EEE 243 Signals and Systems

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Lecture 21: The Laplace Transform

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School of Engineering
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Inspiring Excellence

INTRODUCTION

The continuous-time Fourier transform provides us with a representation for signals as linear combinations of complex exponentials of the form e^{st} with $s = jw$. *However,* for arbitrary values of s (i.e., not only those values that are purely imaginary as in the Fourier transform), the observation leads to a **generalization of the continuous-time Fourier transform**, known as the **Laplace transform**, which we develop in this lecture.

9.1 THE LAPLACE TRANSFORM

Recall that the response of a linear time-invariant system with an impulse response $h(t)$ to a complex exponential input of the form e^{st} is

$$y(t) = H(s)e^{st} \quad (9.1)$$

where

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \quad (9.2)$$

For s imaginary (i.e., $s = j\omega$), the integral in eq. (9.2) corresponds to the ***Fourier transform*** of $h(t)$.

For **general values of the complex variables**, it is referred to as the ***Laplace transform*** of the impulse response $h(t)$.

The Laplace transform of a general signal $x(t)$ is defined as

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt \quad (9.3)$$

The transform defined by eq. (9.3) is often called the *bilateral Laplace transform,* to distinguish it from the unilateral Laplace transform. The bilateral transform in eq. (9.3) involves an integration from $-\infty$ to $+\infty$, while the unilateral transform has a form similar to that in eq. (9.3), but with limits of integration from 0 to $+\infty$.

Note that The complex variables can be written as $s = \sigma + j\omega$, with σ and ω the real and imaginary parts, respectively. For convenience, we will sometimes denote the Laplace transform in operator form as $L\{x(t)\}$ and denote the transform relationship between $x(t)$ and $X(s)$ as

$$x(t) \xleftrightarrow{L} X(s)$$

When $s = j\omega$, eq. (9.3) becomes

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

which corresponds to the *Fourier transform* of *x(t).*

The Laplace transform also bears a straightforward relationship to the Fourier transform when the complex variables is not purely imaginary. To see this relationship, consider $X(s)$ as specified in eq. (9.3) with s expressed $s = \sigma + j\omega$, so that

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t) e^{-(\sigma)t}] e^{-j\omega t} dt \quad (9.8)$$

We recognize the right-hand side of eq. (9.8) as the **Fourier transform of** $x(t)e^{-(\sigma)t}$ that is,

> the **Laplace transform of *x(t)* can be interpreted as the Fourier transform of *x(t)* after multiplication by a real exponential signal (e.g.,** $x(t)e^{-(\sigma)t}$**).**

The real exponential $e^{-(\sigma)t}$ may be decaying or growing in time, depending on whether σ is positive or negative. ■

To illustrate **the Laplace transform and its relationship to the Fourier transform**, let us consider the following example:

> Note that just as the **Fourier transform does not converge for all signals**, the **Laplace transform may converge for some values of** $\Re e\{s\}$ **and not for others.**

Proof:
In eq. (9.13), the Laplace transform converges only for $\sigma = \Re e\{s\} > -a$. If a is positive, then $X(s)$ can be evaluated at $\sigma = 0$ to obtain

$$X(0 + j\omega) = \frac{1}{j\omega + a} \quad (9.15)$$

As indicated in eq. (9.6), **for** $\sigma = 0$ **the Laplace transform is equal to the Fourier transform**, as is evident in the preceding example by comparing eqs. (9.9) and (9.15).

However, if a is **negative or zero**, **the Laplace transform still exists**, since $x(t) = e^{-at}u(t)$ is not absolutely integrable (i.e., x(t) is not finite). But the **Fourier transform** $x(t) = e^{-at}u(t)$ still **does not (see Figure 12.1).**

Example 9.1

Let the signal $x(t) = e^{-at}u(t)$. From Example 4.1, the Fourier transform $X(j\omega)$ converges for $a > 0$ and is given by

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \frac{1}{j\omega + a}, \quad a > 0. \tag{9.9}$$

From eq. (9.3), the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt, \tag{9.10}$$

or, with $s = \sigma + j\omega$,

$$X(\sigma + j\omega) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}\,dt. \tag{9.11}$$

By comparison with eq. (9.9) we recognize eq. (9.11) as the Fourier transform of $e^{-(\sigma+a)t}u(t)$, and thus,

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0, \tag{9.12}$$

or equivalently, since $s = \sigma + j\omega$ and $\sigma = \mathcal{R}e\{s\}$,

$$X(s) = \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a. \tag{9.13}$$

That is,

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a. \tag{9.14}$$

For example, for $a = 0$, $x(t)$ is the unit step with Laplace transform $X(s) = 1/s$, $\mathcal{R}e\{s\} > 0$.

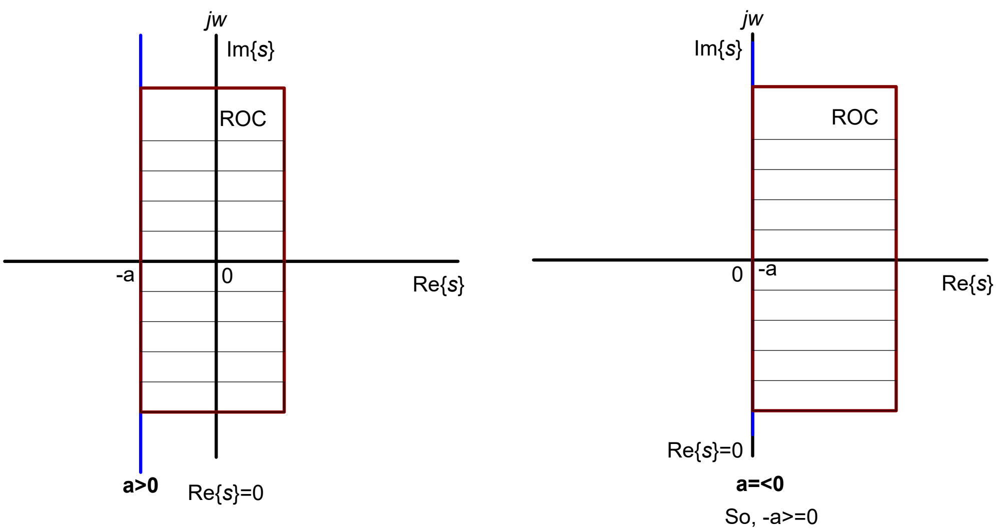


Figure 12.1.

Region of Convergence (ROC)

Note that there are some cases, where the algebraic expression for the Laplace transform of two different signals is identical. However, the set of values of s for which the expression is valid is very different for these signals. This serves to illustrate the fact that, in **specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid are required**.

In general, **the range of values of s for which the integral in eq.(9.3) converges** is referred to as the ***region of convergence*** (which we abbreviate as **ROC**) of the Laplace transform. That is, the ROC consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges.

A convenient way **to display the ROC** is shown in Figure 12.2. The variable s is a complex number, and in the figure, we display the complex plane, generally referred to as the **s-plane**, associated with this complex variable. The coordinate axes are $\Re e\{s\}$ along the horizontal axis and $\mathrm{Im}\{s\}$ along the vertical axis. The horizontal and vertical axes are sometimes referred to as the σ -axis and the $j\omega$ **-axis** is, respectively. The shaded region in Figure 12.2 represents the set of points in the s-plane corresponding to the region of convergence.

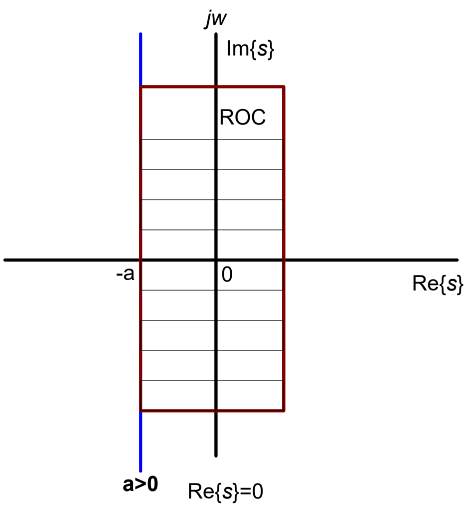


Figure 12.2

Pole-Zero Plot

The Laplace transform is rational, i.e., it is a ratio of polynomials in the complex variables, so that

$$X(s) = \frac{N(s)}{D(s)} \quad (9.31)$$

where N(s) and D(s) are the numerator polynomial and denominator polynomial, respectively.

Note that X(s) will be rational whenever *x(t)* is a linear combination of real or complex exponentials. Except for a scale factor, the numerator and denominator polynomials in a rational Laplace transform can be specified by their roots; thus, marking the locations of the roots of *N(s)* and *D(s)* in the s-plane and indicating the ROC provides a **convenient pictorial way of describing the Laplace transform**.

For example, in Figure 12.3 we show the s-plane representation of the Laplace transform of

$$X(s) = \frac{s-1}{s^2+3s+2} \quad (9.32)$$

with the location of each root of the denominator polynomial indicated with "×" and the location of the root of the numerator polynomial indicated with "o" The **region of convergence** is shaded in the corresponding plot.

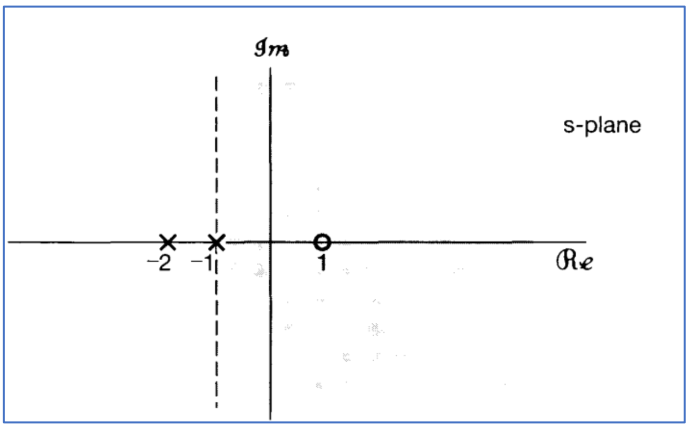


Figure 12.3. s-plane representation of the Laplace transforms for eq. 9.32. Each x in these figures marks the location of a pole-i.e., a root of the denominator. Similarly, each o marks a zero i.e., a root of the numerator. The shaded regions indicate the ROC.

For rational Laplace transforms, the roots of the numerator polynomial are commonly referred to as the ***zeros*** of X(s), since, for those values of *s,* X(s) = 0. The roots of the denominator polynomial are referred to as the ***poles*** of X(s), and for those values of *s,* X(s) is infinite. The poles and zeros of X(s) in the finite s-plane completely characterize the algebraic expression for X(s) to within a scale factor. The representation of X(s) through its poles and zeros in the s-plane is referred to as the ***pole-zero plot*** of X(s).

However, since knowledge of the algebraic form of X (s) does not by itself identify the ROC for the Laplace transform, a **complete specification**, to within a scale factor, of a rational Laplace transform **consists of the pole-zero plot of the transform, together with its ROC** (which is commonly shown as a shaded region in the s-plane).

Also, while they are not needed to specify the algebraic form of a rational transform X(s), it is sometimes convenient to refer to **poles or zeros of X(s) at infinity**.

> Specifically, if the order of the denominator polynomial is **greater than** the order of the numerator polynomial, then X(s) will become zero as *s* approaches infinity. Conversely, if the order of the numerator polynomial is **greater** than the order of the denominator, then X(s) will become unbounded as *s* approaches infinity. This behavior can be interpreted as **zeros or poles at infinity.** For example, the Laplace transform in eq. (9.32) has a denominator of order 2 and a numerator of order only 1, so in this case X(s) has one zero at infinity.

In general, if the order of the denominator exceeds the order of the numerator by *k,* X(s) will have ***k* zeros at infinity**. Similarly, if the order of the numerator exceeds the order of the denominator by *k,* X(s) will have ***k* poles at infinity**.

> Note that for $s = j\omega$, the Laplace transform corresponds to the Fourier transform. However, **if the ROC of the Laplace transform does not include the** $j\omega -$axis **, (i.e., if** $\Re e\{s\} = 0$**), then the Fourier transform does not converge.**

In general, we refer to the ***order*** of a pole or zero as the number of times it is repeated at a given location.

For Example, in $X(s) = \dfrac{(s-1)^2}{(s+1)(s-2)}$ (9.35), there is a second-order zero at s = 1 and two first-order poles, one at s = -1, the other at s = 2. In this example the ROC lies to the right of the rightmost pole (**Figure 12.4**).

In general, for rational Laplace transforms, there is a **close relationship between the locations of the poles and the possible ROCs that can be associated with a given pole-zero plot**.

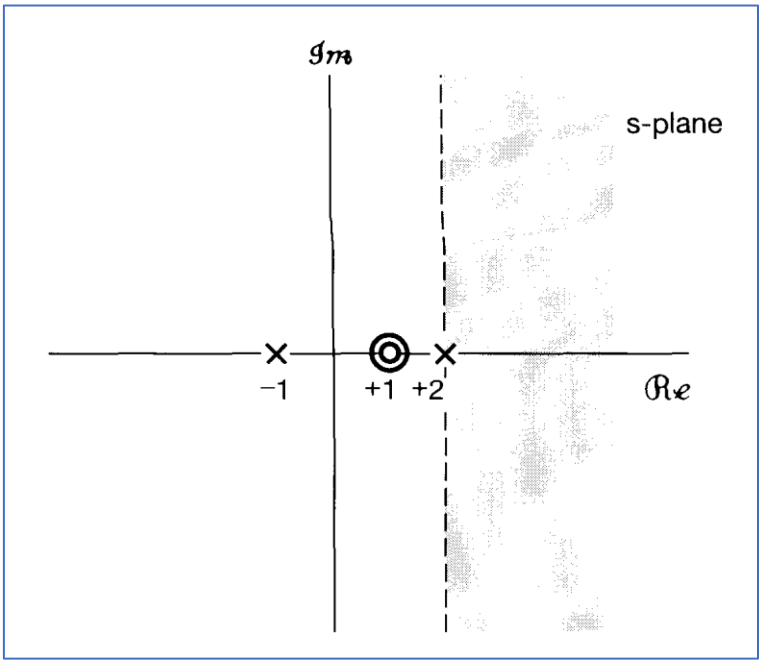


Figure 12.4. Pole-zero plot and ROC for eq. 9.35.

THE REGION OF CONVERGENCE FOR LAPLACE TRANSFORMS

Recall that *a complete specification of the Laplace transform requires not only the algebraic expression for X(s), but also the associated region of convergence.*

Though **two very different signals** can have identical algebraic expressions for X(s), their **Laplace transforms are distinguishable *only* by the region of convergence**.

In this section, we explore **some specific constraints on the ROC** for various classes of signals. As we will see, an understanding of these constraints often permits us to specify implicitly or to reconstruct the ROC from knowledge of only the algebraic expression for X(s) and certain general characteristics of x(t) in the time domain.

Property 1: The ROC of X(s) consists of strips parallel to the $j\omega-$axis in the s-plane (Figure 12.5).

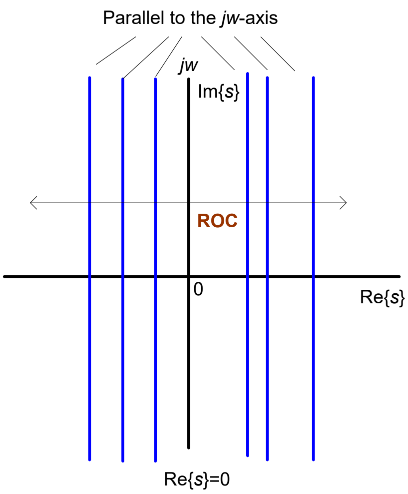


Figure 12.5. Property 1.

Proof: The validity of this property stems from the fact that the ROC of X(s) consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-(\sigma)t}$ converges. That is, the ROC of the Laplace transform of *x(t)* consists of those values of *s* for which $x(t)e^{-(\sigma)t}$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} |x(t)| e^{-\sigma t} dt < \infty$$

Since the above condition depends only on σ, the real part of s (irrespective of the value of $j\omega$), ROC of X(s) consists of strips parallel to the $j\omega-$axis in the s-plane.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

Proof: Property 2 is easily observed by the fact that X (s) is infinite at a pole. For example, for the following rational Laplace transform, $X(s) = \infty$ at s=2 and s=-1.

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}$$

Since $X(s) = \infty$ at a pole, the integral in eq. (9.3) does not converge at a pole. Hence, the ROC cannot contain values of *s* that are poles.

Property 3: If *x(t)* is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Proof: The intuition behind this result is suggested in Figures 9.4 and 9.5. Specifically, a finite-duration signal has the property that it is zero outside an interval of finite duration, as illustrated in Figure 9.4.

In Figure 9.5(a), we have shown *x(t)* of Figure 9.4 multiplied by a decaying exponential, and in Figure 9.5(b) the same signal multiplied by a growing exponential.

Since the interval over which *x(t)* is nonzero **is finite**, the **exponential weighting is never unbounded**, and consequently, *it is reasonable that the integrability of x(t) not be destroyed by this exponential weighting (i.e.,* $\int_{-\infty}^{+\infty} |x(t)| e^{-\sigma t} dt < \infty$ *).*

In other words, there is no pole in the *s*-plane such that the integrability of *x(t)* can be destroyed. Hence, ROC is the entire *s*-plane.

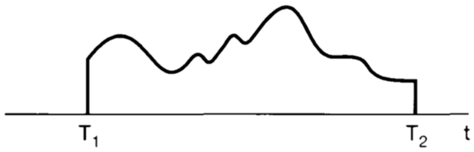


Figure 9.4 Finite-duration signal.

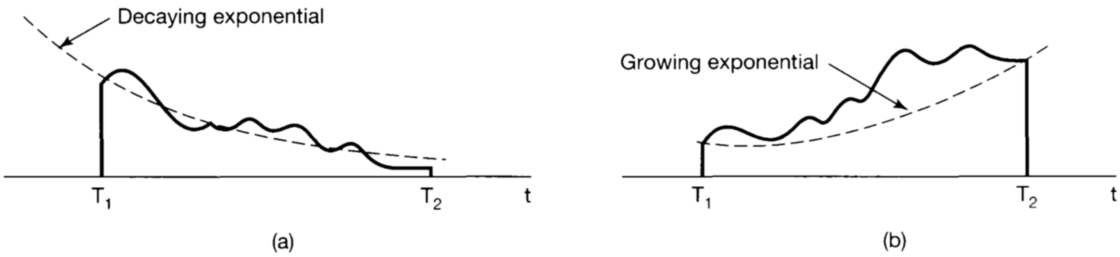


Figure 9.5 (a) Finite-duration signal of Figure 9.4 multiplied by a decaying exponential; (b) finite-duration signal of Figure 9.4 multiplied by a growing exponential.

Property 4: If *x(t)* is right sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of *s* for which $\Re e\{s\} > \sigma_0$ will also be in the ROC.

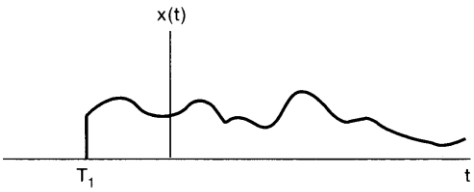


Figure 9.6 Right-sided signal.

Property 5: If *x(t)* is left sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of *s* for which $\Re e\{s\} < \sigma_0$ will also be in the ROC.

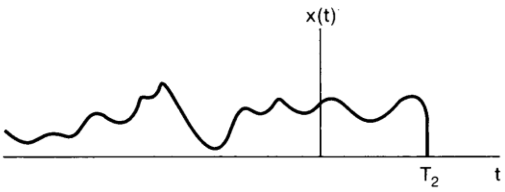


Figure 9.8 Left-sided signal.

Property 6: If *x(t)* is two sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re e\{s\} = \sigma_0$.

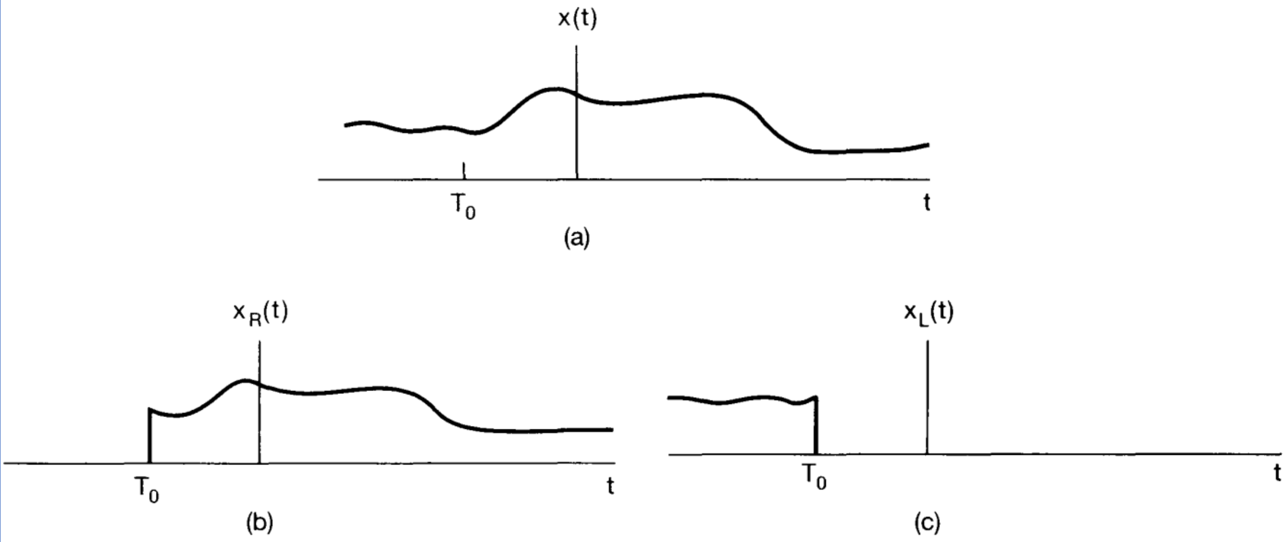


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided signal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 for $t > T_0$.

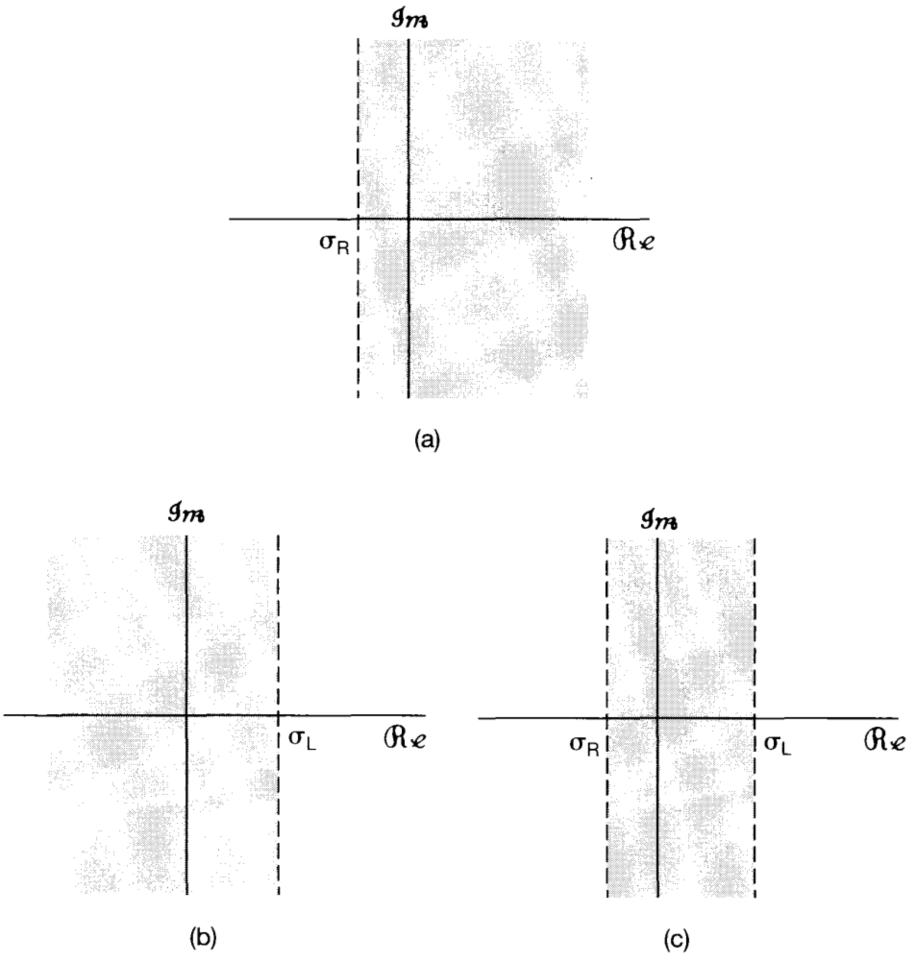


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Figure 9.9; (c) the ROC for $x(t) = x_R(t) + x_L(t)$, assuming that the ROCs in (a) and (b) overlap.

Property 7: If the Laplace transform X(s) of *x(t)* is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(s) are contained in the ROC.

Property 8: If the Laplace transform X(s) of *x(t)* is rational, then if *x(t)* is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If *x(t)* is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

To illustrate how different ROCs can be associated with the same pole-zero pattern, let us consider the following example:

$$X(s) = \frac{1}{(s+1)(s+2)} \quad (9.52)$$

with the associated pole-zero pattern in Figure 9.13(a). As indicated in Figures 9.13(b)-(d), there are **three possible ROCs** that can be associated with this algebraic expression, corresponding to three distinct signals.

The signal associated with the pole-zero pattern **in Figure 9.13(b) is right sided. Since the ROC includes the jw-axis, the Fourier transform of this signal converges.**

Figure 9.13(c) corresponds to a left -sided signal and Figure 9.13(d) to a two-sided signal. **Neither of these two signals have Fourier transforms, since their ROCs do not include the jw-axis**.

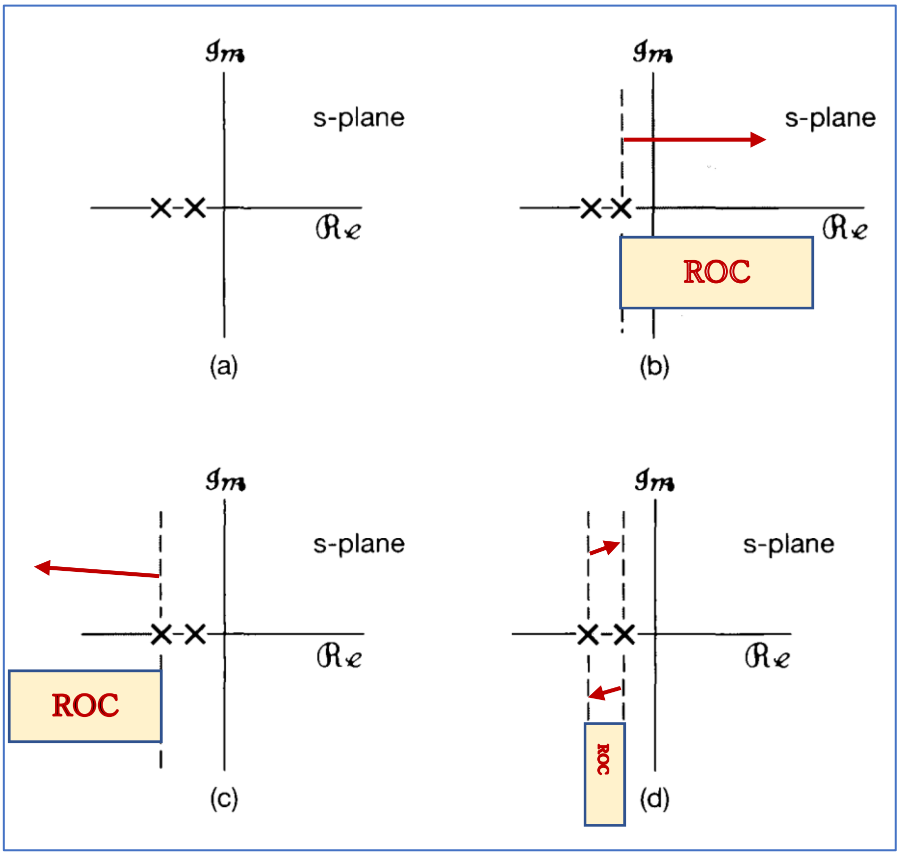


Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a **right-sided** sequence; (c) ROC corresponding to a **left-sided** sequence (Property 8); (d) ROC corresponding to a **two-sided** sequence (Property 7).

9.3 THE INVERSE LAPLACE TRANSFORM

Recall that we discussed the interpretation of the Laplace transform of a signal as the **Fourier transform of an exponentially weighted version of the signal**; that is, with s expressed as $s = \sigma + j\omega$, the Laplace transform of a signal *x(t)* is

$$X(\sigma + j\omega) = F\left\{x(t)e^{-\sigma t}\right\} = \int_{-\infty}^{+\infty} [x(t)e^{-(\sigma)t}]e^{-j\omega t}dt \quad (9.53)$$

for values of $s = \sigma + j\omega$ in the ROC. We can invert this relationship using the inverse Fourier transform as given in eq. (4.9). We have

$$x(t)e^{-\sigma t} = F^{-1}\left\{X(\sigma + j\omega)\right\} = \frac{1}{2\Pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$, we obtain

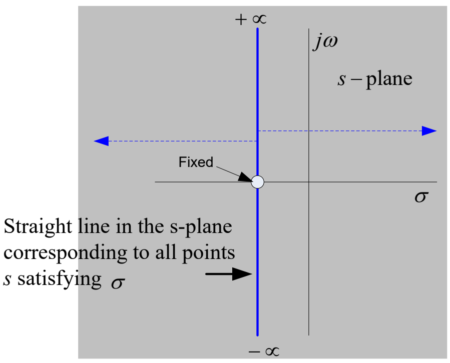
$$x(t) = \frac{1}{2\Pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

That is, we can recover *x(t)* from its Laplace transform evaluated for a set of values of $s = \sigma + j\omega$ in the ROC, with σ fixed and ω varying from $-\infty$ to $+\infty$.

Changing the variable of integration in eq. (9.55) from ω to s and using the fact that σ is constant, so that $ds = jd\omega$, we can get the basic **inverse Laplace transform** equation:

$$x(t) = \frac{1}{2\Pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds \quad (9.56)$$

This equation states that *x(t)* can be represented as a **weighted integral of complex exponentials**. **The contour of integration in eq. (9.56) is the straight line in the s-plane corresponding to all points s satisfying $\Re e\{s\} = \sigma$. This line is parallel to the jw-axis**. Furthermore, we can choose any such line in the ROC-i.e., we can choose any value of σ such that $X(\sigma + j\omega)$ converges.



*Note that the formal evaluation of the integral for a general X(s) requires the use of contour integration in the complex plane, a topic that **we will not consider here**.*

Alternatively, however, for the class of rational transforms, the inverse Laplace transform can be determined without directly evaluating eq. (9.56) by using the technique of partial fraction expansion to determine the inverse Fourier transform.

Basically, the procedure consists of expanding the rational algebraic expression into a linear combination of lower order terms. For example, assuming no multiple-order poles, and assuming that the order of the denominator polynomial is greater than the order of the numerator polynomial, we can expand X(s) in the form

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s+a_i} \quad (9.57)$$

From the ROC of X(s), the ROC of each of the individual terms in eq. (9.57) can be inferred, and then, the inverse Laplace transform of each of these terms can be determined. There are two possible choices for the inverse transform of each term $\frac{A_i}{s+a_i}$ in the equation.

- If the ROC is to the right of the pole at $s = -a_i$, then the inverse transform of this term is $A_i e^{-a_i t} u(t)$, a right-sided signal.

- If the ROC is to the left of the pole at $s = -a_i$, then the inverse transform of the term is $-A_i e^{-a_i t} u(-t)$), a left-sided signal.

Adding the inverse transforms of the individual terms in eq. (9.57) then yields the inverse transform of X(s). The details of this procedure are best presented through a number of **examples as follows**.

Example 9.9

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.58}$$

To obtain the inverse Laplace transform, we first perform a partial-fraction expansion to obtain

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}. \tag{9.59}$$

As discussed in the appendix, we can evaluate the coefficients A and B by multiplying both sides of eq. (9.59) by $(s+1)(s+2)$ and then equating coefficients of equal powers of s on both sides. Alternatively, we can use the relation

$$A = [(s+1)X(s)]\big|_{s=-1} = 1, \tag{9.60}$$

$$B = [(s+2)X(s)]\big|_{s=-2} = -1. \tag{9.61}$$

Thus, the partial-fraction expansion for $X(s)$ is

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}. \tag{9.62}$$

From Examples 9.1 and 9.2, we know that there are two possible inverse transforms for a transform of the form $1/(s+a)$, depending on whether the ROC is to the left or the right of the pole. Consequently, we need to determine which ROC to associate with each of the individual first-order terms in eq. (9.62). This is done by reference to the properties of the ROC developed in Section 9.2. Since the ROC for $X(s)$ is $\mathcal{R}e\{s\} > -1$, the ROC for the individual terms in the partial-fraction expansion of eq. (9.62) includes $\mathcal{R}e\{s\} > -1$. The ROC for each term can then be extended to the left or right (or both) to be bounded by a pole or infinity. This is illustrated in Figure 9.14. Figure 9.14(a) shows the pole-zero plot and ROC for $X(s)$, as specified in eq. (9.58). Figure 9.14(b) and 9.14(c) represent the individual terms in the partial-fraction expansion in eq. (9.62). The ROC for the sum is indicated with lighter shading. For the term represented by Figure 9.14(c), the ROC for the sum can be extended to the left as shown, so that it is bounded by a pole.

Since the ROC is to the right of both poles, the same is true for each of the individual terms, as can be seen in Figures 9.14(b) and (c). Consequently, from Property 8 in the preceding section, we know that each of these terms corresponds to a right-sided signal. The inverse transform of the individual terms in eq. (9.62) can then be obtained by reference to Example 9.1:

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1, \tag{9.63}$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2. \tag{9.64}$$

We thus obtain

$$[e^{-t} - e^{-2t}]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.65}$$

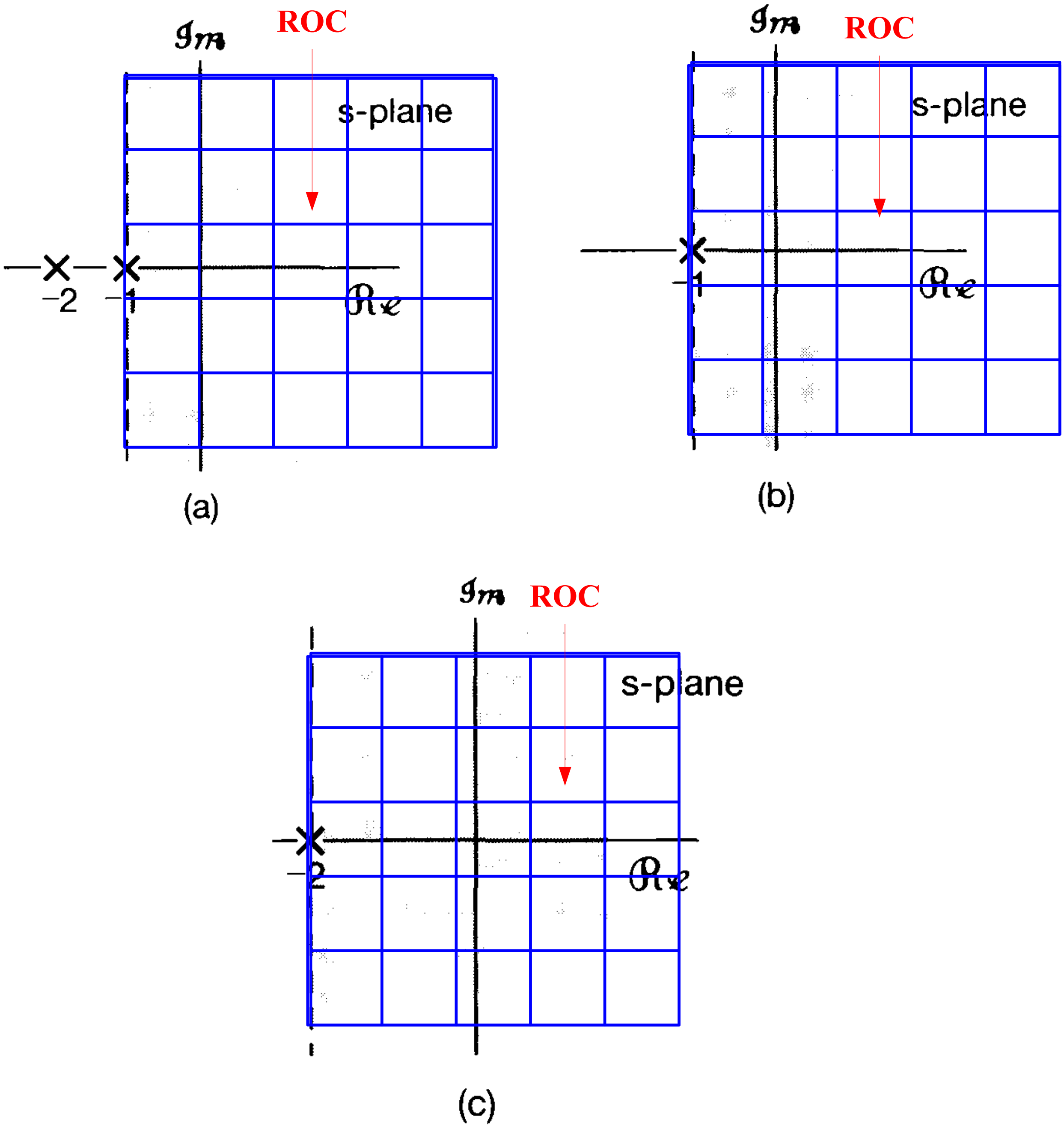


Figure 9.14 Construction of the ROCs for the individual terms in the partial-fraction expansion of *X(s)* in Example 9.8: (a) pole-zero plot and ROC for *X(s)*; (b) pole at s = -1 and its ROC; (c) pole at s = -2 and its ROC.

Example 9.10 $X(s)=\frac{1}{(S+1)(S+2)}, \quad \Re e\{s\}<-2$

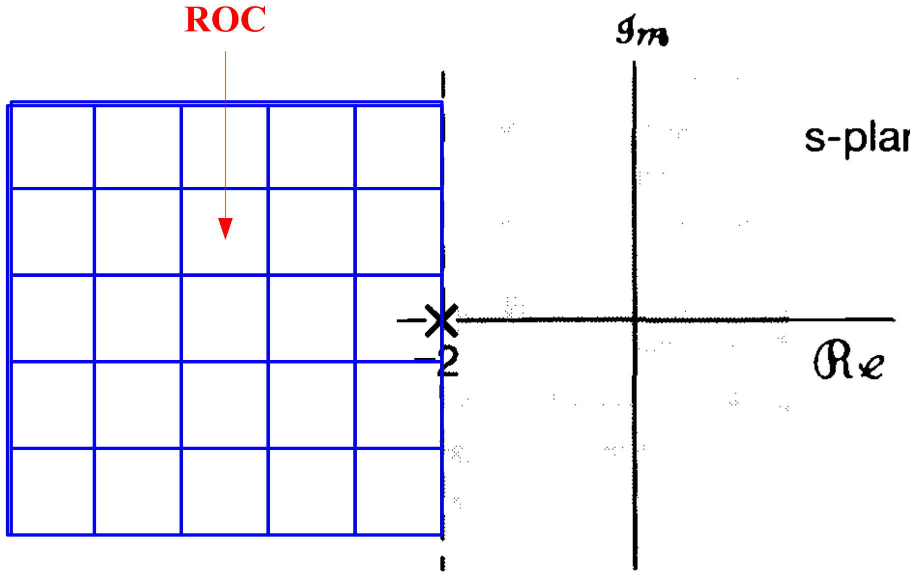
Let us now suppose that the algebraic expression for $X(s)$ is again given by eq. (9.58), but that the ROC is now the left-half plane $\Re e\{s\} < -2$. The partial-fraction expansion for $X(s)$ relates only to the algebraic expression, so eq. (9.62) is still valid. With this new ROC, however, the ROC is to the *left* of both poles and thus, the same must be true for each of the two terms in the equation. That is, the ROC for the term corresponding to the pole at $s = -1$ is $\Re e\{s\} < -1$, while the ROC for the term with pole at $s = -2$ is $\Re e\{s\} < -2$. Then, from Example 9.2,

$$-e^{-t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \Re e\{s\} < -1, \tag{9.66}$$

$$-e^{-2t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \Re e\{s\} < -2, \tag{9.67}$$

so that

$$x(t) = [-e^{-t} + e^{-2t}]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \Re e\{s\} < -2. \tag{9.68}$$

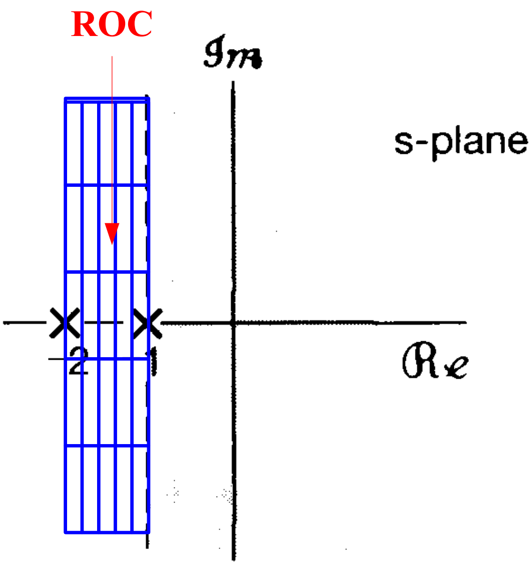


(d)

Example 9.11 $X(s)=\dfrac{1}{(S+1)(S+2)}, \quad -2<\Re e\{s\}<-1$

Finally, suppose that the ROC of $X(s)$ in eq. (9.58) is $-2 < \mathcal{R}e\{s\} < -1$. In this case, the ROC is to the left of the pole at $s = -1$ so that this term corresponds to the left-sided signal in eq. (9.66), while the ROC is to the right of the pole at $s = -2$ so that this term corresponds to the right-sided signal in eq. (9.64). Combining these, we find that

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad -2 < \mathcal{R}e\{s\} < -1. \tag{9.69}$$



(e)

Hence, the **inverse Laplace transform can be easily evaluated by decomposing *X* (s) into a linear combination of simpler terms, the inverse transform of each of which can be recognized.**

Listed in Table 9.2 are a number of useful Laplace transform pairs.

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

9.5 PROPERTIES OF THE LAPLACE TRANSFORM

The derivations of many of these properties are analogous to those of the corresponding properties for the Fourier transform. Consequently, we will not present the derivations in detail.

9.5.1 Linearity of the Laplace Transform

If

$x_1(t) \xleftrightarrow{L} X_1(s)$ with a region of convergence that will be denoted as R_1

and

$x_2(t) \xleftrightarrow{L} X_2(s)$ with a region of convergence that will be denoted as R_2

then

$ax_1(t) + bx_2(t) \xleftrightarrow{L} aX_1(s) + bX_2(s)$ with a region of convergence containing $R_2 \cap R_2$.

Hence, the region of convergence of X(s) is at least the intersection of R_1 and R_2, which could be empty, in which case **X(s) has no region of convergence**-i.e., ***x(t)* has no Laplace transform.**

9.5.2 Time Shifting

If

$x(t) \xleftrightarrow{L} X(s)$ with ROC R

then

$x(t-t_0) \xleftrightarrow{L} e^{-st_0} X(s)$ with ROC R

9.5.3 Shifting in the s-Domain

If

$x(t) \xleftrightarrow{L} X(s)$ with ROC R

then

$e^{s_0 t} x(t) \xleftrightarrow{L} X(s-s_0)$ with ROC $= R + \Re e\{s_0\}$

That is, the ROC associated with $X(s-s_0)$ is that of X(s), shifted by $\Re e\{s_0\}$. Thus, for any values that is in *R*, the values $s+\Re e\{s_0\}$ will be in R1. This is illustrated in Figure 9.23.

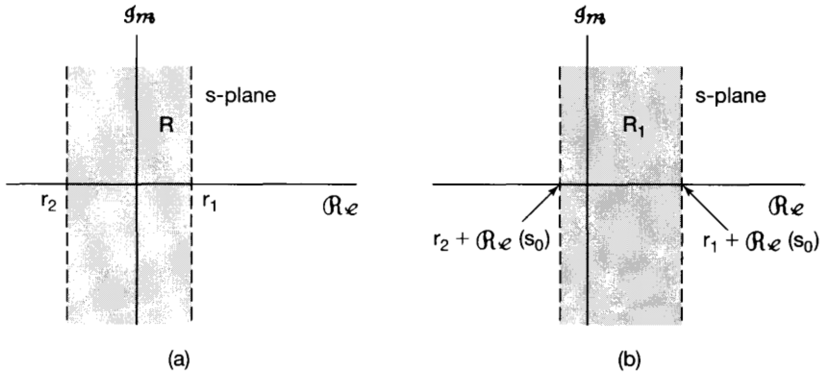


Figure 9.23 Effect on the ROC of shifting in the s-domain: (a) the ROC of X(s); (b) the ROC of *X(s-*s0).

9.5.4 Time Scaling

If

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC } R$$

then

$$x(at) \xleftrightarrow{L} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC} = R_1 = aR$$

That is, for any value *s* in *R* [which is illustrated in Figure **9.24(a)**], the value as will be in R_1.

For a **positive** value of *a* :

- For **0 <*a*< 1**, there is a **compression** in the size of the ROC of X(s) by a factor of *a,* as depicted in Figure **9.24(b),** while
- For **a>1**, the ROC **is expanded** by a factor of *a* as depicted in Figure **9.24(d)**

Also, For a **negative** value of *a* (the ROC undergoes **a reversal plus a scaling):** In particular,

- **For 0 >*a* > -1**, the ROC of $\frac{1}{|a|} X\left(\frac{s}{a}\right)$ involves a reversal about the *jw*-axis, together with a change in the size of the ROC by a factor of $|a|$ as depicted in **Figure 9.24(c)**·

Thus, **time reversal** of *x*(*t*) results in a **reversal of the ROC**. That is,

$$x(-t) \xleftrightarrow{L} X(-s) \quad \text{with ROC} = -R$$

Proof: For a= -1

$$x(-1\times t) \xleftrightarrow{L} \frac{1}{|-1|} X\left(\frac{s}{-1}\right) \quad \text{with ROC} = (-1)\times R$$

$$x(-t) \xleftrightarrow{L} X(-s) \quad \text{with ROC} = -R$$

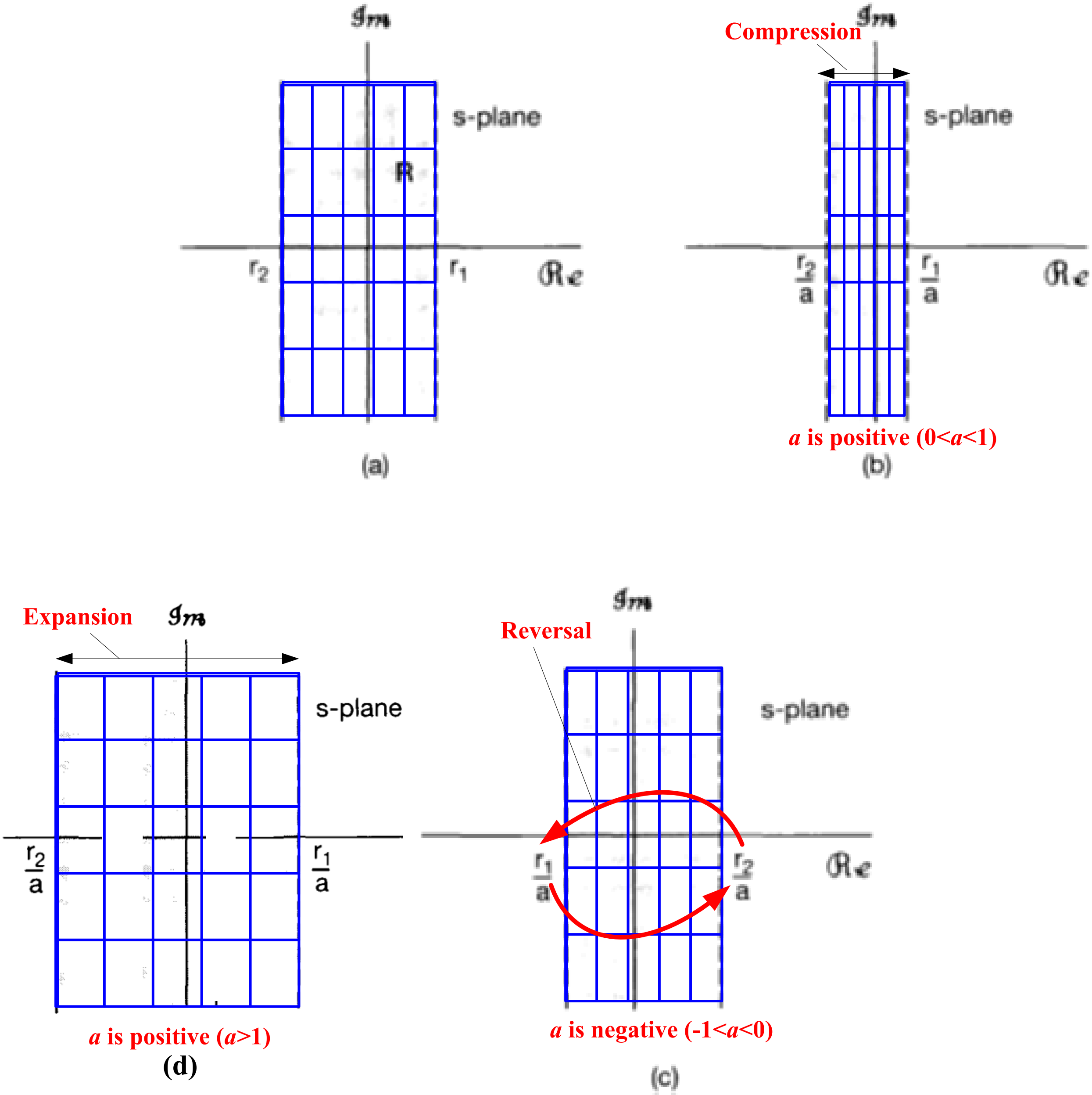


Figure 9.24 Effect on the ROC of time scaling: **(a)** ROC of X(s); **(b)** ROC of $\frac{1}{|a|}X\left(\frac{s}{a}\right)$ for **0 <a< 1**; **(d)** ROC of $\frac{1}{|a|}X\left(\frac{s}{a}\right)$ for **a>1**; **(c)** ROC of $\frac{1}{|a|}X\left(\frac{s}{a}\right)$ for **0>a>-1**.

9.5.5 Conjugation

If

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC } R$$

then

$$x^*(t) \xleftrightarrow{L} X^*(s^*) \quad \text{with ROC } R$$

9. 5. 6 Convolution Property

If

$$x_1(t) \xleftrightarrow{L} X_1(s) \quad \text{with ROC}= R_1$$

and

$$x_2(t) \xleftrightarrow{L} X_2(s) \quad \text{with ROC}= R_2$$

then

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s)X_2(s) \quad \text{with ROC containing } R_2 \cap R_2 .}$$

So, the ROC of $X_1(s)X_2(s)$ includes the intersection of the ROCs of $X_1(s)$ and $X_2(s)$ and may be larger if **pole-zero cancellation occurs in the product**. For example, if

$$X_1(s) = \frac{s+1}{s+2}, \quad \Re e\{s\} > -2$$

and

$$X_2(s) = \frac{s+2}{s+1}, \quad \Re e\{s\} > -1$$

Then $X_1(s)X_2(s) = 1$ (since the product is constant, i.e., bounded, all poles exist in the infinity) and hence, its ROC is the entire *s*-plane.

Note that here both poles and zeros of $X_1(s)X_2(s)$ cancel each other and the ROC is larger than $R_2 \cap R_2$.

9.5.7 Differentiation in the Time Domain

If

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC} = R$$

then

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s) \quad \text{with ROC containing } R.$$

Proof: This property follows by differentiating both sides of the inverse Laplace transform as expressed in equation (9.56). Specifically, let

$$x(t) = \frac{1}{2\Pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\Pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} \left[sX(s)\right]e^{st}\,ds$$

So, $\frac{dx(t)}{dt}$ is the inverse Laplace transform of sX(s).

9.5.8 Differentiation in the s-Domain

If

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC} = R$$

then

$$-tx(t) \xleftrightarrow{L} \frac{dX(s)}{ds} \quad \text{with ROC} = R.$$

Proof: Differentiating both sides of the Laplace transform equation (9.3), i.e.,

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} \left[(-t)x(t)\right]e^{-st}\,dt$$

9.5.9 Integration in the time domain

If

$x(t) \xleftrightarrow{L} X(s)$ with ROC= R

then

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{L} \frac{1}{s}X(s), \quad \text{with ROC containing } R \cap \{\Re e\{s\} > 0\}$$

9.5.10 The Initial- and Final-Value Theorems

Under the specific constraints that $x(t) = 0$ for $t < 0$ and that $x(t)$ contains no impulses or higher order singularities at the origin, one can directly calculate, from the Laplace transform, **the initial value** $x(0^+)$ (i.e., $\boldsymbol{x(t)}$ **as** $\boldsymbol{t}$ **approaches zero from positive values of** t). Specifically, the ***initial-value theorem*** states that

$$x(0^+) = \lim_{s \to \infty} sX(s) \quad (9.110)$$

Also, if $x(t) = 0$ for $t < 0$ and, in addition, $x(t)$ has a finite limit as $t \to \infty$, then the ***final value theorem*** **says that**

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \quad (9.111)$$

9. 7. 1 Causality

For a causal LTI system, the impulse response is zero for $t < 0$ and thus is right sided. Consequently, the ROC associated with the system function for a causal system is a right-half plane.

Note, however, that the **converse of this statement is not necessarily true**. That is, **an ROC to the right of the rightmost pole does not guarantee** that a system is causal; rather, it **guarantees only that the impulse response is right sided**.

However, **if *H* (s) is *rational,* then, we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane**.

Specifically, **For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole**.

Example 9.17

Consider a system with impulse response

$$h(t) = e^{-t}u(t). \tag{9.113}$$

Since $h(t) = 0$ for $t < 0$, this system is causal. Also, the system function can be obtained from Example 9.1:

$$H(s) = \frac{1}{s+1}, \qquad \Re e\{s\} > -1. \tag{9.114}$$

In this case, the system function is rational and the ROC in eq. (9.114) is to the right of the rightmost pole, consistent with our statement that causality for systems with rational system functions is equivalent to the ROC being to the right of the rightmost pole.

Example 9.18

Consider a system with impulse response

$$h(t) = e^{-|t|}.$$

Since $h(t) \neq 0$ for $t < 0$, this system is not causal. Also, from Example 9.7, the system function is

$$H(s) = \frac{-2}{s^2 - 1}, \qquad -1 < \Re e\{s\} < +1.$$

Thus, $H(s)$ is rational and has an ROC that is *not* to the right of the the rightmost pole, consistent with the fact that the system is not causal.

CONCEPT OF ANTICAUSALITY

In an exactly analogous manner, we can deal with the **concept of anticausality**.

A system is *anticausal* if its impulse response *h(t)* = 0 for *t* > 0. Since in that case *h(t)* would be left sided, the ROC of the system function *H(s)* would have to be a left-half plane.

Again, in general, **the converse is not true**. **That is, if the ROC of *H(s)* is a left-half plane, all we know is that *h(t)* is left sided.**

However, if ***H(s)* is rational, then having an ROC to the left of the leftmost pole is equivalent to the system being anticausal.**

9.7.2 Stability

The stability of an LTI system is equivalent to its impulse response being absolutely integrable, in which case the **Fourier transform of the impulse response converges**. Since the Fourier transform of a signal equals the Laplace transform evaluated along the *jw-axis*, we have the following:

An LTI system is stable if and only if the ROC of its system function H(s) includes the entire *jw-axis* [i.e., $\Re e\{s\}=0$].

Example 9.20

Let us consider an LTI system with system function

$$H(s) = \frac{s-1}{(s+1)(s-2)}. \tag{9.119}$$

Since the ROC has not been specified, we know from our discussion in Section 9.2 that there are several different ROCs and, consequently, several different system impulse responses that can be associated with the algebraic expression for $H(s)$ given in eq. (9.119).

If, however, we have information about the causality or stability of the system, the appropriate ROC can be identified. For example, if the system is known to be *causal*, the ROC will be that indicated in Figure 9.25(a), with impulse response

$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t). \tag{9.120}$$

Causal
Unstable

Note that this particular choice of ROC does not include the $j\omega$-axis, and consequently, the corresponding system is unstable (as can be checked by observing that $h(t)$ is not absolutely integrable). On the other hand, if the system is known to be *stable*, the ROC is that given in Figure 9.25(b), and the corresponding impulse response is

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t),$$

Noncausal
Stable

which is absolutely integrable. Finally, for the ROC in Figure 9.25(c), the system is anticausal and unstable, with

$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t).$$

Anticausal
Unstable

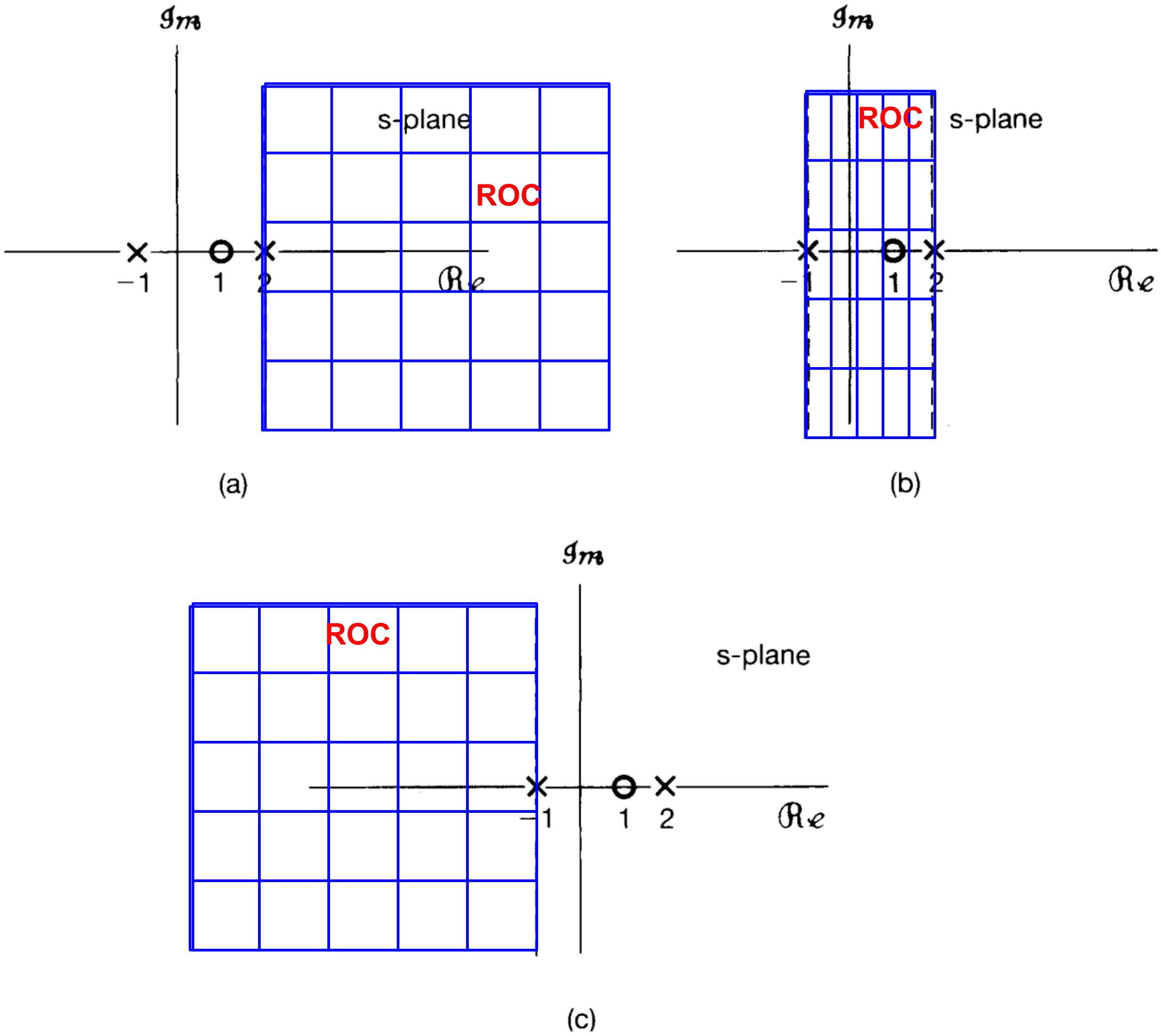


Figure 9.25 Possible ROCs for the system function of Example 9.20 with poles at $s = -1$ and $s = 2$ and a zero at $s = 1$: (a) **causal, unstable system**; (b) **noncausal, stable system**; (c) **anticausal, unstable** system.

It is perfectly **possible, of course, for a system to be stable (or unstable) and have a system function that is not rational.**

For example, the system function in eq. (9.115) is **not rational**, and its impulse response in eq. (9 .118) is **absolutely integrable**, indicating that the system is **stable**.

$$H(s) = \frac{e^{s}}{s+1}, \quad \Re e\{s\} > -1 \quad \textbf{(9.115)}$$

$$h(t) = e^{-(t+1)}u(t+1) \quad \textbf{(9.118)}$$

However, for **systems with rational system functions**, **stability is easily interpreted in terms of the poles of the system.**

For example, for the pole-zero plot in Figure 9.25, **stability corresponds to the choice of an ROC that is between the two poles, so that the jw-axis is contained in the ROC**.

BOTH CAUSALITY AND STABILITY CONDITRIONS

For **one particular and very important class of systems**, **stability can be characterized very simply in terms of the locations of the poles**.

Specifically, consider a **causal** LTI system with a **rational** system function *H(s)*.

- Since the system is **causal**, the **ROC is to the right of the rightmost pole**.
- For this system to be **stable** (i.e., **for the ROC to include the jw-axis**), **the rightmost pole of *H(s)* must be to the *left* of the *jw*-axis**.

That is, **A causal system with rational system function *H(s)* is stable if and only if all of the poles of *H(s)* lie in the left-half of the s-plane-i.e., all of the poles have negative real parts**.

Example 9.21

Consider again the causal system in Example 9.17. The impulse response in eq. (9.113) is absolutely integrable, and thus the system is stable. Consistent with this, we see that the pole of $H(s)$ in eq. (9.114) is at $s = -1$, which is in the left-half of the s-plane. In

$$h(t) = e^{-t}u(t). \tag{9.113}$$

$$H(s) = \frac{1}{s+1}, \qquad \Re e\{s\} > -1. \tag{9.114}$$

In contrast, the causal system with impulse response

$$h(t) = e^{2t}u(t)$$

is unstable, since $h(t)$ is not absolutely integrable. Also, in this case

$$H(s) = \frac{1}{s-2}, \qquad \boxed{\Re e\{s\} > 2,}$$

so the system has a pole at $s = 2$ in the right half of the s-plane.

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