Instituto Politécnico Nacional,
Escuela Superior de Cómputo

REPORTE: Diagramas de Bruijn y ecuaciones

Sistemas Complejos

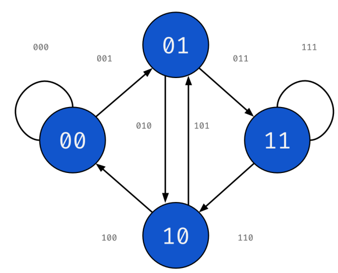
Oscar Andrés Rosas Hernandez

5 de abril de 2020

Índice general

Capítulo 1

Diagramas de Bruijn



Los diagramas de Bruijn proporcionan una manera conveniente de describir las configuraciones de autómatas celulares (CA).

Sea Σ el alfabeto y $s \geq 1$ un número, y el grafo de Bruijn $B(s, \Sigma)$ como sigue: $B(s, \Sigma)$ tiene un conjunto de vértices Σ^s y vertices (ax, xb) para todo $a, b \in \Sigma, x \in \Sigma^s$.

Para calcular estos tuve que mostrar las reglas de una manera que me permitiera facilmente crear los diagramas, para eso cree este pequeño programa en `C++` que me permite mostrarlas de una manera comoda ss:

```cpp
#include <bitset>
#include <cstdint>
#include <iostream>
using namespace std;

auto show_rules(const uint8_t rule_id) {
  const auto rule = bitset<8>{rule_id};

  cout << "Rule " << rule.to_ulong() << endl;
  for (auto i = 0; i < 8; ++i)
    cout << bitset<3>(i) << " -> " << rule[i] << endl;

  cout << endl;
}

auto main() -> int {
  const auto rules = {15, 22, 30, 54, 90, 110};
  for (const auto rule : rules)
    show_rules(rule);

  return 0;
}
```

Dando este resultado:

```
Rule 15
000 -> 1
001 -> 1
010 -> 1
011 -> 1
100 -> 0
101 -> 0
110 -> 0
111 -> 0
```

```
Rule 22
000 -> 0
001 -> 1
010 -> 1
011 -> 0
100 -> 1
101 -> 0
110 -> 0
111 -> 0
```

```
Rule 30
000 -> 0
001 -> 1
010 -> 1
011 -> 1
100 -> 1
101 -> 0
110 -> 0
111 -> 0
```

```
Rule 54
000 -> 0
001 -> 1
010 -> 1
011 -> 0
100 -> 1
101 -> 1
110 -> 0
111 -> 0
```

```
Rule 90
000 -> 0
001 -> 1
010 -> 0
011 -> 1
100 -> 1
101 -> 0
110 -> 1
111 -> 0
```

```
Rule 110
000 -> 0
001 -> 1
010 -> 1
011 -> 1
100 -> 0
101 -> 1
110 -> 1
111 -> 0
```

Podemos editarlo un poco para saber el valor de las conexiones del diagrama:

```cpp
#include <bitset>
#include <cstdint>
#include <iostream>
using namespace std;

auto show_rules(const uint8_t rule_id) {
  const auto rule = bitset<8>{rule_id};

  cout << "Rule " << rule.to_ulong() << endl;
  for (auto i = 0; i < 8; ++i) {
    const auto total = bitset<3>(i).to_string();
    const auto start = total.substr(0, 2);
    const auto end = total.substr(1, 2);
    cout << start << " to " << end << " -> " << rule[i] << endl;
  }
  cout << endl;
}

auto main() -> int {
  const auto rules = {15, 22, 30, 54, 90, 110};
  for (const auto rule : rules)
    show_rules(rule);

  return 0;
}
```

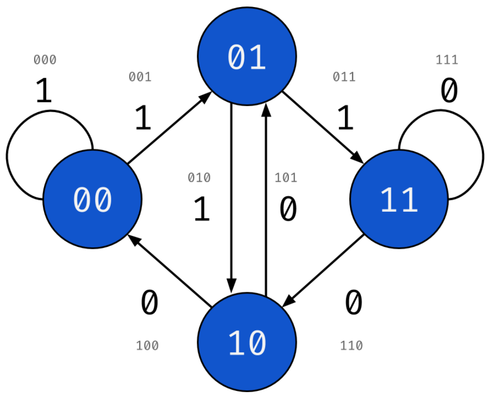
```
Rule 15
00 to 00 -> 1
00 to 01 -> 1
01 to 10 -> 1
01 to 11 -> 1
10 to 00 -> 0
10 to 01 -> 0
11 to 10 -> 0
11 to 11 -> 0

Rule 22
00 to 00 -> 0
00 to 01 -> 1
01 to 10 -> 1
01 to 11 -> 0
10 to 00 -> 1
10 to 01 -> 0
11 to 10 -> 0
11 to 11 -> 0
```

```
Rule 30
00 to 00 -> 0
00 to 01 -> 1
01 to 10 -> 1
01 to 11 -> 1
10 to 00 -> 1
10 to 01 -> 0
11 to 10 -> 0
11 to 11 -> 0

Rule 54
00 to 00 -> 0
00 to 01 -> 1
01 to 10 -> 1
01 to 11 -> 0
10 to 00 -> 1
10 to 01 -> 1
11 to 10 -> 0
11 to 11 -> 0
```

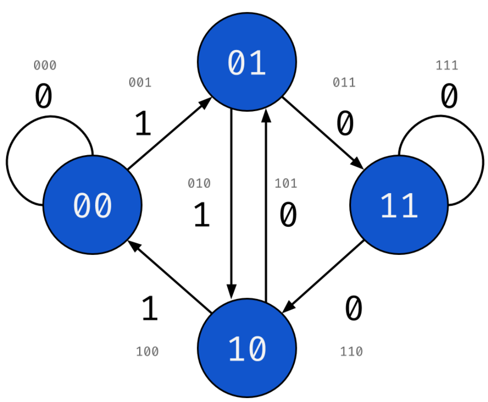
```
Rule 90
00 to 00 -> 0
00 to 01 -> 1
01 to 10 -> 0
01 to 11 -> 1
10 to 00 -> 1
10 to 01 -> 0
11 to 10 -> 1
11 to 11 -> 0

Rule 110
00 to 00 -> 0
00 to 01 -> 1
01 to 10 -> 1
01 to 11 -> 1
10 to 00 -> 0
10 to 01 -> 1
11 to 10 -> 1
11 to 11 -> 0
```

Con esto listo podemos calcular facilmente los diagramas de Bruijn.

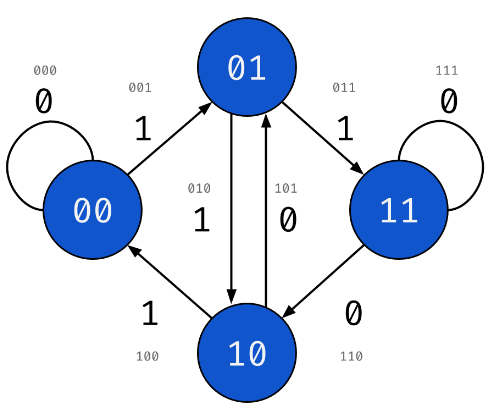
1.1. Regla 15



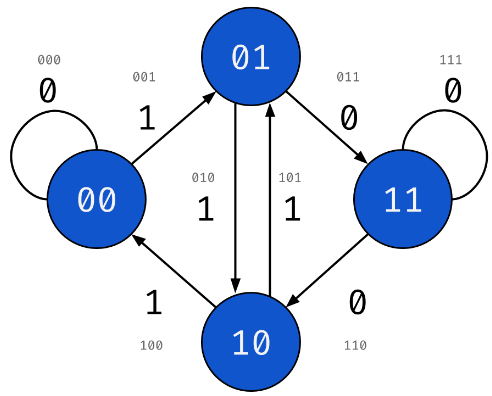
1.2. Regla 22



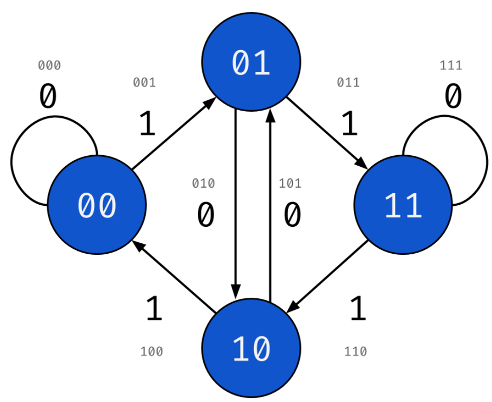
1.3. Regla 30



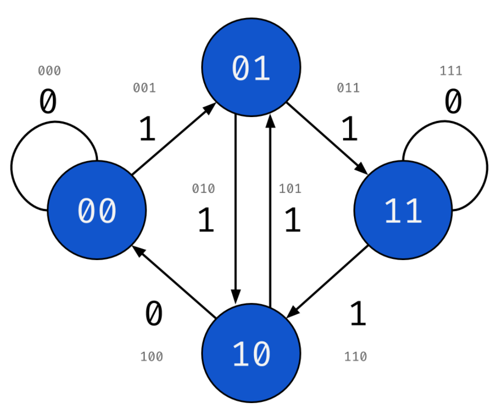
1.4. Regla 54



1.5. Regla 90



1.6. Regla 110



Capítulo 2

Ecuaciones simbólicas (expresiones regulares)

Una vez que construyeron los diagramas podemos calcular todas las ecuaciones simbólicas (expresiones regulares). Hasta calcular la ecuación que representa todo el autómata.

La expresión mas importante es:

$$R_{i,j}^{k} = R_{i,j}^{k-1} + \ R_{i,k}^{k-1} \ (R_{k,k}^{k-1})^{*} \ R_{k,j}^{k-1} \tag{2.1}$$

Podemos crear un programa que se encarge de generar estas ecuaciones para la regla 22 por ejemplo:

```
const nonEmpty = (x: string) => (x === "" ? "∅" : x);

const parentesis = (x: string) => {
  if (x.length < 2) return x;
  if (x[x.length - 1] === ")") return x;
  if (x.split("").every((e) => e === "1" || e === "0")) return x;

  return `(${x})`;
};

const or = (x: Array<string>) => {
  const elements = x.filter((e) => e).filter((e) => e !== "∅");
  return [...new Set(elements)].join(" + ");
};

const and = (x: Array<string>) => {
  if (x.some((e) => e === "∅")) return "∅";
  return x
    .filter((e) => e)
    .filter((e) => e !== "ϵ")
    .filter((e) => e !== "ϵ^*")
    .map((e) => parentesis(e))
    .join("");
};
```

```
const kleeneClosure = (x: string) => {
  const data = x.split(" + ");
  const result = data.filter(e => e !== "ε").join(" + ");

  if (!result) return "";

  return `(${result}^*)`;
};

const path = [
  ["", "", "", ""],
  ["", "", "", ""],
  ["", "", "", ""],
  ["", "", "", ""]
];

const ruleID = 30;
// @ts-ignore
const rule = Number(ruleID).toString(2).padStart(8, "0");

path[0][0] = rule[7];
path[0][1] = rule[6];
path[1][2] = rule[5];
path[1][3] = rule[4];
path[2][0] = rule[3];
path[2][1] = rule[2];
path[3][2] = rule[1];
path[3][3] = rule[0];

path[1][1] = "ε";
path[2][2] = "ε";

path[0][2] = "∅";
path[0][3] = "∅";
path[1][0] = "∅";
path[2][3] = "∅";
path[3][0] = "∅";
path[3][1] = "∅";

path[0][0] = path[0][0] + " + ε";
path[3][3] = path[3][3] + " + ε";

function R(i: number, j: number, k: number): string {
  if (k === 0) return path[i][j];

  //console.log(`${a}+${b}(${c})^*${d}`);
  const a = R(i, j, k - 1);
  let b = parentesis(R(i, k, k - 1));
  let c = kleeneClosure(R(k, k, k - 1));
  let d = parentesis(R(k, j, k - 1));

  if (c === `(${b}^*)`) b = "";
  if (c === `(${d}^*)`) d = "";

  let concat = and([b, c, d]);
  return nonEmpty(or([a, concat]));
}

for (let k = 0; k < 4; ++k) {
  for (let i = 0; i < 4; ++i) {
    for (let j = 0; j < 4; ++j) {
      console.log(`R(${i}, ${j}, ${k}) = ` + R(i, j, k));
    }
  }
  console.log();
```

```
}
```

Logrando estas ecuaciones para la regla 15:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 1
R(1, 3, 0) = 0
R(2, 0, 0) = 1
R(2, 1, 0) = 0
R(2, 2, 0) = ϵ
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 0
R(3, 3, 0) = 0 + ϵ

R(0, 0, 1) = 0 + ϵ
R(0, 1, 1) = 1
R(0, 2, 1) = 11
R(0, 3, 1) = 10
R(1, 0, 1) = ∅
R(1, 1, 1) = ϵ
R(1, 2, 1) = 1
R(1, 3, 1) = 0
R(2, 0, 1) = 1
R(2, 1, 1) = 0
R(2, 2, 1) = ϵ + 01
R(2, 3, 1) = 00
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
R(3, 2, 1) = 0
R(3, 3, 1) = 0 + ϵ

R(0, 0, 2) = 0 + ϵ + 11(01^*)1
R(0, 1, 2) = 1 + 11(01^*)0
R(0, 2, 2) = 11 + 11(01^*)(ϵ + 01)
R(0, 3, 2) = 10 + 11(01^*)00
R(1, 0, 2) = 1(01^*)1
R(1, 1, 2) = ϵ + 1(01^*)0
R(1, 2, 2) = 1 + 1(01^*)(ϵ + 01)
R(1, 3, 2) = 0 + 1(01^*)00
R(2, 0, 2) = 1 + (ϵ + 01)(01^*)1
R(2, 1, 2) = 0 + (ϵ + 01)(01^*)0
R(2, 2, 2) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01)
R(2, 3, 2) = 00 + (ϵ + 01)(01^*)00
R(3, 0, 2) = 0(01^*)1
R(3, 1, 2) = 0(01^*)0
R(3, 2, 2) = 0 + 0(01^*)(ϵ + 01)
R(3, 3, 2) = 0 + ϵ + 0(01^*)00

R(0, 0, 3) = 0 + ϵ + 11(01^*)1 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(0, 1, 3) = 1 + 11(01^*)0 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(0, 2, 3) = 11 + 11(01^*)(ϵ + 01) + (10 + 11(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ
    + 01)
R(0, 3, 3) = 10 + 11(01^*)00 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ + 0(01^*)00)
R(1, 0, 3) = 1(01^*)1 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(1, 1, 3) = ϵ + 1(01^*)0 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(1, 2, 3) = 1 + 1(01^*)(ϵ + 01) + (0 + 1(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ + 01)
R(1, 3, 3) = 0 + 1(01^*)00 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ + 0(01^*)00)
```

```
R(2, 0, 3) = 1 + (ϵ + 01)(01^*)1 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(2, 1, 3) = 0 + (ϵ + 01)(01^*)0 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(2, 2, 3) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01) + (00 + (ϵ + 01)(01^*)00)(0 +
    0(01^*)00^*)0 + 0(01^*)(ϵ + 01)
R(2, 3, 3) = 00 + (ϵ + 01)(01^*)00 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ +
    0(01^*)00)
R(3, 0, 3) = 0(01^*)1 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(3, 1, 3) = 0(01^*)0 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(3, 2, 3) = 0 + 0(01^*)(ϵ + 01) + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ
    + 01)
R(3, 3, 3) = 0 + ϵ + 0(01^*)00 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ +
    0(01^*)00)
```

Logrando estas ecuaciones para la regla 22:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 1
R(1, 3, 0) = 0
R(2, 0, 0) = 1
R(2, 1, 0) = 0
R(2, 2, 0) = ϵ
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 0
R(3, 3, 0) = 0 + ϵ

R(0, 0, 1) = 0 + ϵ
R(0, 1, 1) = 1
R(0, 2, 1) = 11
R(0, 3, 1) = 10
R(1, 0, 1) = ∅
R(1, 1, 1) = ϵ
R(1, 2, 1) = 1
R(1, 3, 1) = 0
R(2, 0, 1) = 1
R(2, 1, 1) = 0
R(2, 2, 1) = ϵ + 01
R(2, 3, 1) = 00
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
R(3, 2, 1) = 0
R(3, 3, 1) = 0 + ϵ

R(0, 0, 2) = 0 + ϵ + 11(01^*)1
R(0, 1, 2) = 1 + 11(01^*)0
R(0, 2, 2) = 11 + 11(01^*)(ϵ + 01)
R(0, 3, 2) = 10 + 11(01^*)00
R(1, 0, 2) = 1(01^*)1
R(1, 1, 2) = ϵ + 1(01^*)0
R(1, 2, 2) = 1 + 1(01^*)(ϵ + 01)
R(1, 3, 2) = 0 + 1(01^*)00
R(2, 0, 2) = 1 + (ϵ + 01)(01^*)1
R(2, 1, 2) = 0 + (ϵ + 01)(01^*)0
R(2, 2, 2) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01)
R(2, 3, 2) = 00 + (ϵ + 01)(01^*)00
R(3, 0, 2) = 0(01^*)1
R(3, 1, 2) = 0(01^*)0
R(3, 2, 2) = 0 + 0(01^*)(ϵ + 01)
```

```
R(3, 3, 2) = 0 + ϵ + 0(01^*)00

R(0, 0, 3) = 0 + ϵ + 11(01^*)1 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(0, 1, 3) = 1 + 11(01^*)0 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(0, 2, 3) = 11 + 11(01^*)(ϵ + 01) + (10 + 11(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ
    + 01)
R(0, 3, 3) = 10 + 11(01^*)00 + (10 + 11(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ + 0(01^*)00)
R(1, 0, 3) = 1(01^*)1 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(1, 1, 3) = ϵ + 1(01^*)0 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(1, 2, 3) = 1 + 1(01^*)(ϵ + 01) + (0 + 1(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ + 01)
R(1, 3, 3) = 0 + 1(01^*)00 + (0 + 1(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ + 0(01^*)00)
R(2, 0, 3) = 1 + (ϵ + 01)(01^*)1 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(2, 1, 3) = 0 + (ϵ + 01)(01^*)0 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(2, 2, 3) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01) + (00 + (ϵ + 01)(01^*)00)(0 +
    0(01^*)00^*)0 + 0(01^*)(ϵ + 01)
R(2, 3, 3) = 00 + (ϵ + 01)(01^*)00 + (00 + (ϵ + 01)(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ +
    0(01^*)00)
R(3, 0, 3) = 0(01^*)1 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0(01^*)1)
R(3, 1, 3) = 0(01^*)0 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0(01^*)0)
R(3, 2, 3) = 0 + 0(01^*)(ϵ + 01) + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)0 + 0(01^*)(ϵ
    + 01)
R(3, 3, 3) = 0 + ϵ + 0(01^*)00 + (0 + ϵ + 0(01^*)00)(0 + 0(01^*)00^*)(0 + ϵ +
    0(01^*)00)
```

Logrando estas ecuaciones para la regla 30:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 1
R(1, 3, 0) = 1
R(2, 0, 0) = 1
R(2, 1, 0) = 0
R(2, 2, 0) = ϵ
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 0
R(3, 3, 0) = 0 + ϵ

R(0, 0, 1) = 0 + ϵ
R(0, 1, 1) = 1
R(0, 2, 1) = 11
R(0, 3, 1) = 11
R(1, 0, 1) = ∅
R(1, 1, 1) = ϵ
R(1, 2, 1) = 1
R(1, 3, 1) = 1
R(2, 0, 1) = 1
R(2, 1, 1) = 0
R(2, 2, 1) = ϵ + 01
R(2, 3, 1) = 01
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
R(3, 2, 1) = 0
R(3, 3, 1) = 0 + ϵ

R(0, 0, 2) = 0 + ϵ + 11(01^*)1
R(0, 1, 2) = 1 + 11(01^*)0
R(0, 2, 2) = 11 + 11(01^*)(ϵ + 01)
R(0, 3, 2) = 11 + 11(01^*)
```

```
R(1, 0, 2) = 1(01^*)1
R(1, 1, 2) = ϵ + 1(01^*)0
R(1, 2, 2) = 1 + 1(01^*)(ϵ + 01)
R(1, 3, 2) = 1 + 1(01^*)
R(2, 0, 2) = 1 + (ϵ + 01)(01^*)1
R(2, 1, 2) = 0 + (ϵ + 01)(01^*)0
R(2, 2, 2) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01)
R(2, 3, 2) = 01 + (ϵ + 01)(01^*)
R(3, 0, 2) = 0(01^*)1
R(3, 1, 2) = 0(01^*)0
R(3, 2, 2) = 0 + 0(01^*)(ϵ + 01)
R(3, 3, 2) = 0 + ϵ + 0(01^*)

R(0, 0, 3) = 0 + ϵ + 11(01^*)1 + 11 + 11(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(0, 1, 3) = 1 + 11(01^*)0 + 11 + 11(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(0, 2, 3) = 11 + 11(01^*)(ϵ + 01) + 11 + 11(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ϵ + 01)
R(0, 3, 3) = 11 + 11(01^*) + 11 + 11(01^*)(0 + 0(01^*)^*)0 + ϵ + 0(01^*)
R(1, 0, 3) = 1(01^*)1 + 1 + 1(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(1, 1, 3) = ϵ + 1(01^*)0 + 1 + 1(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(1, 2, 3) = 1 + 1(01^*)(ϵ + 01) + 1 + 1(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ϵ + 01)
R(1, 3, 3) = 1 + 1(01^*) + 1 + 1(01^*)(0 + 0(01^*)^*)0 + ϵ + 0(01^*)
R(2, 0, 3) = 1 + (ϵ + 01)(01^*)1 + 01 + (ϵ + 01)(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(2, 1, 3) = 0 + (ϵ + 01)(01^*)0 + 01 + (ϵ + 01)(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(2, 2, 3) = ϵ + 01 + (ϵ + 01)(01^*)(ϵ + 01) + 01 + (ϵ + 01)(01^*)(0 + 0(01^*)^*)0 +
    0(01^*)(ϵ + 01)
R(2, 3, 3) = 01 + (ϵ + 01)(01^*) + 01 + (ϵ + 01)(01^*)(0 + 0(01^*)^*)0 + ϵ + 0(01^*)
R(3, 0, 3) = 0(01^*)1 + 0 + ϵ + 0(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(3, 1, 3) = 0(01^*)0 + 0 + ϵ + 0(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(3, 2, 3) = 0 + 0(01^*)(ϵ + 01) + 0 + ϵ + 0(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ϵ + 01)
R(3, 3, 3) = 0 + ϵ + 0(01^*) + 0 + ϵ + 0(01^*)(0 + 0(01^*)^*)0 + ϵ + 0(01^*)
```

Logrando estas ecuaciones para la regla 54:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 1
R(1, 3, 0) = 0
R(2, 0, 0) = 1
R(2, 1, 0) = 1
R(2, 2, 0) = ϵ
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 0
R(3, 3, 0) = 0 + ϵ

R(0, 0, 1) = 0 + ϵ
R(0, 1, 1) = 1
R(0, 2, 1) = 11
R(0, 3, 1) = 10
R(1, 0, 1) = ∅
R(1, 1, 1) = ϵ
R(1, 2, 1) = 1
R(1, 3, 1) = 0
R(2, 0, 1) = 1
R(2, 1, 1) = 1
R(2, 2, 1) = ϵ + 11
R(2, 3, 1) = 10
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
```

```
R(3, 2, 1) = 0
R(3, 3, 1) = 0 + ϵ

R(0, 0, 2) = 0 + ϵ + (11^*)1
R(0, 1, 2) = 1 + (11^*)1
R(0, 2, 2) = 11 + (11^*)(ϵ + 11)
R(0, 3, 2) = 10 + (11^*)10
R(1, 0, 2) = 1(11^*)1
R(1, 1, 2) = ϵ + 1(11^*)1
R(1, 2, 2) = 1 + 1(11^*)(ϵ + 11)
R(1, 3, 2) = 0 + 1(11^*)10
R(2, 0, 2) = 1 + (ϵ + 11)(11^*)1
R(2, 1, 2) = 1 + (ϵ + 11)(11^*)1
R(2, 2, 2) = ϵ + 11 + (ϵ + 11)(11^*)(ϵ + 11)
R(2, 3, 2) = 10 + (ϵ + 11)(11^*)10
R(3, 0, 2) = 0(11^*)1
R(3, 1, 2) = 0(11^*)1
R(3, 2, 2) = 0 + 0(11^*)(ϵ + 11)
R(3, 3, 2) = 0 + ϵ + 0(11^*)10

R(0, 0, 3) = 0 + ϵ + (11^*)1 + (10 + (11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(0, 1, 3) = 1 + (11^*)1 + (10 + (11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(0, 2, 3) = 11 + (11^*)(ϵ + 11) + (10 + (11^*)10)(0 + 0(11^*)10^*)0 + 0(11^*)(ϵ + 11)
R(0, 3, 3) = 10 + (11^*)10 + (10 + (11^*)10)(0 + 0(11^*)10^*)(0 + ϵ + 0(11^*)10)
R(1, 0, 3) = 1(11^*)1 + (0 + 1(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(1, 1, 3) = ϵ + 1(11^*)1 + (0 + 1(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(1, 2, 3) = 1 + 1(11^*)(ϵ + 11) + (0 + 1(11^*)10)(0 + 0(11^*)10^*)0 + 0(11^*)(ϵ + 11)
R(1, 3, 3) = 0 + 1(11^*)10 + (0 + 1(11^*)10)(0 + 0(11^*)10^*)(0 + ϵ + 0(11^*)10)
R(2, 0, 3) = 1 + (ϵ + 11)(11^*)1 + (10 + (ϵ + 11)(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(2, 1, 3) = 1 + (ϵ + 11)(11^*)1 + (10 + (ϵ + 11)(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(2, 2, 3) = ϵ + 11 + (ϵ + 11)(11^*)(ϵ + 11) + (10 + (ϵ + 11)(11^*)10)(0 +
    0(11^*)10^*)0 + 0(11^*)(ϵ + 11)
R(2, 3, 3) = 10 + (ϵ + 11)(11^*)10 + (10 + (ϵ + 11)(11^*)10)(0 + 0(11^*)10^*)(0 + ϵ +
    0(11^*)10)
R(3, 0, 3) = 0(11^*)1 + (0 + ϵ + 0(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(3, 1, 3) = 0(11^*)1 + (0 + ϵ + 0(11^*)10)(0 + 0(11^*)10^*)(0(11^*)1)
R(3, 2, 3) = 0 + 0(11^*)(ϵ + 11) + (0 + ϵ + 0(11^*)10)(0 + 0(11^*)10^*)0 + 0(11^*)(ϵ
    + 11)
R(3, 3, 3) = 0 + ϵ + 0(11^*)10 + (0 + ϵ + 0(11^*)10)(0 + 0(11^*)10^*)(0 + ϵ +
    0(11^*)10)
```

Logrando estas ecuaciones para la regla 90:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 0
R(1, 3, 0) = 1
R(2, 0, 0) = 1
R(2, 1, 0) = 0
R(2, 2, 0) = ϵ
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 1
R(3, 3, 0) = 0 + ϵ

R(0, 0, 1) = 0 + ϵ
R(0, 1, 1) = 1
R(0, 2, 1) = 10
R(0, 3, 1) = 11
```

```
R(1, 0, 1) = ∅
R(1, 1, 1) = ϵ
R(1, 2, 1) = 0
R(1, 3, 1) = 1
R(2, 0, 1) = 1
R(2, 1, 1) = 0
R(2, 2, 1) = ϵ + 00
R(2, 3, 1) = 01
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
R(3, 2, 1) = 1
R(3, 3, 1) = 0 + ϵ

R(0, 0, 2) = 0 + ϵ + 10(00^*)1
R(0, 1, 2) = 1 + 10(00^*)0
R(0, 2, 2) = 10 + 10(00^*)(ϵ + 00)
R(0, 3, 2) = 11 + 10(00^*)01
R(1, 0, 2) = 0(00^*)1
R(1, 1, 2) = ϵ + 0(00^*)0
R(1, 2, 2) = 0 + 0(00^*)(ϵ + 00)
R(1, 3, 2) = 1 + 0(00^*)01
R(2, 0, 2) = 1 + (ϵ + 00)(00^*)1
R(2, 1, 2) = 0 + (ϵ + 00)(00^*)0
R(2, 2, 2) = ϵ + 00 + (ϵ + 00)(00^*)(ϵ + 00)
R(2, 3, 2) = 01 + (ϵ + 00)(00^*)01
R(3, 0, 2) = 1(00^*)1
R(3, 1, 2) = 1(00^*)0
R(3, 2, 2) = 1 + 1(00^*)(ϵ + 00)
R(3, 3, 2) = 0 + ϵ + 1(00^*)01

R(0, 0, 3) = 0 + ϵ + 10(00^*)1 + (11 + 10(00^*)01)(0 + 1(00^*)01^*)(1(00^*)1)
R(0, 1, 3) = 1 + 10(00^*)0 + (11 + 10(00^*)01)(0 + 1(00^*)01^*)(1(00^*)0)
R(0, 2, 3) = 10 + 10(00^*)(ϵ + 00) + (11 + 10(00^*)01)(0 + 1(00^*)01^*)1 + 1(00^*)(ϵ
    + 00)
R(0, 3, 3) = 11 + 10(00^*)01 + (11 + 10(00^*)01)(0 + 1(00^*)01^*)(0 + ϵ + 1(00^*)01)
R(1, 0, 3) = 0(00^*)1 + (1 + 0(00^*)01)(0 + 1(00^*)01^*)(1(00^*)1)
R(1, 1, 3) = ϵ + 0(00^*)0 + (1 + 0(00^*)01)(0 + 1(00^*)01^*)(1(00^*)0)
R(1, 2, 3) = 0 + 0(00^*)(ϵ + 00) + (1 + 0(00^*)01)(0 + 1(00^*)01^*)1 + 1(00^*)(ϵ + 00)
R(1, 3, 3) = 1 + 0(00^*)01 + (1 + 0(00^*)01)(0 + 1(00^*)01^*)(0 + ϵ + 1(00^*)01)
R(2, 0, 3) = 1 + (ϵ + 00)(00^*)1 + (01 + (ϵ + 00)(00^*)01)(0 + 1(00^*)01^*)(1(00^*)1)
R(2, 1, 3) = 0 + (ϵ + 00)(00^*)0 + (01 + (ϵ + 00)(00^*)01)(0 + 1(00^*)01^*)(1(00^*)0)
R(2, 2, 3) = ϵ + 00 + (ϵ + 00)(00^*)(ϵ + 00) + (01 + (ϵ + 00)(00^*)01)(0 +
    1(00^*)01^*)1 + 1(00^*)(ϵ + 00)
R(2, 3, 3) = 01 + (ϵ + 00)(00^*)01 + (01 + (ϵ + 00)(00^*)01)(0 + 1(00^*)01^*)(0 + ϵ +
    1(00^*)01)
R(3, 0, 3) = 1(00^*)1 + (0 + ϵ + 1(00^*)01)(0 + 1(00^*)01^*)(1(00^*)1)
R(3, 1, 3) = 1(00^*)0 + (0 + ϵ + 1(00^*)01)(0 + 1(00^*)01^*)(1(00^*)0)
R(3, 2, 3) = 1 + 1(00^*)(ϵ + 00) + (0 + ϵ + 1(00^*)01)(0 + 1(00^*)01^*)1 + 1(00^*)(ϵ
    + 00)
R(3, 3, 3) = 0 + ϵ + 1(00^*)01 + (0 + ϵ + 1(00^*)01)(0 + 1(00^*)01^*)(0 + ϵ +
    1(00^*)01)
```

Logrando estas ecuaciones para la regla 110:

```
R(0, 0, 0) = 0 + ϵ
R(0, 1, 0) = 1
R(0, 2, 0) = ∅
R(0, 3, 0) = ∅
R(1, 0, 0) = ∅
R(1, 1, 0) = ϵ
R(1, 2, 0) = 1
R(1, 3, 0) = 1
R(2, 0, 0) = 1
R(2, 1, 0) = 0
```

```
R(2, 2, 0) = ε
R(2, 3, 0) = ∅
R(3, 0, 0) = ∅
R(3, 1, 0) = ∅
R(3, 2, 0) = 0
R(3, 3, 0) = 0 + ε

R(0, 0, 1) = 0 + ε
R(0, 1, 1) = 1
R(0, 2, 1) = 11
R(0, 3, 1) = 11
R(1, 0, 1) = ∅
R(1, 1, 1) = ε
R(1, 2, 1) = 1
R(1, 3, 1) = 1
R(2, 0, 1) = 1
R(2, 1, 1) = 0
R(2, 2, 1) = ε + 01
R(2, 3, 1) = 01
R(3, 0, 1) = ∅
R(3, 1, 1) = ∅
R(3, 2, 1) = 0
R(3, 3, 1) = 0 + ε

R(0, 0, 2) = 0 + ε + 11(01^*)1
R(0, 1, 2) = 1 + 11(01^*)0
R(0, 2, 2) = 11 + 11(01^*)(ε + 01)
R(0, 3, 2) = 11 + 11(01^*)
R(1, 0, 2) = 1(01^*)1
R(1, 1, 2) = ε + 1(01^*)0
R(1, 2, 2) = 1 + 1(01^*)(ε + 01)
R(1, 3, 2) = 1 + 1(01^*)
R(2, 0, 2) = 1 + (ε + 01)(01^*)1
R(2, 1, 2) = 0 + (ε + 01)(01^*)0
R(2, 2, 2) = ε + 01 + (ε + 01)(01^*)(ε + 01)
R(2, 3, 2) = 01 + (ε + 01)(01^*)
R(3, 0, 2) = 0(01^*)1
R(3, 1, 2) = 0(01^*)0
R(3, 2, 2) = 0 + 0(01^*)(ε + 01)
R(3, 3, 2) = 0 + ε + 0(01^*)

R(0, 0, 3) = 0 + ε + 11(01^*)1 + 11 + 11(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(0, 1, 3) = 1 + 11(01^*)0 + 11 + 11(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(0, 2, 3) = 11 + 11(01^*)(ε + 01) + 11 + 11(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ε + 01)
R(0, 3, 3) = 11 + 11(01^*) + 11 + 11(01^*)(0 + 0(01^*)^*)0 + ε + 0(01^*)
R(1, 0, 3) = 1(01^*)1 + 1 + 1(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(1, 1, 3) = ε + 1(01^*)0 + 1 + 1(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(1, 2, 3) = 1 + 1(01^*)(ε + 01) + 1 + 1(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ε + 01)
R(1, 3, 3) = 1 + 1(01^*) + 1 + 1(01^*)(0 + 0(01^*)^*)0 + ε + 0(01^*)
R(2, 0, 3) = 1 + (ε + 01)(01^*)1 + 01 + (ε + 01)(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(2, 1, 3) = 0 + (ε + 01)(01^*)0 + 01 + (ε + 01)(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(2, 2, 3) = ε + 01 + (ε + 01)(01^*)(ε + 01) + 01 + (ε + 01)(01^*)(0 + 0(01^*)^*)0 +
    0(01^*)(ε + 01)
R(2, 3, 3) = 01 + (ε + 01)(01^*) + 01 + (ε + 01)(01^*)(0 + 0(01^*)^*)0 + ε + 0(01^*)
R(3, 0, 3) = 0(01^*)1 + 0 + ε + 0(01^*)(0 + 0(01^*)^*)(0(01^*)1)
R(3, 1, 3) = 0(01^*)0 + 0 + ε + 0(01^*)(0 + 0(01^*)^*)(0(01^*)0)
R(3, 2, 3) = 0 + 0(01^*)(ε + 01) + 0 + ε + 0(01^*)(0 + 0(01^*)^*)0 + 0(01^*)(ε + 01)
R(3, 3, 3) = 0 + ε + 0(01^*) + 0 + ε + 0(01^*)(0 + 0(01^*)^*)0 + ε + 0(01^*)
```

[3]

Bibliografía

[1] *Cellular Automata*. Jarkko Kari, Spring 2013
`https://www.cs.tau.ac.il/~nachumd/models/CA.pdf`

[2] *A New Kind of Science*. Wolfram Stephen, 2002

[3] *On Patterns and Dynamics of Rule 22 Cellular Automaton*. Genaro J. Martínez, Andrew Adamatzky, Rolf Hoffmann, Rennes, France, Ivan Zelinka

[4] *Introduccion a los automatas celulares elementales*. Carlos Zacarias Reyes Martinez
Escuela Superior de Computo