Instituto Politécnico Nacional,
Escuela Superior de Cómputo

Autómata Celulares

Sistemas Complejos

Oscar Andrés Rosas Hernandez

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Parte I

Marco Teórico

Capítulo 1

Autómata celulares

1.1. Definición

Un autómata celular es un sistema dinámico discreto que consiste en una red regular de autómatas (celdas) de estado finito que cambian sus estados dependiendo de los estados de sus vecinos (y del mismo), de acuerdo con una función de transferencia.

Todas las células cambian su estado simultáneamente usando la misma regla de actualización. El proceso se repite en pasos de tiempo discretos. Resulta que con reglas de actualización sorprendentemente simples se pueden producir dinámicas extremadamente complejas como en el famoso Juego de la vida de John Conway. [1]

1.1.1. Caracteristicas

- Son discretos tanto en tiempo como en espacio
- Son homogeneos tanto en tiempo como en espacio (la misma regla es aplicada a todas las celulas al mismo tiempo)
- Sus interacciones son locales

Para especificar una autómata celular, debemos especificar los siguientes elementos (algunos de los cuales pueden ser claros por el contexto):

- La dimensión $d \in Z^+$,
- El conjunto de estados finitos S
- Una vecindad N de celdas
- La función de activación $f : S^m \to S$

Por lo tanto, definimos formalmente a un autómata celular correspondiente como la 4-tupla $A = (d, S, N, f)$.

1.2. Autómata celulares elementales

Los autómatas elementales son autómatas celulares unidimensionales con dos estados y una vecindad de radio 1: $d = 1$, $S = \{ 0, 1 \}$, $N = (-1, 0, 1)$.

Se diferencian entre sí solo en la elección de la regla f. Hay 256 autómata elementales.

1.3. Clases de Wolfram

S.Wolfram trabajó en los años 80 con los autómatas elementales y basándose en observaciones empíricas de su comportamiento en configuraciones iniciales aleatorias, las clasificó en cuatro clases.

Estas se conocen como clases Wolfram. Las definiciones no son matemáticamente rigurosas, y desde entonces se han propuesto clasificaciones más precisas.

Wolfram definió las clases de la siguiente manera:

- (W1): Casi todas las configuraciones iniciales conducen a la misma configuración uniforme de punto fijo,
- (W2): Casi todas las configuraciones iniciales conducen a una configuración que se repite periódicamente,
- (W3): Casi todas las configuraciones iniciales conducen a un comportamiento esencialmente aleatorio,
- (W4): Surgen estructuras localizadas con interacciones complejas.

[1]

Es importante recalcar que Wolfram las definia de tal manera que la regla 2 estuviera contenida en al regla 1 y asi con la regla 3 y la 2 y la 4 y la 3.

Wolfram conjeturó que los autómatas celulares de esa clase (W4) son computacionalmente universales. (cosa que se ha probado para la regla 110).

Capítulo 2

La reducción de 256 reglas a solo 88

En su libro A new Kind of Science, S.Wolfram [2] logra demostrar de manera bastante trivial que las 356 reglas se pueden reducir a solo 88 reglas irreducibles, para hacerlo se basa en una idea bastante sencilla:

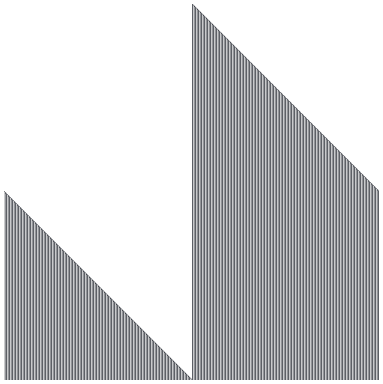
Tienes una regla n entonces:

- La regla que cambia todos los unos por cero no cambia la naturaleza de la regla
- La regla que cambia derecha por izquierda no cambia la naturaleza de la regla
- La regla que cambia las dos anteriores al mismo tiempo no cambia la naturaleza de la regla

Asi podemos hacer grupos de 4, este programa en C++ nos permite obtener las clases de manera sencilla:

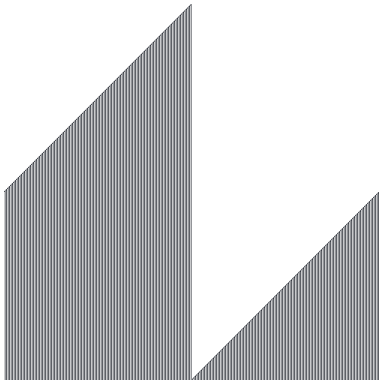
Mira por ejemplo estas 4 clases:

(a) 28



(b) 199

(c) 70



(d) 157

Estas son las clases equivalentes:

- 0, 255, 0, 255
- 1, 127, 1, 127
- 2, 191, 16, 247
- 3, 63, 17, 119
- 4, 223, 4, 223
- 5, 95, 5, 95
- 6, 159, 20, 215
- 7, 31, 21, 87
- 8, 239, 64, 253
- 9, 111, 65, 125
- 10, 175, 80, 245
- 11, 47, 81, 117
- 12, 207, 68, 221
- 13, 79, 69, 93
- 14, 143, 84, 213
- 15, 15, 85, 85
- 16, 247, 2, 191
- 17, 119, 3, 63
- 18, 183, 18, 183
- 19, 55, 19, 55
- 20, 215, 6, 159
- 21, 87, 7, 31
- 22, 151, 22, 151
- 23, 23, 23, 23
- 24, 231, 66, 189
- 25, 103, 67, 61
- 26, 167, 82, 181
- 27, 39, 83, 53
- 28, 199, 70, 157
- 29, 71, 71, 29
- 30, 135, 86, 149
- 31, 7, 87, 21
- 32, 251, 32, 251
- 33, 123, 33, 123
- 34, 187, 48, 243
- 35, 59, 49, 115
- 36, 219, 36, 219
- 37, 91, 37, 91
- 38, 155, 52, 211
- 39, 27, 53, 83
- 40, 235, 96, 249
- 41, 107, 97, 121
- 42, 171, 112, 241
- 43, 43, 113, 113
- 44, 203, 100, 217
- 45, 75, 101, 89
- 46, 139, 116, 209
- 47, 11, 117, 81
- 48, 243, 34, 187
- 49, 115, 35, 59
- 50, 179, 50, 179
- 51, 51, 51, 51
- 52, 211, 38, 155
- 53, 83, 39, 27
- 54, 147, 54, 147
- 55, 19, 55, 19
- 56, 227, 98, 185
- 57, 99, 99, 57
- 58, 163, 114, 177
- 59, 35, 115, 49
- 60, 195, 102, 153
- 61, 67, 103, 25
- 62, 131, 118, 145
- 63, 3, 119, 17
- 64, 253, 8, 239
- 65, 125, 9, 111
- 66, 189, 24, 231
- 67, 61, 25, 103
- 68, 221, 12, 207
- 69, 93, 13, 79
- 70, 157, 28, 199
- 71, 29, 29, 71
- 72, 237, 72, 237
- 73, 109, 73, 109
- 74, 173, 88, 229
- 75, 45, 89, 101

- 76, 205, 76, 205
- 77, 77, 77, 77
- 78, 141, 92, 197
- 79, 13, 93, 69
- 80, 245, 10, 175
- 81, 117, 11, 47
- 82, 181, 26, 167
- 83, 53, 27, 39
- 84, 213, 14, 143
- 85, 85, 15, 15
- 86, 149, 30, 135
- 87, 21, 31, 7
- 88, 229, 74, 173
- 89, 101, 75, 45
- 90, 165, 90, 165
- 91, 37, 91, 37
- 92, 197, 78, 141
- 93, 69, 79, 13
- 94, 133, 94, 133
- 95, 5, 95, 5
- 96, 249, 40, 235
- 97, 121, 41, 107
- 98, 185, 56, 227
- 99, 57, 57, 99
- 100, 217, 44, 203
- 101, 89, 45, 75
- 102, 153, 60, 195
- 103, 25, 61, 67
- 104, 233, 104, 233
- 105, 105, 105, 105
- 106, 169, 120, 225
- 107, 41, 121, 97
- 108, 201, 108, 201
- 109, 73, 109, 73
- 110, 137, 124, 193
- 111, 9, 125, 65
- 112, 241, 42, 171
- 113, 113, 43, 43
- 114, 177, 58, 163
- 115, 49, 59, 35
- 116, 209, 46, 139
- 117, 81, 47, 11
- 118, 145, 62, 131
- 119, 17, 63, 3
- 120, 225, 106, 169
- 121, 97, 107, 41
- 122, 161, 122, 161
- 123, 33, 123, 33
- 124, 193, 110, 137
- 125, 65, 111, 9
- 126, 129, 126, 129
- 127, 1, 127, 1
- 128, 254, 128, 254
- 129, 126, 129, 126
- 130, 190, 144, 246
- 131, 62, 145, 118
- 132, 222, 132, 222
- 133, 94, 133, 94
- 134, 158, 148, 214
- 135, 30, 149, 86
- 136, 238, 192, 252
- 137, 110, 193, 124
- 138, 174, 208, 244
- 139, 46, 209, 116
- 140, 206, 196, 220
- 141, 78, 197, 92
- 142, 142, 212, 212
- 143, 14, 213, 84
- 144, 246, 130, 190
- 145, 118, 131, 62
- 146, 182, 146, 182
- 147, 54, 147, 54
- 148, 214, 134, 158
- 149, 86, 135, 30
- 150, 150, 150, 150
- 151, 22, 151, 22
- 152, 230, 194, 188
- 153, 102, 195, 60
- 154, 166, 210, 180

- 155, 38, 211, 52
- 156, 198, 198, 156
- 157, 70, 199, 28
- 158, 134, 214, 148
- 159, 6, 215, 20
- 160, 250, 160, 250
- 161, 122, 161, 122
- 162, 186, 176, 242
- 163, 58, 177, 114
- 164, 218, 164, 218
- 165, 90, 165, 90
- 166, 154, 180, 210
- 167, 26, 181, 82
- 168, 234, 224, 248
- 169, 106, 225, 120
- 170, 170, 240, 240
- 171, 42, 241, 112
- 172, 202, 228, 216
- 173, 74, 229, 88
- 174, 138, 244, 208
- 175, 10, 245, 80
- 176, 242, 162, 186
- 177, 114, 163, 58
- 178, 178, 178, 178
- 179, 50, 179, 50
- 180, 210, 166, 154
- 181, 82, 167, 26
- 182, 146, 182, 146
- 183, 18, 183, 18
- 184, 226, 226, 184
- 185, 98, 227, 56
- 186, 162, 242, 176
- 187, 34, 243, 48
- 188, 194, 230, 152
- 189, 66, 231, 24
- 190, 130, 246, 144
- 191, 2, 247, 16
- 192, 252, 136, 238
- 193, 124, 137, 110
- 194, 188, 152, 230
- 195, 60, 153, 102
- 196, 220, 140, 206
- 197, 92, 141, 78
- 198, 156, 156, 198
- 199, 28, 157, 70
- 200, 236, 200, 236
- 201, 108, 201, 108
- 202, 172, 216, 228
- 203, 44, 217, 100
- 204, 204, 204, 204
- 205, 76, 205, 76
- 206, 140, 220, 196
- 207, 12, 221, 68
- 208, 244, 138, 174
- 209, 116, 139, 46
- 210, 180, 154, 166
- 211, 52, 155, 38
- 212, 212, 142, 142
- 213, 84, 143, 14
- 214, 148, 158, 134
- 215, 20, 159, 6
- 216, 228, 202, 172
- 217, 100, 203, 44
- 218, 164, 218, 164
- 219, 36, 219, 36
- 220, 196, 206, 140
- 221, 68, 207, 12
- 222, 132, 222, 132
- 223, 4, 223, 4
- 224, 248, 168, 234
- 225, 120, 169, 106
- 226, 184, 184, 226
- 227, 56, 185, 98
- 228, 216, 172, 202
- 229, 88, 173, 74
- 230, 152, 188, 194
- 231, 24, 189, 66
- 232, 232, 232, 232
- 233, 104, 233, 104

- 234, 168, 248, 224
- 235, 40, 249, 96
- 236, 200, 236, 200
- 237, 72, 237, 72
- 238, 136, 252, 192
- 239, 8, 253, 64
- 240, 240, 170, 170
- 241, 112, 171, 42
- 242, 176, 186, 162
- 243, 48, 187, 34
- 244, 208, 174, 138
- 245, 80, 175, 10
- 246, 144, 190, 130
- 247, 16, 191, 2
- 248, 224, 234, 168
- 249, 96, 235, 40
- 250, 160, 250, 160
- 251, 32, 251, 32
- 252, 192, 238, 136
- 253, 64, 239, 8
- 254, 128, 254, 128
- 255, 0, 255, 0

Eligiendo la regla mas pequeña de cada clase podemos general el conjunto de reglas que vamos a analizar: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 50, 51, 54, 56, 57, 58, 60, 62, 72, 73, 74, 76, 77, 78, 90, 94, 104, 105, 106, 108, 110, 122, 126, 128, 130, 132, 134, 136, 138, 140, 142, 146, 150, 152, 154, 156, 160, 162, 164, 168, 170, 172, 178, 184, 200, 204, 232}.

El código realizado fue:

```cpp
#include <algorithm>
#include <bitset>
#include <cstdint>
#include <iostream>
#include <set>

using namespace std;
using num = uint8_t;

auto flip_bits(const num n) -> num {
  auto bits = bitset<8>{n};
  auto buffer = bits;

  buffer[0] = not bits[7];
  buffer[1] = not bits[6];
  buffer[2] = not bits[5];
  buffer[3] = not bits[4];
  buffer[4] = not bits[3];
  buffer[5] = not bits[2];
  buffer[6] = not bits[1];
  buffer[7] = not bits[0];

  return buffer.to_ulong();
}

auto flip_dir(const num n) -> num {
  auto bits = bitset<8>{n};
  auto buffer = bits;

  buffer[1] = bits[4];
  buffer[3] = bits[6];
  buffer[6] = bits[3];
  buffer[4] = bits[1];

  return buffer.to_ulong();
}

auto get_copies(const num n) -> tuple<num, num, num> {
  const auto ones = flip_bits(n);
  const auto dirs = flip_dir(n);
  const auto crazy = flip_dir(flip_bits(n));

  return {ones, dirs, crazy};
}
```

```
auto add_id(const set<num> &seen, set<int> &ids, num a, num b, num c, num d) {
  auto representant = 256;

  if (not seen.count(a))
    representant = min<int>(representant, a);
  if (not seen.count(b))
    representant = min<int>(representant, b);
  if (not seen.count(c))
    representant = min<int>(representant, c);
  if (not seen.count(d))
    representant = min<int>(representant, d);

  if (representant != 256)
    ids.insert(representant);
}

auto main() -> int {
  auto seen = set<num>{};
  auto representants = set<int>{};

  auto i = 0;
  for (auto i = 0; i < 256; ++i) {
    num a = i;
    auto [b, c, d] = get_copies(a);

    add_id(seen, representants, a, b, c, d);
    printf("%d %d %d %d\n", a, b, c, d);

    seen.insert({a, b, c, d});
  }

  cout << endl;
  for (auto key : representants) {
    cout << key << endl;
  }

  return 0;
}
```

Compilado con:

```
  g++ -std=c++17 Equivalence.cpp && ./a.out
```

Es trivial comprobar que el codigo funciona comparando con los resultados obtenido por Wolfram [2].

Capítulo 3

Mi clasificación de los autómatas celulares

3.1. Codigo

Para esto ocupamos dos versiones una creada en Typescript para crear un simulador web bastante bonito:

```
type bits = Uint8Array;

class CellularAutomata {
  readonly histogram: Array<number>;
  epoch: number;
  data: bits;
  buffer: bits;

  constructor(init: Array<number>) {
    this.data = new Uint8Array(new ArrayBuffer(init.length));
    this.buffer = new Uint8Array(new ArrayBuffer(init.length));
    this.histogram = [];
    this.epoch = 0;

    let ones = 0;
    for (let i = 0; i < this.data.length; ++i) {
      this.data[i] = init[i];
      if (this.data[i]) ++ones;
    }

    this.histogram[this.epoch] = ones;
  }

  newEpoch(rulesID: number): void {
    this.epoch += 1;
    let ones = 0;

    const rule = this.getRules(rulesID);
    for (let i = 0; i < this.buffer.length; ++i) {
      this.buffer[i] = rule(this.data, i);
      if (this.buffer[i]) ++ones;
    }

    this.histogram[this.epoch] = ones;
    [this.data, this.buffer] = [this.buffer, this.data];
  }

  createNewEpoch(rulesID: number): bits {
    const rule = this.getRules(rulesID);
    return this.data.map((_, i) => rule(this.data, i));
  }

  getRules = (rulesID: number) => (data: bits, index: number): number => {
    const limit = data.length - 1;
    const n1 = index === 0 ? limit : index - 1;
```

```
    const n2 = index === limit ? 0 : index + 1;

    const id = (data[n1] << 2) + (data[index] << 1) + (data[n2] << 0);
    return (rulesID >> id) & 1;
  };

  get average(): number {
    let total = 0;
    for (let i = 0; i < this.histogram.length; ++i) total += this.histogram[i];

    return total / this.histogram.length;
  }

  get variance(): number {
    const average = this.average;
    let variance = 0;

    for (let i = 0; i < this.histogram.length; ++i)
      variance += (this.histogram[i] - average) * (this.histogram[i] - average);

    return variance / this.histogram.length;
  }
}

export default CellularAutomata;
```

Y otra creada en Python para poder tomar velocidad a la hora de crear las gráficas:

```
from PIL import Image
import os
import random
import plotly.io as pio
from math import sqrt

n = iterations = 500


def graph(histogram, average, variance, path):
    global n

    desviation = sqrt(variance)

    trace1 = {"type": "bar", "y": histogram, "opacity": 1,
              "name": "Histogram", "marker": {"color": "rgb(158,202,225)"}, }

    trace2 = {"type": "scatter", "y": [average, average], "x": [0, len(histogram) - 1], "opacity": 0.5,
              "name": "Average", "marker": {"color": "#FC7A7A"}, }

    trace3 = {"type": "scatter", "y": [average + desviation, average + desviation], "x": [0, len(histogram) - 1],
     "opacity": 0.5,
              "name": "sqrt derivation +", "marker": {"color": "#009999"}, }

    trace4 = {"type": "scatter", "y": [average - desviation, average - desviation], "x": [0, len(histogram) - 1],
     "opacity": 0.5,
              "name": "sqrt derivation -", "marker": {"color": "#009999"}, }

    fig = { "data": [trace1, trace2, trace3, trace4],  "layout": {"title": {"text": "Ones"}} }

    pio.write_image(fig, path)


def get_rules(rules_id):
    def rule(s, i):
        limit = len(s) - 1
        n1 = limit if i == 0 else i - 1
        n2 = 0 if i == limit else i + 1

        id = (s[n1] << 2) + (s[i] << 1) + (s[n2] << 0)
        return (rules_id >> id) & 1

    return rule


def print_evolution(steps, s, rules_id, pixels, path):
    global n
    histogram = []

    rules = get_rules(rules_id)
    current, temporal = list(s), list(s)
    limit = len(s)

    one, zero = (255, 255, 255), (40, 44, 52)

    for step in range(steps):
        histogram.append(sum(current))
```

```
        for i in range(limit):
            temporal[i] = rules(current, i)
        for j, val in enumerate(temporal):
            pixels[j, step] = one if val else zero

        current = list(temporal)

    average = sum(histogram) / n

    variance = 0
    for val in histogram:
        variance += (val - average) * (val - average)

    variance = variance / n

    graph(histogram, average, variance, path)


rules = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29,
         30, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 50, 51, 54, 56, 57, 58, 60, 62, 72, 73,
         74, 76, 77, 78, 90, 94, 104, 105, 106, 108, 110, 122, 126, 128, 130, 132, 134, 136, 138, 140, 142,
         146, 150, 152, 154, 156, 160, 162, 164, 168, 170, 172, 178, 184, 200, 204, 232]

inits = []
random.seed(10)


def oneMiddle():
    s = [0] * n
    s[n // 2] = 1
    return s


def getPercentage(percentage):
    s = [0] * n
    for i in range(len(s)):
        s[i] = 1 if percentage > random.random() else 0

    return s


inits = [oneMiddle(), getPercentage(0.08), getPercentage(.5), getPercentage(.87)]

for rule in [28, 199, 70, 157]:
    print(rule)
    for i, init in enumerate(inits):
        letter = chr(ord("a") + i)
        img = Image.new('RGB', (500, 500), "black")
        pixels = img.load()


        diagram = f"../Images/{rule}/{letter}.png"
        path = f"../Images/{rule}/dia-{letter}.png"

        print_evolution(iterations, init, rule, pixels, path)
        img.save(diagram)
```

3.2. W1

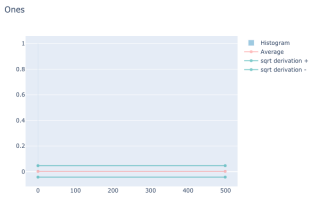
3.2.1. 160

(a) One

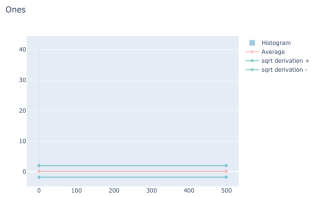
(b) 8 %

(c) 50 %

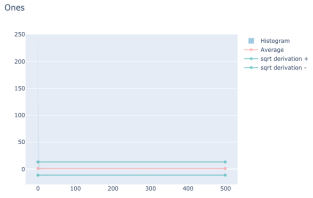
(d) 87 %



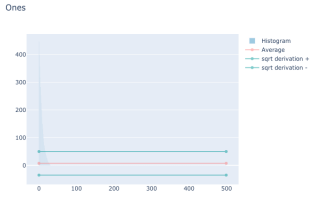
(a) One



(b) 8 %



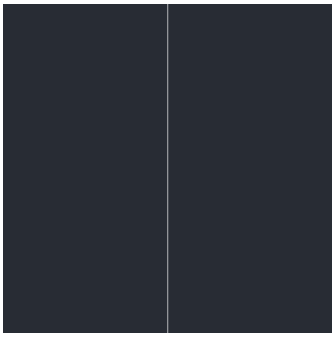
(c) 50 %



(d) 87 %

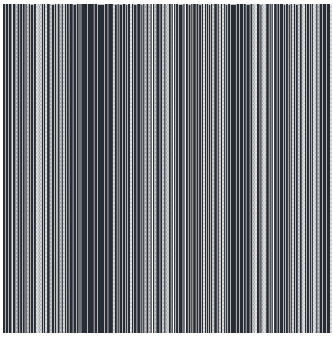
3.3. W2

3.3.1. 108



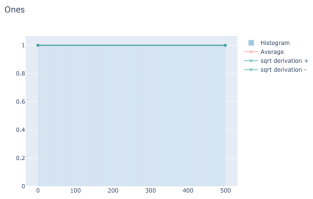
(a) One

(b) 8 %

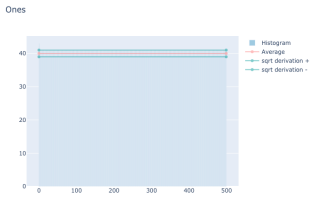


(c) 50 %

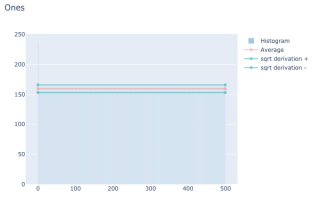
(d) 87 %



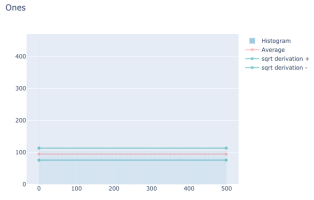
(a) One



(b) 8 %



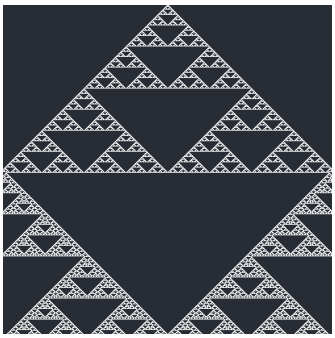
(c) 50 %



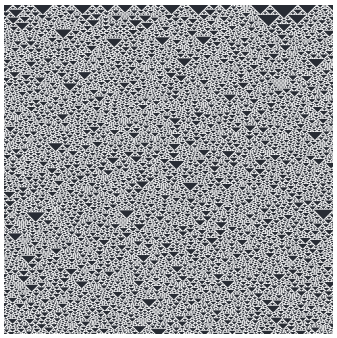
(d) 87 %

3.4. W3

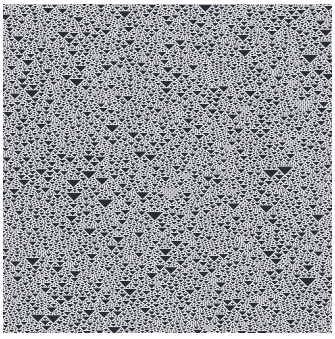
3.4.1. 126



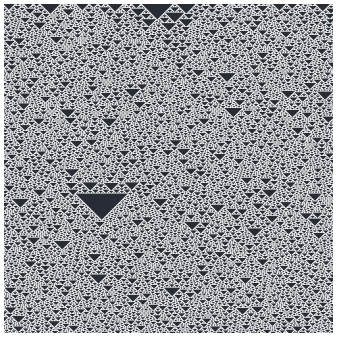
(a) One



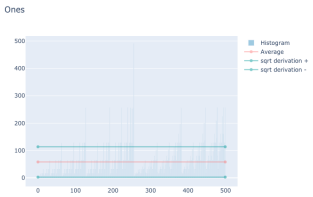
(b) 8 %



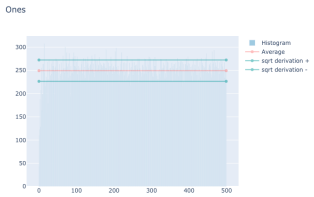
(c) 50 %



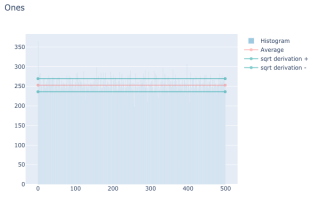
(d) 87 %



(a) One



(b) 8 %



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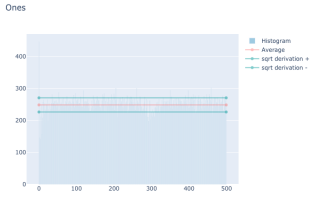
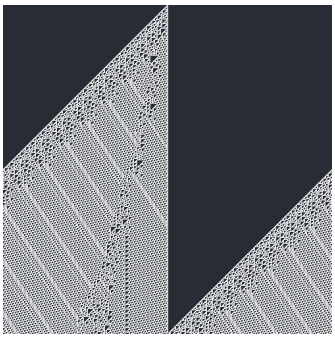


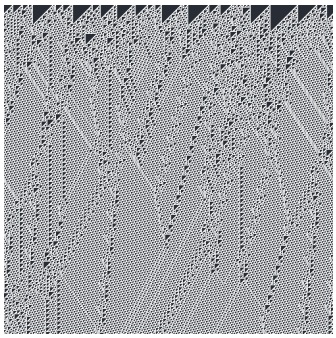
(d) 87 %

3.5. W4

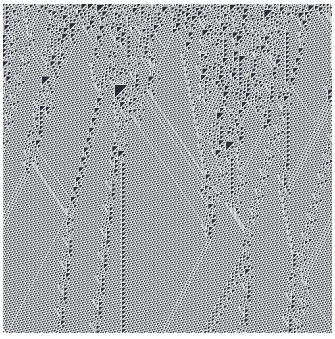
3.5.1. 110



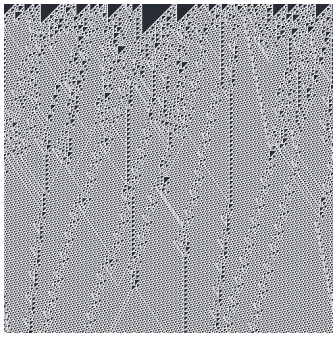
(a) One



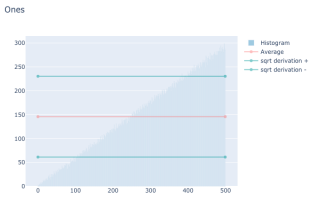
(b) 8 %



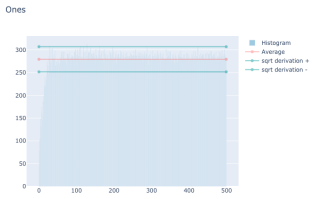
(c) 50 %



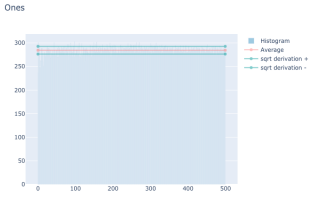
(d) 87 %



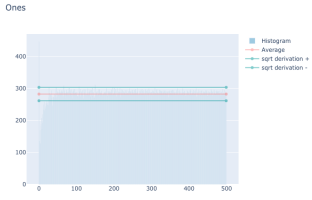
(a) One



(b) 8 %

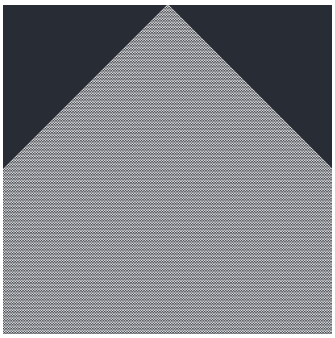


(c) 50 %

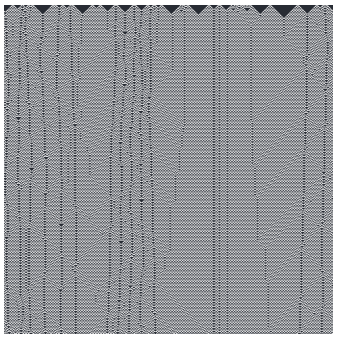


(d) 87 %

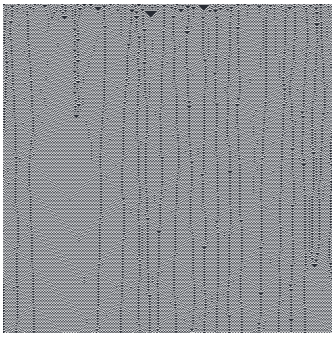
3.5.2. 54



(a) One



(b) 8 %



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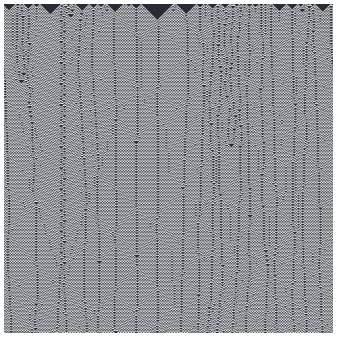
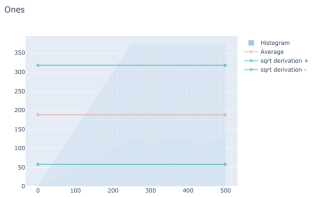
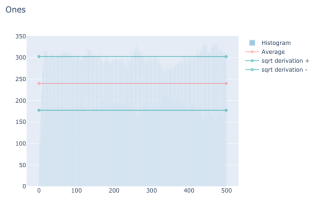


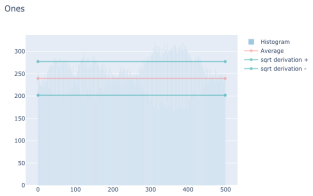
(d) 87 %



(a) One



(b) 8 %



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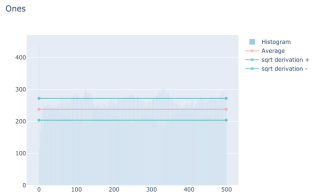

(d) 87 %

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