

What is Probability

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Outline

Paradox of Probability Theory

Galton Board

Natural Sciences and Mathematics

Rolling Dice

More Examples

Predicting Unpredictable

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- tossing a coin

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(0=head,1=tail)

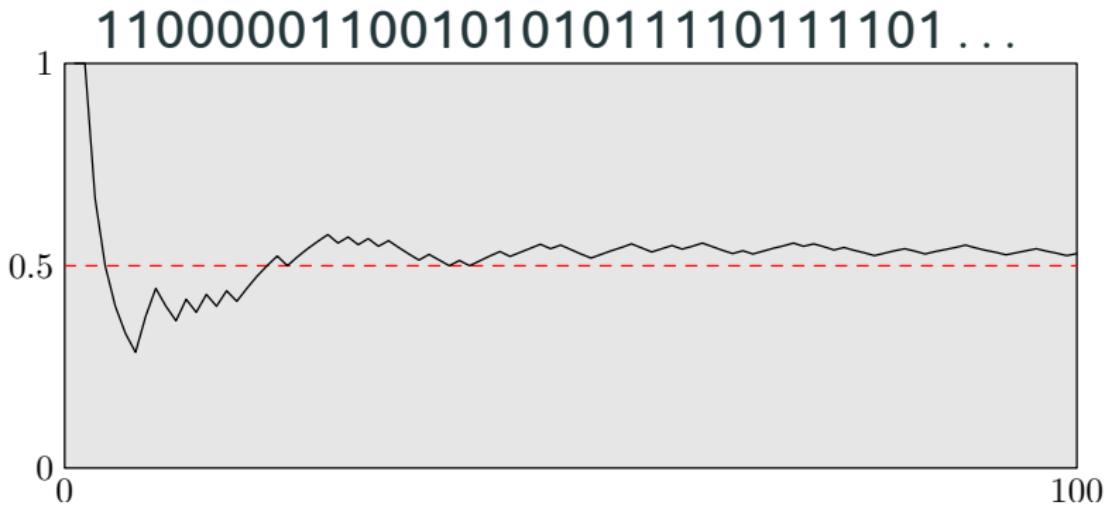
Predicting Unpredictable

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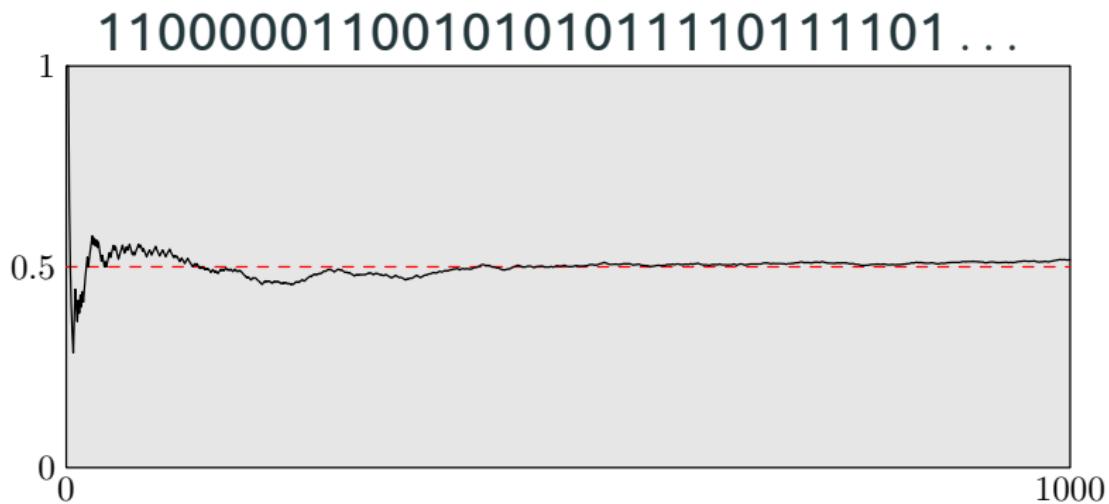
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- *frequency of 1s:* (#ones)/(length) $\approx 1/2$

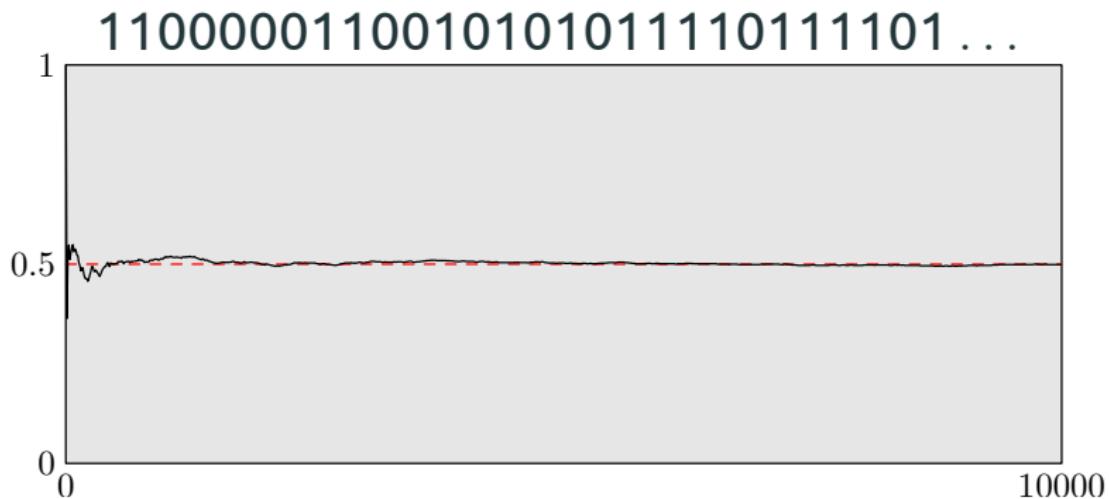
Hundred Random Bits



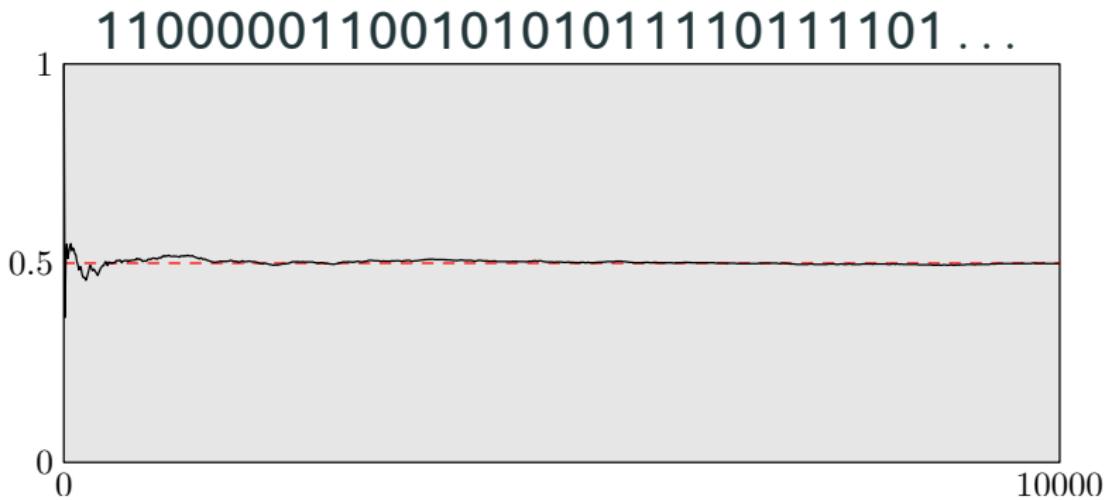
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Ten Thousands Random Bits



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(produced by George Marsaglia, 1995)

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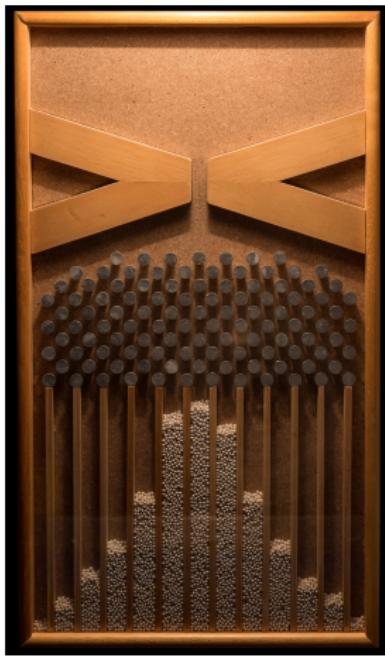
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Bean Machine



https://en.wikipedia.org/wiki/Bean_machine

Sir Francis Galton (1822–1911)

Galton Board: animation

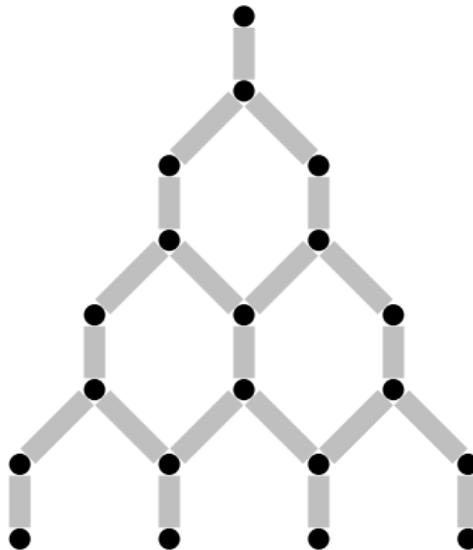
TO BE REPLACED BY VIDEO FROM

https://en.wikipedia.org/wiki/File:Galton_box.webm

Analysis

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Assume that at each level the beans are splitted evenly.



Galton and Pascal

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Pascal triangle/ 2^n :

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Pascal triangle/ 2^n : $\binom{n}{k}/2^n$

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 - mathematics: the implications of the model

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- Dan: in fact, Alice is right for the real coins

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- among the beans that go left [right] at level 1, half go left [right] at level 2, etc.

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 - 2 out of 6 (1, 2, 3, 4, 5, 6)

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$P = \frac{15}{36}$

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- simultaneous and sequential setting
- equiprobable model is usually OK for both settings

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- $\sum_{k \in [0.4n, 0.6n]} \binom{n}{k} / 2^n$

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