EEE 243 Signals and Systems

2022

Lecture 07: Problems on Elementary Signals

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Signals Described in Math Form

Consider the network of Figure 1.1 where the switch is closed at time $t = 0$.

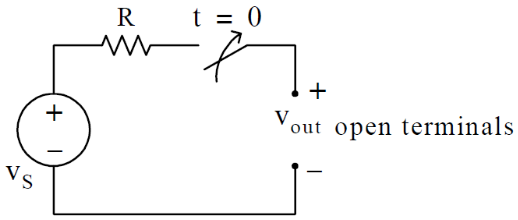

Figure 1.1. A switched network with open terminals

We wish to describe v_{out} in a math form for the time interval $-\infty < t < +\infty$. To do this, it is convenient to divide the time interval into two parts, $-\infty < t < 0$, and $0 < t < \infty$.

For the time interval $-\infty < t < 0$, the switch is open and therefore, the output voltage v_{out} is zero. In other words,

$$v_{out} = 0 \quad \text{for} \quad -\infty < t < 0 \tag{1.1}$$

For the time interval $0 < t < \infty$, the switch is closed. Then, the input voltage v_S appears at the output, i.e.,

$$v_{out} = v_S \quad \text{for} \quad 0 < t < \infty \tag{1.2}$$

Combining (1.1) and (1.2) into a single relationship, we obtain

$$\boxed{v_{out} = \begin{cases} 0 & -\infty < t < 0 \\ v_S & 0 < t < \infty \end{cases}} \tag{1.3}$$

We can express (1.3) by the waveform shown in Figure 1.2.

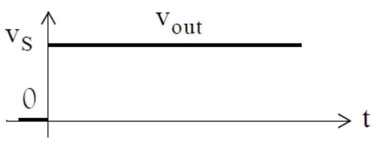

Figure 1.2. Waveform for v_{out} as defined in relation (1.3)

The waveform of Figure 1.2 is an example of a discontinuous function. A function is said to be *discontinuous* if it exhibits points of discontinuity, that is, the function jumps from one value to another without taking on any intermediate values.

The Unit Step Function

A well known discontinuous function is the *unit step function* $u_0(t)$ * which is defined as

$$\boxed{u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}} \tag{1.4}$$

It is also represented by the waveform of Figure 1.3.

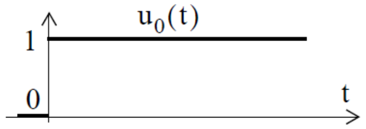

Figure 1.3. Waveform for $u_0(t)$

In the waveform of Figure 1.3, the unit step function $u_0(t)$ changes abruptly from 0 to 1 at $t = 0$. But if it changes at $t = t_0$ instead, it is denoted as $u_0(t - t_0)$. In this case, its waveform and definition are as shown in Figure 1.4 and relation (1.5) respectively.

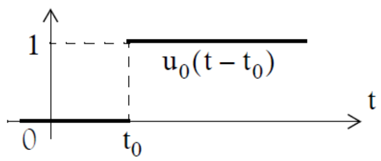

Figure 1.4. Waveform for $u_0(t - t_0)$

$$u_0(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \tag{1.5}$$

If the unit step function changes abruptly from 0 to 1 at $t = -t_0$, it is denoted as $u_0(t + t_0)$. In this case, its waveform and definition are as shown in Figure 1.5 and relation (1.6) respectively.

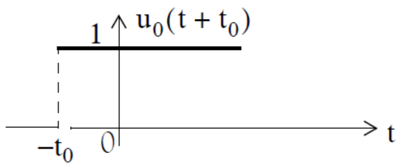

Figure 1.5. Waveform for $u_0(t + t_0)$

$$u_0(t + t_0) = \begin{cases} 0 & t < -t_0 \\ 1 & t > -t_0 \end{cases} \tag{1.6}$$

Example 1.1

Consider the network of Figure 1.6, where the switch is closed at time $t = T$.

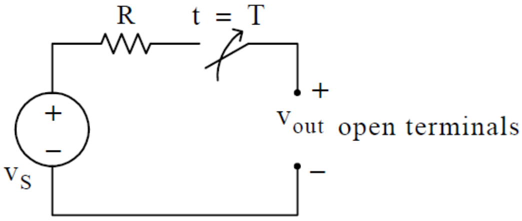

Figure 1.6. Network for Example 1.1

Express the output voltage v_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution:

For this example, the output voltage $v_{out} = 0$ for $t < T$, and $v_{out} = v_S$ for $t > T$. Therefore,

$$v_{out} = v_S u_0(t - T) \tag{1.7}$$

and the waveform is shown in Figure 1.7.

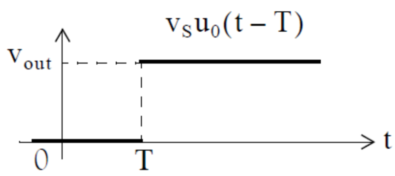

Figure 1.7. Waveform for Example 1.1

Other forms of the unit step function are shown in Figure 1.8.

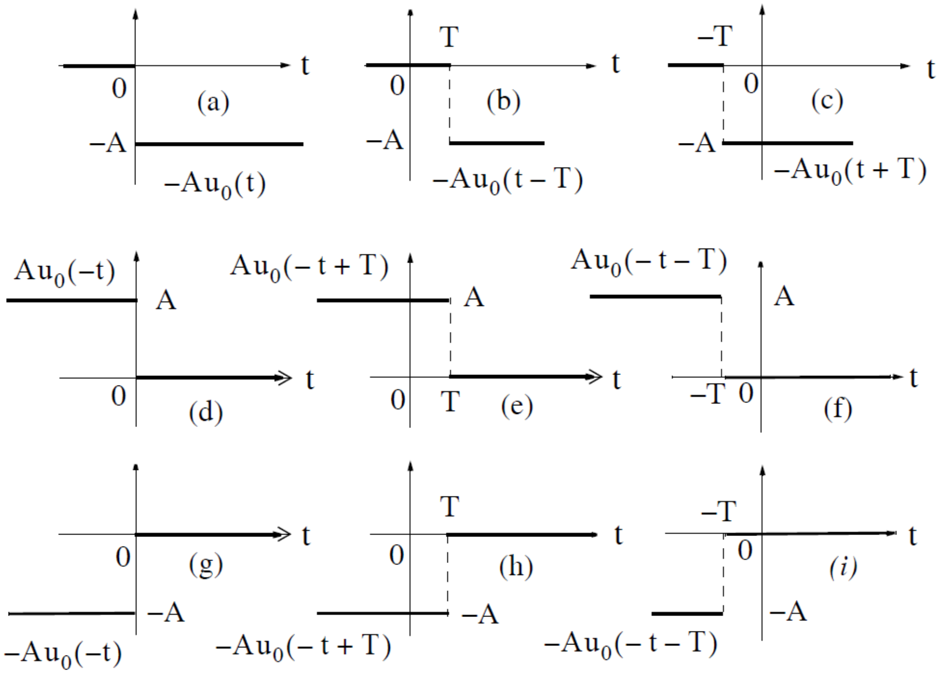

Figure 1.8. Other forms of the unit step function

Unit step functions can be used to represent other time–varying functions such as the rectangular pulse shown in Figure 1.9.

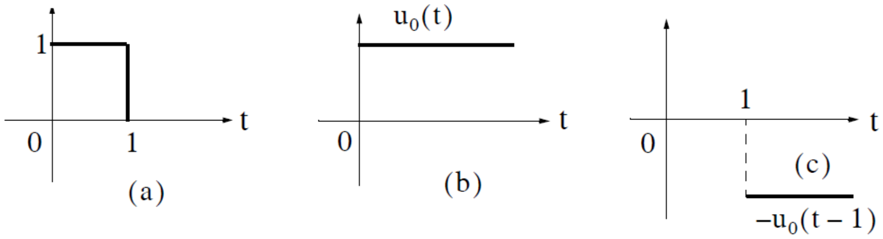

Figure 1.9. A rectangular pulse expressed as the sum of two unit step functions

Thus, the pulse of Figure 1.9(a) is the sum of the unit step functions of Figures 1.9(b) and 1.9(c) and it is represented as $u_0(t) - u_0(t-1)$.

The unit step function offers a convenient method of describing the sudden application of a voltage or current source. For example, a constant voltage source of $24\ \text{V}$ applied at $t = 0$, can be denoted as $24u_0(t)\ \text{V}$. Likewise, a sinusoidal voltage source $v(t) = V_m \cos\omega t\ \text{V}$ that is applied to a circuit at $t = t_0$, can be described as $v(t) = (V_m \cos\omega t)u_0(t - t_0)\ \text{V}$. Also, if the excitation in a circuit is a rectangular, or triangular, or sawtooth, or any other recurring pulse, it can be represented as a sum (difference) of unit step functions.

Express the square waveform of Figure 1.10 as a sum of unit step functions. The vertical dotted lines indicate the discontinuities at $T, 2T, 3T$, and so on.

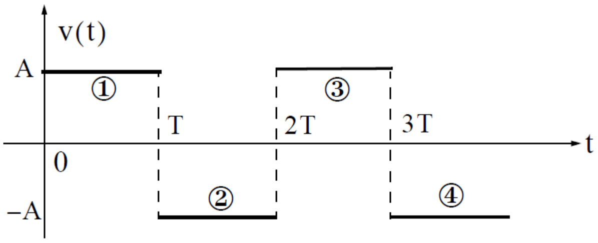

Figure 1.10. Square waveform for Example 1.2

Solution:

Line segment ① has height A, starts at $t = 0$, and terminates at $t = T$. Then, as in Example 1.1, this segment is expressed as

$$v_1(t) = A[u_0(t) - u_0(t-T)] \tag{1.8}$$

Line segment ② has height $-A$, starts at $t = T$ and terminates at $t = 2T$. This segment is expressed as

$$v_2(t) = -A[u_0(t-T) - u_0(t-2T)] \tag{1.9}$$

Line segment ③ has height A, starts at $t = 2T$ and terminates at $t = 3T$. This segment is expressed as

$$v_3(t) = A[u_0(t-2T) - u_0(t-3T)] \tag{1.10}$$

Line segment ④ has height $-A$, starts at $t = 3T$, and terminates at $t = 4T$. It is expressed as

$$v_4(t) = -A[u_0(t-3T) - u_0(t-4T)] \tag{1.11}$$

Example 1.2

Express the square waveform of Figure 1.10 as a sum of unit step functions. The vertical dotted lines indicate the discontinuities at $T, 2T, 3T$, and so on.

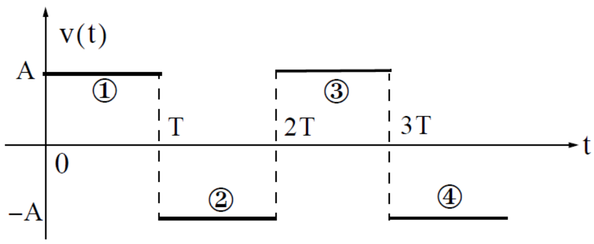

Figure 1.10. Square waveform for Example 1.2

Solution:

Line segment ① has height A, starts at $t = 0$, and terminates at $t = T$. Then, as in Example 1.1, this segment is expressed as

$$v_1(t) = A[u_0(t) - u_0(t-T)] \tag{1.8}$$

Line segment ② has height $-A$, starts at $t = T$ and terminates at $t = 2T$. This segment is expressed as

$$v_2(t) = -A[u_0(t-T) - u_0(t-2T)] \tag{1.9}$$

Line segment ③ has height A, starts at $t = 2T$ and terminates at $t = 3T$. This segment is expressed as

$$v_3(t) = A[u_0(t-2T) - u_0(t-3T)] \tag{1.10}$$

Line segment ④ has height $-A$, starts at $t = 3T$, and terminates at $t = 4T$. It is expressed as

$$v_4(t) = -A[u_0(t-3T) - u_0(t-4T)] \tag{1.11}$$

Thus, the square waveform of Figure 1.10 can be expressed as the summation of (1.8) through (1.11), that is,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) + v_4(t) \\ &= A[u_0(t) - u_0(t-T)] - A[u_0(t-T) - u_0(t-2T)] \\ &\quad + A[u_0(t-2T) - u_0(t-3T)] - A[u_0(t-3T) - u_0(t-4T)] \end{aligned} \tag{1.12}$$

Combining like terms, we obtain

$$v(t) = A[u_0(t) - 2u_0(t-T) + 2u_0(t-2T) - 2u_0(t-3T) + \dots] \tag{1.13}$$

Example 1.3

Express the symmetric rectangular pulse of Figure 1.11 as a sum of unit step functions.

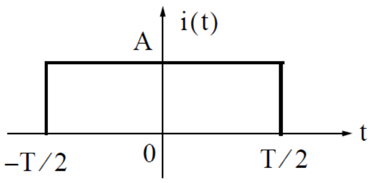

Figure 1.11. Symmetric rectangular pulse for Example 1.3

Solution:

This pulse has height A, starts at $t = -T/2$, and terminates at $t = T/2$. Therefore, with reference to Figures 1.5 and 1.8 (b), we obtain

$$i(t) = Au_0\left(t + \frac{T}{2}\right) - Au_0\left(t - \frac{T}{2}\right) = A\left[u_0\left(t + \frac{T}{2}\right) - u_0\left(t - \frac{T}{2}\right)\right] \tag{1.14}$$

Example 1.4

Express the symmetric triangular waveform of Figure 1.12 as a sum of unit step functions.

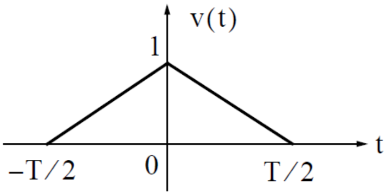

Figure 1.12. Symmetric triangular waveform for Example 1.4

Solution:

We first derive the equations for the linear segments ① and ② shown in Figure 1.13.

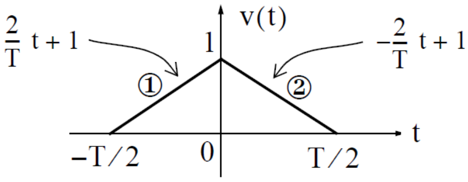

Figure 1.13. Equations for the linear segments of Figure 1.12

For line segment ①,

$$v_1(t) = \left(\frac{2}{T}t + 1\right)\left[u_0\left(t + \frac{T}{2}\right) - u_0(t)\right] \tag{1.15}$$

and for line segment ②,

$$v_2(t) = \left(-\frac{2}{T}t + 1\right)\left[u_0(t) - u_0\left(t - \frac{T}{2}\right)\right] \tag{1.16}$$

Combining (1.15) and (1.16), we obtain

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= \left(\frac{2}{T}t + 1\right)\left[u_0\left(t + \frac{T}{2}\right) - u_0(t)\right] + \left(-\frac{2}{T}t + 1\right)\left[u_0(t) - u_0\left(t - \frac{T}{2}\right)\right] \end{aligned} \tag{1.17}$$

Example 1.5

Express the waveform of Figure 1.14 as a sum of unit step functions.

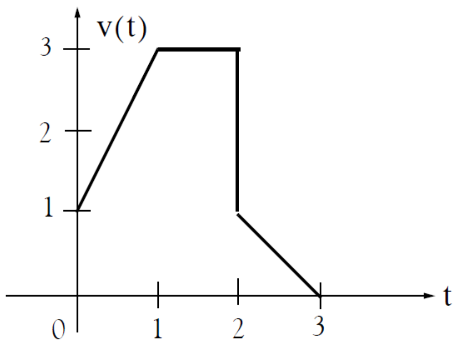

Figure 1.14. Waveform for Example 1.5

Solution:

As in the previous example, we first find the equations of the linear segments linear segments ① and ② shown in Figure 1.15.

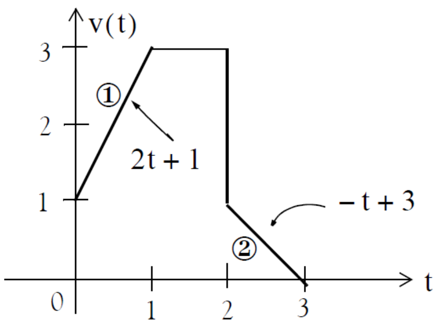

Figure 1.15. Equations for the linear segments of Figure 1.14

Following the same procedure as in the previous examples, we obtain

$$\begin{aligned} v(t) &= (2t+1)[u_0(t) - u_0(t-1)] + 3[u_0(t-1) - u_0(t-2)] \\ &\quad + (-t+3)[u_0(t-2) - u_0(t-3)] \end{aligned}$$

Multiplying the values in parentheses by the values in the brackets, we obtain

$$\begin{aligned} v(t) &= (2t+1)u_0(t) - (2t+1)u_0(t-1) + 3u_0(t-1) \\ &\quad - 3u_0(t-2) + (-t+3)u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

$$\begin{aligned} v(t) &= (2t+1)u_0(t) + [-(2t+1) + 3]u_0(t-1) \\ &\quad + [-3 + (-t+3)]u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

and combining terms inside the brackets, we obtain

$$v(t) = (2t+1)u_0(t) - 2(t-1)u_0(t-1) - tu_0(t-2) + (t-3)u_0(t-3) \tag{1.18}$$

The ramp function

The *unit ramp function*, denoted as $u_1(t)$, is defined as (shown in Figure below)

$$\boxed{u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}}$$

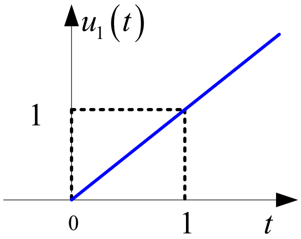

Figure. The ramp function.

From the above figure, we can obtain the ramp function by integrating the unit step function, which can be given by,

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau$$

where τ is a dummy variable.

Since $u_1(t)$ is the integral of $u_0(t)$, then $u_0(t)$ must be the derivative of $u_1(t)$, i.e.,

$$\boxed{\frac{d}{dt}u_1(t) = u_0(t)}$$

Example 1.6

In the network of Figure 1.16 i_S is a constant current source and the switch is closed at time $t = 0$. Express the capacitor voltage $v_C(t)$ as a function of the unit step.

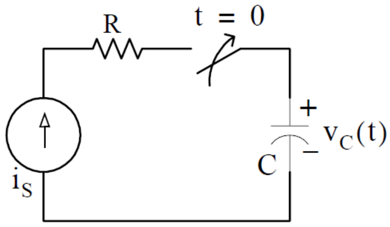

Figure 1.16. Network for Example 1.6

Solution:

The current through the capacitor is $i_C(t) = i_S = \text{constant}$, and the capacitor voltage $v_C(t)$ is

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau^{*} \tag{1.19}$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_C(t)$ as

$$i_C(t) = i_S\, u_0(t) \tag{1.20}$$

and assuming that $v_C(t) = 0$ for $t < 0$, we can write (1.19) as

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_S\, u_0(\tau)d\tau = \underbrace{\frac{i_S}{C}\int_{-\infty}^{0} u_0(\tau)\,d\tau}_{0} + \frac{i_S}{C}\int_{0}^{t} u_0(\tau)d\tau \tag{1.21}$$

or

$$\boxed{v_C(t) = \frac{i_S}{C}\,tu_0(t)} \tag{1.22}$$

Therefore, we see that when a capacitor is charged with a constant current, the voltage across it is a linear function and forms a *ramp* with slope i_S / C as shown in Figure 1.17.

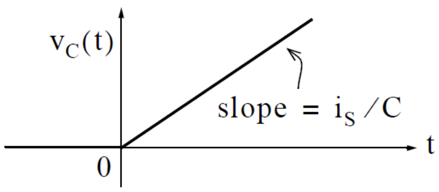

Figure 1.17. Voltage across a capacitor when charged with a constant current source

Example 1.7

In the network of Figure 1.19, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function.

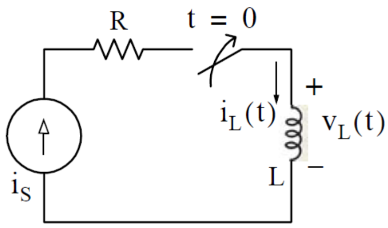

Figure 1.19. Network for Example 1.7

Solution:

The voltage across the inductor is

$$v_L(t) = L\frac{di_L}{dt} \tag{1.30}$$

and since the switch closes at $t = 0$,

$$i_L(t) = i_S u_0(t) \tag{1.31}$$

Therefore, we can write (1.30) as

$$\boxed{v_L(t) = Li_S\frac{d}{dt}u_0(t)} \tag{1.32}$$

Example 1.9

Express the voltage waveform $v(t)$ shown in Figure 1.21 as a sum of unit step functions for the time interval $-1 < t < 7$ s.

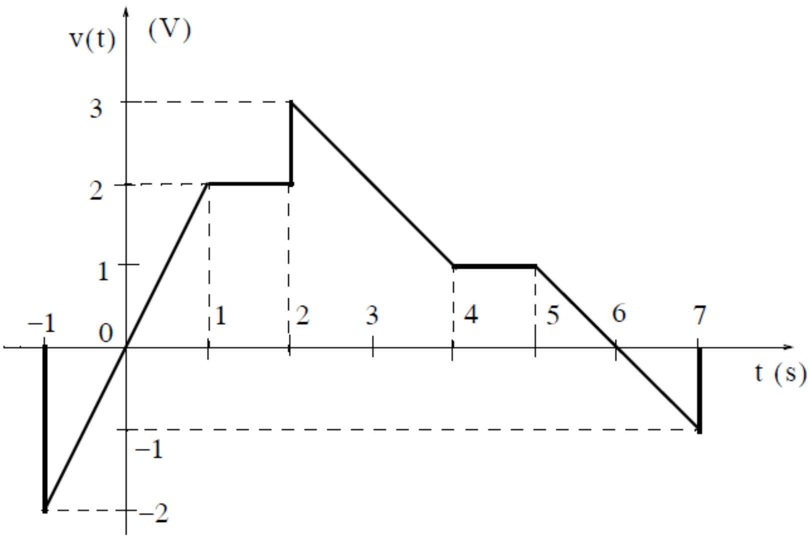

Figure 1.21. Waveform for Example 1.9

Solution:

We begin with the derivation of the equations for the linear segments of the given waveform as shown in Figure 1.22.

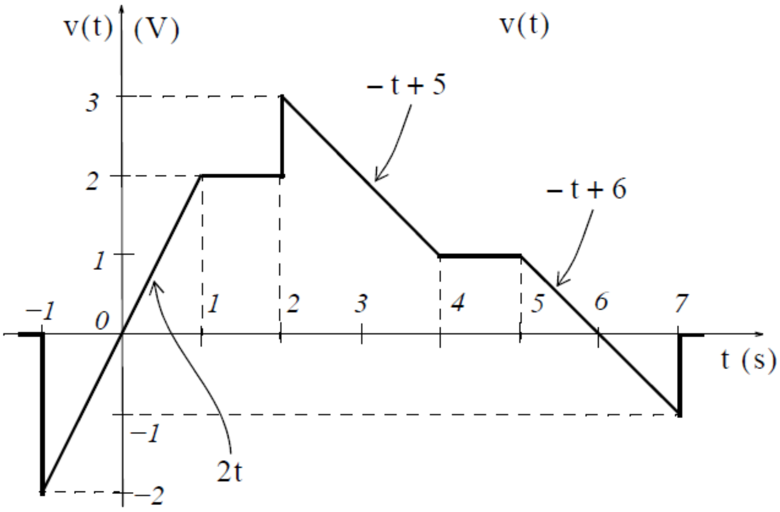

Figure 1.22. Equations for the linear segments of Figure 1.21

Next, we express $v(t)$ in terms of the unit step function $u_0(t)$, and we obtain

$$\begin{aligned} v(t) \;=\; & 2t[u_0(t+1) - u_0(t-1)] + 2[u_0(t-1) - u_0(t-2)] \\ & + (-t+5)[u_0(t-2) - u_0(t-4)] + [u_0(t-4) - u_0(t-5)] \\ & + (-t+6)[u_0(t-5) - u_0(t-7)] \end{aligned} \tag{1.52}$$

Example 1.10

Consider the following signum function (i.e., sgn) shown in the following figure. The unit sgn function is defined by

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

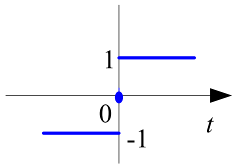

Figure. sgn function.

Express the above function in terms of the unit step function.

Solution

The above sgn function in terms of the unit step function can be expressed as follows.

$$\operatorname{sgn} t = -1 + 2u(t)$$

Summary.

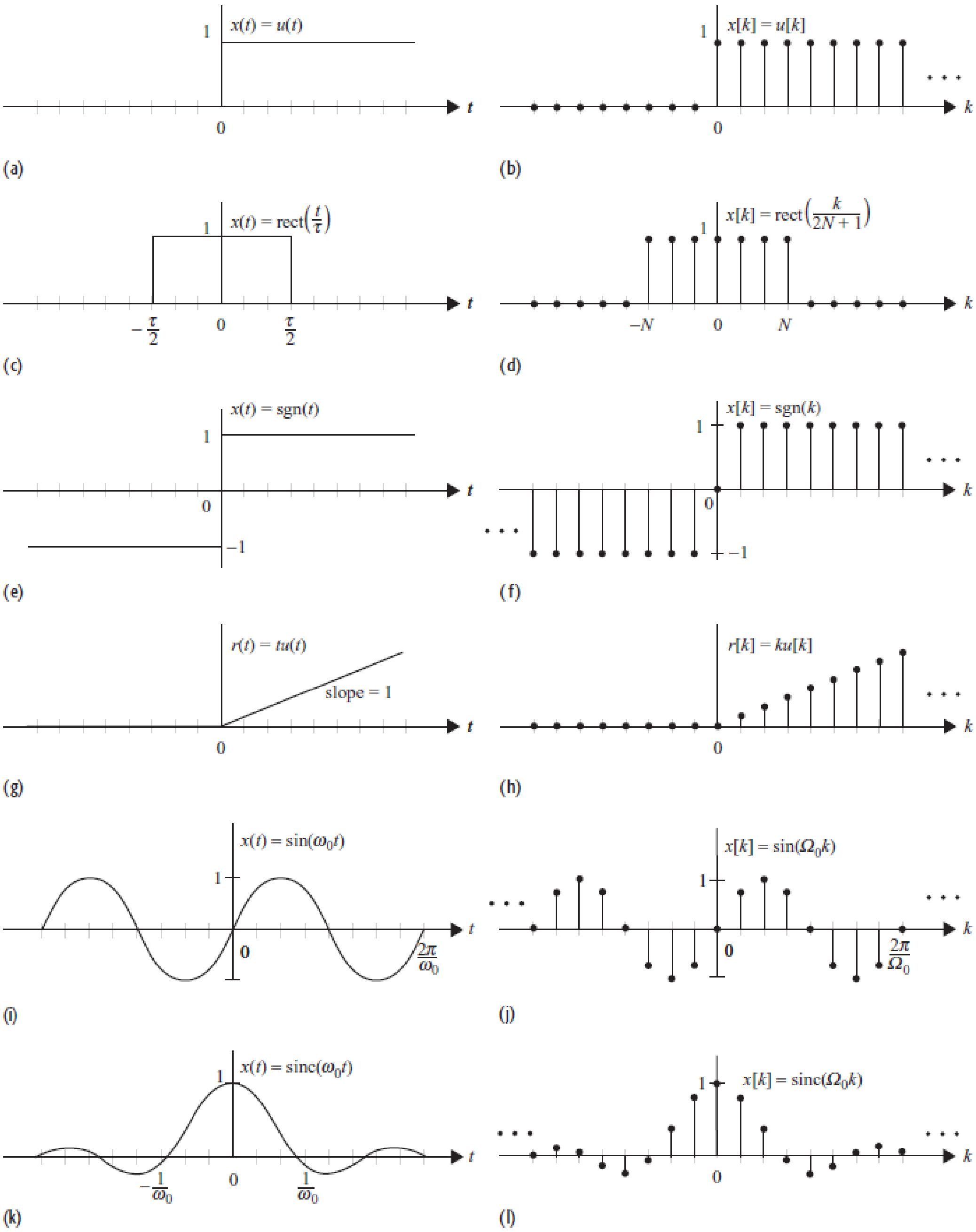

Fig. 1.12. CT and DT elementary functions. (a) CT and (b) DT unit step functions. (c) CT and (d) DT rectangular pulses. (e) CT and (f) DT signum functions. (g) CT and (h) DT ramp functions. (i) CT and (j) DT sinusoidal functions. (k) CT and (l) DT sinc functions.

References

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