EEE 243 Signals and Systems

2022

Lecture 08: Linear Time-Invariant (LTI) Systems- The Convolution Integral

Department of Electrical and Electronic Engineering
School of Engineering
BRAC University

Linear Time-Invariant (LTI) Systems

A **linear time-invariant system** (**LTI system**) is a system that produces an output signal from any input signal subject to the constraints of linearity and time-invariance. These properties apply to many important physical systems, in which case the response $y(t)$ of the system to an arbitrary input ***x(t)*** can be found directly using convolution: $y(t) = x(t) * h(t)$ where $h(t)$ is called the system's impulse response and $*$ represents convolution (described below in detail).

A good example of an LTI system is any electrical circuit consisting of resistors, capacitors, inductors, and linear amplifiers. In general, LTI systems have direct applications in electrical circuit analysis and design, signal processing and filter design, control theory, mechanical engineering, image processing, etc.

The Usefulness of LTI Systems

- There are systematic methods for solving any LTI system (i.e., determining $h(t)$) since systems not meeting both linear and time-invariance properties are generally more difficult (or impossible) to solve analytically.

- Any LTI system can be characterized entirely by a single function called the system's impulse response (presented with a brief input signal, called an impulse). The output of the system is simply the <u>convolution</u> of the input to the system with the system's impulse response .

- LTI systems can also be characterized in the frequency domain by the system's transfer function, which is the Laplace transform of the system's impulse response. As a result, the output of the system in the frequency domain is the product of the transfer function and the transform of the input. In other words, convolution in the time domain is equivalent to multiplication in the frequency domain.

- LTI systems cannot produce frequency components that are not in the input. In this regard, since sinusoids are a sum of complex exponentials with complex-conjugate frequencies, if the input to the system is a sinusoid, then the output of the system will also be a sinusoid, perhaps with a different amplitude and a different phase, but always with the same frequency upon reaching steady-state.

- LTI system theory is good at describing many important systems. Most LTI systems are considered "easy" to analyze, at least compared to the time-varying and/or nonlinear case. Any system that can be modeled as a linear differential equation with constant coefficients is an LTI system. Examples of such systems are electrical circuits made up of resistors, inductors, and capacitors (RLC circuits).

- Most LTI system concepts are similar between the continuous-time and discrete-time (linear shift-invariant) cases. In this regard, in image processing, the time variable is replaced with two space variables, and the notion of time invariance is replaced by two-dimensional shift invariance.

The Convolutional Integral

Convolution is a mathematical operation on two functions (e.g., f and g), producing another function $f*g$ that expresses how the shape of one is modified by the other. It is defined as the integral of the product of the two functions after one is reversed and shifted. The integral is evaluated for all values of shift, producing the convolution function expressed as follows.

$$(f*g)(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Equivalently,

$$(f*g)(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

Note that even though the symbol t is used, it does not necessarily need to represent time. Hence, the convolution formula can be described as the area under the function $f(\tau)$ weighted by the function $g(-\tau)$ shifted by amount t. As t changes, the weighting function $g(t-\tau)$ emphasizes different parts of the input function $f(\tau)$.

Visual representations of the convolutional integral of f and g :

Figure 2 shows the convolution integral of functions $f(\tau)$ (in blue) as the red-colored "pulse", $g(t-\tau)$ (in red colored pulse) moves across it for some value of parameter t (which is arbitrarily defined as the distance from the $\tau=0$ axis to the center of the red pulse). The amount of yellow is the area of the product $f(\tau)g(t-\tau)$ computed by the convolution integral due to continuously changing t and recomputing the integral. The result is shown in black (which is a function of t (but is plotted on the same axis as τ for convenience and comparison.

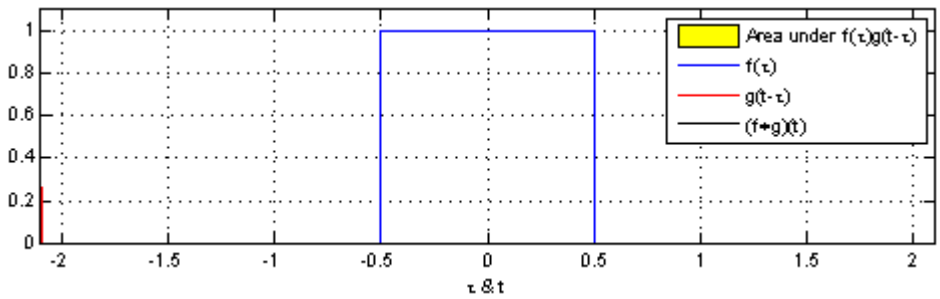


Figure 2. Visual explanation of computing convolutional integral of two functions.

Graphical Approach to Computing the Convolutional Integral

Often one or both of the signals in convolution integral is defined in piecewise fashion, and graphical interpretation of convolution becomes helpful. Steps to compute convolution integral using graphical approach is described in what follows.

Step 1. Express each function in terms of a <u>dummy variable</u>, e.g. τ.
Step 2. Reflect one of the functions, e.g., $g(\tau) \rightarrow g(-\tau)$.
Step 3. Add a time offset, *t*, which allows $g(t-\tau)$ sliding along the τ-axis.
Step 4. Start *t* at $-\infty$ and slide it all the way to $+\infty$. Wherever the two functions intersect, find the integral of their product. In other words, at time *t*, compute the area under the function $f(\tau)$ weighted by the weighting function $g(t-\tau)$.
Step 5. The resulting <u>waveform</u> is the convolution of functions f and g.

Figure 1 shows the above steps of how to generate the convolution integral of two functions f (shown in blue) and g (shown in red).

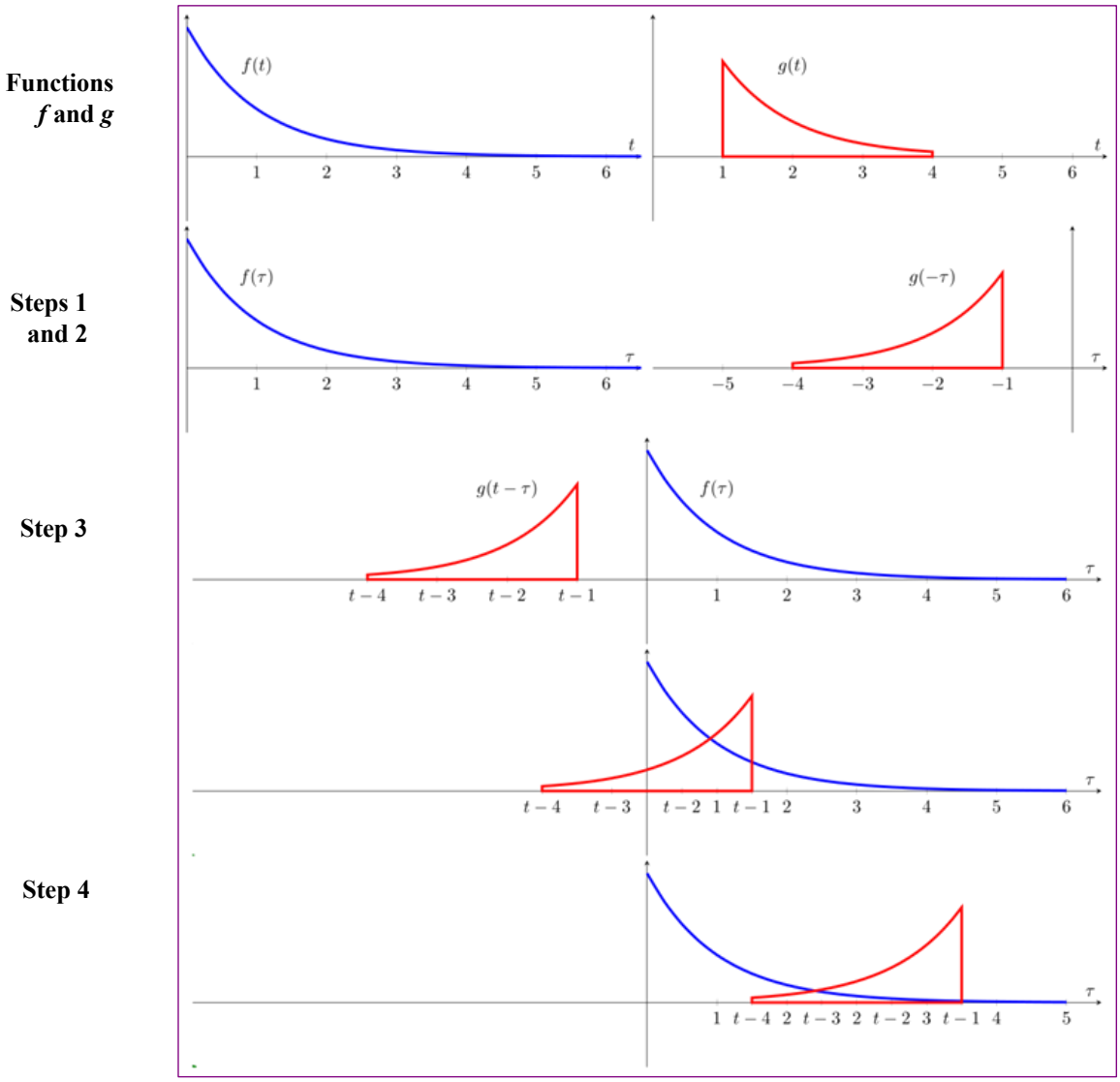


Figure 1. Stepwise approach to computing Convolutional integral.

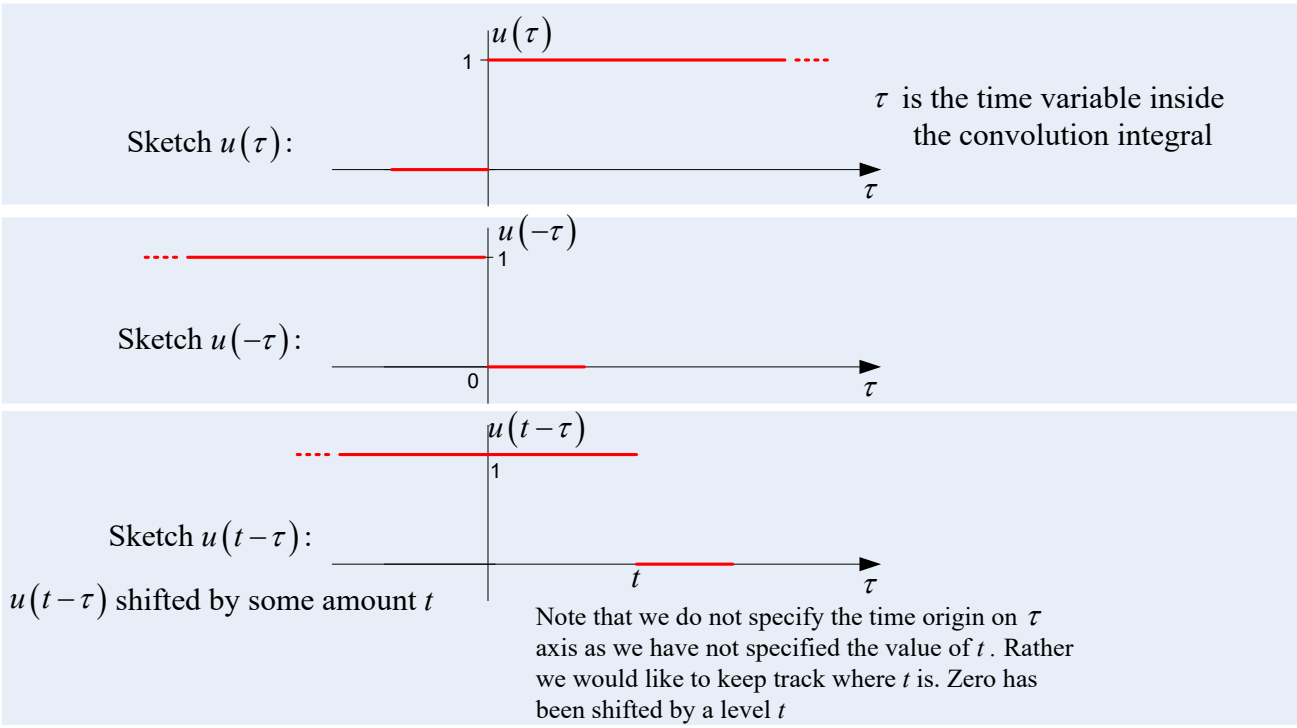
Example 01

Compute the convolution of two unit-step functions, i.e., $y(t)=u(t)*u(t)$.

Solution

$$y(t)=u(t)*u(t)$$
$$y(t)=\int_{-\infty}^{\infty}u(\tau)u(t-\tau)d\tau$$

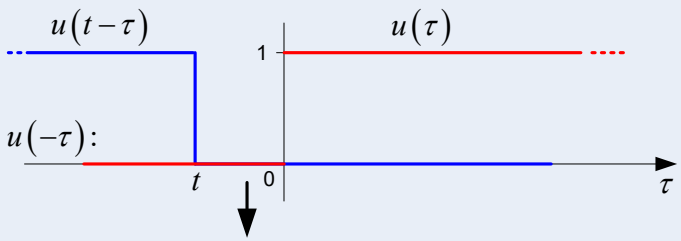
Let's sketch to see how each of the individual component looks like.



Now let's check the pieces of the integrand looks like

Case 1: $t < 0$

Sketch $u(-\tau)$:

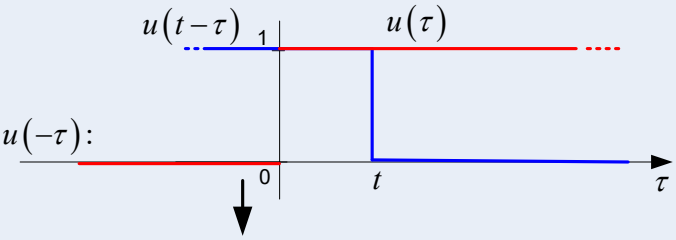


$$u(\tau)u(t-\tau)=0 \quad \text{[no overlapping]}$$

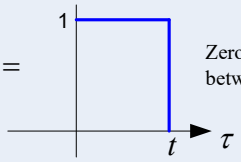
$$\int_{\tau=-\infty}^{\tau=\infty} u(\tau)u(t-\tau)\,d\tau = \int_{\tau=-\infty}^{\tau=\infty} 0\,d\tau = 0$$

Case 2: $t \geq 0$

Sketch $u(-\tau)$:



$u(\tau)u(t-\tau)=$



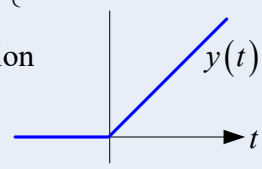
Zero everywhere except for time between 0 and t

$$\int_{\tau=-\infty}^{\tau=\infty} u(\tau)u(t-\tau)\,d\tau = \int_{\tau=0}^{\tau=t} 1\,d\tau = t$$

Combining both cases

$$y(t)=u(\tau)*u(t)=\begin{cases}0 & t<0\\ t & t\geq 0\end{cases} \qquad \textit{Piecewise equation}$$

Sketch $y(t)$: A ramp function



Simplifying $y(t)=u(\tau)*u(t)=tu(t)$

GP-02

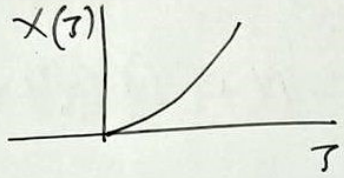
let $x(t) = t^3 u(t)$ and $y(t) = t^2 u(t)$. [polynomial]

Compute $z(t) = x(t) * y(t)$.

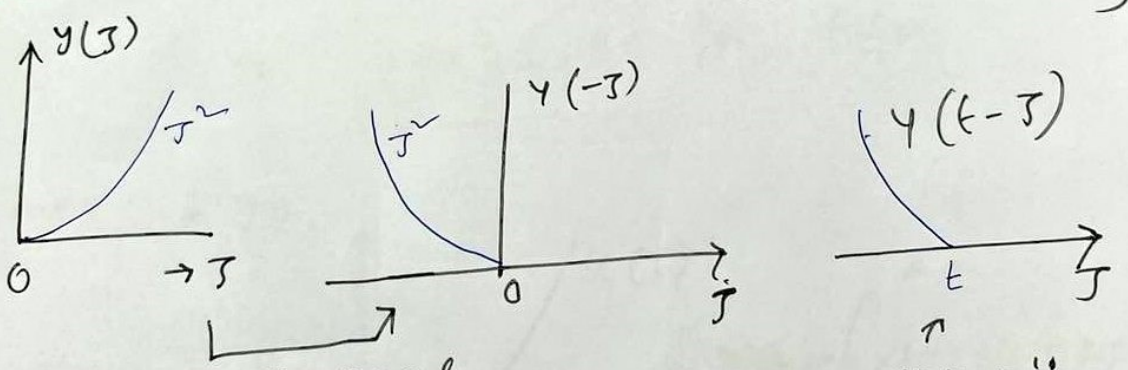
Sol2

$$z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

// Sketch $x(\tau) = \tau^3 u(\tau)$



// Sketch $y(t-\tau) = (t-\tau)^2 u(t-\tau)$ [Replace t with (t−τ)]



Time Reversed [flip everything]

Starts at 't' [time-shifted of y(−τ) by 't']

Piecewise Method to find $y(t-\tau)$

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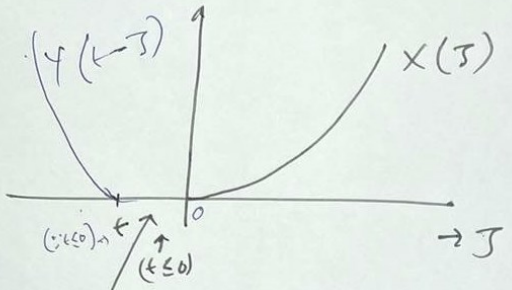
// Case 1 $t \le 0$



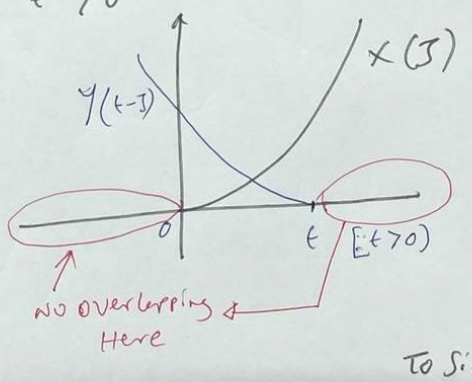
$\Rightarrow$ NO time overlapping

So, $x(\tau)\, y(t-\tau) = 0$

So, $$\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = 0.$$

So, $\boxed{z(t) = 0}$ for $t \le 0$

// Case 2: $t > 0$



$$z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \tau^3 u(\tau)\, (t-\tau)^2 u(t-\tau)\, d\tau$$

To simplify, lets look at the graph.

So, overlapping exists only when τ is between '0' and 't'

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So, $z(t) = \int_{-\infty}^{\infty} \tau^3 u(\tau)(t-\tau)^2 u(t-\tau)\, d\tau$ becomes

$$z(t) = \int_0^t \tau^3 (t-\tau)^2 d\tau$$

$$= \int_0^t \tau^3 [t^2 - 2t\tau + \tau^2]\, d\tau$$

$$= \int_0^t \tau^3 t^2 - 2\tau^4 t + \tau^5\, d\tau$$

$$= \frac{\tau^4 t^2}{4} - \frac{2\tau^5 t}{5} + \frac{\tau^6}{6} \Big|_0^t$$

$$= \frac{t^6}{4} - \frac{2t^6}{5} + \frac{t^6}{6}$$

$$= \frac{30t^6 - 48t^6 + 20t^6}{120}$$

$$= \frac{2t^6}{120}$$

$$z(t) = \tfrac{1}{60} t^6, \quad t > 0$$

$$\boxed{z(t) = \frac{1}{6} t^6 u(t)}$$

So, Combining Both,

$$z(t) = \begin{cases} 0, & t \le 0 \\ \tfrac{1}{6} t^6, & t > 0 \end{cases} \Rightarrow \boxed{z(t) = \tfrac{1}{6} t^6 u(t)}$$ Ans. ☒

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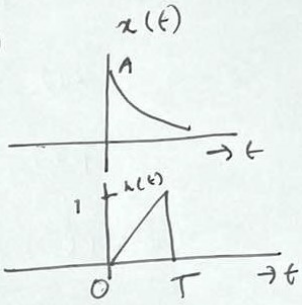
Exp 03 Consider the following signals, where

$$x(t) = A\exp[-t],\ 0 \le t < \infty$$

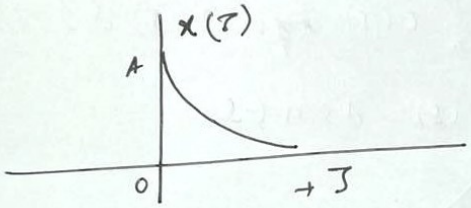
$$h(t) = t/T,\ 0 \le t \le T$$

Compute $y(t) = x(t) * h(t)$

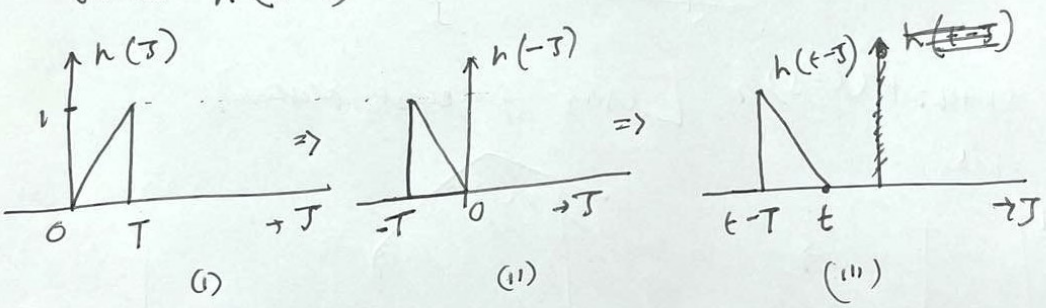


Soln

Sketch $x(\tau) = A\exp[-\tau]$

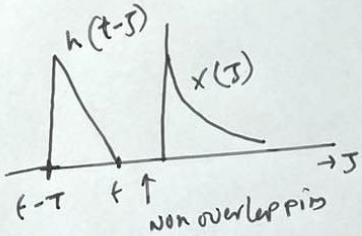


Sketch $h(t-\tau)$



(i) (ii) (iii)

Case I. $t \le 0$



So, $x(\tau)\,h(t-\tau) = 0$

Hence, $\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = 0$

$y(t) = 0$ //.

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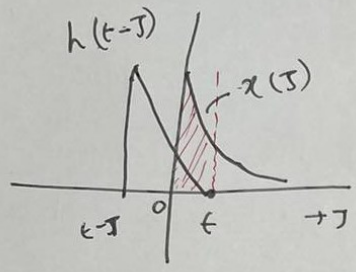
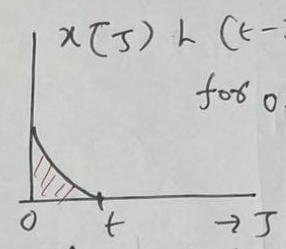
Case II.A $t > 0$, $h(t-\tau)$ overlaps $x(\tau)$



$\Rightarrow$



↑ overlapping Area is completely defined by t!

So, for $0 \le t \le T$ [when $h(t-\tau)$ is totally in the $+T$]

$$y(t) = x(t) * h(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

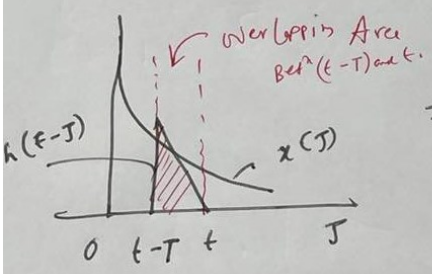
Replace t with τ — Replace t with $(t-\tau)$

$$= \int_0^t A \exp[-\tau] * \frac{(t-\tau)}{T}\, d\tau$$

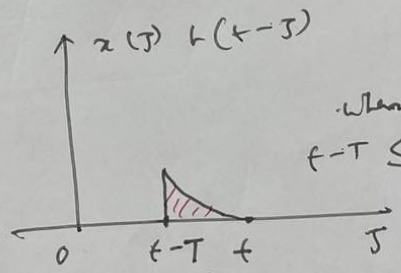
$$= A/T \left[t - 1 + \exp[-t] \right]$$

when $0 < t \le T$

~~when~~ Case II.B $t > T$



$\Rightarrow$



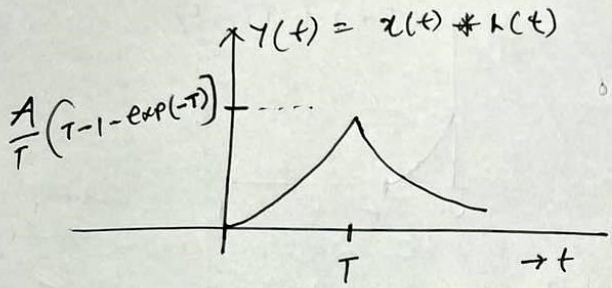
when $t-T \le \tau \le t$.

↑ overlapping Area is defined by $(t-T)$ and t.

So, $$y(t) = \int_{t-T}^{t} A \exp[-\tau] \frac{t-\tau}{T}\, d\tau$$

$$= A/T \left[T - 1 + \exp[-T] \right] \exp[-(t-T)] \quad ; t > T$$

So: Combining all the above Responses



$$y(t) = x(t) * h(t) = \begin{cases} 0, & t \leq 0 \\ A/T\,[t-1+\exp(-t)], & 0 < t \leq T \\ A/T\,[T-1+\exp[-T]]\,\exp[-(t-T)], & t > 0 \end{cases}$$

Ans

Example 04.

Let us determine the convolution

$$y(t) = \text{rect}\,(t/2a) * \text{rect}\,(t/2a)$$

Figure 2.3.4 illustrates the overlapping of the two rectangular pulses for different values of t and the resulting signal $y(t)$. The result is expressed analytically as

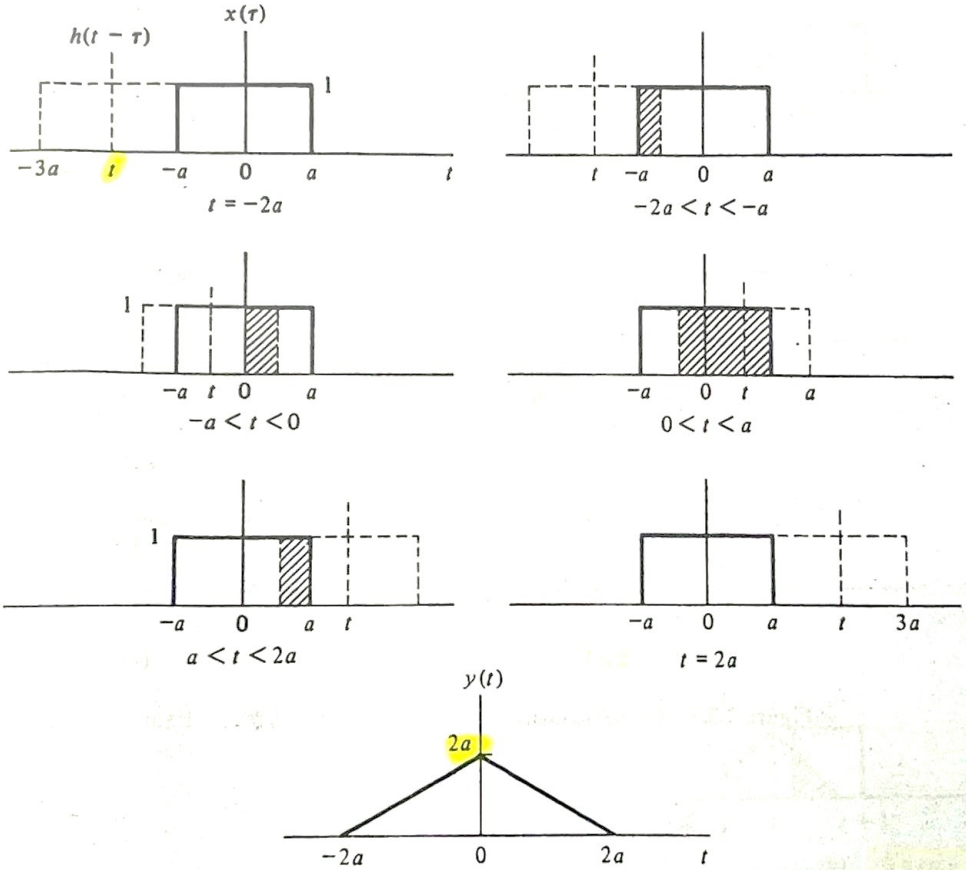
$$y(t) = \begin{cases} 0, & t < -2a \\ t + 2a, & -2a \le t < 0 \\ 2a - t, & 0 \le t < 2a \\ 0, & t \ge 2a \end{cases}$$

or, in more compact form,

$$y(t) = \begin{cases} 2a - |t|, & |t| \le 2a \\ 0, & |t| \ge 2a \end{cases}$$

$$= 2a\,\Delta\,(t/2a)$$

This signal appears frequently in our discussion and is called the triangular signal. We use the notation $\Delta(t/2a)$ to denote the triangular signal that is of unit height, centered around $t = 0$, and has base of length $4a$.

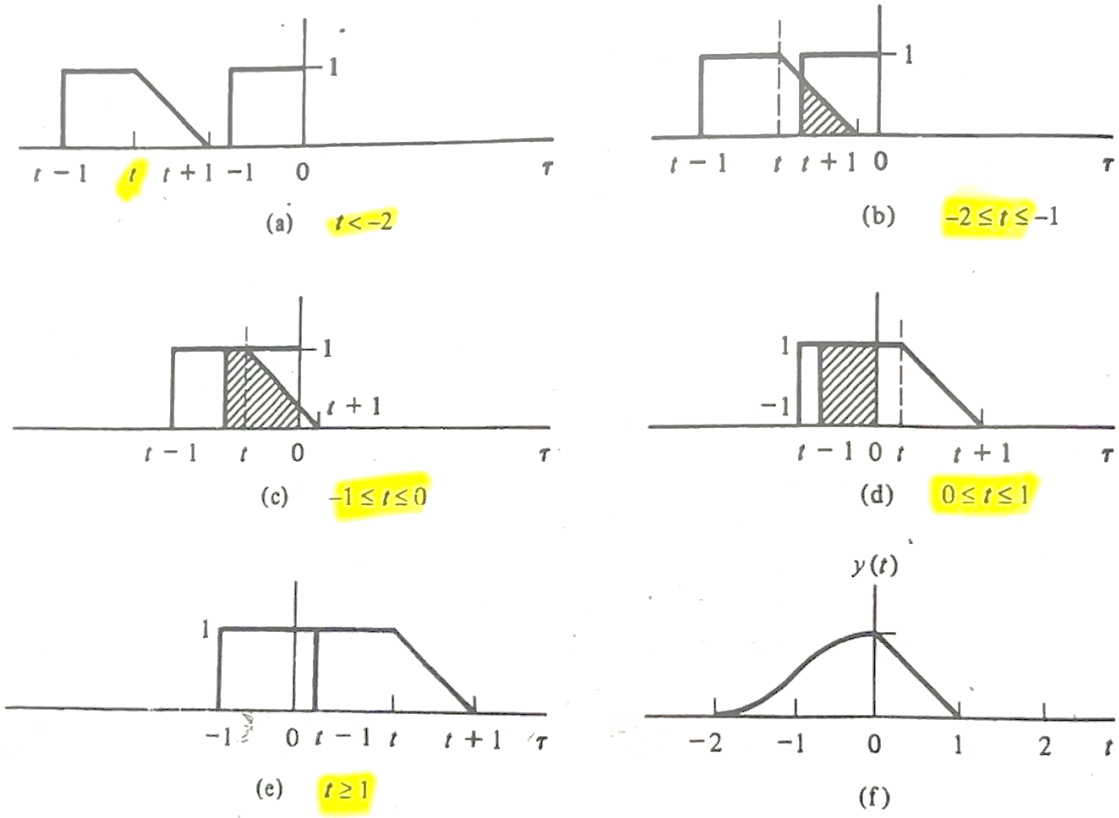


Example 05.

Let us compute the convolution $x(t) * h(t)$, where $x(t)$ and $h(t)$ are as follows:

Figure 2.3.5 demonstrates the overlapping of the two signals $x(\tau)$ and $h(t-\tau)$. We can see that for $t < -2$, the product $x(\tau)h(t-\tau)$ is always zero. For $-2 \le t < -1$, the product is a triangle with base $t+2$ and height $t+2$; therefore, the area is

$$y(t) = \tfrac{1}{2}(t+2)^2 \qquad -2 \le t < -1$$



For $-1 \le t < 0$, the product is shown in Figure 2.3.5(c), and the area is

$$y(t) = 1 - \tfrac{1}{2}t^2, \qquad -1 \le t < 0$$

For $0 \le t < 1$, the product is a rectangle with base $1 - t$ and height 1; therefore, the area is

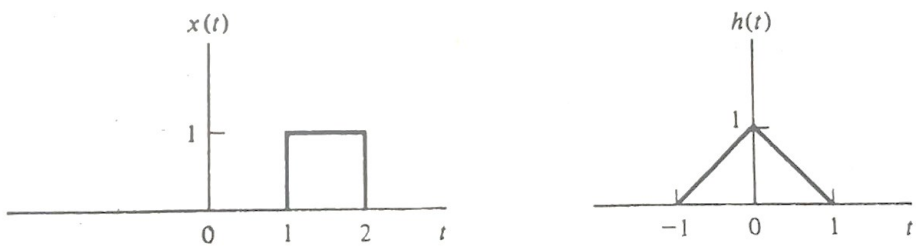
$$y(t) = 1 - t, \qquad 0 \le t < 1$$

For $t > 1$, the product is always zero. Summarizing, we have

$$y(t) = \begin{cases} 0, & t < -2 \\ \dfrac{(t+2)^2}{2}, & -2 \le t < -1 \\ 1 - \dfrac{t^2}{2}, & -1 \le t < 0 \\ 1 - t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

Example 06.

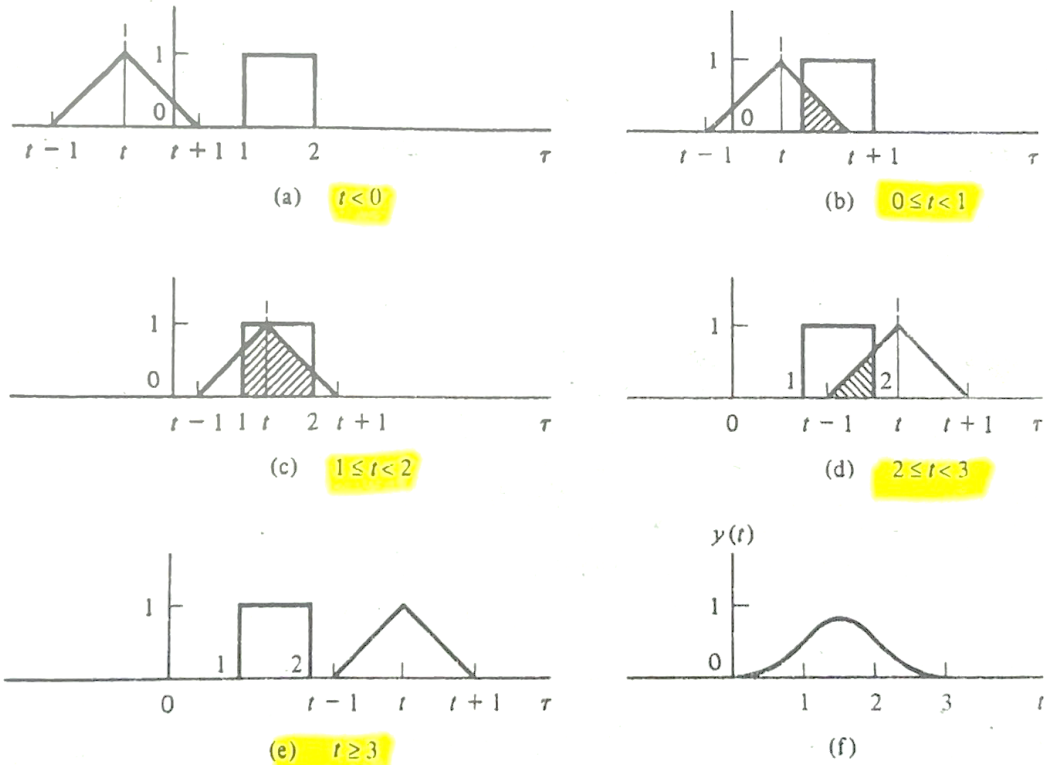
The convolution of the two signals shown in the following figure is evaluated using graphical interpretation.



From Figure 2.3.6, we can see that for $t < 0$, the product $x(\tau)h(t - \tau)$ is always zero for all t; therefore, $y(t) = 0$. For $0 \le t < 1$, the product is a triangle with base t and height t; therefore, $y(t) = t^2/2$. For $1 \le t < 2$, the area under the product is equal to

$$y(t) = 1 - [\tfrac{1}{2}(t-1)^2 + \tfrac{1}{2}(2-t)^2], \qquad 1 \le t < 2$$

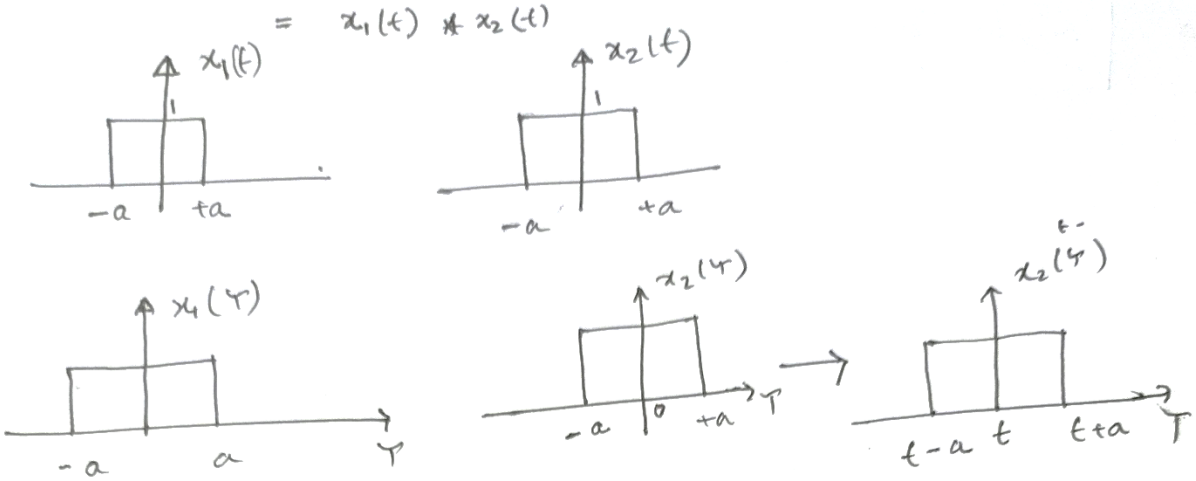
For $2 \le t < 3$, the product is a triangle with base $3 - t$ and height $3 - t$; therefore, $y(t) = (3 - t)^2/2$. For $t \ge 3$, the product $x(\tau)h(t - \tau)$ is always zero. Summarizing, we have



$$y(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^2}{2}, & 0 \le t < 1 \\ 3t - t^2 - \dfrac{3}{2}, & 1 \le t < 2 \\ \dfrac{(3-t)^2}{2}, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Exp 04

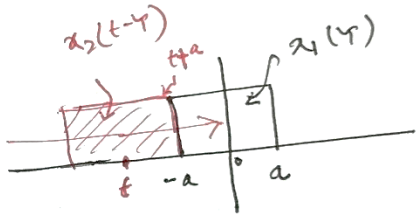
$$y(t) = \text{rect}(t/2a) * \text{rect}(t/2a)$$
$$= x_1(t) * x_2(t)$$



points of overlaps between $x_1(t)$ and $x_2(t)$:

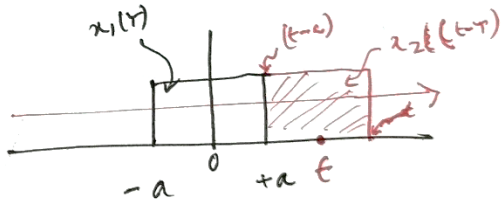
point-1: $(t+a) = -a$
[of $x_2(t-\tau)$] [of $x_1(\tau)$]

$\Rightarrow \boxed{t = -2a}$



point-2: $(t-a) = a$

$\Rightarrow \boxed{t = 2a}$



So, an overlap exists when $-2a \leq t \leq +2a$

when $-2a \leq t \leq 0$

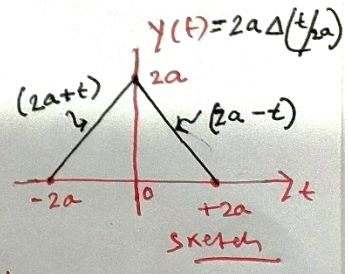
$$\int_{-2a}^{t} x_1(\tau)\, x_2(t-\tau)\, d\tau$$
$$= \int_{-2a}^{t} 1 * 1\, d\tau$$
$$= \tau \Big|_{-2a}^{t}$$
$$= t - (-2a)$$
$$= t + 2a$$

when $0 < t \leq 2a$

$$\int_{t}^{2a} x_1(\tau)\, x_2(t-\tau)\, d\tau$$
$$= \int_{t}^{2a} 1 \times 1\, d\tau$$
$$= 2a - t$$

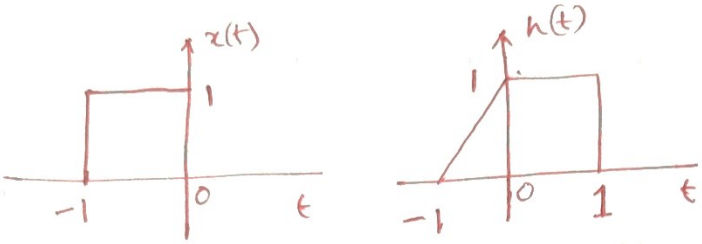
so, $y(t) = \begin{cases} 2a - |t|, & |t| \leq 2a \\ 0, & |t| \geq 2a \end{cases}$

$y(t) = 2a\,\Delta(t/2a)$ Ans.



Exp 05

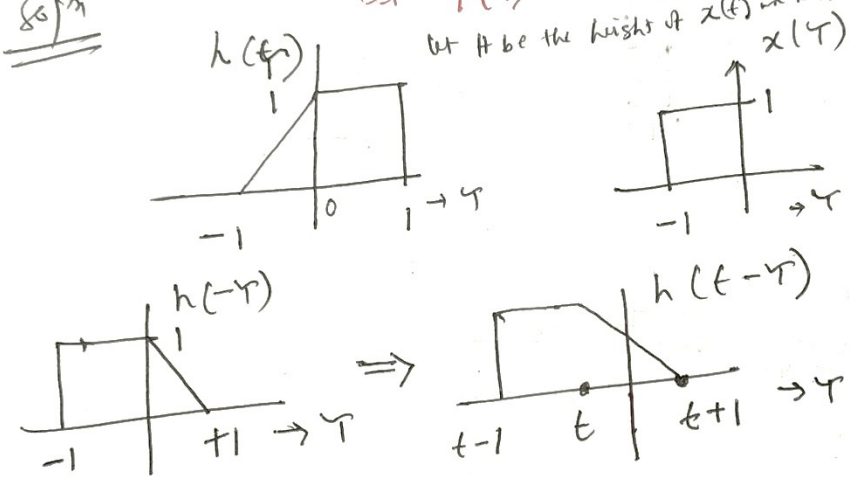
Compute the Convolution $x(t) * h(t)$, where $x(t)$ and $h(t)$ are as follow:



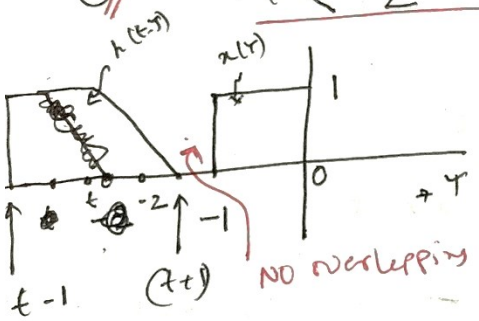
Soln

let $y(t) = x(t) * h(t)$

let H be the height of $x(t)$ and $h(t)$



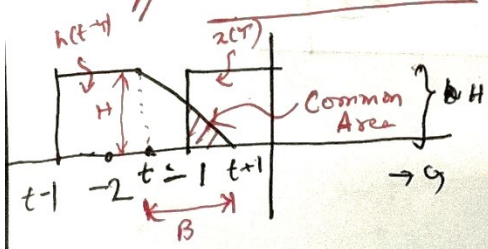
(a) $t < -2$



$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = 0}$$ [As there is no overlapping Area]

(b) $-2 \le t \le -1$



$$y(t) = \int_{-2}^{-1} x(\tau)\, h(t-\tau)\, d\tau$$

Here, at $t=-2$, $H=0$ and $t=-1$, $H=1$.

So, represent H in terms of 't', we can write

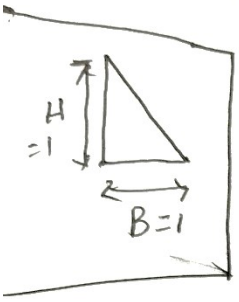
$$\boxed{H = t+2}$$ [Note: $H = \begin{cases} 0, & t=-2 \\ 1, & t=-1 \end{cases}$]

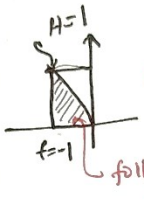
Since the height and base of the triangle are the same

i.e, $H = B = 1$

So, Base, $B = t+2$ [in terms of t]

Hence, The Area of overlapping $y(t) = \int_{-2}^{-1} h(t-\tau)\, x(\tau)\, d\tau$

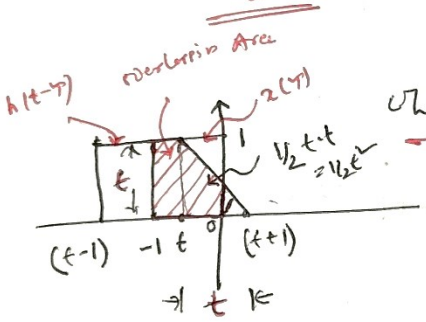




$= \frac{1}{2}(t+2) * (t+2)$

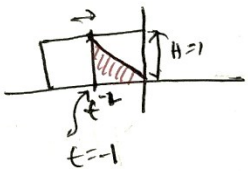
$\boxed{y(t) = (t+2)^2/2}$, for $-2 \le t \le -1$

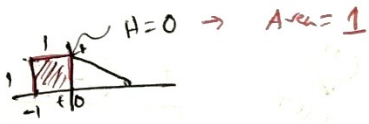
(c) When $-1 \le t \le 0$



at $t=-1$, effective height, $H = 1$

$t = 0$, " " $H = 0$



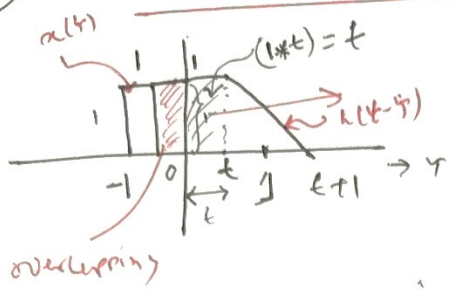


So, Area $= \int_{-1}^{0} h(t-\tau)\, x(\tau)\, d\tau$

$= 1 - (1/2\, t \cdot t)$

$\boxed{y(t) = 1 - t^2/2}$

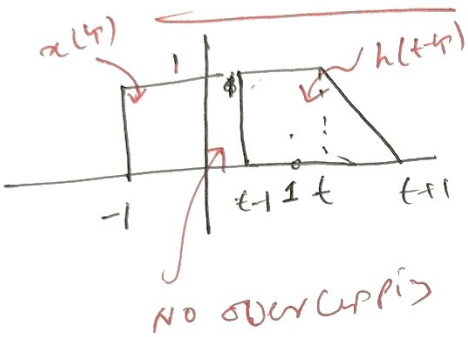
(d) $0 \le t \le 1$



$$\text{Area} = y(t) = \int_0^1 x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = 1 - t}$$

(e) $t \geq 1$



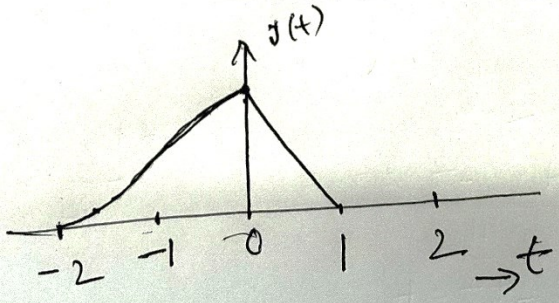
$$\text{Area} = y(t) = \int_1^t x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = 0}$$

So, $\boxed{y(t) = x(t) * h(t)}$

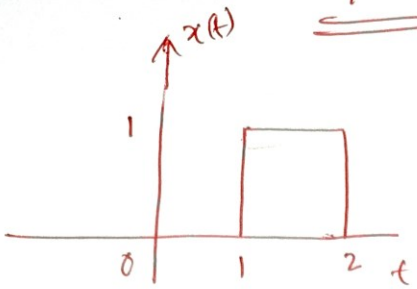
$$y(t) = \begin{cases} 0, & t < -2 \\ \frac{(t+2)^2}{2}, & -2 \le t < -1 \\ (1 - t^2/2), & -1 \le t < 0 \\ (1-t), & 0 \le t < 1 \\ 0 & t \geq 1 \end{cases}$$

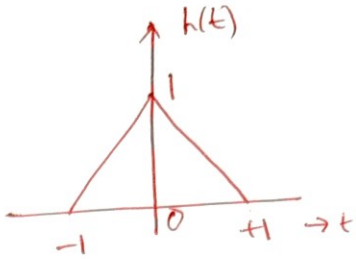
sketch

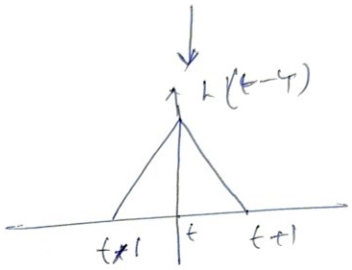


Exp 06

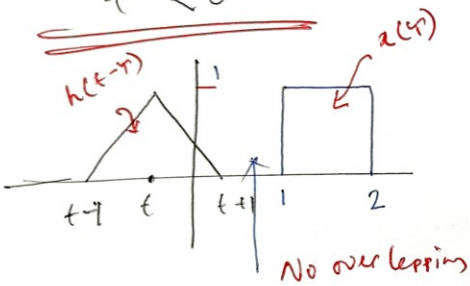
Compute $y(t) = x(t) * h(t)$







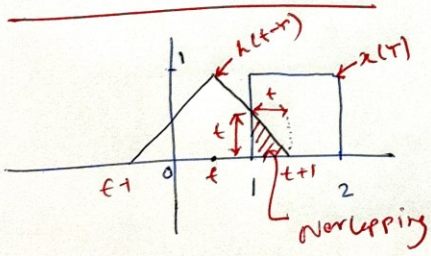
$t < 0$



So, The product of $x(t)$ and $h(t)$ is zero,

Hence, $\boxed{y(t) = 0}$ for $t < 0$

$0 \le t < 1$



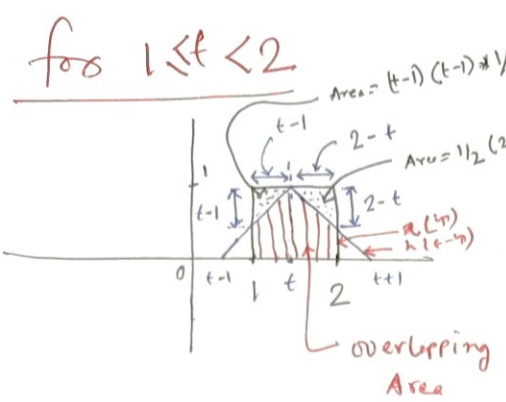
The product is a Triangle with base t and height t

So, $y(t) = \int_0^1 x(\tau)\, h(t-\tau)\, d\tau$

$= \frac{1}{2} \cdot t \cdot t$

$\boxed{y(t) = t^2/2}$; $0 \le t < 1$

for $1 \le t < 2$



Area Under the Product

$$y(t) = \underbrace{\left[1 * (2-1)\right]}_{\text{Area of the Square}} - \left[\frac{1}{2}(t-1)^2 + \frac{1}{2}(2-t)^2\right]$$

$$= 1 - \frac{1}{2}\left[(t-1)^2 + (2-t)^2\right]$$

$$= 1 - \frac{1}{2}\left[t^2 - 2t + 1 + 4 - 4t + t^2\right]$$

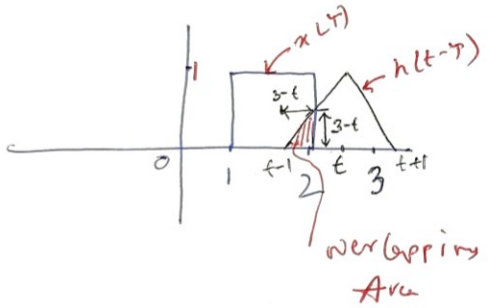
$$\boxed{y(t) = 1 - \frac{1}{2}\left[2t^2 - 6t + 5\right]} \quad 1 \le t < 2$$

$$= \frac{(2 - 2t^2 + 6t - 5)}{2} = \frac{1}{2}\left[-2t^2 + 6t - 3\right]$$

$$= -t^2 + 3t - 3/2$$

$$\boxed{y(t) = 3t - t^2 - 3/2}$$

for $2 \le t < 3$

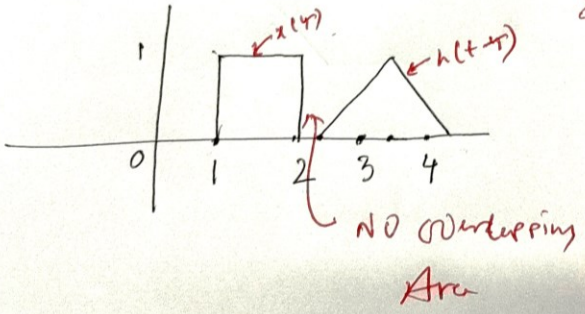


Area Under the Product

$$y(t) = \frac{1}{2}(3-t)(3-t)$$

$$\boxed{y(t) = (3-t)^2/2} \quad , \quad 2 \le t < 3$$

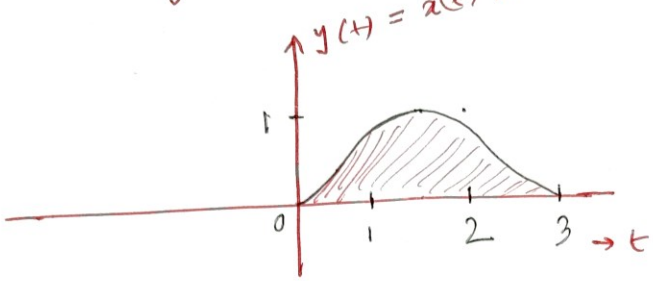
for $t \geq 3$



So, the Area Under the Product

$$\boxed{y(t) = 0} \quad , \quad t > 3$$

So, Summarizing, we have



~~And~~, $y(t) = \begin{cases} 0, & t<0 \\ t^2/2, & 0 \leq t < 1 \\ 3t - t^2 - 3/2, & 1 \leq t < 2 \\ \dfrac{(3-t)^2}{2}, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

Ans

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