Chapter 9

Further Applications of Integration

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9.1

Slope Fields and Separable Differential Equations

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General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane.

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A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx} y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution.

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As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

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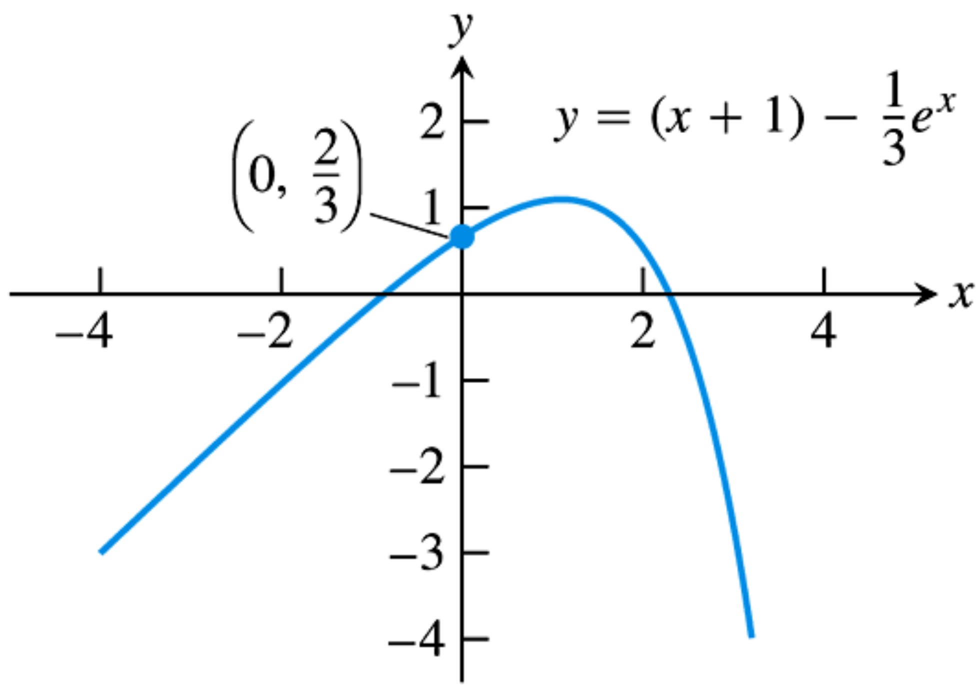

FIGURE 9.1 Graph of the solution $y = (x + 1) - \frac{1}{3}e^x$ to the differential equation $dy/dx = y - x$, with initial condition $y(0) = \frac{2}{3}$ (Example 2).

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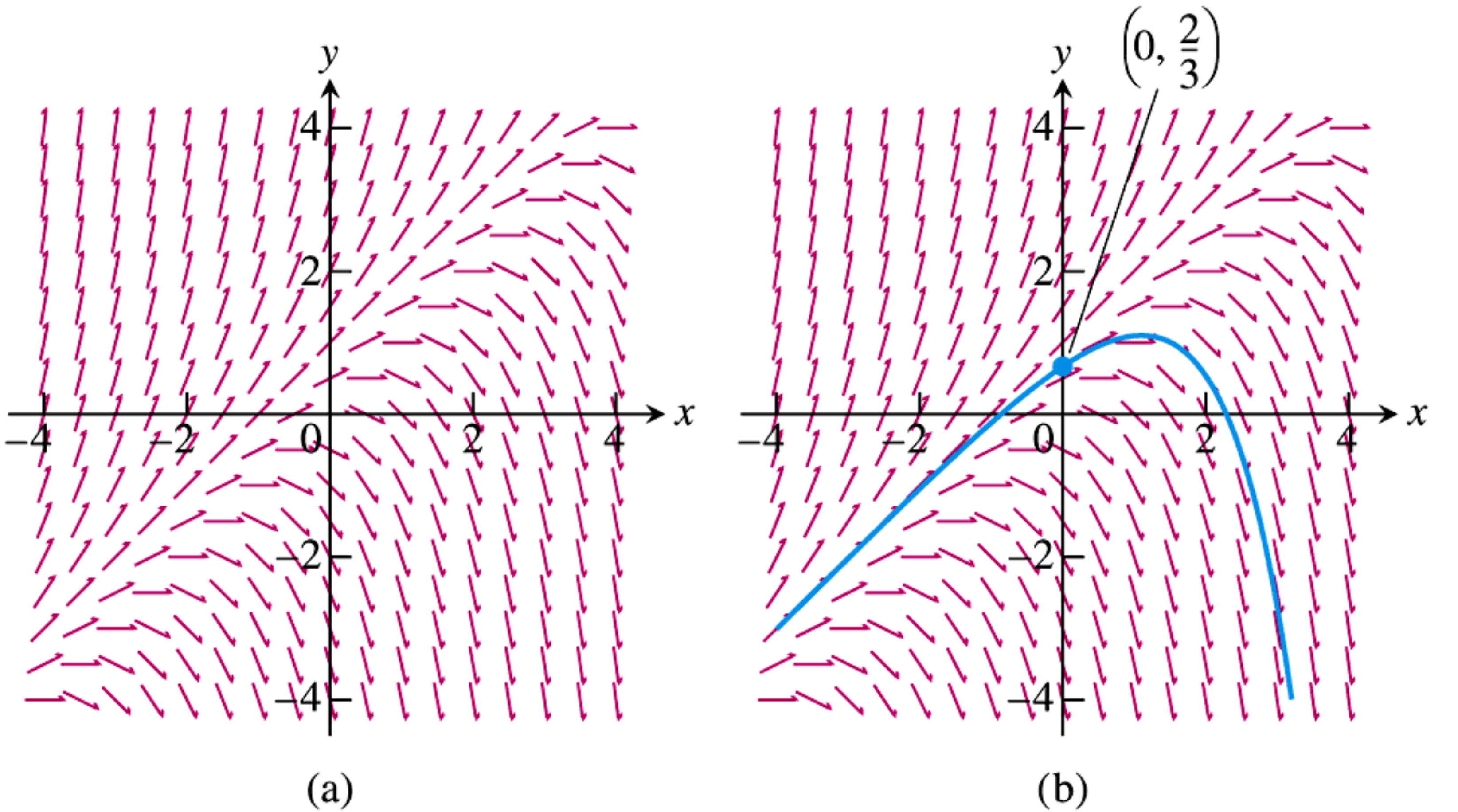

FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

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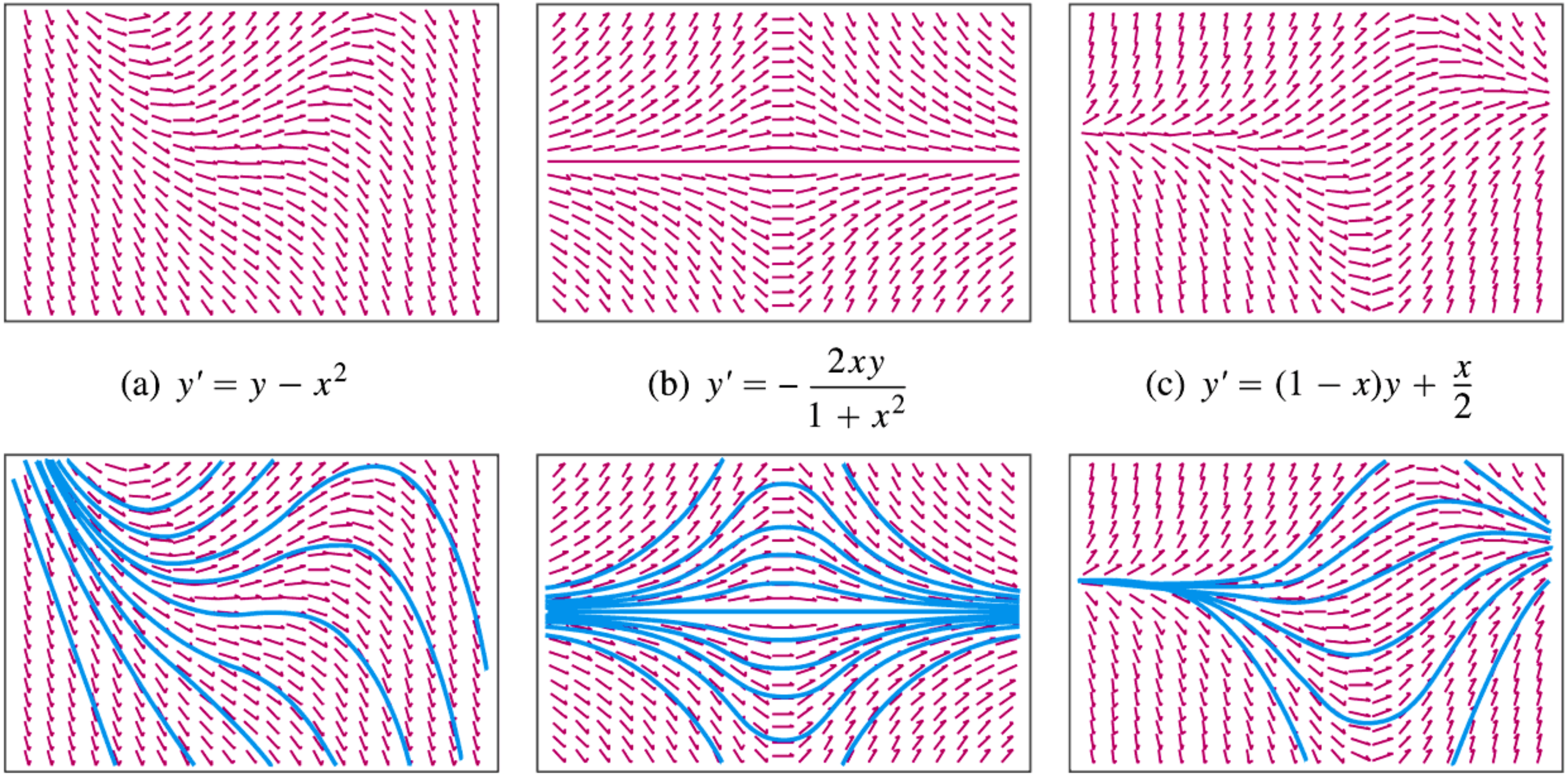

FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

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Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad H(y) = \frac{1}{h(y)}$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{2}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

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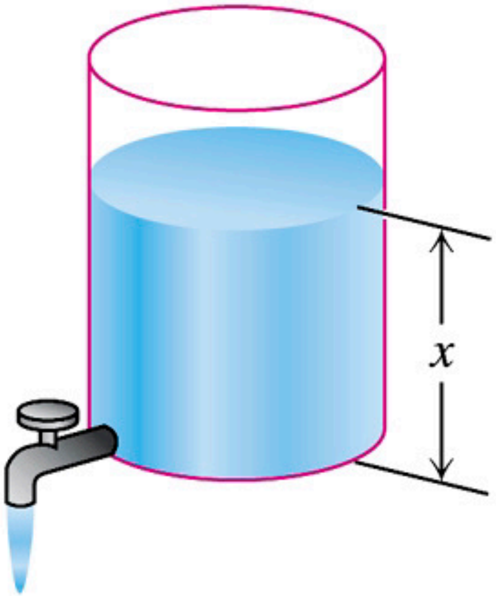

FIGURE 9.4 The rate at which water runs out is $k\sqrt{x}$, where k is a positive constant. In Example 5, $k = 1/2$ and x is measured in feet.

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9.2

First-Order Linear Differential Equations

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A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**.

Since the exponential growth/decay equation can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y)

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To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

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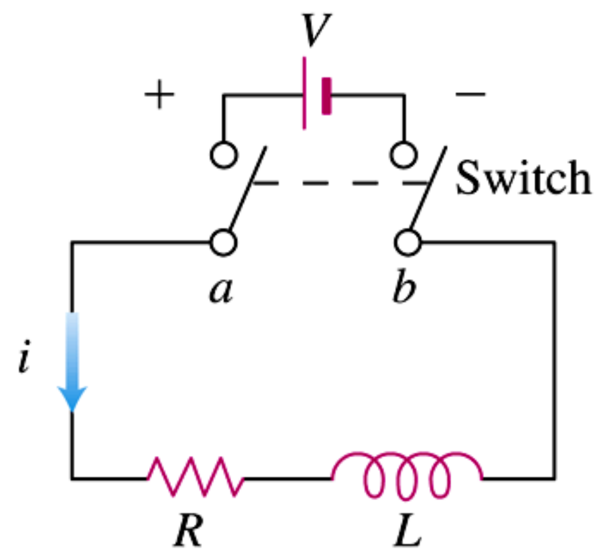

FIGURE 9.5 The *RL* circuit in Example 5.

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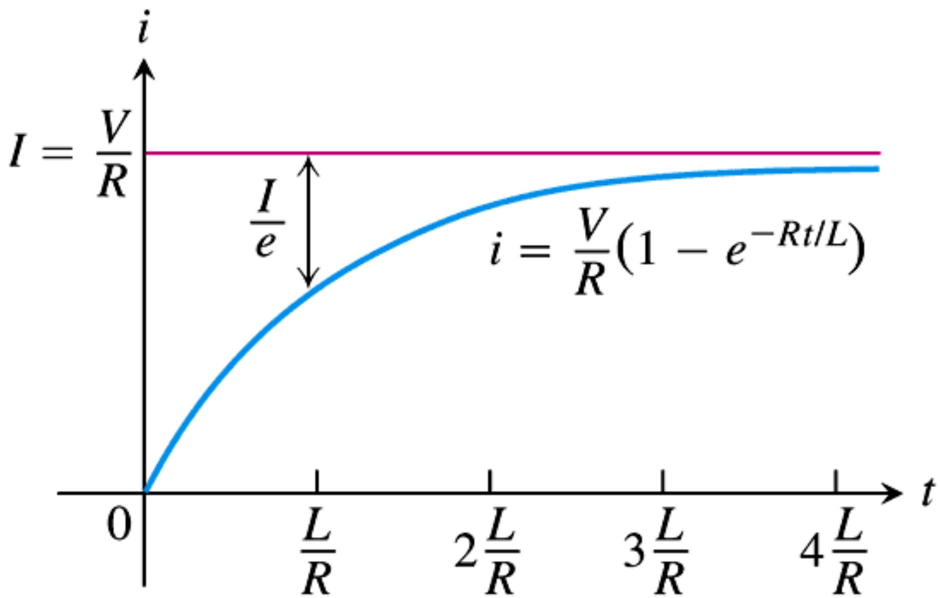

FIGURE 9.6 The growth of the current in the *RL* circuit in Example 5. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 31).

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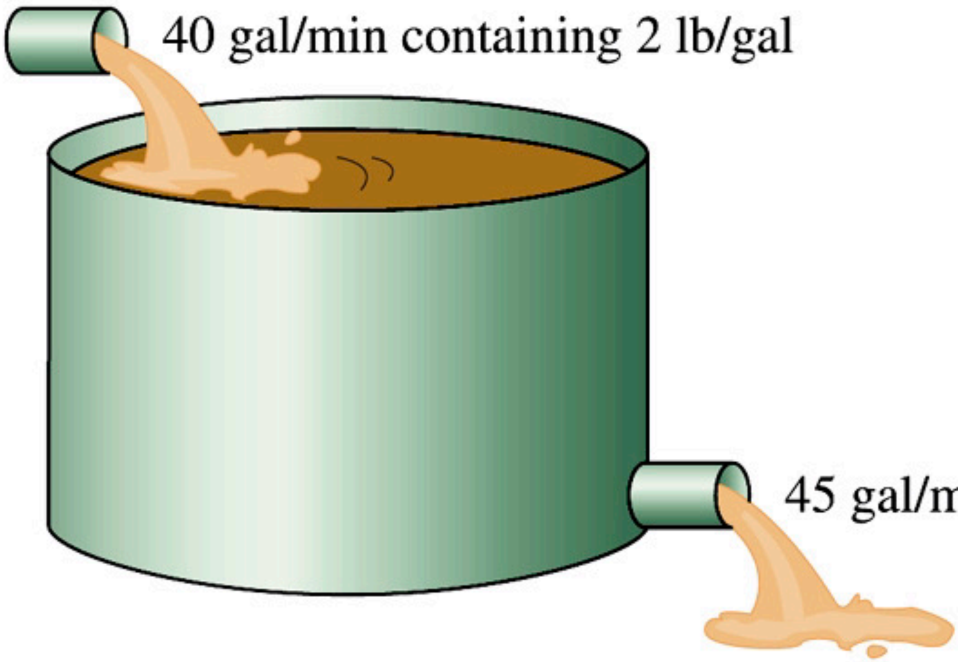

FIGURE 9.7 The storage tank in Example 6 mixes input liquid with stored liquid to produce an output liquid.

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9.3

Euler's Method

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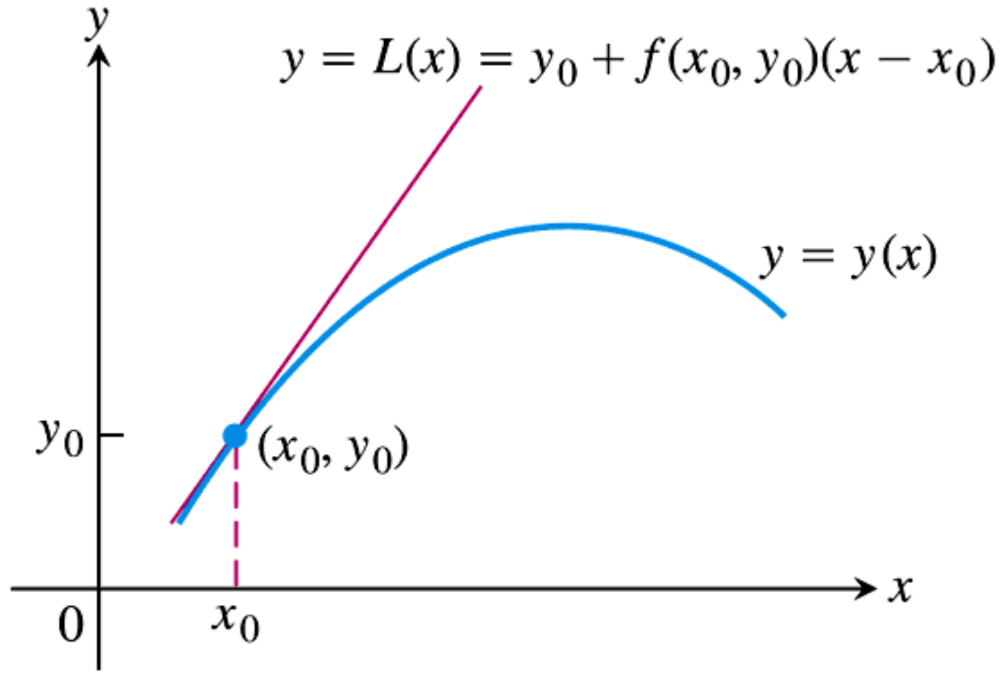

FIGURE 9.8 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

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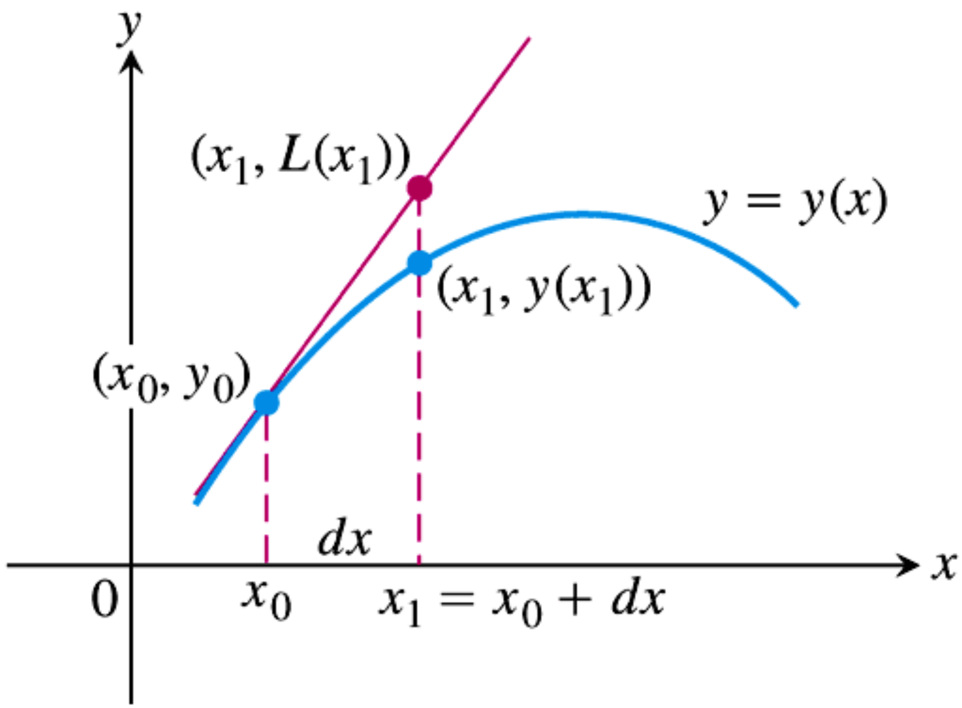

FIGURE 9.9 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

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$$
\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx .
\end{aligned}
\qquad
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\, dx \\
y_2 &= y_1 + f(x_1, y_1)\, dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\, dx .
\end{aligned}
$$

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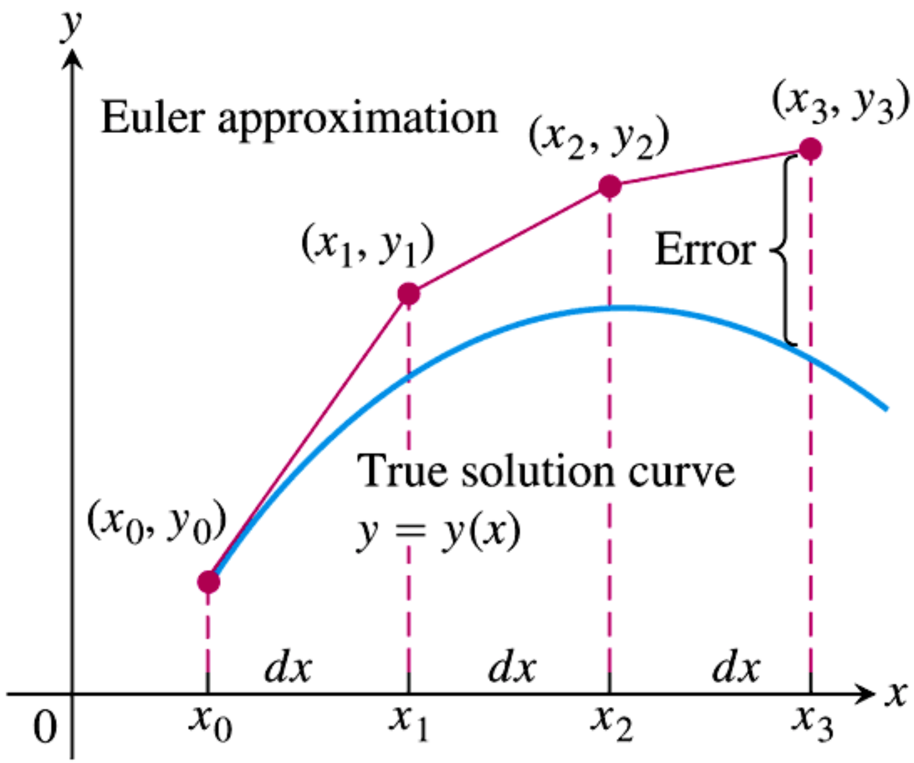

FIGURE 9.10 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

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TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

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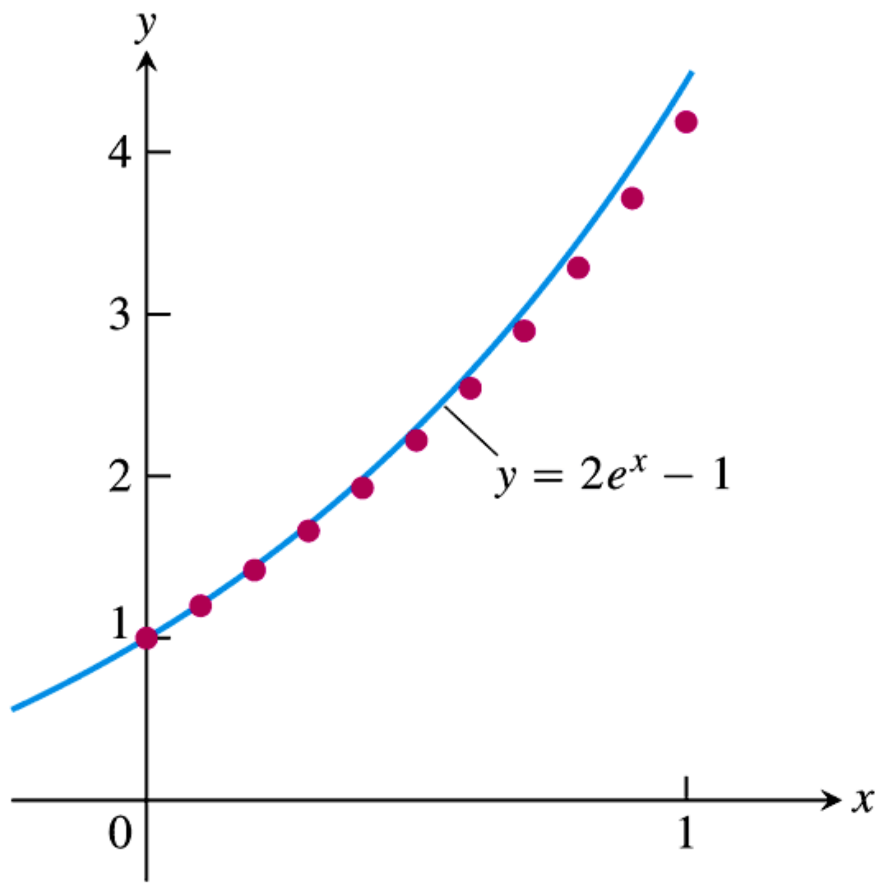

FIGURE 9.11 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 2).

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9.4

Graphical Solutions of Autonomous Differential Equations

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DEFINITION   Equilibrium Values

If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

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EXAMPLE 2 Drawing a Phase Line and Sketching Solution Curves

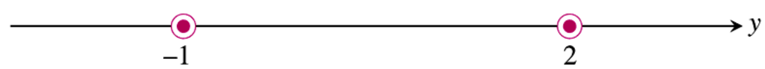
Draw a phase line for the equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

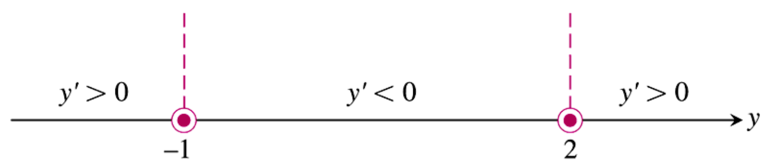
and use it to sketch solutions to the equation.

Solution

1. *Draw a number line for y and mark the equilibrium values $y = -1$ and $y = 2$, where $dy/dx = 0$.*



2. *Identify and label the intervals where $y' > 0$ and $y' < 0$.* This step resembles what we did in Section 4.3, only now we are marking the y-axis instead of the x-axis.



We can encapsulate the information about the sign of y' on the phase line itself. Since $y' > 0$ on the interval to the left of $y = -1$, a solution of the differential equation with a y-value less than -1 will increase from there toward $y = -1$. We display this information by drawing an arrow on the interval pointing to -1.

Similarly, $y' < 0$ between $y = -1$ and $y = 2$, so any solution with a value in this interval will decrease toward $y = -1$.

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For $y > 2$, we have $y' > 0$, so a solution with a y-value greater than 2 will increase from there without bound.

In short, solution curves below the horizontal line $y = -1$ in the xy-plane rise toward $y = -1$. Solution curves between the lines $y = -1$ and $y = 2$ fall away from $y = 2$ toward $y = -1$. Solution curves above $y = 2$ rise away from $y = 2$ and keep going up.

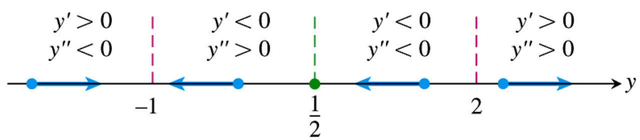
3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.* To find y'', we differentiate y' *with respect to* x, using implicit differentiation.

$$y' = (y + 1)(y - 2) = y^2 - y - 2 \qquad \text{Formula for } y' \ldots$$

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - y - 2) \\ &= 2yy' - y' \\ &= (2y - 1)y' \\ &= (2y - 1)(y + 1)(y - 2). \end{aligned}$$

differentiated implicitly with respect to x.

From this formula, we see that y'' changes sign at $y = -1$, $y = 1/2$, and $y = 2$. We add the sign information to the phase line.



4. *Sketch an assortment of solution curves in the xy-plane.* The horizontal lines $y = -1$, $y = 1/2$, and $y = 2$ partition the plane into horizontal bands in which we know the signs of y' and y''. In each band, this information tells us whether the solution curves rise or fall and how they bend as x increases (Figure 9.12).

The "equilibrium lines" $y = -1$ and $y = 2$ are also solution curves. (The constant functions $y = -1$ and $y = 2$ satisfy the differential equation.) Solution curves that cross the line $y = 1/2$ have an inflection point there. The concavity changes from concave down (above the line) to concave up (below the line).

As predicted in Step 2, solutions in the middle and lower bands approach the equilibrium value $y = -1$ as x increases. Solutions in the upper band rise steadily away from the value $y = 2$. ■

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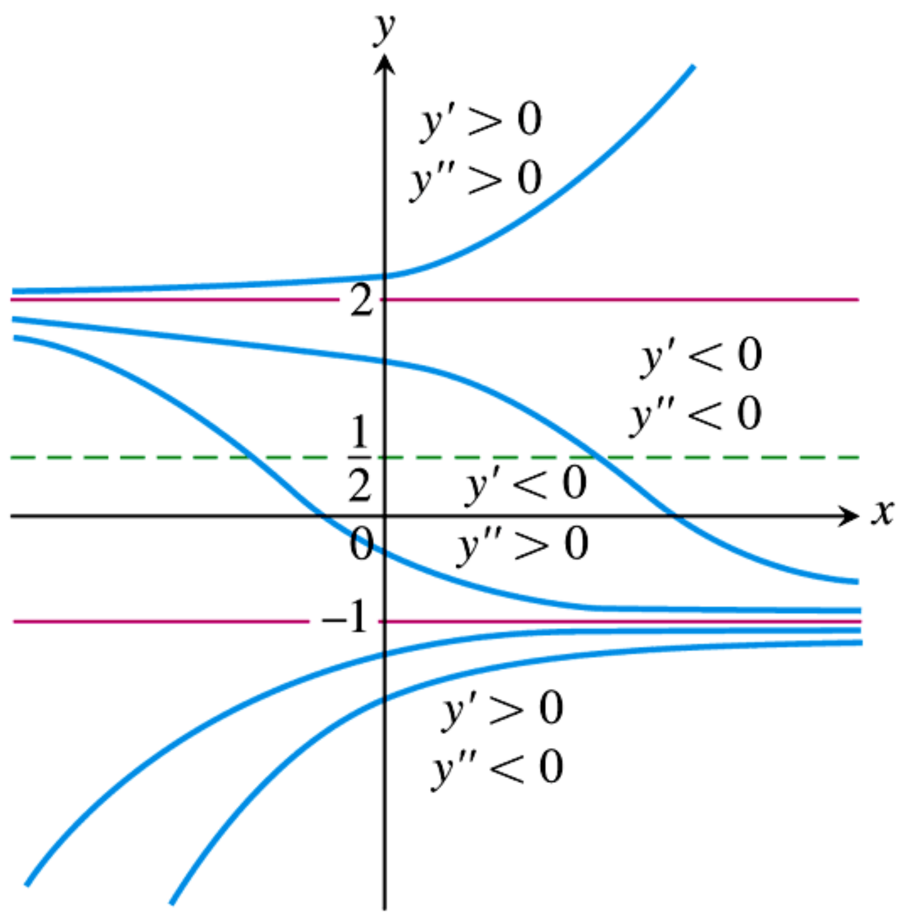

FIGURE 9.12 Graphical solutions from Example 2 include the horizontal lines $y = -1$ and $y = 2$ through the equilibrium values.

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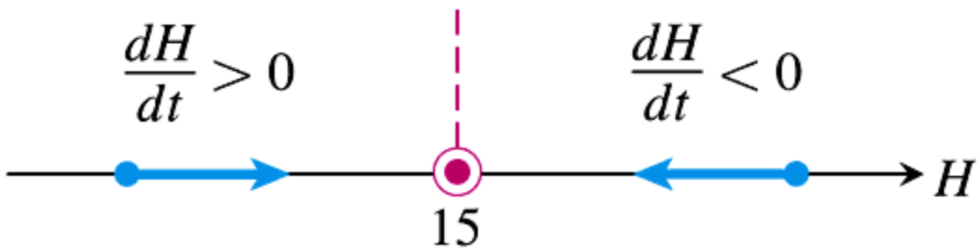

FIGURE 9.13 First step in constructing the phase line for Newton's law of cooling in Example 3. The temperature tends towards the equilibrium (surrounding-medium) value in the long run.

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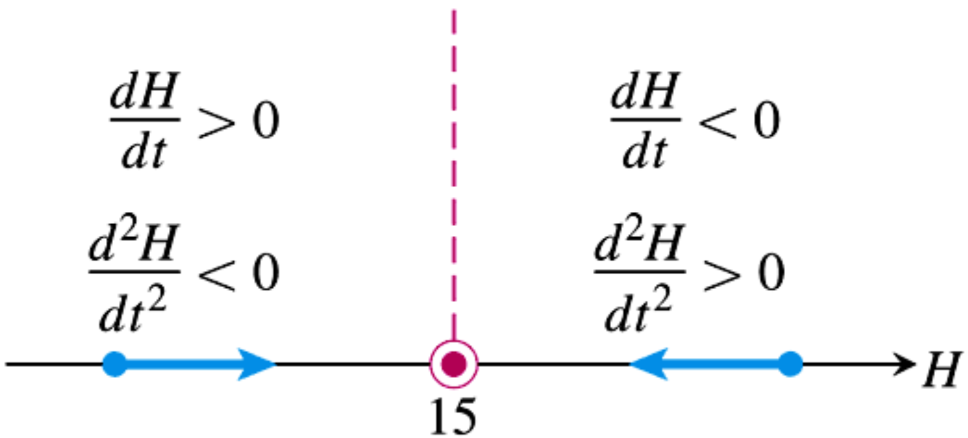

FIGURE 9.14 The complete phase line for Newton's law of cooling (Example 3).

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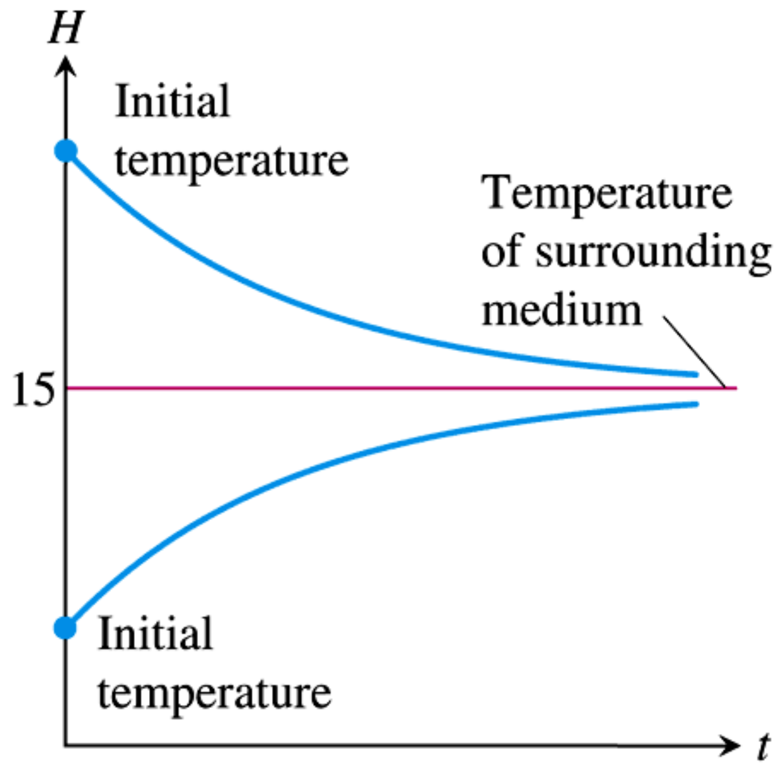

FIGURE 9.15 Temperature versus time. Regardless of initial temperature, the object's temperature $H(t)$ tends toward 15°C, the temperature of the surrounding medium.

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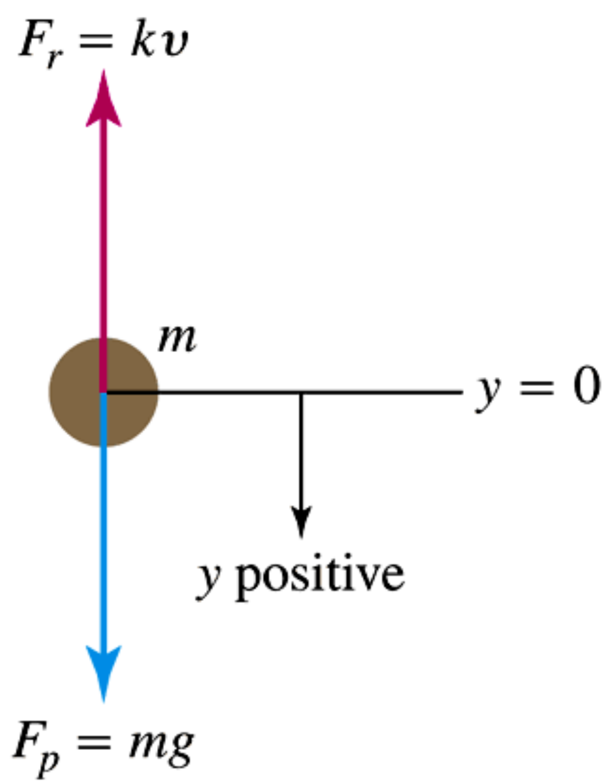

FIGURE 9.16 An object falling under the influence of gravity with a resistive force assumed to be proportional to the velocity.

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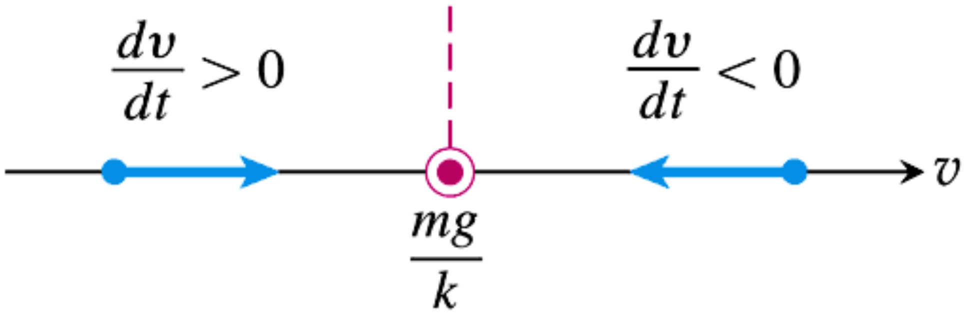

FIGURE 9.17 Initial phase line for Example 4.

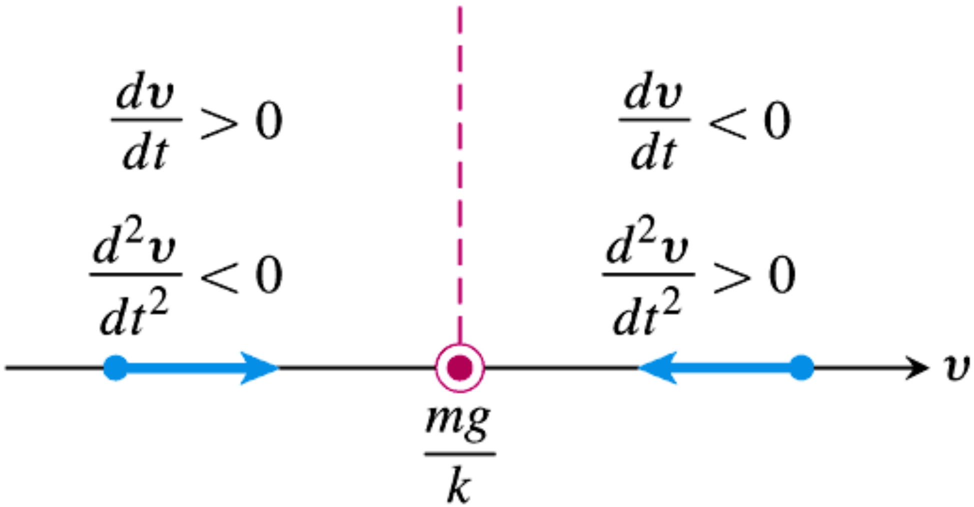

FIGURE 9.18 The completed phase line for Example 4.

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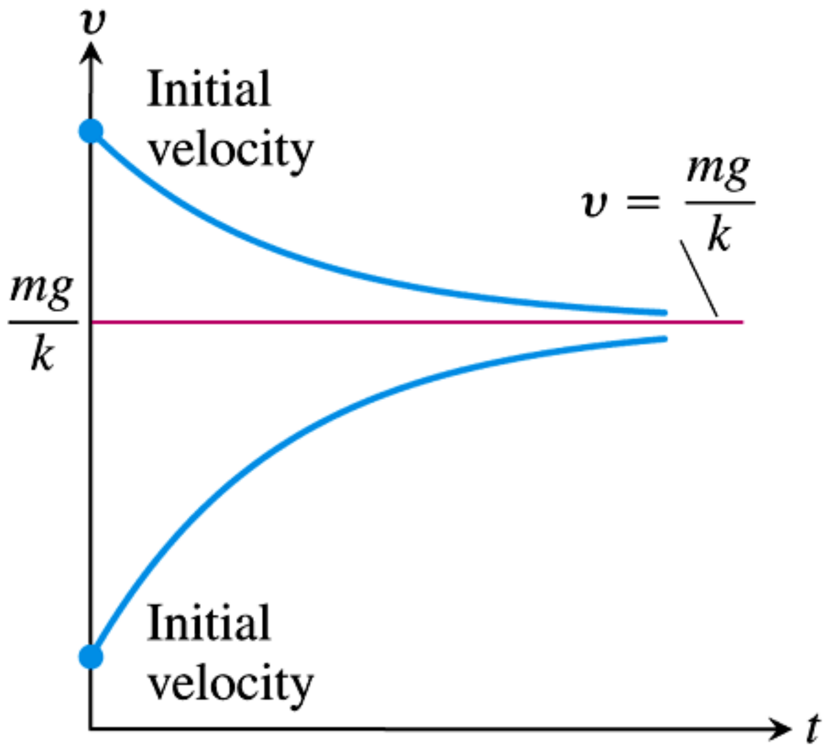

FIGURE 9.19 Typical velocity curves in Example 4. The value $v = mg/k$ is the terminal velocity.

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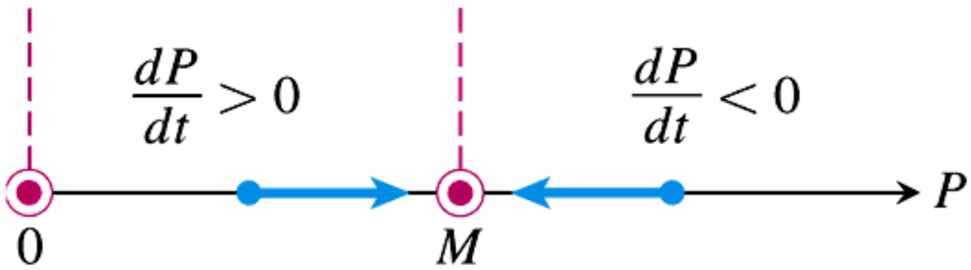

FIGURE 9.20 The initial phase line for Equation 6.

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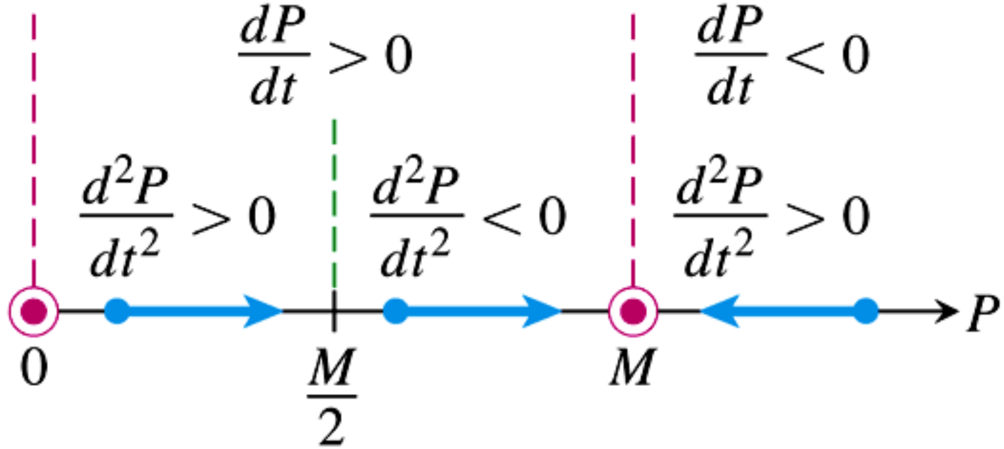

FIGURE 9.21 The completed phase line for logistic growth (Equation 6).

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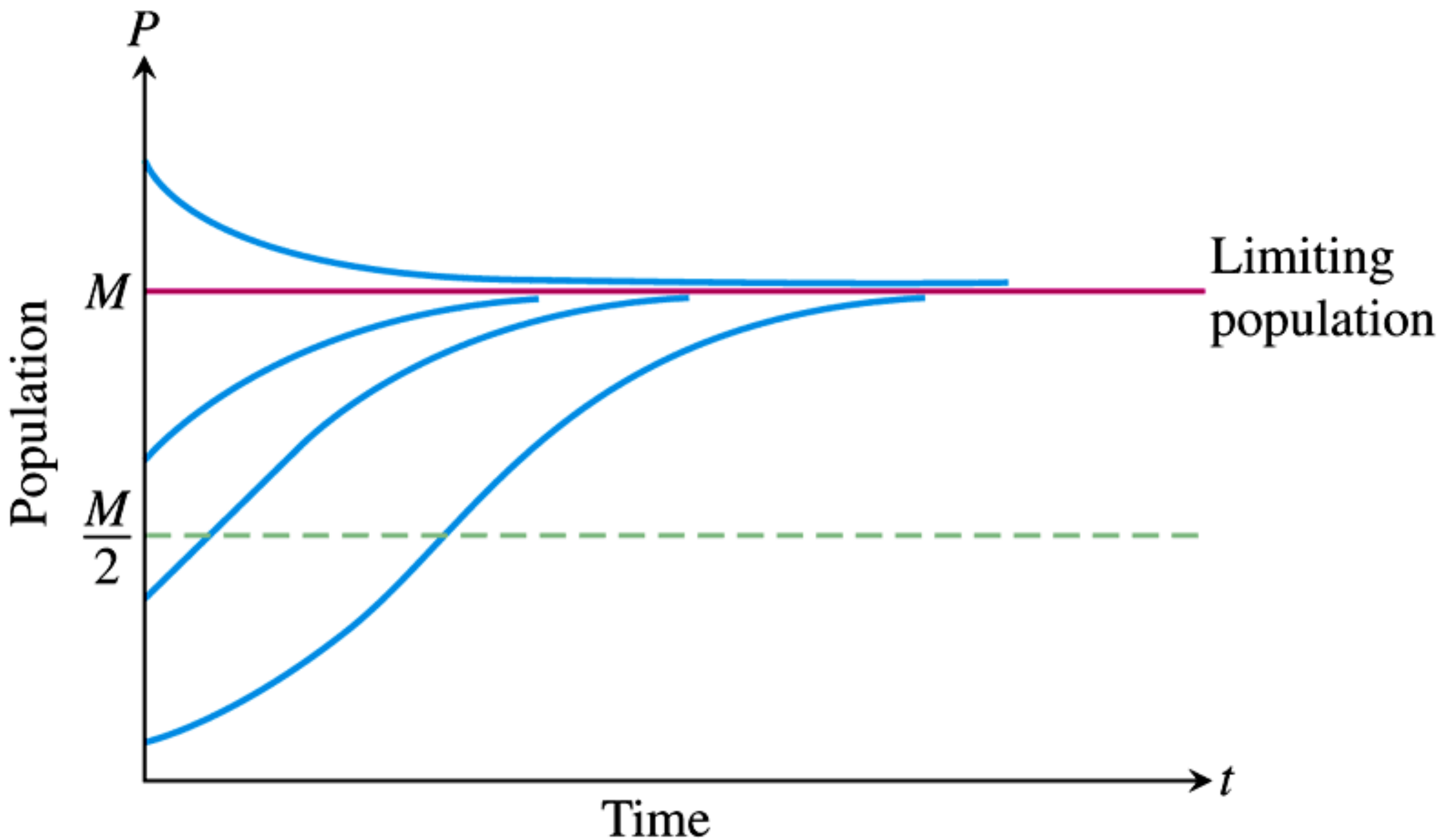

FIGURE 9.22 Population curves in Example 5.

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9.5

Applications of First-Order Differential Equations

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TABLE 9.4 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 1999): www.census.gov/ipc/www/worldpop.html.

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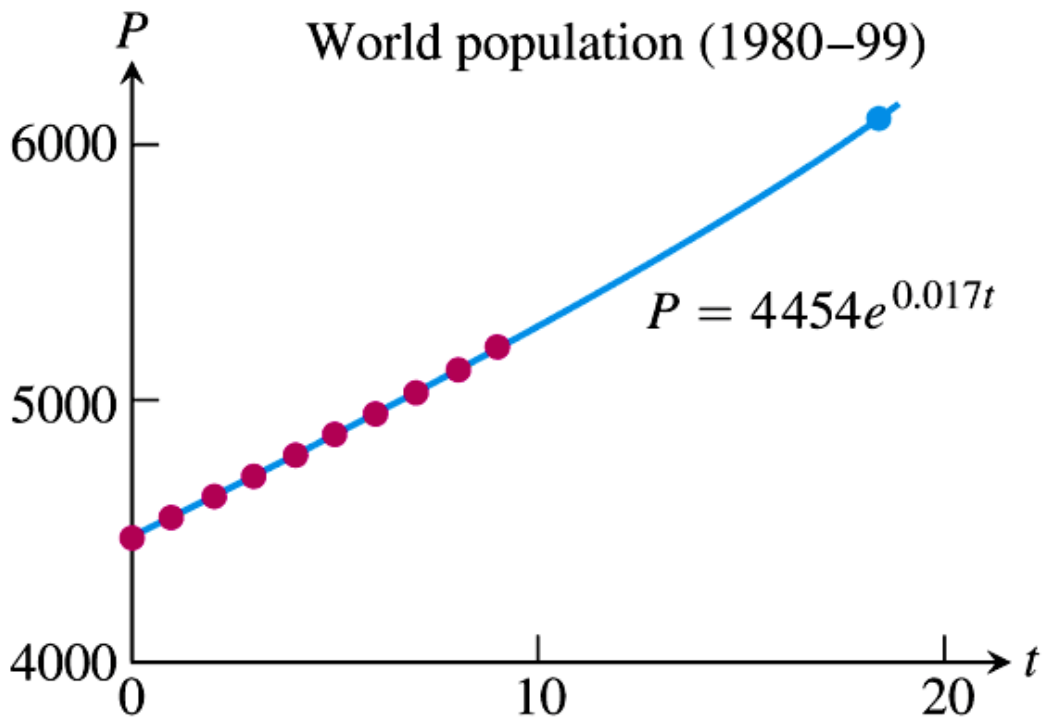

FIGURE 9.23 Notice that the value of the solution $P = 4454e^{0.017t}$ is 6152.16 when $t = 19$, which is slightly higher than the actual population in 1999.

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TABLE 9.5 Recent world population

Year	Population (millions)	$\Delta P/P$
1990	5275	$84/5275 \approx 0.0159$
1991	5359	$84/5359 \approx 0.0157$
1992	5443	$81/5443 \approx 0.0149$
1993	5524	$81/5524 \approx 0.0147$
1994	5605	$80/5605 \approx 0.0143$
1995	5685	$79/5685 \approx 0.0139$
1996	5764	$80/5764 \approx 0.0139$
1997	5844	$79/5844 \approx 0.0135$
1998	5923	$78/5923 \approx 0.0132$
1999	6001	$78/6001 \approx 0.0130$
2000	6079	$73/6079 \approx 0.0120$
2001	6152	$76/6152 \approx 0.0124$
2002	6228	?
2003	?	

Source: U.S. Bureau of the Census (Sept., 2003): www.census.gov/ipc/www/worldpop.html.

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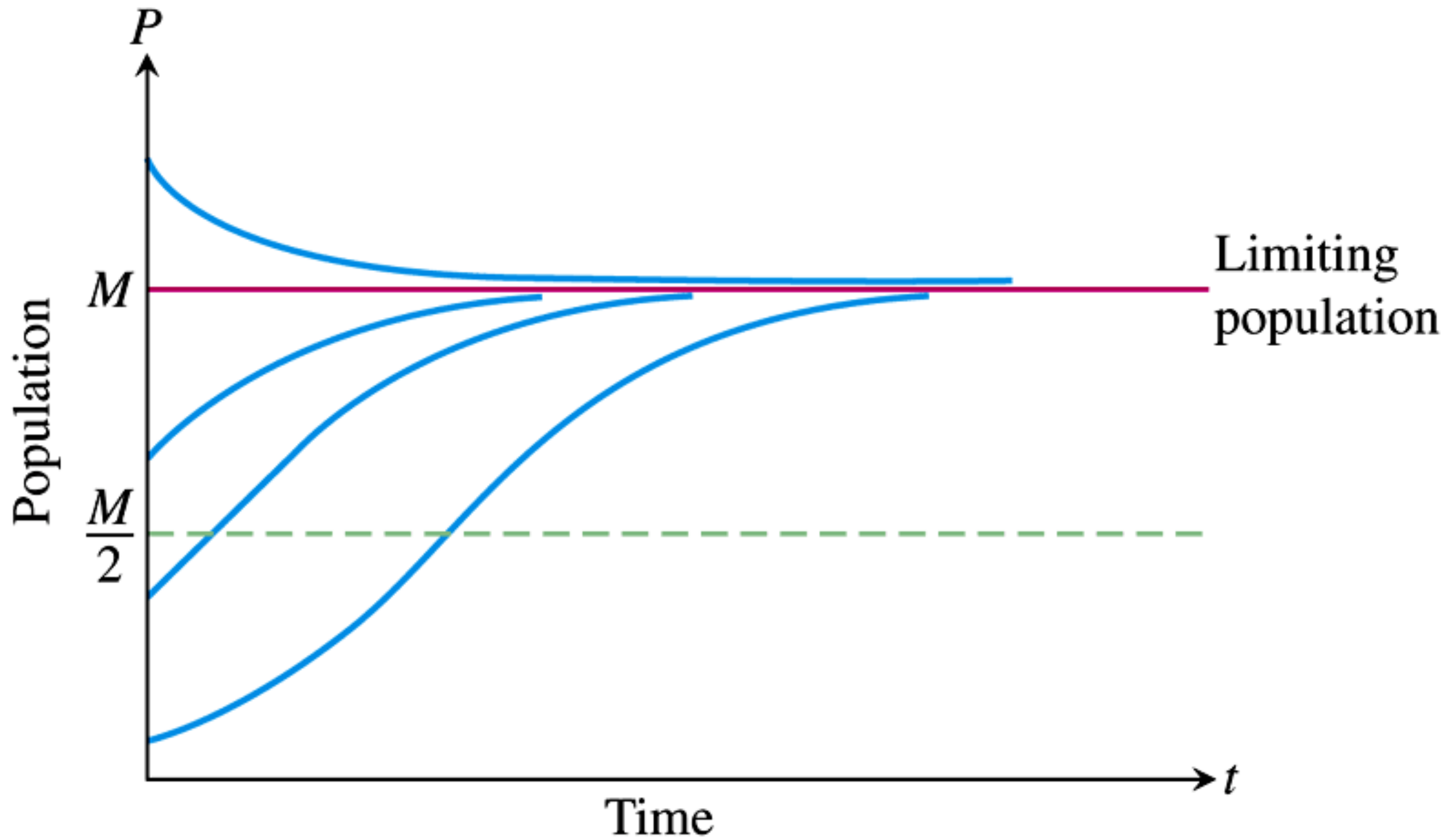

FIGURE 9.24 Solution curves to the logistic population model $dP/dt = r(M - P)P$.

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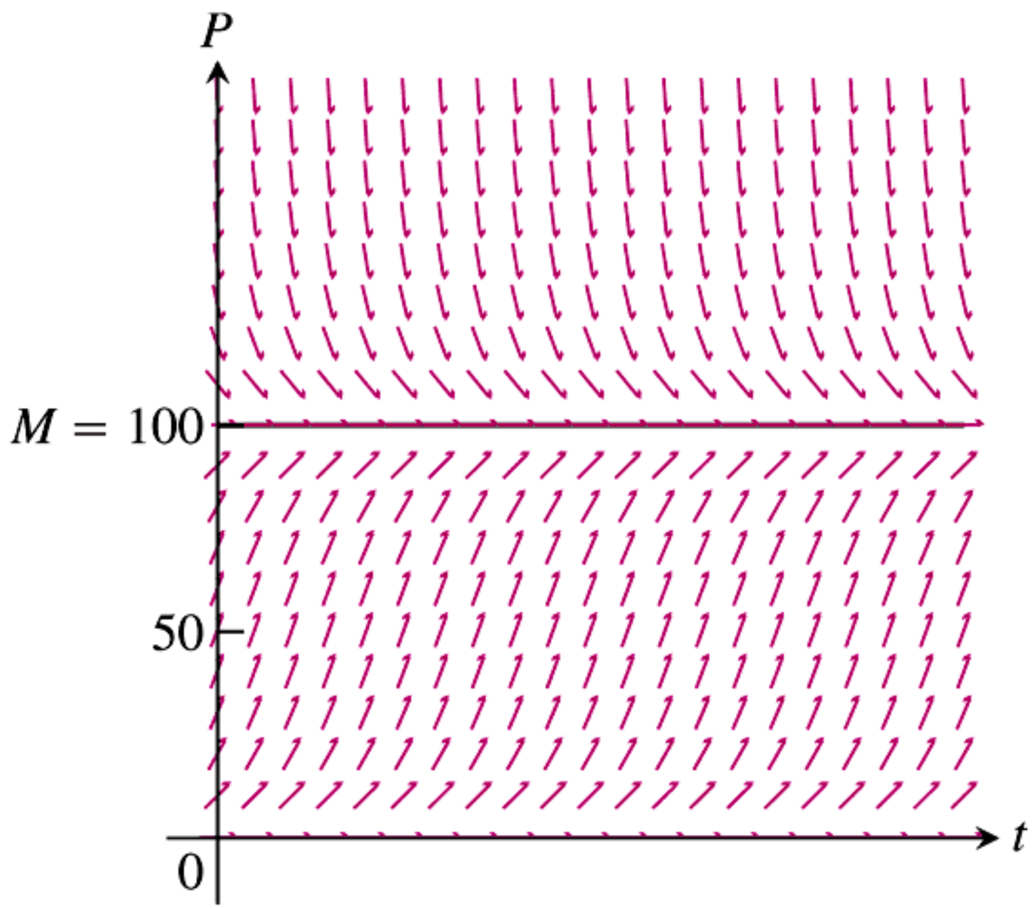

FIGURE 9.25 A slope field for the logistic differential equation $dP/dt = 0.001(100 - P)P$ (Example 2).

TABLE 9.6 Euler solution of $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$

t	P (Euler)	t	P (Euler)
0	10		
1	10.9	11	24.3629
2	11.8712	12	26.2056
3	12.9174	13	28.1395
4	14.0423	14	30.1616
5	15.2493	15	32.2680
6	16.5417	16	34.4536
7	17.9222	17	36.7119
8	19.3933	18	39.0353
9	20.9565	19	41.4151
10	22.6130	20	43.8414

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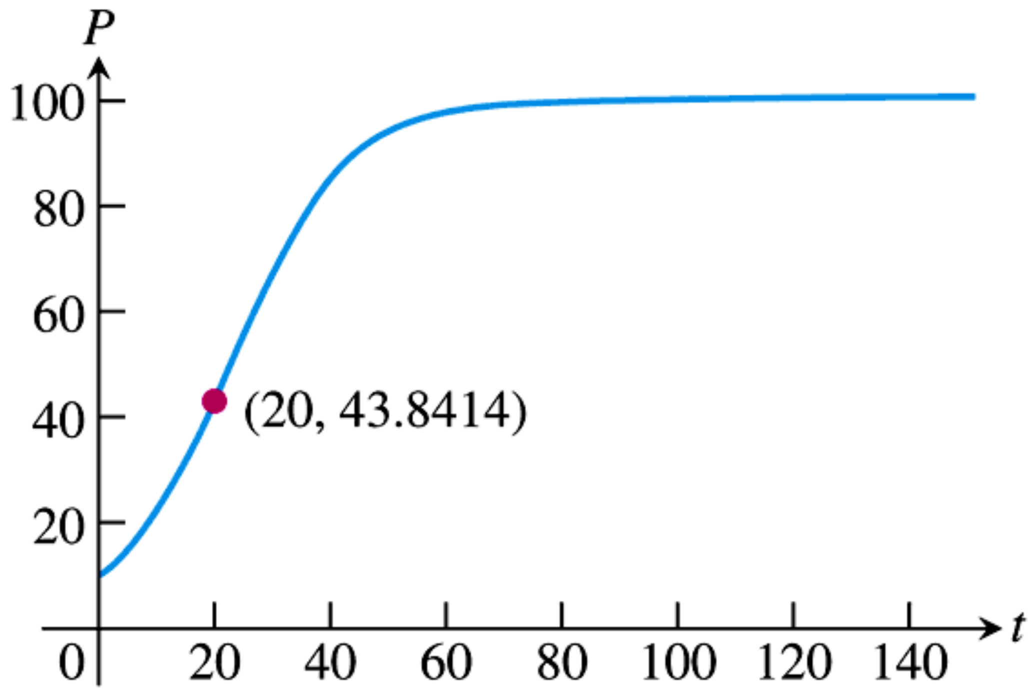

FIGURE 9.26 Euler approximations of the solution to $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$.

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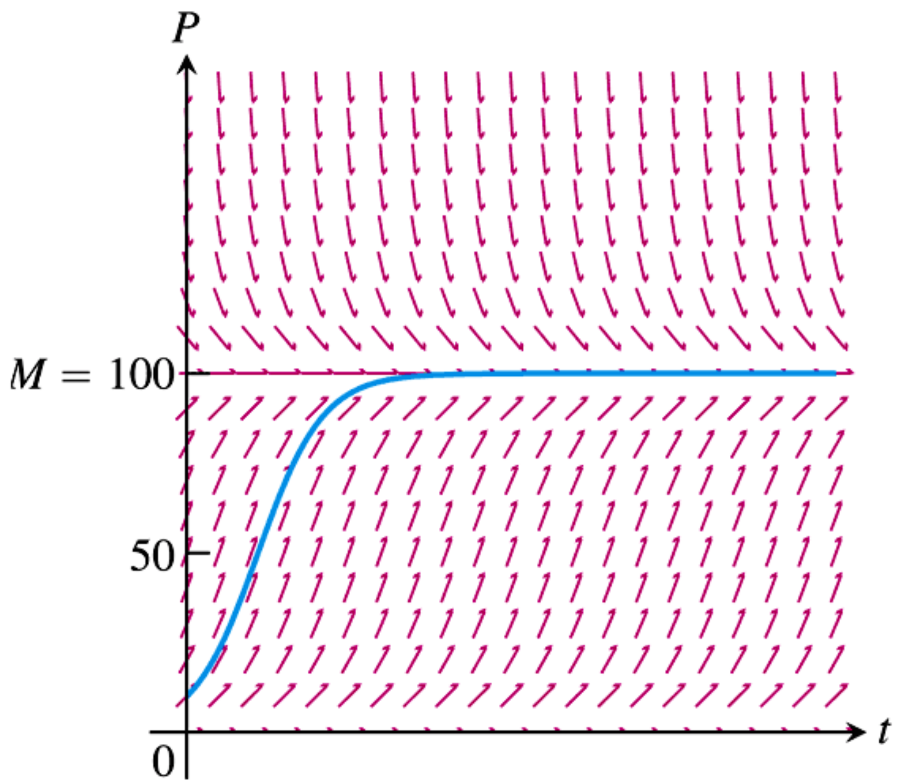

FIGURE 9.27 The graph of

$$P = \frac{100}{1 + 9e^{-0.1t}}$$

superimposed on the slope field in Figure 9.25 (Example 2).

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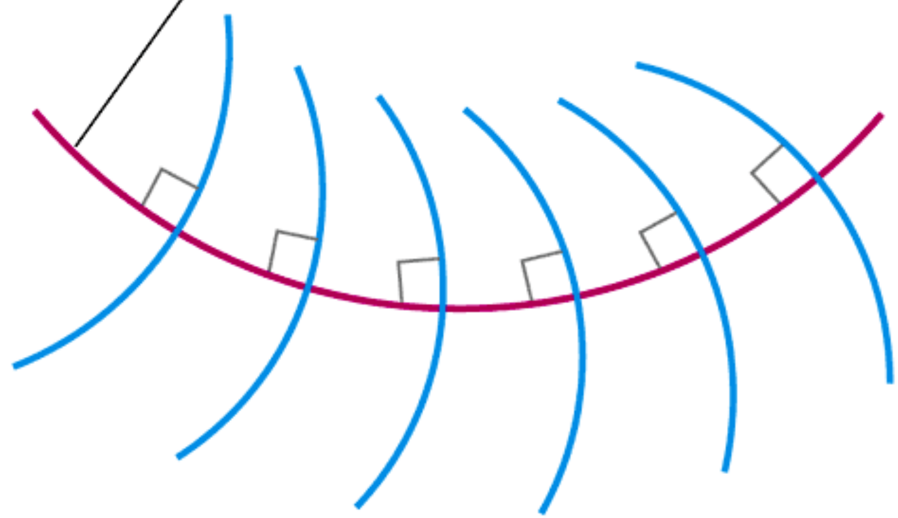

FIGURE 9.28 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

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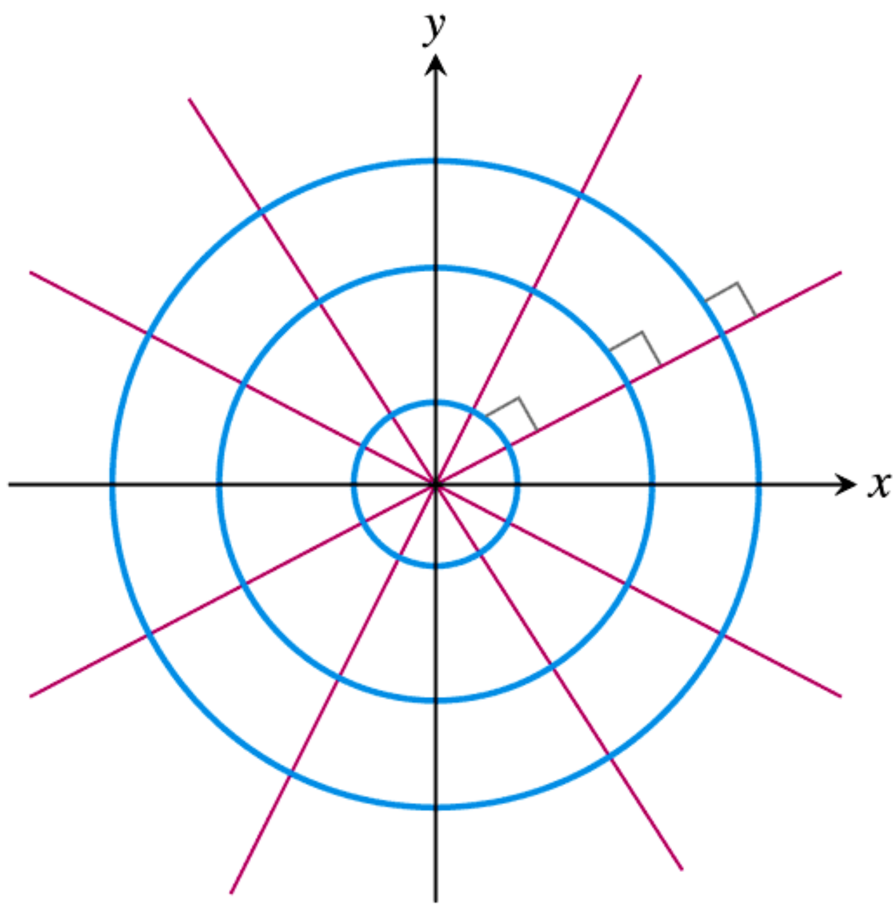

FIGURE 9.29 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

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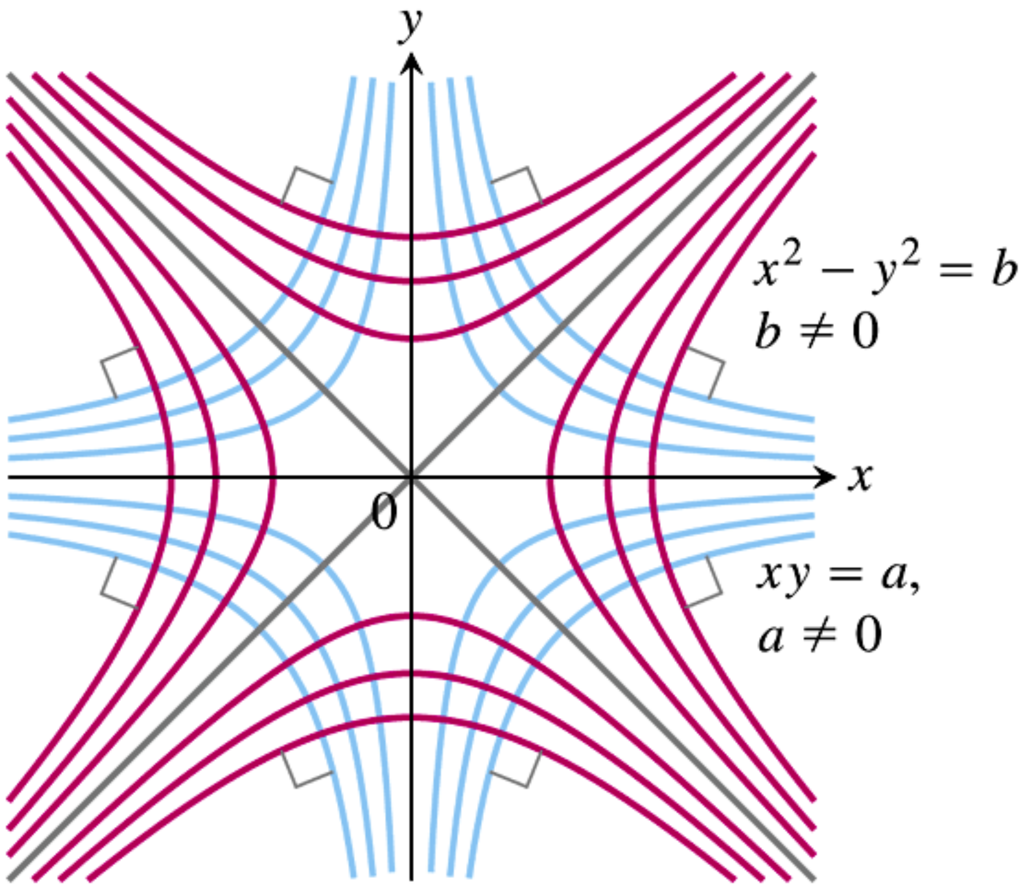

FIGURE 9.30 Each curve is orthogonal to every curve it meets in the other family (Example 3).

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