Chapter 7

Transcendental Functions

7.1

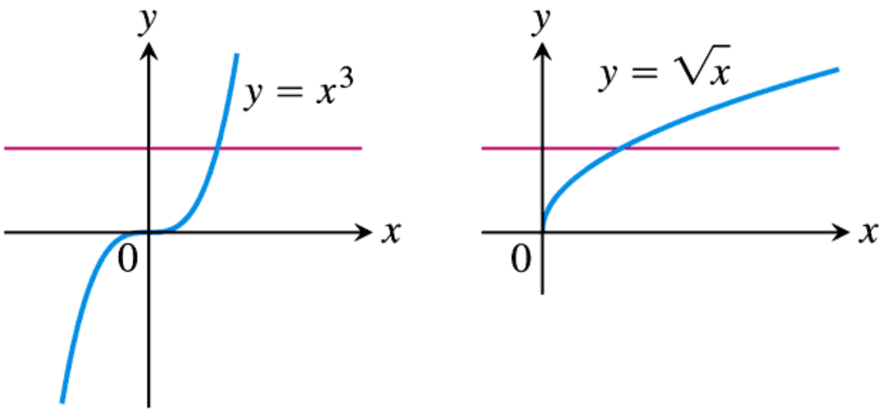
Inverse Functions and Their Derivatives

DEFINITION One-to-One Function

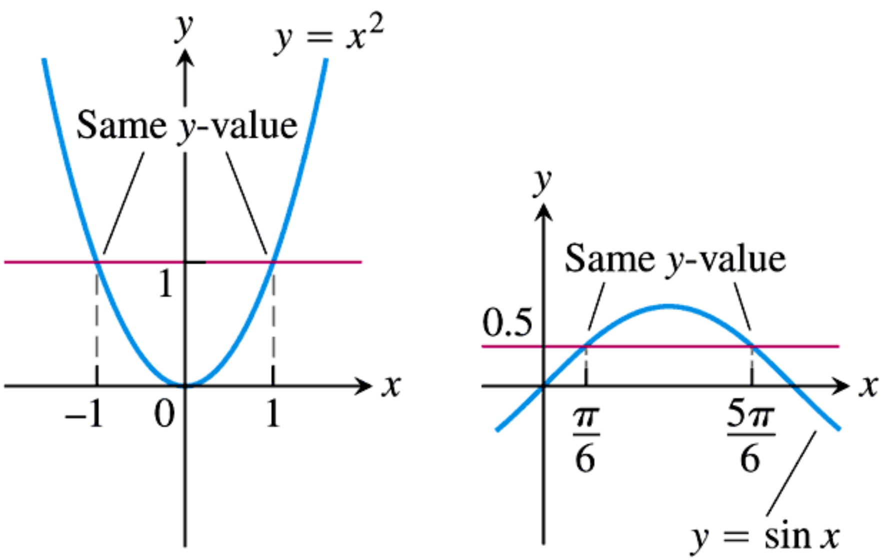
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

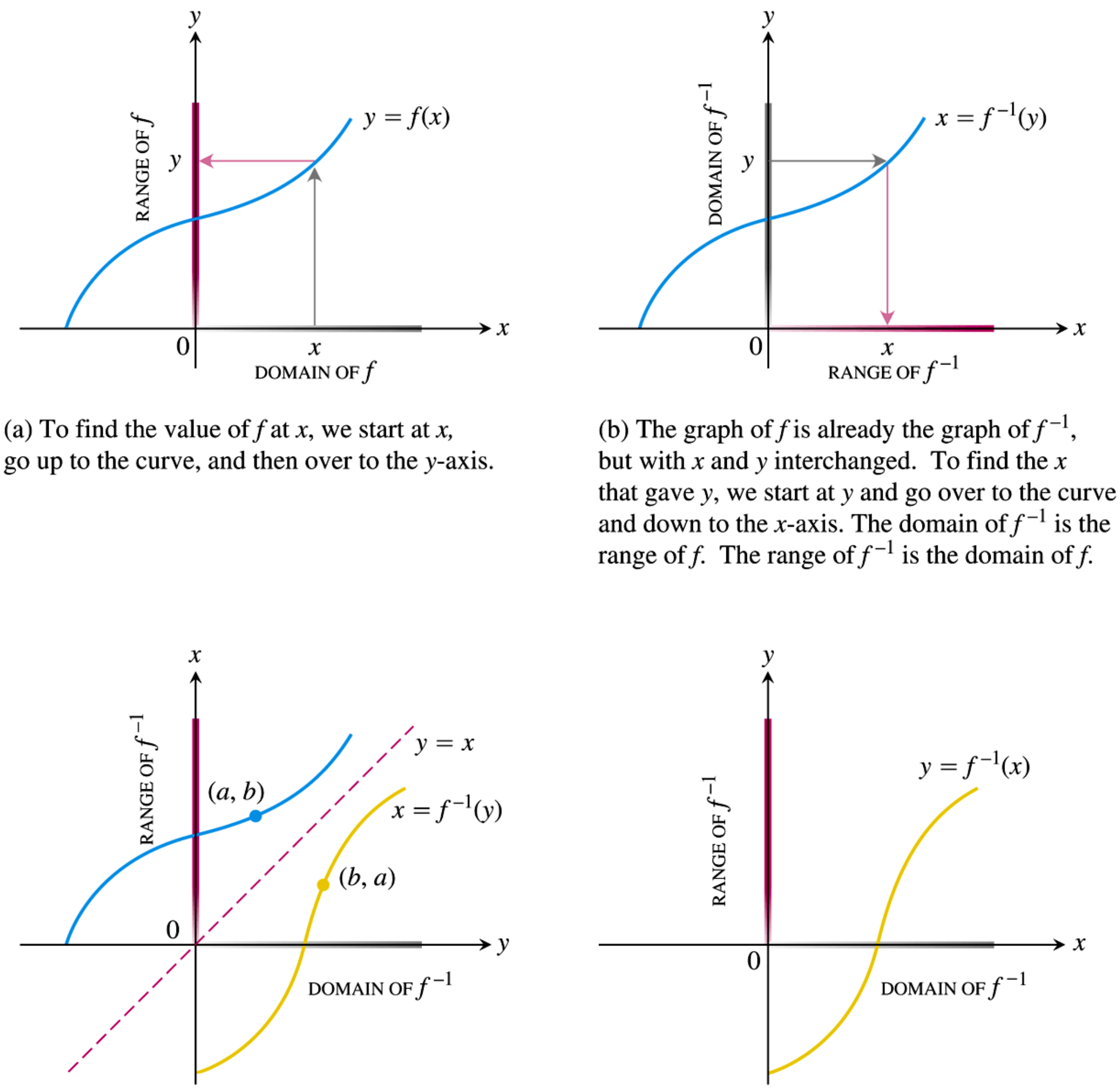
FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

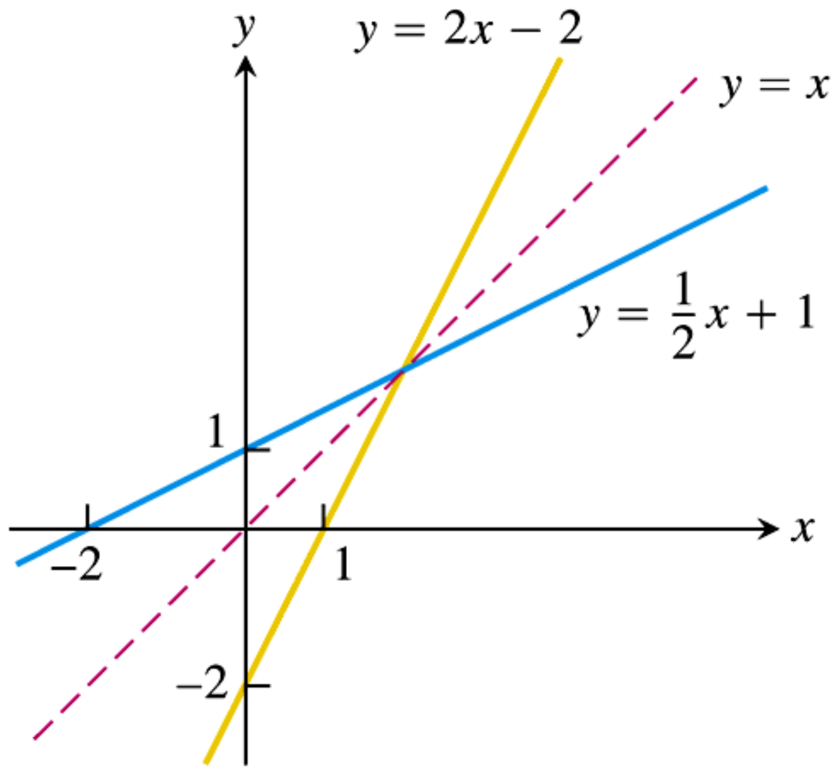


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

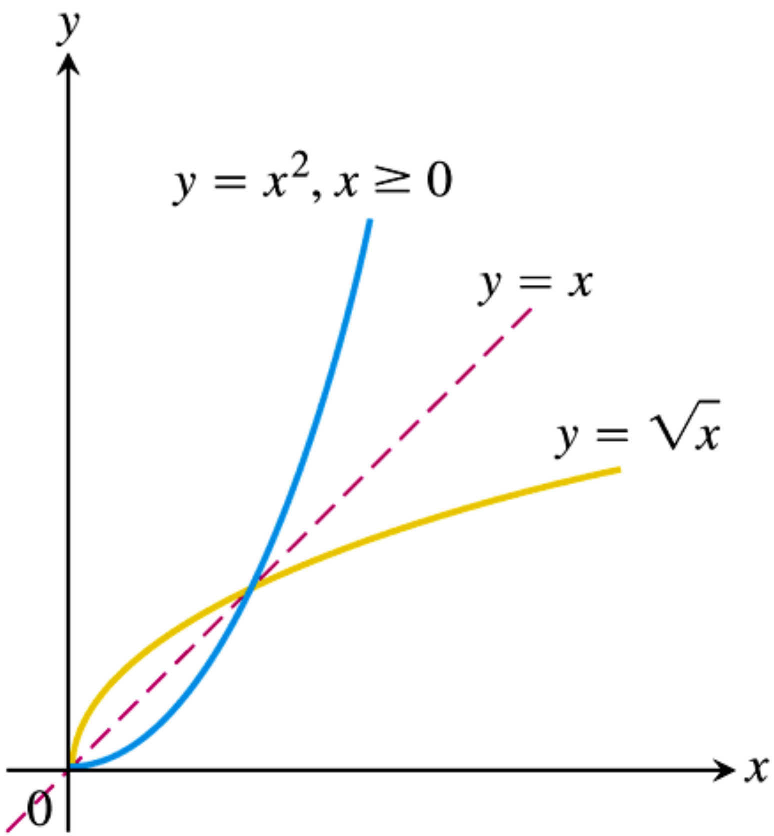


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

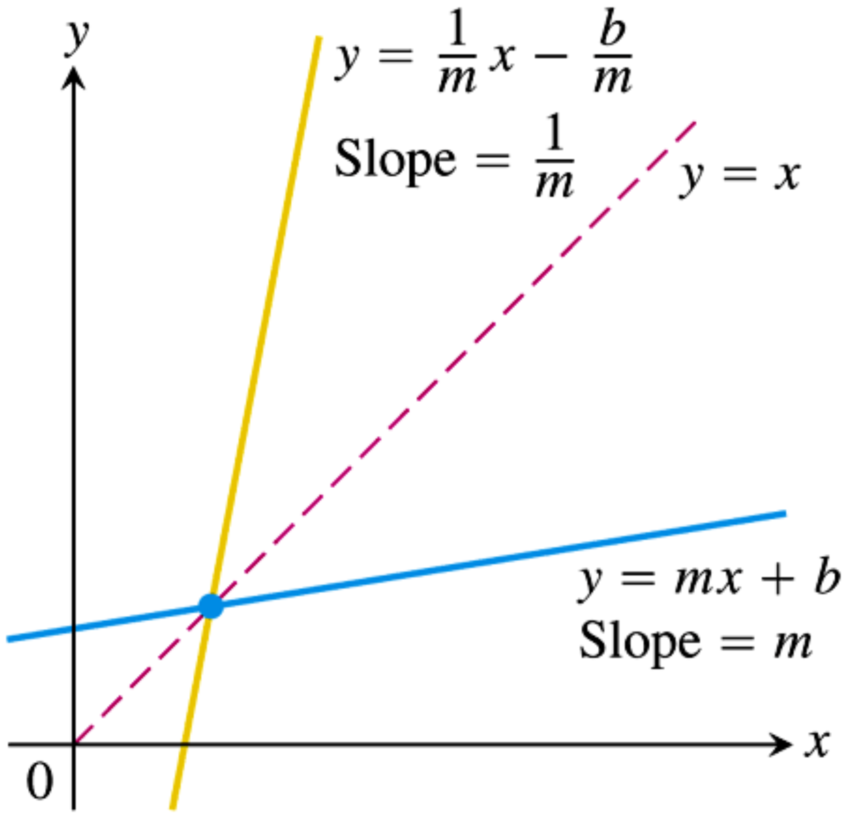
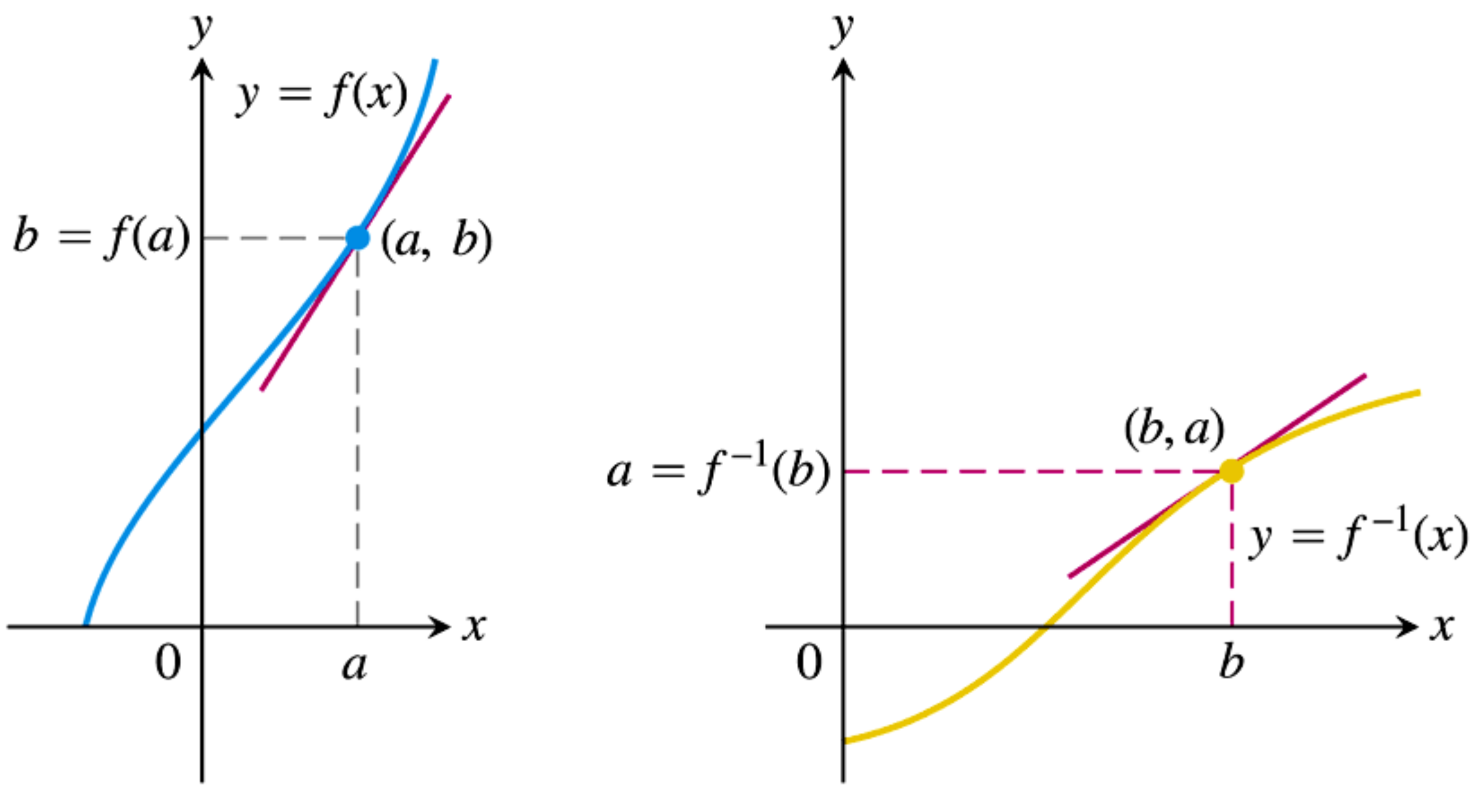


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

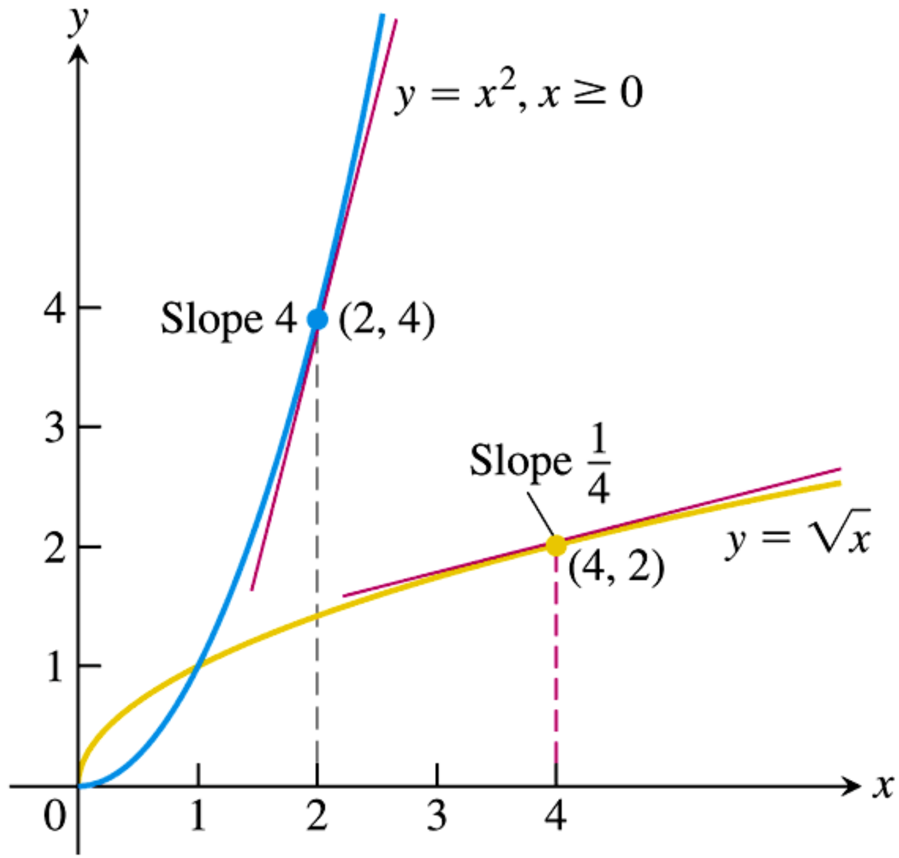


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

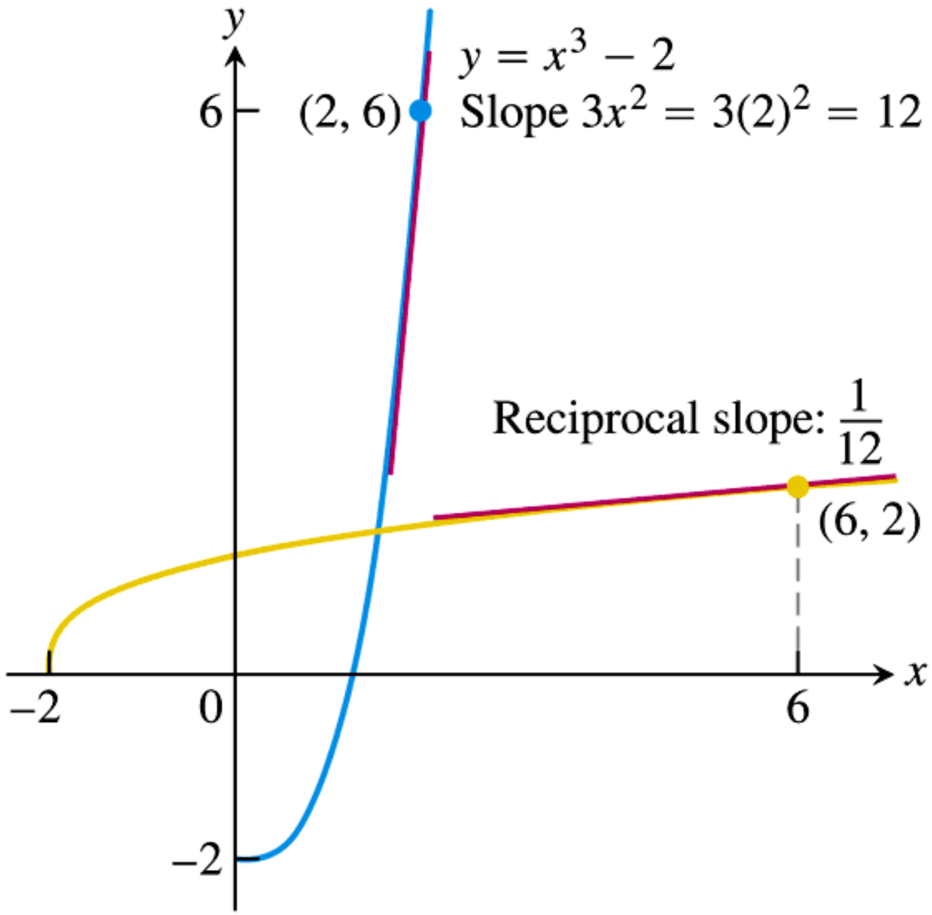


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

7.2

Natural Logarithms

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

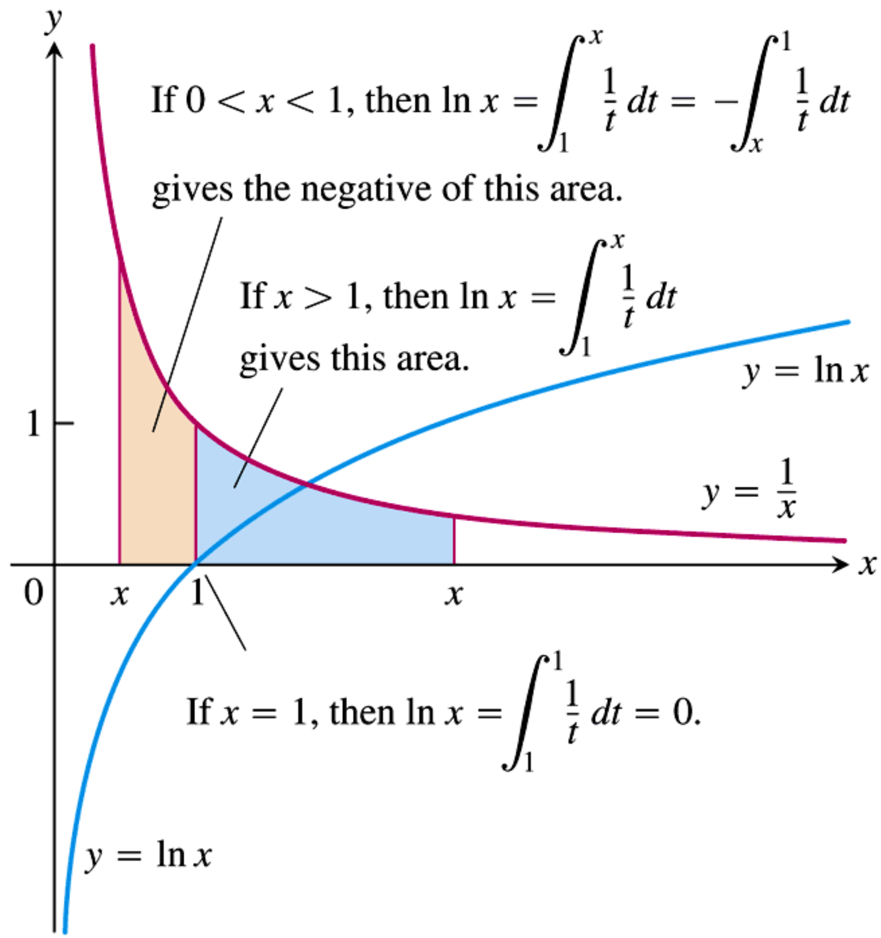


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

$$\ln x = \int_1^x \frac{1}{x}\,dx$$

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

DEFINITION **The Number *e***

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \tag{1}$$

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ r rational

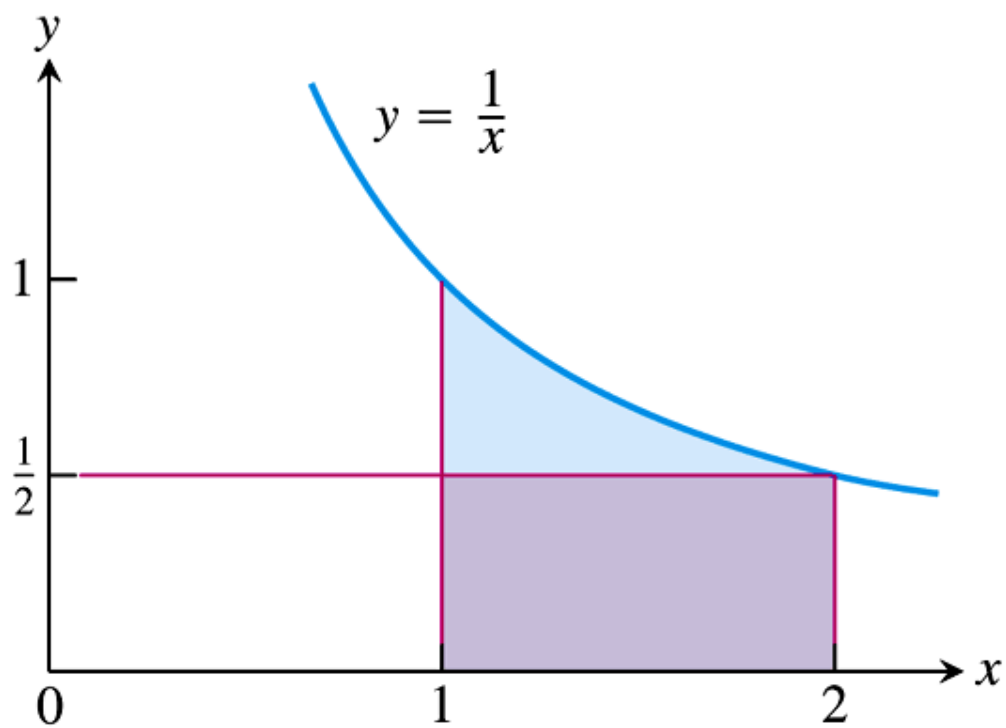


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \leq x \leq 2$.

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

$$\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln\left((x^2 + 1)(x + 3)^{1/2}\right) - \ln(x - 1) && \text{Rule 2} \\
&= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Rule 1} \\
&= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). && \text{Rule 3}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

7.3

The Exponential Function

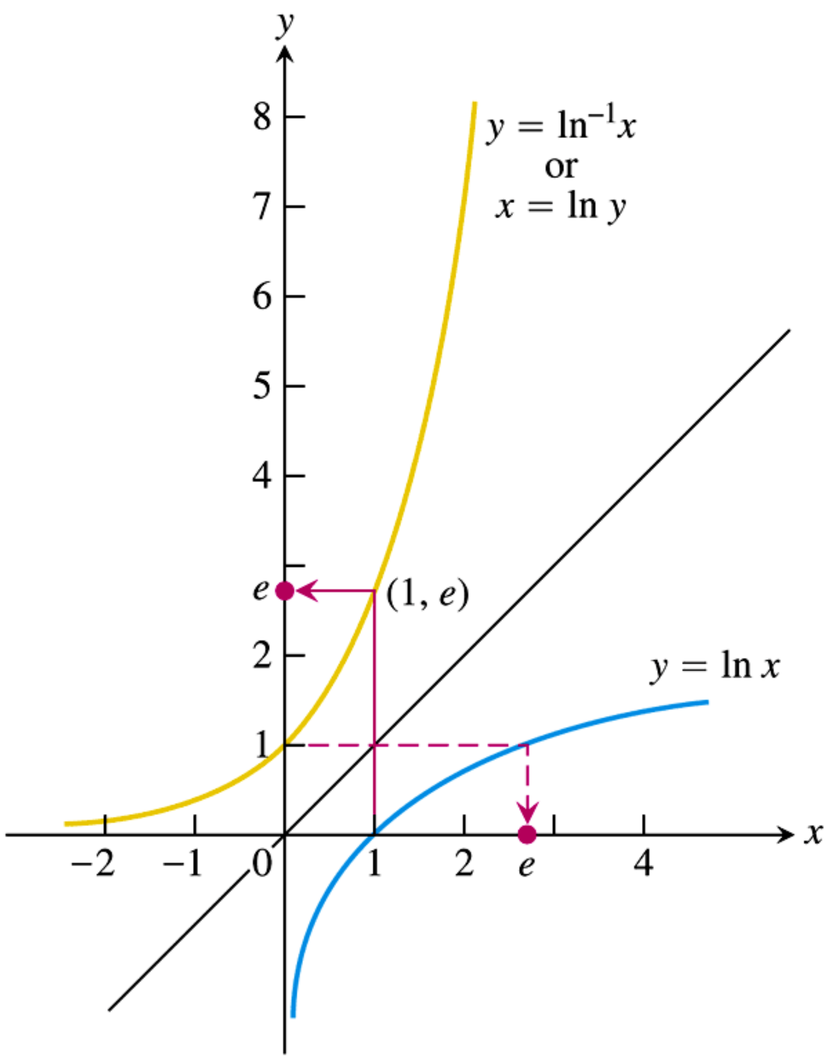


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$.

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

THEOREM 3 **Laws of Exponents for e^x**

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

$$\boxed{\frac{d}{dx} e^x = e^x \qquad (5)}$$

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{6}$$

$$\int e^u \, du = e^u + C.$$

THEOREM 4 The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}.$$

7.4

a^x and $\log_a x$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{1}$$

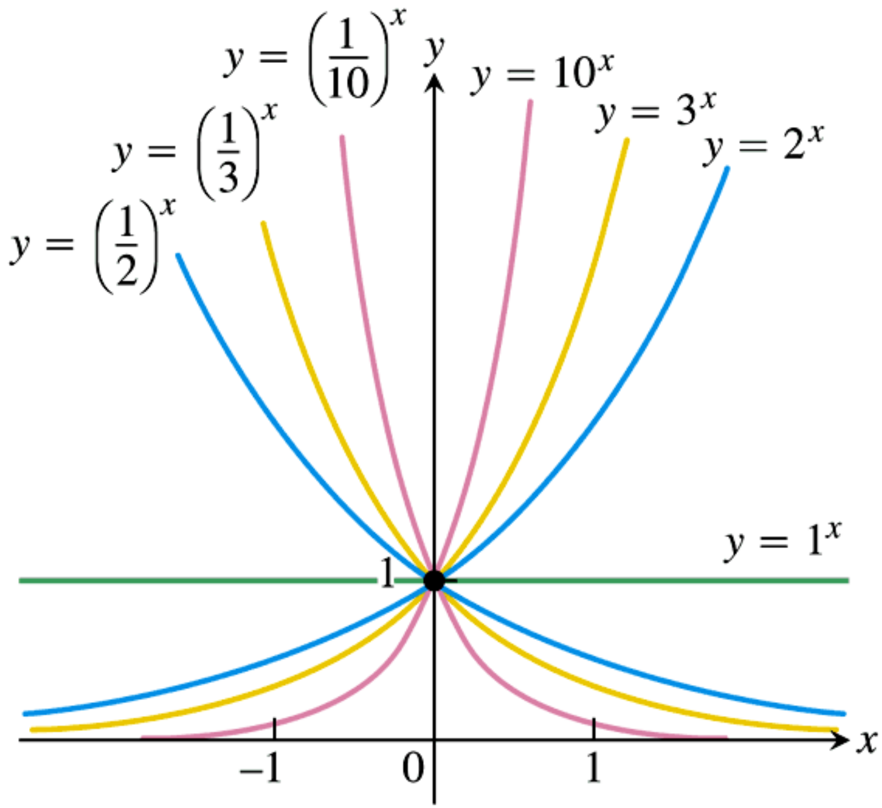


FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

$$\int a^u \, du = \frac{a^u}{\ln a} + C. \qquad (2)$$

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x .$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{3}$$

$$\log_a(a^x) = x \qquad (\text{all } x) \tag{4}$$

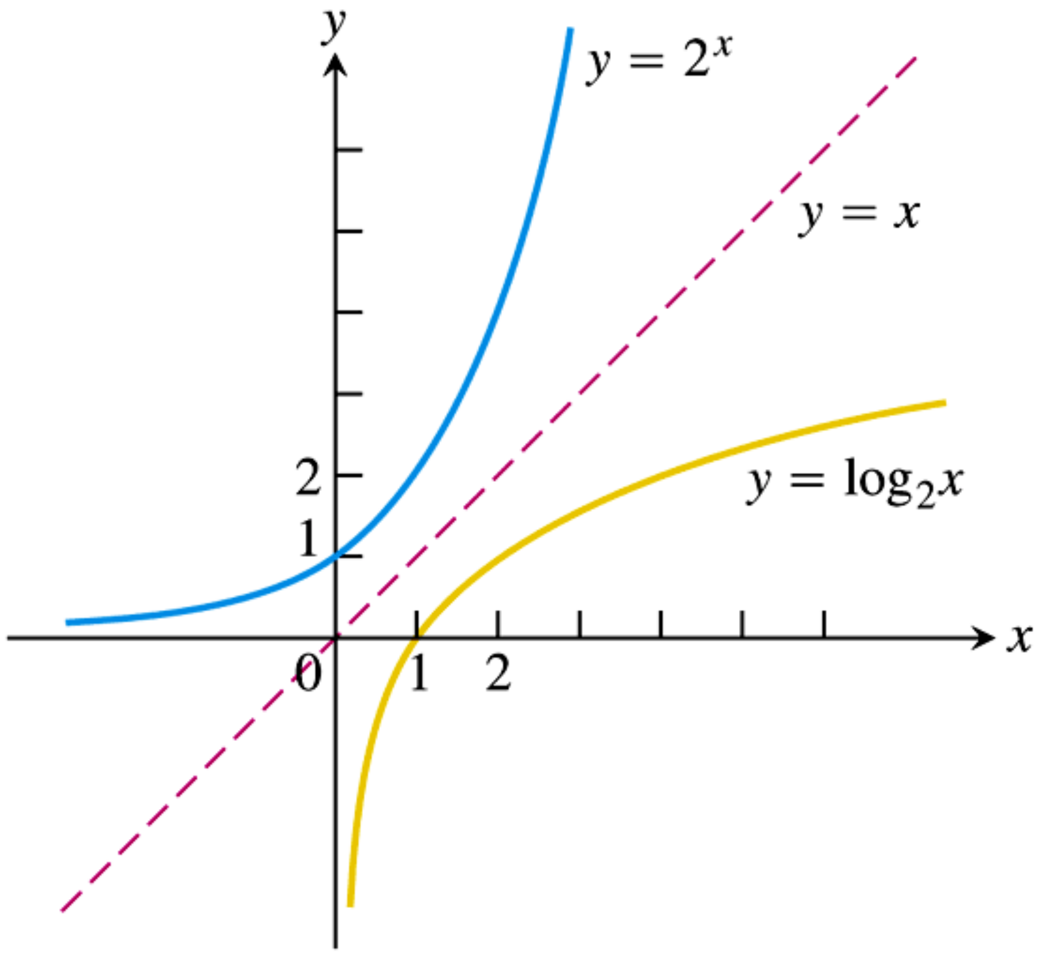


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \qquad (5)$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
   $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
   $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
   $\log_a x^y = y \log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

7.5

Exponential Growth and Decay

The Law of Exponential Change

$$y = y_0 e^{kt} \tag{2}$$

$$\text{Growth:} \quad k > 0 \qquad \text{Decay:} \quad k < 0$$

The number k is the **rate constant** of the equation.

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt}. \qquad (5)$$

Interest paid according to this formula is called compounded interest rate.

$$\lim_{k\to\infty} A_t = \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$$

$$= A_0 \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{\frac{k}{r}\cdot rt}$$

$$= A_0\left[\lim_{\frac{r}{k}\to 0}\left(1+\frac{r}{k}\right)^{\frac{k}{r}}\right]^{rt} \qquad \text{As } k\to\infty, \frac{r}{k}\to 0$$

$$= A_0\left[\lim_{x\to 0}(1+x)^{1/x}\right]^{rt} \qquad \text{Substitute } x=\frac{r}{k}$$

$$= A_0 e^{rt} \qquad \text{Theorem 4}$$

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0 e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously**. The number r is called the **continuous interest rate.** The amount of money after t years is calculated with the law of exponential change given in Equation (6).

$$y = y_0 e^{-kt}, \qquad k > 0$$

This formula calculates the radioactive decay still present at any time t.

$$H - H_S = (H_0 - H_S)e^{-kt}, \qquad (9)$$

where H_0 is the temperature at $t = 0$.

This is the equation for Newton's Law of Cooling.

7.6

Relative Rates of Growth

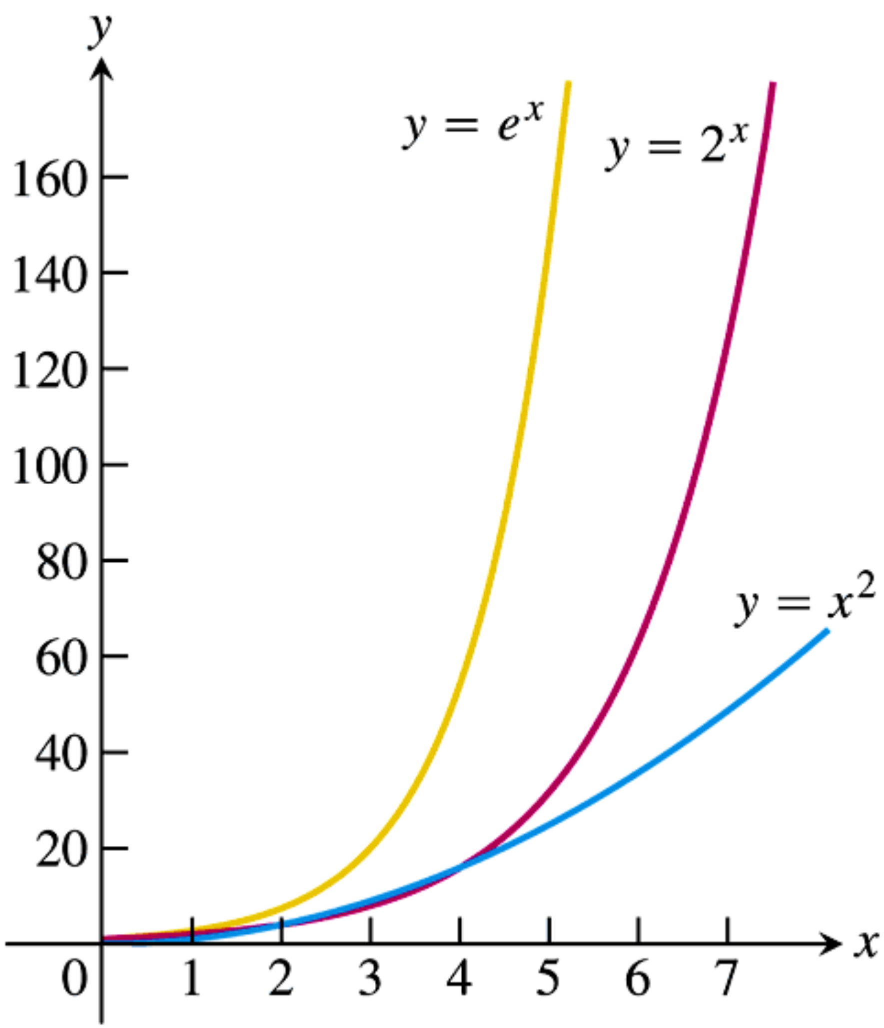


FIGURE 7.14 The graphs of e^x, 2^x, and x^2.

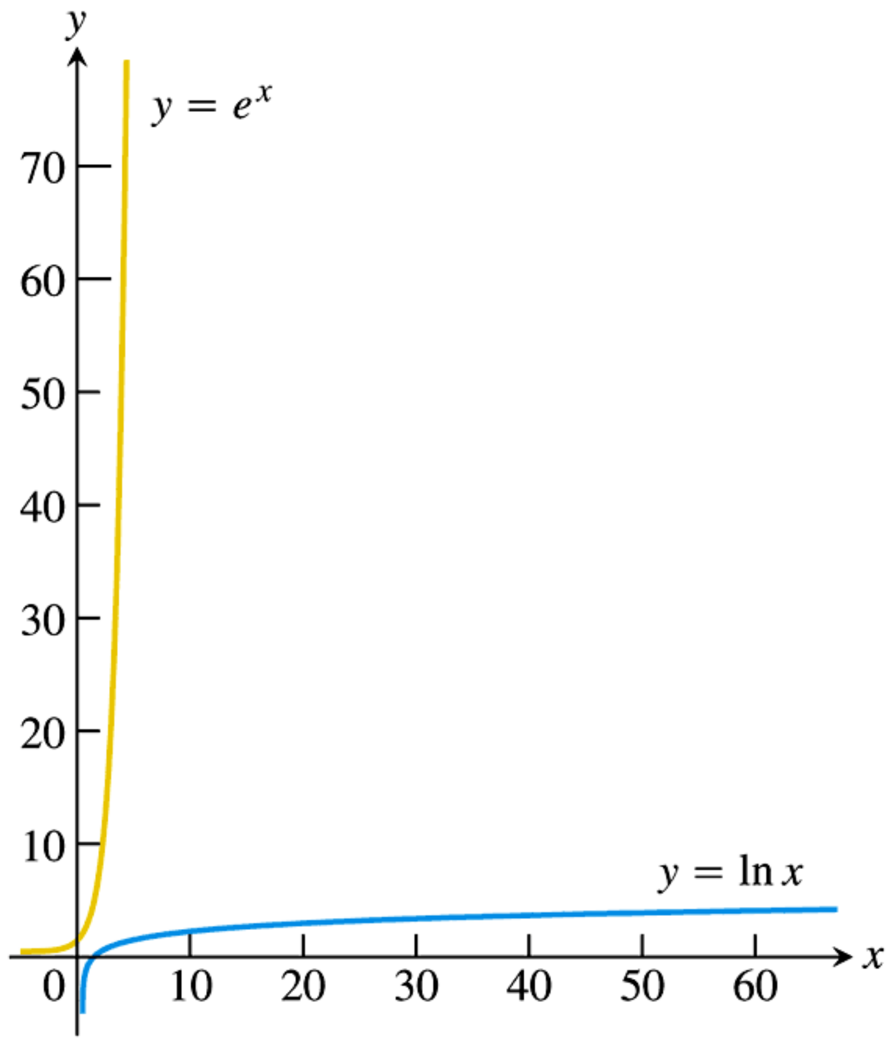


FIGURE 7.15 Scale drawings of the graphs of e^x and $\ln x$.

DEFINITION **Rates of Growth as $x \to \infty$**

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

DEFINITION **Little-oh**

A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

DEFINITION Big-oh

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

7.7

Inverse Trigonometric Functions

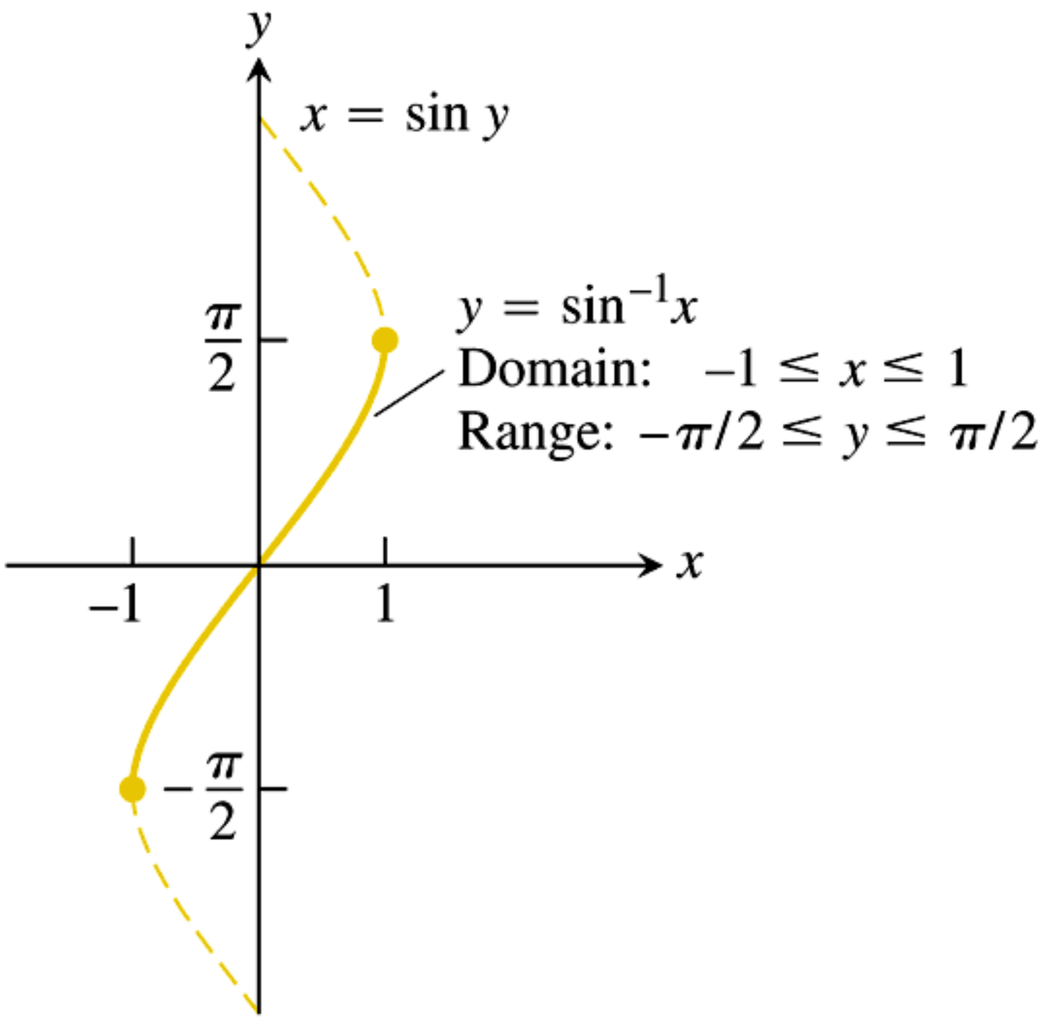


FIGURE 7.16 The graph of $y = \sin^{-1} x$.

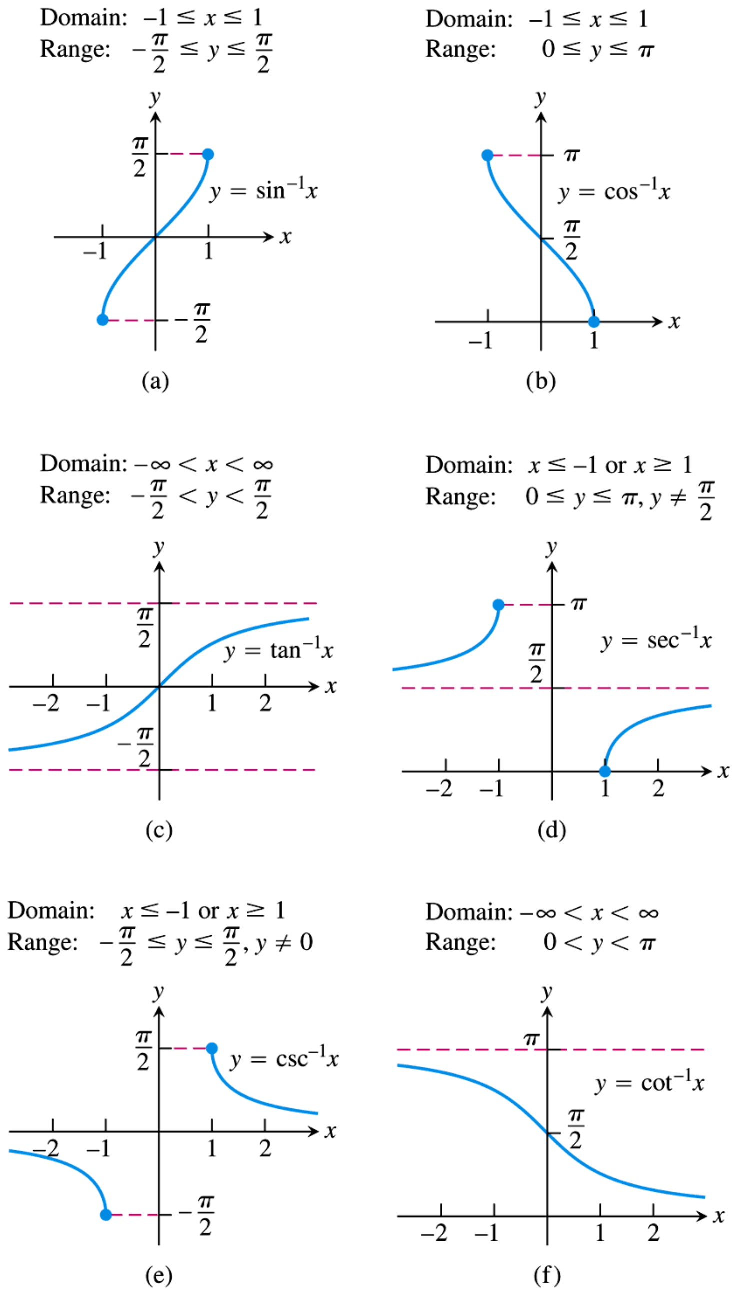


FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

DEFINITION **Arcsine and Arccosine Functions**

$\boldsymbol{y} = \mathbf{sin}^{-1}\boldsymbol{x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y} = \mathbf{cos}^{-1}\boldsymbol{x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

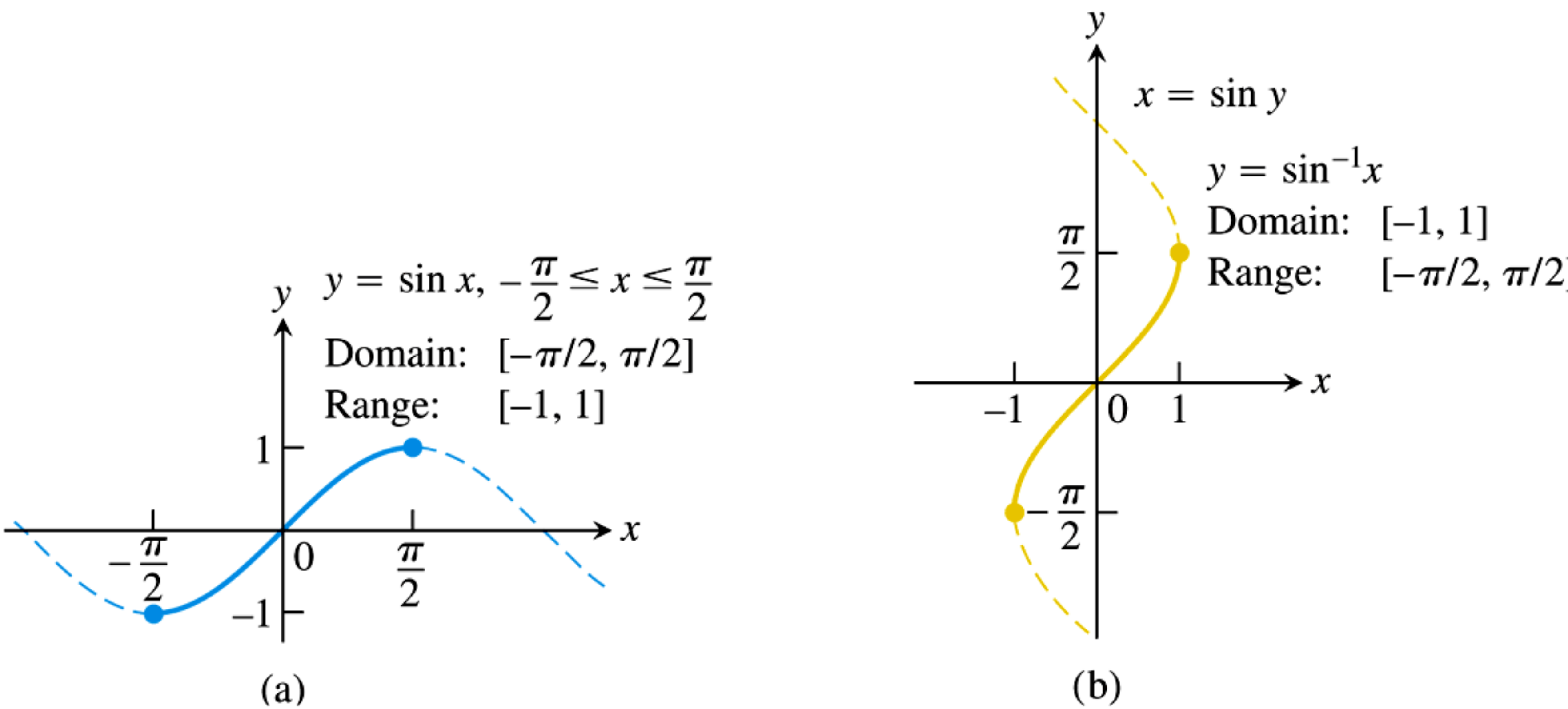


FIGURE 7.18 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

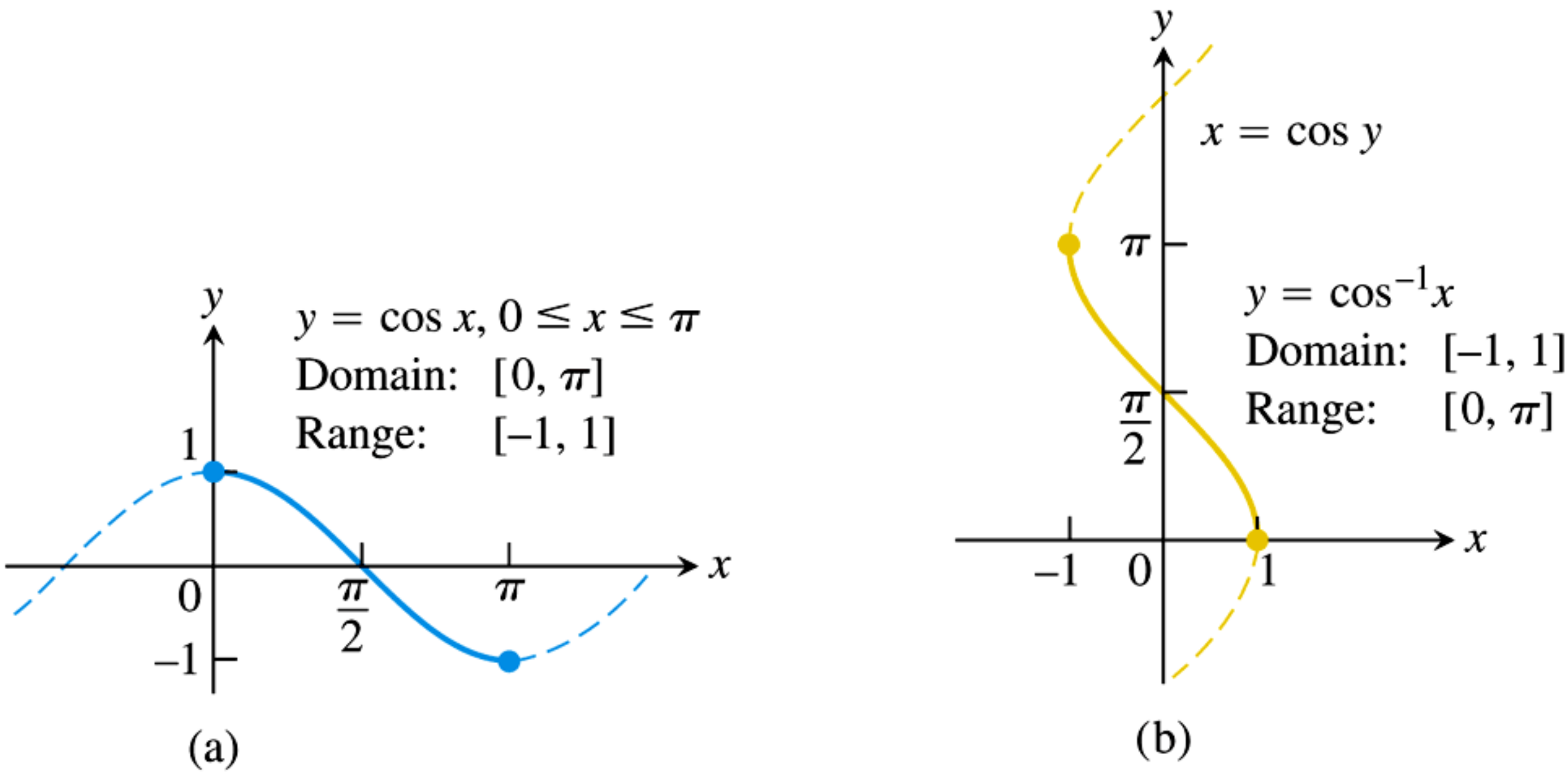
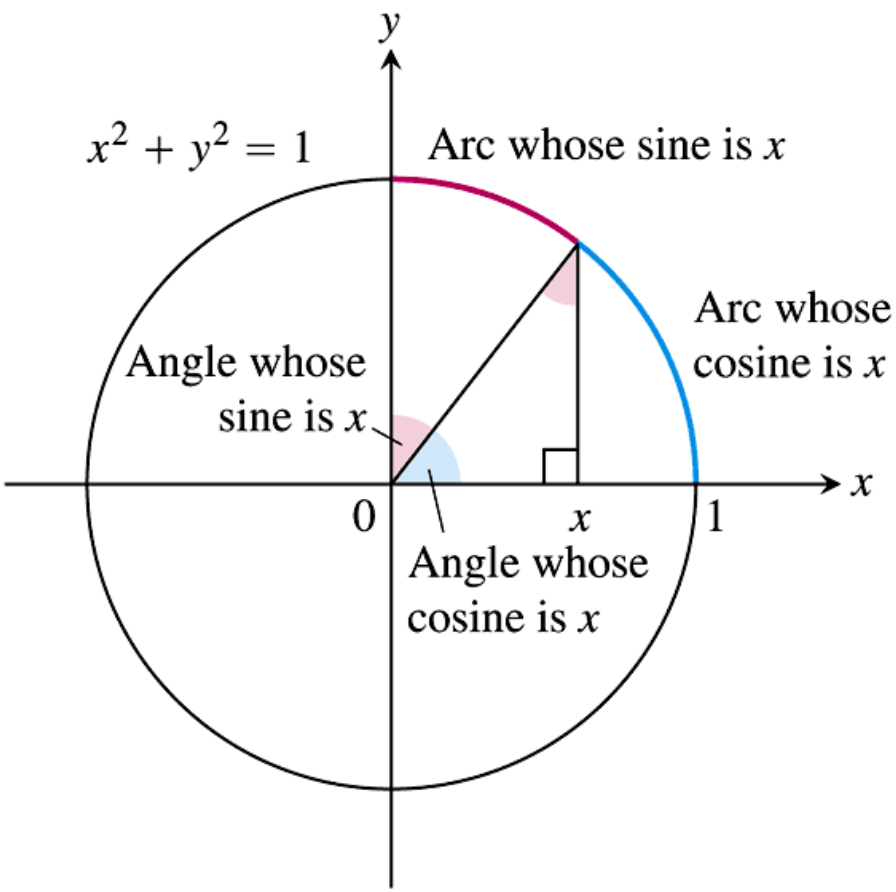


FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.



The "Arc" in Arc Sine and Arc Cosine

The accompanying figure gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for radian angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

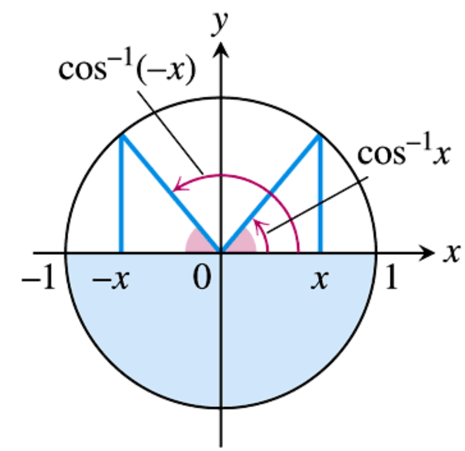
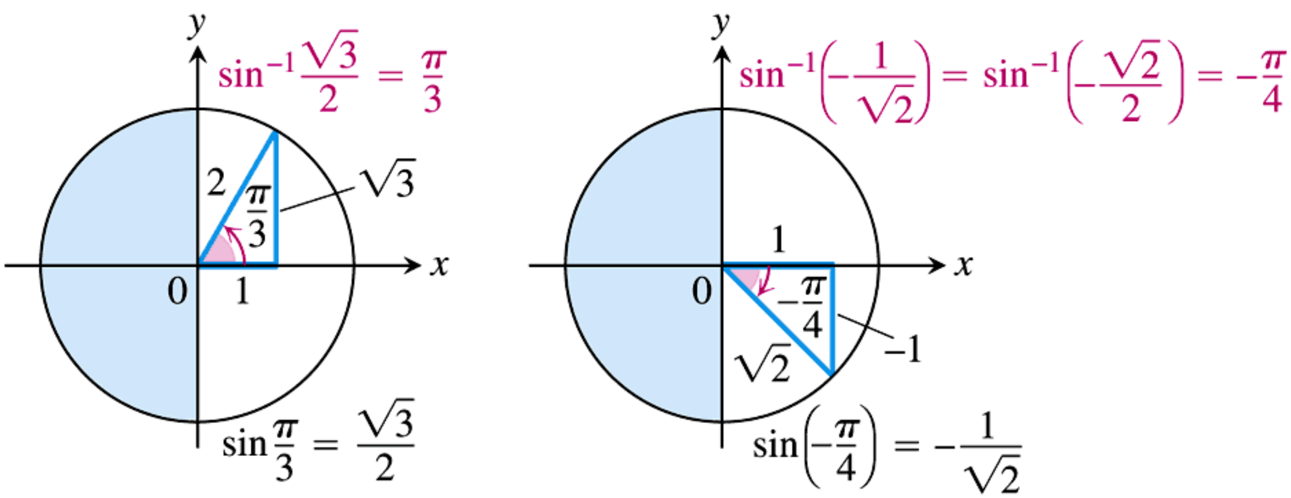


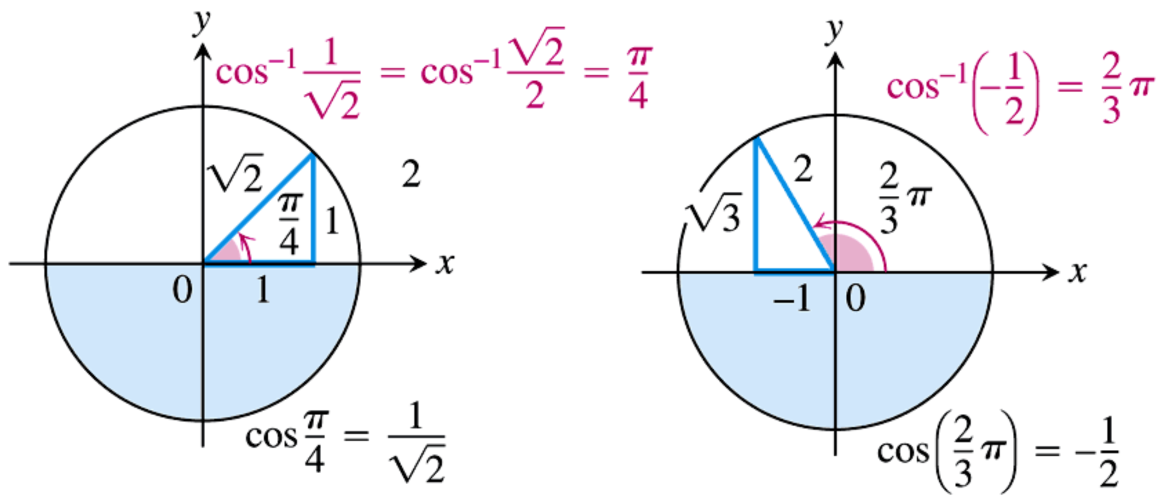
FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

EXAMPLE 1 Common Values of $\sin^{-1} x$



The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.

EXAMPLE 2 Common Values of $\cos^{-1} x$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$.

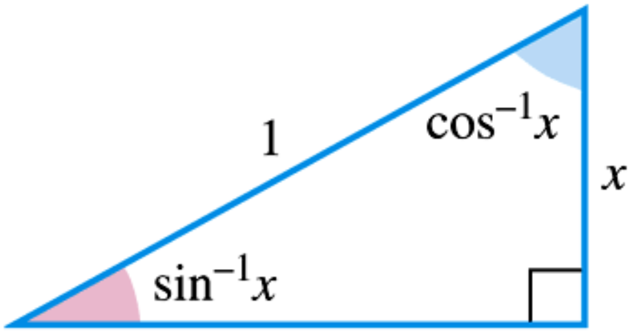


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

DEFINITION Arctangent and Arccotangent Functions

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

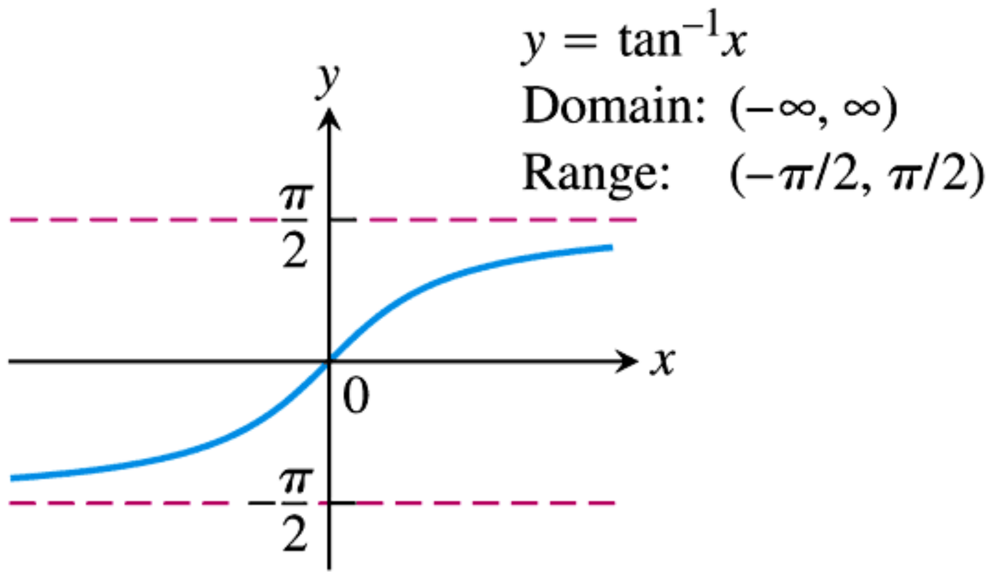


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

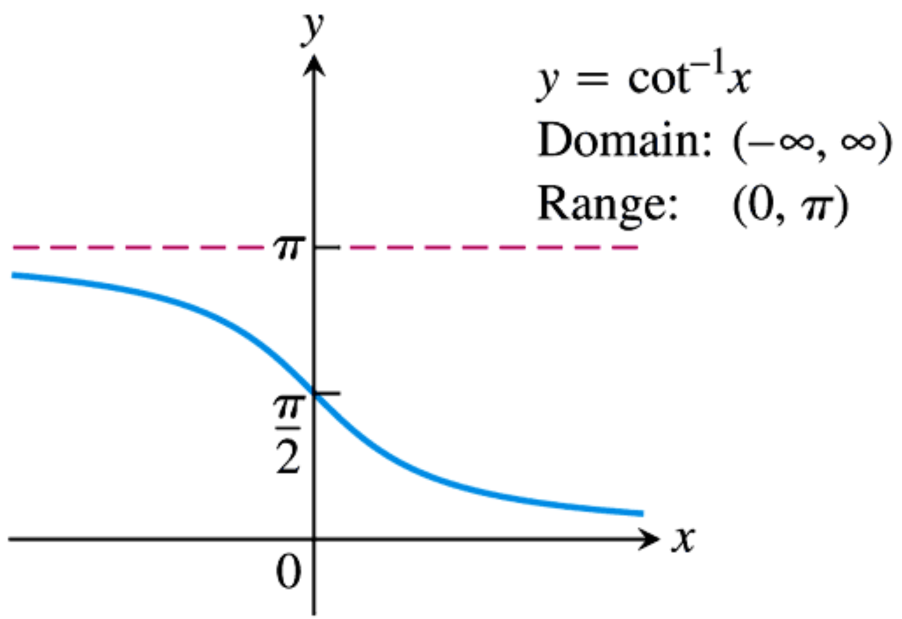


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

$y = \sec^{-1} x$

Domain: $|x| \geq 1$

Range: $[0, \pi/2) \cup (\pi/2, \pi]$

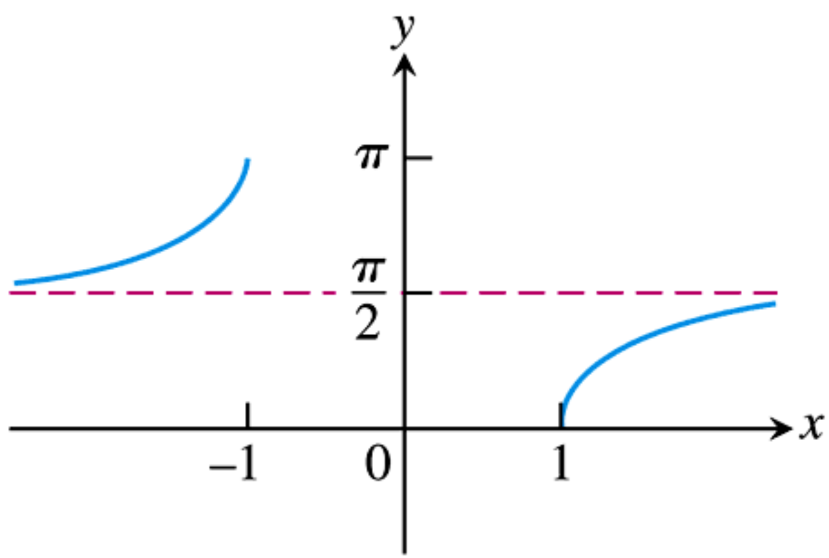


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

$y = \csc^{-1} x$

Domain: $|x| \geq 1$

Range: $[-\pi/2, 0) \cup (0, \pi/2]$

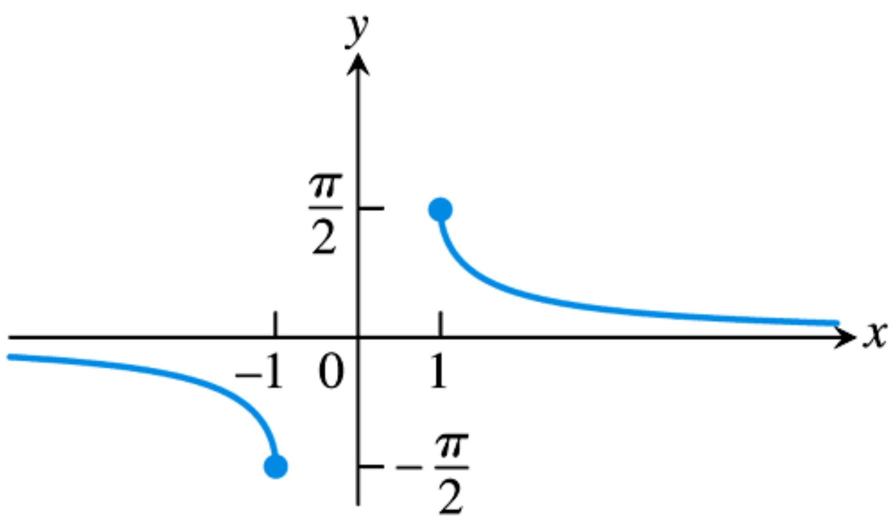


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

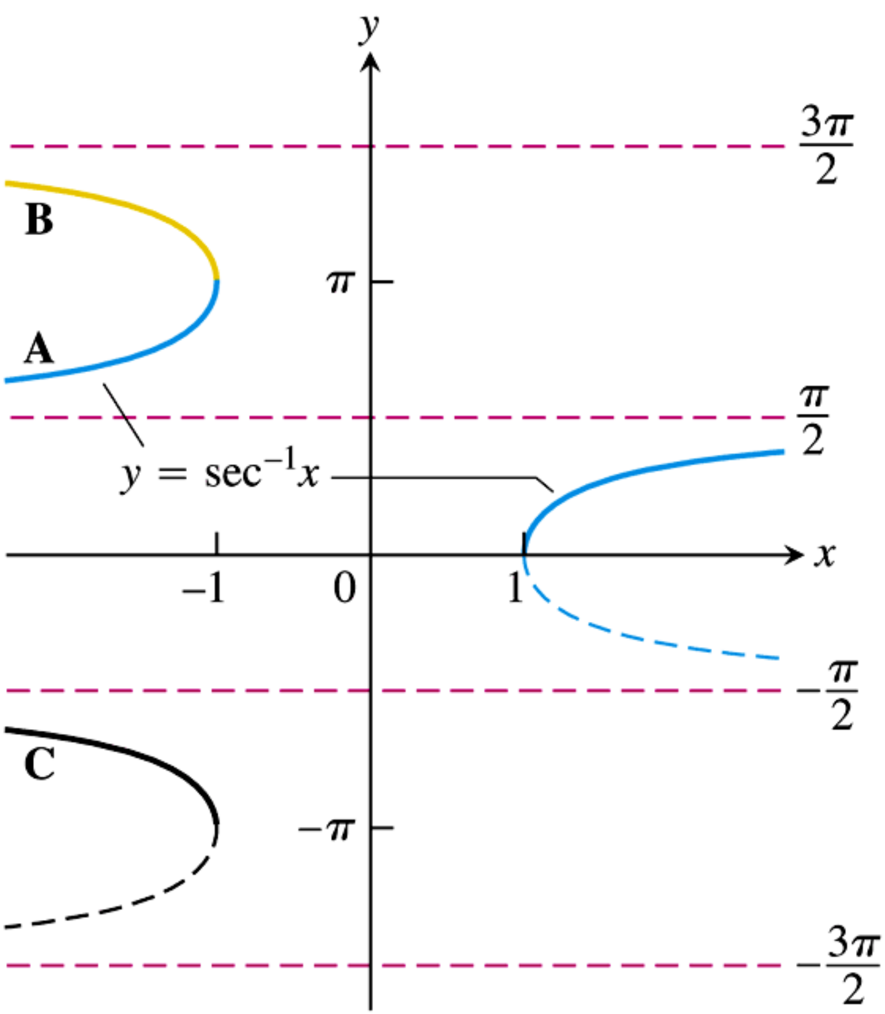
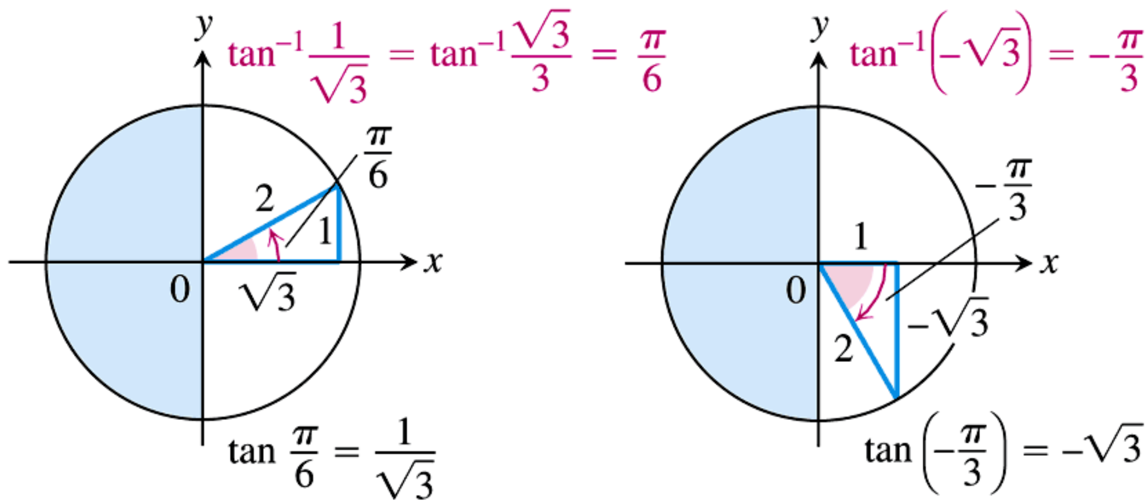


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 3 Common Values of $\tan^{-1} x$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$.

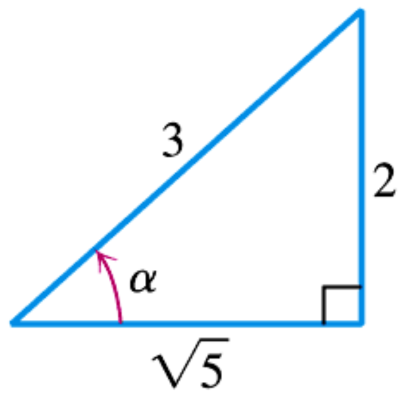


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

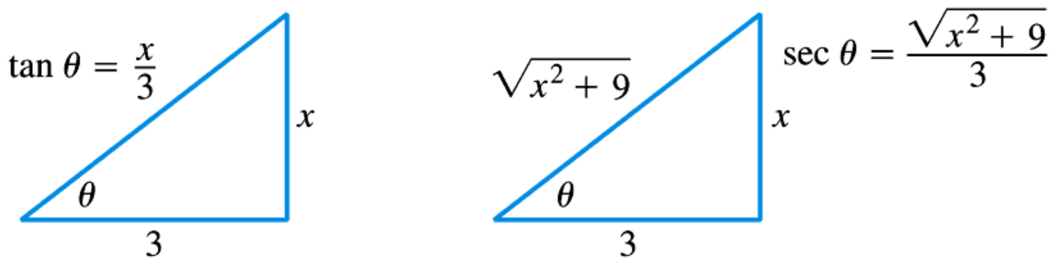
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2 + 9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

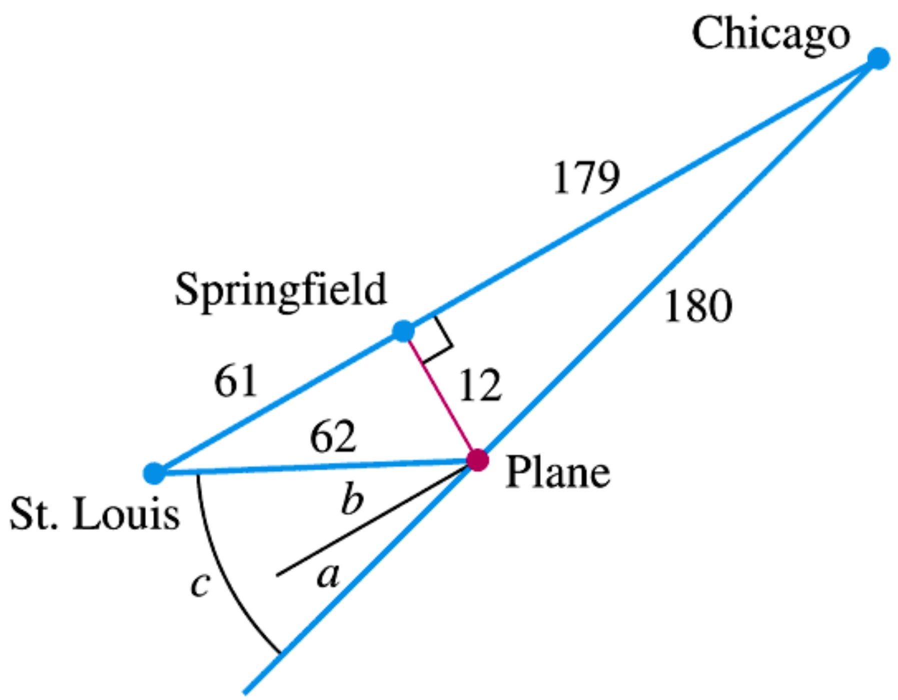


FIGURE 7.28 Diagram for drift correction (Example 6), with distances rounded to the nearest mile (drawing not to scale).

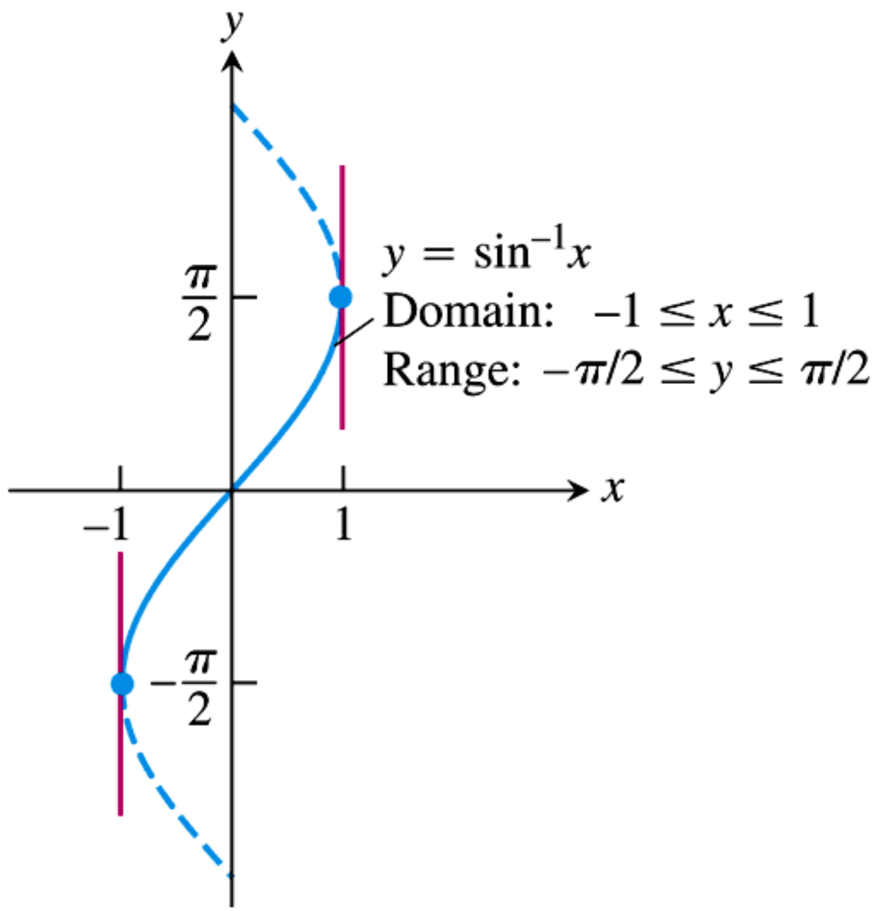


FIGURE 7.29 The graph of $y = \sin^{-1} x$.

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

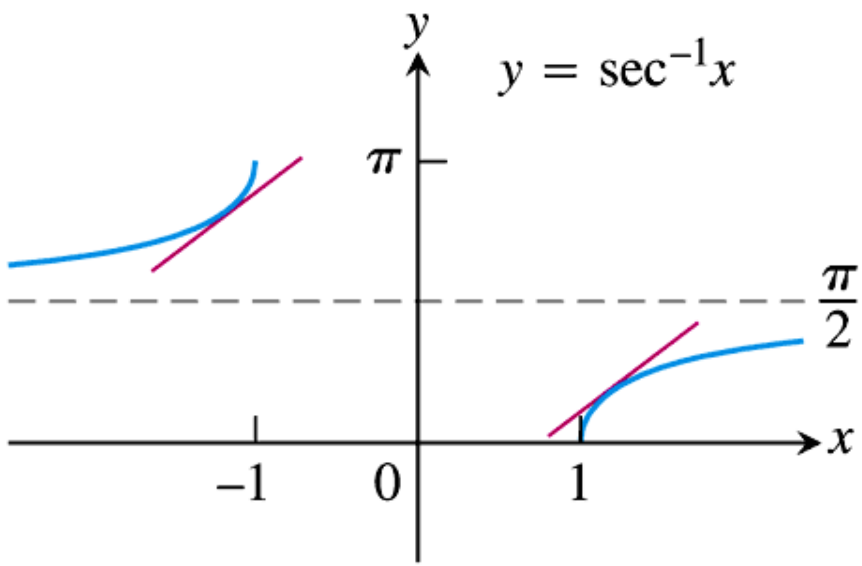


FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1 + u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1 + u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

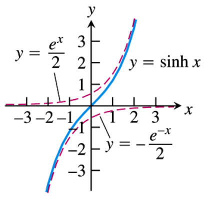
1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

7.8

Hyperbolic Functions

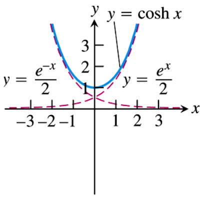
TABLE 7.5 The six basic hyperbolic functions **FIGURE 7.31**

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$



(a)

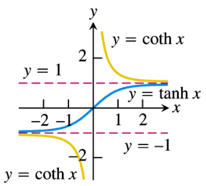
Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$



(b)

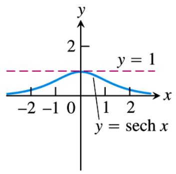
Hyperbolic tangent: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



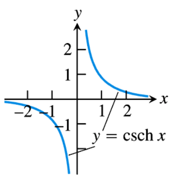
(c)

Hyperbolic secant: $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$



(e)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

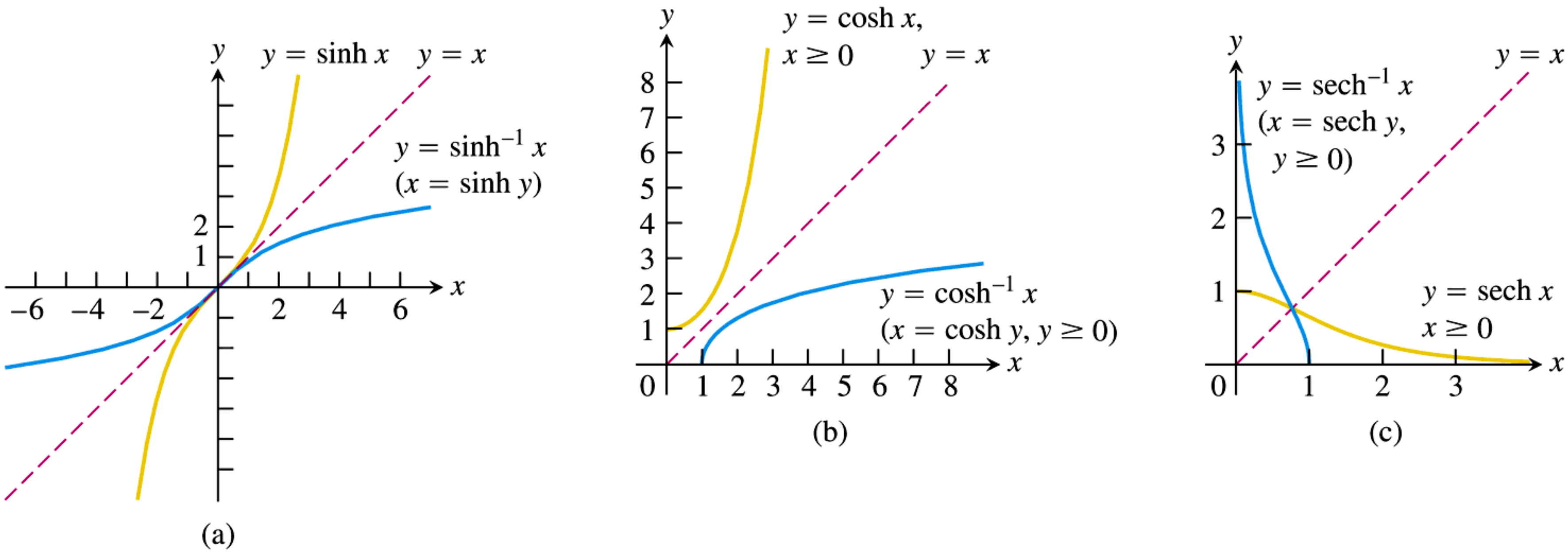


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

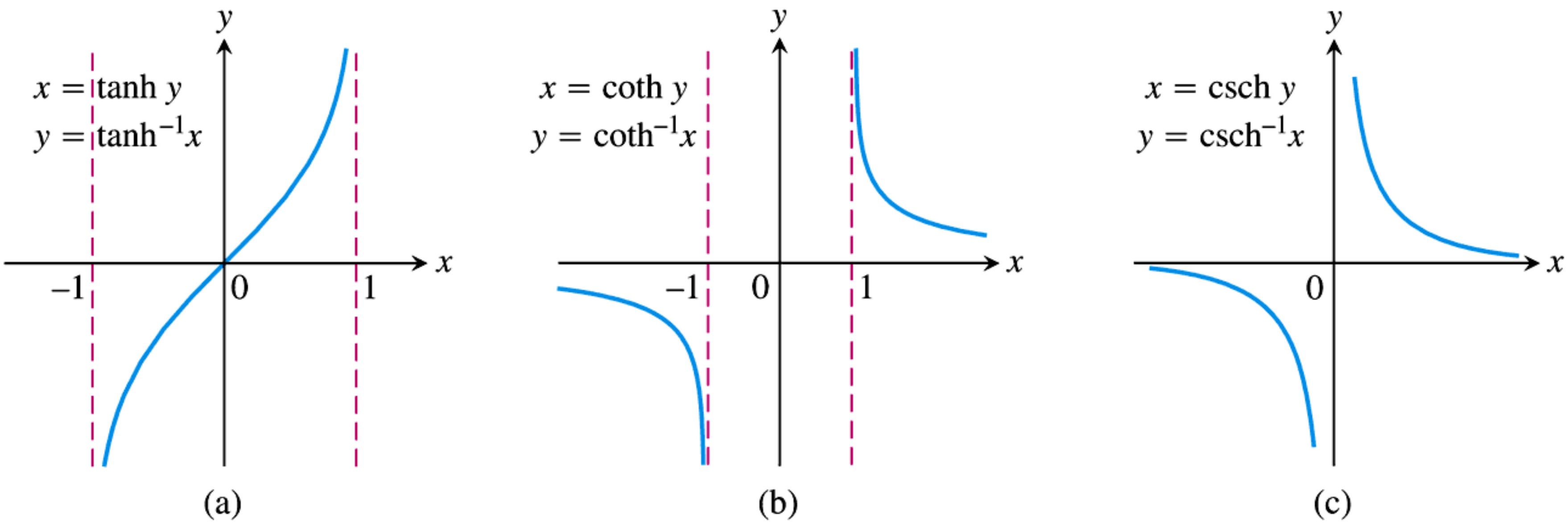


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1 - u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1 + u^2}}, \qquad u \neq 0$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1.	$\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\displaystyle\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$	
4.	$\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C,$	$u \neq 0$ and $a > 0$