EEE 243 Signals and Systems

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Lecture 06: Problems on Classifications of Systems

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Linearity

If an input consists of the weighted sum of several signals, then the output is the superposition, i.e., the weighted sum of the responses of the system to each of those signals.

Let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and $y_2(t)$ be the output corresponding to the input $x_2(t)$. Let $y_3(t)$ be the response to the weighted sum of $x_1(t)$ and $x_2(t)$, i.e., $x_3(t) = ax_1(t) + bx_2(t)$. Then, to verify whether a system is linear or not, carry out the following operations.

Operation 1. Find the signal $x_3(t)$, which is the weighted sum of $x_1(t)$ and $x_2(t)$ (i.e., $x_3(t) = ax_1(t) + bx_2(t)$).

Operation 2. Find the response to the signal $x_3(t)$ (i.e., $H(x_3(t)) = H(ax_1(t) + bx_2(t))$.

Operation 3. Now, find the responses of the system to $x_1(t)$ (i.e., $H(x_1(t))$) and $x_2(t)$ (i.e., $H(x_2(t))$)

Operation 4. Then, find the weighted sum of these responses in operation 3 (i.e., $aH(x_1(t)) + bH(x_2(t))$).

Operation 5. Now, check: $H(ax_1(t) + bx_2(t)) = aH(x_1(t)) + bH(x_2(t))$

If operation 5 exists, then the system is linear. Otherwise, it is a non-linear system.

Problem 01.

Consider a system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$
$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output may be expressed as

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

We conclude that the system S is linear.

Problem 02

Let us apply the linearity-checking procedure of the previous example to another system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

Defining $x_1(t)$, $x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \to y_1(t) = x_1^2(t)$$
$$x_2(t) \to y_2(t) = x_2^2(t)$$

and

$$\begin{aligned} x_3(t) \to y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

For $x_1(t)=1, x_2(t)=0, a=2, b=0$

$$H(x_3(t)) = x_3(t)^2 = (ax_1(t)+bx_2(t))^2 = a^2x_1(t)^2 + b^2x_2(t)^2 + 2abx_1(t)x_2(t)$$
$$H(ax_1(t)+bx_2(t)) = (2^2 \times 1^2) + 0 + 0 = 4$$

Whereas,

$$aH(x_1(t)) + bH(x_2(t)) = ax_1(t)^2 + bx_2(t)^2$$
$$aH(x_1(t)) + bH(x_2(t)) = 2 \times 1^2 + 0 = 2$$

Since $H(ax_1(t)+bx_2(t)) \neq aH(x_1(t)) + bH(x_2(t))$, the system is <u>non-linear.</u>

Note that, we can proof it as well by carrying out operations 3 and 4, i.e.,

$aH(x_1(t)) + bH(x_2(t)) = ax_1(t)^2 + bx_2(t)^2$, which is different from

$H(ax_1(t)+bx_2(t)) = a^2x_1(t)^2 + b^2x_2(t)^2 + 2abx_1(t)x_2(t)$.

Time Invariance

To demonstrate the time-invariance property, we have to show that the following two operations exist.

Step 1. Operation 1 (Delay in the output) Take an input signal $x_1(t)$ and find its output $y_1(t)$. Now delay the output $y_1(t)$ by *T,* which results in an output signal $y_1(t-T)$.

Step 2. Operation 2 (Delay in the input) Take the delayed version of the input signal $x_2(t)$, i.e., $x_2(t) = x_1(t-T)$, and find its (i.e., $x_2(t)$) corresponding output $y_2(t)$.

Step 3. Now check whether the following satisfies or not.

Output from operation 1=output from operation 2

i.e., $$y_1(t-T) = y_2(t)$$

If the above satisfies, the system is time-invariant. If otherwise, the system is time-varying.

Problem 01. Consider the system

$$\boldsymbol{y(t) = x(2t)}$$

Determine whether or not it is a time-invariant system?

Solution.

This system represents a time scaling. That is, y(t) is a time-compressed (by a factor of 2) version of x(t).

Intuitively, then, any time shift in the input will also be compressed by a factor of 2, and it is for this reason that the system is not time-invariant.

Proof: Let's do operation 1 (Delay in the output).

Consider the input $x_1(t)$ shown in Figure 1(a) and the resulting output $y_1(t)$ depicted in Figure 1(b).

Now delay the output $y_1(t)$ by $T=2$, which results in an output signal $y_1(t-2)$ as shown in Figure 1(c).

Now, do operation 2 (Delay in the input).

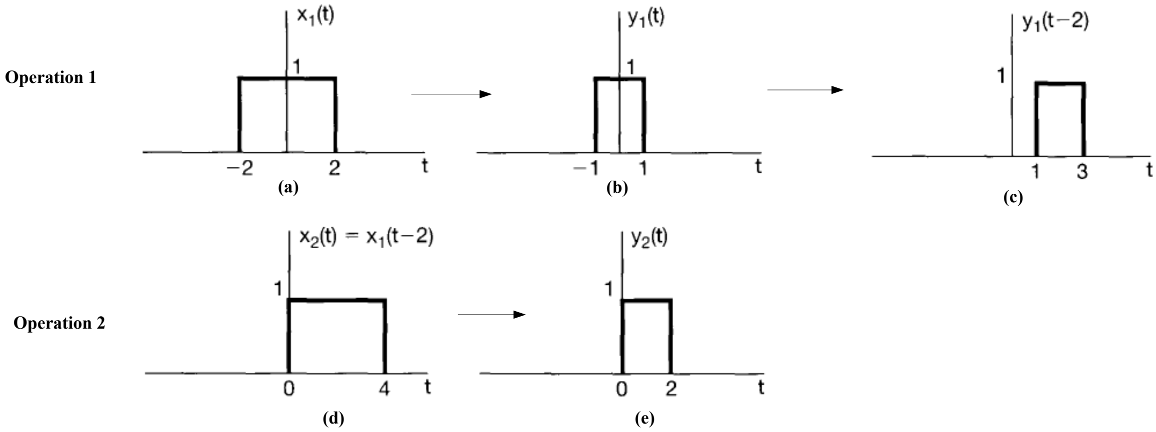
Shift the input $x_1(t)$ by 2, which results in an output $x_2(t) = x_1(t-2)$, as shown in Figure 1(d).

Now take the output $y_2(t)$ corresponding to $x_2(t)$ (passing through the system) as shown in figure 1(e).

Now if we compare the outputs of these two operations, we will see the following.

y2(t) $y_2\left(t\right) \neq y_1\left(t-2\right)$

This implies that the system is not time-invariant.



Time scaling operation :
Outputs $y_1(t)$ and $y_2(t)$ are scaled
(compressed by 2) since $y(t)=x(2t)$

Figure 1.

Stability

If we suspect that a system is unstable, then

(1) **Approach 1**. a useful strategy to verify this is to look for a specific bounded input that leads to an unbounded output. Finding one such example enables us to conclude that the given system is unstable.

(2) **Approach 2.** If such an example does not exist or is difficult to find, we must check for stability by using a method that does not utilize specific examples of input signals.

Problem 3. Let us check the stability of two systems,

(i) $S_1 : y(t) = t\,x(t)$ **(1.109) [using Approach 1]**

(ii) $S_2 : y(t) = e^{x(t)}$ **(1.110) [using Approach 2]**

Solution.

In seeking a specific counterexample in order to disprove stability, we might try simple bounded inputs such as a constant or a unit step. For system S_1 in eq. (1.109), a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded, since no matter what finite constant we pick, $|y(t)|$ will exceed that constant for some t. We conclude that system S_1 is unstable.

For system S_2, which happens to be stable, we would be unable to find a bounded input that results in an unbounded output. So we proceed to verify that all bounded inputs result in bounded outputs. Specifically, let B be an arbitrary positive number, and let $x(t)$ be an arbitrary signal bounded by B; that is, we are making no assumption about $x(t)$, except that

$$|x(t)| < B, \tag{1.111}$$

or

$$-B < x(t) < B, \tag{1.112}$$

for all t. Using the definition of S_2 in eq. (1.110), we then see that if $x(t)$ satisfies eq. (1.111), then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}. \tag{1.113}$$

We conclude that if any input to S_2 is bounded by an arbitrary positive number B, the corresponding output is guaranteed to be bounded by e^B. Thus, S_2 is stable.

Causality

A system is causal if the output at any time depends only on values of the input at the present time and in the past. All memory less systems are causal, since the output responds only to the current value of the input.

Problem 04. Determine whether the following system is causal or not

$$y[n] = x[-n].$$

Solution

Note that the output $y[n_0]$ at a positive time n_o depends only on the value of the input signal $x[-n_o]$ at time $(-n_o)$, which is negative and therefore in the past of n_o. We may be tempted to conclude at this point that the given system is causal.

However, we should always be careful to check the input-output relation for *all* times. In particular, for $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence, the system is not causal.

Note: It is also important to distinguish carefully the effects of the input from those of any other functions used in the definition of the system. For example, consider the system in Problem 05.

Problem 05. Determine whether the following system is causal or not

$$y(t) = x(t)\cos(t+1).$$

Solution

In this system, the output at any time t equals the input at that same time multiplied by a number that varies with time. Specifically, we can rewrite as

$$y(t) = x(t)g(t)$$

where $g(t)$ is a time-varying function, namely $g(t) = \cos(t+1)$.

Thus, only the current value of the input $x(t)$ influences the current value of the output $y(t)$, and we conclude that this system is causal (and, in fact, memoryless).

Practice problem 01. Determine the causality of the following systems

(i) $y(t) = x(t+1)$

(ii) $y[n] = x[n] - x[n+1]$

Invertibility

If a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system (Figure 2a).

Problem 06. Determine the inverse system for the invertibility of the following systems.

(i) $y(t) = 2x(t)$

(ii) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(iii) $y[n] = 0$

Solution

(i) The system is invertible, and the inverse system $w(t) = \frac{1}{2}y(t) = x(t)$ (Figure 2b)

(ii) For this system, the difference between two successive values of the output is precisely the last input value. Therefore, the system is invertible and in this case the inverse system is $w[n] = y[n] - y[n-1]$ as shown in (Figure 2c).

(iii) since the system produces the zero-output sequence for any input sequence, we cannot retrieve the original input signal irrespective of the inverse system. Hence, $y[n] = 0$ is noninvertible.

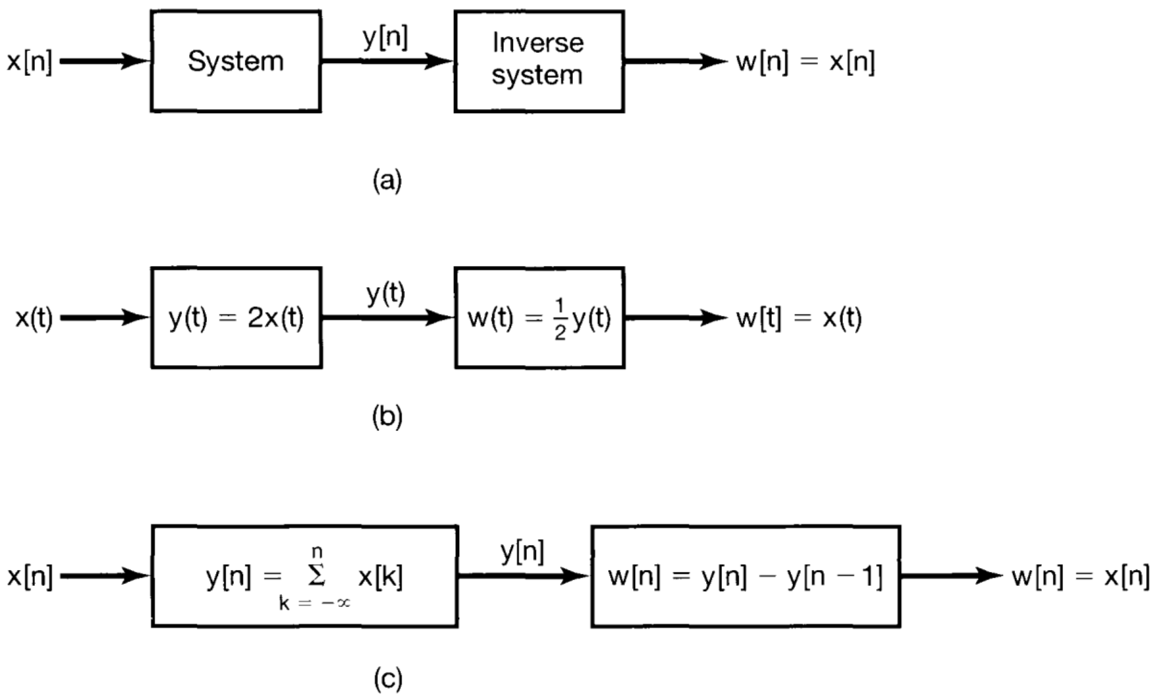


Figure 2.

Memoryless

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent only on the input at that same time.

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times.

Problem 07. Determine whether or not the following systems are memoryless.

[1] $y[n] = \left(2x[n] - x^2[n]\right)^2$

[2] $y(t) = Rx(t)$

[3] $y[n] = x[n]$

[4] $y[n] = \sum_{k=-\infty}^{n} x[k]$

[5] $y[n] = x[n-1]$

[6] $y(t) = \dfrac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$

Solution.

[1] Memoryless, as the value of *y[n]* at any particular time n_0 depends only on the value of *x[n]* at that time.
[2] A resistor is a memoryless system; with the input *x(t)* taken as the current and with the voltage taken as the output *y(t).*
[3] Memoryless system since the output is identical to its input.
[4] it is a discrete-time system with memory, also called an *accumulator* or *summer.*
[5] with memory since the output is the delayed version of the input.
[6] A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage. Hence, the system is with memory.