1. Vector Algebra and Spatial Geometry

1.2 Dot Product

1. Definition of Dot Product

2. Length of Vector

Rigorous definition
Notation
Unit Vector (i, j, k)
Vector Composition

3. Angle between Vectors

Convention 'angle'
Notation
Properties of zero vector (5 properties)
General formula

4. Direction Angle and Direction Cosine

Definition in 2D, 3D:
    Angle between two arrow

5. Projection

Definition (it's a scalar)
Notation

6. Arithmetic Rules

Dot product rules
Projection rules

7. Parallel and Orthogonal

Definition
Equivalent definitions

1.3 Cross Product

1. Determinant and Geometry Semantic

Area of parallelogram with direction

2. Vector Product

Definition: by determinant
Geometry meaning of cross product

3. Arithmetic Properties

3 properties

4. Triple Product

Rule
Meaning

5. Properties of Triple Product

1.4 Spatial Plane and Equation

1. Directional and Normal Vector

Definition of Direction Vector of a Line
Definition of Normal Vector of a Plane

2. Point-normal equation of a plane

> $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$,
> where $M = (x_0, y_0, z_0)$ is a point on the plane, $n = (A, B, C)$ is the normal vector

Geometry meaning

Derivation: M and M0 are orthogonal

3. General equation of a plane

> $Ax + By + Cz + D = 0$, where $n = (A, B, C)$ is a normal vector
> $-D = Ax_0 + By_0 + Cz_0$

Special planes:
Plane through origin
Plane parallel to axis
Plane parallel to axis plane

4. Intercept form equation

> $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad a \neq 0, b \neq 0, c \neq 0$

Geometry meaning

5. Parametric Equation of Spatial Plane

Given three non-colinear points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$ on a same plane. The equation of the plane is

$$\begin{cases} x = a_1 + k_1(b_1 - a_1) + k_2(c_1 - a_1) \\ y = a_2 + k_1(b_2 - a_2) + k_2(c_2 - a_2) \\ z = a_3 + k_1(b_3 - a_3) + k_2(c_3 - a_3) \end{cases}, \quad k_1, k_2 \in \mathbb{R}$$

Further, let

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \overrightarrow{AB} = B - A = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}, \quad \overrightarrow{AC} = C - A = \begin{pmatrix} c_1 - a_1 \\ c_2 - a_2 \\ c_3 - a_3 \end{pmatrix}$$

the equation could be written as

$$\boldsymbol{x} = \boldsymbol{A} + k_1\overrightarrow{AB} + k_2\overrightarrow{AC}, \quad k_1, k_2 \in \mathbb{R}$$

- Normal vector of plane given by parametric equation

6. Angle between Planes

Definition and theorem
$0 \leq \theta \leq \frac{\pi}{2}$

7. Distance from Point to Plane

> $$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Geometry meaning

Derivation

8. Exercise

Q: $M(1,1,1)$ and $M(0,1,-1)$ on plane P. P and $x+y+z=0$ are orthogonal. Find P

1.5 Spatial Line and Equation

1. General Equation

Definition
$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

Geometry meaning

General equation of l is not unique

Direction vector of line l

2. Point-direction equation

$$\begin{cases} x - x_0 = km \\ y - y_0 = kn \\ z - z_0 = kp \end{cases}, \quad k \in \mathbb{R}$$

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}, \quad m \neq 0, n \neq 0, p \neq 0$$

- What if $m = 0$

3. Parametric Equation

$$\boldsymbol{M} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \boldsymbol{s} = \begin{pmatrix} m \\ n \\ p \end{pmatrix}$$

$$\begin{cases} x = x_0 + km \\ y = y_0 + kn \\ z = z_0 + kp \end{cases}, \quad k \in \mathbb{R}$$

Let $x = (x, y, z)$, it could be written as $\boldsymbol{x} = \boldsymbol{M} + k\boldsymbol{s}, \quad k \in \mathbb{R}$

4. Exercise

Find point-direction form and parametric form of $\begin{cases} x + y + z + 1 = 0 \\ 2x - y + 3z + 4 = 0 \end{cases}$

5. Angle between spatial lines

Definition of two lines

Theorem

> s_1, s_2 are direction vector of l_1, l_2 respectively
>
> $$\cos\theta = \frac{|\boldsymbol{s}_1 \cdot \boldsymbol{s}_2|}{\|\boldsymbol{s}_1\|\|\boldsymbol{s}_2\|} = \frac{|m_1m_2 + n_1n_2 + p_1p_2|}{\sqrt{m_1^2 + n_1^2 + p_1^2}\sqrt{m_2^2 + n_2^2 + p_2^2}}, \quad 0 \le \theta \le \frac{\pi}{2}$$
>
> Note that θ $(0 \le \theta \le \frac{\pi}{2})$

Ex. Find the angle between $l_1 : \dfrac{x-1}{1} = \dfrac{y}{-4} = \dfrac{z+3}{1}$ and $l_2 : \dfrac{x}{2} = \dfrac{y+2}{-2} = \dfrac{z}{-1}$

6. Angle between line and plane

Definition
Theorem

7. Plane cluster of spatial line

Definition

直线 l 的平面束方程： $\begin{cases}\text{这是一个平面}\\ \text{该平面始终包含直线 } l\text{，不论 } \lambda \text{ 如何变化}\\ \text{该平面代表了所有包含直线 } l \text{ 的平面，除平面 } A_2x+B_2y+C_2z+D_2=0\end{cases}$

Theorem

> Given $l:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$
>
> Then $p: A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0, \quad \lambda\in\mathbb{R}$ is the plane cluster

Geometry meaning

- Ex. $l:\begin{cases}x+y-z-1=0\\ x-y+z+1=0\end{cases}$, find the projection on $x+y+z=0$

  Ans: line <- intersection <- plane <- cluster

8. Distance from Point to Line

1.6 Curve Surface and Equation

1. Sphere Equation

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=R \quad \text{or} \quad (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$$

2. Rotational Surface

Generating line

Axis of rotational surface

3. Instances of Rotational Surface

Find generation line $\Rightarrow$ Substitute

- Cone surface
- Hyperboloid of one sheet
- Hyperboloid of two sheet
- Rotational paraboloid

4. Cylindrical Surface

Definition of generating line, base line
Different type (direction) of cylinder

5. Quadratic Surface

1.6 Spatial Curves and Equation

1. General Equation

Definition by intersection
$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

2. Parametric Equation of Spatial Curve

Definition by trajectory

$$\Big(f(t), g(t), h(t)\Big) \implies \begin{cases} x=f(t) \\ y=g(t) \\ z=h(t) \end{cases} \implies \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}, \quad t\in T$$

3. Parametric Equation fo Curve Surface

Vector valued function

$$x(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{h}$$

- Ex. rotate $x(t)=(1,t,2t)$ with respect of z axis
  Ans: $(\sqrt{1+t^2}\cos\theta, \sqrt{1+t^2}\sin\theta, 2t), \quad t\in\mathbb{R}, \theta\in[0,2\pi]$
- Ex. find the equation of a sphere
  Ans: $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}, \quad 0\le\varphi\le\pi, 0\le\theta\le 2\pi$

4. Projection to Coordinate Plane

Ex. $z=\sqrt{4-x^2-y^2}$ and $z=\sqrt{3(x^2+y^2)}$ compose a object, find its projection on xOy

Ans: $\begin{cases} x^2 + y^2 \le 1 \\ z = 0 \end{cases}$

Ex. Find the projection of $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x^2 + (y-1)^2 + (z-1)^2 = 1 \end{cases}$ on xOy

Ans: $\begin{cases} x^2 + 2y^2 - 2y = 0 \\ z = 0 \end{cases}$ $\text{or} \quad (t, \frac{1}{2} \pm \sqrt{-\frac{t^2}{2} + \frac{1}{2}}, 0)$

2. Derivative of Multivariable Function

2.1 Intro of Multivariable Function

1. Set of Real Numbers

2. Bivariable Function and Multivariable Function

Definition

Domain, Range

3. Neighborhood

Definition

Notation

4. Boundary and Boundary Point

Interior, exterior, boundary

5. Open Set and Close Set

6. Connected Set, Open Region and Closed Region

7. Bounded Set and Unbounded Set

2.2 Limit and Continuity

1. Definition of Limit

Definition:
Let (x_0, y_0) be a point in the domain of f. Then, we say that the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is L, denoted by:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for any number $\varepsilon > 0$, there exists a number $\delta > 0$ that if $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ (deleted neighborhood), then $|f(x, y) - L| < \varepsilon$.

- Geometry meaning

- Properties
  $\lim \left[f(x) \pm g(x)\right] = \lim f(x) \pm \lim g(x) = A \pm B$
  $\lim \left[f(x) \cdot g(x)\right] = \lim f(x) \cdot \lim g(x) = A \cdot B$
  $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} = \frac{A}{B}, B \neq 0$

- Additional statement:

1. The *path* is irrelevant to the limit
2. $f(x, y)$ does **NOT** need to be defined on $P(a, b)$

- Theorem:

1. If $f(x, y)$ is polynomial, then its limit always exists and $\lim \Rightarrow f(a, b)$
2. If $f(x, y) = \frac{p(x, y)}{q(x, y)}$ is rational, then its limit exist exists when $q(x, y) \neq 0$, and $\lim = \frac{p(a, b)}{q(a, b)}$. Moreover if $\lim p(x, y) = L \neq 0 \text{ while} \lim q(x, y) = 0$, the limit does not exist.

- Techniques

1. Different Path : $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ (2 ways)
2. Substitute (polar coordinate)

2. Exercise

1. $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\sin\dfrac{1}{x^2+y^2} = 0$
2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} \quad \text{Does not exist}$
3. $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin(xy)}{x}$
4. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{2-e^{xy}}-1}$

3. Continuity

Definition

A function $f(x,y)$ of two variables is said to be continuous at a point (x_0,y_0) in its domain if the following conditions are met:

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
2. $f(x,y)$ is defined at (x_0,y_0).
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

In other words, $f(x,y)$ is continuous at (x_0,y_0) if the limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is equal to the value of $f(x,y)$ at (x_0,y_0).

- Theorem

1. If f, g both continuos on domain, then $(f \odot g)\ \text{or} f(g(x,y))$ is continuous as well
2. If $f_{xy} and f_{yx} are continuous on an open set S, then$f_{xy}=f_{yx}$ at each point of S.

4. Properties of Continuity

Boundedness and Extreme Value Theorem

Intermediate Value Theorem

2.3 Partial Derivative, total Derivative

1. Partial Derivative

Definition

$$\frac{\partial f}{\partial x}_{y=y_0} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y_0) - f(x, y_0)}{\Delta x}$$

$$\frac{\partial f}{\partial y}_{x=x_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y+\Delta y) - f(x_0, y)}{\Delta y}$$

- Geometry Meaning

2. Differentiability of a function

Differential is linear asymptote

Definition: partial derivative
We say $f(x, Y)$ is locally linear at (a, b)
if $f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + \varepsilon$
where $\varepsilon_2(h_1, h_2) \Rightarrow 0, \varepsilon_1(h_1, h_2) \Rightarrow 0$

- Question : show the direction vector of two partial derivatives of $f(x, y)$
- The normal vector of total derivative
- Further, the tangent plane could be represented

3. Exercise

- Find partial derivative of $x^2 sin 2y$

4. Higher order derivative and Mixed partial derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

Theorem
If two mixed partial derivatives of $f(x, y)$ continuous in D,
then $f_{xy}(x,y) = f_{yx}(x,y), \quad (x,y) \in D$

2.4 Total Differentiability

1. Total differential

Partial differential -> total differential

$$\mathrm{dz} = A\mathrm{dx} + B\mathrm{dy}$$
$$z - f(x_0, y_0) = A(x - x_0) + B(y - y_0)$$

$$A = f_x(x_0, y_0), B = f_y(x_0, y_0)$$

- Theorem:
  if $f(x,y)$ is differentiable, then $f_x(x_0,y_0), f_y(x_0,y_0)$ exist, and &z=f(x,y)$hastotaldifferential$ \mathrm{d}z=f_x(x_0,y_0)\mathrm{d}x+f_y(x_0,y_0)\mathrm{d}yat(x_0,y_0)$

2. Exercise

1. Find differential of $z=e^{xy}$ $(2,1)$
   Ans: $\mathrm{d}z|_{(x,y)=(2,1)}=e^2\mathrm{d}x+2e^2\mathrm{d}y$
2. $u=x+\sin\dfrac{y}{2}+e^{yz}$
   Ans: $\mathrm{d}u=\mathrm{d}x+\left(\frac{1}{2}\cos\frac{y}{2}+ze^{yz}\right)\mathrm{d}y+ye^{yz}\mathrm{d}z$
3. if both two partial differential of $f(x,y)$ exist, does $f(x,y)$ differentiable
4. Asymptote

3. Relation with continuity

$f(x,y)$ differentiable at $(x_0,y_0)\Rightarrow f(x,y)$ continuous
$$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\text{ continuous at}(x_0,y_0)\implies f(x,y)\text{ differentiable at }(x_0,y_0)$$
$$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\text{ continuous at}(x_0,y_0)\not\Leftarrow f(x,y)\text{ differentiable at }(x_0,y_0)$$

2.5 Higher Order and Composition of Multivariable Function

1. $z=f(\phi(t),\psi(t))$

If $x=\phi(t), y=\psi(t)$ have derivative at t_0, and $z=f(x,y)$ has continuous partial derivative, then f is differentiable
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$
$\frac{dz}{dt}$ is total differential

Ex. $x=\cos t, y=\sin t, z=xy$, find $\frac{dz}{dt}$
Ans: $cos(2t)$

2. Chain rule

$w=f(x,y,z)=f[\varphi(t),\psi(t),\omega(t)]$
$$\frac{\mathrm{d}w}{\mathrm{d}t}=\frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Ex.$w=xy+z, x=\cos t, y=\sin t, z=t$, find $\dfrac{dw}{dt}$
Ans: $1+\cos 2t$

Ex. $w = e^{x^2+y^2+z^2}$, $z = x^2 \sin y$, find $\dfrac{\partial w}{\partial x}$

Ans:
$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 2xe^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2}\cdot 2x\sin y \\ &= 2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}\end{aligned}$$

2.6 Derivative of Implicit Function

1. Existence Theorem

If implicit function $F(x,y)=0$ has continuous partial derivative at (x_0,y_0)

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{F_x}{F_y}$

2. Existence Theorem 2

$$F(x_0,y_0,z_0)=0,\quad F_z(x_0,y_0,z_0)\neq 0$$
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Ex. $x^2+y^2+z^2-4z=0$, find $\dfrac{\partial^2 z}{\partial x^2}$

Ans: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{x}{2-z}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2} = \frac{(2-z)+x\dfrac{x}{2-z}}{(2-z)^2} = \frac{(2-z)^2+x^2}{(2-z)^3}$$

2.7 Direction Derivative and Gradient

1. Direction derivative

Say the unit direction vector is $\boldsymbol{e_u} = \begin{pmatrix}\cos\alpha \\ \cos\beta\end{pmatrix}$. If the following limit exist, then the direction derivative is

$$\left.\frac{\partial f}{\partial \boldsymbol{u}}\right|_{\substack{x=x_0 \\ y=y_0}} = \lim_{t\to 0^+}\frac{f(x_0+t\cos\alpha, y_0+t\cos\beta)-f(x_0,y_0)}{t}$$

2. Exercise

$f(x,y)=xy$ find $\left.\dfrac{\partial f}{\partial \boldsymbol{u}}\right|_{(x,y)\Rightarrow(1,2)}$ where $\vec{u}=(1,1)$

3. Find direction derivative when differentiable

$u = (u_1, u_2)$ is a unit vector
$\frac{\partial f}{\partial \boldsymbol{u}}\Big|_{\substack{x = x_0 \\ y = y_0}} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$

4. Gradient

$\nabla f = f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j}$

Geometry Meaning

5. Exercise

$f(x, y) = \dfrac{1}{2}(x^2 + y^2)$, find:

1. the direction where decrease most quickly at (1,1), and its direction derivative

6. Gradient and Contour Lines

2.8 Extreme Values

1. Definition

2. Conditions

extreme value $(x_0, y_0) \Rightarrow f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

- Stationary point

Hessian Matrix
$H = \frac{\partial^2 z}{\partial (x,y)^2} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$
If $|H| > 0$, we can determine the concavity or convexity. $f_{xx} > 0$ concave, $f_{xx} < 0$ convex.

Given point $p(x_0, y_0)$, if f is continuous and its Hessian Matrix on p's neighbor, and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ (stationary point).
Then

1. $|H| > 0, f_{xx} > 0$
2. $|H| > 0, f_{xx} < 0$
3. $|H| < 0$
4. $|H|=0

Ex. Find the extreme value of $f(x, y) = x^3 - y$3+3x^{2+3y}2-9x$

Ans:

$$f_x = 3x^2 + 6x - 9, \quad f_y = -3y^2 + 6y$$

$$\begin{cases} f_x = 3x^2 + 6x - 9 = 0 \implies x = 1, -3 \\ f_y = -3y^2 + 6y = 0 \implies y = 0, 2 \end{cases}$$

$$f_{xx} = 6x + 6, \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = -6y + 6$$

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x+6 & 0 \\ 0 & -6y+6 \end{pmatrix} \implies |H| = 36(-xy + x - y + 1)$$

2.9 Conditional Extreme

Ex. The minimum of distance from point on $x^2y = 3$ to origin

$$\nabla h = \begin{pmatrix} h_x \\ h_y \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}} \end{pmatrix}, \quad \nabla g = \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$$

$$\begin{cases} g(x_0, y_0) = 3 \\ \nabla h(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases} \implies \begin{cases} x_0^2 y_0 = 3 \\ \begin{pmatrix} \dfrac{x_0}{\sqrt{x_0^2+y_0^2}} \\ \dfrac{y_0}{\sqrt{x_0^2+y_0^2}} \end{pmatrix} = \lambda \begin{pmatrix} 2x_0y_0 \\ x_0^2 \end{pmatrix} \end{cases} \implies \begin{cases} x_0^2 y_0 = 3 \\ \dfrac{x_0}{\sqrt{x_0^2+y_0^2}} = \lambda 2x_0y_0 \\ \dfrac{y_0}{\sqrt{x_0^2+y_0^2}} = \lambda x_0^2 \end{cases}$$

$$\implies x_0 = \pm\sqrt[6]{18}, \quad y_0 = \frac{3}{\sqrt[3]{18}}, \quad \lambda \approx 0.22$$