, but this time as samples of the envelope of Ta_k as specified in eq. (4.2).

- From the figure, we see that as T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\Pi/T$ decreases, the envelope is sampled with a closer and closer spacing.

- As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse (i.e., all that remains in the time domain is an aperiodic signal corresponding to one period of the square wave).

- Also, the Fourier series coefficients, multiplied by *T*, become **more and more closely spaced samples** of the envelope, so that in some sense, the set of Fourier series coefficients approaches the envelope function as $T \to \propto$.

This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Specifically, we think of **an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large**.

In particular, consider a signal $x(t)$ that is of *finite* duration. That is, for some number T_1, $x(t) = 0, \quad \text{if } |t| > T_1$ as illustrated in **Figure 4.3(a).** From this **aperiodic signal**, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period, as indicated in **Figure 4.3(b).** As we choose the period T to be larger, $\tilde{x}(t)$ is identical to $x(t)$ over a longer interval, and as $T \to \propto$, $\tilde{x}(t)$ is equal to $x(t)$ for any finite value *of t.*

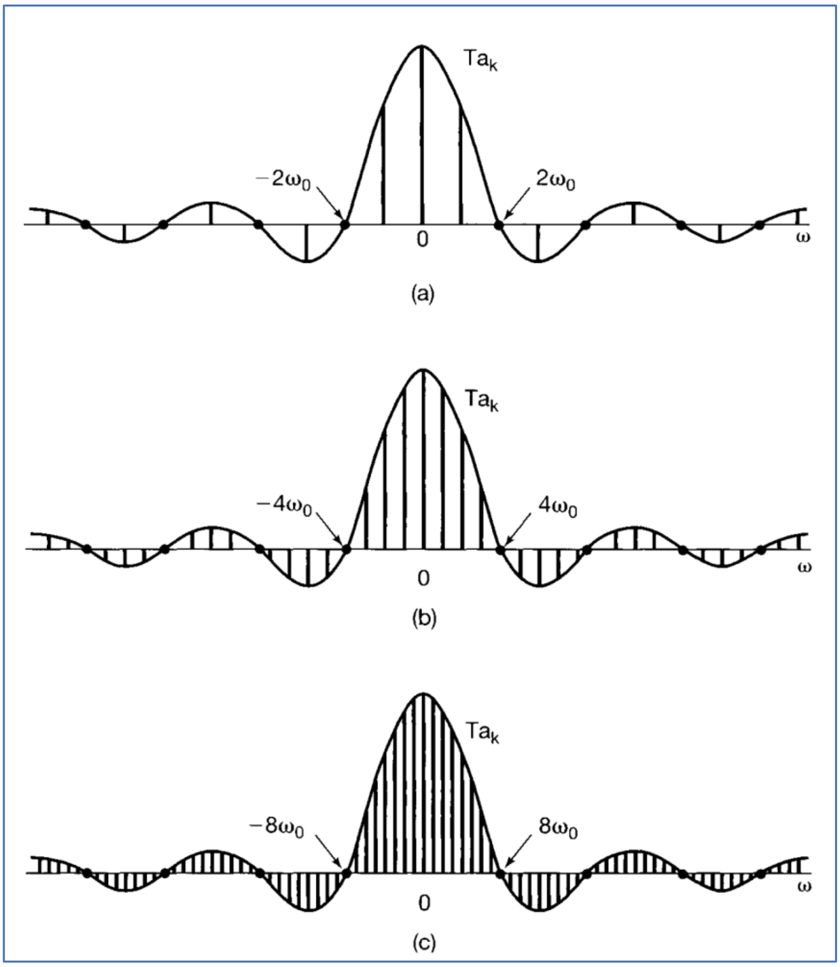

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8\ T_1$: (c) $T = 16\ T_1$.

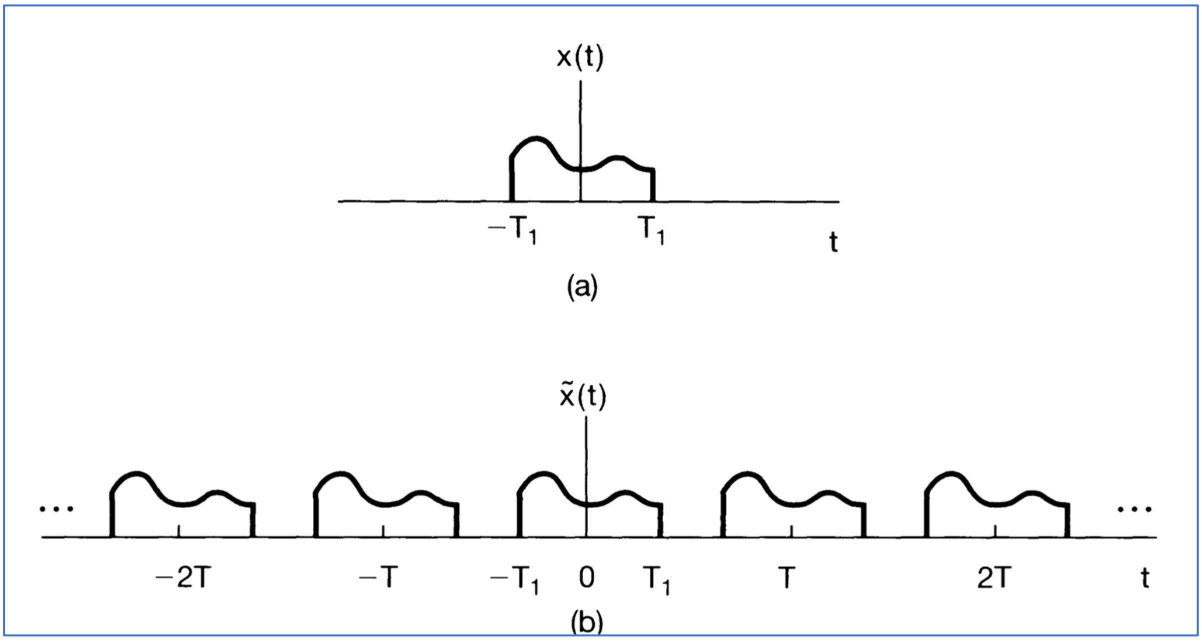

Figure 4.3 (a) Aperiodic signal *x(t);* (b) periodic signal *x(t),* constructed to be equal to *x(t)* over one period.

Let us now examine the effect of this on the Fourier series representation of $\tilde{x}(t)$. Rewriting eqs. (3.38) and (3.39) here for convenience, with the integral in eq. (3.39) carried out over the interval $-T/2 \le t \le T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$
$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \qquad \text{(4.3)- (4.4)}$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$
$$a_k = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad (4.5)$$

Hence the coefficients a_k,

$$\frac{1}{T} X(jk\omega_0) = a_k \qquad (4.6)$$

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or equivalently, since $\omega_0 = 2\Pi/T$,

$$\tilde{x}(t) = \frac{1}{2\Pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \qquad (4.7)$$

Figure 4.4 shows the graphical interpretation of eq. (4.7).

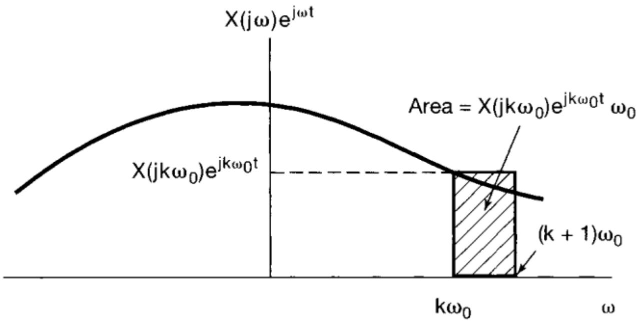

Figure 4.4 Graphical interpretation of eq. (4.7).

From Figure 4.4 and eq. (4.7), we have the following.

- As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit eq. (4.7) becomes a representation of $x(t)$.
- Furthermore, $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of eq. (4.7) passes to an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 4.4.
- Each term in the summation on the right-hand side is the area of a rectangle of height $X(jk\omega_0 t)$ and width ω_0 . (Here, *t* is regarded as fixed.)
- As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{jk\omega t}$.

Therefore, using the fact that $\tilde{x}(t) \to x(t)$ as $T \to \infty$ we see that eqs. (4.7) and (4.5) respectively become

$$x(t) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad (4.8)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad (4.9)$$

Equations (4.8) and (4.9) are referred to as the **Fourier transform pair,** with the function $X(j\omega)$ referred to as the **Fourier Transform or Fourier integral of** $x(t)$ and eq. (4.8) as the **inverse Fourier transform** equation.

For **aperiodic signals**, the complex exponentials occur at a continuum of frequencies unlike periodic signals where these complex exponentials have amplitudes $\{a_k\}$, occur at a discrete set of harmonically related frequencies $k\omega_0, k = 0, \pm 1, \pm 2, \ldots$ *kw0, k* = 0, .± 1, ±2, ... In analogy with the terminology used for the Fourier series coefficients of a periodic signal, the transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the *spectrum* of $x(t)$.

Note 1: Based on the above development, or equivalently on a comparison of eq. (4.9) and eq. (3.39), we also note that the Fourier coefficients a_k of a periodic signal $\tilde{x}(t)$ can be expressed in terms of equally spaced *samples* of the Fourier transform of one period of $\tilde{x}(t)$.

Specifically, suppose that $\tilde{x}(t)$ is a periodic signal with period T and Fourier coefficients a_k. Let *x(t)* be a finite-duration signal that is equal to $\tilde{x}(t)$ over exactly one period-say, for $s \le t \le s+T$ for some value of *s*-and that is zero otherwise. Then, since eq. (3.39) allows us to compute the Fourier coefficients of $\tilde{x}(t)$ by integrating over any period, we can write

$$a_k = \frac{1}{T}\int_{s}^{s+T} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{s}^{s+T} x(t)e^{-jk\omega_0 t}dt$$

Since *x(t)* is zero outside the range $s \le t \le s+T$ we can equivalently write

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt$$

Comparing with eq. (4.9) we conclude that

$$a_k = \frac{1}{T}X(j\omega)\Big|_{\omega=k\omega_0} \qquad (4.10)$$

where $X(j\omega)$ is the Fourier transform of *x(t).* Equation 4.10 states that

"The Fourier coefficients of $\tilde{x}(t)$ are proportional to samples of the Fourier transform of one period of $\tilde{x}(t)$"

This fact, which is often of use in practice.

4.1.2 Convergence of Fourier Transforms

Just as with periodic signals, there is an alternative **set of conditions which are sufficient to ensure that $\tilde{x}(t)$ is equal to *x(t)* for any *t* except at a discontinuity**, where it is equal to the average of the values on either side of the discontinuity. These conditions, again referred to as **the Dirichlet conditions**, require that:

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty \qquad (4.13)$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.

3. $x(t)$ have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have a finite number of discontinuities **have Fourier transforms.**

4.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

While our attention in the previous section was focused on aperiodic signals, we can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context.

In fact, as we will see, *we can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of the impulses proportional to the Fourier series coefficients*.

Let us consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2Π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\Pi\delta(\omega - \omega_0) \quad (4.21)$$

To determine the signal $x(t)$ for which this is the Fourier transform, we can apply the inverse transform relation, eq. (4.8), to obtain

$$x(t) = \frac{1}{2\Pi}\int_{-\infty}^{+\infty} 2\Pi\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

$$x(t) = e^{j\omega_0 t} \qquad \text{Note: F. T. of } \delta(\omega - \omega_0) = \int_{-\infty}^{+\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t} \quad (4.21.\text{A})$$

More generally, if $X(j\omega)$ is of the form of **a linear combination of impulses** equally spaced in frequency, that is,

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\Pi\, a_k\, \delta(\omega - k\omega_0) \quad (4.22)$$

then the application of eq. (4.8) and (using (4.21) and (4.21.A)) yields

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \quad (4.23)$$

Note that eq. **(4.23) corresponds exactly to the Fourier *series* representation of a periodic signal**, as specified by eq. (3.38).

Thus, the **Fourier transform of a periodic signal** with Fourier series coefficients $\{a_k\}$ can be interpreted as **a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the *k*th harmonic frequency $k\omega_0$ is 2Π times the *k*th Fourier series coefficient a_k.**

4.3 PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

As was the case for the Fourier series representation of periodic signals, these properties provide us with a significant amount of insight into the transform and into the **relationship between the time-domain and frequency-domain descriptions of a signal.**

In addition, many of the properties are often useful **in reducing the complexity** of the evaluation of Fourier transforms or inverse transforms.

Furthermore, as described in the preceding section, there is **a close relationship between the Fourier series and Fourier transform representations of a periodic signal,** and using this relationship, we can translate many of the Fourier transform properties into corresponding Fourier series properties (See, in particular, Section 3.5 and Table 3.1.)

A detailed listing of these properties is given in **Table 4.1** in Section 4.6.

As developed in Section 4.1, a signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by the Fourier transform synthesis and analysis equations,

$$x(t) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad (4.24)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \quad (4.25)$$

We will sometimes refer to $X(j\omega)$ with the notation $F\{x(t)\}$ and to $x(t)$ with the notation $F^{-1}\{X(j\omega)\}$. We will also refer to $x(t)$ and $X(j\omega)$ as a **Fourier transform pair** with the notation

$$x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$$

4.3. 1 Linearity

If $\quad x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$

and $\quad y(t) \stackrel{F}{\longleftrightarrow} Y(j\omega)$

then $\quad ax(t) + by(t) \stackrel{F}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$

The linearity property is easily **extended to a linear combination** of an arbitrary number of signals.

4.3.2 Time Shifting

If $\quad x(t) \xleftrightarrow{F} X(j\omega)$

then $\quad x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega) \quad (4.27)$

Proof.

$$x(t) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Replacing t by t-t0

$$x(t-t_0) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega$$
$$x(t-t_0) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} \left(X(j\omega) e^{-j\omega t_0} \right) e^{j\omega t} d\omega$$
$$F\{x(t-t_0)\} = X(j\omega) e^{-j\omega t_0}$$

One consequence of the time-shift property is that a signal which is shifted in time **does not have the magnitude** of its Fourier transform altered. However, the effect of a time shift on a signal introduces into its transform **a phase shift, namely,** $(-\omega t_0)$ **.**

4.3.3 Conjugation and Conjugate Symmetry

If $\quad x(t) \xleftrightarrow{F} X(j\omega)$

then $\quad x^*(t) \xleftrightarrow{F} X^*(-j\omega)$

4.3.4 Differentiation and Integration

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then, by differentiating both sides of the Fourier transform synthesis equation (4.24), we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\Pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega$$

So, $$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega) \quad (4.31)$$

This is a particularly important property, as it replaces the operation of **differentiation** in the time domain with that of **multiplication** by $j\omega$ in the frequency domain.

Since differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain, one might conclude that **integration** should involve **division** by $j\omega$ in the frequency domain. This is indeed the case, but it is only one part of the picture. The precise relationship is

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega)+\Pi X(0)\delta(\omega) \quad (4.32)$$

The impulse term on the right-hand side of eq. (4.32) reflects the de or average value (i.e., $X(0)\delta(\omega)$) that can result from integration.

4.3.5 Time and Frequency Scaling

If $\qquad x(t) \xleftrightarrow{F} X(j\omega)$

then $\qquad x(at) \xleftrightarrow{F} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \quad (4.34)$

where a is a nonzero real number.

Proof:

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

Putting $\tau = at$, we obtain

$$F\{x(at)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a>0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a<0 \end{cases}$$

which corresponds to eq. (4.34). Thus, aside from the amplitude factor $1/|a|$, a linear scaling in time by a factor of a corresponds to a linear scaling in frequency by a factor of $1/a$, and vice versa.

Also, letting a = -1, we see from eq. (4.34) that

$$x(-t) \xleftrightarrow{F} X(-j\omega) \qquad (4.35)$$

That is, **reversing** a signal in time also reverses its Fourier transform.

Example: A common illustration of eq. (4.34) is the effect on frequency content that results when an audiotape is recorded at one speed and played back at a different speed. If the playback speed is **higher than the recording speed**, **corresponding to compression in time (i.e., *a* > 1)**, then the spectrum is expanded in frequency (i.e., the audible effect is that the playback frequencies are higher).

Conversely, the signal played back will be **scaled down in frequency if the playback speed is slower than the recording speed (0 < *a* < 1).** For example, if a recording of the sound of a small bell ringing is played back at a reduced speed, the result will sound like the chiming of a larger and deeper sounding bell.

4.3.6 Duality

By comparing the transform and inverse transform relations given in eqs. (4.24) and (4.25), as follows, we observe that these equations are similar, but not quite identical, in form. This **symmetry** leads to a property of the Fourier transform referred to as **duality**.

$$x(t) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad (4.24)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \quad (4.25)$$

For example, consider the Fourier transform pairs

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega} \quad (4.36)$$

And

$$x_2(t) = \frac{2\sin Wt}{\Pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W_1 \end{cases} \quad (4.37)$$

The two Fourier transform pairs and the relationship between them are depicted in **Figure 4.17.**

The symmetry exhibited by these two examples extends to Fourier transforms in general. Specifically, because of the **symmetry between eqs. (4.24) and (4.25),**

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

$$t \longleftrightarrow f \; [\text{Time and frequency interchanged}]$$

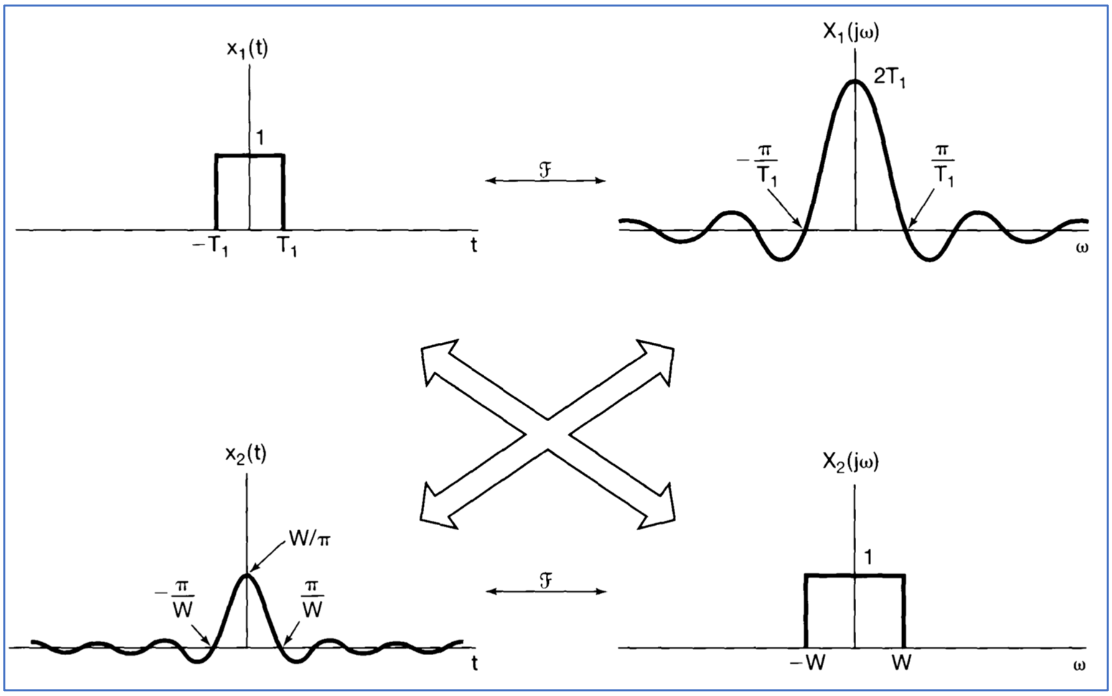

Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

Note: Let's **replace** ω **with** ***t*** of the Fourier Transform in frequency-domain, say $X_1(jw)$, of a signal in time-domain, say $x_1(t)$. Let the resulting new signal due to replacing is $x_2(t)$ in time-domain. Then, the Fourier Transform in frequency-domain of $x_2(t)$, say $X_2(jw)$, is given by the expression of the signal $x_1(t)$ in time-domain.

Note: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} \left[-jt\, x(t)\right] e^{-j\omega t} dt$$

So,

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} \left[-jt\, x(t)\right] e^{-j\omega t} dt$$

$$\boxed{-jt\, x(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega} \quad (4.40)}$$

Similarly, we can derive the dual properties of eq. (4.27):

$$\boxed{X(j(\omega-\omega_0)) \xleftrightarrow{F} e^{j\omega t_0} x(t) \quad (4.41)}$$

4.3.7 Parseval's Relation

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\Pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega \quad (4.43)$$

The term on the left-hand side of eq. (4.43) is the **total energy** in the signal *x(t).*

> Parseval's relation says that *this total energy may be determined either by computing the energy per unit time (*$|x(t)|^2$*) and integrating over all time or by computing the energy per unit frequency (*$|X(j\omega)|^2/2\Pi$*) and integrating over all frequencies.*

For this reason, $|X(j\omega)|^2$ is often referred to as the ***energy-density spectrum*** of the signal *x(t).*

Note that Parseval's relation **for finite-energy signals** is the direct counterpart of Parseval's relation **for periodic signals** (eq. 3.67), which states that the average *power* of a periodic signal equals the sum of the average powers of its individual harmonic components, which in tum are equal to the squared magnitudes of the Fourier series coefficients.

4.4 THE CONVOLUTION PROPERTY

The Fourier transform maps the **convolution** of two signals in time-domain into the **product** of their Fourier transforms in frequency-domain.

$$y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega) \quad (4.56)$$

THE MULTIPLICATION PROPERTY

The convolution property states that convolution in the *time* domain corresponds to multiplication in the *frequency* domain. Because of **duality between the time and frequency domains**, we would expect a dual property also to hold (i.e., that **multiplication in the time domain corresponds to convolution in the frequency domain**). Specifically,

$$r(t) = s(t)p(t) \xleftrightarrow{F} R(j\omega) = \frac{1}{2\Pi} \int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta \quad (4.70)$$

Multiplication of one signal by another can be thought of as using one signal to scale or *modulate* the amplitude of the other, and consequently, the multiplication of two signals is often referred to as *amplitude modulation*. For this reason, eq. (4.70) is sometimes referred to as the ***modulation property***.

TABLE 4. 1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

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