A Unified Control-Theoretic Framework for Stabilized Navier–Stokes Dynamics: From Vanishing Feedback in Two Dimensions to Entropy-Regulated Three-Dimensional Flows

Paul M. Roe
TecKnows Inc., Maine, USA
E-mail: TecKnows.Maine@gmail.com
ORCID: 0009-0005-6159-947X

Abstract

This paper develops a unified analytical and computational framework for stabilized incompressible Navier–Stokes dynamics by introducing a general class of feedback operators that augment the nonlinear convective term. Three contributions are made.

(1) A rigorous, parameterized family of stabilizing operators is introduced and shown to satisfy coercivity, local boundedness, weak sequential continuity, a vanishing-feedback property, and (in many cases) monotonicity on bounded sets. These assumptions guarantee global existence and—for monotone operators—uniqueness for the controlled two-dimensional Navier–Stokes equations.

(2) A vanishing-feedback limit theorem is proven: as the feedback strength $\varepsilon \to 0$, solutions of the controlled two-dimensional equations converge strongly to the unique strong solution of the classical uncontrolled system. This provides a mathematically clean bridge between controlled and physical flows.

(3) Motivated by entropy-production and dissipation-balance considerations in turbulent flows, a class of entropy-motivated stabilizing operators is introduced and shown to fit within the same analytical framework. Extensions to the three-dimensional equations are discussed, and numerical experiments demonstrate suppression of high-frequency energy transfer and improved regularity in controlled dynamics.

Taken together, these results form a coherent foundation for viewing nonlinear feedback—including variationally derived and dissipation-driven mechanisms—as analytically tractable and computationally effective tools for stabilizing Navier–Stokes dynamics.

1 Introduction

The incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0, \tag{1.1}$$

serve as both a mathematical model for fluid motion and a testbed for stabilization and control. In two dimensions, global strong solutions are classical; in three dimensions, global regularity remains open. A wide range of regularization and feedback techniques—Leray filtering, Voigt-type smoothing, spectral damping, and entropy-based dissipation—modify the equations to suppress high-frequency growth. Yet these approaches are typically studied in isolation, and most alter the physical structure of the PDE.

This work introduces a unified operator-theoretic framework for controlled Navier–Stokes dynamics. We study nonlinear feedback operators $F_\varepsilon[u] : H^1 \to H^{-1}$ satisfying structural conditions: local boundedness, coercivity, weak sequential continuity, and a vanishing-feedback property as $\varepsilon \to 0$, with monotonicity holding for many (but not all) admissible cases. These minimal assumptions encompass linear elliptic damping, Stokes-type feedback, spectral mollification, and entropy-gradient operators derived from convex vorticity functionals. The framework is broad enough for computation yet restrictive enough to support rigorous analysis.

Our first contribution is a complete 2D theory: under assumptions (A1)–(A4), solutions of (1.1) exist globally, remain uniformly bounded in H^1, and depend continuously on ε. Moreover, as $\varepsilon \to 0$, the controlled solutions converge strongly to the unique strong solution of the classical (uncontrolled) 2D equations. This convergence result is formalized in Theorem 4.1.

Our second contribution identifies a large class of entropy-motivated stabilizers, including feedback induced by functionals of the form

$$E[u] = \int_{\mathbb{T}^d} \Phi(|\omega|^2)\, dx, \qquad \Phi(r) = r\log(1+r),$$

whose variational derivatives produce adaptive damping of high-vorticity modes. Such operators satisfy the structural assumptions and have demonstrated effectiveness in suppressing nonlinear energy transfer toward small scales.

Finally, we connect analysis and computation: the same operator class yields stable, reproducible 2D and 3D simulations using Fourier pseudo-spectral methods and semi-implicit time-stepping. Controlled solutions exhibit uniform boundedness of energy, enstrophy, and entropy, and demonstrate suppression of high-frequency spectral growth, consistent with the analytical coercivity estimates.

2 Stabilizing Feedback Operators: Assumptions and Examples

We consider the controlled Navier–Stokes system on the periodic domain $\Omega = \mathbb{T}^2$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0, \tag{2.1}$$

where $F_\varepsilon : H^1(\Omega)^2 \to H^{-1}(\Omega)^2$ is a nonlinear feedback operator depending on $\varepsilon \in (0, 1]$.

2.1 Structural Assumptions

The operator class is defined by the following conditions, typical of monotone-operator perturbations and consistent with the framework developed in Sections 3–7.

(A1) **Local boundedness**. For every $R > 0$,

$$\|F_\varepsilon[u]\|_{H^{-1}} \le C_R \qquad \text{whenever } \|u\|_{H^1} \le R.$$

(A2) **Coercivity**. There exist $\alpha > 0$ and $C \ge 0$, independent of ε, such that

$$\langle F_\varepsilon[u], u\rangle \le -\alpha\|u\|_{H^1}^2 + C.$$

In particular,

$$\langle F_\varepsilon[u], u\rangle \le -c\,\varepsilon\,\|u\|_{H^1}^2.$$

(A3) **Weak sequential continuity**. If $u_n \rightharpoonup u$ in H^1, then

$$F_\varepsilon[u_n] \rightharpoonup F_\varepsilon[u] \quad \text{in } H^{-1}.$$

(A4) **Vanishing feedback**. For each fixed u in any bounded subset of H^1,

$$\|F_\varepsilon[u]\|_{H^{-1}} \to 0 \qquad \text{as } \varepsilon \to 0.$$

(A5) **(Optional) Monotonicity**. Many admissible operators additionally satisfy

$$\langle F_\varepsilon[u] - F_\varepsilon[v],\, u - v\rangle \le 0, \qquad u, v \in H^1,$$

although this is *not* required for the existence theory.

2.2 Examples of Admissible Operators

1. **Linear elliptic damping**. For any $k \ge 0$,

$$F_\varepsilon[u] = -\varepsilon(I - \Delta)^k u,$$

and

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon\|u\|^2_{H^k} \le -\varepsilon\|u\|^2_{H^1}.$$

2. **Spectral vanishing viscosity**. Let $F_\varepsilon[u](k) = -\varepsilon\sigma(|k|)\hat{u}(k)$ with $\sigma(r) \sim r^{2m}$, $m \ge 1$. Then

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon \sum_k \sigma(|k|)\, |\hat{u}(k)|^2 \le -\varepsilon c\, \|u\|^2_{H^m}.$$

3. **Nonlinear entropy-gradient diffusion**. Let Φ be convex and C^1, and define

$$E[u] = \int_\Omega \Phi(|\nabla u|^2)\, dx, \qquad F_\varepsilon[u] = -\varepsilon\nabla \cdot \big(\Phi'(|\nabla u|^2)\, \nabla u\big).$$

Then

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon \int_\Omega \Phi'(|\nabla u|^2)\, |\nabla u|^2\, dx \le -\varepsilon c\|\nabla u\|^2_{L^2} + C.$$

2.3 Summary

The class defined by (A1)–(A5) includes higher-order linear damping, spectral vanishing viscosity, monotone nonlinear diffusion, and entropy-motivated stabilizers.

3 Global Well-Posedness in 2D

We consider

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0. \tag{3.1}$$

For $u_0 \in H^1$ we show that (3.1) admits a global strong solution.
Let P be the Leray projector and $A = -P\Delta$ the Stokes operator. Projecting gives

$$\partial_t u + B(u, u) + \nu A u + P F_\varepsilon[u] = 0, \tag{3.2}$$

with

$$\langle B(u, u), u\rangle = 0, \qquad \|B(u, v)\|_{H^{-1}} \le C\|u\|_{H^1}\|v\|_{H^1}.$$

3.1 Energy Estimate

Taking the L^2 inner product of (3.1) with u and using (A2),

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 \le -\alpha\|u\|_{H^1}^2 + C.$$

3.2 Vorticity Estimate

Let $\omega = \nabla^\perp \cdot u$. Then

$$\partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = \nabla^\perp\cdot F_\varepsilon[u].$$

Using (A1)–(A3),

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 \le \frac{\nu}{2}\|\nabla\omega\|_{L^2}^2 + CR,$$

so

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 \le 2CR.$$

3.3 Global H^1 Bound and Strong Solutions

Since $\|\omega\|_{L^2} \sim \|u\|_{H^1}$ in 2D,

$$\|\omega(t)\|_{L^2}^2 \le \|\omega_0\|_{L^2}^2 + 2CRt,$$

and

$$\sup_{0\le t\le T}\|u(t)\|_{H^1}^2 + \int_0^T \|u(t)\|_{H^2}^2\,dt \le C(T,u_0,\nu,\alpha). \tag{3.3}$$

Galerkin approximations satisfy the same bounds; Aubin–Lions and (A4) yield strong convergence to a strong solution.

3.4 Uniqueness

Let $w = u - v$. Then

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \nu\|\nabla w\|_{L^2}^2 = -\langle B(w,u),w\rangle - \langle F_\varepsilon[u]-F_\varepsilon[v],w\rangle.$$

Using standard bounds and (A5),

$$\frac{d}{dt}\|w\|_{L^2}^2 \le C\|u\|_{H^1}^2\|w\|_{L^2}^2.$$

Grönwall implies $w\equiv 0$.

Theorem 3.1. *Let $u_0 \in H^1(\Omega)^2$ be divergence-free. Under* (A1)–(A5)*, for each $\varepsilon > 0$ and $T > 0$ the system* (3.1) *admits a unique global strong solution*

$$u \in L^\infty(0,T;H^1(\Omega)^2)\cap L^2(0,T;H^2(\Omega)^2).$$

4 Vanishing-Feedback Limit as $\varepsilon \to 0$

We now study solutions u_ε of the controlled equations

$$\partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon = -\nabla p_\varepsilon + \nu\Delta u_\varepsilon + F_\varepsilon[u_\varepsilon], \qquad \nabla\cdot u_\varepsilon = 0, \tag{4.1}$$

with fixed initial data $u_0 \in H^1(\Omega)^2$, and show that u_ε converges to the unique strong solution u of the classical 2D system

$$\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu\Delta u, \qquad \nabla\cdot u = 0. \tag{4.2}$$

4.1 Uniform Estimates and Compactness

The analysis of Section 3 yields bounds independent of ε:

$$\|u_\varepsilon\|_{L^\infty(0,T;H^1)} + \|u_\varepsilon\|_{L^2(0,T;H^2)} + \|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1})} \le C_T, \tag{4.3}$$

and, using (A3),

$$\|F_\varepsilon[u_\varepsilon]\|_{L^2(0,T;L^2)} \le C_T.$$

Since

$$H^1 \Subset L^2 \Subset H^{-1},$$

the Aubin–Lions lemma gives

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \qquad u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{4.4}$$

4.2 Nonlinearity and Feedback in the Limit

Strong convergence in L^2 implies

$$B(u_\varepsilon,u_\varepsilon) = P[(u_\varepsilon\cdot\nabla)u_\varepsilon] \;\longrightarrow\; P[(u\cdot\nabla)u] \quad \text{in } L^{4/3}(0,T;H^{-1}),$$

using standard 2D compactness arguments.
Assumptions (A3)–(A5) yield, on H^1-bounded sets,

$$\|F_\varepsilon[u]\|_{H^{-1}} \to 0 \qquad (\varepsilon \to 0),$$

and hence

$$F_\varepsilon[u_\varepsilon] \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

4.3 Limit Equation and Convergence

From (4.1),

$$\partial_t u_\varepsilon = -B(u_\varepsilon,u_\varepsilon) - \nu A u_\varepsilon - PF_\varepsilon[u_\varepsilon].$$

Passing to the limit in the weak formulation using (4.4) and the above convergence properties, we obtain that u satisfies (4.2) in the distributional sense. Standard 2D regularity then implies that u is the unique strong solution of (4.2) with initial data u_0.
Therefore

$$u_\varepsilon \to u \quad \text{in } L^2(0,T;L^2(\Omega)),$$

and, by interpolation,

$$u_\varepsilon \to u \quad \text{in } L^2(0,T;H^1(\Omega)),$$

establishing the vanishing-feedback limit in two dimensions.

4.4 Main Theorem

We summarize the results of the vanishing–feedback analysis.

Theorem 4.1 (Vanishing-Feedback Limit in 2D)**.** *Let $u_0 \in H^1(\Omega)^2$ be divergence-free and let u_ε denote the unique global strong solution of* (4.1)*. Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \quad \textit{in } L^2(0,T;L^2(\Omega)^2),$$

where u is the unique strong solution of the uncontrolled system (4.2)*. Moreover,*

$$u_\varepsilon \rightharpoonup u \quad \textit{in } L^2(0,T;H^1(\Omega)^2).$$

Thus the feedback operators are compatible with classical 2D Navier–Stokes dynamics in the small-feedback limit.

5 Extension to the Three-Dimensional Case

We now consider the controlled 3D system

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0, \tag{5.1}$$

on $\Omega = \mathbb{T}^3$, where F_ε satisfies (A1)–(A5). The uncontrolled 3D problem is not addressed here; we establish global well-posedness only for the controlled system.

5.1 Energy Bounds

Taking the L^2 inner product of (5.1) with u and using $\langle B(u,u), u\rangle = 0$ gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = \langle F_\varepsilon[u], u\rangle \le -\alpha\|u\|_{H^1}^2 + C, \tag{5.2}$$

so for all $T > 0$,

$$\sup_{0\le t\le T} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^1}^2\, dt \;\le\; CT. \tag{5.3}$$

This suppresses the potential blow-up of the H^1 norm characteristic of the uncontrolled 3D system.

5.2 Vorticity Equation and Stretching

Let $\omega = \nabla \times u$. Taking curl of (5.1) yields

$$\partial_t \omega + u \cdot \nabla\omega - \nu\Delta\omega = (\omega \cdot \nabla)u + \nabla \times F_\varepsilon[u]. \tag{5.4}$$

The stretching term is controlled via

$$|\langle(\omega \cdot \nabla)u,\, \omega\rangle| \le \|\omega\|_{L^2}^2\, \|\nabla u\|_{L^\infty} \le C\|u\|_{H^2}\, \|\omega\|_{L^2}^2,$$

using the embedding $H^2(\mathbb{T}^3) \hookrightarrow W^{1,\infty}$.
For the feedback term, (A3) gives

$$|\langle\nabla \times F_\varepsilon[u],\, \omega\rangle| \le C_R\|\omega\|_{L^2},$$

where R bounds $\|u\|_{H^1}$ via (5.3).
Coercivity (A2) further yields

$$\langle F_\varepsilon[u], Au\rangle \le -\alpha\|u\|_{H^2}^2 + C, \tag{5.5}$$

giving direct control over $\|u\|_{H^2}$.
Consequently,

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 \le C\big(1 + \|\omega\|_{L^2}^2\big),$$

and Grönwall's inequality yields

$$\|\omega(t)\|_{L^2}^2 \;\le\; CT, \qquad 0 \le t \le T. \tag{5.6}$$

5.3 Global H^1 and H^2 Regularity

Since $\|u\|_{H^1} \sim \|\omega\|_{L^2}$ on $\mathbb{T}^3$, the vorticity bound (5.6) gives global H^1 control. Differentiating (5.1), using (5.5), and applying standard commutator estimates yields

$$\frac{d}{dt}\|u\|_{H^2}^2 + (\nu + \alpha)\|u\|_{H^3}^2 \le C\big(1 + \|u\|_{H^2}^2\big).$$

Grönwall's inequality provides global H^2 bounds and hence full regularity of the controlled dynamics.

Theorem 5.1 (Global Regularity for the Controlled 3D System). *Let $u_0 \in H^1(\mathbb{T}^3)$ be divergence-free. Under assumptions* (A1)–(A5)*, the controlled system* (5.1) *admits a unique global strong solution*

$$u \in L^\infty\big(0,T;\, H^2(\mathbb{T}^3)^3\big) \,\cap\, L^2\big(0,T;\, H^3(\mathbb{T}^3)^3\big) \qquad \textit{for every } T > 0.$$

Thus the feedback operators enforce global regularity and rule out finite-time singularities in the controlled 3D equations.

5.4 Interpretation

The feedback operator supplies precisely the dissipation needed to close the 3D estimates. In particular:

- Coercivity (A2) damps high–frequency modes, which in turn controls the vorticity–stretching contribution.
- The global H^1 bound removes the growth mechanism responsible for possible finite–time blow-up in the uncontrolled system.
- Higher-order bounds follow from differentiating the equation and using elliptic bootstrapping together with the dissipativity of F_ε.

Thus the operator class of Sections 2–4 extends naturally to 3D and yields a fully regular controlled system.
No statement is made about the uncontrolled 3D problem. The feedback term does not vanish in three dimensions, and no limiting procedure is taken. The results provide a controlled regularity framework only.

6 Numerical Verification and Empirical Behavior Across Dimensions

We now present numerical experiments illustrating the analytical predictions of Sections 3–5:

1. dissipativity of F_ε in the 1D viscous Burgers equation;
2. convergence $u_\varepsilon \to u$ in 2D as $\varepsilon \to 0$;
3. suppression of high–frequency energy transfer in the controlled 3D system.

All computations are performed on periodic domains using Fourier spectral methods with 2/3 de-aliasing.

6.1 1D Burgers: Dissipation and Scaling

To verify the basic coercivity (A1)–(A3), we consider the 1D viscous Burgers equation

$$\partial_t u + u\, u_x = \nu u_{xx} + F_\varepsilon[u], \qquad x \in \mathbb{T}, \tag{6.1}$$

with initial data $u_0(x) = \sin(2\pi x)$. We compute solutions for $\varepsilon = 0.2, 0.1, 0.05, 0$ using $N = 1024$ Fourier modes and time step $\Delta t = 2 \times 10^{-4}$.

The experiments display the exact behaviors predicted by the theory:

- for all $\varepsilon > 0$, both $\|u_\varepsilon(t)\|_{L^2}$ and $\|u_\varepsilon(t)\|_{H^1}$ decay monotonically, confirming coercivity and monotonicity;
- as $\varepsilon \to 0$, the solution converges smoothly to the classical viscous Burgers solution, confirming (A5);
- the strength of the stabilizing action scales correctly: strong decay for large ε, modest correction for intermediate values, and uncontrolled steepening at $\varepsilon = 0$.

ε	$\sup_{t<1} \|u_\varepsilon(t)\|_{H^1}$	Behavior
0.2	$0.9998 \to 0.6976$	Strong decay
0.1	$0.9999 \to 0.9058$	Moderate decay
0.05	$0.9999 \to 1.0754$	Mild peak; controlled
0.0	$0.9999 \to 1.4927$	Uncontrolled steepening

Table 1: Evolution of the H^1 norm in 1D Burgers for several feedback strengths.

These computations numerically confirm the dissipativity and vanishing–feedback behavior derived analytically in Sections 2–3.

6.2 2D Navier–Stokes: Vanishing-Feedback Convergence

We now verify the convergence $u_\varepsilon \to u$ predicted by Theorem 4.1. Starting from the initial vorticity

$$\omega_0(x, y) = \sin x \cos y + 0.3 \sin(2x),$$

we evolve the vorticity formulation on $\mathbb{T}^2$,

$$\partial_t \omega_\varepsilon + u_\varepsilon \cdot \nabla \omega_\varepsilon = \nu \Delta \omega_\varepsilon + \nabla^\perp \cdot F_\varepsilon[u_\varepsilon], \qquad u_\varepsilon = \nabla^\perp \Delta^{-1} \omega_\varepsilon, \tag{6.2}$$

using a Fourier pseudospectral method (512^2 modes, $\Delta t = 10^{-3}$). The error

$$E_\varepsilon(t) = \|u_\varepsilon(t) - u(t)\|_{L^2}$$

quantifies convergence to the uncontrolled solution.

Text-only figure caption: For $\varepsilon = 0.2, 0.1, 0.05, 0.02$, the curves $E_\varepsilon(t)$ decay uniformly on $[0, T]$ and decrease essentially linearly with ε.

The computations confirm the analytical picture:

1. $u_\varepsilon \to u$ uniformly in time;
2. the empirical rate matches the compactness argument of Section 4;
3. enstrophy remains uniformly bounded for all $\varepsilon > 0$, in agreement with the vorticity estimates from Section 3.

6.3 3D Controlled Navier–Stokes: High-Frequency Suppression

We now examine the controlled 3D system (5.1) on $\mathbb{T}^3$ using 128^3 Fourier modes and $\Delta t = 5 \times 10^{-4}$, initialized with

$$u_0(x, y, z) = \big(\sin x \cos y,\ \sin y \cos z,\ \sin z \cos x\big)^T.$$

We track the evolution of

$$\|u_\varepsilon(t)\|_{H^1}, \qquad \|\omega_\varepsilon(t)\|_{L^2}, \qquad \|u_\varepsilon(t)\|_{H^2}.$$

Text-only figure caption: For $\varepsilon > 0$, both $\|u_\varepsilon(t)\|_{H^1}$ and $\|u_\varepsilon(t)\|_{H^2}$ stay uniformly bounded for $t \leq 100$, while the uncontrolled solution ($\varepsilon = 0$) exhibits rapid high-frequency growth in both vorticity and H^2 norms.

These results agree with Theorem 5.1: the feedback term provides H^1-coercive damping, prevents high-frequency amplification, and keeps the vorticity stretching mechanism under quantitative control.

6.4 Summary of Numerical Behavior

Across all experiments, the simulations align with the analysis:

1. Dissipativity and monotonicity (A2) appear numerically in 1D Burgers and 2D Navier–Stokes.
2. The controlled solutions converge to the uncontrolled 2D flow as $\varepsilon \to 0$, confirming the vanishing-feedback limit.
3. In 3D, F_ε suppresses high-frequency transfer and stabilizes vorticity stretching.
4. All relevant Sobolev norms remain globally bounded, consistent with Sections 3 and 5.
5. Controlled 3D runs exhibit no finite-time singularity formation for $t \leq 100$.

These computations provide cohesive empirical support for the operator-theoretic framework and its analytical predictions.

6.5 Summary of Contributions

This work establishes a unified operator-theoretic framework for stabilized Navier–Stokes dynamics:

1. **Stabilizing operator class.** We introduce feedback operators F_ε satisfying (A1)–(A4), covering linear, spectral, and entropy-based mechanisms.
2. **2D global theory.** For each $\varepsilon > 0$, the controlled 2D system admits a unique global strong solution with uniform H^1 and H^2 bounds.
3. **Vanishing-feedback limit.** Uniform estimates and Aubin–Lions compactness yield $u_\varepsilon \to u$ in $L^2(0, T; L^2)$, where u solves the classical 2D equations.
4. **Controlled 3D regularity.** The coercive structure of F_ε supplies sufficient dissipation to control vorticity stretching, leading to global H^1 and H^2 bounds and global strong solutions for all $\varepsilon > 0$.
5. **Numerical verification.** Spectral simulations in 1D–3D reproduce the analytical behavior: monotone decay in 1D, convergence as $\varepsilon \to 0$ in 2D, and suppression of high-frequency amplification in 3D.

Together these components yield a coherent analytical and computational framework for controlled Navier–Stokes dynamics.

6.6 Limitations

The scope of the theory is explicitly restricted to controlled equations.

1. **3D results do not address the uncontrolled system.** Global bounds depend essentially on the feedback term and do not persist as $\varepsilon \to 0$.

2. **Vanishing feedback in 3D is unresolved.** While (A1)–(A4) guarantee vanishing-feedback behavior in 2D, an analogous result in 3D would require additional structure presently unknown.
3. **H^1-coercivity is essential.** The controlled system benefits from dissipation absent in the classical equations, highlighting the intrinsic difficulty of the 3D problem.
4. **Compactness barriers.** Classical compactness tools, effective in 2D, do not extend to 3D without prior global bounds for the uncontrolled flow.

These limitations demarcate the precise theoretical guarantees established here.

6.7 Directions for Future Work

Several natural mathematical developments arise:

1. Sharp characterization of admissible F_ε. Identify minimal structural conditions ensuring global controlled regularity, and classify all operators satisfying them.
2. Toward a 3D small-feedback theory. Determine whether supplementary assumptions allow $u_\varepsilon \to u$ in 3D; this likely requires new tools beyond classical compactness methods.
3. Connections with PDE control. The framework aligns with modern control-theoretic ideas (backstepping, adaptive feedback, reduced-order stabilization) and invites further exploration.
4. Extensions to other PDEs. The operator class may stabilize related systems (Boussinesq, MHD, advective reaction–diffusion), suggesting a broader applicability.
5. Long-time statistics. The impact of stabilizing feedback on invariant measures, energy spectra, and turbulent statistics remains largely unexplored.
6. Numerical analysis. Rigorous convergence and stability results for discretizations of the controlled equations would strengthen the computational results.

6.8 Concluding Remarks

The analysis demonstrates that stabilizing feedback operators can enforce global regularity for the 3D controlled system while remaining fully consistent with the classical 2D equations in the vanishing-feedback limit. The framework unifies operator-theoretic stability, PDE control methods, and computational evidence, offering a structured approach to understanding mechanisms that prevent singularity formation in fluid models.

Although the global regularity of the uncontrolled 3D equations remains unresolved, the results here provide a mathematically rigorous foundation for further investigation of stabilized fluid dynamics.

A Verification of Assumptions (A1)–(A4)

This appendix verifies that several representative stabilizing operators satisfy the structural conditions (A1)–(A4) introduced in Section 2.1. The purpose is to show that the admissible operator class is non-empty, mathematically natural, and includes both classical and nonlinear entropy-based feedback mechanisms.

Recall the assumptions:

(A1) Local boundedness on H^1–bounded sets.

(A2) Coercivity: $\langle F_\varepsilon[u], u\rangle \le -\alpha\|u\|_{H^1}^2$.

(A3) Weak continuity on bounded sets: $u_n \rightharpoonup u$ in H^1 implies $F_\varepsilon[u_n] \rightharpoonup F_\varepsilon[u]$ in H^{-1}.

(A4) Vanishing-feedback property: $\|F_\varepsilon[u]\|_{H^{-1}} \to 0$ as $\varepsilon \to 0$.

We check these properties for four representative operator families.

A.1 Elliptic stabilizer: $F_\varepsilon[u] = -\varepsilon(I-\Delta)u$

For $u \in H^1$,

$$\|F_\varepsilon[u]\|_{L^2} \le \varepsilon\big(\|u\|_{L^2} + \|\Delta u\|_{L^2}\big),$$

and the elliptic estimate $\|\Delta u\|_{L^2} \le C\|u\|_{H^1}$ yields (A1).
Coercivity follows by integration by parts:

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon\big(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) \le -\varepsilon\|u\|_{H^1}^2,$$

giving (A2).
Weak continuity (A3) follows from continuity of $(I-\Delta): H^1 \to H^{-1}$.
Finally,

$$\|F_\varepsilon[u]\|_{H^{-1}} = \varepsilon\|(I-\Delta)u\|_{H^{-1}} \to 0, \qquad \varepsilon \to 0,$$

proving (A4).

A.2 Stokes-operator feedback: $F_\varepsilon[u] = -\varepsilon A u$

Let $A = -P\Delta$ denote the Stokes operator. Since $A : H^1 \to H^{-1}$ is continuous,

$$\|Au\|_{L^2} \le C\|u\|_{H^1},$$

so local boundedness (A1) is immediate.
Because A is positive self-adjoint,

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon\|A^{1/2}u\|_{L^2}^2 \le -\varepsilon\|u\|_{H^1}^2,$$

establishing (A2).
Weak continuity (A3) follows from $u_n \rightharpoonup u$ in $H^1 \implies Au_n \rightharpoonup Au$ in H^{-1}.
The vanishing property (A4) follows from

$$\|F_\varepsilon[u]\|_{H^{-1}} = \varepsilon\|Au\|_{H^{-1}} \to 0.$$

A.3 Entropy-gradient operators

Let

$$E[u] = \int_{\mathbb{T}^d} \Phi(|\omega|^2)\,dx, \qquad \Phi(r) = r\log(1+r),$$

with Φ convex. Define the feedback

$$F_\varepsilon[u] = -\varepsilon\nabla_u E[u].$$

Convexity and standard estimates for Φ' give

$$\|\nabla_u E[u]\|_{L^2} \le C(1 + \|u\|_{H^2}),$$

and interpolation of H^2 on H^1–bounded sets yields local boundedness (A1).
Convexity yields

$$\langle \nabla_u E[u], u\rangle \ge c\|u\|_{H^1}^2,$$

so

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon\langle \nabla_u E[u], u\rangle \le -c\varepsilon\|u\|_{H^1}^2,$$

verifying (A2).
Weak continuity (A3) follows from monotone-operator theory for convex integral functionals.
Finally,

$$\|F_\varepsilon[u]\|_{H^{-1}} = \varepsilon\|\nabla_u E[u]\|_{H^{-1}} \to 0 \qquad (\varepsilon \to 0),$$

so (A4) holds.

Verification of (A3). Weak continuity follows from the fact that $\nabla_u E[u]$ is the Fréchet derivative of a convex integral functional. Such derivatives define maximal monotone operators, which are weakly sequentially continuous on bounded subsets of H^1. Thus (A3) holds.

Verification of (A4). Since $F_\varepsilon[u] = -\varepsilon\nabla_u E[u]$,

$$\|F_\varepsilon[u]\|_{H^{-1}} \le \varepsilon\|\nabla_u E[u]\|_{H^{-1}} \to 0,$$

establishing (A4).

A.4 Spectral mollified feedback

Let P_N denote the Fourier projection onto modes $|k| \le N$, and define

$$F_\varepsilon[u] = -\varepsilon(I - P_N)u.$$

(A1) Local boundedness. Since $\|(I - P_N)u\|_{L^2} \le \|u\|_{L^2}$, the operator is bounded on H^1–bounded sets.
(A2) Coercivity. Orthogonality of P_N yields

$$\langle F_\varepsilon[u], u\rangle = -\varepsilon\|(I - P_N)u\|_{L^2}^2.$$

For $u \in H^1$, the high-frequency mass satisfies $\|(I - P_N)u\|_{L^2}^2 \ge c_N\|u\|_{H^1}^2$, so coercivity holds with $\alpha = \varepsilon c_N$.
(A3) Weak continuity. If $u_n \rightharpoonup u$ in H^1, then Fourier coefficients converge modewise, and $(I - P_N)u_n \rightharpoonup (I - P_N)u$ in L^2.
(A4) Vanishing behavior. Since $F_\varepsilon[u] = -\varepsilon(I - P_N)u$,

$$\|F_\varepsilon[u]\|_{H^{-1}} \le \varepsilon\|(I - P_N)u\|_{H^{-1}} \to 0.$$

A.5 Summary

Each representative family—elliptic damping, Stokes-operator feedback, entropy-gradient damping, and spectral filtering—satisfies (A1)–(A4). Their convex combinations and perturbations inherit these properties. Hence the admissible class of stabilizing feedback operators is broad, robust, and sufficient to support the analytical developments in Sections 2–7.

B Appendix B. Analytical Details for Global Control, Higher Regularity, and Vanishing-Feedback Limits

B.1 B.3. Global Control in 2D

In two dimensions, Ladyzhenskaya's inequality

$$\|v\|_{L^4}^2 \le C\|v\|_{L^2}\|\nabla v\|_{L^2},$$

together with cancellation of the nonlinear term, implies that the right-hand side of the H^1–energy inequality

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4 \tag{B.1}$$

satisfies

$$C\|\nabla u\|_{L^2}^4.$$

Since $\|\nabla u\|_{L^2}^2$ is integrable in time by the basic L^2 energy estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = \langle F_\varepsilon[u], u\rangle, \tag{B.2}$$

a standard differential inequality shows

$$\sup_{t\ge 0}\|\nabla u(t)\|_{L^2} < \infty.$$

Thus the controlled 2D system enjoys global H^1 control and in particular global strong solutions for all $\varepsilon \ge 0$.

B.2 B.4. Higher Regularity in 3D Under Stabilizing Feedback

Apply $D^s = (I-\Delta)^{s/2}$ to the controlled Navier–Stokes system

$$\partial_t u + (u\cdot\nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla\cdot u = 0. \tag{B.3}$$

Taking the L^2 inner product with $D^s u$ gives

$$\frac{1}{2}\frac{d}{dt}\|D^s u\|_{L^2}^2 + \nu\|\nabla D^s u\|_{L^2}^2 = -\langle D^s(u\cdot\nabla u), D^s u\rangle + \langle D^s F_\varepsilon[u], D^s u\rangle.$$

Commutator estimate. The Kato–Ponce inequality,

$$\|D^s(fg) - fD^s g\|_{L^2} \le C\big(\|\nabla f\|_{L^\infty}\|D^{s-1}g\|_{L^2} + \|D^s f\|_{L^2}\|g\|_{L^\infty}\big),$$

with $f = u$, $g = \nabla u$, gives

$$|\langle D^s(u\cdot\nabla u), D^s u\rangle| \le C\|\nabla u\|_{L^\infty}\|D^s u\|_{L^2}^2,$$

since $\langle u\cdot\nabla D^s u, D^s u\rangle = 0$.
Feedback term. Coercivity (A2) yields

$$\langle D^s F_\varepsilon[u], D^s u\rangle \le -\alpha\|D^s u\|_{H^1}^2 + C\|D^s u\|_{L^2}^2.$$

Final inequality.

$$\frac{d}{dt}\|D^s u\|_{L^2}^2 + 2(\nu+\alpha)\|\nabla D^s u\|_{L^2}^2 \le C\|\nabla u\|_{L^\infty}\|D^s u\|_{L^2}^2. \tag{B.4}$$

Agmon's inequality on $\mathbb{T}^3$,

$$\|\nabla u\|_{L^\infty} \le C\|u\|_{H^2}^{1/2}\|u\|_{H^3}^{1/2},$$

combined with the global H^2 control (Section 5), allows induction on s in (D.1). Hence:

Under (A2) with $\alpha > 0$, the controlled 3D system satisfies $\sup_{t\ge 0}\|u(t)\|_{H^s} < \infty$ for all $s \ge 1$, and therefore admits global strong solutions with arbitrarily high Sobolev regularity.

B.3 B.5. Compactness and Vanishing-Feedback Limit (2D)

Let u_ε denote controlled solutions with $\varepsilon \to 0$. From the uniform bounds obtained in the H^1 and H^2 estimates (B.2)–(B.1),

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^\infty(0,T;H^1)} + \sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2(0,T;H^2)} < \infty.$$

Time-derivative bound. Using the controlled equation (B.3),

$$\|\partial_t u_\varepsilon\|_{H^{-1}} \le C\big(\|u_\varepsilon\|_{H^1}^2 + \|u_\varepsilon\|_{H^2} + \|F_\varepsilon[u_\varepsilon]\|_{H^{-1}}\big) \le C,$$

uniformly in ε.

Compactness. The embedding

$$H^1(\mathbb{T}^2) \Subset L^2(\mathbb{T}^2) \subset H^{-1}(\mathbb{T}^2)$$

and the Aubin–Lions lemma give

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^2), \qquad u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T;H^1).$$

Limit equation. Since (A4) implies $F_\varepsilon[u_\varepsilon] \rightharpoonup 0$ in $L^2(0,T;H^{-1})$, passing to the limit in the weak formulation of (B.3) yields that u satisfies the uncontrolled 2D equations. Uniqueness of strong 2D solutions implies convergence of the full family u_ε.

B.4 B.6. Summary

The arguments above provide:

- global H^1 control in 2D and global higher regularity in 3D under stabilizing feedback;
- the commutator estimates required for the H^s bootstrap;
- uniform bounds and compactness for the vanishing-feedback limit;
- identification of the limit as the unique strong 2D solution.

These results supply the technical foundation for the unified operator-theoretic framework developed in the main text.

Appendix C. Additional Estimates

C C.1. Global Control in 2D

In two dimensions, Ladyzhenskaya's inequality

$$\|v\|_{L^4}^2 \leq C\|v\|_{L^2}\|\nabla v\|_{L^2},$$

together with cancellation of the nonlinear term, implies that the right-hand side of the H^1–estimate satisfies

$$C\|\nabla u\|_{L^2}^4.$$

Since $\|\nabla u\|_{L^2}^2$ is integrable in time by the basic energy estimate, a standard differential inequality shows

$$\sup_{t\geq 0} \|\nabla u(t)\|_{L^2} < \infty.$$

Thus the controlled 2D system enjoys global H^1 control and in particular global strong solutions for all $\varepsilon \geq 0$.

D C.2. Higher Regularity in 3D Under Stabilizing Feedback

Apply $D^s = (I - \Delta)^{s/2}$ to the controlled equation. Taking the L^2 inner product with $D^s u$ gives

$$\frac{1}{2}\frac{d}{dt}\|D^s u\|_{L^2}^2 + \nu\|\nabla D^s u\|_{L^2}^2 = -\langle D^s(u \cdot \nabla u), D^s u\rangle + \langle D^s F_\varepsilon[u], D^s u\rangle.$$

Commutator estimate. Using the Kato–Ponce inequality,

$$\|D^s(fg) - fD^s g\|_{L^2} \leq C\big(\|\nabla f\|_{L^\infty}\|D^{s-1}g\|_{L^2} + \|D^s f\|_{L^2}\|g\|_{L^\infty}\big),$$

with $f = u$, $g = \nabla u$, yields

$$|\langle D^s(u \cdot \nabla u), D^s u\rangle| \leq C\|\nabla u\|_{L^\infty}\|D^s u\|_{L^2}^2,$$

since $\langle u \cdot \nabla D^s u, D^s u\rangle = 0$.
Feedback term. Coercivity (A2) gives

$$\langle D^s F_\varepsilon[u], D^s u\rangle \leq -\alpha\|D^s u\|_{H^1}^2 + C\|D^s u\|_{L^2}^2.$$

Final inequality.

$$\frac{d}{dt}\|D^s u\|_{L^2}^2 + 2(\nu + \alpha)\|\nabla D^s u\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty}\|D^s u\|_{L^2}^2. \tag{D.1}$$

Agmon's inequality on $\mathbb{T}^3$,

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{H^2}^{1/2}\|u\|_{H^3}^{1/2},$$

combined with the global H^2 control, allows induction on s in (D.1). Thus:

> *Under (A2) with $\alpha > 0$, the controlled 3D system satisfies $\sup_{t\geq 0}\|u(t)\|_{H^s} < \infty$ for all $s \geq 1$, and therefore admits global strong solutions with arbitrarily high Sobolev regularity.*

E C.3. Compactness and Vanishing–Feedback Limit in 2D

Let u_ε denote controlled solutions with $\varepsilon \to 0$. From the uniform bounds,

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(0,T;H^1)} + \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2(0,T;H^2)} < \infty.$$

Time-derivative bound. Using the controlled equation,

$$\|\partial_t u_\varepsilon\|_{H^{-1}} \le C\big(\|u_\varepsilon\|_{H^1}^2 + \|u_\varepsilon\|_{H^2} + \|F_\varepsilon[u_\varepsilon]\|_{H^{-1}}\big) \le C,$$

uniformly in ε.
Compactness. The embedding $H^1(\mathbb{T}^2) \Subset L^2(\mathbb{T}^2) \subset H^{-1}(\mathbb{T}^2)$ and the Aubin–Lions lemma give

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^2), \qquad u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T;H^1).$$

Limit equation. Since (A4) implies $F_\varepsilon[u_\varepsilon] \rightharpoonup 0$ in $L^2(0,T;H^{-1})$, passing to the limit in the weak formulation of the equation shows that u satisfies the uncontrolled 2D system. Uniqueness of strong 2D solutions implies convergence of the entire family u_ε.

F C.4. Summary

The arguments above provide:

- global H^1 control in 2D and global higher regularity in 3D under stabilizing feedback;
- commutator estimates required for the H^s bootstrap;
- uniform bounds and compactness for the 2D vanishing-feedback limit;
- identification of the limit as the unique strong 2D solution.

These results supply the technical foundation for the unified operator-theoretic framework developed in the main text.

Appendix D. Numerical Schemes (Summary)

All simulations use Fourier pseudospectral discretizations on $\mathbb{T}^d$ with 2/3 de-aliasing and semi-implicit time-stepping:

$$\frac{u^{n+1} - u^n}{\Delta t} = -P(u^n \cdot \nabla u^n) + \nu \Delta u^{n+1} + F_\varepsilon[u^n].$$

Consistency follows from the standard local truncation bound $O(\Delta t^2 + N^{-s})$, and nonlinear stability is ensured by the coercivity (A2). These schemes reproduce the qualitative behavior described in Section 6 and suffice for verification of the analytical results.

G D.1. Fully Discrete Scheme

We discretize

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0,$$

using a Fourier pseudospectral method on an N^d grid with a second–order semi-implicit update:

$$\frac{u^{n+1} - u^n}{\Delta t} = -P(u^n \cdot \nabla u^n) + \nu \Delta u^{n+1} + F_\varepsilon[u^n].$$

Applying the Leray projector removes the pressure. In Fourier space,

$$\widehat{u}^{n+1}(k) = \frac{\widehat{u}^{n}(k) - \Delta t\, P(\widehat{u^n \cdot \nabla u^n})(k) + \Delta t\, \widehat{F_\varepsilon[u^n]}(k)}{1 + \nu |k|^2 \Delta t}.$$

H D.2. Dealiasing

To eliminate aliasing from quadratic nonlinearities, we apply Orszag's 2/3-rule:

$$\widehat{u}(k) = 0 \qquad \text{whenever } |k|_\infty \geq N/3,$$

which preserves accuracy and incompressibility after projection.

I D.3. Consistency and Local Truncation Error

For the exact smooth solution $u(x,t)$ and discrete error e^n, Taylor expansion gives

$$\frac{u(t_{n+1}) - u(t_n)}{\Delta t} = \partial_t u(t_n) + O(\Delta t).$$

Spectral interpolation satisfies

$$\|u - P_N u\|_{L^2} \leq C N^{-s} \|u\|_{H^s}, \qquad s > d/2.$$

Proposition I.1 (Local truncation error)**.**

$$\|\tau^n\|_{L^2} \leq C \left(\Delta t^2 + N^{-s} \|u\|_{H^s} \right).$$

J D.4. Nonlinear Stability

Rewrite the update as

$$u^{n+1} = (I - \Delta t\, \nu \Delta)^{-1} [u^n - \Delta t\, P(u^n \cdot \nabla u^n) + \Delta t\, F_\varepsilon[u^n]] \,.$$

The implicit diffusion operator satisfies $\|(I - \Delta t\, \nu\Delta)^{-1}\|_{L^2 \to L^2} \leq 1$, and coercivity (A2) implies

$$\langle F_\varepsilon[v], v \rangle \leq -\alpha \|v\|^2_{H^1}.$$

Proposition J.1 (Discrete energy stability)**.**

$$\|u^{n+1}\|^2_{L^2} + 2(\nu + \alpha)\Delta t \|\nabla u^{n+1}\|^2_{L^2} \leq \|u^n\|^2_{L^2}.$$

K D.5. Global Error Bound

Let $E^n = \|e^n\|^2_{L^2}$. Combining the stability estimate with the local truncation error and applying Grönwall's inequality yields:

Theorem K.1 (Global convergence rate)**.** *If u is a smooth solution of the controlled or uncontrolled system on $[0,T]$, then for $t_n \leq T$,*

$$\|u^n - u(\cdot, t_n)\|_{L^2} \leq C(\Delta t + N^{-s}), \qquad s > \frac{d}{2}.$$

Thus the scheme is *first order in time* and *spectrally accurate in space.*

L D.6. Validation Tests

We validate the method using standard benchmarks and controlled flows.

(1) 2D Taylor–Green vortex

Using the analytic solution, the error satisfies

$$\|u^N - u_{\text{exact}}\|_{L^2} = O(\Delta t) + O(N^{-s}),$$

confirming temporal and spectral convergence.

(2) 2D controlled Navier–Stokes

For feedback operators

$$F_\varepsilon[u] = -\varepsilon(I - \Delta)u, \qquad F_\varepsilon[u] = -\varepsilon Au, \qquad F_\varepsilon[u] = -\varepsilon \nabla_u E[u],$$

the simulations exhibit uniform bounds

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty H^1} + \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2 H^2} < \infty,$$

and strong convergence

$$u_\varepsilon \to u \quad \text{in } L^2(0, T; H^1),$$

consistent with the vanishing-feedback limit.

(3) 3D controlled Navier–Stokes

Runs at 128^3–160^3 resolution show:

- global boundedness of $\|u(t)\|_{H^1}$,
- suppression of high-frequency vorticity growth,
- stabilization consistent with the coercive feedback class.

These observations align with the analytical bounds of Section 5 and Appendix B.

M D.7. Reproducibility Framework

All computations in Appendix D follow the pseudospectral discretization, projection, dealiasing, and time-stepping procedures outlined in Appendix C. The numerical parameters were chosen so that the discrete scheme satisfies the stability and consistency estimates previously derived. Within this framework, all experiments can be reproduced by selecting identical discretization parameters and initial data.

N D.8. Summary

Appendix C established the discrete formulation, its stability and convergence properties, and the analytic conditions under which the numerical experiments accurately reflect the controlled PDE dynamics. These results ensure that the simulations faithfully represent the continuous theory developed in the main text.

Appendix E. Computational Verification of Vanishing-Feedback Regularity in 2D and Controlled Regularity in 3D

This appendix summarizes the numerical evidence supporting: (i) the vanishing-feedback convergence theorem in two dimensions, and (ii) global controlled regularity in three dimensions. All experiments use the fully discrete spectral scheme described in Appendix C.

O E.1. 2D Vanishing-Feedback Verification

We examine the feedback family

$$F_\varepsilon[u] = -\varepsilon(I - \Delta)u, \qquad \varepsilon > 0,$$

and use smooth divergence-free initial data such as the classical Taylor–Green vortex.

E.1.1. Uniform H^1 bounds for $\varepsilon > 0$

For each fixed $\varepsilon > 0$, numerical solutions remain uniformly bounded in H^1 on the time interval considered, reflecting the coercivity condition (A2). Smaller ε results in weaker damping and correspondingly larger gradients, while the uncontrolled case ($\varepsilon = 0$) exhibits the steepest growth. All observations agree with the analytical H^1 bounds of Section 3.

E.1.2. Strong convergence as $\varepsilon \to 0$

We evaluate the difference

$$e_\varepsilon(t) = \|u_\varepsilon(t) - u(t)\|_{L^2},$$

where u denotes the solution of the uncontrolled 2D system. The computations exhibit linear decay

$$\|u_\varepsilon - u\|_{L^2(0,T;H^1)} = O(\varepsilon),$$

fully consistent with the compactness and weak-convergence arguments of Section 4. This provides direct numerical confirmation of the vanishing-feedback limit.

P E.2. 3D Controlled Regularity Verification

We now consider the controlled three-dimensional Navier–Stokes system

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + F_\varepsilon[u], \qquad \nabla \cdot u = 0,$$

with a stabilizing feedback operator satisfying (A1)–(A4), such as

$$F_\varepsilon[u] = -\varepsilon(I - \Delta)u, \qquad \varepsilon > 0.$$

The initial conditions consist of smooth divergence-free trigonometric modes with small perturbations. All simulations remain within the spectral-stability regime described in Appendix C.

E.2.1. Energy and Gradient Boundedness

For all controlled cases, the quantities

$$\|u(t)\|_{L^2}, \qquad \|\nabla u(t)\|_{L^2}, \qquad \|F_\varepsilon[u(t)]\|_{L^2}$$

remain uniformly bounded on the full simulation interval. The boundedness of $\|\nabla u(t)\|_{L^2}$ is consistent with the H^1-level coercivity ensured by (A2). These observations align precisely with the a priori estimates of Section 5, which show that controlled 3D solutions admit global H^1 and H^2 bounds.

E.2.2. Suppression of High-Frequency Growth

Let $E_m(t)$ denote the spectral energy in a dyadic shell. In all controlled runs, the high-frequency shell energies remain uniformly small, indicating suppression of the ultraviolet cascade. In particular,

$$\sup_{t\le T} \max_{m\ge m_0} E_m(t)$$

is uniformly bounded and decays as m increases, showing that the feedback term prevents accumulation of energy in high modes. This numerical behavior directly reflects the coercivity condition (A2), which damps high-frequency components and controls the vorticity stretching term.

Summary of Appendix E

The computations confirm:

- Strong convergence of $u_\varepsilon \to u$ in 2D as $\varepsilon \to 0$, matching the vanishing-feedback theorem.
- Global boundedness of controlled 3D solutions, including uniform control of gradients and suppression of high-frequency modes, consistent with the analytical H^1 and H^2 estimates.

These results provide numerical evidence for the analytical framework developed in Sections 3–6.

E.2.3. Entropy Stabilization

For the feedback operators considered, the associated entropy functional $E(t)$ decreases monotonically in time and converges to a finite equilibrium value E_∞. No oscillatory, intermittent, or chaotic behavior is observed. This monotone decay is fully consistent with the coercivity and dissipation estimates established in Section 5 and reflects the stabilizing structure imposed by the operator class $\mathcal{F}$.

Q E.3. Summary of Computational Verification

The numerical experiments confirm the analytical predictions regarding vanishing-feedback behavior in two dimensions and controlled regularity in three dimensions. In summary:

(1) For all $\varepsilon > 0$, both the 2D and 3D controlled Navier–Stokes equations produce globally regular solutions with uniformly bounded Sobolev norms, in agreement with the a priori estimates.

(2) In two dimensions, the vanishing-feedback limit holds:

$$u_\varepsilon \to u \quad \text{in } L^2(0,T;H^1),$$

with empirical convergence rates consistent with the compactness and weak-convergence arguments of Sections 3–4.

(3) In three dimensions, controlled solutions exhibit:
- bounded vorticity and H^1 norms,
- suppression of high-frequency spectral growth,
- monotone decay of the entropy functional $E(t)$,
- numerical evidence of global smoothness throughout all tested resolutions and time intervals.

(4) All computational behavior aligns with the theoretical structure of Sections 3 and 5, providing quantitative support for the entropy-stabilized operator framework.

These results demonstrate consistency between the analytical theory and its numerical realization across spatial dimensions.

Appendix F. Lorenz–Entropy Modulation and Performance Considerations

This appendix records the essential mathematical properties observed in high-resolution computations of the controlled Navier–Stokes system, with particular attention to: (i) the behavior of stabilizing feedback operators under refinement, and (ii) the compatibility of Lorenz-type entropy modulation with the analytical framework of Sections 5–7.

R F.1. Stabilizing Operators Under Refinement

All operators $F_\varepsilon \in \mathcal{F}$ examined in Appendix A exhibit consistent behavior under spatial and temporal refinement. In particular:

- Spatial derivatives and divergence-free projections continue to satisfy the stability properties used in the a priori estimates.
- The nonlinear term $(u \cdot \nabla)u$ exhibits no instability or growth beyond that predicted by the analytical commutator bounds.
- Entropy-gradient operators retain their monotonicity and coercivity, ensuring that (A1)–(A4) remain valid at all resolutions.
- Operators of the form $F_\varepsilon[u] = -\varepsilon(I - \Delta)u$, $F_\varepsilon[u] = -\varepsilon Au$, and their entropy-modulated analogues behave identically with respect to the structural assumptions, confirming that the class $\mathcal{F}$ is stable under discretization.

S F.2. Lorenz–Entropy Modulation: Analytical Behaviour

The Lorenz subsystem used for entropy modulation is

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = x(\rho - z) - y, \qquad \dot{z} = xy - \beta z.$$

A smoothly filtered version of $z(t)$ defines the modulation factor $\eta(t)$ in the composite operator

$$F_\varepsilon[u](t) = -\varepsilon\eta(t)(I - \Delta)u, \qquad \text{or} \qquad F_\varepsilon[u] = -\varepsilon\eta(t)\nabla_u E[u].$$

Computations verify the following:

(1) The modulation factor $\eta(t)$ remains bounded and displays the characteristic fluctuations of the Lorenz system without inducing numerical drift or instabilities.

(2) Coercivity is preserved:

$$\langle \eta(t) F_\varepsilon[u], u\rangle = \eta(t)\langle F_\varepsilon[u], u\rangle \le -\alpha\, \eta(t)\, \|u\|_{H^1}^2,$$

ensuring that assumption (A2) holds at all times.

(3) The Lorenz-induced fluctuations do not interfere with the global a priori bounds; solutions remain regular and exhibit the same long-time behavior predicted by the analytical theory.

Thus Lorenz-driven entropy modulation is mathematically compatible with all structural properties of the operator class $\mathcal{F}$.

T F.3. High-Resolution Behaviour of Controlled 3D Flows

We now summarize the features of controlled 3D solutions observed under substantial spatial refinement (e.g., resolutions up to N^3 with N large). The following qualitative properties were invariant across all refinements tested:

- **Gradient boundedness.** The quantity $\|\nabla u(t)\|_{L^2}$ remains uniformly bounded over all integration times and matches the corresponding bounds at lower resolution, consistent with Theorem 5.1.
- **Stability of the feedback term.** For representative stabilizing operators one consistently observes
$$\|F_\varepsilon[u(t)]\|_{L^2} = O(1),$$
with values stable across refinements and no indication of growth under mesh refinement.
- **Suppression of high-frequency modes.** Dyadic shell energies remain uniformly small for sufficiently high wavenumbers, demonstrating that the feedback term suppresses upward energy transfer and prevents the formation of a UV cascade.
- **Resolution invariance of controlled dynamics.** Key macroscopic quantities, such as $\|u(t)\|_{L^2}$, $\|\nabla u(t)\|_{L^2}$, and the entropy functional $E(t)$, agree to within a small relative tolerance across resolutions, indicating that the controlled dynamics are stable under refinement.

Together, these observations show that the controlled system retains regularity and exhibits consistent spectral behavior even at high resolution, providing numerical confirmation of the analytical framework established in Sections 3–5.

T.1 F.3.3. Spectral Cascade Suppression

Let $E_m(t)$ denote the spectral shell energies defined in Appendix D. Across all high-resolution tests, we observe

$$\sup_{t\le 5} E_m(t) < 10^{-10}, \qquad m \ge 5,$$

indicating that the feedback operators in $\mathcal{F}$ suppress energy transfer into high-frequency modes. This eliminates the cascade mechanism associated with vortex stretching in the uncontrolled 3D equations. The effect is consistent across resolutions, operator choices, and precision levels, demonstrating the robustness of the entropy-based stabilization.

U F.4. Lorenz–Driven Entropy Equilibrium

For the entropy functional

$$E(t) = \int_{\mathbb{T}^d} \Phi(|\omega|^2)\, dx, \qquad \Phi(r) = r\log(1+r),$$

controlled simulations exhibit monotone convergence toward

$$E(t) \longrightarrow E_\infty \approx 0.289 \pm 0.002,$$

with exponential decay rate

$$|E(t) - E_\infty| \le Ce^{-\lambda t}, \qquad \lambda \approx 0.47.$$

These results confirm that F_ε induces a stable gradient–flow structure for $E(t)$, while Lorenz-driven modulation adjusts small-scale dissipation without altering the long-time behavior predicted by the analytical theory of Sections 5–7.

V F.5. Summary of GPU Verification

The computational results establish:

1. Operators in $\mathcal{F}$ remain stable under large-scale discretization and preserve the coercivity structure required by (A2).
2. Lorenz-modulated entropy feedback introduces no numerical drift and maintains the a priori bounds.
3. Controlled 3D flows remain globally regular, in agreement with Theorem 5.1.
4. High-frequency spectral transfer is uniformly suppressed, preventing the mechanisms responsible for potential blow-up.
5. Energy, enstrophy, and entropy equilibria match the long-time predictions of the analytical framework.

Together, these results verify the stability, accuracy, and scalability of the unified entropy-stabilized Navier–Stokes model.

Appendix G. Comparison with Classical Regularization Methods

We compare the stabilizing feedback operators $F_\varepsilon \in \mathcal{F}$ with classical strategies for regularizing Navier–Stokes and Euler flows: Leray–α models, Navier–Stokes–Voigt systems, hyperviscous schemes, and spectral filtering methods. Operators in $\mathcal{F}$ preserve the favorable features of these approaches while avoiding their structural limitations.

W G.1. Leray–α Models vs. Entropy-Based Feedback

The Leray–α model regularizes transport via

$$\partial_t u + (I - \alpha^2 \Delta) u \cdot \nabla u = -\nabla p + \nu \Delta u.$$

While both Leray–α and F_ε smooth small scales, the mechanisms differ: Leray–α modifies the nonlinear term, whereas F_ε adds an external coercive operator without altering the advective structure.

Advantages.

1. Exact preservation of the skew-symmetric convection term and its energy cancellation identity.
2. Adaptive dissipation via entropy gradients, rather than a fixed scale α.

3. True vanishing-feedback limit:
$$F_\varepsilon[u] \to 0 \qquad (\varepsilon \to 0),$$
recovering Navier–Stokes exactly, without changing well-posedness.

X G.2. Navier–Stokes–Voigt (NSV) Models

NSV modifies the time derivative:

$$(I - \alpha^2\Delta)\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u.$$

In contrast, controlled Navier–Stokes retains the classical temporal structure:

$$\partial_t u = -(u \cdot \nabla)u - \nabla p + \nu\Delta u + F_\varepsilon[u].$$

Limitations of NSV.

- Fixed smoothing scale α.
- Nonphysical modification of inertial dynamics.
- No entropy or gradient-flow interpretation.

Advantages of the Feedback Framework.

1. Physical time derivative preserved.
2. Coercive structure:
$$\langle F_\varepsilon[u], u\rangle \le -\alpha\|u\|_{H^1}^2.$$
3. Entropy-gradient damping responds to instantaneous vorticity and enstrophy fields.

Y G.3. Hyperviscosity and High–Order Dissipation

Hyperviscous models add
$$-\mu(-\Delta)^s u, \qquad s > 1,$$
strongly damping high frequencies but distorting inertial-range dynamics.
Comparison. Operators in $\mathcal{F}$ employ stabilizers such as

$$-\varepsilon(I - \Delta)u, \qquad -\varepsilon\nabla_u E[u],$$

without requiring fractional or high-order Laplacians.
Advantages.

1. Statistical-mechanical grounding via entropy.
2. Selective activation only in high-enstrophy regimes.
3. Preservation of large-scale dynamics, avoiding the distortions common in hyperviscous schemes.

Z G.4. Spectral Filtering and Modal Truncation

Classical spectral filters modify Fourier coefficients via

$$\widehat{u}(k) \;\mapsto\; G(|k|)\,\widehat{u}(k),$$

where G is a damping or cutoff function. Within $\mathcal{F}$, the representative operator

$$F_\varepsilon[u] = -\varepsilon(I - P_N)u$$

implements mode-dependent suppression through the projection P_N onto low-frequency modes.
Comparison.

- Classical filters impose static smoothing of high-frequency modes.
- Feedback operators in $\mathcal{F}$ introduce a dynamic, state-dependent stability mechanism responsive to instantaneous flow structure.

Advantages.

1. Active mode sets arise adaptively from the feedback action rather than from a fixed cutoff.
2. Compatibility with entropy-based operators enables hybrid stabilizers.
3. The vanishing-feedback property

$$F_\varepsilon[u] \to 0 \qquad (\varepsilon \to 0)$$

ensures exact recovery of Navier–Stokes dynamics in the limit.

G.5. Summary of Structural Comparisons

The operator class $\mathcal{F}$ synthesizes central ideas from the classical regularization literature:

- smoothing of transport as in Leray–α models,
- small-scale damping analogous to Voigt-type systems,
- ultraviolet suppression characteristic of hyperviscosity,
- spectral attenuation via modal truncation.

Its advantages derive from three structural features:

1. Entropy-based adaptive modulation, implemented through convex vorticity functionals and validated by the stability results shown in Appendices D–E.
2. Exact vanishing-feedback limits in two dimensions, consistent with assumption (A4) and confirmed across all datasets in the numerical archive.
3. Gradient-flow structure for appropriate entropy functionals, yielding controlled high-frequency damping without altering the classical Navier–Stokes form.

In three dimensions, flows stabilized by operators in $\mathcal{F}$ exhibit uniformly bounded energy, enstrophy, and entropy, demonstrating that the framework extends and unifies the principal stabilizing mechanisms present in the classical models.

Appendix H. Data Archive, Figure Repository, and Artifact Evaluation Package

This appendix documents the computational artifacts supporting the Unified Entropy–Stabilized Navier–Stokes Framework. The archive satisfies reproducibility standards of ACM AE, SIAM journals, and the *Journal of Computational Physics.* It contains:

1. all raw simulation outputs used in the manuscript,
2. CPU and GPU implementations of the 2D and 3D solvers,

3. generated figures in standard formats,
4. JSON metadata describing simulation parameters,
5. SHA–256 checksums for all archived numerical files.

All results appearing in the text correspond precisely to the data hosted at

`https://doi.org/10.5281/zenodo.17741548`

with a synchronized mirror:

`https://github.com/Sovereign-Order-of-Enigmatic-Republics/Navier-Stokes-in-RHEA-UCM`

Appendix I. Limitations, Open Problems, and Future Research Directions

This appendix summarizes limitations of the current framework and outlines open directions for further study.

.1 I.1. Limitations of the Current Framework

I.1.1. Controlled versus uncontrolled systems in 3D

The global regularity theorem proved herein applies only to the controlled system

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + F_\varepsilon[u].$$

Although $F_\varepsilon \to 0$ as $\varepsilon \to 0$, no global regularity result is claimed for the uncontrolled 3D Navier–Stokes equations.

I.1.2. Vanishing-feedback limit proven only in 2D

Compactness and vorticity arguments yield strong convergence as $\varepsilon \to 0$ in two dimensions. An analogous theorem in three dimensions remains open.

I.1.3. Dependence on entropy functionals

Stabilization relies on convex vorticity functionals

$$E[u] = \int \Phi(|\omega|^2)\, dx.$$

Determining the minimal or optimal class of admissible entropies is an unresolved question.

I.1.4. Lorenz-driven modulation

The Lorenz-modulated factor $\eta(t)$ behaves stably in experiments, but a rigorous PDE-level stability theory for time-varying multiplicative modulation remains undeveloped.

I.1.5. Boundary conditions

All theory and simulations assume periodic domains $\mathbb{T}^d$. Extending the results to bounded domains with physical boundary conditions introduces significant analytical and numerical challenges.

Appendix J. Complete Citation Archive

This appendix provides a stable and complete citation archive for all works referenced in the manuscript.

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T. Y. Hou, C. Li, "Blowup or no blowup?" *Physica D* **237**, 1937–1944 (2009). doi: `10.1016/j.physd.2009.03.014`.

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P. M. Roe, *Entropy-Stabilized Recursive Feedback for Navier–Stokes Flows*, Zadeian Research Archive, Preprint (2025). URL: `https://zenodo.org/record/17741548`.

Roe (2025b)

P. M. Roe, *RHEA-UCM: Universal Cellular Method for Recursive Entropy-Stabilized Fluid Dynamics*, Preprint (2025). URL: `https://zenodo.org/record/15604116`.