MAT 216

Linear Algebra & Fourier Analysis

Summer 2020
Week 1 Lecture 1
Lecture Note

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Contents:

- ***Matrices***
- ***Matrix Operations & Properties***
- ***Vectors***
- ***Vector Operations***

Reference Book:

Introduction to Linear Algebra, 5th Ed. Gilbert Strang

Matrices

Matrix:

A matrix is a rectangular array of number

$$\begin{pmatrix} 1 & 2 & 5 & 2 \\ 0 & 3 & 9 & 4 \\ 8 & 3 & 9 & 8 \end{pmatrix}$$

Why do we need matrix?

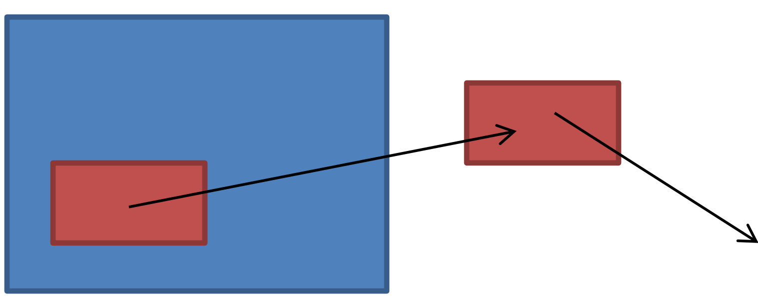
Matrices can store information structurally and efficiently in a way.

Storing information in a Matrix:

When we look at a picture in a computer, we are watching a graphical representation of a data.

Data in **computers** is **stored** and transmitted as a series of ones and zeros (also known as Binary). To store an image on a **computer**, the image is broken down into tiny elements called pixels. A pixel (short for **picture** element) represents one color.

Every pixel can have a value from 0 to 255. We can store every value of every pixel in a rectangular matrix. That is how the pictures are stored in our computer or we can say this is one of the ways to store data.



15	**45**	**123**	**211**	**…**	**…**
0	**212**	**21**	**5**	**…**	**…**
10	**12**	**200**	**156**	**…**	**…**
11	**21**	**12**	**21**	**…**	**…**
…	**…**	**…**	**…**	**…**	**…**

Consider an example:

Harry, Rom and Hermione live in the same street of a neighborhood. In last week they had conversation with each other in the following manner.

Hermione didn’t call Ron at all, as she is mad at him

Harry called Hermione twice

Hermione called Harry once

Harry called Ron twice

Ron called Harry thrice

Ron called Hermione once

If we convert their conversation strategy in matrix it turns out:

	Harry	**Ron**	**Hermione**
Harry	**0**	**2**	**2**
Ron	**3**	**0**	**1**
Hermione	**1**	**0**	**0**

We can operate on this data using various matrix operations

<u>Matrix Operations:</u>

Addition/Subtraction: The dimensions of the matrices must be equal

$$\begin{pmatrix} 5 & 5 \\ 0 & -8 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 9 & -3 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} 5+6 & 5+0 \\ 0+9 & -8+(-3) \\ 3+7 & 1+(-4) \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 9 & -11 \\ 10 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 6 & 3 & -9 \\ 4 & 6 & 3 \end{pmatrix} + \begin{pmatrix} -8 & 3 & 4 \\ 6 & -3 & 7 \end{pmatrix} = \boldsymbol{Not\ possible\ to\ evaluate}$$

$$\begin{pmatrix} 5 & 5 \\ 0 & -8 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 9 & -3 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} 5-6 & 5-0 \\ 0-9 & -8-(-3) \\ 3-7 & 1-(-4) \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -9 & -5 \\ -5 & 5 \end{pmatrix}$$

Elementary Row Operations:

Three elementary row operations for matrices:

- Interchanging Rows

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 6 & 8 \\ 0 & 4 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 4 & 9 \\ 5 & 6 & 8 \\ 1 & 0 & 2 \end{pmatrix} \mathrm{R}_1, \mathrm{R}_3 \text{ interchanged}$$

- Multiplying a row with a scalar

$$\begin{pmatrix} 0 \times 2 & 4 \times 2 & 9 \times 2 \\ 5 & 6 & 8 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 8 & 18 \\ 5 & 6 & 8 \\ 1 & 0 & 2 \end{pmatrix}$$

- Adding or Subtracting and multiple of two rows

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 6 & 8 \\ 0 & 4 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 5-5\times 1 & 6-5\times 0 & 8-5\times 2 \\ 0 & 4 & 9 \end{pmatrix} R_2 \rightarrow R_2 - 5R_1$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 6 & -2 \\ 0 & 4 & 9 \end{pmatrix}$$

Multiplication:

Matrix multiplication is an operation that produces a matrix from two matrices. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$\begin{array}{c} \\ \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{pmatrix} \end{array} \begin{array}{c} \boldsymbol{R_1} \\ \boldsymbol{R_2} \\ \boldsymbol{R_3} \end{array} \times \begin{pmatrix} \overbrace{\boldsymbol{G}}^{\boldsymbol{C_1}} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \boldsymbol{R_1 C_1} \\ \boldsymbol{R_2 C_1} \\ \boldsymbol{R_3 C_1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A.G + B.H} \\ \boldsymbol{C.G + D.H} \\ \boldsymbol{E.G + F.H} \end{pmatrix} \text{ It acts like dot products}$$

We have $(3 \times \mathbf{2})$ matrix multiply with $(\mathbf{2} \times 1)$ matrix. Here number of **columns** in the first matrix must be equal to the number of **rows** in the second matrix.

$$\begin{bmatrix}1 & 2 & -3\\0 & 4 & 5\\7 & 0 & 6\end{bmatrix}\begin{bmatrix}1 & 0\\-2 & 3\\1 & 1\end{bmatrix}=\begin{bmatrix}-6 & 3\\-3 & 17\\13 & 6\end{bmatrix}$$

$$3\times 3 \qquad 3\times 2 \qquad = 3\times 2$$

$$[\quad][\quad] = ?$$

Not possible

$$(2\times 2).(3\times 2)$$

If A is a matrix and B is another matrix and consider that it is possible to evaluate AB and BA,

But $AB \neq BA$

$$\boldsymbol{AB} = \begin{pmatrix}\mathbf{1} & \mathbf{2}\\\mathbf{3} & \mathbf{4}\end{pmatrix}\begin{pmatrix}\mathbf{4} & \mathbf{3}\\\mathbf{2} & \mathbf{1}\end{pmatrix} = \begin{pmatrix}\mathbf{8} & \mathbf{5}\\\mathbf{20} & \mathbf{13}\end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix}\mathbf{4} & \mathbf{3}\\\mathbf{2} & \mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \mathbf{2}\\\mathbf{3} & \mathbf{4}\end{pmatrix} = \begin{pmatrix}\mathbf{13} & \mathbf{20}\\\mathbf{5} & \mathbf{8}\end{pmatrix}$$

Working with the information stored in a matrix:

$$2x + 3y + 4z = 0$$
$$x - 2y + 2z = 0$$
$$y + z = 0$$

These are linear equations since all the variables have power 1. Together they call a system of linear equation. This system of linear equation can be stored in a matrix

$$\begin{pmatrix}2 & 3 & 4\\1 & -2 & 2\\0 & 1 & 1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Rotation of Matrix:

To rotate counterclockwise by:

<u>90 degrees</u>   **<u>180 degrees</u>**   **<u>270 degrees</u>**

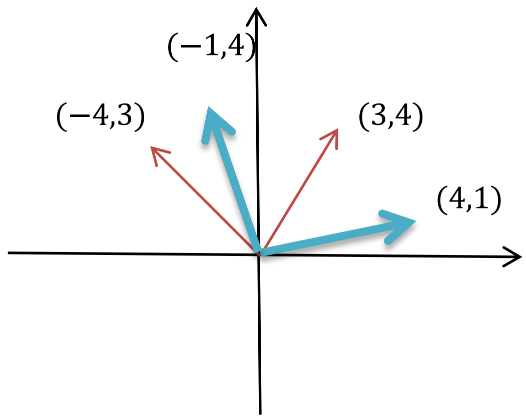
Multiply by:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Example:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

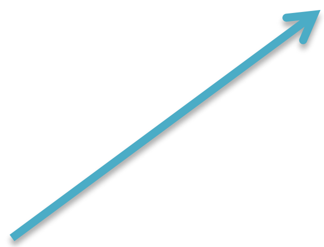


This is one of the properties of ***Linear Transformation*** which will be discussed later in this course.

Vectors

According to Physicist:

A vector is a quantity that has both a magnitude and a direction.



Let's say the magnitude of this arrow is 5 cm and the direction is North-East. As long as the direction and the magnitude are the same, it does not matter where the vector is. A vector can start anywhere as long as it goes to the same direction with same length represents the same vector.

According to Computer Scientist:

A vector is an ordered list of numbers. Consider a computer with the following specifications:

Ram	32 GB
HDD	2 TB
GPU	8 GB

The vector of the above information will be $\begin{bmatrix} 32 \\ 2048 \\ 8 \end{bmatrix}$.

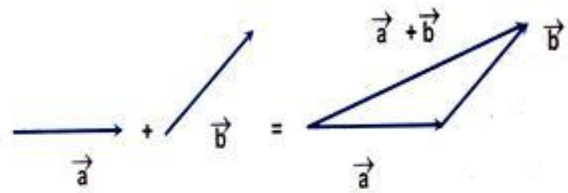
Please note $\begin{bmatrix} 32 \\ 2048 \\ 8 \end{bmatrix} \neq \begin{bmatrix} 2048 \\ 8 \\ 32 \end{bmatrix}$. So the order is important.

$\begin{bmatrix} 2048 \\ 8 \\ 32 \end{bmatrix}$ indicates $\left(\begin{array}{c|c} \text{Ram} & 2\ TB \\ \text{HDD} & 8\ GB \\ \text{GPU} & 32\ GB \end{array}\right)$, which not the fact is.

<u>Vector Operations:</u>

- **Adding Two Vectors**

Graphical representation of vector addition:

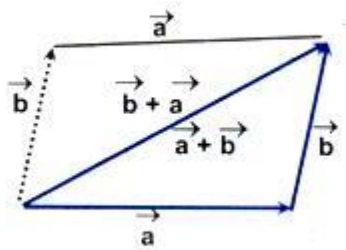


Example:

$\begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}9\\7\end{bmatrix}$,

$\begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}5\\4\end{bmatrix} = \begin{bmatrix}10\\4\end{bmatrix}$ etc.

Also note: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$



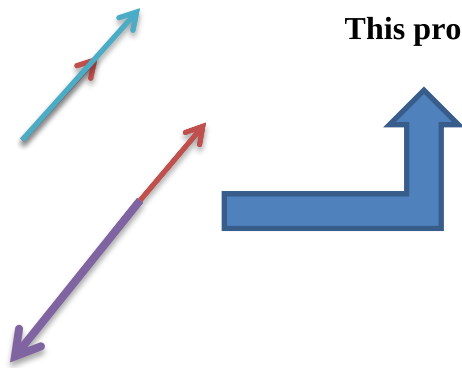
- **Scaling and Scalars**

A real number is sometimes called a **scalar**. **Scaling** a geometrical **vector** means keeping its orientation the same but changing its length by a **scale** (factor). It is like changing the **scale** of a picture. The objects may expand or shrink, but the directions remain the same.

$\mathbf{1.5} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 6 \end{bmatrix}$

This process is known as scaling

Scalars



$\mathbf{-2} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$

<u>Magnitude of a Vector</u>

If $\vec{V} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ then the magnitude of the vector is $\left\|\vec{V}\right\| = \sqrt{2^2 + 3^2 + 5^2}$.

Hence if $\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ then the magnitude of the vector is $\left\|\vec{V}\right\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$

Also $\left\|\vec{V}\right\| = \sqrt{\vec{V}.\vec{V}}$

<u>Product of Two Vector: Dot Product or Inner Product</u>

Let $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, then $\vec{V}.\vec{U} = v_1.u_1 + v_2.u_2 + v_3.u_3$

Also $\vec{V}.\vec{V} = v_1.v_1 + v_2.v_2 + v_3.v_3 \rightarrow$ Square of the magnitude or **norm** of $\vec{V}$.

$\vec{V}.\vec{U} = \left\|\vec{V}\right\|.\left\|\vec{U}\right\|.cos\theta \quad \rightarrow \quad \theta$ is the angle between these two vectors.

- If the two vectors have the same direction, dot product gives the maximum value.
- As the angle increases, the value of the dot product decreases. At 90 degree the value reaches0.

- If the angle exits 90 degree on either side, the value of the dot product becomes negative.

  If two vectors are closer to each other they have higher value of dot product.

  $cos\theta = \frac{\vec{V}.\vec{U}}{||\vec{V}||.||\vec{U}||}$ $\rightarrow$ This relationship is known as the **Cosine Similarity**.

  <u>Example of Cosine Similarity:</u>

  Two very short texts to compare:

  Text 1: Julia loves me more than Linda loves me.

  Text 2: Jane likes me more than Julia loves me.

We want to know how similar these texts are, purely in terms of word counts (and ignoring word order). We begin by making a list of the words from both texts:

Me, Julia, loves, Linda, than, more, likes, Jane

Now we count the number of times each of these words appears in each text:

	Text 1	Text 2
Me	2	2
Julia	1	1
Loves	2	1
Linda	1	0
Than	1	1
More	1	1
Likes	0	1
Jane	0	1

We are not interested in the words themselves. We are interested only in those two vertical vectors of counts. For instance, there are two instances of 'Me' in each text. We are going to decide how close these two texts are to each other by calculating one function of those two vectors, namely the cosine of the angle between them.

The two vectors are:

$\vec{V} = \{2, 1, 2, 1, 1,1,0,0\}$, $\vec{U} = \{2, 1, 1, 0, 1, 1, 1, 1\}$

The cosine of the angle between them is about 0.822.

These vectors are **8-dimensional**. A virtue of using cosine similarity is clearly that it converts a question that is beyond human ability to visualize to one that can be. In this case we can think of this as the angle of about 35 degrees which is some 'distance' from zero.