Differentiable Overlap

We consider closed shape as a sequence of smooth curves $\mathscr{C}$. Define the indicator function I denoting whether a pixel is Inside a given closed shape.

$$I(q_x, q_y) = \sum_{L \in \mathscr{C}} \oint_L H(x_t - q_x) \, . \, \delta(y_t - q_y) \, . \, dy \tag{1}$$

Here, (q_x, q_y) is the coordinates of the pixel of interest, H is the heavy-sided function and δ is the Dirac Delta function.

We will consider approximation of H, δ to make I smooth. To that effect, equation 1. can be rewritten as:

$$I(q_x, q_y) = \sum_{L \in \mathscr{C}} \alpha \, . \oint_L \frac{exp\Big(-\frac{1}{2}\big(\frac{y - q_y}{\sigma}\big)^2 \Big)}{1 - exp\big(-\frac{(x - q_x)}{\lambda} \big)} \, . \, dy \tag{2}$$

δ is approximated with Gaussian, and H is approximated with sigmoid. The quantities σ, λ is for tuning the approximation.

The overlap function of two curves $\mathscr{C}_1$ and $\mathscr{C}_2$ is defined as:

$$\Omega(\mathscr{C}_1, \mathscr{C}_2) = \sum_{q_x, q_y} I_1(q_x, q_y) \, . \, I_2(q_x, q_y)$$

Where I_1, I_2 is the indicator function for $\mathscr{C}_1, \mathscr{C}_2$, respectively.

We are interested in the gradient of Ω with respect to change in the control points of one of the curve, say $\mathscr{C}_1$. In essence we aim to compute $\dfrac{\partial \Omega}{\partial \theta}$, where θ represent the variable parameter that controls the curve. Which boils down to computation of

$$\frac{\partial I(q_x, q_y)}{\partial \theta} = \sum_L \oint_L \frac{\partial H}{\partial \theta} \, . \, \delta \, . \, dy + H \, . \, \frac{\partial \delta}{\partial \theta} \, . \, dy + H \, . \, \delta \, . \, \frac{\partial dy}{\partial \theta} \tag{3}$$

We have,

$$\frac{\partial H(x - q_x)}{\partial \theta} = \delta(x - q_x) \, . \, \frac{\partial (x - q_x)}{\partial \theta} \tag{4}$$

And,

$$\frac{\partial \delta(y - q_y)}{\partial \theta} = -\frac{1}{\sigma^2} \, . \, \delta(y - q_y) \, . \, (y - q_y) \, . \, \frac{\partial y}{\partial \theta}.$$ (assuming δ to be Gaussian with s.d. σ, mean 0)

Lets look at each of the derivatives under the line integral with respect to curve L:

$$D1 = \oint_L \delta(x_t - q_x) \,.\, \delta(y_t - q_y) \,.\, \frac{\partial x_t}{\partial \theta} \,.\, dy \tag{6}$$

For (q_x, q_y) far from the curve, I.e. $|x - q_x| \gg \sigma$ and $|y_t - q_y| \gg \sigma$ the value of D_1 is close to zero. Hence the effective contribution of D_1 to the line integral happens if (q_x, q_y) is in a close band spanning about L.

$$D_2 = -\frac{1}{\sigma^2} \oint_L H(x - q_x) \,.\, \delta(y - q_y) \,.\, (y - q_y) \,.\, \frac{\partial y}{\partial \theta} \,.\, dy \tag{7}$$

If $|y - q_y| \gg \sigma$, D2 is rendered zero. For $|y - q_y| \sim \sigma$ and $x \gg q_x$ we get,

$D_2 = -\dfrac{1}{\sigma^2} \oint \delta(y - q_y) \,.\, (y - q_y) \,.\, \dfrac{\partial y}{\partial \theta} \,.\, dy$, in the close region around q_y. Else if (q_x, q_y) is close to L, then D_2 is as equation as (7).

The idea is to run a gradient descent on Ω by varying θ which maximises the overlap.