EEE 243 Signals and Systems

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Lecture 09: Fourier Series Representation and Practice Problems

Department of Electrical and Electronic Engineering
School of Engineering
BRAC University

Introduction

Often, It is convenient to choose a set of **orthogonal waveforms** as the **basic signals**. This is because

- Many of the calculations involving signals are simplified by such representation, i.e. representing **an arbitrary signal as a weighted sum of orthogonal waveforms**.

- It is possible to **visualize the signal as a vector in an orthogonal coordinate system**, with the orthogonal waveforms being coordinates.

- The representation in terms of **orthogonal basis functions provides a convenient means of solving** for the response of linear systems to arbitrary inputs.

For periodic signals, a convenient choice for an orthogonal basis is the **set of harmonically related complex exponentials.**

The **representation of a periodic signal in terms of complex exponentials or equivalently, in terms of sine and cosine waveforms**, leads to **the Fourier series,** which is named after

the French physicist **JEAN BAPTISTE FOURIER** who first suggested that

Periodic Signals could be represented by a sum of sinusoids.

9.1 FOURIER SERIES REPRESENTATION OF CONTINUOUS-TIME PERIODIC SIGNALS

9.1.1 Linear Combinations of Harmonically Related Complex Exponentials

A signal is periodic if, for some positive value of *T,*

$$x(t) = x(t+T) \qquad \text{for all } t \quad (9.1)$$

where T is the fundamental period of $x(t)$ and $\omega_0 = \frac{2\Pi}{T}$ is referred to as the fundamental frequency $x(t)$.

Note that the sinusoidal signal $x(t) = \cos\omega_0 t$ and the periodic complex exponential $x(t) = e^{j\omega_0 t}$ are two **basic periodic signals** with fundamental frequency ω_0 and fundamental period $T = \frac{2\Pi}{\omega_0}$. Moreover, a set of **harmonically related complex exponentials** as given below are **associated** with the signal $x(t) = e^{j\omega_0 t}$.

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\Pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots \quad (9.2)$$

Each of these signals has a fundamental frequency that is a multiple of ω_0 , and therefore, each is periodic with period *T.* Then, a **linear combination of harmonically related complex exponentials** of the form given below is also periodic with period *T.*

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\Pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots \quad (9.3)}$$

In eq. (9.3), the term for $k = 0$ is a **constant**. The terms for $k = +1$ and $k = -1$ both have fundamental frequency equal to ω_0 and are collectively referred to as the **fundamental components** or the first harmonic components. More generally, the components for $k = +N$ and $k = -N$ are referred to as the ***Nth* harmonic components**.

The representation of a periodic signal in the form of eq. (9.3) is referred to as the **Fourier series representation**.

Alternative form (Real Periodic signal): If $x(t)$ is **real**, then $x^*(t) = x(t)$.

Then using eq. (9.3), we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t} \quad (9.4)$$

Replacing k by - k in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t} \quad (9.5)$$

Comparing eq. (9.3) with eq. (9.5), we get

$$a_k = a_{-k}^*$$

Equivalently, $a_k^* = a_{-k}$ (9.6)

Now, to derive the **alternative forms** of the Fourier series, lets rearrange the summation in eq. (9.3) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

Substituting $a_{-k} = a_k^*$ from eq. (9.6), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right]$$

Since the two terms inside the summation are **complex conjugates of each other**, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{ a_k e^{jk\omega_0 t} \right\} \quad (9.7)$$

Alternative form 1 (Real Periodic signal): If a_k is expressed **in polar** form as

$$a_k = A_k e^{j\theta_k}$$

Then from eq. (9.7), we get

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{ A_k e^{jk\omega_0 t + \theta_k} \right\}$$

Hence, $$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right) \quad (9.8)$$

Equation (9.8) is one commonly encountered form for the Fourier series of **real periodic signals** in continuous time.

Alternative form 2 (Real Periodic signal): If a_k is expressed **in rectangular form** as

$$a_k = B_k + jC_k$$

where B_k and C_k are both real. Then eq. (9.7) becomes

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \Re e\left[\left(B_k + jC_k\right)\left(\cos\left(k\omega_0 t\right) + j\sin\left(k\omega_0 t\right)\right)\right]$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \Re e\left[B_k \cos\left(k\omega_0 t\right) + jB_k \sin\left(k\omega_0 t\right) + jC_k \cos\left(k\omega_0 t\right) + j^2 C_k \sin\left(k\omega_0 t\right)\right]$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \Re e\left[B_k \cos\left(k\omega_0 t\right) - C_k \sin\left(k\omega_0 t\right) + j\left(B_k \sin\left(k\omega_0 t\right) + C_k \cos\left(k\omega_0 t\right)\right)\right]$$

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} B_k \cos\left(k\omega_0 t\right) - C_k \sin\left(k\omega_0 t\right) \quad (9.9)}$$

Thus, **for real periodic** functions, the Fourier series in terms of **complex exponentials**, as given in eq. (9.3), is mathematically equivalent to either of the two forms in eqs. (9.8) and (9.9) that use **trigonometric functions**.

9.1.2. Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Assuming that a given periodic signal can be represented with the series of eq. (9.3). To **determine the coefficients** a_k, multiplying both sides of **eq. (9.3)** by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating both sides from 0 to T, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt \quad (9.10)$$

$$\text{Here, } \int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos\left(k-n\right)\omega_0 t\, dt + j\int_0^T \sin\left(k-n\right)\omega_0 t\, dt \quad (9.11)$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos\left(\frac{2\Pi}{T/(k-n)}\right)t\, dt + j\int_0^T \sin\left(\frac{2\Pi}{T/(k-n)}\right)t\, dt$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are **periodic sinusoids** with fundamental period $T/|k-n|$. Since the integral may be viewed as measuring the **total area** under the functions over the interval, it can be easily found that for $k \neq n$, both of the integrals on the right-hand side of eq. (9.11) are **zero.**

But for $k = n$, $$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T 1\,dt = T$$

such that we can write the following

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} 0, & k \neq n \\ T, & k = n \end{cases}$$

and consequently, the right side of (9.10) becomes

$$\sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt = T a_k$$

Hence, from eq. (9.10), we obtain for $k = n$

$$\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = T a_n$$

$$a_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt \quad (9.12)$$

$$a_n = \frac{1}{T}\int_T x(t)\,e^{-jn\omega_0 t}\,dt \quad (9.13)$$

where $\int_T$ denotes integration over ***any*** interval.

Equation (9.13) provides the equation for **determining the coefficients,** a_n.

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. (9.3)], then the coefficients are given by eq. (9.13).

This pair of equations, then, defines the **Fourier series of a periodic continuous-time signal**:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\Pi/T)t}, \qquad (9.14)$$ refered to as systhesis equation

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)\,e^{-jk(2\Pi/T)t}\,dt. \qquad (9.15)$$ refered to as analysis equation

where the coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. (9.15) with k = 0. That is,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

which is the **average value** of $x(t)$ over one period.

Convergence of the Fourier Series

Fourier believed that any periodic signal could be expressed as a sum of sinusoids. Although this is not quite true, the class of functions which can be represented by a Fourier series is large and sufficiently general that most conceivable periodic signals arising in Engineering applications do have a Fourier-series representation.

There are **two somewhat different classes of conditions** that **a periodic signal** can satisfy to guarantee that it can be represented by a Fourier series.

Approach 01

One class of periodic signals that are representable through the Fourier series is those signals which have **finite energy over a single period**, i.e., signals for which

$$\int_T |x(t)|^2\,dt < \infty$$

Let $x_N(t)$ be the approximation to $x(t)$ obtained by using the coefficients a_k in eq. (9.3) for $|k| \le N$:

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} \qquad (9.16)$$

Let $e_N(t)$ denote the approximation error such that

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Now, the energy in the error over one period is given by

$$E_N = \int_T |e_N(t)|^2\,dt$$

Then, it can be guaranteed that as we add more and more terms (in eq. (9.16)), i.e., as $N \to \infty$ such that the approximation error is defined as

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (9.17)$$

Then, we obtain

$$\int_T |e(t)|^2 \, dt = 0 \quad (9.18)$$

Since physical systems respond to signal energy, $x(t)$ and its Fourier series representation are indistinguishable. Because most of the periodic signals that we consider do have finite energy over a single period, they **have Fourier series representations**.

Approach 02

Moreover, **an alternative set of conditions**, developed by P. L. Dirichlet and also satisfied by essentially all of the signals with which we will be concerned, guarantees that $x(t)$ *equals* its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ converges to the average of the values on either side of the discontinuity.

The **Dirichlet conditions** are as follows:

For the Fourier series to converge, the signal $x(t)$ must posses the following properties, which are known as the **Dirichlet conditions**, over any period:

Condition 1. Over any period, $x(t)$ must be absolutely integrable; that is,

$$\int_T |x(t)| \, dt < \infty$$

Condition 2. $x(t)$ has only a finite number of maxima and minima. In other words, in any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3. The number of discontinuities in $x(t)$ must be finite, i.e., in any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

These conditions are sufficient, but not necssary. Thus, if a signal $x(t)$ satisfies the Dirichlet conditions, then the coresponding Fourier series is convergent anf its sum is $x(t)$, except at any point t_o at which $x(t)$ is discontinuous. At the points of discontinuity, the sum of the series is the average of the left- and right-hand limits of $x(t)$ at t_o, that is,

$$x(t_0) = \frac{1}{2}\left[x(t_0^+) + x(t_0^-)\right]$$

Example signals that violets above conditions

Signals violeting condition 1

A periodic signal $x(t)$ with period 1 that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1$$

This signal is illustrated in Figure 9.1(a).

Signals violeting condition 2

An example of a function that meets Condition 1 but **not** Condition 2 is

$$x(t) = \sin\left(\frac{2\Pi}{t}\right), \quad 0 < t \le 1$$

The function is illustrated in Figure 9.1(b). For this function, which is periodic with $T = 1$, the condition 1 is satisfied since $\int_0^1 |x(t)|\,dt < 1$.

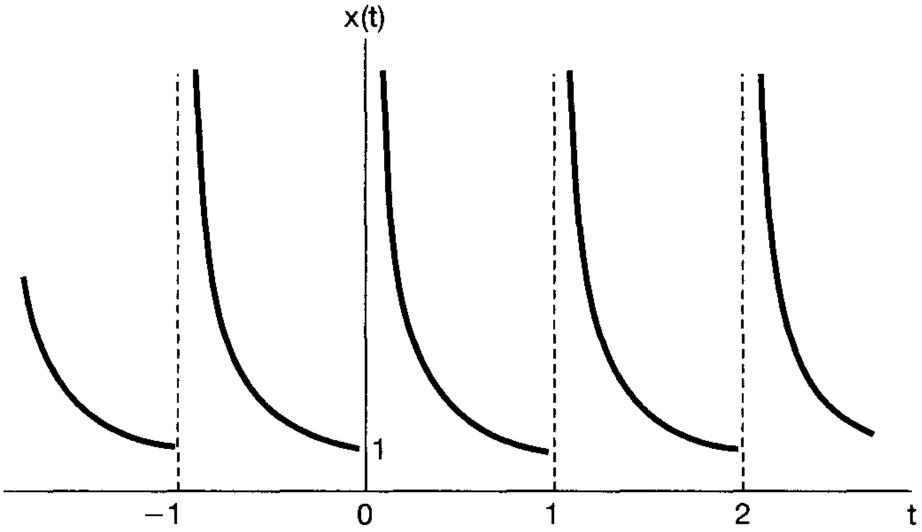
The function has, however, an infinite number of maxima and minima in the interval (in a time period). So, it violets condition 2.

Signals violeting condition 3

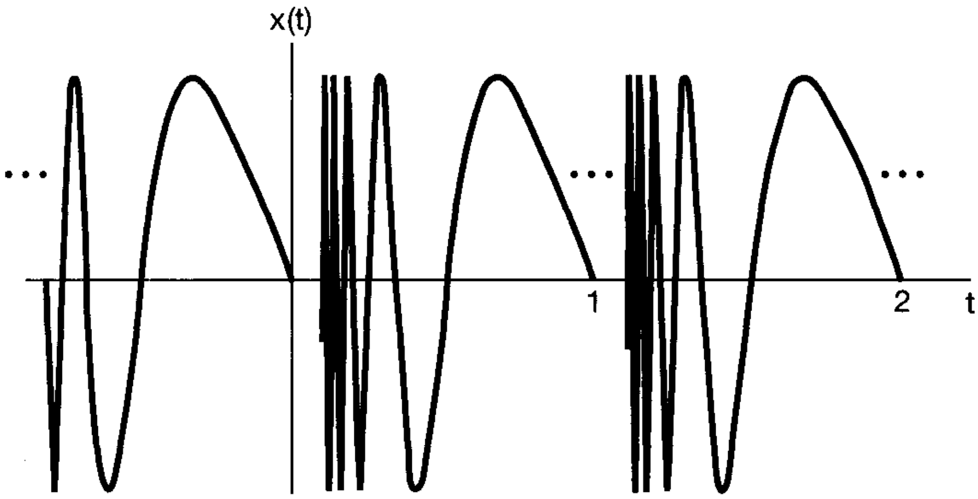
An example of a function that violates Condition 3 is illustrated in Figure 9.1(c).

The signal, of period $T = 8$, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8 (thus, satisfying condition 1).

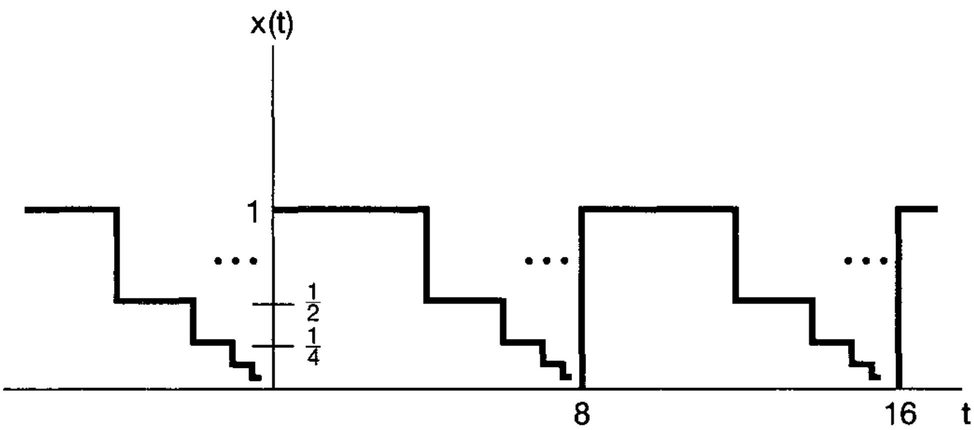
However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.



(a)



(b)



(c)

Figure 9.1. Signals that violate the Dirichlet conditions: (a) condition 1 (b) condition 2, and (c) condition 3.

Analysis

As can be seen from the examples given in Figure 9.1, signals that do not satisfy the Dirichlet conditions are generally **pathological in nature** and consequently **do not typically arise in practical contexts**. For this reason, the **question of the convergence of Fourier series will not play** a particularly significant role.

For a periodic signal that has **no discontinuities**, the Fourier series representation **converges and equals the original signal** at every value of *t*.

For a periodic signal **with a finite number of discontinuities** in each period, the Fourier series representation equals the signal everywhere *except at the isolated points of discontinuity*, at which the *series converges to the average value of the signal on either side of the discontinuity*.

In this case the **difference** between the original signal and its Fourier series representation contains no energy, and consequently, the two signals can be thought of as being the same for all practical purposes

Specifically, since the signals *differ only at isolated points*, the integrals of both signals over any interval *are* identical. For this reason, the two signals behave identically under convolution and consequently **are identical from the standpoint of the analysis of LTI systems**.

Convergence of Fourier series of a Square Wave - Gibbs Phenomenon

In 1898, an **American physicist, Albert Michelson**, constructed a harmonic analyzer, a device that, for any periodic signal $x(t)$, would compute the truncated Fourier series approximation of eq. (9.16) as follows for values of ***N* up to 80**.

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Michelson *tested his device* on many functions, with the expected result that $x_N(t)$ looked very much like $x(t)$.

- However, **when he tried the square wave**, he obtained an important and, to him, **very surprising result**. Michelson was concerned about the behavior he observed and thought that his device might have had **a defect**.

- He **wrote about the problem** to the famous mathematical physicist **Josiah Gibbs**, who investigated it and *reported his explanation in 1899*.

- What Michelson **had observed** is illustrated in **Figure 2**, where we have shown $x_N(t)$ for **several values of *N*** for $x(t)$, a symmetric square wave. In each case, the *partial sum is superimposed* on the original square wave.

- Since the **square wave satisfies the Dirichlet conditions**, the limit as $N \to \infty$ of $x_N(t)$ at the discontinuities should be the average value of the discontinuity. We see from the figure that this is in fact the case, since for any *N*, $x_N(t)$ has exactly that value at the discontinuities.

- Furthermore, for any other value of *t*, say, $t = t_1$, we are guaranteed that $\lim_{N\to\infty} x_N(t_1) = x(t_1)$.

- Therefore, the squared error in the Fourier series representation of the square wave has zero area, as in eqs. (9.17) and (9.18).

For this example, **the interesting effect that Michelson observed** is that the behavior of the partial sum in the vicinity of the discontinuity exhibits **ripples** and that the **peak amplitude of these ripples *does not seem to decrease with increasing N***.

Gibbs showed that these are in fact the case.

- Specifically, for a discontinuity of **unity height**, the partial sum exhibits **a maximum value of 1.09** (i.e., **an overshoot of 9% of the height of the discontinuity**), **no matter how large *N* becomes**.

- As **stated before,** for any *fixed* value of *t*, say, $t = t_1$, the *partial sums will converge to the correct value, and at the discontinuity they will converge to one-half the sum of the values of the signal* **on either side of the discontinuity**.

However, **the closer t_1 is chosen to the point of discontinuity, the larger *N* must be in order to reduce the error below a specified amount**. *Thus, as N increases, the ripples in the partial sums become compressed toward the discontinuity*, *but for any finite value of N, the peak amplitude of the ripples **remains constant***.

This **behavior has come to be known** as the ***Gibbs phenomenon***.

The **implication** is that

- the **truncated Fourier series** approximation $x_N(t)$ of a discontinuous signal $x(t)$ will *in general exhibit high-frequency ripples and overshoot* $x(t)$ *near the discontinuities*.
- If *such an approximation is used* in practice, **a large enough value of *N* should be chosen** so as to guarantee that the *total energy in these ripples is insignificant*.
- In the limit, of course, we know that *the energy in the approximation error vanishes* and that **the Fourier series representation of a discontinuous signal such as the square wave converges**.

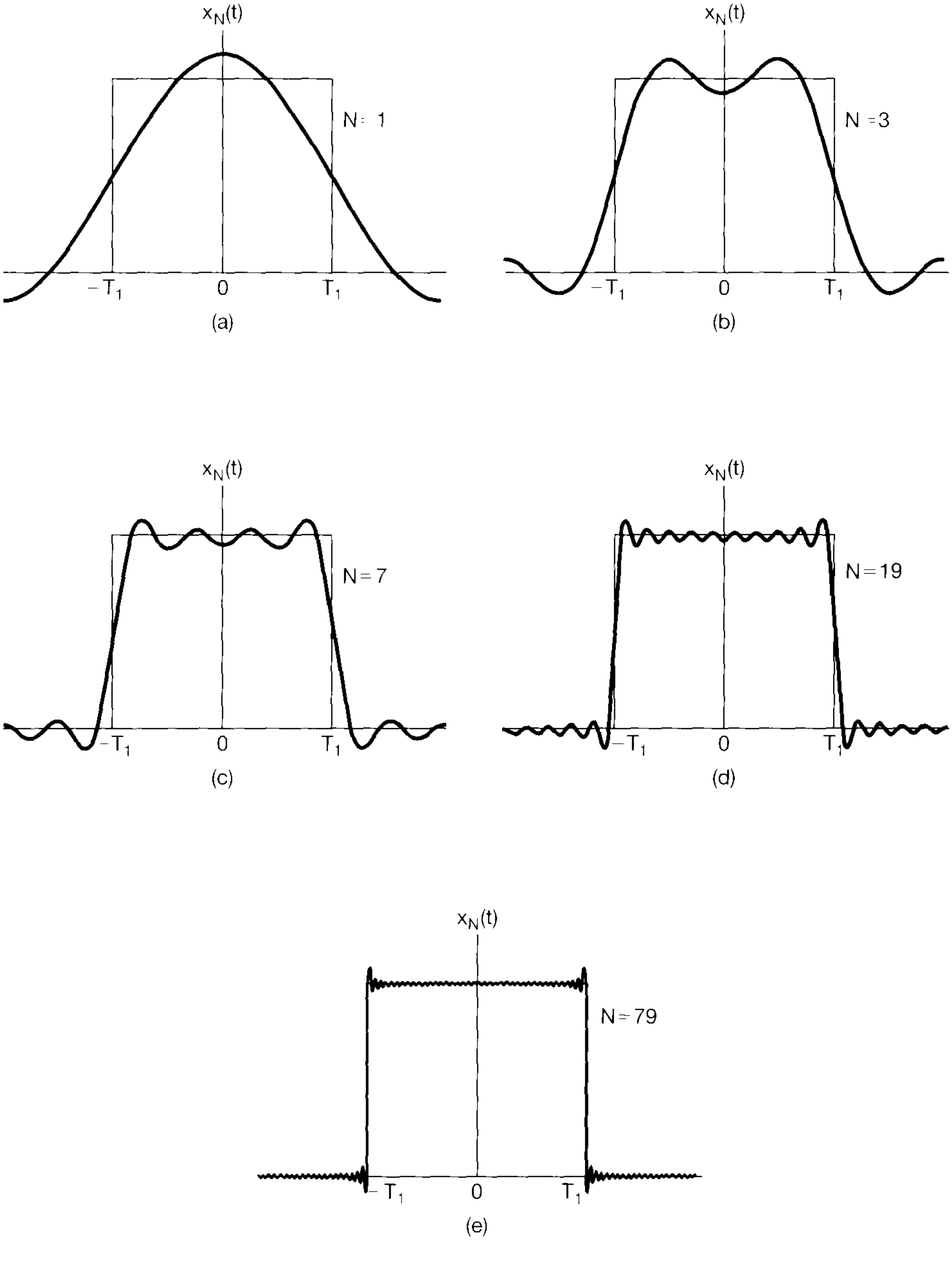


Figure 9.2 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, the finite series approximation $x_N(t)=\sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$ for several values of *N* has been depicted.

Properties of Continuous-Time Fourier Series

Fourier series representations **possess a number of important properties** that are useful for developing conceptual insights into such representations, and they can also help to reduce the complexity of the evaluation of the Fourier series of many signals.

In what follows, we use **a shorthand notation** to indicate the relationship between a periodic signal and its Fourier series coefficients.

Specifically, suppose that $x(t)$ is a **periodic signal** with period T and fundamental frequency $\omega_0 = \dfrac{2\Pi}{T}$. Then if the **Fourier series coefficients** of $x(t)$ are denoted by a_k, we use the following notation to signify the pairing of a periodic signal with its Fourier series coefficients.

$$x(t) \xleftrightarrow{FS} a_k$$

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, respectively. That is,

$$x(t) \xleftrightarrow{FS} a_k,$$
$$y(t) \xleftrightarrow{FS} b_k.$$

Let c_k be the Fourier series coefficients of the **linear combination** of $x(t)$ and $y(t)$, i.e., $z(t) = Ax(t) + By(t)$.

Since $x(t)$ and $y(t)$ **have the same period** T, it follows that any linear combination of the two signals will also be periodic with period T. Then, c_k can be **given by the same linear combination of the Fourier series coefficients** for $x(t)$ and $y(t)$. That is,

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

Time Shifting

Let b_k be the Fourier series coefficients of the resulting (time shifted) signal $y(t) = x(t - t_0)$. Then, b_k can be expressed as follows.

$$b_k = \frac{1}{T}\int_T x(t - t_0) e^{-jk\omega_0 t} dt.$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration *T*, we obtain

$$b_k = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau = e^{-jk\omega_0 t_0}a_k$$

$$b_k = e^{-jk\left(\frac{2\Pi}{T}\right)t_0}a_k$$

Hence, if

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(t-t_0) \xleftrightarrow{FS} b_k = e^{-jk\omega_0 t_0}a_k = e^{-jk\left(\frac{2\Pi}{T}\right)t_0}a_k$$

Note the following:

- When a periodic signal **is shifted in time**, only the phases of its Fourier series coefficients changes, and the magnitudes remain *unaltered*, i.e., $|a_k| = |b_k|$.
- Moreover, when a time shift is applied to a periodic signal $x(t)$, the period T of the signal is *preserved*.

Time Reversal

To determine the Fourier series coefficients of a time reversed signal, i.e., $y(t) = x(-t)$, let us consider the **effect of time reversal** on the synthesis equation:

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\left(\frac{2\Pi}{T}\right)t}$$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\left(\frac{2\Pi}{T}\right)t} \quad \text{, let } k = -m$$

Note that the right-hand side of this equation has the form of a Fourier series synthesis equation for $y(t) = x(-t)$ where the Fourier series coefficients b_k are as follows.

$$b_k = a_{-k}$$

Hence, if

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(-t) \xleftrightarrow{FS} b_k = a_{-k}$$

- So, time reversal applied to a continuous-time signal *results in a time reversal* of the corresponding sequence of Fourier series coefficients.
- The period T of a periodic signal $x(t)$ *remains unchanged* when the signal undergoes time reversal.

- Moreover, if $x(t)$ is *even*, that is, if $x(-t)=x(t)$, then its Fourier series coefficients are also *even*, i.e., $a_{-k}=a_k$. Similarly, if $x(t)$ is *odd*, that is, if $x(-t)=-x(t)$, then its Fourier series coefficients are also *odd*, i.e., $a_{-k}=-a_k$.

Time Scaling

Time scaling is an operation that in general *changes the period* of the underlying signal. Particularly, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0=\frac{2\Pi}{T}$, then $x(\alpha t)$, where α is a positive real number, is periodic with period $\frac{T}{\alpha}$ and fundamental frequency $\alpha\omega_0$.

If $x(t)$ has the Fourier series representation as follows.

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Then, the Fourier series representation of $x(\alpha t)$ is given by,

$$x(\alpha t)=\sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

Note that,

- Fourier coefficients for each of the harmonic components of $x(t)$ *remain the same.*
- While **the Fourier coefficients have *not changed***, the **Fourier series representation *has* changed** because of the *change in the fundamental frequency*.

Multiplication

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and that

$$x(t)\xleftrightarrow{FS} a_k,$$
$$y(t)\xleftrightarrow{FS} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, *the* Fourier series coefficients h_k of the product can be expressed in terms of those for $x(t)$ and $y(t)$ as follows.

$$x(t)y(t)\xleftrightarrow{FS} h_k=\sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

One way to derive this relationship is to *multiply* the Fourier series representations of $x(t)$ and to note that the kth harmonic component in the product will have a coefficient which is the sum of terms of the form $a_l b_{k-l}$.

Note that the sum on the right-hand side of the above equation may be interpreted as the *discrete-time convolution* of the sequence representing the Fourier coefficients of $x(t)$ and the sequence representing the Fourier coefficients of $y(t)$.

Conjugation and Conjugate Symmetry

Taking the *complex conjugate* of a periodic signal $x(t)$ has the effect of *complex conjugation* and *time reversal* on the corresponding Fourier series coefficients. That is, if

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k,$$
$$x^*(t) \stackrel{FS}{\longleftrightarrow} a_{-k}^*.$$

If $x(t)$ is *real*, that is, when $x(t) = x^*(t)$, the Fourier series coefficients will be *conjugate symmetric,* i.e.,

$$a_{-k} = a_k^*.$$

Parseval's Relation for Continuous-Time Periodic Signals

Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

where the a_k are the Fourier series coefficients of $x(t)$ and T is the period of the signal.

Note that the left-hand side of the above equation is the *average power* (i.e., energy per unit time) in *one period* of the periodic signal $x(t)$.

Also,

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = \frac{1}{T}|a_k|^2 (T-0) = |a_k|^2$$

where $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$.

Thus, *Parseval's relation* states the following:

> *The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.*

Summary of Properties of the Continuous-Time Fourier Series

In Table 9.1, a summary of these and other important properties of continuous-time Fourier series is given.

Table 9.1. Properties of continuous-time Fourier series

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$, $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Fourier series representation of discrete-time periodic signals

While the discussion for the *Fourier series representation of discrete-time periodic signals* closely parallels that of continuous-time periodic signals, **there are some important differences**. In particular,

- the Fourier series representation of a **discrete-time periodic signal is a finite series**, as opposed to the **infinite series representation required for continuous-time** periodic signals.
- As a consequence, **there are no mathematical issues of convergence** such as those discussed in the case of continuous-time periodic signals.

Note that a **discrete-time signal** x[n] is periodic with period N if

$$x[\mathrm{n}] = x[n+N].$$

The fundamental period is the smallest positive integer N for which the above equation holds.

Following the Fourier series representation for continuous-time periodic signals, the **discrete-time Fourier series pair** can be expressed as follows **by replacing** the following parameters of the Fourier series representation for continuous-time periodic signals.

$$\begin{aligned} &T \to N \\ &t \to n \\ &k = -\propto to + \propto \to \langle N \rangle \end{aligned}.$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\Pi/N)n}, \qquad (9.20)$$ refered to as systhesis equation

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\left(2\Pi/N\right)n}. \qquad (9.21)$$ refered to as analysis equation

These **equations play the same role for discrete-time periodic signals** as those play for continuous-time periodic signals, with the former equation the **synthesis** equation and the later the **analysis** equation.

As in continuous-time, the discrete-time Fourier series coefficients a_k **are often referred to as the spectral coefficients of** x[n]. These **coefficients specify a decomposition** of $x[n]$ into a sum of N *harmonically related complex exponentials*.

Properties of Discrete-Time Fourier Series

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 9.2 with their continuous-time counterparts in Table 9.1.

Table 9.2. Properties of discrete-time Fourier series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

The **derivations of many of these properties are very similar to those** of the corresponding properties for continuous-time Fourier series. Consequently, **we limit the discussion in the following subsections to only a few of these properties**, including several that have **important differences** relative to those for continuous time.

As with continuous time, it **is often convenient to use a shorthand notation** to indicate the relationship between a periodic signal and its Fourier series coefficients. Specifically, if $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k then we can write

$$x[n] \xleftrightarrow{FS} a_k.$$

Multiplication

The multiplication property of the Fourier series representation is *one example of a property* that *reflects the difference* between continuous time and discrete time.

From Table 9.1, the *product of two continuous-time signals of period T* results in a periodic signal with period *T* whose sequence of Fourier series coefficients is the **convolution** of the sequences of Fourier series coefficients of the two signals being multiplied.

In discrete time, suppose that

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

And

$$y[n] \overset{FS}{\longleftrightarrow} b_k.$$

are both periodic with period *N*. Then the product $x[n]y[n]$ is also periodic with period *N,* and its Fourier coefficients, d_k are given by

$$x[n]y[n] \overset{FS}{\longleftrightarrow} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

The above equation is analogous to the *definition of convolution*, except that the summation variable is now *restricted to an interval of N* consecutive samples. We refer to this type of operation as a ***periodic convolution*** between the two periodic sequences of Fourier coefficients.

The **usual form of the convolution** sum (where the summation variable ranges from $-\infty$ to $+\infty$) is sometimes referred to as ***aperiodic convolution***, to distinguish it from periodic convolution.

First Difference

The **discrete-time parallel to the differentiation** property of the continuous-time Fourier series involves **the use of the first-difference operation**, which is defined as $x[n]-x[n-1]$. If

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

then the Fourier coefficients corresponding to the first difference of *x[n]* may be expressed as

$$x[n]-x[n-1] \overset{FS}{\longleftrightarrow} \left(1-e^{-jk(2\Pi/N)}\right)a_k$$

A common use of this property is in situations where evaluation of the Fourier series coefficients is easier for the first difference than for the original sequence.

Parseval's Relation for Discrete-Time Periodic Signals

Parseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}^{\infty}|a_k|^2$$

where the a_k are the Fourier series coefficients of $x[n]$ and N is the period of the signal. Once again, ***Parseval's relation states* that the average power in a periodic signal equals the sum of the average powers in all of its harmonic components.**

Fourier Series and LTI Systems (with periodic inputs)

Consider a linear, time-invariant (LTI) continuous-time system with impulse response $h(t)$. Then the output response $y(t)$ resulting from an input $x(t)$ is

$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

For complex **exponential inputs of the form**

$$x(t)=e^{j\omega t}$$

The output of the system is ,

$$y(t)=\int_{-\infty}^{\infty}h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$y(t)=e^{j\omega t}\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

By defining

$$H(\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

We can write,

$$\boxed{y(t)=H(\omega)e^{j\omega t}}$$

$H(\omega)$ is called the **system (transfer) function** and is constant for fixed ω. The magnitude $|H(\omega)|$ is called the magnitude function of the system. $\measuredangle H(\omega)$ is called the **phase function** of the system.

Knowing $H(\omega)$ we can determine whether the system amplifies or attenuates a given sinusoidal component of the input and how much of the phase shift the system adds to that particular component.

Note that the responses of an LTI system to a periodic input with period T is periodic with the same period.

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