EEE 243 Signals and Systems

2022

Lecture 10: Practice Problems on Fourier Series

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Fourier Series and Fourier Coefficients Problems [1]

Example 3.2

Consider a periodic signal $x(t)$, with fundamental frequency 2π, that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \tag{3.26}$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

Rewriting eq. (3.26) and collecting each of the harmonic components which have the same fundamental frequency, we obtain

$$\begin{aligned} x(t) = 1 &+ \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) \\ &+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}). \end{aligned} \tag{3.27}$$

Equivalently, using Euler's relation, we can write $x(t)$ in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t. \tag{3.28}$$

In Figure 3.4, we illustrate graphically how the signal $x(t)$ is built up from its harmonic components.

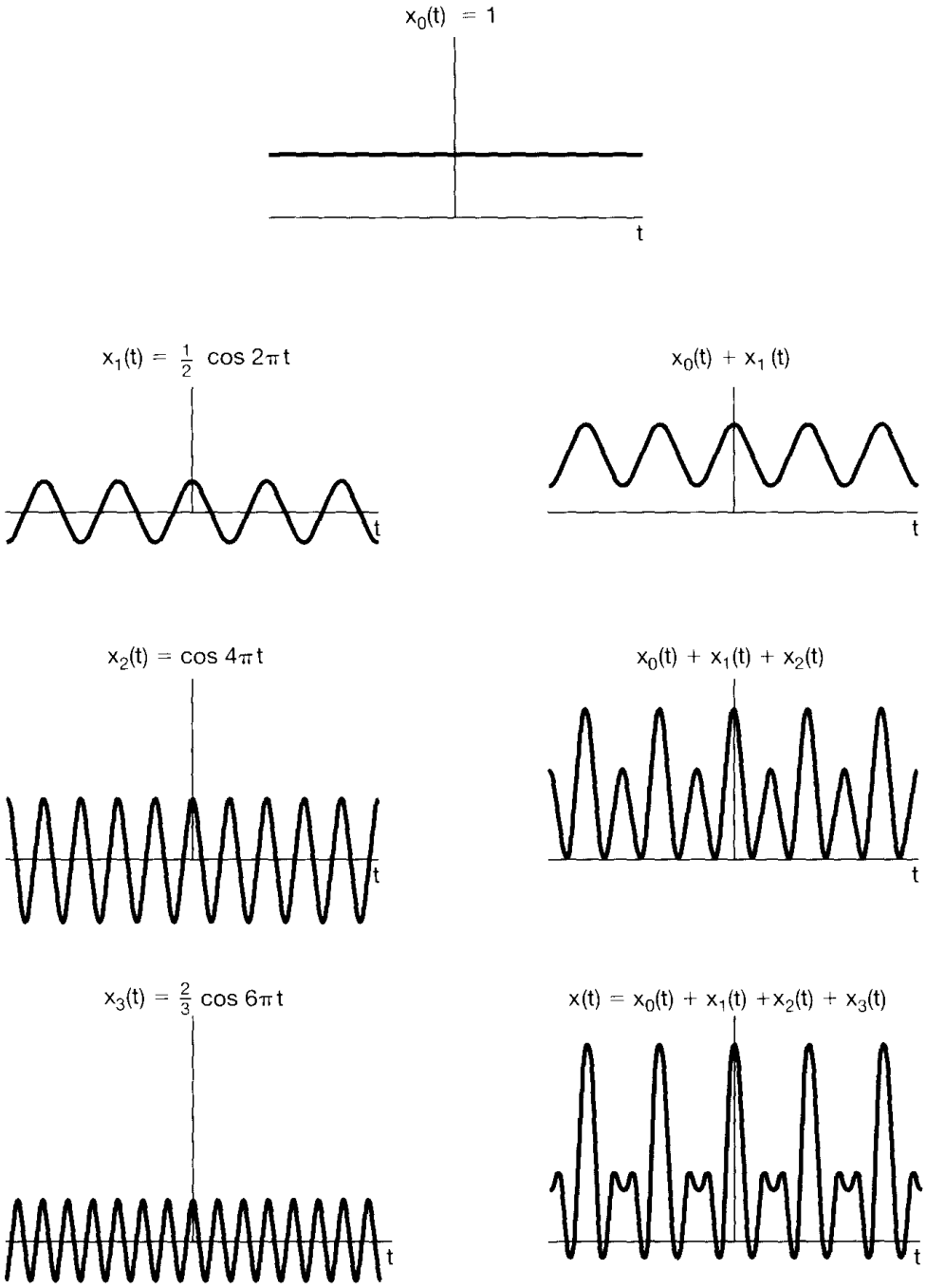


Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

Example 3.3

Consider the signal

$$x(t) = \sin \omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (3.39). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin \omega_0 t$ as

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation and eq. (3.38), we obtain

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 3.4

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3.3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, \ |k| > 2.$$

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k.

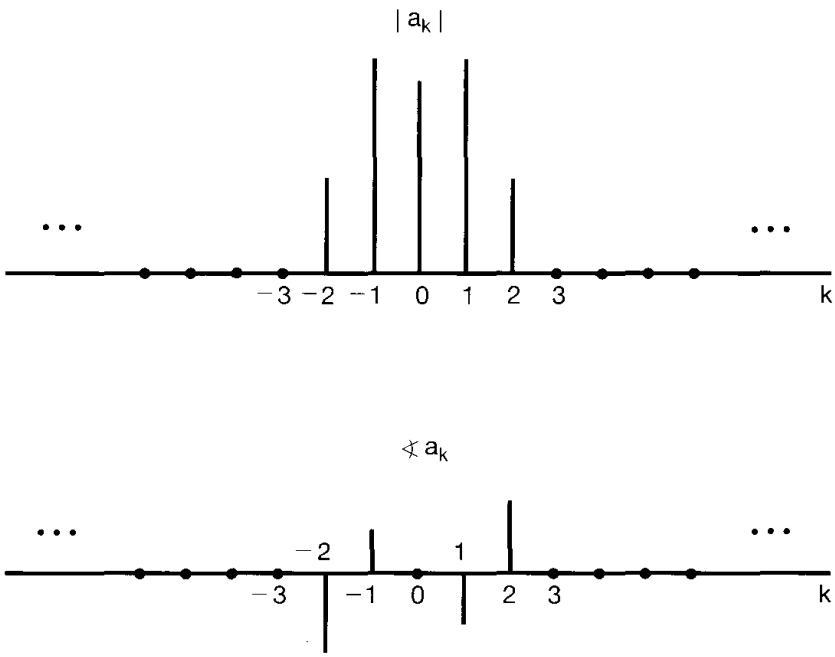


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

EXAMPLE 3.5

The periodic square wave, sketched in Figure 3.6 and defined over one period as

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T/2\end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\Pi/T$ .Determine the **Fourier series coefficients of** $x(t)$.

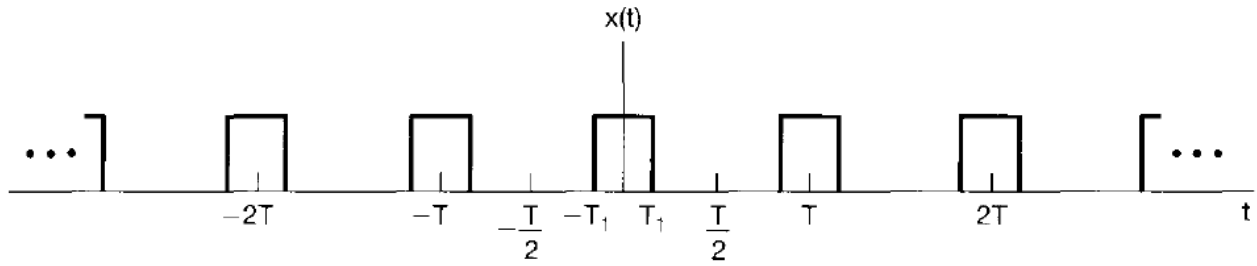


Figure 3.6 Periodic square wave.

Solution

Because of the symmetry of $x(t)$ about $t=0$, it is convenient to choose $-T/2 \le t < T/2$ as the interval over which the integration is performed, although any interval of length T is equally valid and thus will lead to the same result.

For *k*=0,

$$a_0=\frac{1}{T}\int_{-T_1}^{T_1}1\times e^{-j0w_0t}dt=\frac{1}{T}\int_{-T_1}^{T_1}dt=\frac{1}{T}\left[T_1-(-T)_1\right]=\frac{2T_1}{T}$$

For $k\neq 0$,

$$a_k=\frac{1}{T}\int_{-T_1}^{T_1}1\times e^{-jkw_0t}dt=-\frac{1}{jkw_0T}e^{-jkw_0t}\Big|_{-T_1}^{T_1}=\frac{2}{kw_0T}\left[\frac{e^{jkw_0T_1}-e^{-jkw_0T_1}}{2j}\right]=\frac{2}{kw_0T}\left[\sin kw_0T_1\right]$$

$$a_k=\frac{2\sin(kw_0T_1)}{kw_0T}=\frac{2\sin(kw_0T_1)}{k\left(\frac{2\Pi}{T}\right)T}=\frac{\sin(kw_0T_1)}{k\Pi}$$

$$a_k=\frac{\sin(kw_0T_1)}{k\Pi},\qquad k\neq 0$$

Figure 3.7 is a bar graph of the Fourier series coefficients for this example. In particular, the **coefficients are plotted for a fixed value of T_1** and several values of T.

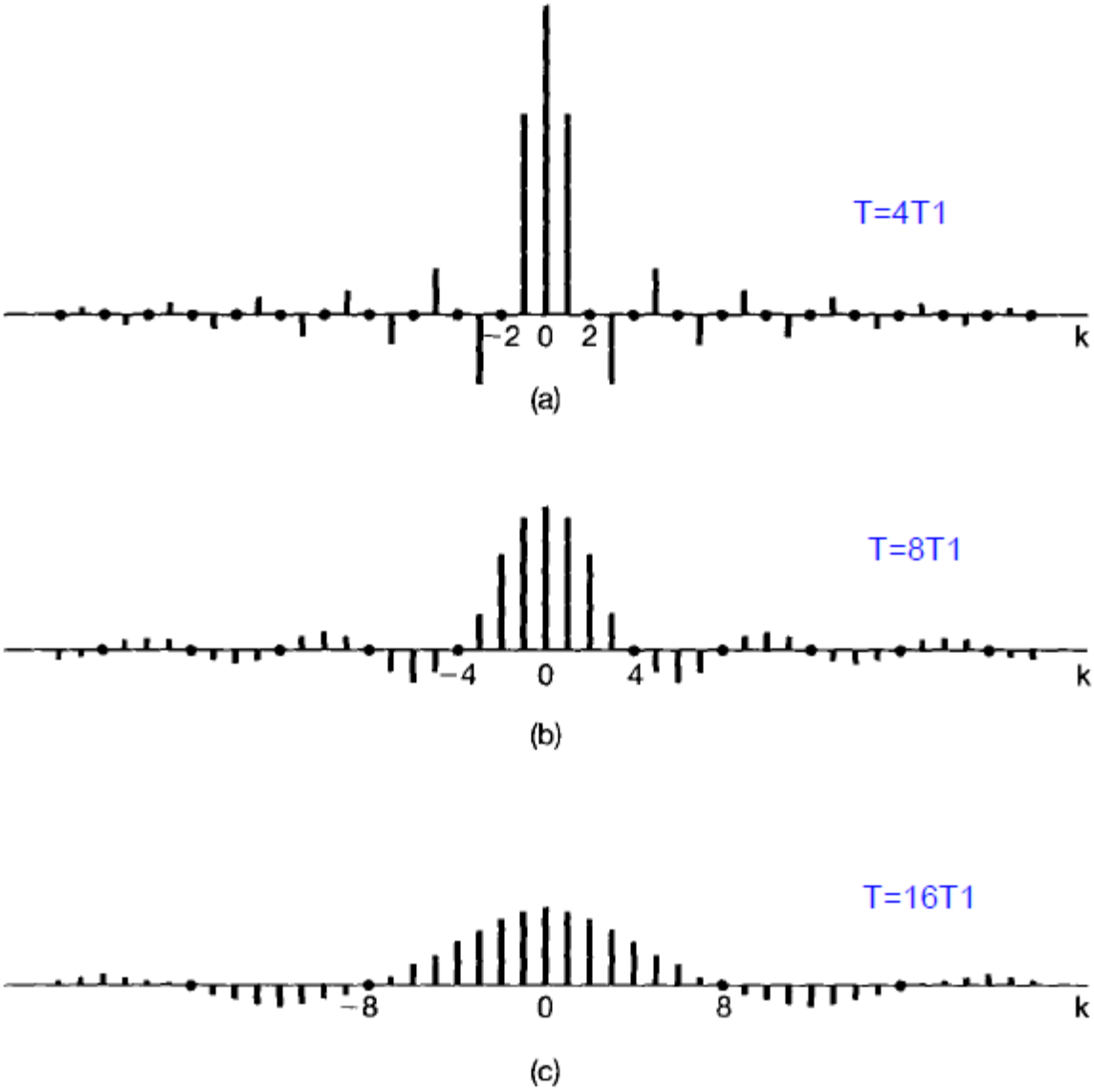


Figure 3.7. Plots of the scaled Fourier series coefficients Ta_k for the periodic square *wave* with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8\ T_1$; (c) $T = 16\ T_1$.

The coefficients are regularly spaced samples of the envelope ($2\sin(wT_1)/w$), where the spacing between samples, $2\Pi/T$ **, decreases as *T* increases**.

Problem 01 [2]

A sinusoidal voltage $E \sin \omega_0 t$ is passed through a half-wave rectifier that clips the negative portion of the waveform, as shown in Figure 3.3.4. Such signals may be encountered

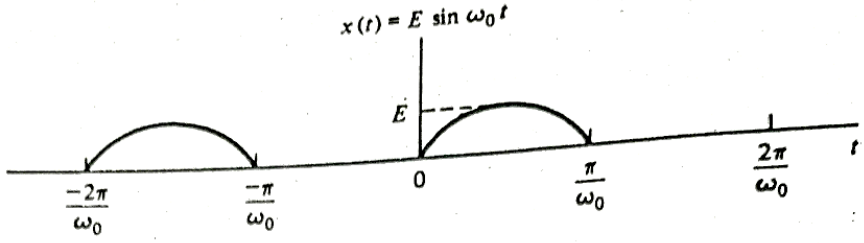


Figure 3.3.4 Signal $x(t)$ for Example 3.3.2.

in rectifier design problems. Rectifiers are circuits that produce direct current (dc) from alternating current (ac).

The analytic representation of $x(t)$ is

$$x(t) = \begin{cases} 0, & \text{when } \frac{-\pi}{\omega_0} < t < 0 \\ E \sin \omega_0 t, & \text{when } 0 < t < \frac{\pi}{\omega_0} \end{cases}$$

and $x(t + 2\pi/\omega_0) = x(t)$. Since $x(t) = 0$ when $-\pi/\omega_0 < t < 0$, we obtain, from Equation (3.3.4),

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T E \sin \omega_0 t \exp\left[-j\frac{2n\pi t}{T}\right] dt \\ &= \frac{E\omega_0}{2\pi}\int_0^{\pi/\omega_0} \frac{1}{2j}[\exp[j\omega_0 t] - \exp[-j\omega_0 t]] \exp[-jn\omega_0 t]\, dt \\ &= \frac{E\omega_0}{4\pi j}\int_0^{\pi/\omega_0} [\exp[-j\omega_0(n-1)t] - \exp[-j\omega_0(n+1)t]\, dt \\ &= \frac{E \exp[-jn\pi/2]}{2\pi(1-n^2)}\left(\exp\left[\frac{-jn\pi}{2}\right] + \exp\left[\frac{jn\pi}{2}\right]\right) \\ &= \frac{E}{\pi(1-n^2)} \cos(n\pi/2) \exp[-jn\pi/2], \quad n \neq \pm 1 \qquad (3.3.15) \\ &= \begin{cases} \frac{E}{\pi(1-n^2)}, & n \text{ even} \\ 0, & n \text{ odd}, \quad n \neq \pm 1 \end{cases} \qquad (3.3.16) \end{aligned}$$

Setting $n = 0$, we obtain the dc component, or the average value of the periodic signal, as $c_0 = E/\pi$. This result can be verified by calculating the area under one half cycle of a sine wave and dividing by T. To determine the coefficients c_1 and c_{-1} which correspond to the first harmonic, we note that we cannot substitute $n = \pm 1$ in Equation (3.3.15), since this yields an indeterminate quantity. So we use Equation (3.3.4) instead with $n = \pm 1$, which results in

$$c_1 = \frac{E}{4j} \quad \text{and} \quad c_{-1} = \frac{-E}{4j}$$

The line spectra of $x(t)$ are displayed in Figure 3.3.5.

In general, a rectifi...

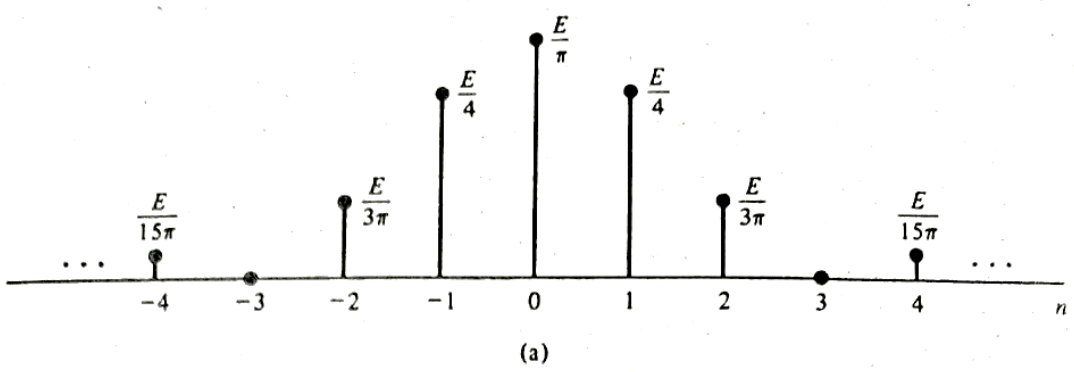


Figure 3.3.5. Magnitude spectrum.

Problem 02

Consider the square-wave signal shown in Figure 3.3.6. The analytic representation of $x(t)$ is

$$x(t) = \begin{cases} 0, & \text{when } \frac{-T}{2} < t < \frac{-\tau}{2} \\ K, & \text{when } \frac{-\tau}{2} < t < \frac{\tau}{2} \\ 0, & \text{when } \frac{\tau}{2} < t < \frac{T}{2} \end{cases}$$

and $x(t + T) = x(t)$. Signals of this type can be produced by pulse generators and are used extensively in radar and sonar systems. From Equation (3.3.4), we obtain

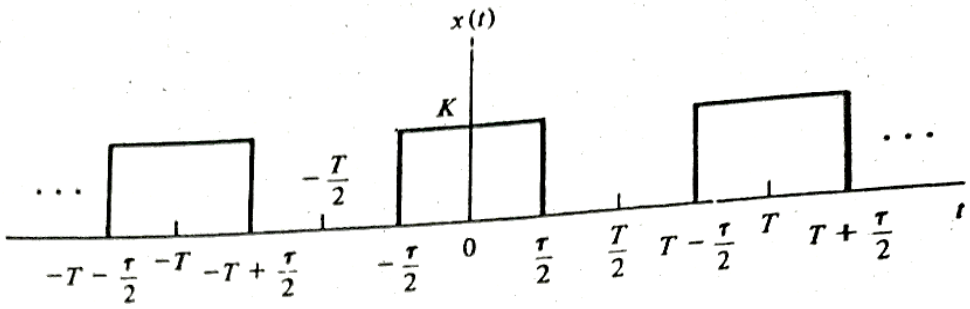


Figure 3.3.6 Signal $x(t)$ for Example 3.3.3.

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left[\frac{-j2n\pi t}{T}\right]dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} K\exp\left[\frac{-j2n\pi t}{T}\right]dt \\
&= \frac{K}{j2n\pi}\left[\exp\left[\frac{jn\pi\tau}{T}\right] - \exp\left[\frac{-jn\pi\tau}{T}\right]\right] \\
&= \frac{K}{n\pi}\sin\frac{n\pi\tau}{T} \\
&= \frac{K\tau}{T}\operatorname{sinc}\frac{n\tau}{T} \qquad (3.3.17)
\end{aligned}
$$

where $\operatorname{sinc}(\lambda) = \sin(\pi\lambda)/\pi\lambda$. The sinc function plays an important role in Fourier analysis and in the study of LTI systems. It has a maximum value at $\lambda = 0$ and approaches zero as λ approaches infinity, oscillating through positive and negative values. It goes through zero at $\lambda = \pm 1, \pm 2, \ldots$.

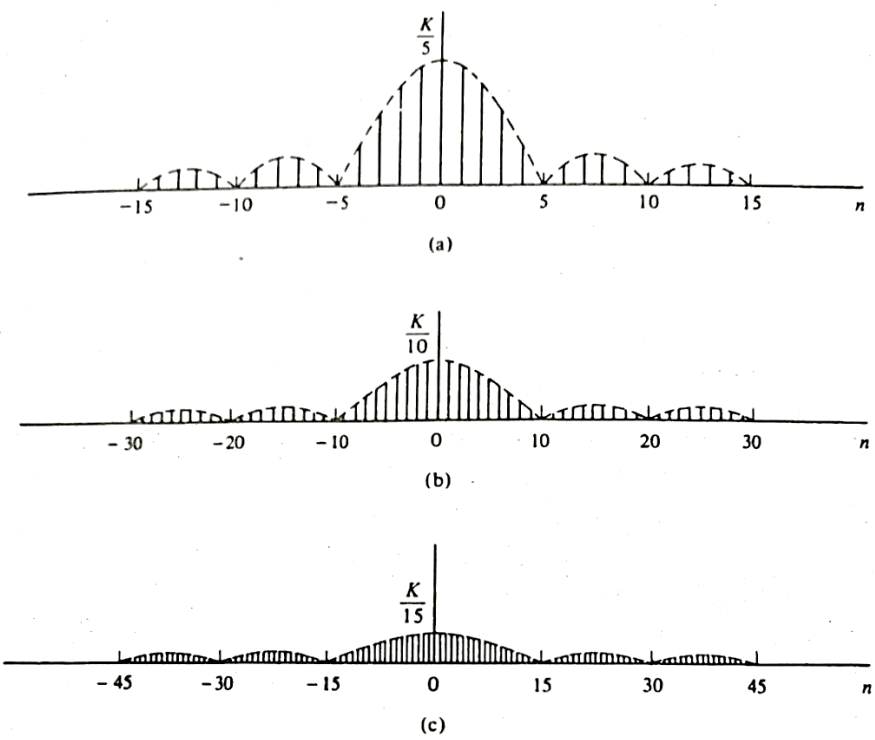


Figure 3.3.7 Line spectra for the $x(t)$ in Example 3.3.3. (a) Magnitude spectrum for $\tau = 1$ and $T = 5$. (b) Magnitude spectrum for $\tau = 1$ and $T = 10$. (c) Magnitude spectrum for $\tau = 1$ and $T = 15$.

Example 3.10

Consider the signal

$$x[n] = \sin \omega_0 n, \tag{3.99}$$

which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3.3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \tag{3.100}$$

Comparing eq. (3.100) with eq. (3.94), we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \tag{3.101}$$

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).

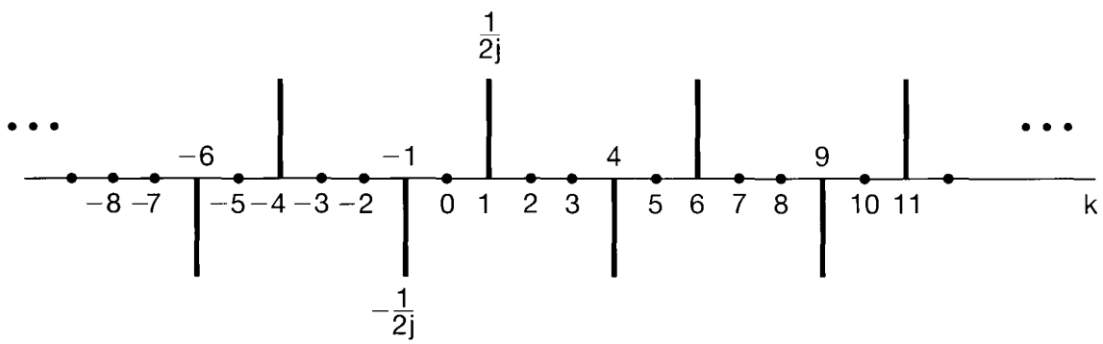


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Practice Problems on Properties of Fourier Series [1]

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 3.6

Consider the signal g(t) with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of g(t) directly from the analysis equation (3.39) in [1]. Instead, we will use the relationship of g(t) to the symmetric periodic square wave x(t) in Example 3.5 in [1]. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

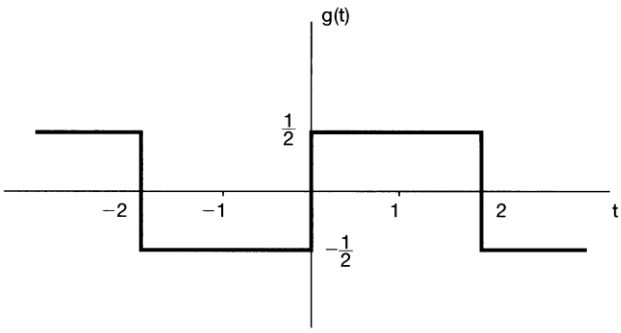


Figure 3.10 Periodic signal for Example 3.6.

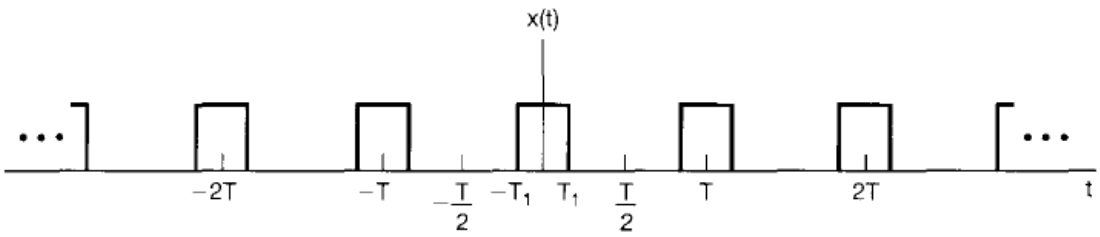


Figure 3.6 Periodic square wave for Example 3.5.

$$g(t) = x(t-1) - 1/2 \quad (3.69)$$

The time-shift property (in Table 3.1) indicates that, if the Fourier Series coefficients of *x(t)* are denoted by a_k , the Fourier coefficients of *x(t* - 1) may be expressed as

$$b_k = a_k e^{-jk\Pi/2}$$

Note 1: here, from Example 3.5 in [1], t_0=1, T=4 such that $e^{-jk\left(\frac{2\Pi}{T}\right)t_0} = e^{-jk\left(\frac{2\Pi}{4}\right)\times 1} = e^{-\frac{jk\Pi}{2}}$

The Fourier coefficients of the *de offset* in g(t)-i.e., the term -1/2 on the right-hand side of eq. (3.69)- are given by

$$c_k = \begin{cases} 0, & k \neq 0 \\ -1/2, & k = 0 \end{cases}$$

Applying the linearity property in Table 3.1, we conclude that the coefficients for *g(t)* may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\Pi/2}, & k \neq 0 \\ a_0 = -1/2, & k = 0 \end{cases}$$

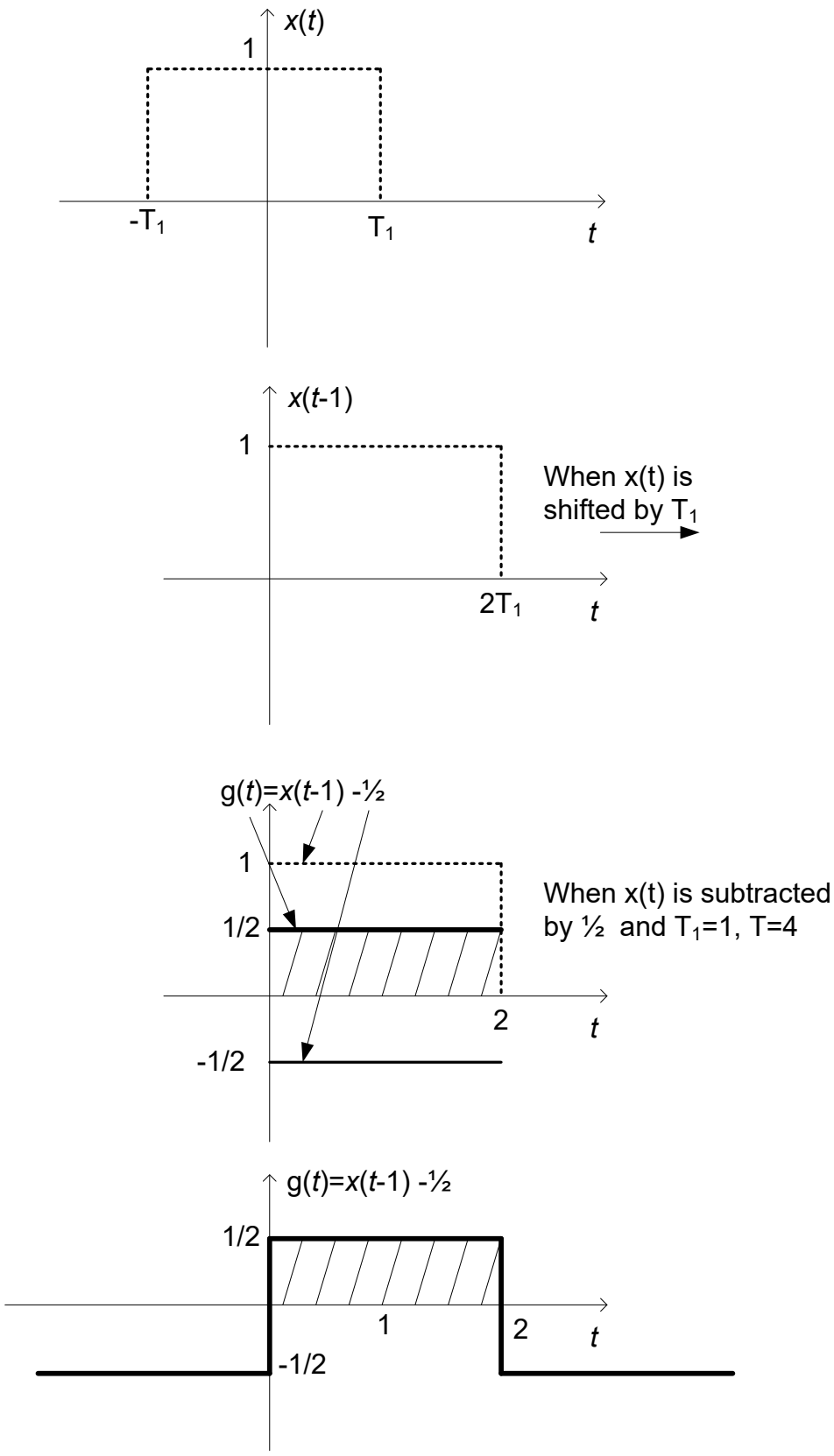


Figure x1. $g(t) = x(t-1) - 1/2$

Note 2: From Eq. (3.46)

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\Pi}; k \neq 0$$

$$a_k = \frac{\sin\left(k\frac{2\Pi}{T}T_1\right)}{k\Pi}; k \neq 0$$

$$a_k = \frac{\sin\left(k\frac{2\Pi}{4}\times 1\right)}{k\Pi}; k \neq 0$$

$$a_k = \frac{\sin\left(k\frac{\Pi}{2}\right)}{k\Pi}; k \neq 0$$

So, we can now find the following,

$$d_k = \begin{cases} \frac{\sin(k\Pi/2)}{k\Pi} e^{-jk\Pi/2}, & k \neq 0 \\ -1/2, & k = 0 \end{cases} \qquad (3.72)$$

□

Example 3.7

Consider the triangular wave signal *x(t)* with period $T = 4$ and fundamental frequency $\omega_0 = \Pi/2$ shown in Figure 3.11. The derivative of this signal is the signal *g(t)* in Example 3.6.

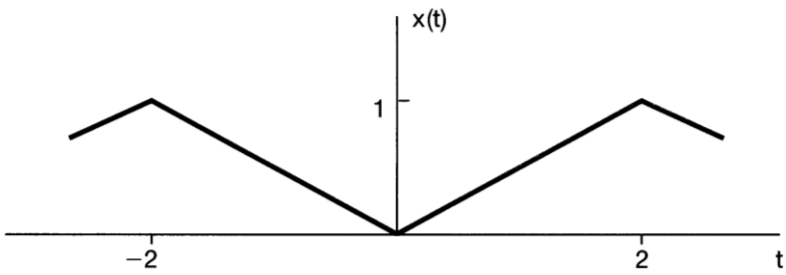


Figure 3.11 Triangular wave signal in Example 3.7.

Denoting the Fourier coefficients of g(t) by d_k, and those of *x(t)* by e_k, we see that the differentiation property in Table 3.1 indicates that

$$d_k = jk(\Pi/2)e_k \qquad (3.73)$$

Note 1:

$$\frac{dx(t)}{dt} \rightarrow jk\frac{2\Pi}{T}e_k \quad \text{[using differentiation property]}$$
$$= jk\frac{2\Pi}{4}e_k = jk\frac{\Pi}{2}e_k$$

This equation can be used to express e_k in terms of d_k except when $k = 0$. Specifically, from eq. (3.72),

$$e_k = \frac{2d_k}{jk\Pi} = \frac{2\sin(k\Pi/2)}{j(k\Pi)^2}e^{-jk\Pi/2}, \quad k \neq 0 \qquad (3.74)$$

For $k = 0$, e_0 can be determined by finding the area under one period of *x(t)* and dividing by the length of the period as follows.

$$e_0 = \frac{area}{T} \quad [\text{from Figure 3.11}]$$
$$e_0 = \frac{\left[\left(\frac{1}{2}\times 2\times 1\right)+\left(\frac{1}{2}\times 2\times 1\right)\right]}{(2-(-2))}$$
$$e_0 = \frac{[1+1]}{4}$$
$$e_0 = \frac{1}{2}$$

Note that the e_k are the F.S. coefficients expressions for the triangular wave in Figure 3.11. □

Example 3.8

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or *impulse train.* This signal and its representation in terms of complex exponentials will play an important role in sampling operations.

Part I. Find Fourier Series Coefficients of a periodic train of impulses

The impulse train with period T may be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad (3.75)$$

It is illustrated in Figure 3.12(a).

To determine the Fourier series coefficients a_k of $x(t)$, we use the original FS coefficients expression in eq. (3.39) and select the interval of integration to be $-\frac{T}{2} \le t \le \frac{T}{2}$ (avoiding the placement of impulses at the integration limits).

Within this interval, x(t) is the same as $\delta(t)$, and it follows that

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\Pi/T} dt$$

$$a_k = \frac{1}{T}$$

In other words, all the Fourier series coefficients of the impulse train are **identical**. These coefficients are also real valued and even (with respect to the index *k).* This is to be expected, since, according to Table 3.1, any real and even signal (such as our impulse train) should have real and even Fourier coefficients. □

Part II. Find Fourier Series Coefficients of impulse responses from square wave signals

The impulse train also has a straightforward relationship to square-wave signals such as g(t) in Figure 3.6, which we repeat in Figure 3.12(b).

The derivative of g(t) is the signal q(t) illustrated in Figure 3.12(c). We may interpret q(t) as the difference between two shifted versions of the impulse train x(t). That is,

$$q(t) = x(t + T_1) - x(t - T_1) \quad (3.77)$$

Using the properties of the Fourier series, we can now compute the Fourier series coefficients of q(t) and g(t) without any further direct evaluation of the Fourier series analysis equation.

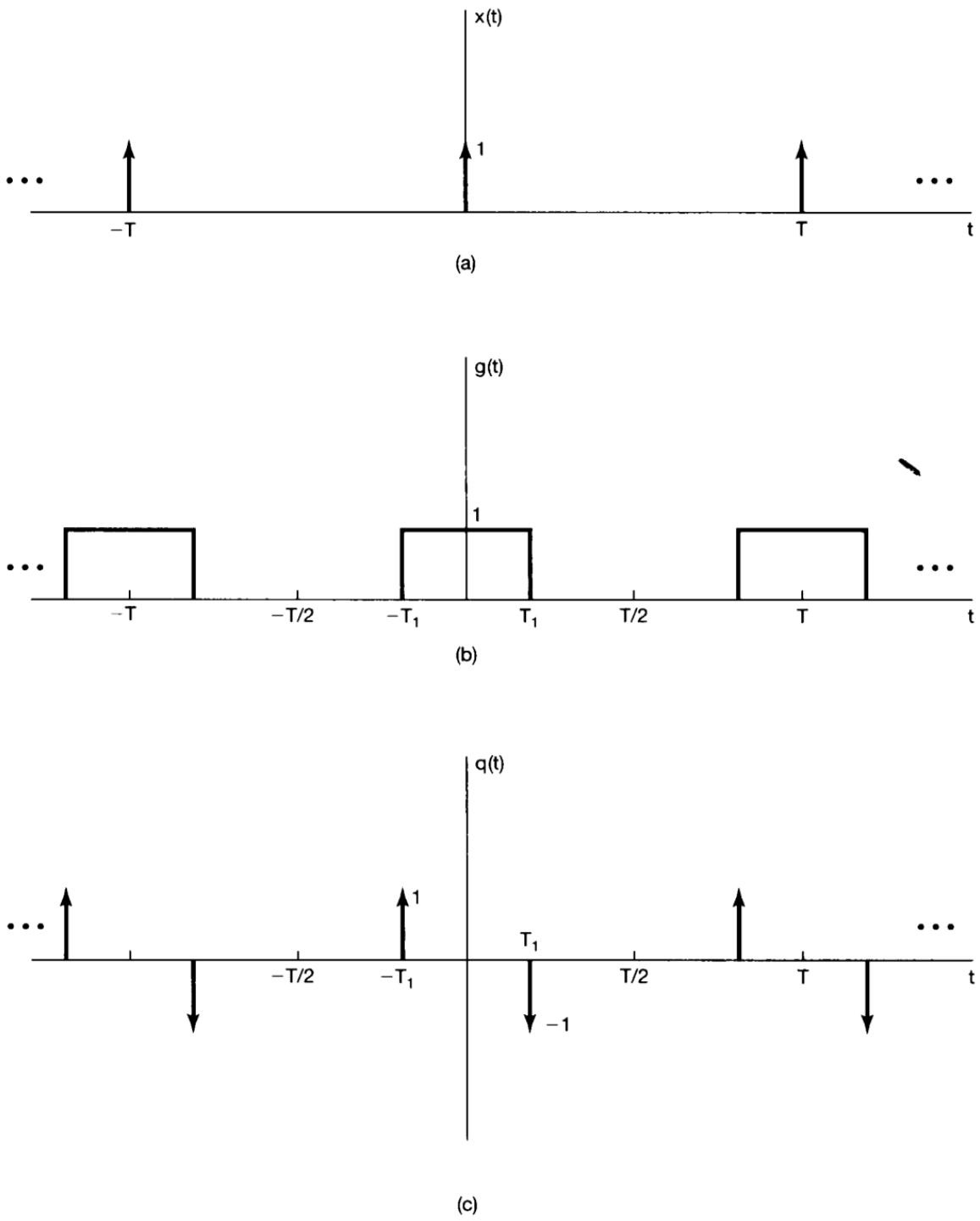


Figure 3.12 (a) Periodic train of impulses; (b) periodic square wave; (c) derivative of the periodic square wave in (b).

First, from the **time-shifting** and **linearity** properties, we see from eq. (3.77) that the Fourier series coefficients b_k of q(t) may be expressed in terms of the Fourier series coefficients a_k of x(t*);* that is,

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

where $\omega_0 = 2\Pi/T$. Using eq. (3.76) (putting the value of a_k), we then have

$$b_k = \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] = \frac{2j\sin\left(k\omega_0 T_1\right)}{T}$$

The above is the expression for the Fourier series coefficients of q(t). □

Part III. [Reverse operation] Find Fourier Series Coefficients of square wave signals from impulse responses

Finally, since *q(t)* is the derivative of *g(t),* we can use the differentiation property in Table 3.1 to write

$$b_k = jk\omega_0 c_k$$

Where the c_k are the Fourier series coefficients of *g(t).* Thus,

$$\frac{b_k}{jk\omega_0} = c_k$$

$$c_k = \left(\frac{2j\sin\left(k\omega_0 T_1\right)}{T}\right) \Big/ jk\omega_0$$

$$c_k = \sin\left(k\omega_0 T_1\right)/k\Pi, \quad k \neq 0$$

where we have used the fact that $\omega_0 = 2\Pi/T$. Note that eq. (3.79) is valid for $k \neq 0$, since we cannot solve for c_0 from eq. (3.78) with k = 0. However, since c_0 is just the average value of *g(t)* over one period, we can determine it by inspection from Figure 3.12(b):

$$c_0 = \frac{T_1 - \left(-T_1\right)}{T} = \frac{2T_1}{T}$$

The above are the expressions for the Fourier series coefficients of g(t).

Note that Eqs. (3.80) and (3.79) are identical to eqs. (3.42) and (3.44), respectively, for the Fourier series coefficients of the **square wave** derived in Example 3.5. □

LTI Systems with Periodic Inputs and Related Problems [2]

Example 3.6.1

Consider the system described by the input/output differential equation

$$y^{(n)}(t) + \sum_{i=0}^{n-1} p_i y^{(i)}(t) = \sum_{i=0}^{m} q_i x^{(i)}(t)$$

For input $x(t) = \exp[j\omega t]$, the corresponding output is $y(t) = H(\omega)\exp[j\omega t]$. Since every input and output should satisfy the system differential equation, substituting into the latter yields

$$\left[(j\omega)^n + \sum_{i=0}^{n-1} p_i (j\omega)^i\right] H(\omega)\exp[j\omega t] = \sum_{i=0}^{m} q_i (j\omega)^i \exp[j\omega t]$$

Solving for $H(\omega)$, we obtain

$$\boxed{H(\omega) = \frac{\sum_{i=0}^{m} q_i (j\omega)^i}{(j\omega)^n + \sum_{i=0}^{n-1} p_i (j\omega)^i}}$$

xample 3.6.2

Let us find the output voltage $y(t)$ of the system shown in Figure 3.6.1 if the input voltage is the periodic signal

$$x(t) = 4\cos t - 2\cos 2t$$

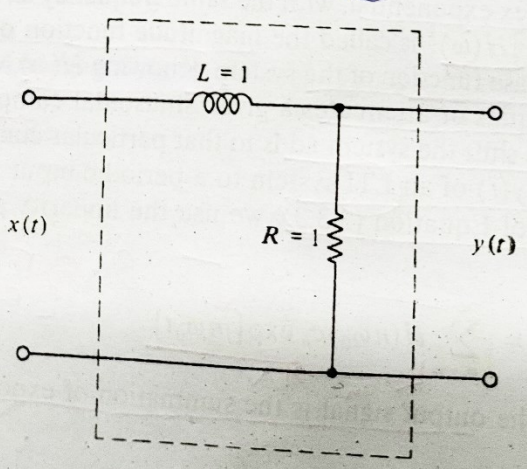


Figure 3.6.1 System for Example 3.6.2.

Applying Kirchhoff's voltage law to the circuit yields

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

If we set $x(t) = \exp[j\omega t]$ in this equation, the output voltage is $y(t) = H(\omega)\exp[j\omega t]$. Using the system differential equation, we obtain

$$j\omega H(\omega)\exp[j\omega t] + \frac{R}{L}H(\omega)\exp[j\omega t] = \frac{R}{L}\exp[j\omega t]$$

Solving for $H(\omega)$ yields

$$H(\omega) = \frac{R/L}{R/L + j\omega}.$$

$\frac{dy(t)}{dt} = \frac{d}{dt}(H(\omega)\cdot\exp(j\omega t)) = j\omega H(\omega)\cdot\exp$

At any frequency $\omega = n\omega_0$, the system function is

$$H(n\omega_0) = \frac{R/L}{R/L + jn\omega_0}$$

For this example, $\omega_0 = 1$ and $R/L = 1$, so that the output is

$$y(t) = \frac{2}{1+j1}\exp[jt] + \frac{2}{1-j1}\exp[-jt] - \frac{1}{1+j2}\exp[j2t] - \frac{1}{1-j2}\exp[-j2t]$$

$$= 2\sqrt{2}\cos(t - 45°) - \frac{2}{\sqrt{5}}\cos(2t - 63°)$$

Example 3.6.3

Consider the circuit shown in Figure 3.6.2. The differential equation governing the system is

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R}$$

For an input of the form $i(t) = \exp[j\omega t]$, we expect the output $v(t)$ to be $v(t) = H(\omega)\exp[j\omega t]$. Substituting into the differential equation yields

$$\exp[j\omega t] = Cj\omega H(\omega)\exp[j\omega t] + \frac{1}{R}H(\omega)\exp[j\omega t]$$

Canceling the $\exp[j\omega t]$ term and solving for $H(\omega)$, we have

$$H(\omega) = \frac{1}{1/R + j\omega C}$$

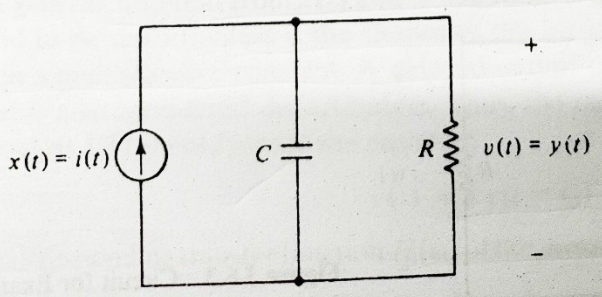


Figure 3.6.2 Circuit for Example 3.6.3.

Example 3.6.4

Consider the system shown in Figure 3.6.3. Applying Kirchhoff's voltage law, we find that the differential equation describing the system is

$$x(t) = L\frac{d}{dt}\left(\frac{y(t)}{R} + C\frac{dy(t)}{dt}\right) + y(t)$$

which can be written as

$$LC\,y''(t) + \frac{L}{R}y'(t) + y(t) = x(t)$$

For an input in the form $x(t) = \exp[j\omega t]$, the output voltage is $\boxed{y(t) = H(\omega)\exp[j\omega t].}$ Using the system differential equation, we obtain

$$\cancel{\exp(j\omega t)}\left[(j\omega)^2 LC\,H(\omega) + j\omega\frac{L}{R}H(\omega) + H(\omega)\right] = 1$$

$\cancel{\exp(j\omega t)}$

$\frac{dy}{dt} = j\omega\, H(\omega)\cdot\exp(j\omega t)$

$\frac{d^2y}{dt^2} = (j\omega)^2 H(\omega)\cdot\exp(j\omega t)$

Solving for $H(\omega)$ yields

$$H(\omega) = \frac{1}{1 + j\omega L/R - \omega^2 LC}$$

with

$$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + (\omega L/R)^2}}$$

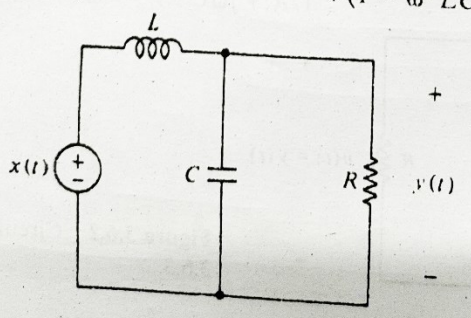


Figure 3.6.3 Circuit for Example 3.6.4.

and

$$\measuredangle H(\omega) = -\tan^{-1}\frac{\omega L/R}{1-\omega^2 LC}$$

Example 3.6.6

Let $x(t)$ and $y(t)$ be the input and the output, respectively, of the simple RC circuit shown in Figure 3.6.5. Applying Kirchhoff's voltage law, we obtain

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

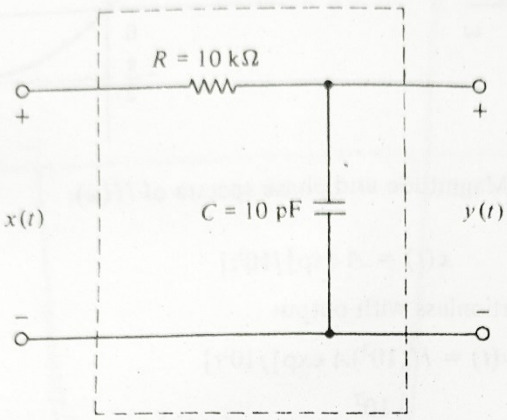


Figure 3.6.5 Circuit for Example 3.6.6.

Setting $x(t) = \exp[j\omega t]$ and recognizing that $y(t) = H(\omega)\exp[j\omega t]$, we have

$$j\omega H(\omega)\exp[j\omega t] + \frac{1}{RC}H(\omega)\exp[j\omega t] = \frac{1}{RC}\exp[j\omega t]$$

Solving for $H(\omega)$ yields

$$H(\omega) = \frac{1/RC}{1/RC + j\omega} = \frac{\eta}{\eta + j\omega}$$

where

$$\eta = \frac{1}{10^4 \times 10^{-11}} = 10^7\, s^{-1}$$

Hence,

$$|H(\omega)| = \frac{\eta}{\sqrt{\eta^2 + \omega^2}}$$

$$\measuredangle H(\omega) = -\tan^{-1}\frac{\omega}{\eta}$$

The amplitude and phase spectra of $H(\omega)$ are shown in Figure 3.6.6. Note that for $\omega \ll \eta$,

$$H(\omega) \simeq 1$$

and

$$\measuredangle H(\omega) \simeq -\frac{\omega}{\eta}$$

That is, the magnitude and phase characteristics are practically ideal. For example, for input

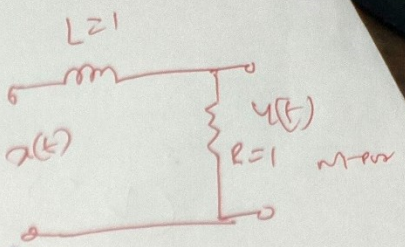
Detailed Solutions to 3.6.2, 3.6.3, 3.6.4, 3.6.6

See → 3.6.1 Page-136

3.6.2

Part (a) $x(t) = e^{j\omega t}$

Applying KVL.



$$x(t) = y(t) + L\frac{di}{dt}$$

$$x(t) = y(t) + L\frac{d}{dt}\left(\frac{y(t)}{R}\right)$$

$$x(t) = y(t) + \frac{L}{R}\frac{dy(t)}{dt}$$

$$\Rightarrow \frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{R}{L}x(t) \quad \text{--- (1)}$$

Here Input voltage $x(t) = e^{j\omega t}$

So, output voltage, $y(t) = H(\omega)\cdot x(t)$
$= H(\omega)\cdot e^{j\omega t}$.

From (1) $\Rightarrow \frac{d}{dt}\left(H(\omega)\cdot e^{j\omega t}\right) + \frac{R}{L}\cdot H(\omega)\cdot e^{j\omega t} = \frac{R}{L}\cdot e^{j\omega t}$

$$\Rightarrow (j\omega)H(\omega)e^{j\omega t} + \frac{R}{L}H(\omega)e^{j\omega t} = R/L\, e^{j\omega t}$$

$$\Rightarrow H(\omega)\left[j\omega + R/L\right] = R/L$$

$$\Rightarrow \boxed{H(\omega) = \frac{R/L}{R/L + j\omega}}$$

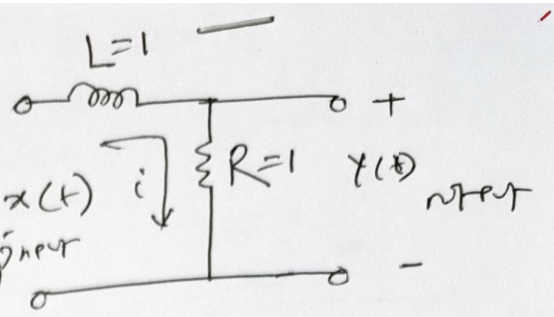
$|H(\omega)|$ ← magnitude

$$|H(\omega)| = \frac{R/L}{\sqrt{(R/L)^2 + \omega^2}}$$

Ans → $\angle H(\omega) = -\tan^{-1}(\omega/(R/L))$

Updated [24NOV2022]

P. 3.6.2



$$x(t) = 4\cos t - 2\cos 2t$$

Sol $\Rightarrow$ $x(t) = 4\left[\frac{e^{jt}+e^{-jt}}{2}\right] - 2\left[\frac{e^{j2t}+e^{-j2t}}{2}\right]$

$$x(t) = 2\left[e^{jt}+e^{-jt}\right] - \left[e^{j2t}+e^{-j2t}\right]$$

Here, $H(\omega) = \frac{R/L}{R/L + j\omega} = \frac{1/1}{1/1 + j\omega} = \frac{1}{1+j\omega}$.

So, $y(t) = H(\omega)\, x(t)$

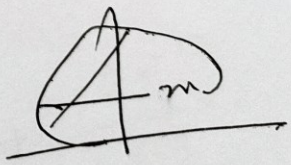
$$= \frac{1}{1+j\omega}\left[2\left[e^{jt}+e^{-jt}\right]\right] - \frac{1}{1+j\omega}\left[e^{j2t}+e^{-j2t}\right]$$

$$= \left[\frac{1 * 2e^{jt}}{1+j(1)} + \frac{1*2\, e^{-jt}}{1+j(-1)}\right] - \left[\frac{1 * e^{j2t}}{1+j(2)} + \frac{1 * e^{-j2t}}{1+j(-2)}\right]$$

Here $\omega = 1$ (+1) (-1)

Here. $\omega = 2$ $\omega = -2$

$$y(t) = \frac{2e^{jt}}{1+j1} + \frac{2e^{-jt}}{1-j1} - \frac{e^{j2t}}{1+j2} - \frac{e^{-j2t}}{1-j2}$$

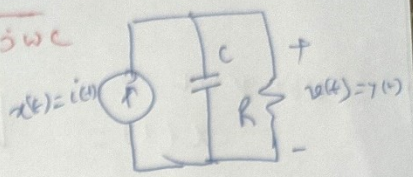


Ex 3-6-3

②

$$\mathcal{H}(\omega) = \frac{R \times \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{R}{1 + RC j\omega}$$

$$H(\omega) = \frac{R \cdot 1}{1/R + j\omega C}$$



output,

output $y(t) = v(t) = H(\omega) \exp(j\omega t)$

$$= \frac{\exp(j\omega t)}{1/R + j\omega C}$$

$$e^{j\omega t} = C \frac{dv(t)}{dt} + \frac{v(t)}{R}$$

$$= C \cdot \frac{d}{dt}\left[H(\omega) \cdot \exp(j\omega t)\right] + \frac{H(\omega) \cdot \exp(j\omega t)}{R}$$

$$= C \cdot H(\omega) \cdot j\omega \cdot \exp(j\omega t) + \frac{1}{R} H(\omega) \cdot \exp(j\omega t)$$

$$\cancel{e^{j\omega t}} = \cancel{e^{j\omega t}} \left[C \cdot H(\omega)\, j\omega + 1/R \cdot H(\omega)\right]$$

$$1 = H(\omega)\left[j\omega C + 1/R\right]$$

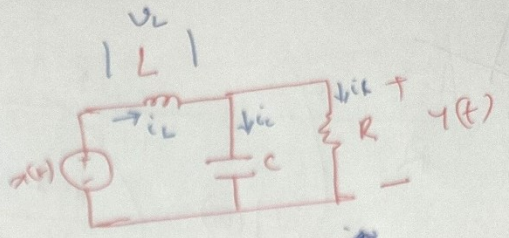
$$\Rightarrow \boxed{H(\omega) = 1 \Big/ 1/R + j\omega C}$$

Ans

So, output, $y(t) = v(t) = \mathcal{H}(\omega) * i(t)$ ($i(t) \to e^{j\omega t}$)

$$\boxed{v(t) = \frac{e^{j\omega t}}{1/R + j\omega C}}$$

3.64



using KVL

$$x(t) = L \cdot \frac{di_L}{dt} + y(t)$$

$$= L \frac{d}{dt}\left[i_c + i_R\right] + y(t)$$

$$= L \frac{d}{dt}\left[C \cdot \frac{dy(t)}{dt} + \frac{y(t)}{R}\right] + y(t)$$

$$= LC \frac{d^2 y(t)}{dt^2} + \frac{L}{R} \cdot \frac{dy(t)}{dt} + y(t)$$

$$x(t) = LC\, y''(t) + \frac{L}{R} y'(t) + y(t) \quad \text{— (1)}$$

for an input, $x(t) = \exp(j\omega t)$

the output voltage, $y(t) = H(\omega) \cdot \exp[j\omega t]$

from eqn (1)

$$e^{j\omega t} = LC\left[H(\omega)\, e^{j\omega t}\right]'' + \frac{L}{R}\left[H(\omega) \cdot e^{j\omega t}\right]' + H(\omega) \cdot e^{j\omega t}$$

$$= LC\left[(j\omega)^2 \cdot H(\omega) \cdot e^{j\omega t}\right] + \frac{L}{R} \cdot j\omega\, H(\omega) \cdot e^{j\omega t} + H(\omega)\, e^{j\omega t}$$

$$1 = LC(j\omega)^2 H(\omega) + \frac{L}{R}\cdot(j\omega)H(\omega) + H(\omega).$$

$$1 = \left[-\omega^2 LC + \frac{L}{R}j\omega + 1\right] H(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{1}{1 + j\frac{\omega L}{R} - \omega^2 LC}}$$ Ans

So, $$|H(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}}$$ ← sin a denominator

(minus)

$$\boxed{\angle H(\omega) = -\tan^{-1}\left[\frac{(\omega L/R)}{1-\omega^2 LC}\right]}$$ Ans.

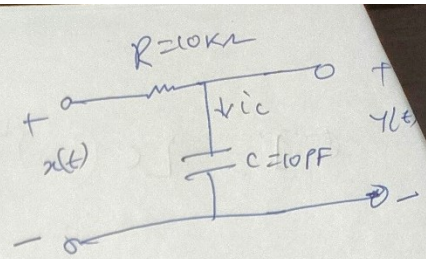
$-\tan^{-1}[\ ]$ →

Hence, output, $y(t) = H(\omega)\cdot e^{j\omega t}$

$$\boxed{\cancel{y(t) = \frac{(\omega L/R)\, e^{j\omega t}}{(1-\omega^2 LC)}}}$$ Ans

$$\boxed{y(t) = \frac{e^{j\omega t}}{1 + j\frac{\omega L}{R} - \omega^2 LC}}$$ Ans

Exp. 3.66)



~~Applying KVL~~

$C\frac{dy(t)}{dt}$ Here, $i_c = C \cdot \frac{dy(t)}{dt}$.

So, using KVL.

$$x(t) = R\, i_c + y(t)$$

$$\Rightarrow x(t) = R \cdot C \cdot \frac{dy(t)}{dt} + y(t)$$

$$\Rightarrow \boxed{\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{x(t)}{RC}} \quad \text{--- (1)}$$

given, $x(t) = e^{j\omega t}$

Also, $y(t) = H(\omega)\, e^{j\omega t}$

$$(1) \Rightarrow \frac{d}{dt}\left(H(\omega)\, e^{j\omega t}\right) + \frac{H(\omega)\, e^{j\omega t}}{RC} = \frac{e^{j\omega t}}{RC}$$

$$\Rightarrow (j\omega)\, H(\omega)\, e^{j\omega t} + \frac{H(\omega)}{RC} e^{j\omega t} = \frac{e^{j\omega t}}{RC}$$

$$\Rightarrow j\omega\, H(\omega) + \frac{1}{RC} H(\omega) = 1/RC.$$

$$\Rightarrow \boxed{H(\omega) = \frac{1/RC}{1/RC + j\omega}}$$

Here, $\frac{1}{RC} = \frac{1}{10 \times 10^3 \times 10 \times 10^{-12}\,F} = 10^{7}\ s^{-1}$

So, $|H(\omega)| =$ Let, $\eta = 1/RC$

4

$|H(\omega)|\phi$

So, $\{H(\omega)\} = \dfrac{\eta}{\eta + j\omega}$

Hence $\boxed{|H(\omega)| = \dfrac{\eta}{\sqrt{\eta^2 + \omega^2}}}$

$\angle H(\omega) = -\tan^{-1}(\omega/\eta)$

$= -\tan^{-1}\left(\dfrac{\omega}{1/RC}\right)$

$\boxed{\angle H(\omega) = -\tan^{-1}(\omega RC)}$

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