Pareto Optimality

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Abstract

This article briefly describes the concept of pareto optimality in game theory and economics and also details some examples which describe the characteristics of a pareto optimal solution

Introduction

When trying to analyse the outcomes of games, there are certain outcomes that are deemed to be better than other outcomes. A game is a situation where multiple agents are involved. Hence the best decision for each agent depends on the choices/strategies of other agents.

An outcome is said to be **pareto optimal** if there is no outcome that makes at least one agent better off without making any agent worse off.

Formally, a pure strategy profile $s = (s_1, s_2, \ldots, s_n)$ is said to be Pareto optimal if for any player i, its strategy s_i is replaced by s'_i, then there exists some player j such that,
$\pi_j(s_{-i}, s_i) > \pi_j(s_{-i}, s'_i)$

Examples of Pareto Optimal Solutions

Let us consider some commonly studied games and see how the pareto optimal states in the game can be understood.

Prisoner's Dilemma:

This is one of most commonly studied games in game theory. Let's take an example of this scenario. Two thieves plan to rob a store. As they approach the entrance in the night, the police arrest them for trespassing. The cops suspect that the pair planned to break in but lack of evidence means they cannot be charged for such an accusation. They therefore require a confession to charge the suspects with the greater crime.

The cops then decide to separate both robbers and tell each of them the following:
We are currently charging you with trespassing, which implies a six month jail sentence. We know that you were planning on robbing the store, but since we cannot prove it, we need your confession. In exchange for your cooperation, we will dismiss your trespassing charge, and your partner will be charged to the fullest extent of the law: a 2 year jail sentence.
We are offering your partner the same deal. If both of you confess, your individual testimony is no longer as valuable, and your jail sentence will be one year each.

If both criminals are self-interested and only care about minimizing their jail time, should they take the interrogator's deal?

If we were to build a payoff matrix for the above scenario, it will be as below:

Player A	Player B	
	Stay silent(S)	Confess(C)
Stay Silent(S)	(-0.5, -0.5)	(-2, 0)
Confess(C)	(0, -2)	(-1, -1)

Let's examine each state one by one to see if there are any pareto optimal states here:
(S, S) - There is no other state where both players get a better payoff. Hence this is a pareto optimal state
(S,C) - There is no other state where Player 2 gets a better payoff. Hence this is also a pareto optimal state
(C, S) - On similar lines, there is no other state where Player 1 gets a better payoff. Hence this is also a pareto optimal state.
(C, C) - Here, this state is **pareto dominated** by the state (S, S). Hence we say that this is not a pareto optimal state. This is what is called as a **pareto inefficient state** because by both players changing their strategy, they can move to a better state which increases both their rewards. However, it is important to note that this is the nash equilibrium state and also the dominant strategy equilibrium. Hence once the game reaches this state, it becomes difficult to switch to a different state

When a situation arises like the prisoner's dilemma, where the Pareto efficient outcome is not the actual outcome, occurs in a market economy, that can be an example of a market failure.

Stag Hunt

Two hunters enter a forest range filled with hares and a single stag. Hares being smaller and less strong are easy to capture. The stag, on the other hand, is cunning—the hunters can only catch it by working together.
Without any communication, the hunters independently choose whether to hunt hares or the stag. If they both hunt hares, they each capture hares. If one hunts the stag and the other hunts hares, the stag hunter goes home empty-handed while the hare hunter captures all of the hares. Finally, if both hunt the stag, then each of their shares of the stag is greater than the value of all of the hares.

This can be represented using the below payoff matrix

Player A	Player B	
	Stag(S)	Hare(H)
Stag(S)	(3, 3)	(2, 0)
Hare(H)	(0, 2)	(2, 2)

If we explore the above matrix for any pareto optimal states, we can see that the state (S, S) and (H, H) are both in equilibrium state. However we say that **(S, S) is the pareto optimal** state since the reward for both the players are maximized when you are in this state compared to the (H, H) state.

If the system was in a state(S,H) or (H, S) then there can be a change made to the state (H, H). This is called a **pareto improvement -** an improvement in a situation where no one is worse off and atleast one player is better of.

Battle of the Sexes

A man and a woman want to go on a date. There are only two venues of entertainment in the city that night: Sebastian Bach or Igor Stravinsky concert. The woman wants to go to the Igor Stravinsky concert. The man wants to go see the Sebastian Bach concert. However, they both prefer being together than being alone. As such, both must choose where to go simultaneously and without the ability to communicate with one another. We can draw up the payoff matrix like this:

Player A	Player B	
	Bach(B)	Stravinsky(S)
Bach(B)	(2, 3)	(0, 0)
Stravinsky(S)	(1, 1)	(3, 2)

If we analyze the above matrix, we can see that both the **(B, B) and the (S, S) states are pareto optimal** since neither one dominates over the other.

Matching Pennies

Two players each have a penny. Simultaneously, they choose whether to put the penny on the table with heads or tails facing up. If both of the pennies show heads or both of the pennies show tails (that is, they match), then player A have to pay player B a dollar. But if one shows heads and the other shows tails (that is, they do not match), then player B has to pay player A a dollar.

Player A	Player B	
	Head(H)	Tails(T)
Head(H)	(-1, 1)	(1, -1)
Tails(T)	(1, -1)	(-1, 1)

This game is an example of what is called a **Zero Sum game**. Here the net payoff in any state is zero. One player's gain is equivalent to another player's losses.

If we analyze the payoff matrix above, we can see that **all the four states are pareto optimal states.** Infact we can observe that in general, for most zero-sum games, all the states tend to be pareto optimal states.Here, when the system is in any given state, there is no better state where atleast one player is better of since the loss of one player is what can lead to a gain of another player.

Characteristics of Pareto Optimality

From the above examples we can observe the below characteristics of pareto optimal solutions:

1. **A game can have more than one pareto optimal outcome**. This is because its possible for two outcomes that neither dominates over another like the case with Battle of the Sexes.We can also think of a situation where the all the players get the same payoff in the game. Here, all the states become pareto optimal since no state dominates the other.
2. **Every game must have atleast one pareto optimal outcome**. This can be understood in this way. For a outcome to be not pareto optimal, then it mus be pareto dominated by some other outcome. Hence, in order for a game to have no pareto optimal outcomes, we need to have a cycle of paret dominant outcomes. We can see that a cycle like that cannot be possible since for some outcome to dominate over the other, we need the outcome to be atleast as good for every agent and for some agent it should be better of.
3. The pareto optimal outcomes need not guarantee a fairness in terms of the payoffs for each agent.

Conclusion

In this article, we saw the concept of Pareto Optimality in game outcomes and tried to analyze some commonly studied games to identify these outcomes. We also saw a few characteristics of Pareto Optimality.

References

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