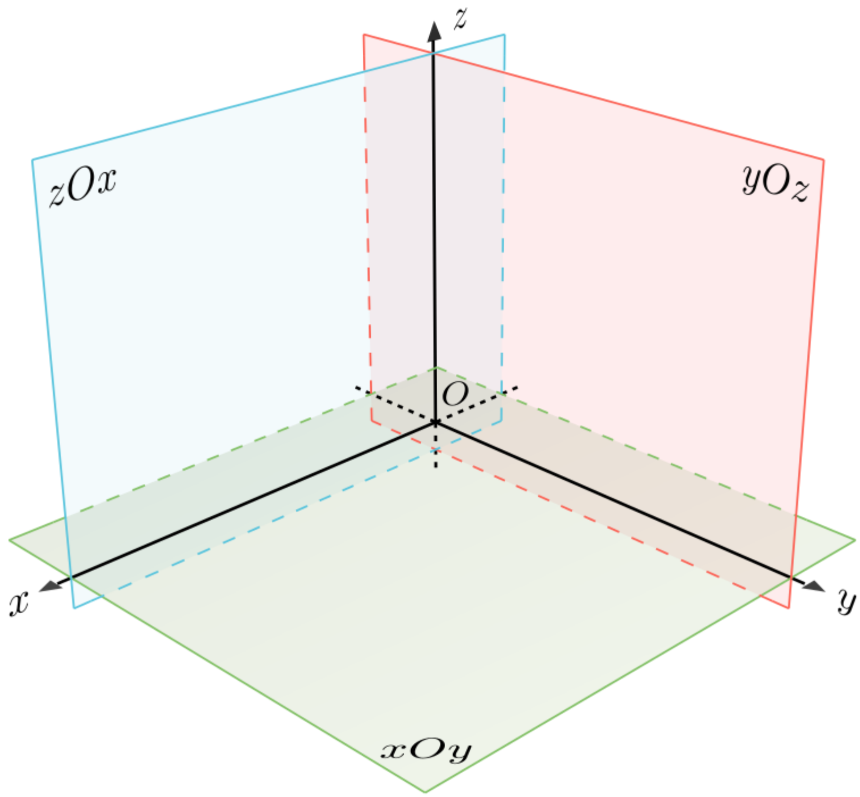
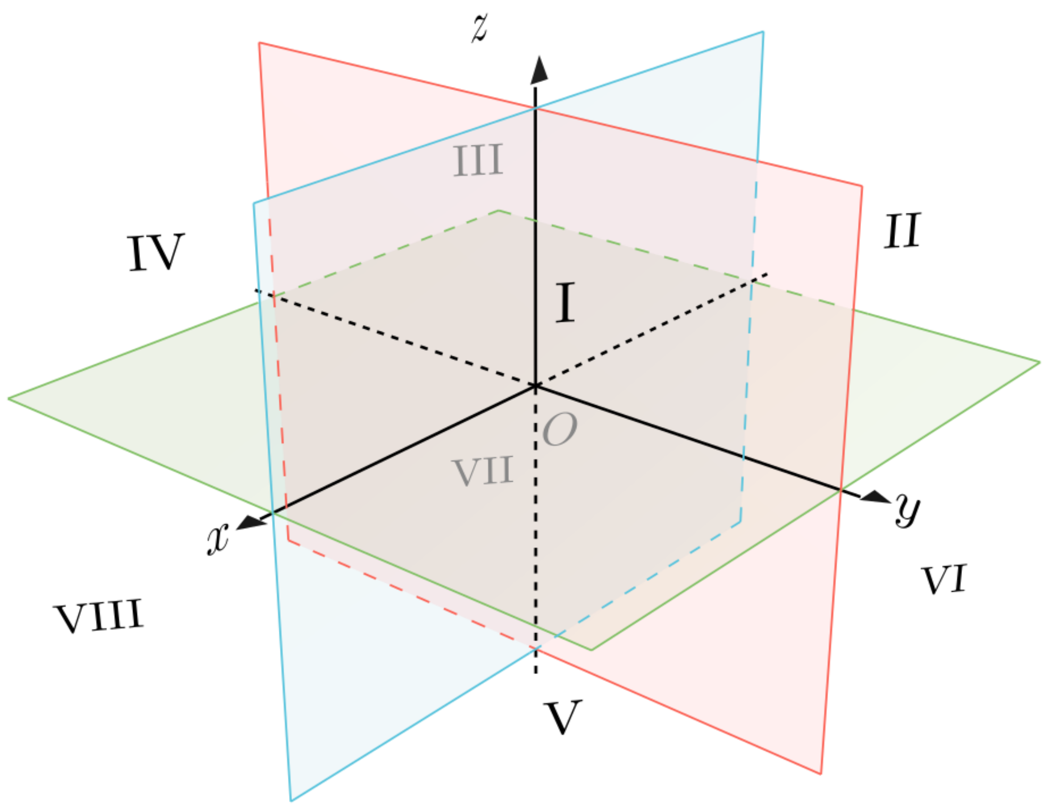
Chapter 11 立体几何与空间向量

11.1 立体直角坐标系

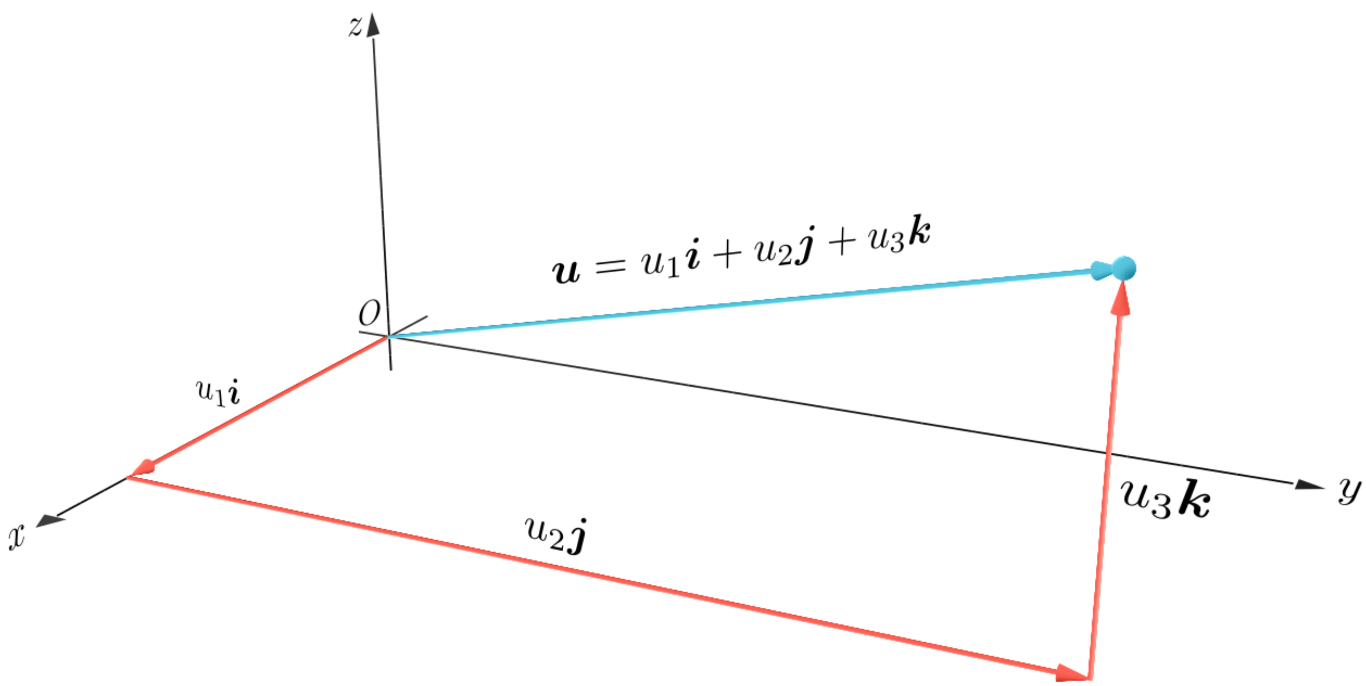




11.2 向量

1. Concepts

- 从低维到高维
- 几何意义
    - 向量 $\Leftarrow$ 空间中的点
    -
- 向量的分量

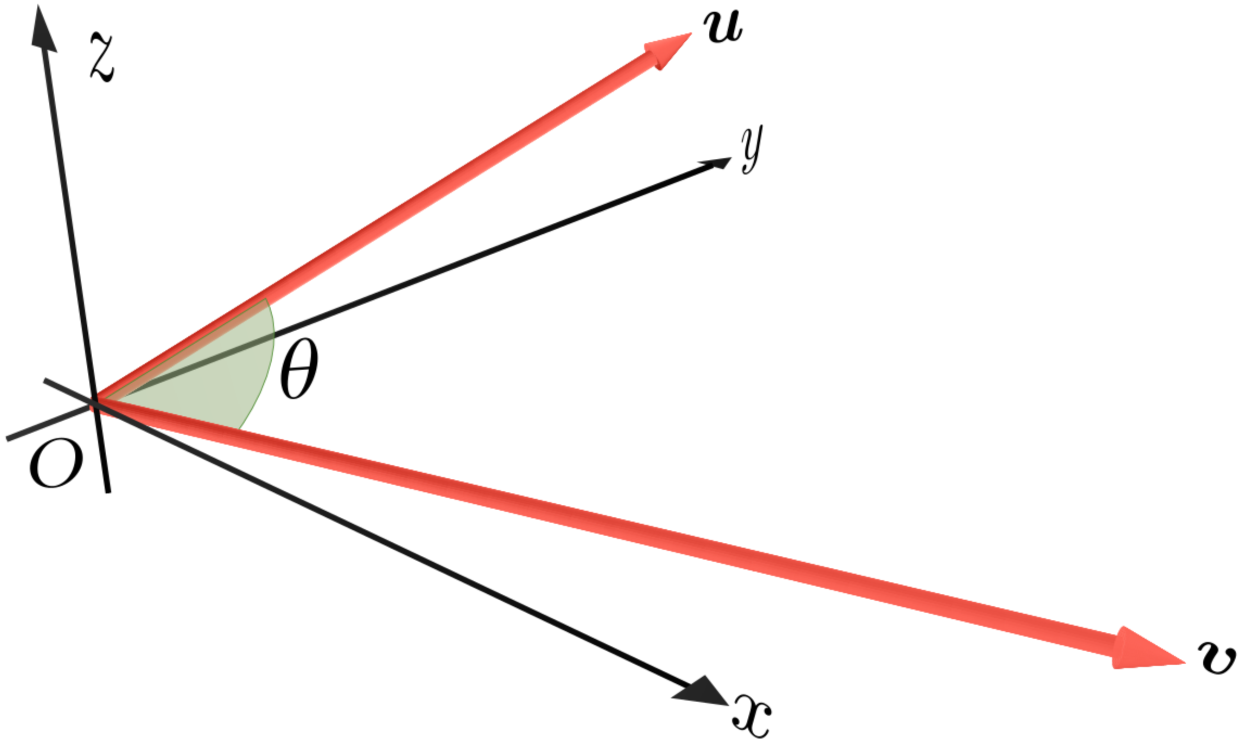


2. 向量加法
3. 向量数乘
4. 向量减法
5. 线性性质：
   加法：交换律、结合律
   数乘：交换律、结合律、分配律
6. 向量长度

11.3 点积（数量积）

a. 定义　$\sum u_i v_i$: 标量
b. 几何意义:一个 **向量的长度** 乘以另一个向量 **投影的长度**

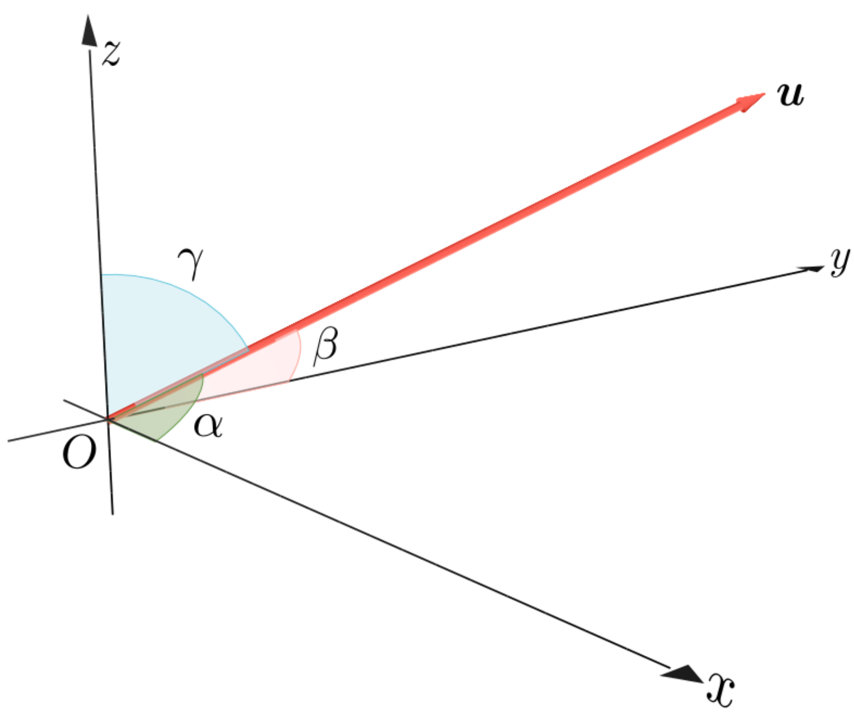
1. 向量夹角



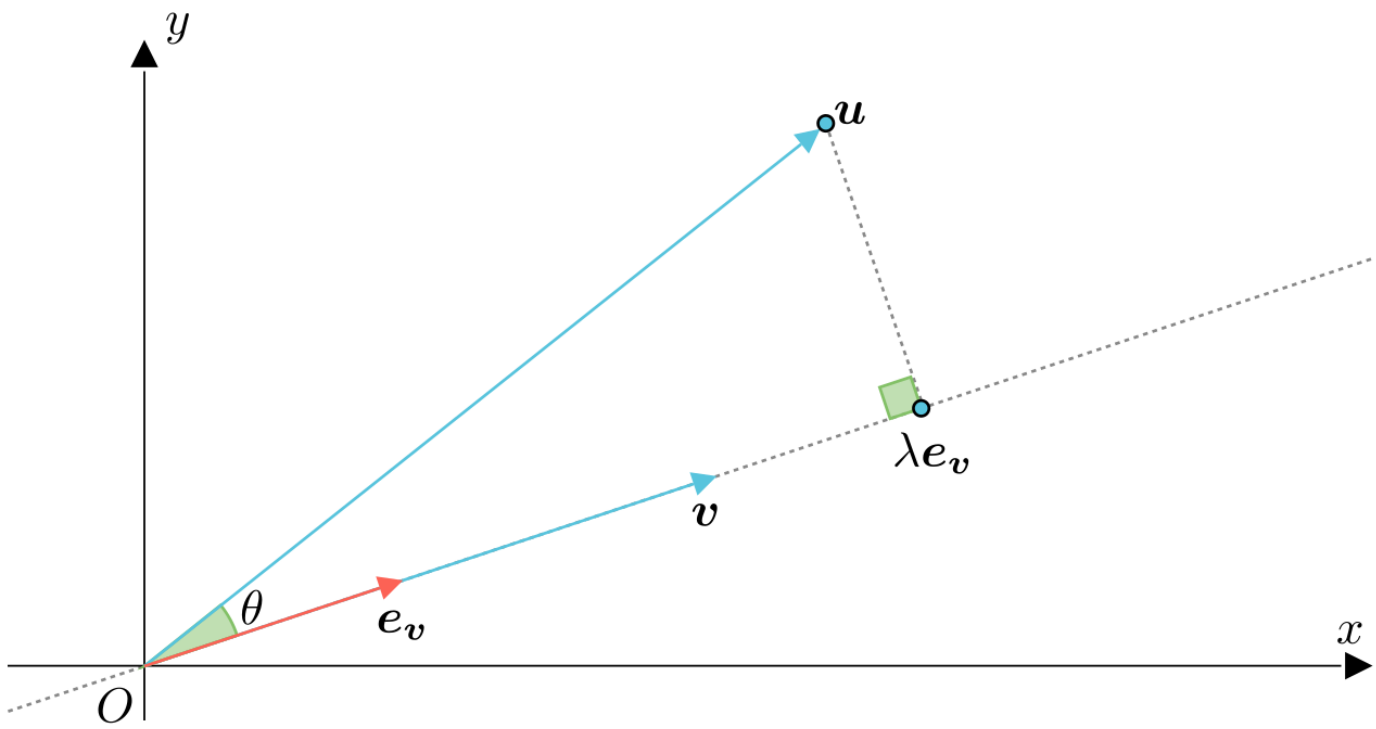
> 若$\boldsymbol{u}$、$\boldsymbol{v}$都不是零向量，其夹角为$(\widehat{\boldsymbol{u},\boldsymbol{v}})=\theta$，则$\cos\theta=\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$。

例:向量与坐标轴的夹角

2. 方向角与方向余弦



投影



3. 点积的性质
   交换律，数乘结合律，**分配律**

4. 正交

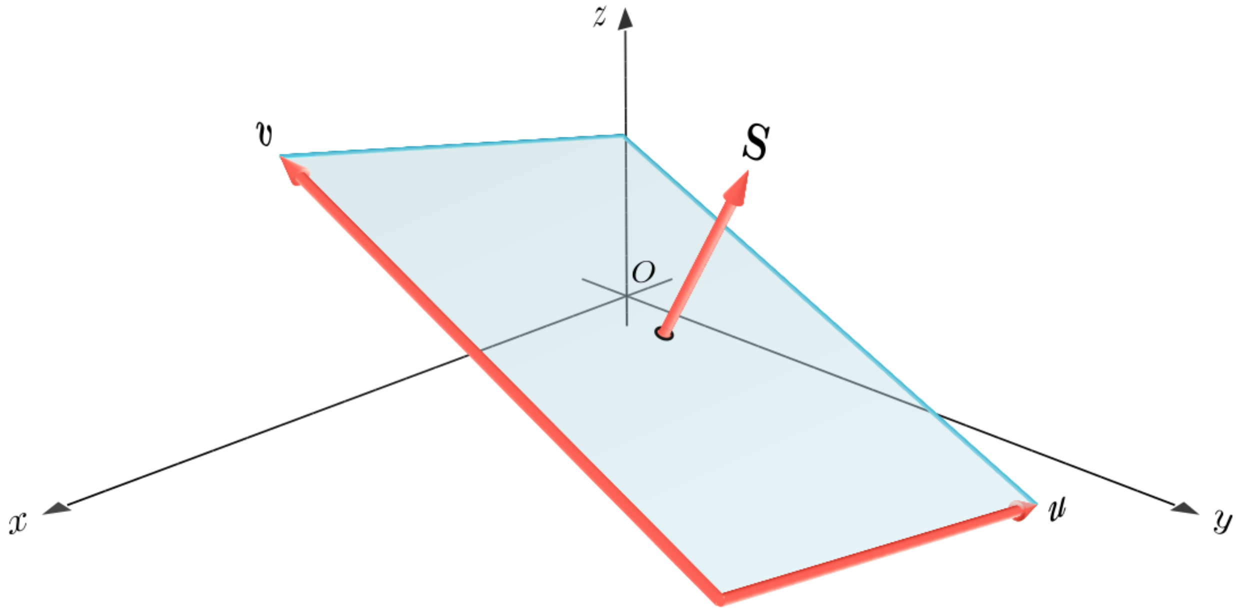
11.4 叉积(向量积)

1. 行列式

> $$\boldsymbol{S}=\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\boldsymbol{k}$$

2. 几何意义

有向面积

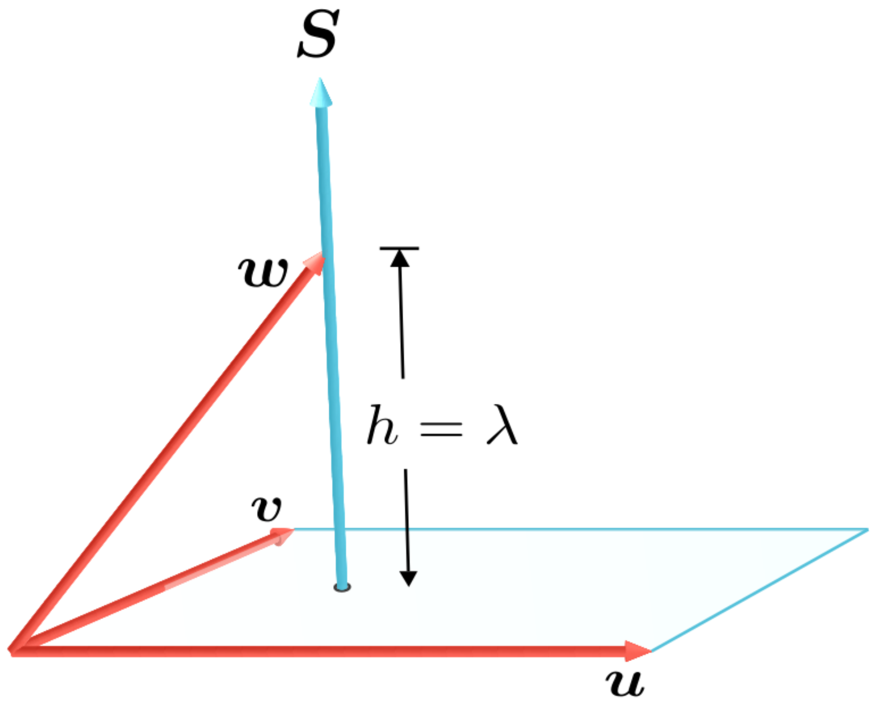


3. 性质

反交换律	$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$
分配律	$\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
数乘结合律	$\lambda(\boldsymbol{u} \times \boldsymbol{v}) = (\lambda \boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (\lambda \boldsymbol{v})$

混合积

$(u \times v) \cdot w$

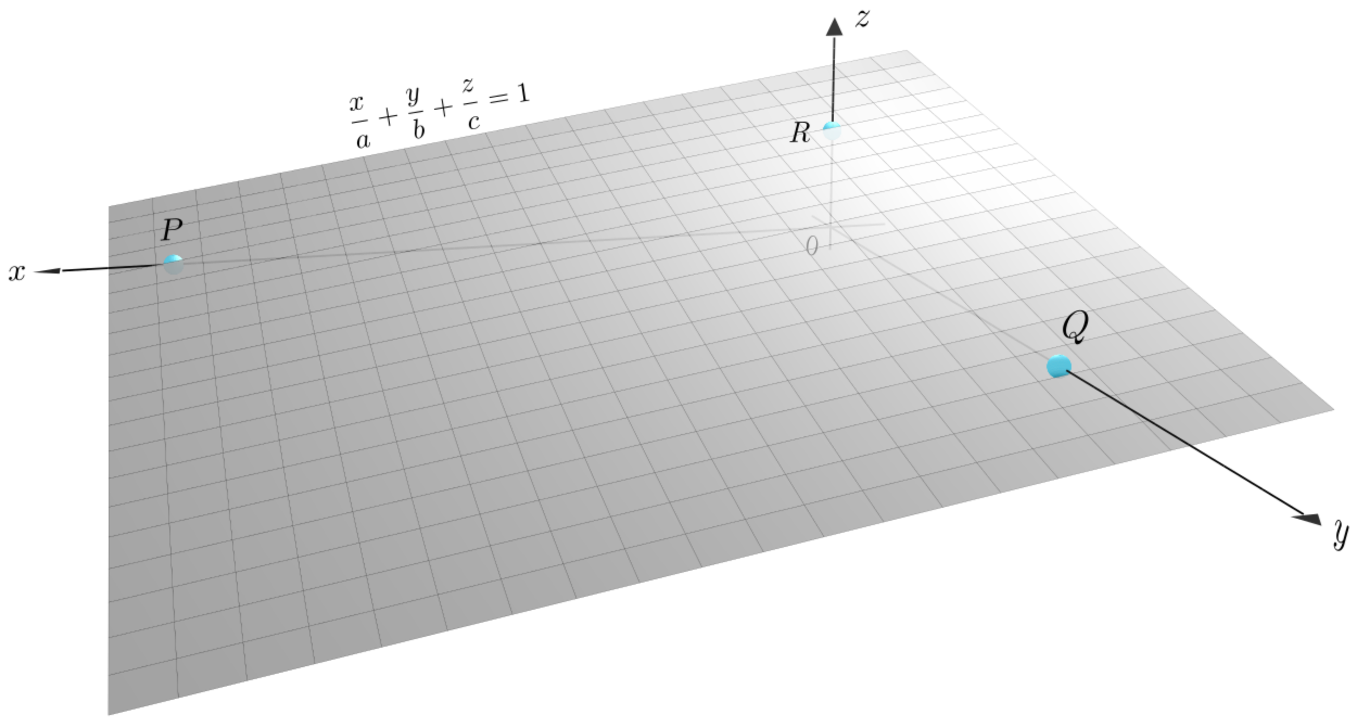


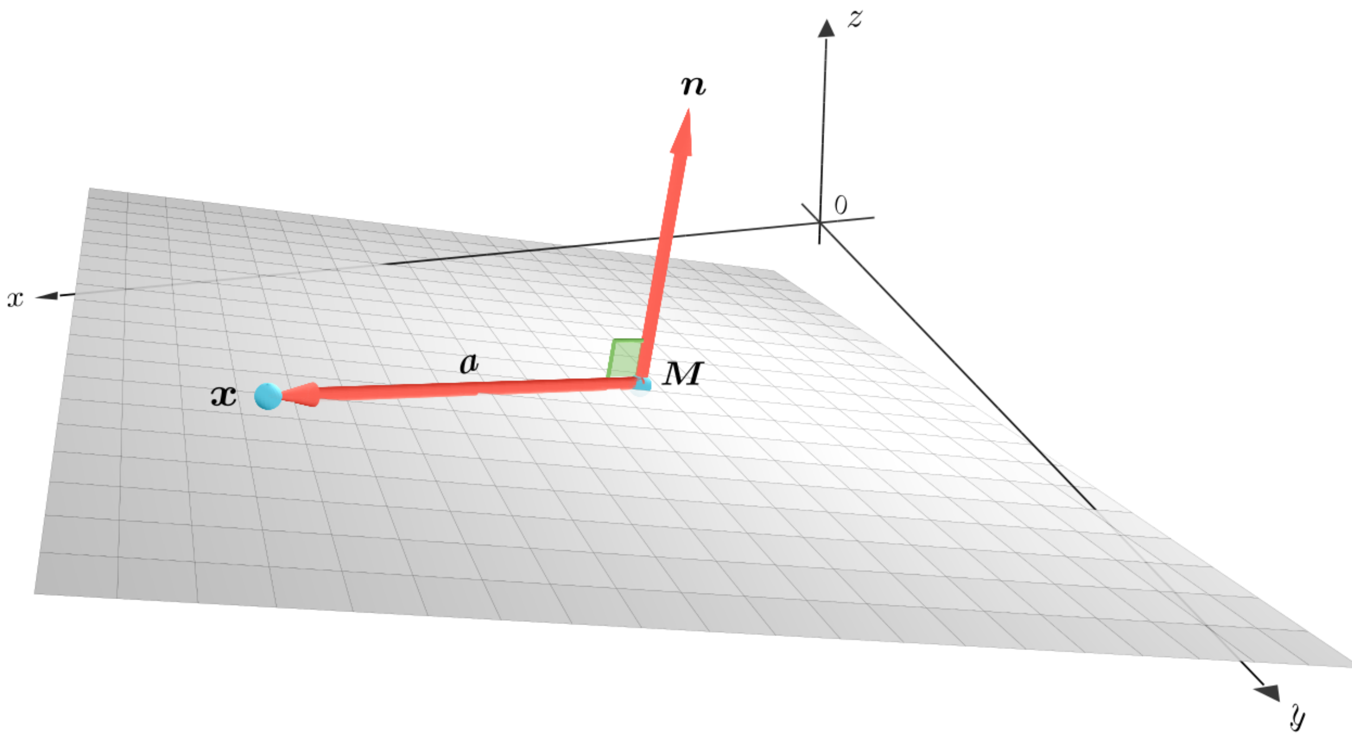
$(u \times v) \cdot w = u \cdot (v \times w)$

$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

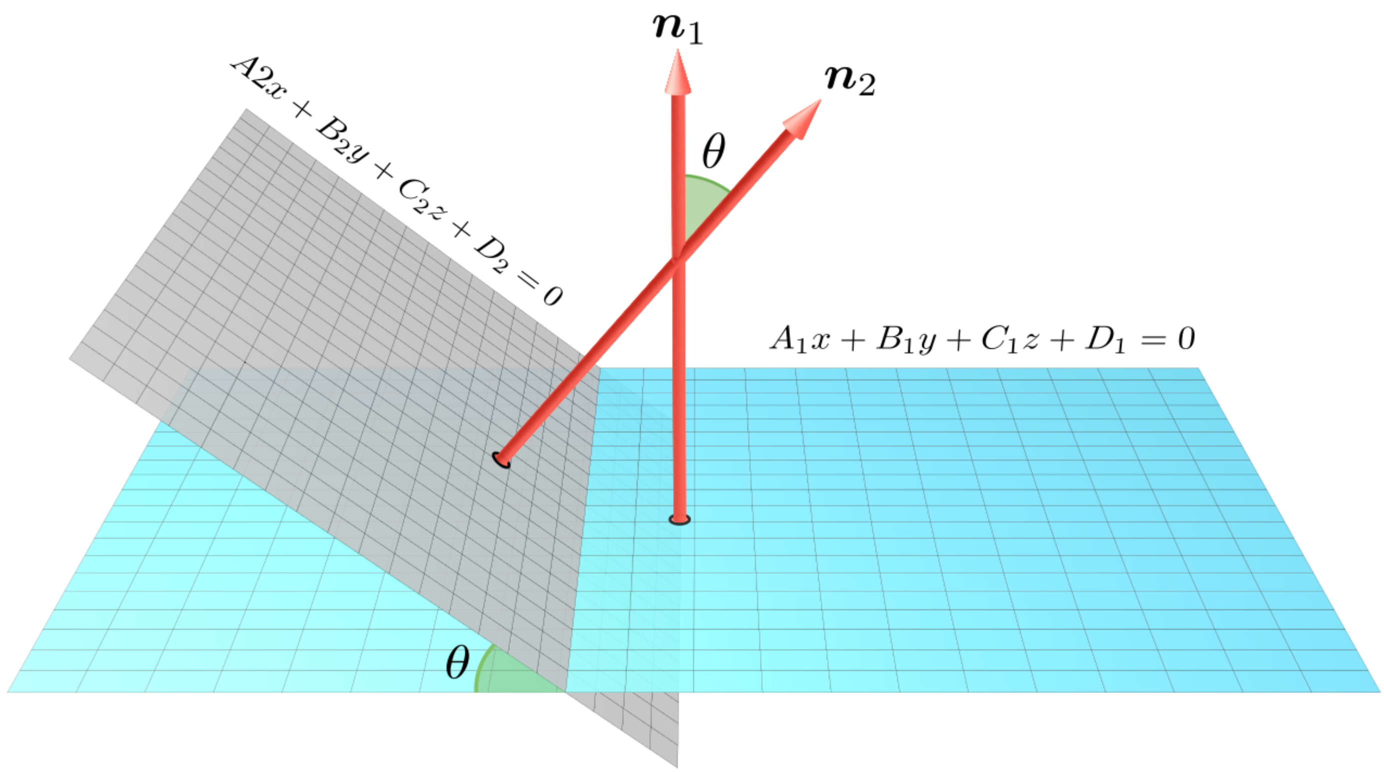
11.5 向量值函数

1. 用方程表示平面
   截距式、点法式、一般式





2. 平面的夹角



3. 点到平面的距离
   复习：点到直线的距离

4. 直线方程
   参数方程⇒向量方程（向量形式）
   对称方程

5. 直线间的夹角
6. 直线与平面的夹角

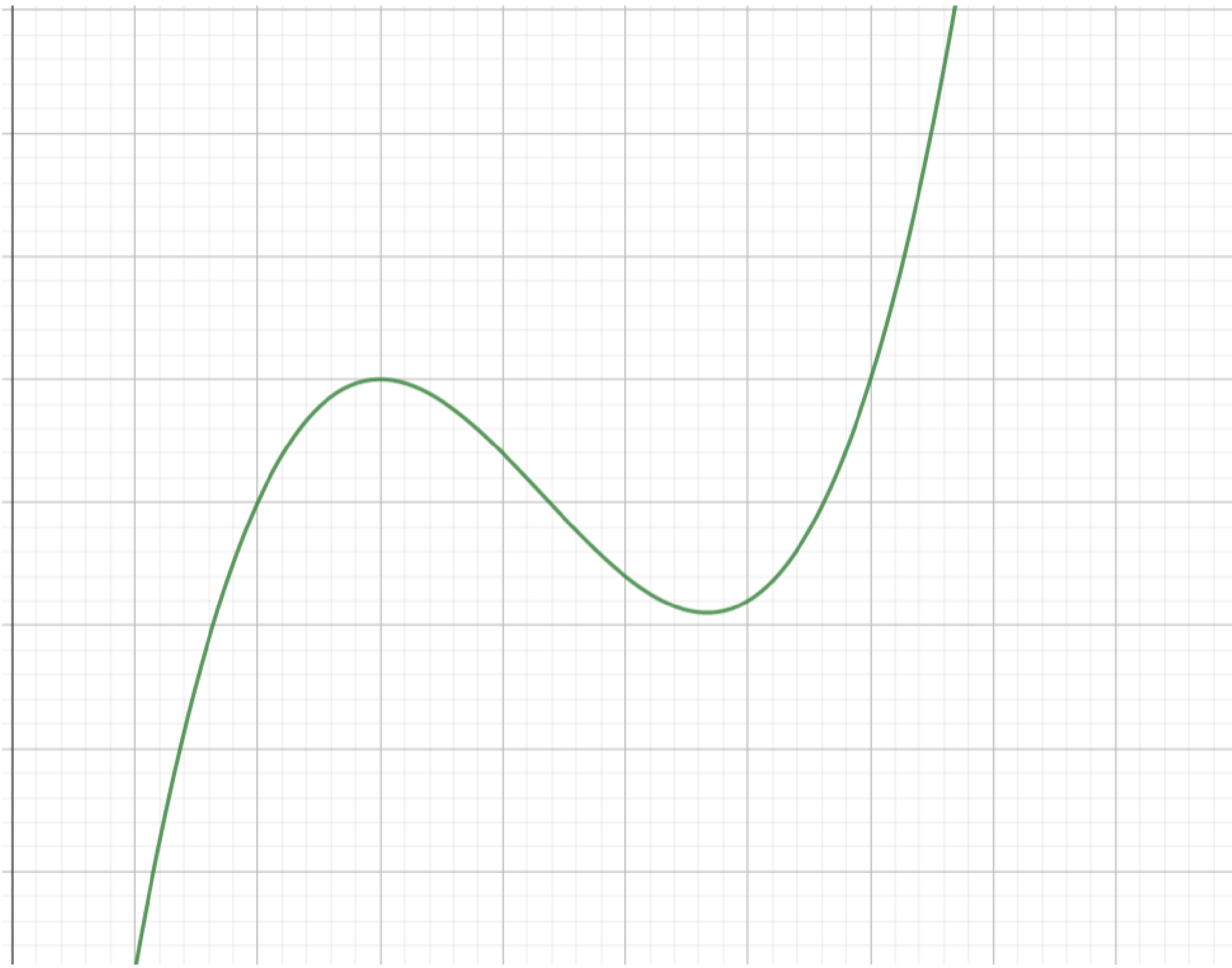
7. **曲线**
   点系⇒曲线
   参数方程⇒向量方程

> $F(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k} =< f(t)\boldsymbol{i}, g(t)\boldsymbol{j}, h(t)\boldsymbol{k} >$

- 曲线的微分：
  - 求导：几何意义(一次、二次)
  - $F'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k} =< f'(t)\boldsymbol{i}, g'(t)\boldsymbol{j}, h'(t)\boldsymbol{k} >$
  - 性质：
    - $D_t[F(t) \times G(t)] = F(t) \times G'(t) + F'(t) \times G(t)$

- 求曲线的长度

> 对于曲线:$l: F(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$
>
> $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt$
>
> $= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2} dx$

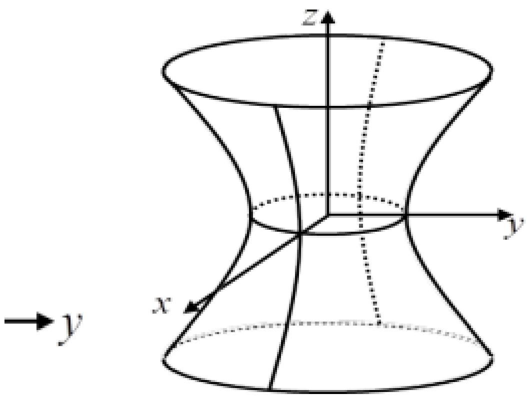


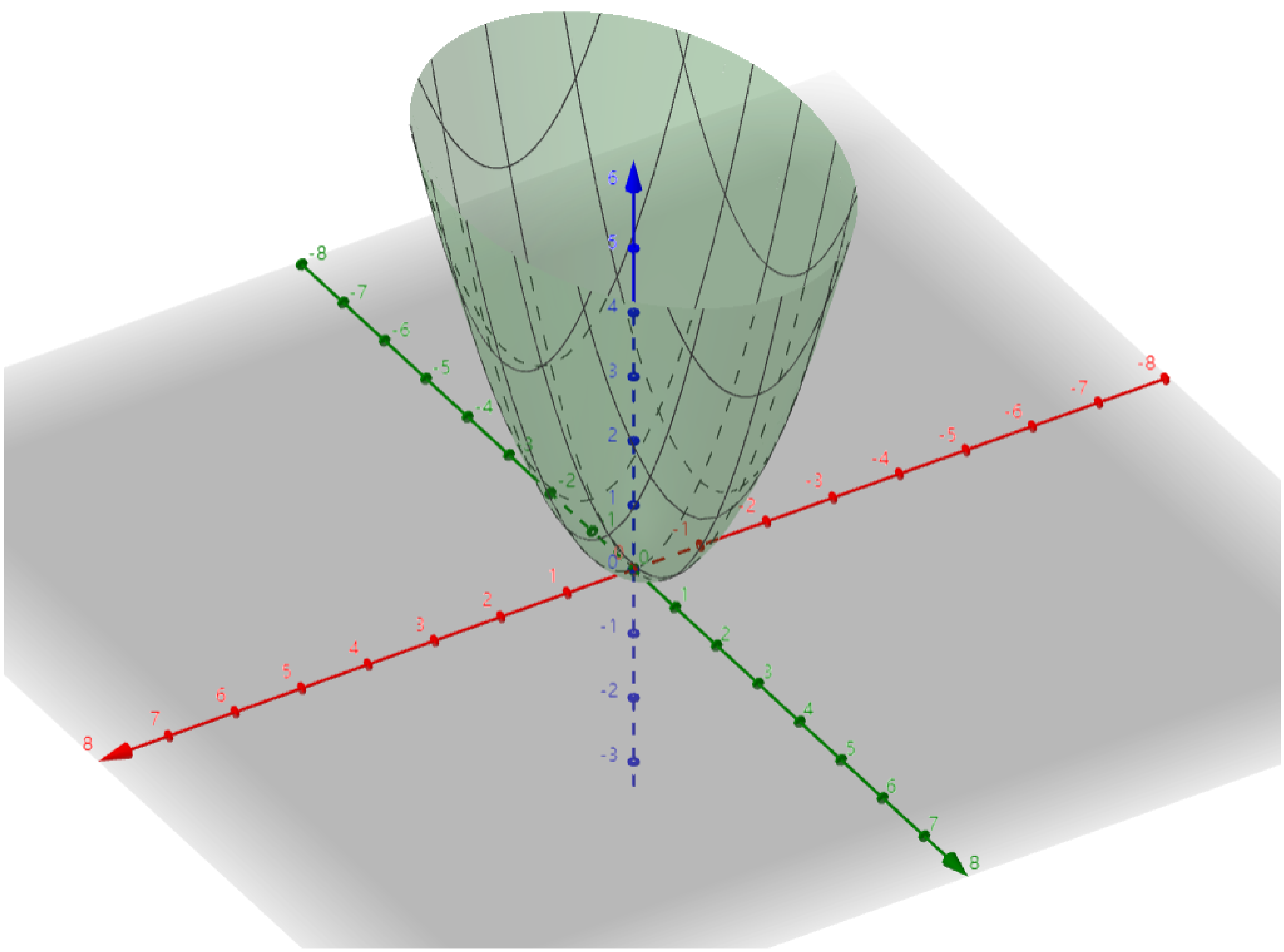
1. **曲面**
   柱面 cylindrical surface
   二次曲面 quadric surface
   例：如何画空间曲面草图? (固定一个变量)

$4x^2 + 4y^2 - z^2 = 4$ 方程

$z = f(x, y) = x^2 + y^2$ 函数

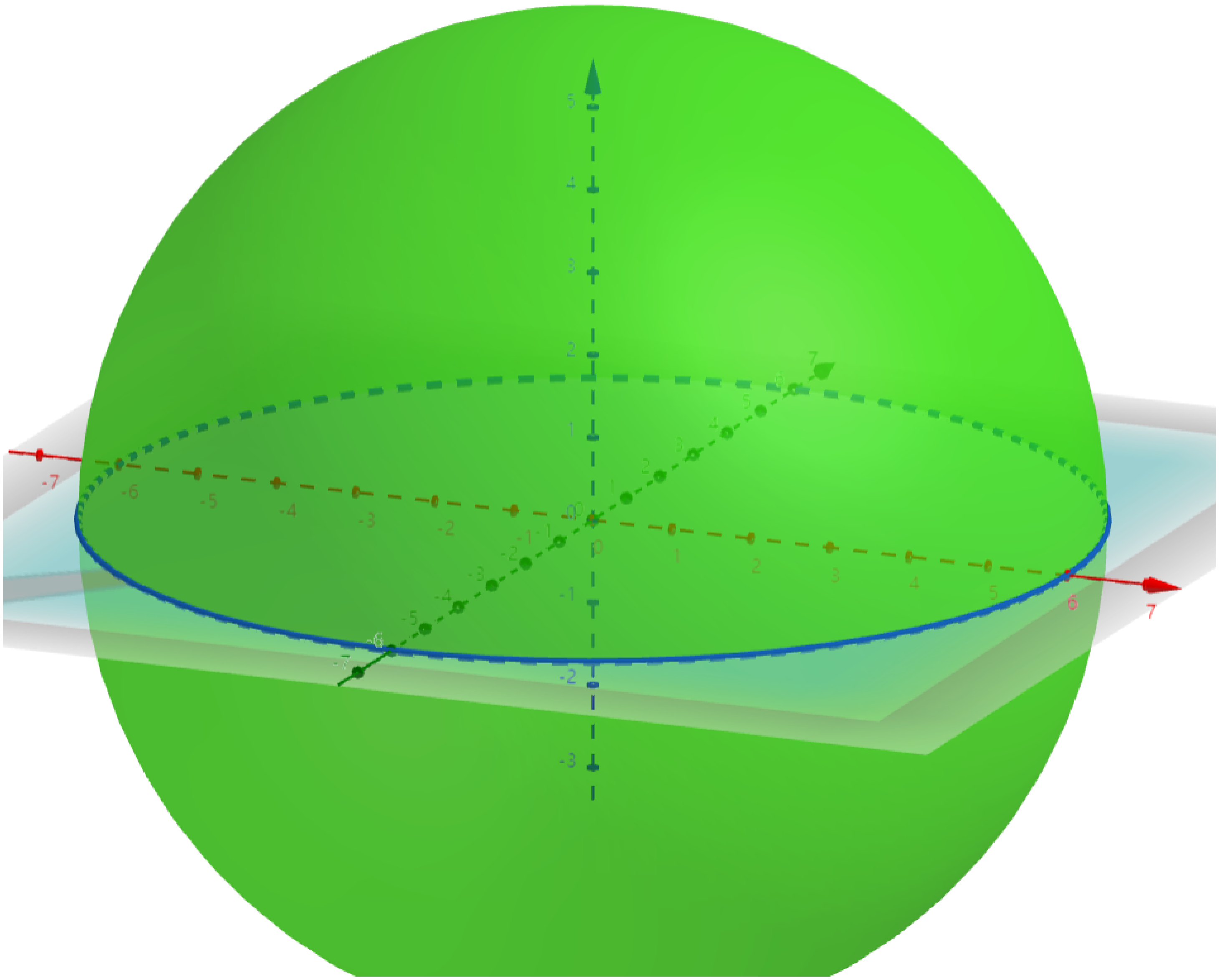
$z = f(x, y) = x^2 + \frac{y^2}{2}$ 函数

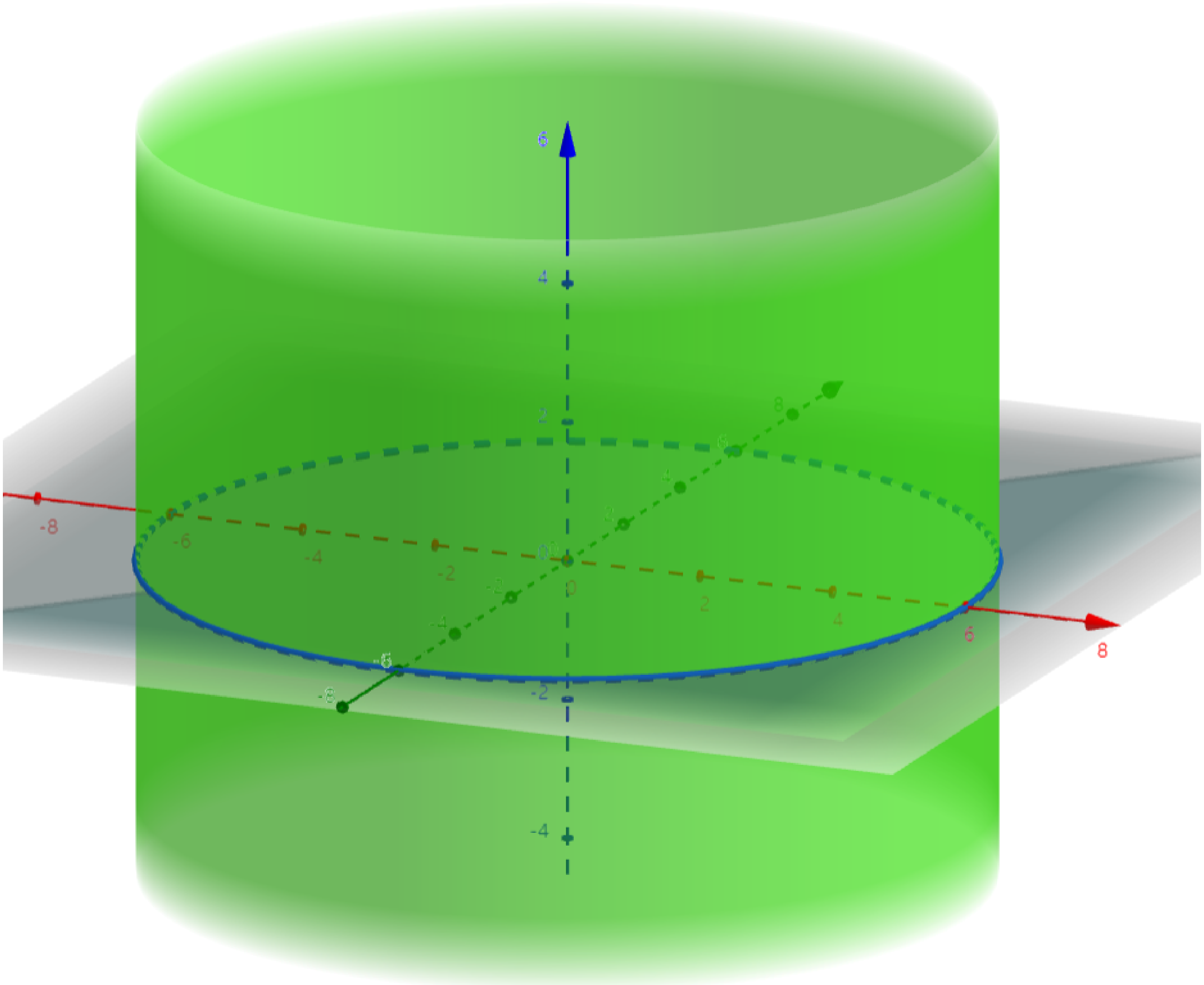




11.9 柱面坐标系和球面坐标系

$(x, y, z) \Rightarrow (\rho, \theta, z)$ or (ρ, θ, ϕ)





例题

1. 求两个平面的交线
2. 求异面直线距离

3. Given two planes $x + y + z = 0$ and $2x - y + z = 2$.

(1) Find the parametric equations of the intersection line of the two planes,

(2) Find the distance from the point $P\,(3, -1, 2)$ to the line in part (1).

3. 求异面直线最近的两点距离

4. 给三个点，求出平面方程

（↓这题不讲了）

2. Given three points P (1, 1, 1), Q (0, 3, 1) and R (0, 1, 4).

(1) Find the area of the triangle PQR.

(2) Find the equation of the plane through P, Q and R, expressed in the form $Ax+By+Cz=D$.

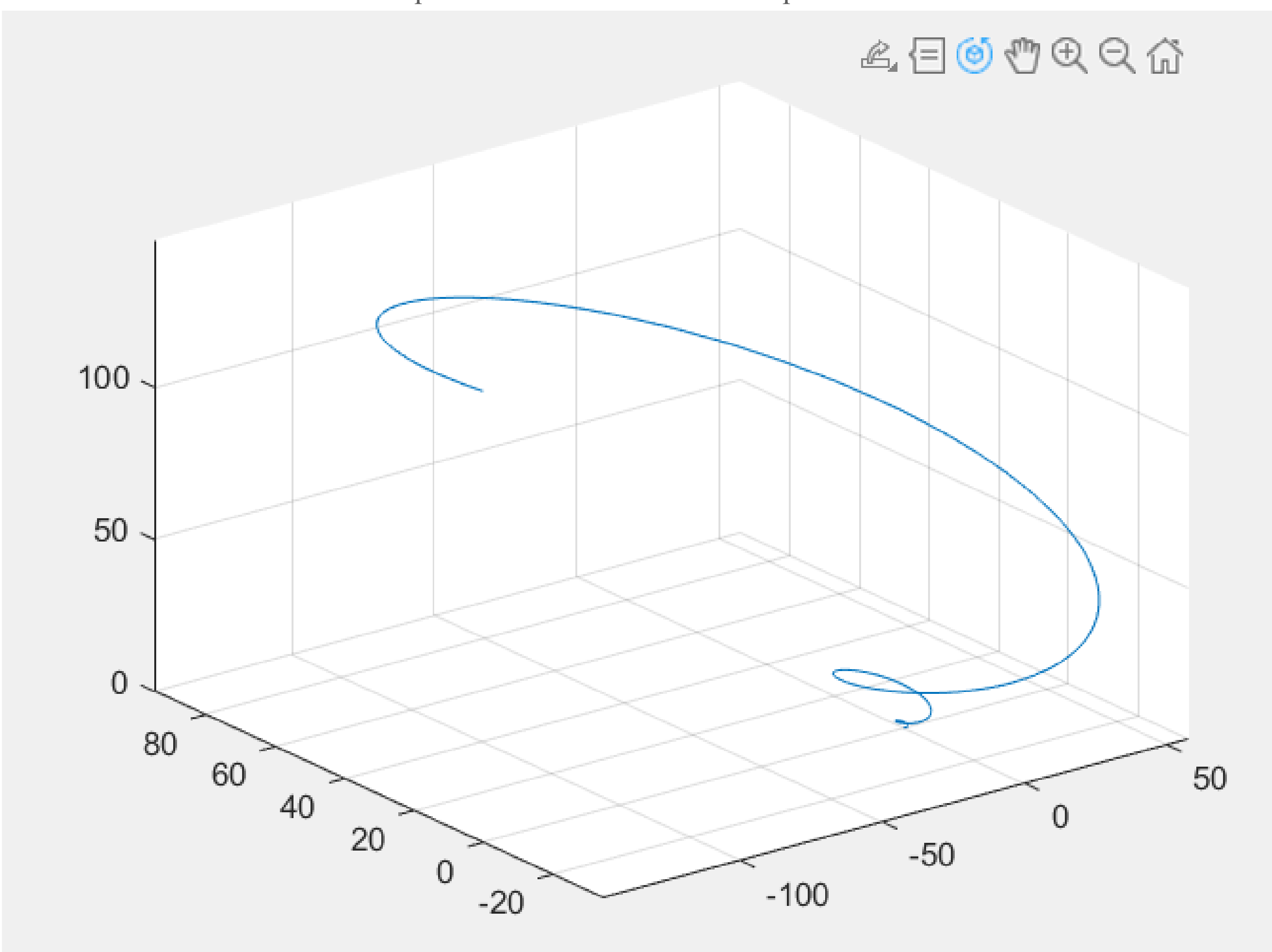
(3) Is the line through (1, 2, 3) and (2, 2, 0) parallel to the plane in part (2)? Explain why or why not?

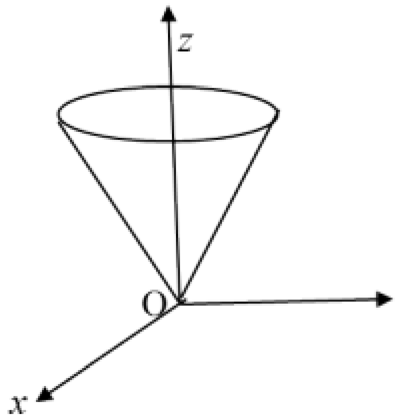
5. 找一个向量在另一个向量上的投影

> The vector projection of $\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ on $\boldsymbol{v}=\boldsymbol{j}+2\boldsymbol{k}$ is
> $-\frac{1}{5}j-\frac{2}{5}k$

6. Given the curve $C: r(t)=(e^t cos\pi t)\boldsymbol{i}+(e^t sin\pi t)\boldsymbol{j}+e^t\boldsymbol{k}$

Show that the curve C lies on a quadric surface and find the equation of the surface.

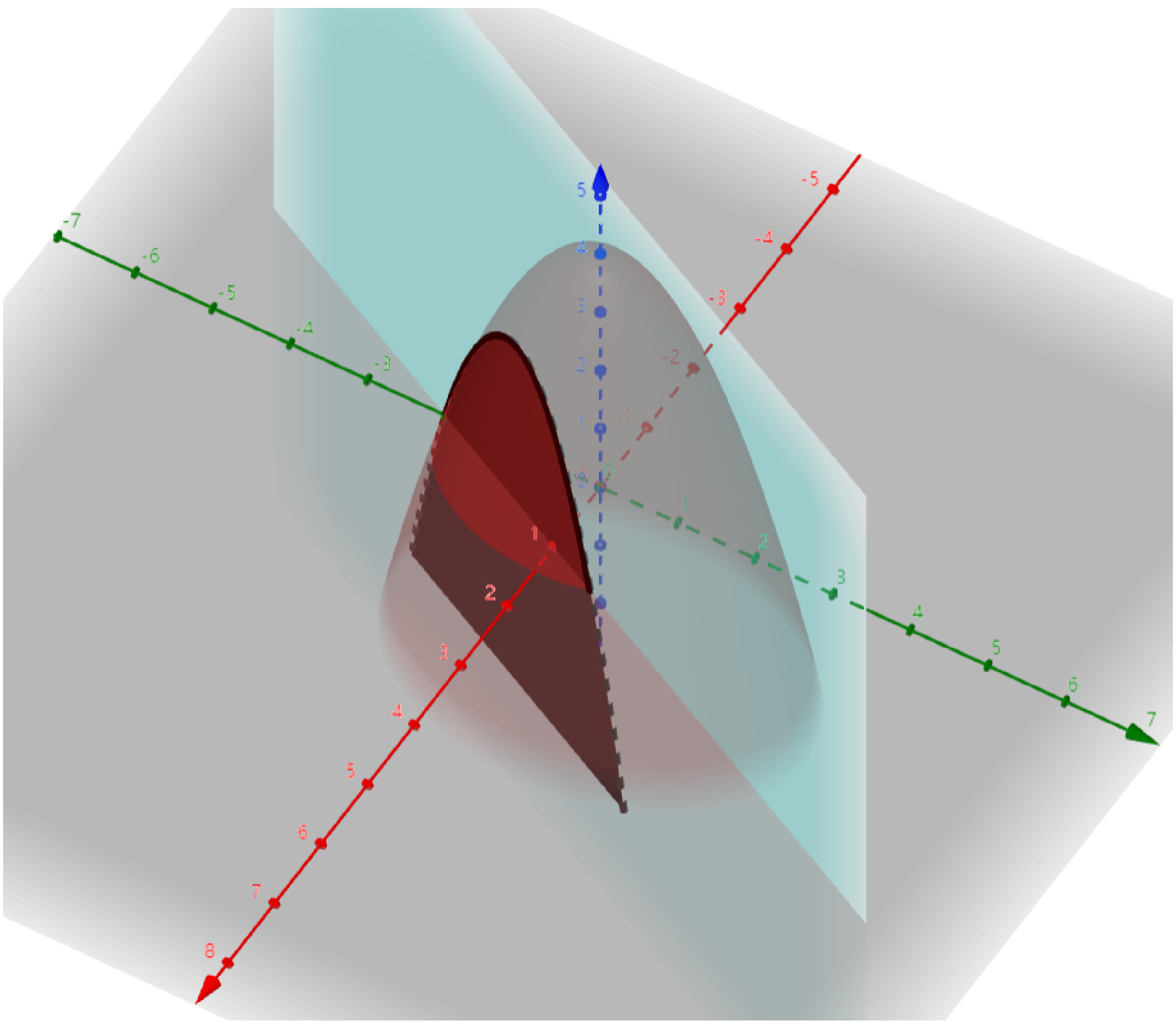




7. 求曲面被切的边缘

Let C be the curve of intersection of the paraboloid $z = 4 - 2x^2 - y^2$ and plane $2x - y = 2$

解方程游戏

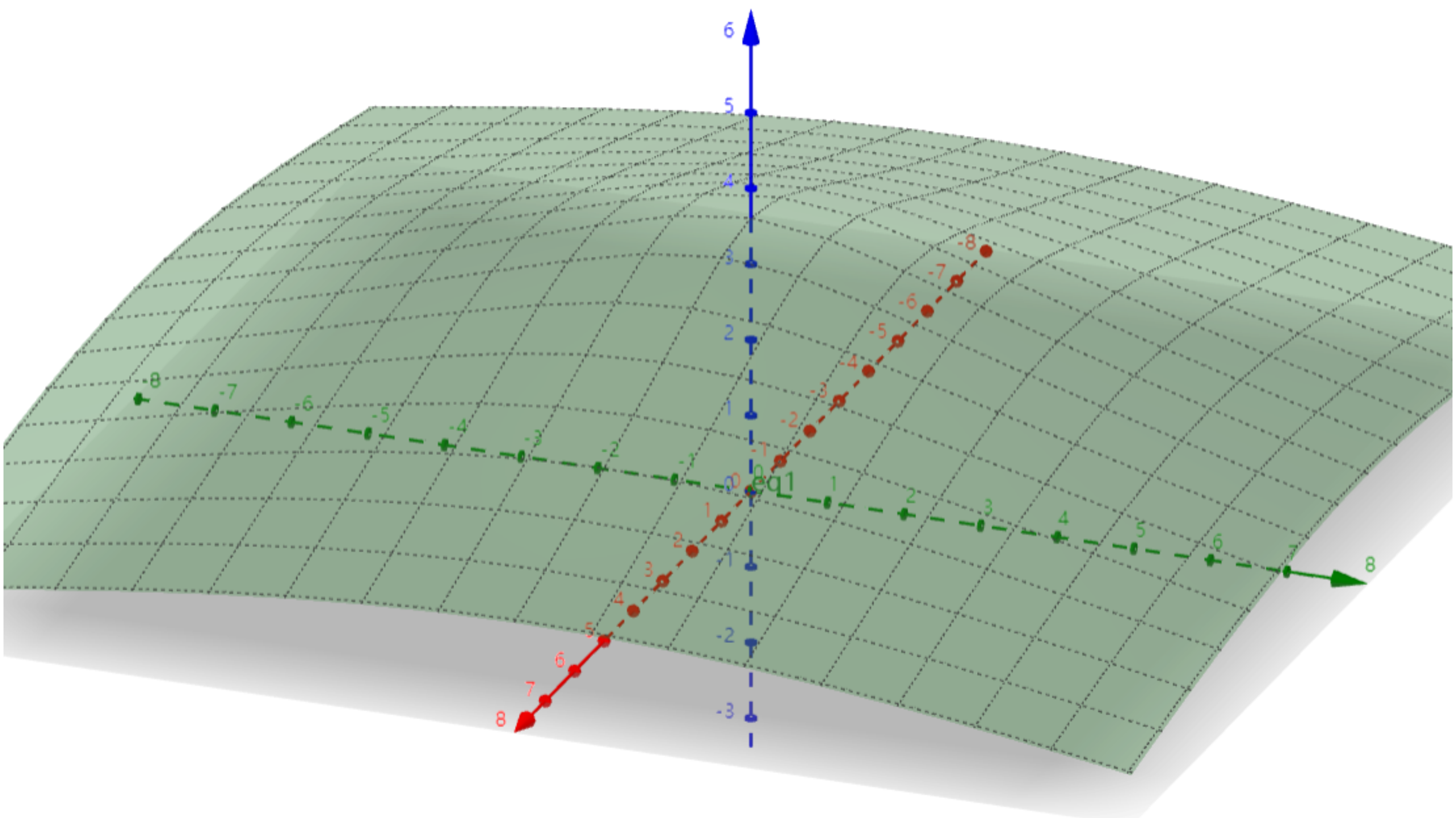


Chapter 12二元变量函数

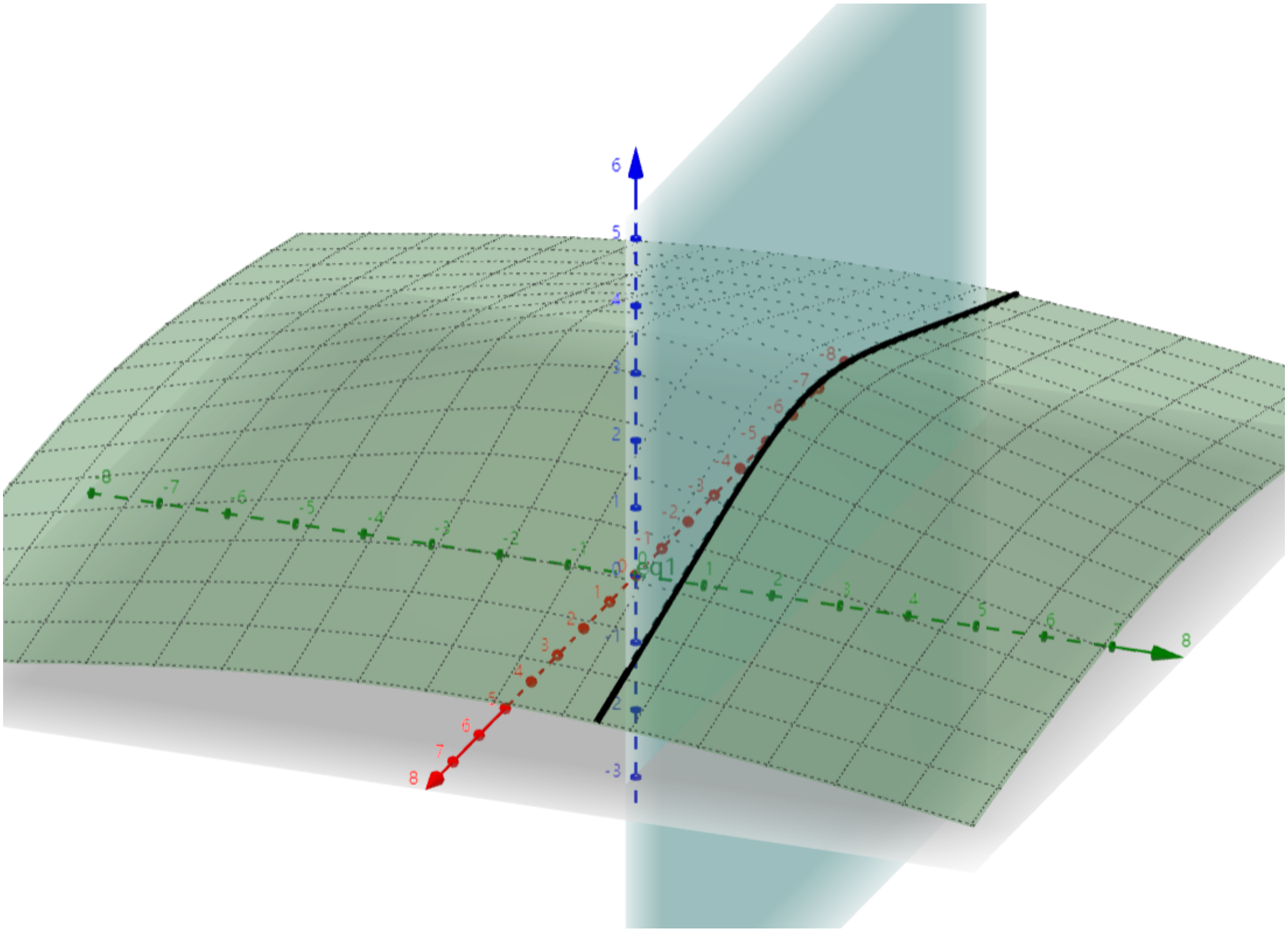
主线：找切平面

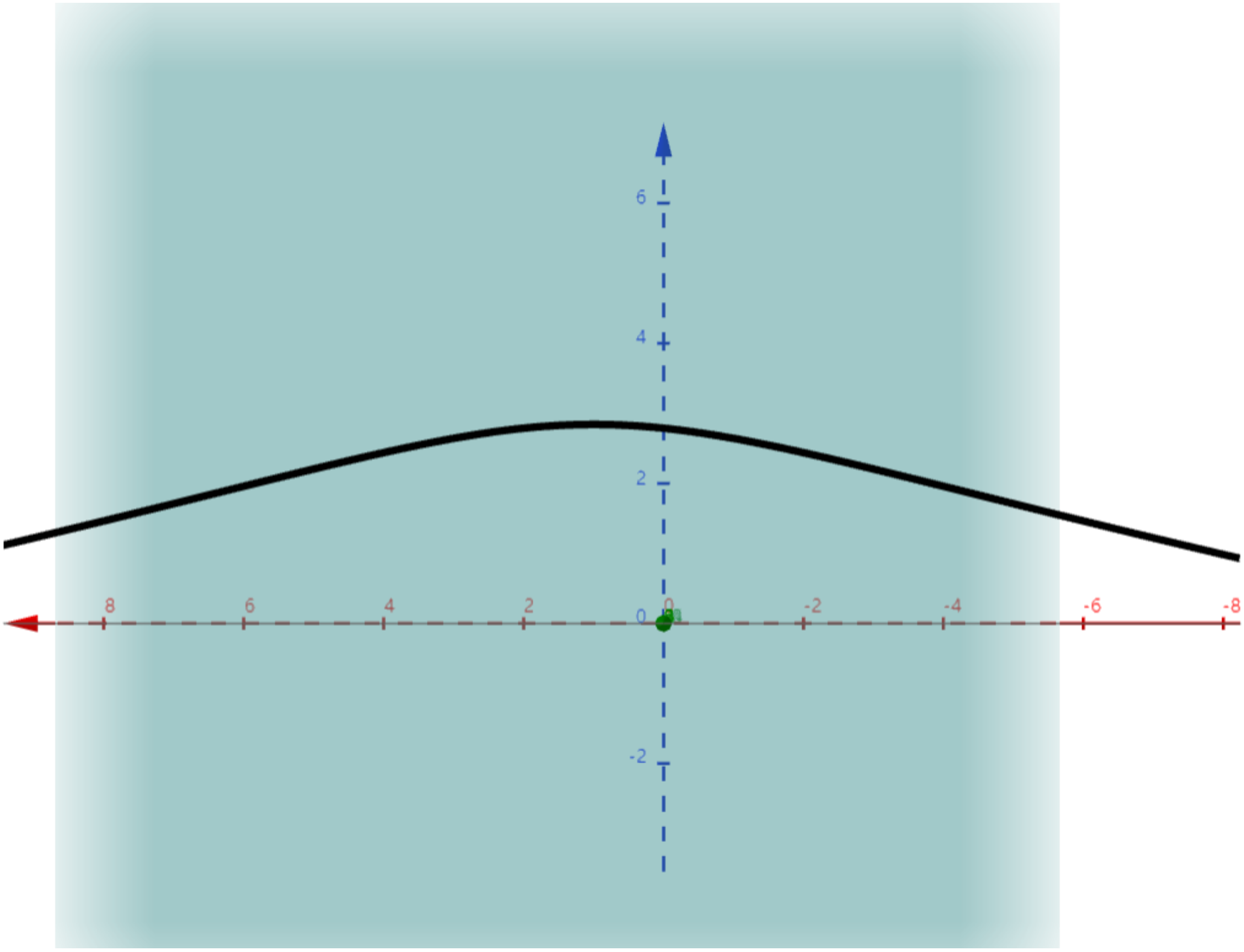
12.1 二元变量函数

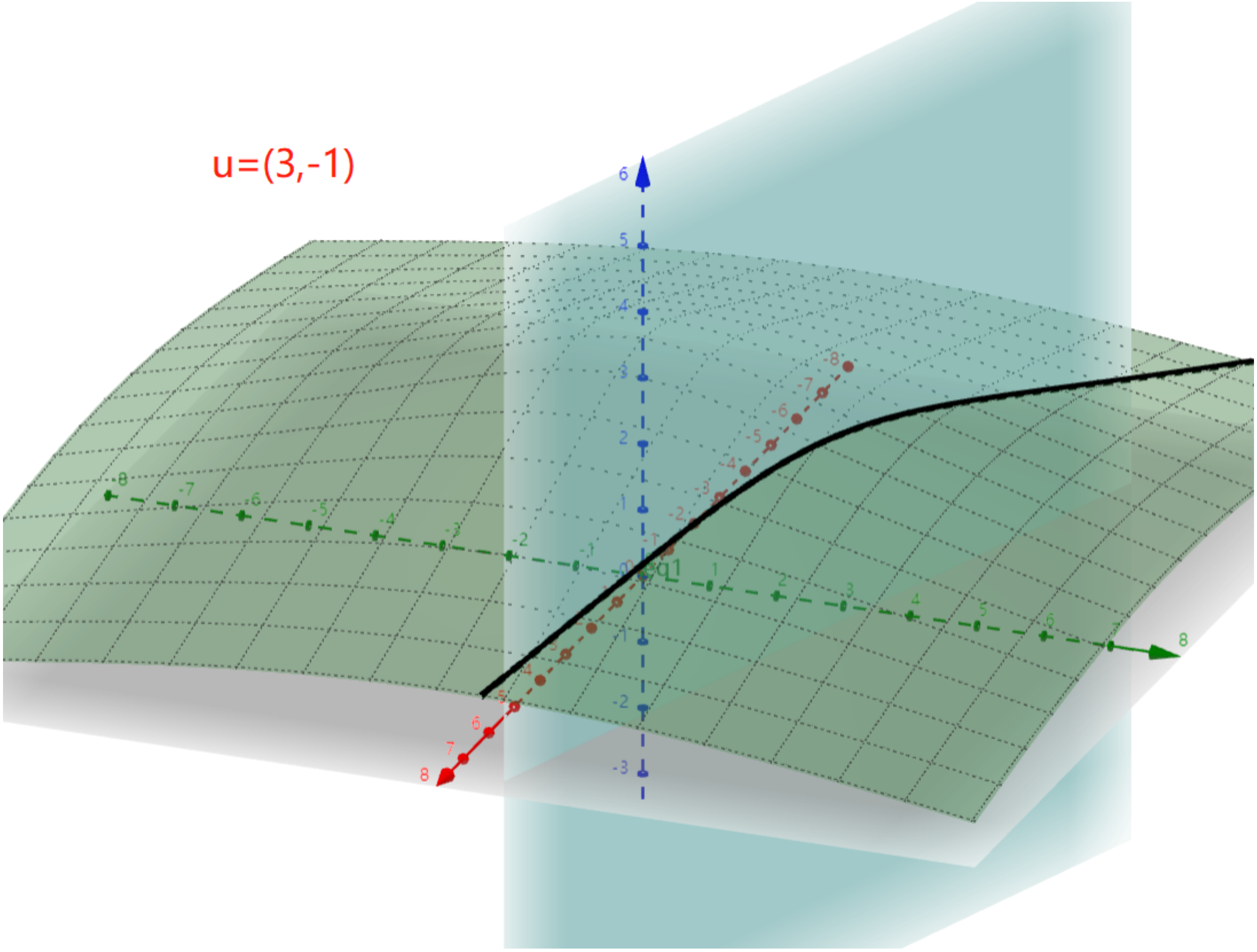
单变量函数⇒双变量函数



12.2 微分(偏导数)




6
4
2
0
-2
8
6
4
2
0
-2
-4
-6
-8

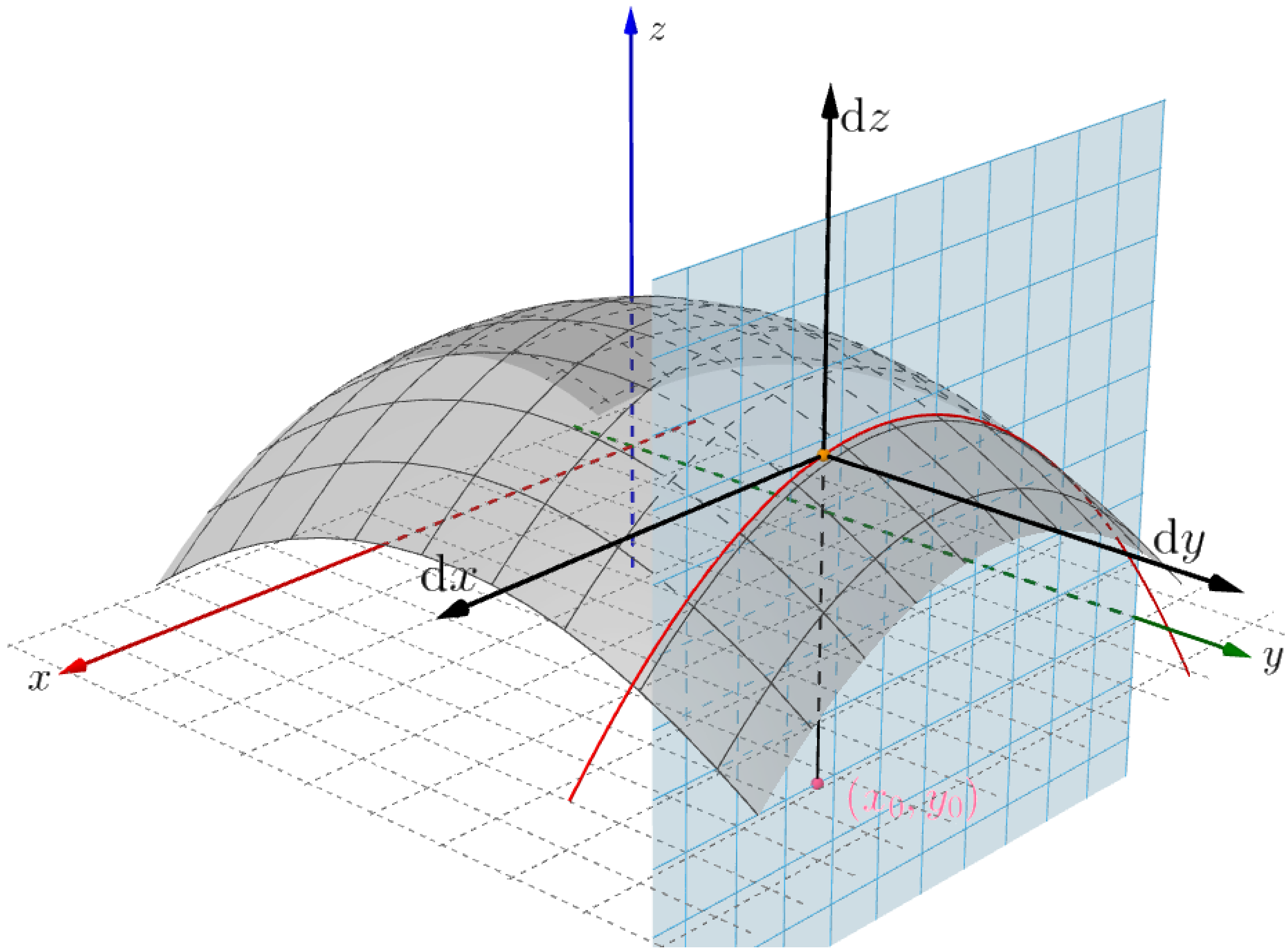


高阶的偏导数

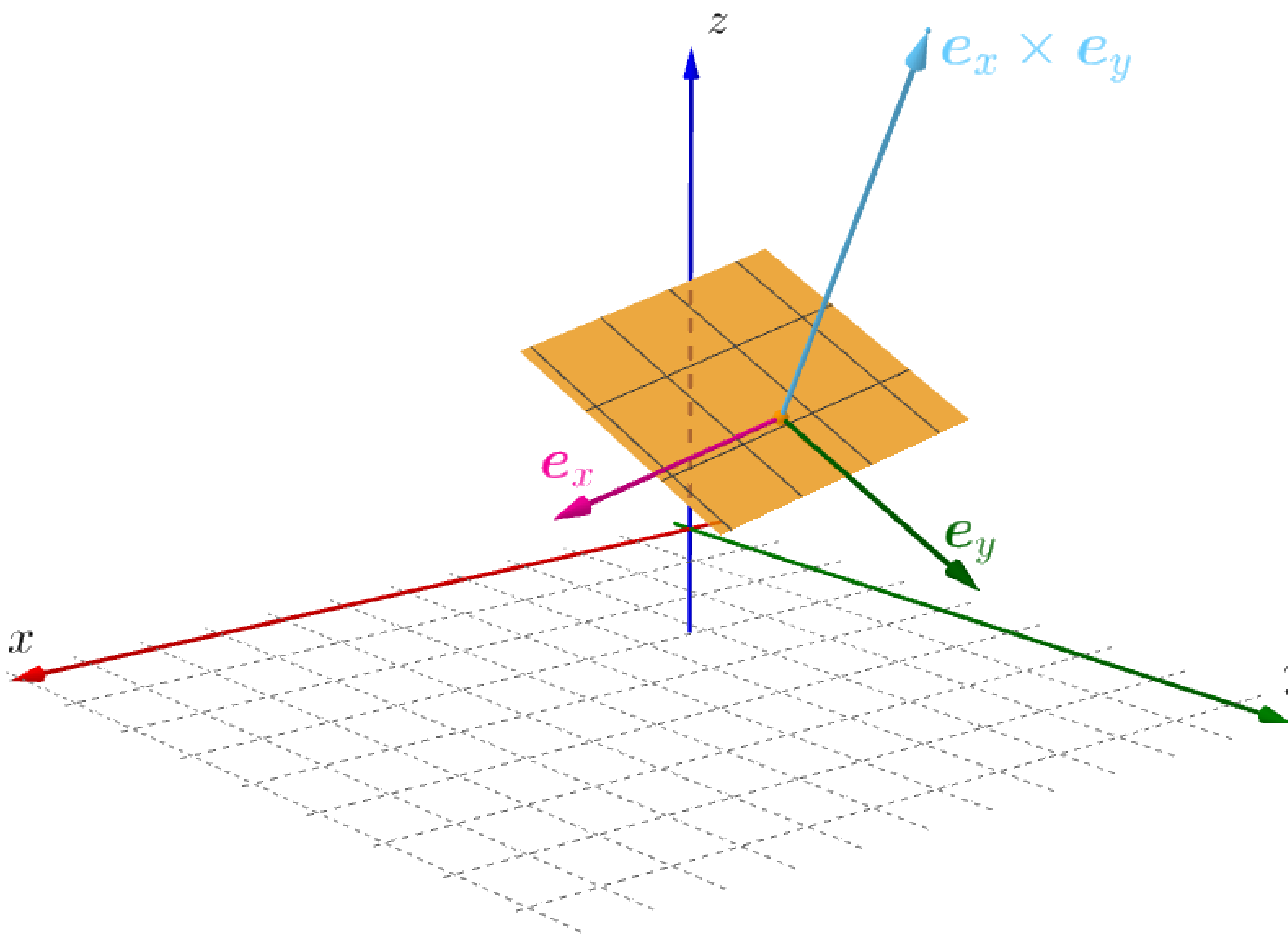
混合的偏导数

求二元函数的切平面

法一：

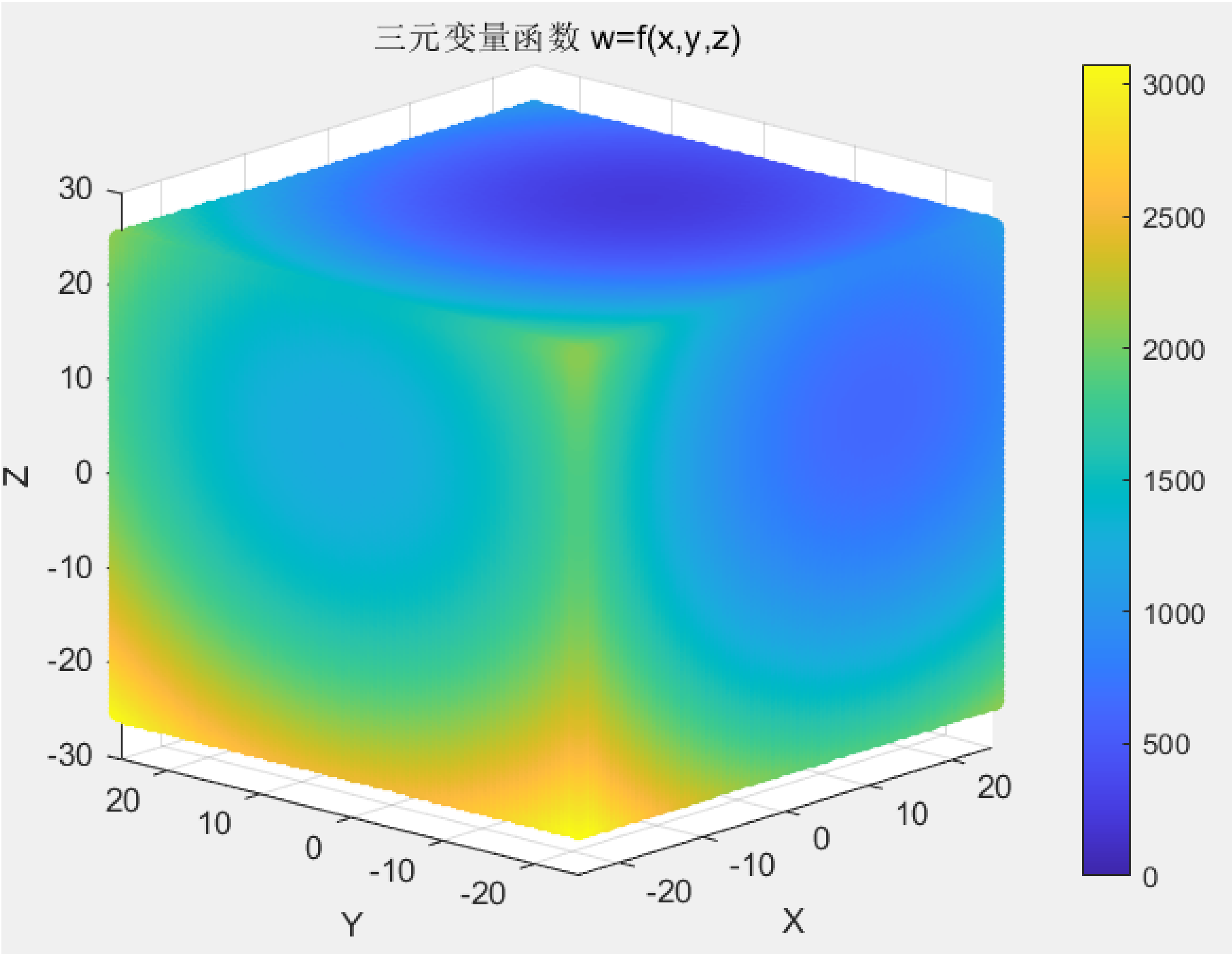


z
$\mathrm{d}z$
$\mathrm{d}x$
$\mathrm{d}y$
x
y
(x_0, y_0)



> $$\boldsymbol{n} = \boldsymbol{e}_x \times \boldsymbol{e}_y = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} = -f_x(x_0, y_0)\boldsymbol{e_1} - f_y(x_0, y_0)\boldsymbol{e_2} + \boldsymbol{e_3}$$

法二：高维函数的等势面(线)
二元函数：等值线
三元函数：等值面



12.3 二元函数的极限

1. 求极限

例：判断极限存在？求极限。

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{x^2-y^2}?$$

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}?$$

计算方法：换元

2. 判断连续：
   1. 整式多项式是连续的
   2. 复合规则：连续的复合连续的依然是连续的

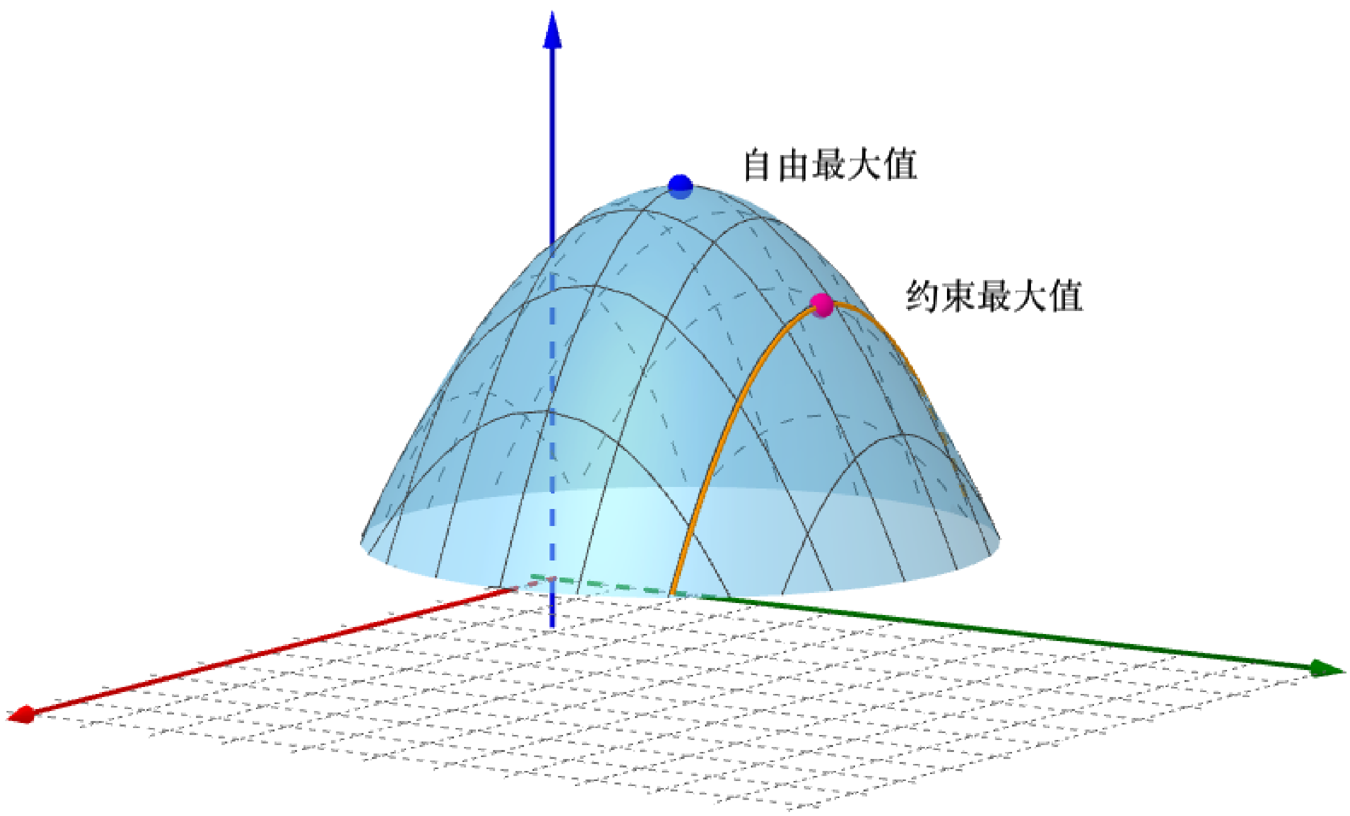
12.4 二元函数的微分

1. 全微分(切平面、线性近似)
   $f(a+\Delta a, b+\Delta b) = f(a,b) + \Delta a f_x(a,b) + \Delta b f_y(a,b) + \epsilon$

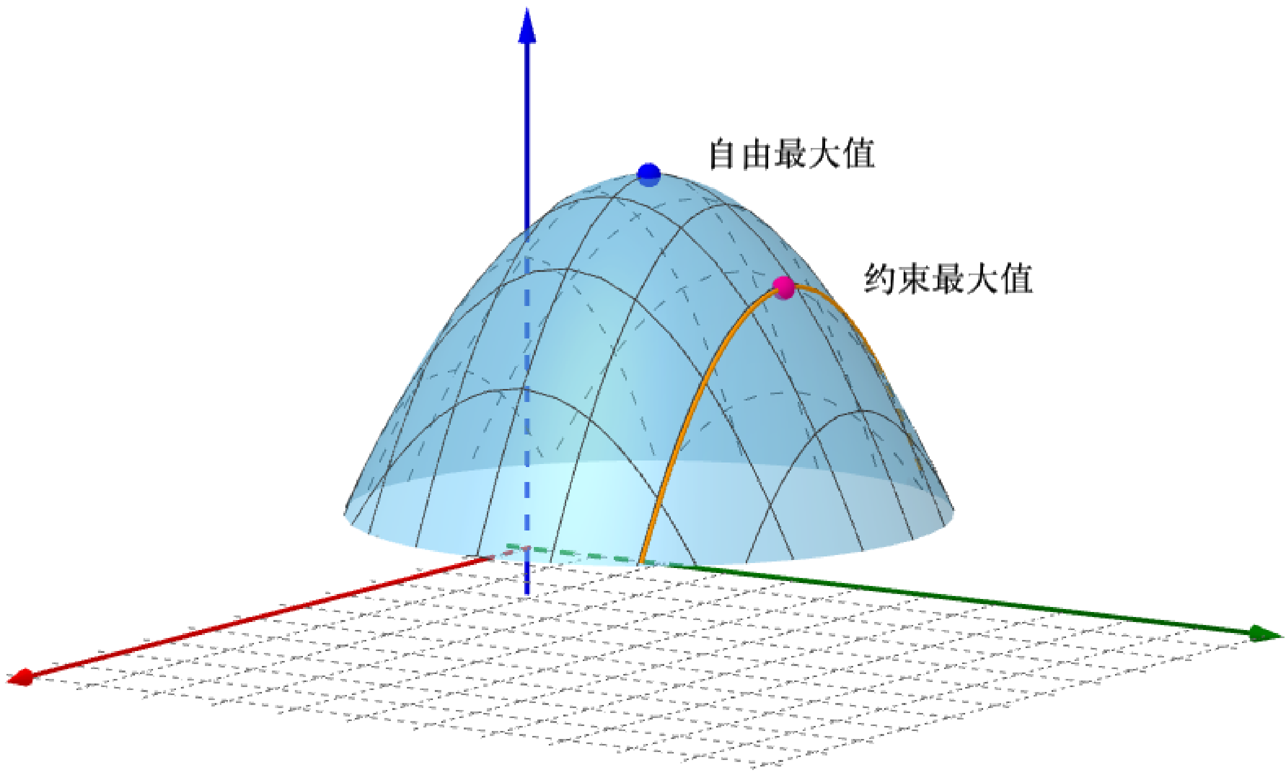
$f(x,y) = f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0) + \epsilon \Rightarrow$ 平面方程

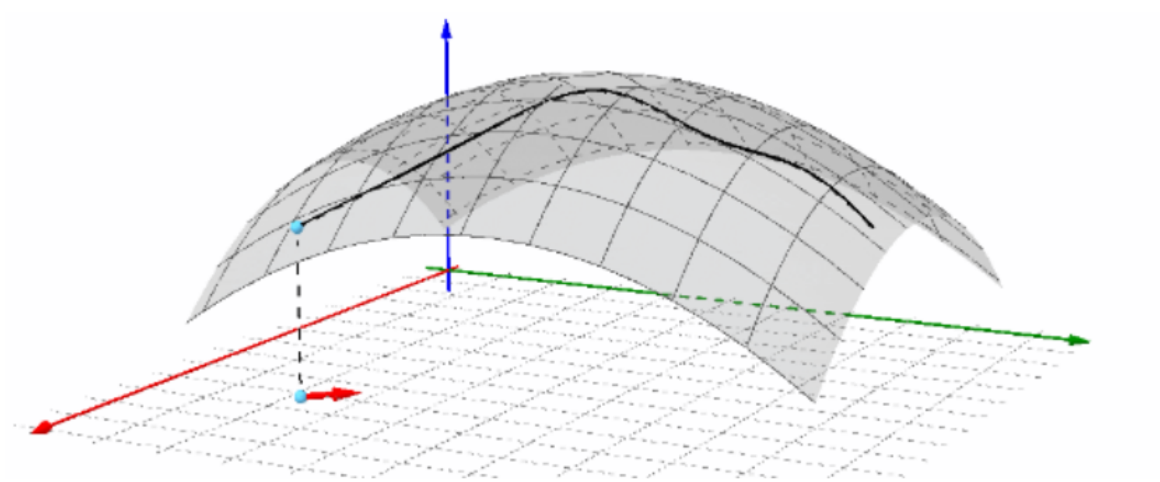
12.5 梯度和方向导数

1. 方向导数
   u:单位向量
   几何意义



2. 梯度
   1. 定义：一个向量，其方向上的坡度是最大的
   2. 几何意义
      指向山峰吗?





3. 方向导数和梯度的关系：
   梯度方向的方向导数最大，$D_u f(p) = u \cdot \nabla f(p)$

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12.6 链式法则（解题法）

例： $w = x^2y + y + xz$, *where* $x = cos\theta, y = sin\theta, z = \theta^2$
Find $\frac{dw}{d\theta}$

经验技术：隐函数求导

Implicit: $F(x,y,z)=0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

12.7 切平面

12.8 找面上的最大值最小值

1. Critical points有以下几种:
    1. Boundary
    2. Stationary point: 两个偏导数(或任意两个方向导数)都是0
    3. Singular point：尖尖

1. 对于stationary points。海森矩阵:二次检验 **stationary point**
    1. 雅可比矩阵

> R S
>
> where $J(u,v)$, called the **Jacobian,** is equal to the determinant
>
> $$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

    2. 海森矩阵

$$H=\frac{\partial^2 z}{\partial (x,y)^2}=\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}$$

看两个:

二、f_{xx}或f_{yy}的正负

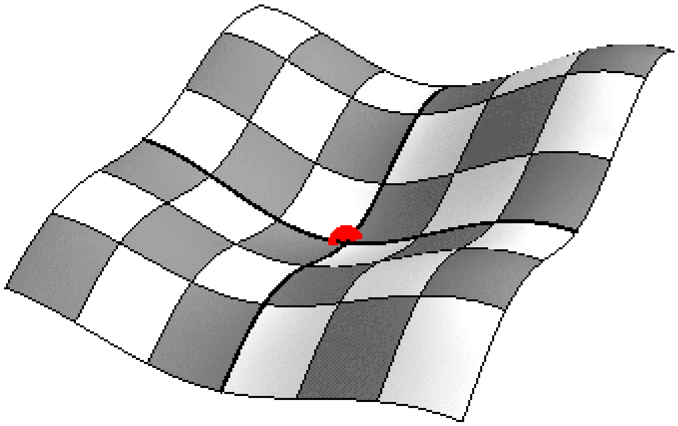
$f_{xx}<0$，该点是极小值

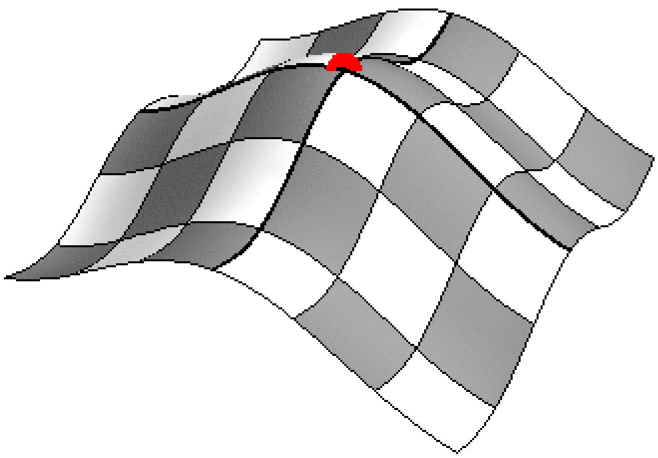
$f_{xx}>0$，该点是极大值

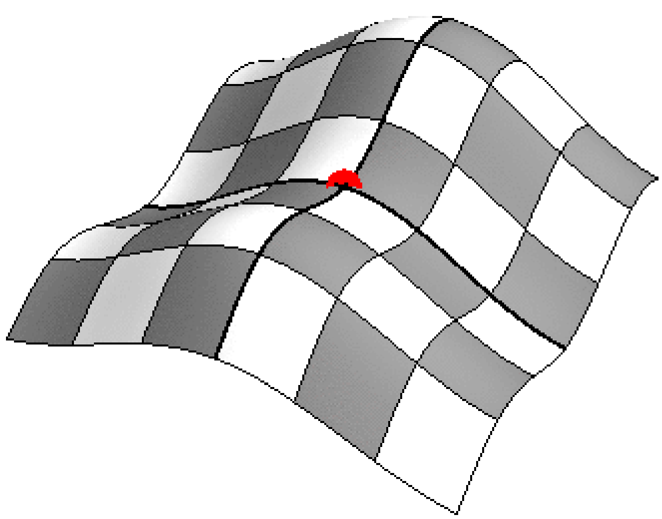
一、$det(H)$的正负

$det(H)>0$，该点是极值

$det(H)<0$，该点不是极值(图三)





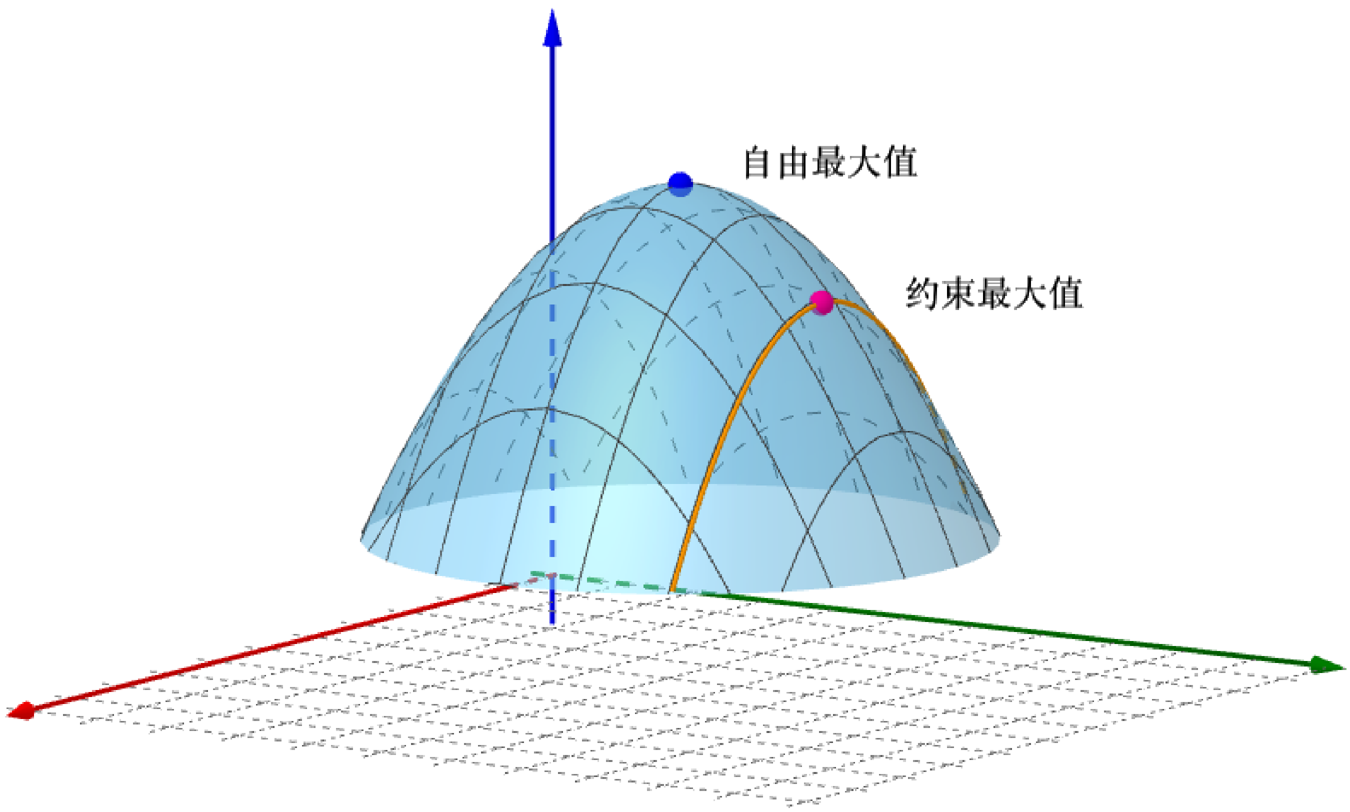


例：函数$f(x,y)=4(x-y)-x^2-y^2$*的极值情况?*

Ans:有一个极大值，值为8

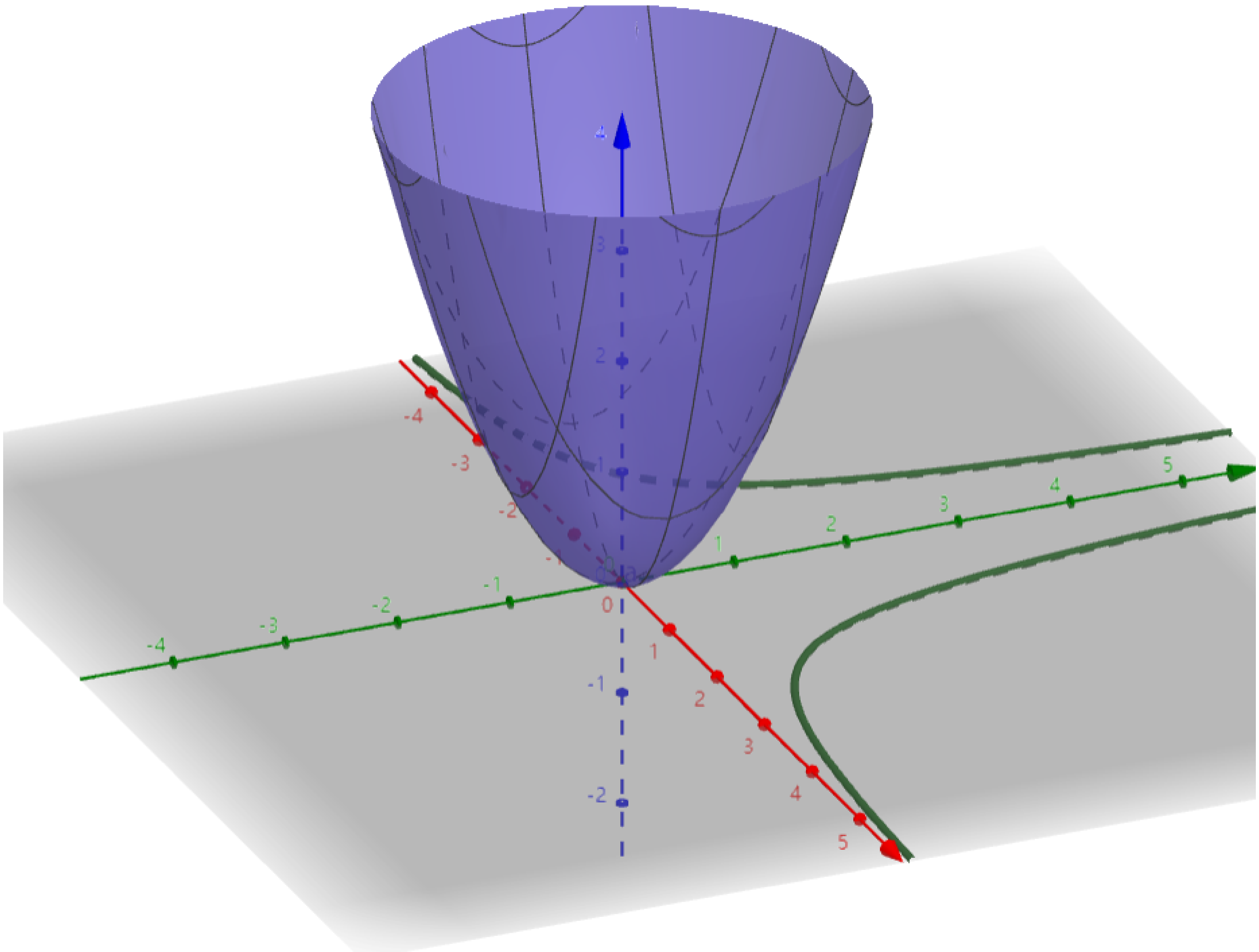
2. 拉格朗日乘子法；检验 **Boundary**

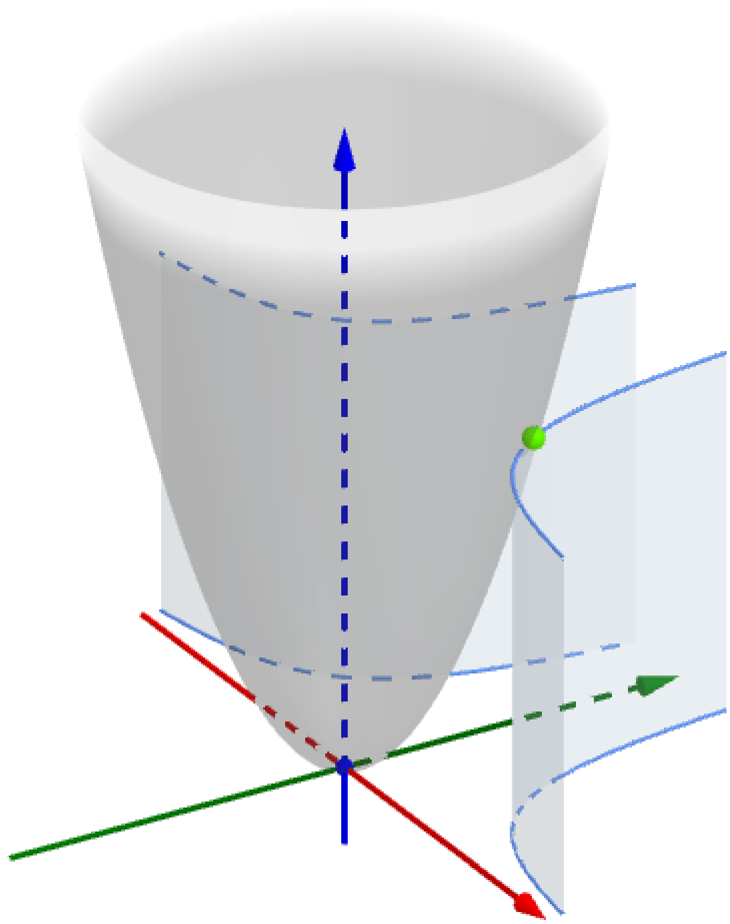
即，在约束条件下求极值

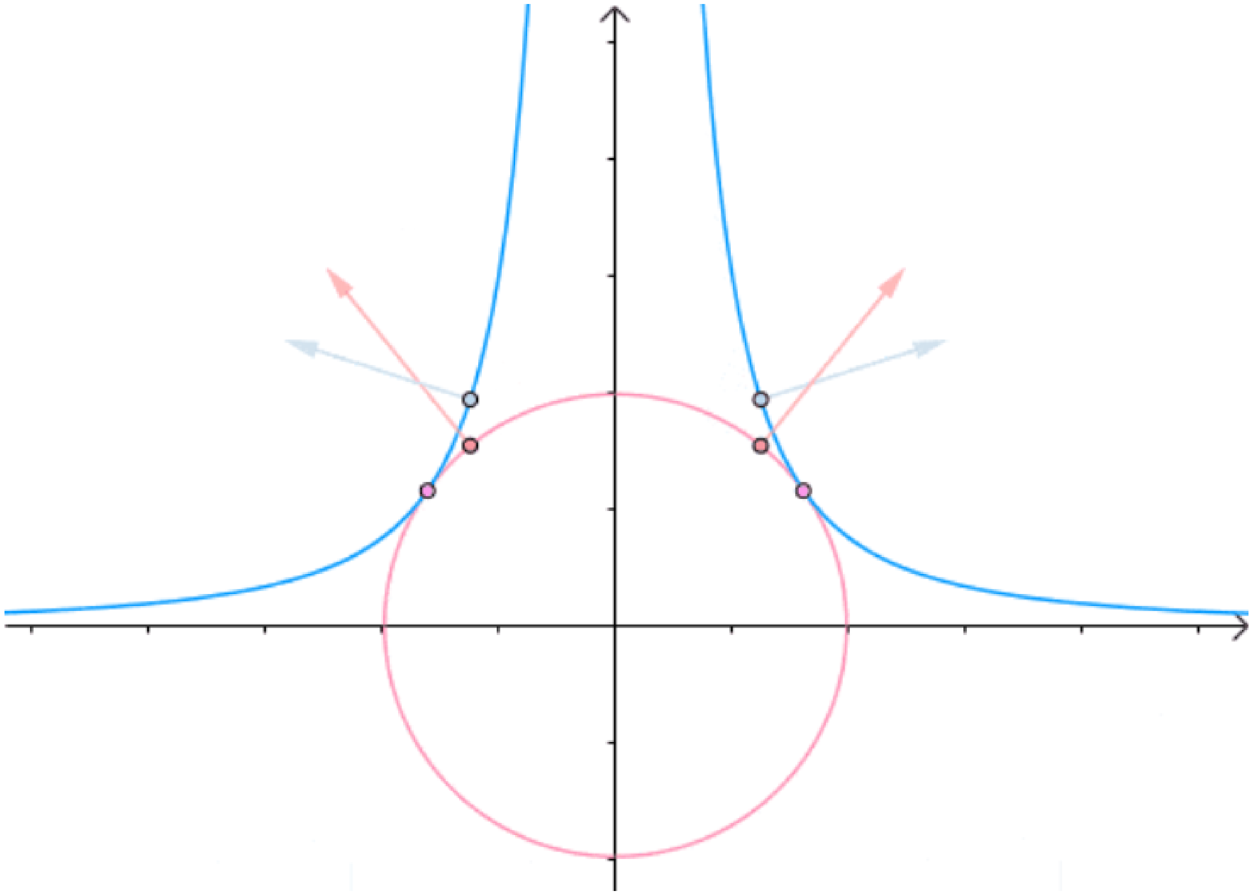


例：求 $z = f(x, y) = x^2 + y^2$

在 $g(x, y) = x^2 y = 3$ 约束下的最小值







在切点处，梯度的方向一致
（复习：梯度是一个方向，可以看作是地上的一个向量）

$$\begin{cases} f(x,y)=x^2+y^2 \\ g(x,y)=x^2y \\ \nabla f=\lambda\nabla g \\ x^2y=3 \end{cases} \Longrightarrow \begin{cases} \begin{pmatrix} 2x \\ 2y \end{pmatrix}=\lambda\begin{pmatrix} 2xy \\ x^2 \end{pmatrix} \\ x^2y=3 \end{cases}$$

例：内接于半径为a的球的长方体，体积最大为？

例题

1. 求偏导
    1. > $$z=\frac{x-y}{x+y},\quad \text{find } \frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial y\partial x}$$

2. $z = \frac{y}{x} arcsin\frac{x}{y}$, find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$. Ans=0

复习一下三角函数求导

$sin(x)', cos(x)', tan(x)', csc(x)', sec(x)', cot(x)'$

$arcsin(x)', arccos(x)', arctan(x)', arccsc(x)', arcsec(x)', arccot(x)'$

$cos(x), -sin(x), sec^2(x), -cot(x)csc(x), tan(x)sec(x), -csc^2(x)$

$\frac{1}{\sqrt{1-x^2}}, \frac{-1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}, \frac{1}{|x|\sqrt{1-x^2}}, \frac{-1}{|x|\sqrt{1-x^2}}, \frac{-1}{1+x^2}$

3. Implicit: $z(x, y)$is defined as $xyz = ln\frac{z}{y}$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

2. 求极限
   1. **10**. Which of the following limits does not exist? 【C】

      (A) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$ (B $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ (C) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ (D) $\lim_{(x,y)\to(0,0)} \frac{x^2 y + xy^2}{x^2+y^2}$

3. 求方向导数和梯度
   1. Given two points $A(3, 0)$ and $B(1, 3)$. Find the directional derivative of the function $f(x, y) = xe^{xy}$ at the point A in the direction of the vector $\vec{AB}$
   2. Let $\boldsymbol{u} = 3\boldsymbol{i} - 4\boldsymbol{j}, \boldsymbol{v} = 4\boldsymbol{i} + 3\boldsymbol{j}$. Suppose that at point $P(x_0, y_0)$,$D_u f(x_0, y_0) = -6$, $D_v f(x_0, y_0) = 17$. Find $\nabla f(x_0, y_0)$. (Ans:$\langle 10, 15\rangle$)
   3. The unit vector in the direction in which $f(x, y, z) = x^2 + 2y^2 - 3z^2$ increases most rapidly at $P(1, 1, 1)$

4. 全微分
   1. Find the tangent planes of the surface $x^2 + 2y^2 + 3z^2 = 21$ that are parallel to the plane $x + 4y + 6z = 0$

   ps：只能求出比例，需再解

   2. Find all points (x, y)at which the tangent plane to the graph $z = x^2 + 10x - 2y^2 + 4y + 2xy$ Ans:(-4,-1)

5. 求极值和最值
   1. Find the local maximum value and local minimum value of $f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$

Solution Solving $f_x = \dfrac{x^2y-2}{x^2} = 0$ and $f_y = \dfrac{xy^2-4}{y^2} = 0$ shows that $(1, 2)$ is the only stationary point of f.

$$f_{xx} = \frac{4}{x^3},\ f_{yy} = \frac{8}{y^3},\ f_{xy} = 1 \text{ and } D = f_{xx}f_{yy} - f_{xy}^2 = \frac{32}{x^3y^3} - 1 \ .$$

At $(1, 2)$, $D(1,2) = 3 > 0,\ f_{xx}(1,2) = 4 > 0$, we conclude that $f(1, 2) = 6$ is a local minimum value .

2. Find the global maximum value and global mimimum value of $f(x, y) = x^2 - y^2 + 2$ on the closed and bounded set $S = \{(x, y) : x^2 + \frac{1}{4}y^2 \leq 1\}$

Not this time

$\sum \frac{1}{n^2-1}$
$\sum[(c - \frac{c}{n+1})^2 - (c - \frac{c}{n})^2]$

The 9 frequently used Maclaurin series

① $\dfrac{1}{1-x}=\sum_{n=0}^{\infty} x^n \qquad x\in(-1,1)$

② $e^x=\sum_{n=0}^{\infty}\dfrac{x^n}{n!} \qquad x\in\mathbb{R}$

③ $\ln(1+x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{n+1}}{n+1} \qquad x\in(-1,1]$

④ $\tan^{-1}x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{2n+1} \qquad x\in[-1,1]$

⑤ $\cosh x=\sum_{n=0}^{\infty}\dfrac{x^{2n}}{(2n)!} \qquad x\in\mathbb{R}$

⑥ $\sinh x=\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!} \qquad x\in\mathbb{R}$

⑦ $\cos x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{(2n)!} \qquad x\in\mathbb{R}$

⑧ $\sin x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad x\in\mathbb{R}$

⑨ $(1+x)^p=\sum_{n=0}^{\infty}\binom{p}{n}x^n \qquad x\in(-1,1)$

By Uniqueness Theorem 51, ①~⑧ obtained by power series operations have the same coefficients as obtained by Taylor formula $c_n=\frac{f^{(n)}(0)}{n!}$.

????? Shy ???? QQ —— Best !!!!!! Yell !!!!!!!!

Theorem 47 ($\frac{d}{dx}\int$ Deteriorate/Improve)

1. Radius of convergence remains but **Conv** set might expand to include endpoint(s) after integration, or shrink to exclude endpoint(s) after differentiation.
2. $f(x)=\sum_{n=0}^{\infty} a_n x^n$ in $(-R,R)$. If f **Cont** and $\sum_{n=0}^{\infty} a_n x^n$ **Conv** both at $x=-R$ or R, then they are equal therein.

????? Shy ???? QQ —— Best !!!!!! Yell !!!!!!!!