Gale's Roundtrip Jeep Problem

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1 author:

Brad Jackson
San Jose State University

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Alan Hausrath*
Department of Mathematics
Boise State University
Boise, Idaho 83725

Bradley Jackson
John Mitchem*
Edward Schmeichel†
Department of Mathematics and Computer Science
San Jose State University
San Jose, California 95192

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Abstract

We solve the following longstanding open problem of David Gale: Maximize the round trip distance that a jeep can travel given x units of fuel shared between 2 depots at opposite ends of the desert. We also solve the analogous problem where the depots can be placed anywhere in the desert.

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In 1947 Fine [FIN] introduced and solved a problem of maximizing the distance a jeep can travel into the desert using n drums of fuel. Subsequently, Phipps [PHI], Alway [ALW], and Gale [GAL] gave other solutions to the original problem or considered related problems. As mentioned in [FIN], the original problem is similiar to one which arose in air transport operations in the China theater during World War II, and it has been suggested that there may be applications to Arctic expeditions and interplanetary travel.

Near the end of [GAL], the author states, "An apparently simple question is the round trip problem in which fuel is available at both ends of the desert, but I must confess ... that I have not been able to find the solution. It is not hard to see that one can do at least as well in this case as in the case of two jeeps making one-way trips, but it may be possible to do better. The difficulty here as with many optimization problems is that there does not appear to be any simple way to determine whether or not a given solution is optimal."

Gale's problem can be interpreted in two equivalent ways. (i) Given unlimited fuel at each end of a desert of given length, find a round trip across the desert which uses as little fuel as possible. (ii) Given a fixed amount of fuel which can be distributed between the two ends of a desert, find the maximum length desert which can be crossed in a round trip using the available fuel. We find it convenient to consider (ii) and give an optimal solution for it. We also describe a solution for the analogous round trip problem where the two allowed depots may be placed anywhere in the desert.

In each of the above problems the jeep can carry exactly 1 drum. It is implicit that the jeep can store whatever fraction of a drum is desired at any point in the desert. (Perhaps the driver carries large plastic bags for fuel storage.) In [DEW], Dewdney proposed an interesting variation of the one-way problem. Although Dewdney's problem was given in terms of drums, gallons, and miles, it can be rephrased as follows: Find the maximum distance a jeep can travel into the desert using n drums of fuel where the jeep can carry 1 drum plus 1/5 of a drum in its tank, but only drums can be stored. That is the jeep can dump at most 5/6 of its fuel capacity in the desert. It is interesting to note that Dewdney's problem has been solved as a linear programming problem; an optimal algorithm for Dewdney's problem appears in [JAC]. But the problems solved in this paper apparently are not easily posed as either linear or dynamic programming problems. In [GAL], Gale also points out that "there is a feeling among many people that the original jeep problem can be solved by the functional equation method of dynamic programming . . . I know of no way of solving the problem by this method."

1 A Brief Description Of The Solution To Gale's Problem

In solving Gale's problem we will start by considering the longest desert which can be crossed in a round trip if there are m drums of fuel at the start S and k drums of fuel at the finish F. Let $D(f)$ denote the length of the optimal one-way trip using f drums of fuel. If $m \leq k$, it is clear that one can do no better than $D(m)$ and should use the S-fuel outbound and the F-fuel returning. For $m > k$, going $D(m)$ outbound will not work as the jeep is unable to return to S. Instead, in order to make full use of the drums at S, on the outbound trip a number of depots are created leaving fuel for the return. Let T denote the location of the depot furthest from S. We prove that the following highly plausible qualitative conditions determine an optimal solution: (i) Use only S-fuel when going from S to F. (ii) Use only F-fuel when returning from F to T. (iii) Use only S-fuel stored at the depots, when returning from T to S. The solution then follows by putting together solutions of previously solved jeep problems. Thus it follows from (ii) that the distance from F to T is $D(k)$, and the distance from T to S is obtained by solving a slight variation of the well-known round trip jeep problem with fuel only at S.

To finish Gale's problem, we need only find the optimal distribution of the available fuel between S and F.

2 Original Problems

We have x drums of fuel available at the edge of the desert and a jeep which can carry at most 1 drum. Here we give the well known algorithm for maximizing the one-way distance, and an algorithm for maximizing the round trip distance for the jeep using x drums of which k must be delivered to F. One unit of distance will be the distance that the jeep can travel on one drum of fuel. We assume that the jeep's efficiency is constant. It does not depend on wind, weather, weight, or depth of the ruts in the sand. The algorithms and their optimality proofs which we give are based on work appearing in [Gal], [Phi], [Fin], and [Max, Section 10.9].

Theorem A. Given $n + f$, $0 \le f < 1$, drums of fuel at the start and a jeep with capacity of 1 drum, the maximum one-way distance which the jeep can travel is $D_1 = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \frac{f}{2n+1}$.

PROOF. We begin with an algorithm which achieves distance D_1. First assume $f = 0$. Repeat n times: Put 1 drum of fuel into the jeep, drive forward $\frac{1}{2n-1}$ units, store $1 - \frac{2}{2n-1}$ units, and return to the previous fuel dump, except on the nth iteration do not return. We now have $(n-1)(1 - \frac{2}{2n-1}) + 1 - \frac{1}{2n-1} = n - 1$ drums of fuel, and the jeep at distance $\frac{1}{2n-1}$ from the previous dump. Iterate this process, replacing n successively by $n-1, n-2, \ldots, 1$.

If $f > 0$, begin the above process by first moving all $n+1$ drums forward $\frac{f}{2n+1}$ units, thus delivering n full drums to the first fuel dump.

In order to show that D_1 is the maximum attainable distance when $f = 0$, we let $x_i, 0 \le i \le n$, denote the point on segment SF such that the total distance traveled on the right of x_i is i. (We emphasize that the distance traveled is on the right, not 'toward' the right.) Then $x_n = S$ and $x_0 = F$. Since the jeep used exactly k units of fuel while traveling on the right of x_k, at least k units must arrive at x_k. Let P be any point between x_{k+1} and x_k.

Then more than k units of fuel must have crossed P, and thus the jeep crossed P while traveling toward the right at least $k+1$ times. It follows that the jeep crossed P at least k times while going toward the left. Hence the jeep crossed P at least $2k+1$ times and the distance between x_{k+1} and x_k, denoted (x_{k+1}, x_k) is at most $\frac{1}{2k+1}$. Thus we have that the distance from S to F is at most $(x_n, x_{n-1}) + (x_{n-1}, x_{n-2}) + \cdots + (x_1, x_0) \le \frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{3} + 1 = D_1$. Similarly when $f > 0$, the distance from S to

x_n is at most $\frac{f}{2n+1}$. This completes the proof of Theorem A.

THEOREM B. Let m,k be integers, $0 \leq f, g < 1$, and $m + f > k + g$. Given $m + f$ drums of fuel at S and a jeep with capacity 1 drum, the maximum round trip distance in which the jeep delivers $k + g$ drums to F is

$$D_2 = \frac{g}{2m+2} + \frac{1}{2m} + \frac{1}{2m-2} + \cdots + \frac{1}{2k+4} + \frac{1-f}{2k+2}.$$

PROOF. Let x_i be the point such that exactly i units of S-fuel are used to the right of that point or delivered to F, $m + g \geq i \geq k + f$. Let P be a point between x_i and x_{i-1}, for $m \geq i \geq k + 2$. Then P is crossed at least $2i$ times using S-fuel and dist $(x_i, x_{i-1}) \leq \frac{1}{2i}$. Also if P is between S and x_m, then P is crossed at least $2m + 2$ times using S-fuel so that dist $(S, x_{n-k-1}) \leq \frac{g}{2m+2}$. Similarly if P is between x_{k+1} and T, then P is crossed at least $2k+2$ times using S-fuel and dist $(x_{k+1}, T) \leq \frac{1-f}{2k+2}$. Thus the distance between S and F is at most D_2.

We now give an algorithm which uses $m + g$ drums of fuel for a round trip of length D_2 which delivers $k+f$ drums of fuel to F. At S, repeat $m+1$ times: Put $\frac{m+g}{m+1}$ units into the jeep, drive forward $\frac{g}{2m+2}$ units and leave all fuel except just enough to return to S. On the last trip do not return to S but leave all fuel at this dump. For each $i, m \leq i \leq k + 2$, fill the jeep i times and each time go forward $\frac{1}{2i}$ units. On each of the first $i - 1$ times dump $1 - \frac{1}{i}$ units and return. The last time dump $1 - \frac{1}{2i}$ units. Now we use $i - 1$ units to go on to the next fuel dump, having left $\frac{1}{2i}$ for the return. Finally, use $1 - f$ units to deliver $k + f$ units for the continuing outbound trip to a dump at distance $\frac{1-f}{2k+2}$ from the current dump plus $\frac{1-f}{2k+2}$ units for the return. This completes the proof of Theorem B.

We observe that when $k + g = 0$, Theorem B gives the well-known maximum length desert which can be crossed in a round trip using only fuel from S.

3 Round Trips With 2 Fuel Depots

We consider two different problems, one in which we have the depots at each end of the desert, and one in which we can put the second depot at any point. Of course the first depot must be at the start. In both problems we want to maximize the length of the desert which can be crossed for a fixed amount of fuel.

We begin with a theorem which gives the maximum length desert which can be crossed given that some fixed amounts of fuel are available at each end of the desert.

Suppose that a total of x drums of fuel are available. One way to proceed is to divide the fuel equally between S and F. Then use the one way algorithm for traveling from S to F and for returning to S from F. We show, however, that it is more efficient to allocate less than half the fuel to F. In fact the maximum distance is achieved when F receives only $k = \left\lfloor \left(\frac{x+1}{2}\right)^{\frac{1}{2}} \right\rfloor$ drums of fuel and $x - k$ drums are available at S.

If $x - k_1(k_1)$ drums of fuel are available at $S(F)$, Theorem 1 establishes the maximum desert length that can be crossed. The algorithm for attaining that distance establishes a point T between S and F. The distance from T to F is the maximum one way distance which can be crossed using k_1 drums of fuel. The jeep will travel from S to T in the same manner as the round trip algorithm given in Theorem B. It will deliver k_1 drums of fuel to T, which will be used on a one way trip from T to F, plus a small amount of extra fuel for the return trip to S. On the return trip the jeep uses the k_1 drums of fuel at F to reach T, and from T uses the fuel at each depot to reach the next depot eventually returning to S. These ideas are illustrated in Figures 1 and 2 for $5\frac{3}{4}$ drums at S and $2\frac{2}{3}$ drums at F. In Figure 1 at each D_i the pair (r, s) denotes r units of fuel delivered to D_i for use to the right of D_i and s units delivered to D_i to be used on the return trip from S. In Figure 2 the number at each D_i is the amount of fuel used on the return trip to the left of D_i.

Before stating and proving Theorem 1 we make a definition which for $m = 5, g = \frac{3}{4}, k = 2, f = \frac{2}{3}$ gives the distance shown in Figure 1 (as well as in Figure 2).

For m, k integers, $0 \leq f, g < 1$, and $m + g \geq k + f$ we define

$$(1) \qquad d(m+g, k+f) = \frac{g}{2m+2} + \frac{1}{2m} + \frac{1}{2m-2} + \cdots + \frac{1}{2k+4} + \frac{1-f}{2k+2} + \frac{f}{2k+1} + \frac{1}{2k-1} + \cdots + \frac{1}{3} + 1.$$

THEOREM 1. Suppose there are $m + g$ drums of fuel at S and $k + f$ drums at F, where m is a positive integer, k is a non-negative integer, $m + g \geq k + f$, and $0 \leq f, g < 1$. Then the maximum distance between S and F is $d(m + g, k + f)$.

Before proving the theorem we make some definitions and prove a lemma. A *feasible solution* to a jeep problem is any trip which obeys the rules of the

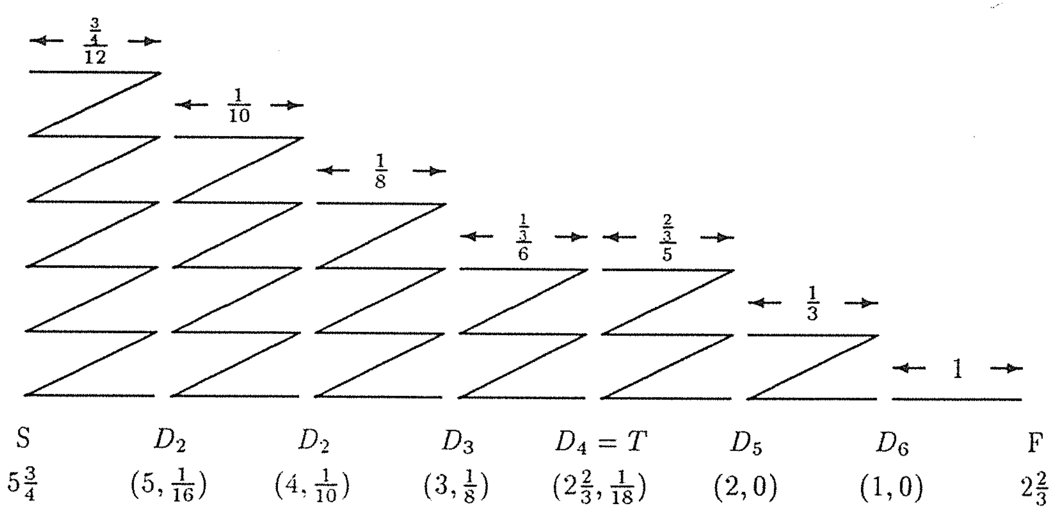

Figure 1. Outbound Trip from S to F

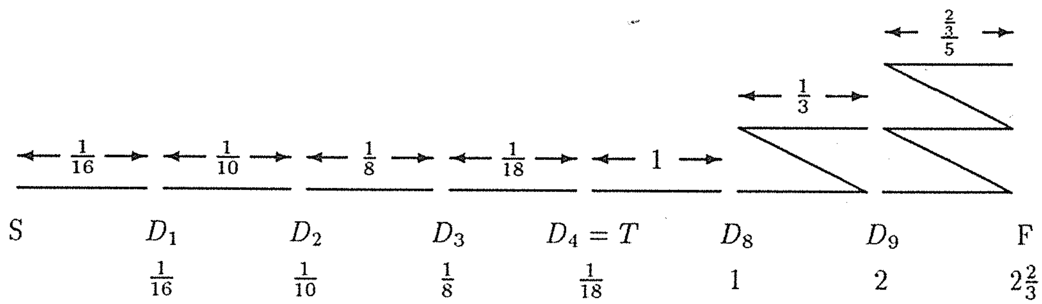

Figure 2.
Return Trip from S to F

problem. The *value* of the feasible solution is the distance from S to F, and an *optimal solution* is any feasible solution with maximum value.

LEMMA 1. For any feasible solution $\mathcal{FS}$ that entails S-fuel arriving by jeep at point F, there is a new feasible solution $\mathcal{FS}'$ at least as large in which no S-fuel arrives at F.

PROOF OF LEMMA 1. Suppose S-fuel arrives at F. Without loss of generality we may assume that when S-fuel first arrives at F, say at time t_0, there is F-fuel still there. After time t_0 each time the jeep leaves F let the percentage of S-fuel in the tank equal the percentage of S-fuel at F. Let M be the minimum amount of F-fuel which leaves F after time t_0. Let $F' =$ max $\{$ $F - M$, the return trip turnaround point closest to F, return trip fuel dump closest to F but different from F.$\}$

In the new solution $\mathcal{FS}'$ the jeep does exactly the same movement as before, but takes no S-fuel to F, instead S-fuel is left at F'. On the return trip S-fuel which was previously taken on at F is added at F'. This proves Lemma 1.

PROOF OF THEOREM 1. By Lemma 1 we may restrict our attention to feasible solutions in which no fuel arrives at F. We define the return trip as all travel after the jeep's first arrival at F. Let T be the point in $[S, F]$ farthest from F which can be reached from F using only F-fuel. By Theorem A, $\mathrm{dist}(T, F) = 1+\frac{1}{3}+\cdots+\frac{1}{2k-1}+\frac{f}{2k+1}$. We use y to denote this distance.

We give now another lemma.

LEMMA 2. Suppose there is a feasible solution $\mathcal{FS}$ in which S-fuel is used to the right of T on the return trip. Then there is a new feasible solution $\mathcal{FS}'$ at least as large in which no S-fuel is used to the right of T on the return trip.

PROOF OF LEMMA 2. Change $\mathcal{FS}$ to $\mathcal{FS}'$ as follows. On the outbound trip the travel is exactly the same as in $\mathcal{FS}$ except any S-fuel stored in $(T, F]$ for use on the return is instead stored at T. In the return trip of $\mathcal{FS}'$ the first travel is that required to cover the distance y from F to T. In traveling from T back to S the jeep travels as it did in $\mathcal{FS}$ taking on fuel at T for travel to the left. This is possible because in $\mathcal{FS}'$ we have at least as much fuel available at T as in $\mathcal{FS}$ and the same fuel between S and T as in $\mathcal{FS}$. This completes the proof of Lemma 2.

By Lemma 2 we may consider only feasible solutions in which dist $(T, F) = y$ and $k + f$ units of S-fuel must arrive at T for the outbound journey. Hence we only need to maximize dist (S, T), given that $k+f$ units

x (total number of drums)	[2,7)	[7,16)	[17,31)	[31,49)	[49,71)	[71,97)
k (number of drums at F)	1	2	3	4	5	6

Table 1

of fuel must arrive at T for the outbound trip, and only $m + g$ units are available at S. An optimal distance and algorithm are given in Theorem B.

This completes the proof of Theorem 1.

Theorem 2 gives the maximum length desert which can be crossed using x drums of fuel divided between S and F. It shows that $k = \left\lfloor \left(\frac{x+1}{2} \right)^{\frac{1}{2}} \right\rfloor$ drums should be placed at F, and thus the distance from T to F will be the maximum one way distance D_i, as given in Theorem A, that can be traveled using k drums of fuel.

Table 1 gives the number of drums at F for various values of x.

In order to prove Theorem 2 it is convenient to first prove another lemma, which shows that we can assume there are an integer number of drums at F.

LEMMA 3. Let $m \geq 1$ and $k \geq 0$ be integers, $0 \leq f, g < 1$, and $m+g \geq k+f$. Then

$$d(m+g, k+f) \leq \max\{d(m+f+g, k), d(m+f+g-1, k+1\}.$$

PROOF. Suppose first that $f+g<1$. Then we have

$$\begin{aligned} &d(m+f+g, k) - d(m+g, k+f) = \\ &f\left(\frac{1}{2m+2} - \frac{1}{2k+1} + \frac{1}{2k+2}\right), \end{aligned} \tag{2}$$

while

$$\begin{aligned} &d(m+f+g-1, k+1) - d(m+g, k+f) = \\ &-(1-f)\left[\frac{1}{2m} - \frac{1}{2k+1} + \frac{1}{2k+2}\right] + g\left[\frac{1}{2m} - \frac{1}{2m+2}\right] \end{aligned} \tag{3}$$

Since $\left[\frac{1}{2m} - \frac{1}{2m+2}\right] > 0$, the right side of either (2) or (3) will be nonnegative unless $\frac{1}{2m+2} < \frac{1}{2k+1} - \frac{1}{2k+2} < \frac{1}{2m}$, or equivalently $\frac{1}{2m+2} < \frac{1}{(2k+1)(2k+2)} < \frac{1}{2m}$. But this forces $(2k+1)(2k+2)$, which is an even integer, to be $2m+1$. This impossibility completes the proof when $f+g<1$.

Suppose next that $1 \leq f+g < 2$. Then we have

$$\begin{aligned} d(m+f+g, k) - d(m+g, k+f) = f&\left[\frac{1}{2m+4} - \frac{1}{2k+1} + \frac{1}{2k+2}\right] \\ +(1-g)&\left[\frac{1}{2m+2} - \frac{1}{2m+4}\right] \end{aligned} \tag{4}$$

and

(5)
$$d(m+f+g-1,k+1) - d(m+g,k+f) = -(1-f)\left[\frac{1}{2m+2} - \frac{1}{2k+1} + \frac{1}{2k+2}\right].$$

Now an argument precisely analogous to the one above shows that the right side of either (4) or (5) is nonnegative, which completes the proof of Lemma 3.

THEOREM 2. Given $x \geq 2$ drums of fuel divided between depots at each end of the desert, the maximum distance which can be crossed in a round trip is $d(x-k,k)$ where $k = \left\lfloor \left(\frac{x+1}{2}\right)^{\frac{1}{2}} \right\rfloor$. Furthermore the algorithm for achieving this distance is given in the proofs of Theorems A and B.

PROOF: By Lemma 3 we may restrict our attention to feasible solutions with an integer number of drums at F. It suffices to show that

(6)
$$\text{if} \quad t < k, \quad \text{then} \quad d(x-t-1,t+1) \geq d(x-t,t)$$

and

(7)
$$\text{if} \quad t > k, \quad \text{then} \quad d(x-t+1,t-1) \geq d(x-t,t)$$

hold for positive integer t.

In the remainder of this proof let $m = \lfloor x \rfloor - t$ and $f = x - \lfloor x \rfloor$. In order to verify (6) we have:

$$(8)\quad d(x-t-1,t+1)-d(x-t,t)=d(m-1+f,t+1)-d(n+f,t)$$
$$=f\left(\frac{1}{2m}-\frac{1}{2m+2}\right)-\left(\frac{1}{2m}-\frac{1}{2t+1}+\frac{1}{2t+2}\right)$$

Since $t<k=\left\lfloor\left(\frac{x+1}{2}\right)^{\frac{1}{2}}\right\rfloor$, we have $t+1\leq\left(\frac{x+1}{2}\right)^{\frac{1}{2}}$ or $\lfloor x\rfloor\geq 2(t+1)^2-1$. Thus $2m=2(\lfloor x\rfloor-t)\geq 2((2(t+1)^2-1)-t)=(2t+1)(2t+2)$ or $\frac{1}{2m}-\frac{1}{2t+1}+\frac{1}{2t+2}=\frac{1}{2m}-\frac{1}{(2t+1)(2t+2)}\leq 0$. This and the fact that $\frac{1}{2m}-\frac{1}{2m+2}>0$ imply that the right side of (8) is nonnegative, as required.

In order to verify (7) we have

$$(9)\quad d(x-t+1,t-1)-d(x-t,t)=d(m+1+f,t-1)-d(m+f,t)$$
$$=-f\left(\frac{1}{2m+2}-\frac{1}{2m+4}\right)+\left(\frac{1}{2m+2}-\frac{1}{2t-1}+\frac{1}{2t}\right)$$
$$=\frac{-2f}{(2m+2)(2m+4)}+\frac{1}{2m+2}-\frac{1}{(2t-1)2t}$$
$$\geq\frac{-2f}{(2m+2)(2m+2f+2)}+\frac{1}{2m+2}-\frac{1}{(2t-1)2t}.$$

Since integer $t>k=\left\lfloor\left(\frac{x+1}{2}\right)^{\frac{1}{2}}\right\rfloor$, we have that $t\geq\left(\frac{x+1}{2}\right)^{\frac{1}{2}}$ or $x\leq 2t^2-1$. Thus $2(m+f)+2=2(x-t)+2\leq 2((2t^2-1)-t)+2=(2t-1)2t$, and so $\frac{1}{2m+2f+2}-\frac{1}{(2t-1)(2t)}\geq 0$. Adding and subtracting $\frac{1}{2m+2}$ gives $\frac{-2f}{(2m+2)(2m+2f+2)}+\frac{1}{2m+2}-\frac{1}{(2t-1)2t}\geq 0$. That is, the right side of (9) is nonnegative as required.

This completes the proof of Theorem 2.

Theorem 2 can be viewed as providing the solution to the equivalent problem: Given unlimited fuel at each end of a desert of fixed length, minimize the amount of fuel required for a round trip across the desert. Since our solution gives a distance roughly one half of the harmonic number $H_{\lfloor x \rfloor - k}$, any length desert can be crossed given the availability of sufficient fuel. It is also interesting to note that although our solution is better than placing half of the fuel at each end of the desert, the difference between the two solutions is not great. In fact it is bounded above by a small constant. It is easy to show, using common identities and estimates for harmonic numbers that this difference is always less than $1 + \ln 2$. Finally, we observe that the number of intermediate fuel depots for our optimal solution is $\lceil x \rceil - 2$.

We next consider the problem of finding the position of two depots so that a desert of maximum length can be crossed on a round trip with n drums of fuel, n an integer, distributed between the two depots. Obviously one depot, D_1, must be at the start; otherwise the jeep cannot move. At the second depot, D_2, suppose that we have k units of fuel, where k is not necessarily an integer. Let r be the amount of fuel at D_2 which is used on the return trip and $t = k - r$ be the amount which is used to continue the trip across the desert. Without loss of generality we can assume that in an optimal solution on the return trip the jeep arrives at D_2 without fuel, for that fuel could be included in the k units stored at D_2 where k need not be an integer. Let $s + f_1$ be the amount of fuel at S, let $r + f_2$ be the amount of fuel at D_2 which is used on the return trip, and $t + f_3$ be the amount at D_2 used to continue the outbound trip where s, r, t are integers and $0 \leq f_i \leq 1$ for $i = 1, 2, 3$ such that $s + f_1 + r + f_2 + t + f_3 = n$. The maximum width desert which can be crossed via a round trip is:

$$(10)\ \frac{f_1}{2s+2}+\frac{1}{2s}+\frac{1}{2s-2}+\cdots+\frac{1}{2r+4}+\frac{1-f_2}{2r+2}+\frac{f_2}{2r+1}+\frac{1}{2r-1}+$$
$$\cdots+\frac{1}{3}+1+\frac{f_3}{2t+2}+\frac{1}{2t}+\frac{1}{2t-2}+\cdots+\frac{1}{4}+\frac{1}{2}.$$

This follows immediately from the fact that we can consider the problem as a round trip from D_1 to D_2 and a round trip onward from D_2.

Thus we need to decide how to partition n drums of fuel into 3 parts so as to maximize (10). It is easy to show, similar to Lemma 3, that for any feasible solution with value given by (10) there is a feasible solution at least as large where each f_i is an integer. We state without proof Theorem 3 which gives, for arbitrary n, an optimal partition, the maximum desert length, and an optimal position for depot D_2. Table 2 shows these optimal partitions and distances for some sample values of n. We also observe that for an optimal solution. D_2 is placed slightly more than half way across the desert from $D_1 = S$. In order to state Theorem 3 we make two definitions:

First define $d(s,r) = \frac{1}{2s}+\frac{1}{2s-2}+\cdots+\frac{1}{2r+2}+\frac{1}{2r-1}+\cdots+\frac{1}{3}+1+\frac{1}{2t}+\cdots\frac{1}{4}+\frac{1}{2}$. Second given n, t, positive integers with $n > t$ we define:

$$D(n,t) = \frac{1}{2s}+\frac{1}{2s-2}+\cdots+\frac{1}{2r+2}+\frac{1}{2r-1}+\cdots+\frac{1}{3}+1+\frac{1}{2t}+\frac{1}{2t-2}$$
$$+\cdots+\frac{1}{4}+\frac{1}{2}, \quad \text{where} \quad s+r=n-t \quad \text{and} \quad r=\left\lfloor\left(\frac{r+s+1}{2}\right)^{1/2}\right\rfloor.$$

THEOREM 3. The maximum width desert which can be crossed by a round trip using $n \geq 3$ drums of fuel is $D(n,t)$, where r is given above and $s = \left\lfloor\frac{n-r}{2}\right\rfloor, t = \left\lceil\frac{n-r}{2}\right\rceil$. The depots D_1 and D_2 are located at S and at distance $d(s,r)$ from S respectively.

n	s	r	t	Distance from S to D_2	Desert Length
3	1	1	1	1	$1\frac{1}{2}$
4	1	1	2	1	$1\frac{3}{4}$
5	2	1	2	$1\frac{1}{4}$	2
6	2	1	3	$1\frac{1}{4}$	$2\frac{1}{6}$
7	3	1	3	$1\frac{5}{12}$	$2\frac{1}{3}$
8	3	1	4	$1\frac{5}{12}$	$2\frac{11}{24}$
9	4	1	4	$1\frac{13}{24}$	$2\frac{7}{12}$
10	4	1	5	$1\frac{13}{24}$	$2\frac{41}{60}$
11	5	1	5	$1\frac{77}{120}$	$2\frac{47}{60}$
12	5	2	5	$1\frac{29}{40}$	$2\frac{13}{15}$
30	14	2	14	$\frac{1}{2}H_{14} + \frac{7}{12}$	$H_{14} + \frac{7}{12}$
31	14	3	14	$\frac{1}{2}H_{14} + \frac{37}{60}$	$H_{14} + \frac{37}{60}$

Table 2
Optimal partitions and distance for various values of n.

Observe that the definition of r is similar to that of k in the previous problem. There are several ways of calculating r and one easy way is given in Theorem 4. The proof of Theorem 3 is somewhat similar to, but more complicated than, that of Theroem 2. Any interested reader may obtain a copy of the proof from the third author.

THEOREM 4. If n is a positive integer and m is the least positive integer such that $4m^2 + 7m \geq n$, then $r = m$.

PROOF. It suffices to show that for $t = \left\lceil \frac{n-m}{2} \right\rceil$, if $n = 4m^2 + 7m$ or $n = 4(m-1)^2 + 7(m-1) + 1$, then $m = \left\lfloor \left(\frac{n-t+1}{2}\right)^{1/2} \right\rfloor$. Let $n = 4m^2 + 7m$, then $t = 2m^2 + 3m$ and $\left\lfloor \left(\frac{n-t+1}{2}\right)^{1/2} \right\rfloor = \left\lfloor \left(m^2 + 2m + \frac{1}{2}\right)^{1/2} \right\rfloor = m$. Let $n = 4(m-1)^2 + 7(m-1) + 1 = 4m^2 - m - 2$. Then $t = 2m^2 - m - 1$ and $\left\lfloor \left(\frac{n-t+1}{2}\right)^{1/2} \right\rfloor - \left\lfloor \left(\frac{2m^2}{2}\right)^{1/2} \right\rfloor = m$. This proves Theorem 4.

Finally from Theorem 4, it follows that $r = \left\lceil \frac{-7 + (49 + 16n)^{1/2}}{8} \right\rceil$.

4 Dewdney's Problem

Suppose a jeep, which achieves 10 miles per gallon of fuel, can carry one 50 gallon drum of fuel in addition to at most 10 gallons in its tank. Dewdney [DEW] asked for an algorithm which maximizes the one-way distance the jeep can attain using n drums available at the start.

It may appear that this problem is the same as, or at least very similar to, the one-way problem in Section 1. Dewdney's problem, however, is somewhat more subtle. Change the units so that the jeep travels 1 unit of distance on one tank of fuel. One drum holds 5 tankfuls of fuel, and the jeep can carry one drum in addition to the fuel in its tank. Fuel can be stored in drums only. Thus at most $\frac{5}{6}$ of the capacity of the jeep can be stored, whereas in Theorem A any fraction of the jeep's capacity can be stored. Theorem A gives an upper bound for the Dewdney problem once an appropriate change of units is made. A somewhat complicated optimal algorithm for the Dewdney problem is given in [JAC]. This algorithm is optimal for all n, but attains the Theorem A bound only for small n.

A friend, Ken Maddex, misunderstood a discussion with one of the authors regarding Dewdney's problem. The Maddex problem: Given an unlimited fuel supply, but only n drums for carrying fuel and only one jeep of the Dewdney kind, maximize the distance into the desert that the jeep can attain. Of course, there are both one way and roundtrip variations of this problem. As with other jeep problems finding a travel algorithm is easy but deciding optimality is apparently not easy.

Finally we note that Brauer and Brauer [BRA] considered a problem similar to Dewdney with their jeep able to carry 1 drum and its tank able to hold 1 drum. They also added the constraint that the tank could refill only when it was empty. They developed a number of algorithms but did not prove any of them optimal except for very small n.

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GRAPHS AND APPLICATIONS

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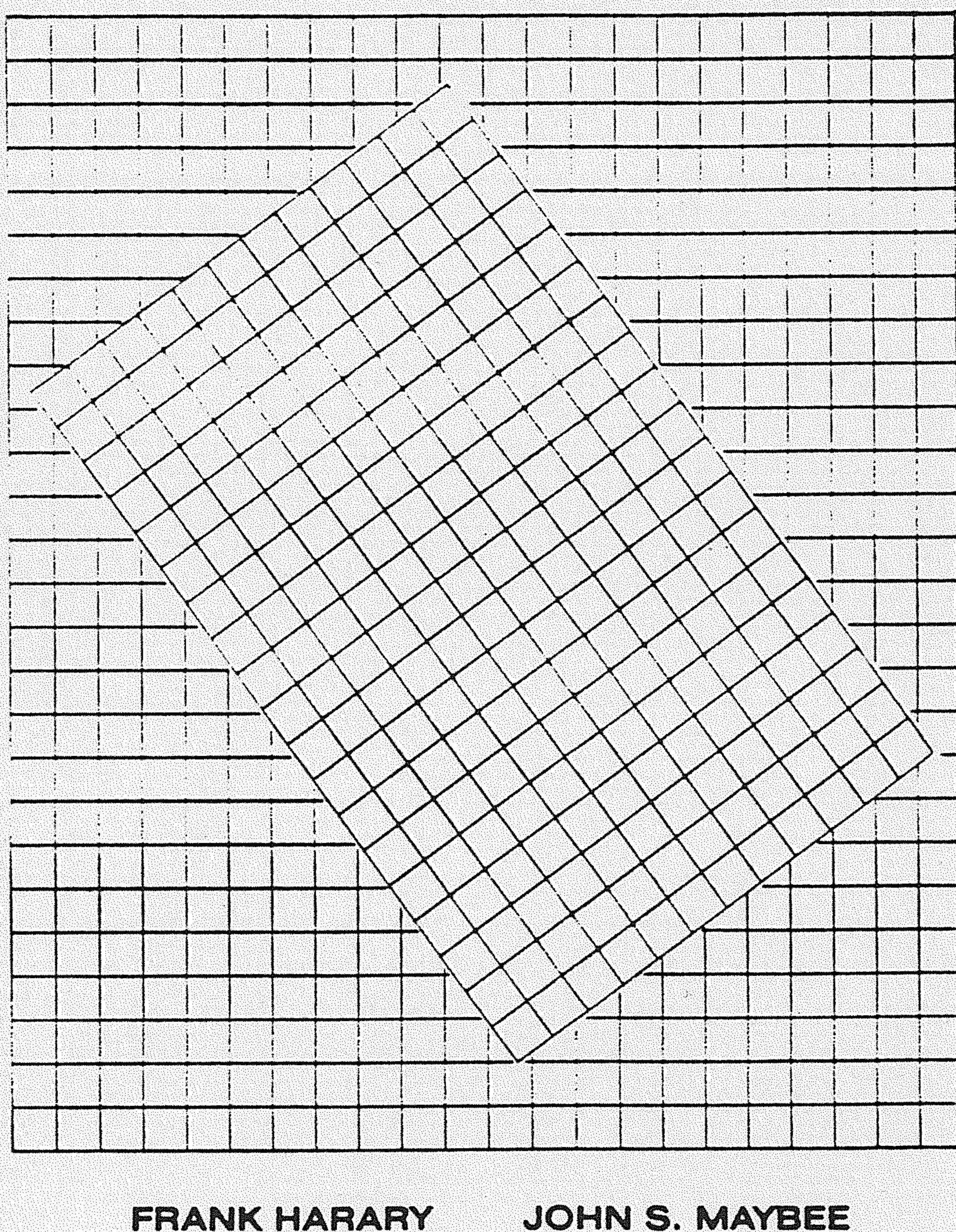

FRANK HARARY JOHN S. MAYBEE

EMPIRE MAPS

Brad Jackson and Gerhard Ringel

Department of Mathematics
University of California
Santa Cruz, CA 95064

ABSTRACT. Heawood generalized his Map Color Problem to empire maps. Solutions are given here for the projective plane and some other surfaces.

The Four Color Theorem says that the vertices of each planar graph can be colored by four colors, or that the countries of each map on the plane are colorable by four colors. Heawood in 1890 could not prove this theorem so he generalized the concept. He did it in two directions. First -- take instead of the plane any other surface, for instance the torus.

Second -- consider maps where certain collections of nonadjacent countries are called empires and each country belongs to exactly one empire. These maps will be called empire maps. Of course in a coloring of an empire map each component of an empire should have the same color and any two adjacent countries have to have different colors.

In the 90 years after Heawood's paper was published, mathematicians worked on the first aspect until the question was finally solved completely. But they almost entirely neglected the second aspect.

Let S be a surface and M a positive integer. By $\chi(S,M)$ we denote the minimum number of colors necessary and sufficient to color every empire map on S, where each empire had no more than M components. This number is called the <u>M-pire chromatic number</u> of S. We call an empire with exactly m components an <u>m-pire</u>.

Before we report general results we consider the special case where S is the plane or the projective plane. The following two theorems [10, p. 24] are often very helpful in coloring problems.

Theorem 1. If G is a planar or projective planar graph, and d(G) is the average of the degrees of the vertices in G then d(G) < 6.

Theorem 2. Let T be a set of graphs and assume that for every graph $G \in T$ each subgraph of G is also in T and that $d(G) < h$ for each $G \in T$. Then the vertices of G are colorable by h colors for each $G \in T$.

If we apply Theorems 1 and 2 to planar graphs, we obtain the result that every planar graph is colorable by 6 colors. This is of course very disappointing. However for other problems Theorem 2 will be very sharp.

Given an empire map we can define a graph where each country, respectively each empire, is represented by a vertex and two vertices are adjacent if and only if the two represented countries, respectively empires, are adjacent in the map. This graph is called the <u>country graph</u>, respectively the <u>empire graph</u>, of the empire map. It is clear that one obtains the empire graph from the country graph by identifying those vertices which represent the same empire. Coloring an empire map is then equivalent to coloring its empire graph.

We consider all empire maps on the plane or projective plane where each empire has no more than M components. Then because of Theorem 1 in the corresponding empire graphs the average degree of the vertices is < 6M. Then Theorem 2 gives:

$$\chi(\text{plane}, M) \le 6M, \tag{1}$$

$$\chi(\text{projective plane}, M) \le 6M. \tag{2}$$

Heawood exhibited a map on the plane with 12 mutually adjacent 2-pires showing that equality holds in (1). Taylor [4] constructed two more maps showing that (1) holds as an equality also for M = 3 and 4. For $M \geq 5$ the matter is unsolved. The maps of Heawood and Taylor are very irregular and have no symmetry, and no pattern. There is not much hope of generalizing them. It seems easier to find empire maps on other surfaces. We give here a short sketch of a proof for equality in (2) for each M.

Consider the example m = 3 given in Figure 1. It shows an empire map on the projective plane consisting of 18 mutually adjacent empires; 17 of them are 3-pires and one is a 1-pire. One can see that the permutation $(0\ 1\ 2\ \cdots\ 16)$ which replaces i by $i+1$ (mod 17) describes an automorphism of the map. Notice that the labels of the neighbors of the 3-pire 0 are given in three cycles

(5,2,8,x,9,11,1), (3,7,6,15), (4,1,16,10,13,12,14)

or written in the form

(5,2,8,x,-8,-6,1), (3,7,6,-2) (4,1,-1,-7,-4,-5,-3)

These three cycles we obtained by travelling

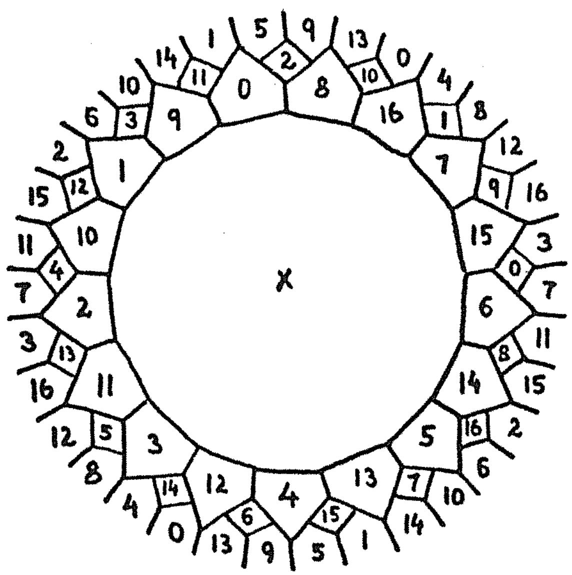

Figure 1.

through the "cascade" of Figure 2. A cascade is a modification of a current graph -- a construction device. For details see Ringel [11, p. 13]. Notice that at each vertex of degree 3 in Figure 2 Kirchoff's Current Law is true: The sum of the incoming currents equals the sum of the outgoing currents. This law guarantees that the constructed map will have cubic vertices only.

The edge with current 4 is a "broken" edge; this makes the current graph a cascade and the generated map non-orientable. Using Euler's formula one can prove that the map defines the projective plane.

The next example m = 7 in Figure 3 shows how the concept can be generalized. For even integers m the cascade is slightly different. These were the ideas used to prove that equality holds in (2).

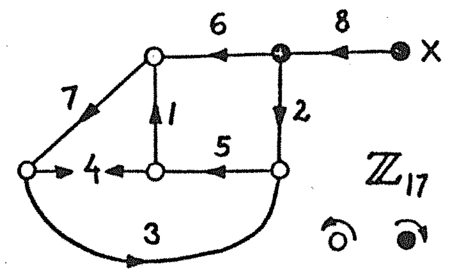

Figure 2.

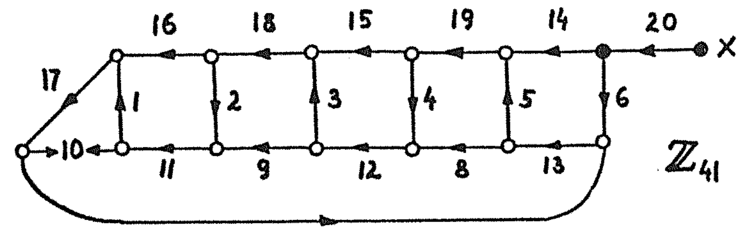

Figure 3.

Now we report the present situation of the general problem of determining the number of colors $\chi(X,M)$, " which is sufficient and necessary to color every empire map on the surface S where each empire has no more than M components. Let E be the Euler characteristic of the surface S.

Heawood in 1890 showed the inequality

$$\chi(S,M) \leq \left\lfloor \frac{1}{2}\left(6M + 1 + \sqrt{(6M+1)^2 - 24E}\right) \right\rfloor \tag{3}$$

for every surface S and every natural number M with one exception ($M = 1$, S = sphere). This exception is the Four Color Theorem which was proved by Haken and Appel [1,2]. So (3) holds in general. Heawood conjectured that equality always holds in (3).

In the following cases equality in (3) has been proven.

(a) $M = 1$, S is a non-orientable surface $\neq$ Klein's bottle. Ringel [9], 1954. If S is the Klein bottle and $M = 1$, then (3) does not hold as an equality. Franklin [3], 1934.

(b) $M = 1$, S is orientable. Ringel, Youngs and others [11], 1968.

(c) $M = 2$, S is the sphere. Heawood [5], 1890.

(d) $M = 3$ or 4, S is the sphere. Taylor [4], 1890.

(e) S = torus. Taylor [12], 1982.

(f) S is the projective plane. Jackson, Ringel [6], 1983.

(g) S is non-orientable and the right-hand side of (3) is congruent to 1, 4, or 7 (mod 12). Jackson, Ringel [8], 1983.

(h) S is orientable, M is even, and the right-hand side of (3) is congruent to 1 (mod 12). Jackson, Ringel [8], 1983.

(i) S is orientable, M is odd, and the right-hand side of (3) is congruent to 4 or 7 (mod 12). Jackson, Ringel [8], 1983.

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