# SIMULATED SWAPTION DELTA–HEDGING IN THE LOGNORMAL FORWARD LIBOR MODEL

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ABSTRACT

Alternative approaches to hedging swaptions are explored and tested by simulation. Hedging methods implied by the Black swaption formula are compared with a lognormal forward LIBOR model approach encompassing all the relevant forward rates. The simulation is undertaken within the LIBOR model framework for a range of swaptions and volatility structures. Despite incompatibilities with the model assumptions, the Black method performs equally well as the LIBOR method, yielding very similar distributions for the hedging profit and loss — even at high rehedging frequencies. This result demonstrates the robustness of the Black hedging technique and implies that — being simpler and generally better understood by financial practitioners — it would be the preferred method in practice.

*Keywords*: term structure of interest rates, hedging, simulation, lognormal forward LIBOR model.

## 1. Introduction

In recent years a lot of interest has focused on models which take market quoted rates, such as forward LIBOR, as lognormal under the corresponding forward probability measures.[1] This type of term structure model gives a theoretical justification for the widespread market practice of analytically pricing cap and floor contracts using the Black formula. A lognormal forward LIBOR model can also be used to price swaptions, but in this case the pricing does not match the market standard Black swaption formula.[2] The differences go beyond the pricing equation — the two approaches also imply different methods for the hedging of swaptions. For example, hedging according to the assumptions of the Black swaption formula requires only two instruments. In the lognormal LIBOR model, the underlying swap rate of the same swaption will be dependent on $n$ forward LIBORs, and

[1] These models were first introduced and extensively studied in [1], [2], [3] and [4].

[2] The Black formula for swaptions is in turn supported by a model in which forward swap rates are assumed to follow lognormal dynamics (see [4]). However, forward LIBOR *and* swap rates cannot be taken to be lognormal simultaneously; i.e. the Black formula can only be consistently applied to either caps/floors or swaptions.

require $n+1$ hedging instruments. This is due to the fact that, contrary to the assumptions of the Black formula, the underlying swap rate does not follow Markovian dynamics; it is Markovian only when all $n$ forward LIBORs are taken as the state variable.

However, Brace et al. [5] argue that in practice the LIBOR model yields swaption prices which are close to those given by the Black formula,[3] so the question arises whether this *closeness* can be exploited in terms of hedging as well. Simulated swaption hedging using both the LIBOR and Black methods can provide an answer to this question. If the results show that Black hedging methods can be used to hedge LIBOR model swaptions, then this would reinforce the notion of the closeness of the models, while also simplifying the use and implementation of the LIBOR model. Conversely, if the results were to show that Black hedging is not effective, then this would imply that the LIBOR model, with its greater complexity, has something extra to offer practitioners in terms of swaption hedging.

Since aim of the study is to judge the effectiveness of hedging swaptions as prescribed by a lognormal forward LIBOR model *vis–à–vis* the simpler approach of hedging according to the Black formula, both hedging strategies are simulated within the framework of the LIBOR model. The simulated hedging will necessarily only be an approximation of the continuous, replicating, self-financing strategy stipulated by the model. Thus, either the replicating or the self-financing property is lost to some extent.[4] However, any conclusions drawn for hedging strategies in practice are not invalidated by this approximation, since such strategies must necessarily also be discrete. Furthermore, a high rehedging frequency in the simulated strategies can greatly reduce any errors from this source. For the hedging strategy prescribed by the Black formula there will be another source of error: the mismatch between the simulated model and the assumptions underlying the hedging strategy. If this error is small compared to the error resulting from discretisation, even at high rehedging frequencies, then it can be concluded that for any practical purposes the simpler Black–type hedging strategies are as effective as the considerably more complicated strategies which are strictly correct in a LIBOR model framework.

The format of the paper is as follows. Section 2 details swaption pricing and hedging using the Black and LIBOR approaches, section 3 explains how these are then simulated, while section 4 presents the results and section 5 concludes.

## 2. Model Background and Hedging Methods

This section gives the background to swap and swaption pricing in general, and for the Black and lognormal forward LIBOR approaches in more detail. Hedging methods for each approach are then presented. The section starts by introducing the basic notation and framework for the study.

Consider an equi-spaced tenor structure defined by

$$T_j = T_0 + j\delta \text{ for } j = 1, \dots, n.$$

Time $t$ values of zero coupon bonds expiring on the tenor dates are expressed as $P(t, T_j)$,

[3] Independently, Rebonato [6] arrives at a similar conclusion.
[4] Here it is chosen to run the portfolio strategy in a self-financing manner — implying that it will replicate only *on average.*

and the time $T$ forward price for a zero coupon bond maturing at $T_j \geq T$ is

$$F_T(t, T_j) = \frac{P(t, T_j)}{P(t, T)}. \tag{1}$$

The forward LIBOR $K(t, T_j)$ is defined as the simple forward interest rate between tenor dates $T_j$ and $T_{j+1}$, and is related to the zero coupon bonds by

$$K(t, T_j) = \frac{1}{\delta}\left(\frac{P(t, T_j)}{P(t, T_{j+1})} - 1\right). \tag{2}$$

Typically, the study considers a *forward payer swap* which is a contract to exchange cash flows based on fixed and floating interest rates respectively. The swap commences at time $T_0$ and the payments are made at each of the times $T_j$, where payments based on the LIBOR prevailing at the *previous* time step $K(T_{j-1}, T_{j-1})$ are swapped for payments based on the fixed rate $\kappa$. The time $t$ value of the contract $\mathrm{Pswap}(t)$ is hence the expected value of these cash flows discounted to time $t$, as in

$$\mathrm{Pswap}(t) = \mathbb{E}\left\{ \delta \sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)} \left(K(T_{j-1}, T_{j-1}) - \kappa\right) \middle| \mathcal{F}_t \right\}$$

where $\beta(t)$ represents the time $t$ value of the *bank account asset*, which accumulates at the risk free instantaneous interest rate, and the expectation is taken under the arbitrage-free spot probability measure $\mathbb{P}$.[5] This can then be shown to reduce to[6]

$$\mathrm{Pswap}(t) = \delta \sum_{j=1}^{n} P(t, T_j)\left(K(t, T_{j-1}) - \kappa\right). \tag{3}$$

The (forward) swap rate $\omega(t)$ is that value of the strike which gives the swap zero worth, and can be found by solving (3) to yield

$$\omega(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{j=1}^{n} P(t, T_j)} = \frac{\sum_{j=1}^{n} P(t, T_j) K(t, T_{j-1})}{\sum_{j=1}^{n} P(t, T_j)}, \tag{4}$$

which, upon substitution into (3), allows the value of a swap to be expressed as

$$\mathrm{Pswap}(t) = \delta \sum_{j=1}^{n} P(t, T_j)\left(\omega(t) - \kappa\right). \tag{5}$$

A swaption $\mathrm{Pswpn}(t)$ is simply an option on a swap, paying, at maturity $T_0$, the discounted positive value of the difference between the swap rate and the swaption strike,

$$\begin{aligned} \mathrm{Pswpn}(T_0) &= \mathrm{Pswap}(T_0)^+ \\ &= \delta \sum_{j=1}^{n} P(T_0, T_j)\left(\omega(T_0) - \kappa\right)^+. \end{aligned} \tag{6}$$

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[5] The arbitrage–free spot probability measure $\mathbb{P}$, or *risk neutral measure,* is the equivalent probability measure associated with taking the bank account asset as the *numeraire;* i.e. by definition any asset discounted by $\beta(\cdot)$ follows a martingale under $\mathbb{P}$. See [9] for a treatment of numeraire assets and associated martingale measures.

[6] Note that (3) is in fact a deterministic, model–independent relationship, and can be derived without the need for defining the bank account or the spot probability measure.

The time $t$ value of the swaption is then given by the discounted expectation of (6), as in

$$\mathrm{Pswpn}(t) = \mathbb{E}\left\{ \delta \frac{\beta(t)}{\beta(T_0)} \sum_{j=1}^{n} P(T_0, T_j) \left(\omega(T_0) - \kappa\right)^+ \middle| \mathcal{F}_t \right\} \tag{7}$$

or equivalently

$$\mathrm{Pswpn}(t) = \mathbb{E}\left\{ \delta \frac{\beta(t)}{\beta(T_0)} \sum_{j=1}^{n} P(T_0, T_j) \left(K(T_0, T_{j-1}) - \kappa\right) \mathbb{I}(A) \middle| \mathcal{F}_t \right\} \tag{8}$$

where $\mathbb{I}(\cdot)$ is the indicator function and $A$ the event that the swaption is exercised,

$$A = \{\omega(T_0) \geq \kappa\} = \{\mathrm{Pswap}(T_0) \geq 0\}$$

## 2.1. *Black–type Pricing and Hedging of Swaptions*

### 2.1.1. *Description*

If the forward swap rate is taken to be lognormal under the *forward swap measure*[7], swaptions are priced by a formula very similar to that of the Black/Scholes call option. Define the *present value of a basis point* of a swap (or *PVBP*) to represent the sum $\delta \sum_{j=1}^{n} P(t, T_j)$. The forward swap measure $\widetilde{\mathbb{P}}$ used in pricing Black swaptions is the one induced by choosing the *PVBP* as the numeraire asset. Under this probability measure, all assets discounted by the *PVBP* will be martingales.

Using $\sigma(t)$ as the (in general multifactor) swap rate volatility, and $\widetilde{W}(t)$ as Brownian motion under $\widetilde{\mathbb{P}}$, the swap rate is modelled by

$$d\omega = \omega\sigma(t) \cdot d\widetilde{W}(t),$$

and the swaption price is given by, from (7),

$$\begin{aligned} \mathrm{Pswpn}(t) &= \mathbb{E}\left\{ \delta \sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)} \left(\omega(T_0) - \kappa\right)^+ |\mathcal{F}_t \right\} \\ &= \delta \sum_{j=1}^{n} P(t, T_j) \widetilde{\mathbb{E}} \left\{ \left(\omega(T_0) - \kappa\right)^+ |\mathcal{F}_t \right\} \\ &= \delta \sum_{j=1}^{n} P(t, T_j) \left(\omega(t)\mathbf{N}(h_1) - \kappa\mathbf{N}(h_2)\right), \end{aligned} \tag{9}$$

where

$$\begin{aligned} h_1 &= \frac{\ln\left(\omega(t)/\kappa\right) + \frac{1}{2}\zeta}{\sqrt{\zeta}}, \ h_2 = h_1 - \sqrt{\zeta}, \\ \zeta &= \int_t^{T_0} |\sigma(s)|^2 \, ds, \end{aligned}$$

[7] See [4].

and $\mathbf{N}(\cdot)$ represents the cumulative standard normal distribution function.

Note that (9) can be expressed in several different ways, notably in terms of zero coupon bonds only, by substituting for $\omega(t)$ from (4), giving

$$\mathrm{Pswpn}(t) = \left(P(t,T_0) - P(t,T_n)\right)\mathbf{N}(h_1) - \delta\kappa\sum_{j=1}^{n} P(t,T_j)\mathbf{N}(h_2). \tag{10}$$

2.1.2. *Hedging Methods*

Below are presented three methods for hedging swaptions within the Black framework. Note that the methods are equivalent, independent of any assumptions on the term structure dynamics, since the hedging instruments used in any one method can be transformed into the hedging instruments used in any other method by a static portfolio argument. In short, it can be said that the methods are *statically equivalent*. It follows that in a numerical simulation all methods provide identical hedging performance, and they differ only in the manner that they would be implemented in practice.

**The Underlying Swap Method** Consider equation (9) for the swaption price. This can be rewritten as

$$\begin{aligned}
\mathrm{Pswpn}(t) &= \delta\sum_{j=1}^{n} P(t,T_j)\left(\omega(t)\mathbf{N}(h_1) - \kappa\mathbf{N}(h_2)\right) \\
&= \delta\sum_{j=1}^{n} P(t,T_j)(\omega(t)-\kappa)\mathbf{N}(h_1) - \delta\kappa\sum_{j=1}^{n} P(t,T_j)\left(\mathbf{N}(h_2)-\mathbf{N}(h_1)\right) \\
&= \mathbf{N}(h_1)\mathrm{Pswap}(t) - \delta\kappa\left(\mathbf{N}(h_2)-\mathbf{N}(h_1)\right)\sum_{j=1}^{n} P(t,T_j) \\
&= \mathbf{N}(h_1)\left(\mathrm{Pswap}(t) + \delta\kappa\sum_{j=1}^{n} P(t,T_j)\right) - \mathbf{N}(h_2)\delta\kappa\sum_{j=1}^{n} P(t,T_j).
\end{aligned} \tag{11}$$

Equation (11) expresses the price of the swaption explicitly as the sum of the underlying swap and a portfolio of zero coupon bonds (the *PVBP*). By virtue of the lognormality assumption on the quotient

$$\frac{\mathrm{Pswap} + \kappa PVBP}{PVBP} = \omega,$$

these two quantities become hedging instruments in analogy to Black/Scholes, and the hedge in terms of these quantities can be read directly from the equation. In this case, the hedge to replicate the swaption will consist of going long $\Delta = \mathbf{N}(h_1)$ units of the underlying swap, and going short $\kappa\left(\mathbf{N}(h_2) - \mathbf{N}(h_1)\right)$ units of the *PVBP*. As with Black/Scholes, the quantity $\Delta$ is called the *option delta* and represents the partial derivative of the option price with respect to the underlying asset.

**The Zero Coupon Bond Method** Swaptions can also be hedged using portfolios consisting purely of zero coupon bonds. In an analogous method to the one described by (11) above, take the zero coupon bond swaption equation (10)

$$\mathrm{Pswpn}(t) = \left(P(t,T_0) - P(t,T_n)\right)\mathbf{N}(h_1) - \delta\kappa\sum_{j=1}^{n} P(t,T_j)\mathbf{N}(h_2)$$

and note immediately that it yields two instruments to use for hedging, both portfolios of zero coupon bonds, the first being $P(t,T_0) - P(t,T_n)$ and the second $\delta\sum_{j=1}^{n} P(t,T_j)$.

**The Forward Swaps Method** Hedging methods involving portfolios of zero coupon bonds can be difficult to implement in practice, as true zero coupon bonds are not common in the market and tend to be more of a theoretical construct than a real physical asset. It is possible, however, to avoid the need for portfolios of zero coupon bonds altogether by using instead two forward swaps with different strikes.

For a swaption with strike $\kappa$, it is chosen to hedge with two forward swaps $\mathrm{Pswap}_1$ and $\mathrm{Pswap}_2$ (of strikes $\kappa_1$ and $\kappa_2$, with a tenor structure identical to the swap underlying the swaption)[8]. This leads to a swaption price equation as follows

$$\mathrm{Pswpn}(t) = \Delta_1\mathrm{Pswap}_1(t) - \Delta_2\mathrm{Pswap}_2(t) \tag{12}$$

with our hedge ratios $\Delta_1$ and $\Delta_2$ defined as

$$\begin{aligned}\Delta_1 &= \frac{\mathbf{N}(h_1)\kappa_2 - \mathbf{N}(h_2)\kappa}{\kappa_2 - \kappa_1},\\ \Delta_2 &= \frac{\mathbf{N}(h_1)\kappa_1 - \mathbf{N}(h_2)\kappa}{\kappa_2 - \kappa_1}.\end{aligned}$$

Equation (12) can be verified by simple substitution of the hedge ratio and swap equations to recover (9).

## 2.2. *LIBOR Model Swaptions*

### 2.2.1. *Description*

The lognormal forward LIBOR model, while yielding a cap/floor formula which matches the one used in the market, does not allow for a lognormality assumption on forward swap rates. Rather, there is no swaption formula available in closed form and the functional form of the approximations proposed by Brace et al. [2] and Brace [10] differs markedly from widespread market practice.

The LIBOR model swaption formula takes the form

$$\mathrm{Pswpn}(t) = \delta\sum_{j=1}^{n} P(t,T_j)\left(K(t,T_{j-1})\mathbf{N}(h_j) - \kappa\mathbf{N}(\bar{h}_j)\right) \tag{13}$$

where the formulae for $h$ and $\bar{h}$ depend on the level of approximation used.[9]

[8] Note that although the choice of the $\kappa_i$ is arbitrary, they should be chosen such that $|\kappa_1 - \kappa_2|$ is not too small to avoid large values of the hedge ratios $\Delta_i$.

[9] See appendix A.1 for details.

The approximation in (13) is two-fold, firstly with deterministic values being substituted for the stochastic drift, and, secondly, a rank $k$ (commonly rank one or two) approximation made to the so-called *swaption covariance matrix* defined by

$$\lambda = (\lambda_{ij}) = \left( \int_t^T \gamma(s, T_{i-1}) \cdot \gamma(s, T_{j-1}) ds \right) \tag{14}$$

where $\gamma(t,T)$ is the (in general multifactor) forward LIBOR volatility. The rank one approximation seems adequate in most cases, with a rank two version appropriate for the more extreme instances.

2.2.2. *LIBOR Model Hedging Method*

A simple Black/Scholes option has a single index of uncertainty — that of the stock price $S$. Since the Black/Scholes stock price dynamics are Markovian, this uncertainty can be hedged with two instruments — the underlying stock itself, and a risk free bank account asset (or, alternatively, a zero coupon bond maturing at option expiry).[10] Now consider a LIBOR model swaption where the underlying forward swap rate can be expressed in terms of $n$ forward LIBORs. This forward swap rate is *not* Markovian under any of the forward measures. Rather, when considering the $n$ forward LIBORs jointly (and thus the forward swap rate as well) under a measure forward to an arbitrary date in the tenor structure of the swap, the dynamics of the $n$–dimensional vector of rates is Markovian.[11] Thus $n+1$ instruments are required to hedge the swaption, with $n$ of these being the component LIBORs. Following the example of the hedging methods in the case of Black swaptions, the $PVBP$ can be used as the final hedging instrument. The next problem, however, is to determine the required hedge ratios into each of the hedging instruments.

In the Black/Scholes model, the hedge into the underlying stock is given by the partial derivative of the option price with respect to the stock price. Taking the same approach in the present case represents a non-trivial problem, since the swaption price given by (13) depends on the forward LIBORs in a manner that is not at all straightforward. The arguments $h_j$ of the cumulative distribution function are determined by the solution to a highly nonlinear fixed point problem. In addition, the discount factors $P(t,T_j)$ are also influenced by the level of the forward rates. Consequently, in the simulation study two methods are considered for determining the swaption hedge in the LIBOR model: the approximate hedge proposed by Brace et al. [5] and numerical evaluation of the partial derivative by difference quotient.

To obtain the former, Brace et al. make the additional simplifying assumption that the derivatives of the discount factors (w.r.t. the forward LIBORs) can be ignored. By writing the swaption price in terms of expectations under the relevant forward measures and differentiating inside the expectation, differentiation of the fixed point problem determining $h_j$ can be avoided and one obtains

$$\Delta_{j-1} \stackrel{def}{=} \frac{\partial \mathrm{Pswpn}(t)}{\partial K(t, T_{j-1})} = \delta P(t,T_j) \mathbf{N}(h_j) \tag{15}$$

[10] This is true even in the multifactor case; see [11].
[11] For a discussion of the Markov property of the lognormal forward LIBOR models, see [12].

where the $h_j$ are calculated as in the rank one swaption formula (see Appendix A.1).[12]

If the value of the swaption is then partitioned into separate hedges for each forward rate, with the rest invested in the $PVBP$, one obtains an approximate hedge for the swaption. Implementing this, one can express the LIBOR model swaption price as the sum of its hedges and hedging instruments, as was done for the Black swaption, giving

$$\mathrm{Pswpn}(t) = \delta \sum_{j=1}^{n} P(t, T_j)\mathbf{N}(h_j)K(t, T_{j-1}) + \theta \sum_{j=1}^{n} P(t, T_j) \tag{16}$$

where $\theta$ is chosen so that the equality holds.

Recalling (2), it is seen immediately that each term within the left-hand summation of (16) can be expressed as the difference of two zero coupon bonds, as in

$$\delta P(t, T_j)K(t, T_{j-1}) = P(t, T_{j-1}) - P(t, T_j)$$

leading to a hedging strategy for the LIBOR model swaption entirely expressed in terms of zero coupon bonds

$$\mathrm{Pswpn}(t) = \sum_{j=1}^{n} \mathbf{N}(h_j)\left(P(t, T_{j-1}) - P(t, T_j)\right) + \theta \sum_{j=1}^{n} P(t, T_j).$$

## 3. Simulation

The purpose of the simulation is to provide a basis for judging the relative effectiveness of the hedging methods described above, i.e. to answer the question whether there is a justification in favouring the more complicated approach to swaption hedging prescribed by a lognormal forward LIBOR model (LFM) over the simpler Black–type hedging. Since the answer to this question is expected to be negative, the LFM is favoured as much as possible by taking it to describe the "true" term structure dynamics underlying the simulation. This means that if LFM hedging is found not to substantially improve upon the effectiveness of the Black hedge, it can be concluded that the additional overhead that LFM hedging entails is not worthwhile.

In the simulation study, four types of hedges are considered:

1. a Black hedge based on the rank 1 swaption price,
2. the (rank 1) LFM hedging method as described in section 2.2,
3. a Black hedge based on the rank 2 swaption price,
4. and a hedge where the positions in the component zero coupon bonds are determined by numerical differentiation of the swaption formula.

[12] Note that the hedging method of Brace et al. assumes that the rank one approximation is adequate and one may therefore expect it to fail in more extreme volatility scenarios. However, in appendix A.2 the approach of Brace et al. is used to derive a hedge for the rank–$k$ swaption price without additional assumptions. Comparing the resulting hedge with (15) shows that the difference between the two will be negligible in all but the most pathological cases.

For the Black hedges, the hedge is constructed in the underlying forward swap and the *PVBP*.[13] One possibility of determining the *option delta*, i.e. the position in the underlying asset, is to imply the Black/Scholes–type volatility of the underlying swap rate from the price given by the approximate formula for the LFM swaption (13) and then construct the hedge as described in section 2.1. The second approach is due to Brace et al. [5], who derive a *delta* position for Black–type hedging in the underlying swap entirely within the LIBOR model framework, using a series of approximating assumptions:

$$\Delta = \frac{\sum_{j=1}^{n} P\left(t, T_j\right) K\left(t, T_{j-1}\right) \mathbf{N}\left(h_j\right) {,}_{j}}{\sum_{j=1}^{n} P\left(t, T_j\right) K\left(t, T_{j-1}\right) {,}_{j}} \tag{17}$$

where $h_j$ and $,_{j}$ are the terms used in the determination of the LIBOR model swaption price (cf. appendix A.1). The simulations show that both methods yield effectively identical hedging performance and, in the results presented below, the former method was used in the rank 2 case (e.g. column 3 of table 2), while the latter was used in the rank 1 case (e.g. column 1). To keep the hedge self-financing, profits and losses are accumulated in the position in the *PVBP*.

The remaining two hedging methods involve considerably more overhead, both computationally and in terms of their practical implementation. Firstly, the study simulates hedges as derived in section 2.2, which exploit the approximating assumptions made in deriving the swaption formula itself. Secondly, the study considers hedges in the component zero coupon bonds where the hedge ratios are calculated by numerical differentiation. This does not make any further use of approximating assumptions at all: assuming that the swaption formula is accurate, then by Ito's lemma the requisite position in each of the zero coupon bonds is given by the first derivative of the swaption price with respect to that bond. In either case, the difference in the value of the resulting portfolio to the current swaption price is then made up by a position in the numeraire asset. Thus the study collects gains and losses resulting from the (imperfect) hedging strategies in a particular asset depending on which probability measure is chosen for the simulation. However, choosing different measures for the simulations has no discernible effect on the results.

To simulate swaption hedging in the forward LIBOR model, stochastic differential equations for the underlying LIBORs are discretised from time $t = 0$ to swaption maturity. Swap rate and forward price information can then be extracted from the LIBORs to successively price and hedge the swaptions.

The simulation is undertaken for a range of swaptions using two volatility structures taken from [5].[14] The first is a single-factor homogeneous parameterisation to historic market data chosen to represent typical market conditions, while the second is a contrived two-factor volatility function designed to represent an extreme, pathological market situation.

### 3.1. *Simulation Routine*

The simulation is carried out on a set of discrete, equi-spaced times between time 0 and swaption maturity $T_0$, defined as $t_i = idt$, for $i = 0, \ldots, N$ and $t_N = Ndt = T_0$. The

[13] The specific Black method used is immaterial, since, as noted in Section 2.1, the methods are in fact statically equivalent.

[14] See Appendix B.

dynamics of the time $T_{j-1}$ forward LIBOR under the time $T_j$ forward measure

$$dK(t, T_{j-1}) = K(t, T_{j-1})\gamma(t, T_{j-1})dW_{T_j}(t) \tag{18}$$

are discretised using a Euler scheme for $\ln(K(t, T_{j-1}))$ i.e.

$$K(t + dt, T_{j-1}) = K(t, T_{j-1}) \exp\left(-\frac{1}{2}\left|\gamma(t, T_{j-1})\right|^2 dt + \gamma(t, T_{j-1})\Delta W_{T_j}(t)\right), \tag{19}$$

where the Brownian motion increments $\Delta W_{T_j}(t) \stackrel{def}{=} W_{T_j}(t + dt) - W_{T_j}(t)$ are normally distributed under the $\mathbb{P}_{T_j}$ forward measure with a mean of zero and a standard deviation of $\sqrt{dt}$. In the Monte Carlo simulation, the Brownian motion increments are generated under one chosen measure $\mathbb{P}_{T_j}$. For all other measures, we the relationship[15]

$$dW_{T_j}(t) = dW_{T_{j-1}}(t) + \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\gamma(t, T_{j-1})dt \tag{20}$$

as

$$\Delta W_{T_j}(t) = \Delta W_{T_{j-1}}(t) + \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\gamma(t, T_{j-1})\Delta t. \tag{21}$$

Iterative use of (19) and (21) starting from the initial LIBORs $K(0, T_j)$ yields sample paths for all requisite forward rates.[16] The use of a simple Euler scheme of this type does introduce some discretisation bias into the simulation, but in the present context this effect is negligible.

### 3.2. *Discounting*

Forward rates are simulated under the forward measure $\mathbb{P}_{T_0}$ corresponding to the maturity of the swaption. Under this measure the discounted time $t$ price of a payoff is given by

$$X_t = P(t, T_0)\mathbb{E}_{\mathbb{P}_{T_0}}\left\{\frac{X_T}{P(T, T_0)}\middle| \mathcal{F}_t\right\} \qquad t < T \leq T_0.$$

By simulating under this measure, one can work in units of the numeraire, namely, $P(\cdot, T_0)$. When expressed in these units, the zero coupon bonds required for the calculation of the hedges and hedging portfolios become time $T_0$ forward prices, which are obtained directly from the simulated LIBORs. The same applies for the simulation within the Black environment, where it is chosen to work in the measure induced by the $PVBP$. No explicit calculation of the $PVBP$ value (other than its initial value) is required, as all the relevant terms are expressed in terms of the $PVBP$. In this case, the discounted value is given by

$$X_t = \sum_{j=1}^{n} P(t, T_j)\mathbb{E}_{\widetilde{\mathbb{P}}}\left\{\frac{X_T}{\sum_{j=1}^{n} P(T, T_j)}\middle| \mathcal{F}_t\right\} \qquad t < T \leq T_1.$$

[15] See [3] for a derivation of the relationship between Brownian motions under forward measures of different maturities.

[16] See also [13] for a more detailed discussion of this simulation algorithm.

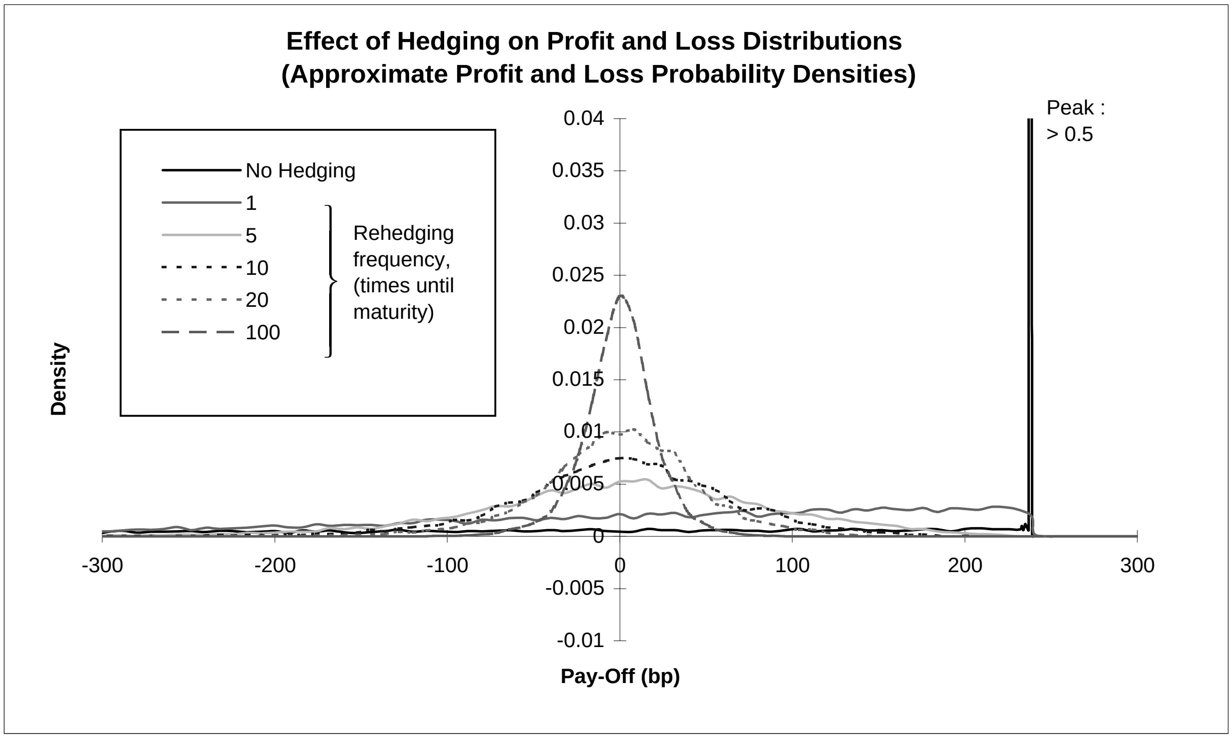



Figure 1: Estimated hedging P/L probability densities as a function of hedging frequency for a 2yr/4yr swaption simulated under the second volatility structure.

## 4. Results

The aim of the simulation study is to determine whether the improvement in hedging accuracy warrants the additional computational overhead associated with hedging swaptions as prescribed by the lognormal forward LIBOR model. As in any implementation of hedging strategies in practice, the hedge portfolio is adjusted at discrete points in time. Thus there will be a hedging error even if all other model assumptions are fulfilled, as the hedge portfolio would need to be adjusted continually in order to achieve perfect self-financing replication of the option. In the absence of any other sources of misspecification, however, this error can be made arbitrarily small by increasing the hedging frequency. Conversely, other sources of misspecification, such as the additional approximating assumptions implicit in applying Black–type hedging methods for swaptions in a lognormal forward LIBOR framework, are effectively irrelevant if the error due to discretization of the hedging strategy dominates for all practically feasible hedging frequencies.

To obtain a measure for the hedging error, 10,000 paths of the forward LIBORs are simulated. For each path, the discrete hedging strategy is calculated and any profit or loss (P/L) accumulated in the chosen numeraire. At maturity, the terminal payoff is subtracted from the value of the portfolio to yield the total profit or loss. From the 10,000 P/L realizations, the shape of the P/L distribution is inferred, in particular its mean and variance. The latter serves to characterize the hedging accuracy, i.e. a lower P/L variance signals a more effective hedge.

| Rebalancing Frequency (Times Until Maturity) | Hedging Profit and Loss (bp) | |
|---|---|---|
| | Mean (95% Error) | Standard Deviation (95% Error) |
| None | -6.7 (7.8) | 392.1 (5.5) |
| 1 | -4.0 (3.7) | 183.2 (2.6) |
| 5 | -3.6 (1.8) | 88.4 (1.2) |
| 10 | -4.1 (1.3) | 63.4 (0.9) |
| 20 | -4.5 (0.9) | 46.0 (0.6) |
| 100 | -4.2 (0.4) | 20.6 (0.3) |

Table 1: Hedging P/L means and standard deviations (with error bounds) as a function of hedging frequency, for the *at–the–money* 2yr/4yr swaption simulated under the second volatility structure.

### 4.1. *Hedging Distributions*

As an example of how the effectiveness of hedging LIBOR model swaptions increases as the hedge portfolio is adjusted more frequently, consider figure 1. It shows estimated probability distributions at different hedging frequencies for the hedging P/L of a 2yr/4yr swaption[17] simulated under the second (extreme) volatility structure, for a Black–type hedge calculated according to equation (17), using the rank 1 approximation to the covariance matrix. The hedging frequency $f$ is defined as the number of times the hedge portfolio is rebalanced throughout the life of the option. The large peak on the right-hand-side of the figure is associated with the case where the swaption is not hedged at all. This point mass corresponds to the occurrence (here approximately 53% of the time) when the swaption matures out-of-the-money and the option writer hence makes a profit equal to the initial (rank 1) swaption price of 237 basis points. The other 47% of the time a wide range of payoffs are possible, including some very large losses. In this case, the risk, which can be measured in terms of the standard deviation of all possible payoffs, will be quite large.[18] This standard deviation can be seen to decrease, however, as a result of the hedging, with its value progressively dropping for greater hedging frequencies. It is interesting to note that even at low hedging frequencies ($f \geq 5$), the hedging P/L distribution exhibits Gaussian characteristics. The means and standard deviations of the distributions are summarised in table 1. The mean P/L values do not appear to be centred around zero, implying that the hedges are producing a slight bias. This point is discussed further in section 4.3.

Equivalent results for the same swaption but with strike values away–from–the–money are presented in appendix C. These results are qualitatively very similar to those given

[17] In the following, European payer swaptions are considered. The first number denotes the option expiry and start of the underlying forward swap; the second number is the length of the swap.

[18] Note that these distributions are under an equivalent martingale measure. In an analogous situation, Wilmott [14] (Chapter 49) argues that the estimation of the payoff distribution of the unhedged case is not a valid indicator of the actual payoff distribution, as the unhedged position is not guaranteed to return at the same rate as the hedged position and the numeraire, since drifts are affected by the measure change from the "real–world" to a martingale measure. Nevertheless, the qualitative features of the distribution will remain unchanged. Furthermore, volatilities are not affected by an equivalent measure change, so that at higher hedging frequencies this distinction becomes negligible, disappearing in the limit.

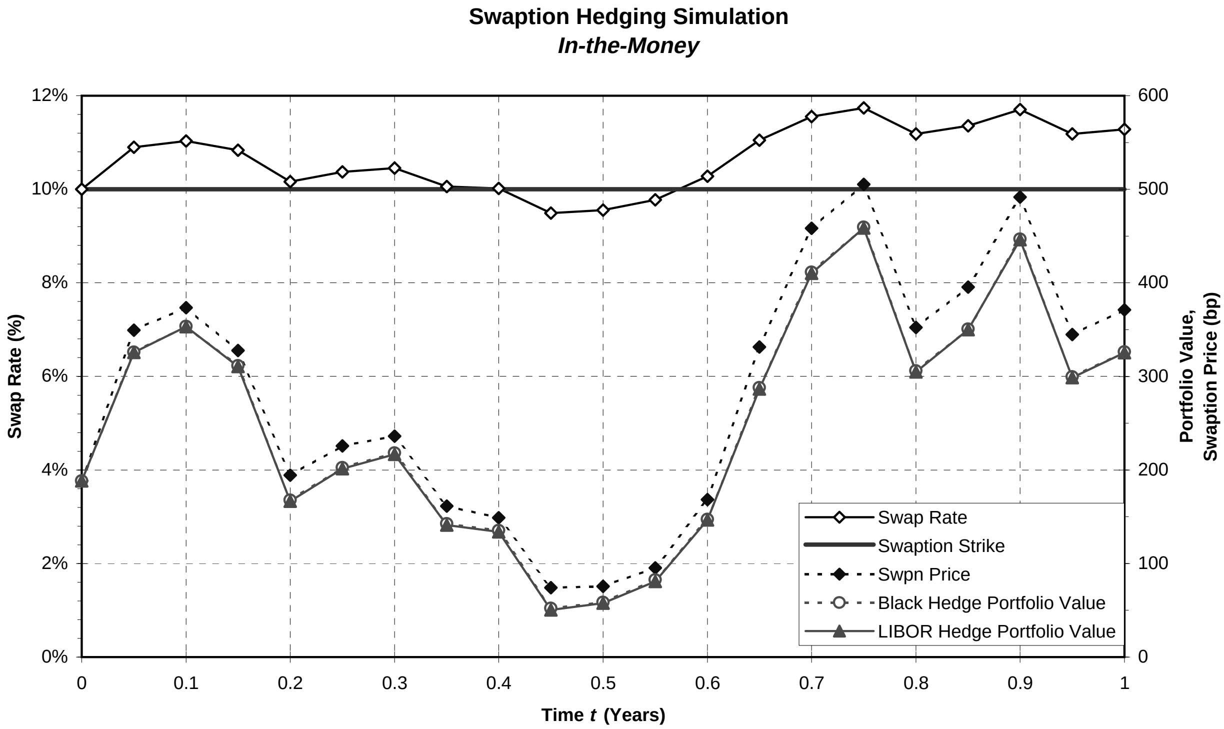



Figure 2: Hedging simulation path for an *at–the–money* 1yr/4yr swaption (under the second volatility structure) rehedged 20 times using the rank 1 Black and LIBOR approximate methods. Swap rate maturing *in–the–money.*

here. Overall the (rank 1) Black hedging method is able to effectively reduce the riskiness of LIBOR model swaptions, albeit at the cost of a slight bias in the average hedging P/L. Note that for the bulk of the results presented in subsequent sections, a hedging frequency of $f = 20$ has been chosen, giving an acceptable compromise between the quality of the hedging result and computational time.

## 4.2. *Hedging Method Comparison*

Figures 2 and 3 show examples of the profiles produced by swap rates, swaption prices and hedging portfolio values during the course of a single yield curve path realisation. Figure 2 depicts the case where the swaption matures *in–the–money,* while figure 3 is for the *out-of–the–money* case. Both cases are for an *at–the–money* 1yr/4yr swaption simulated under the second volatility structure and rehedged 20 times using the rank 1 Black and LIBOR hedging methods. The swaption price and portfolio values are expressed in "forward units", that is, in terms of the numeraire zero coupon bond $P(\cdot, T_0)$.

At each rehedging point, the swap rate can be seen to have changed, inducing a change in the price of the swaption. The value of the hedge portfolios follows that of the swaption price, but not exactly. This discrepancy arises primarily from the fact that the hedge is discrete rather than continuous. It will be different for each yield curve realisation and is, indeed, what leads to the hedging P/L distribution seen in figure 1. The most important point to observe in figures 2 and 3, however, is that the lines representing the Black and LIBOR hedge portfolios are effectively indistinguishable – indicating that the portfolios

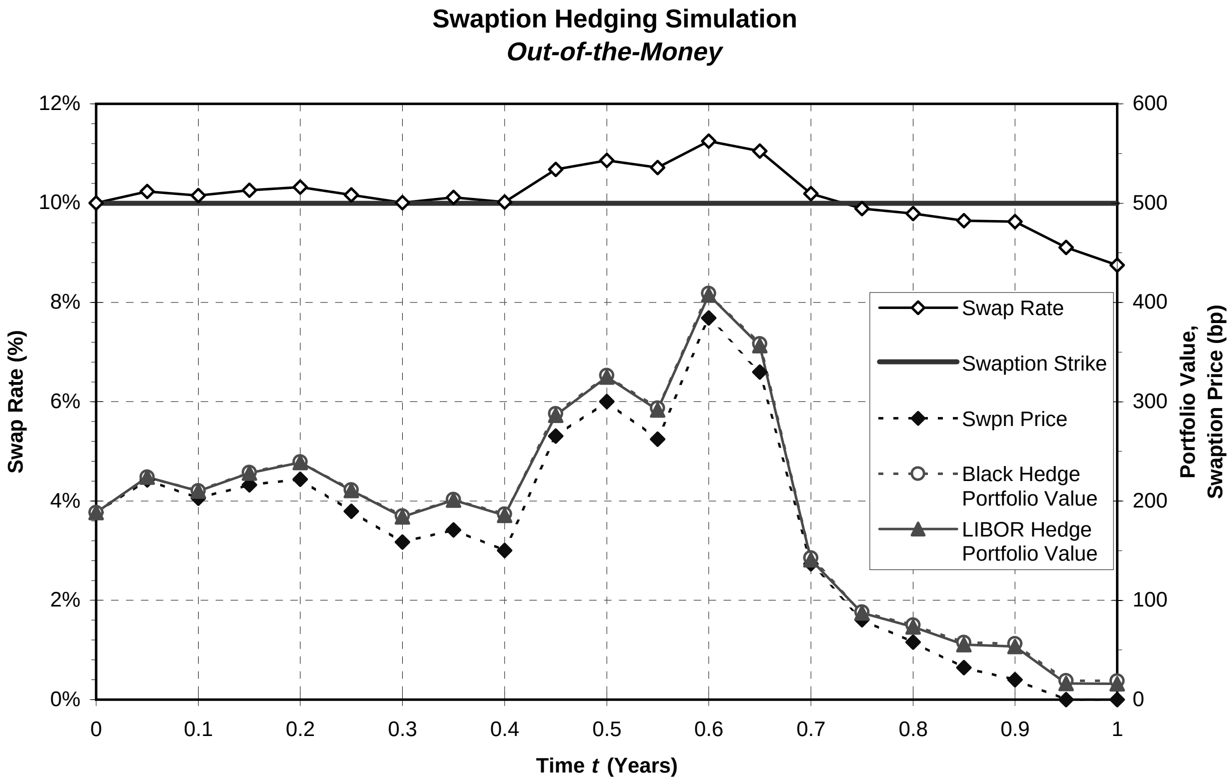



Figure 3: Hedging simulation path for an *at–the–money* 1yr/4yr swaption (under the second volatility structure) rehedged 20 times using the rank 1 Black and LIBOR approximate methods. Swap rate maturing *out–of–the–money.*

behave in a very similar manner and, further, that these hedge techniques are essentially equivalent.

This assertion is tested further by considering the means and standard deviations of the hedging P/L's based on 10,000 such path realisations, repeated for a range of swaptions at different strikes and under two volatility structures. The complete set of results is presented in appendix C, while table 2 shows an extract of the results for a sample of swaptions and a range of strikes for both volatility structures. For each swaption and strike value, the table lists the mean hedging P/L (with 95% error bounds) and standard deviations relative to the (rank 2) swaption price, for each of the four hedging methods. The last column represents the benchmark case, i.e. Black–type hedging of a swaption with value equal to the rank 2 price, but simulated in a lognormal forward *swap* rate model. Thus, the last column indicates the size of the deviation due to discretisation of the hedging strategy, since in this case this is the only source of hedging error. For each volatility structure, the results for the 0.25yr/2yr swaption are typical of the majority of the cases, while the remaining swaptions shown (i.e. the 1/8, 2/8 and 4/4 yr swaptions) represent more extreme cases. Looking at the means and standard deviations for the two rank 1 hedging methods, the values can be seen to be very close in each case. Comparing the standard deviations, there is no indication that the Black hedge is any "riskier" than the LIBOR hedge.

While the numerical results are not as close to each other as for the rank 1 case, the rank 2 hedging methods also appear to behave in a very like manner. It is also interesting

Table 2: Hedging P/L

| Volatility Structure | Swaption Maturity / Length (yrs) | Swaption Strike | **Black (Rank 1 Approx)** Mean P/L (95% Error) (bp) Relative P/L Std Dev$^n$ (%) | **LIBOR (Rank 1 Approx)** Mean P/L (95% Error) (bp) Relative P/L Std Dev$^n$ (%) | **Black (Rank 2 Calib$^n$)** Mean P/L (95% Error) (bp) Relative P/L Std Dev$^n$ (%) | **LIBOR (Rank 2 FD)** Mean P/L (95% Error) (bp) Relative P/L Std Dev$^n$ (%) | **Black Benchmark** Mean P/L (95% Error) (bp) Relative P/L Std Dev$^n$ (%) |
|---|---|---|---|---|---|---|---|
| 1 | 0.25/2 | IN | 0.00 (0.06) / 8.14% | 0.00 (0.06) / 8.14% | 0.00 (0.06) / 8.14% | 0.00 (0.06) / 8.14% | 0.00 (0.06) / 8.16% |
| | | AT | 0.01 (0.07) / 18.96% | 0.01 (0.07) / 18.96% | 0.01 (0.07) / 18.96% | 0.01 (0.07) / 18.96% | 0.01 (0.07) / 18.96% |
| | | OUT | 0.02 (0.06) / 44.13% | 0.02 (0.06) / 44.13% | 0.02 (0.06) / 44.13% | 0.02 (0.06) / 44.13% | 0.02 (0.06) / 44.14% |
| | 1/8 | IN | -0.10 (0.86) / 8.52% | -0.10 (0.86) / 8.51% | -0.04 (0.86) / 8.53% | -0.06 (0.86) / 8.51% | 0.04 (0.87) / 8.55% |
| | | AT | -0.08 (1.03) / 18.66% | -0.08 (1.03) / 18.62% | 0.00 (1.03) / 18.67% | -0.03 (1.03) / 18.62% | -0.04 (1.04) / 18.66% |
| | | OUT | 0.30 (0.92) / 48.05% | 0.30 (0.92) / 47.96% | 0.36 (0.92) / 48.08% | 0.35 (0.92) / 47.96% | 0.09 (0.94) / 48.70% |
| | 2/8 | IN | 0.16 (1.19) / 8.94% | 0.17 (1.19) / 8.92% | 0.33 (1.19) / 8.95% | 0.33 (1.19) / 8.91% | 0.68 (1.19) / 8.88% |
| | | AT | -0.14 (1.41) / 18.47% | -0.13 (1.41) / 18.42% | 0.06 (1.42) / 18.49% | 0.06 (1.41) / 18.41% | 0.35 (1.43) / 18.49% |
| | | OUT | -0.62 (1.34) / 51.98% | -0.61 (1.34) / 51.81% | -0.49 (1.34) / 52.06% | -0.50 (1.34) / 51.84% | -0.38 (1.38) / 53.34% |
| | 4/4 | IN | -1.11 (0.94) / 9.87% | -1.10 (0.94) / 9.87% | -1.01 (0.94) / 9.87% | -0.99 (0.94) / 9.86% | -0.24 (0.93) / 9.77% |
| | | AT | -0.85 (1.15) / 19.28% | -0.82 (1.16) / 19.30% | -0.69 (1.16) / 19.30% | -0.69 (1.15) / 19.28% | 0.14 (1.15) / 19.23% |
| | | OUT | -1.41 (1.08) / 57.37% | -1.38 (1.09) / 57.44% | -1.25 (1.09) / 57.42% | -1.25 (1.08) / 57.38% | -0.83 (1.10) / 58.26% |
| 2 | 0.25/2 | IN | -0.12 (0.21) / 8.44% | -0.11 (0.20) / 8.44% | 0.01 (0.21) / 8.45% | 0.02 (0.20) / 8.44% | 0.02 (0.21) / 8.49% |
| | | AT | -0.16 (0.24) / 19.31% | -0.16 (0.24) / 19.30% | -0.01 (0.24) / 19.32% | -0.01 (0.24) / 19.30% | -0.01 (0.24) / 19.30% |
| | | OUT | -0.19 (0.21) / 47.32% | -0.19 (0.21) / 47.28% | -0.09 (0.21) / 47.32% | -0.08 (0.21) / 47.28% | -0.10 (0.21) / 47.26% |
| | 1/8 | IN | -12.57 (1.12) / 9.31% | -12.56 (1.12) / 9.29% | 0.80 (1.10) / 9.18% | 0.80 (1.10) / 9.18% | 0.20 (1.05) / 8.87% |
| | | AT | -16.22 (1.29) / 18.99% | -16.21 (1.30) / 19.02% | -0.53 (1.28) / 18.77% | -0.53 (1.28) / 18.76% | -0.44 (1.24) / 18.84% |
| | | OUT | -12.90 (1.20) / 49.00% | -12.90 (1.20) / 49.28% | -1.17 (1.18) / 48.40% | -1.17 (1.18) / 48.39% | -0.47 (1.15) / 51.14% |
| | 2/8 | IN | -9.83 (1.40) / 9.55% | -9.80 (1.39) / 9.54% | 1.68 (1.38) / 9.45% | 1.71 (1.38) / 9.43% | 0.46 (1.30) / 9.10% |
| | | AT | -12.74 (1.67) / 18.89% | -12.73 (1.67) / 18.93% | 0.25 (1.66) / 18.75% | 0.30 (1.65) / 18.69% | 0.28 (1.59) / 18.75% |
| | | OUT | -10.55 (1.59) / 51.66% | -10.56 (1.60) / 51.96% | -1.16 (1.58) / 51.33% | -1.08 (1.57) / 51.15% | 0.04 (1.53) / 54.78% |
| | 4/4 | IN | -1.14 (0.87) / 9.96% | -1.15 (0.87) / 9.97% | -0.03 (0.87) / 9.97% | -0.03 (0.87) / 9.97% | -0.28 (0.85) / 9.88% |
| | | AT | -1.51 (1.05) / 18.78% | -1.53 (1.05) / 18.79% | -0.27 (1.05) / 18.80% | -0.27 (1.05) / 18.78% | -0.45 (1.05) / 18.92% |
| | | OUT | -1.00 (1.02) / 58.00% | -1.02 (1.03) / 58.03% | -0.16 (1.02) / 58.01% | -0.15 (1.02) / 57.93% | -0.01 (1.04) / 60.01% |

Hedging simulation results showing the average hedging P/L (with error bounds) and P/L standard deviation relative to swaption price for a range of swaptions, strike values, hedging methods and volatility structures.

to note that the standard deviations of these methods differ only slightly from that for the benchmark swaption, suggesting that, despite the differences between the two frameworks, a hedged LIBOR swaption has the same level of risk as an equivalently hedged Black swaption in its native framework. Overall, then, these results indicate that there is little to distinguish between the Black and LIBOR model hedging methods. Note that for each of the strike values, the hedging P/L standard deviation relative to the swaption price is essentially constant regardless of the maturity or length of the swaption. That is, the relative standard deviation is centred around 18% for all the *at–the–money swaptions,* while the values are around 9% and 50% for swaptions in and *out–of–the–money,* respectively. This means that, relative to their price, *out–of–the–money* swaptions are inherently riskier, and would require a greater hedging frequency than swaptions at or *in–the–money* in order to achieve a set level of acceptable risk (measured in terms of hedging P/L standard deviation). This suggests, quite intuitively, that most risk is due to the need to hedge the time value of the option, as opposed to its intrinsic value, and this risk becomes greater as the leverage of the option increases. Note that some swaptions show a mean hedging P/L which deviates significantly from the expected value of zero. This *hedging bias* is considered in the next section.

### 4.3. *Hedging Bias*

While the hedging P/L standard deviations show little variation between the different hedging methods, this is not always the case for the average of the P/L's. Mean P/L values lying away from the expected value of zero can be seen in table 2, deviations which cannot be attributed to chance since

- they are of greater magnitude than the corresponding 95% statistical error bounds; and

- the benchmark options show no such bias themselves[19].

However, the source of this bias is easily identified. Table 3 compares the hedging bias (with 95% error) from the rank 1 hedging methods to the calculated difference between the rank 1 and 2 swaption prices for all the *at–the–money* swaptions simulated under this volatility structure.[20] Not only are the larger biases associated with swaptions whose covariance matrices show rank 2 characteristics, but the size of the bias is closely related to the difference between the rank 1 and 2 prices. This suggests that, overall, the hedges have been successful, but money has been lost simply because the initial rank 1 price underestimated the true price. Hedging based on the rank 2 price, no matter what the hedging method, reduces these biases to insignificance. Indeed, this can be seen in the mean P/L figures in Table 2, and it is presented graphically in figures 4 and 5 which show the average P/L and error bounds relative to the swaption price for all *at–the–money* swaptions considered under both volatility structures. Figure 4 shows the rank 1 results (here for Black–style hedging, which, from table 2, are very similar to the results of the rank 1 LIBOR hedging), where it is clear that large biases are occurring for highly rank 2 swaptions, as measured by the ratio of the second and first eigenvalues of the swaption

[19] Indeed, it is comforting to note that all the benchmark swaptions show no statistical hedging bias.

[20] Results are very similar for *away–from–the–money* swaptions.

| Swaption Mat/Len (yrs) | Eigenvalue Ratio (%) | Difference R1/R2 Price (bp) | Black R1 Approx Mean Hedging P/L (95% Error) (bp) |
|---|---|---|---|
| 0.25/1 | 0.5% | 0.00 | -0.02 (0.14) |
| 0.25/2 | 4.4% | -0.15 | -0.16 (0.24) |
| 0.25/4 | 30.8% | -0.23 | -0.32 (0.38) |
| 0.25/8 | 20.5% | -10.39 | -10.47 (0.68) |
| 1/1 | 1.0% | -0.01 | 0.09 (0.28) |
| 1/2 | 6.7% | -0.24 | -0.03 (0.48) |
| 1/4 | 27.9% | -1.09 | -0.90 (0.71) |
| 1/8 | 13.0% | -15.73 | -16.22 (1.29) |
| 2/1 | 1.6% | -0.01 | 0.07 (0.39) |
| 2/2 | 8.2% | -0.04 | -0.22 (0.58) |
| 2/4 | 17.3% | -4.24 | -4.42 (0.93) |
| 2/8 | 7.9% | -13.07 | -12.74 (1.67) |
| 4/1 | 1.6% | 0.00 | -0.16 (0.31) |
| 4/2 | 4.8% | -0.06 | -0.31 (0.56) |
| 4/4 | 6.4% | -1.25 | -1.51 (1.05) |

Table 3: Comparison of rank 1 hedging biases against the calculated difference between rank 1 and 2 *at–the–money* swaption prices for the second volatility structure.

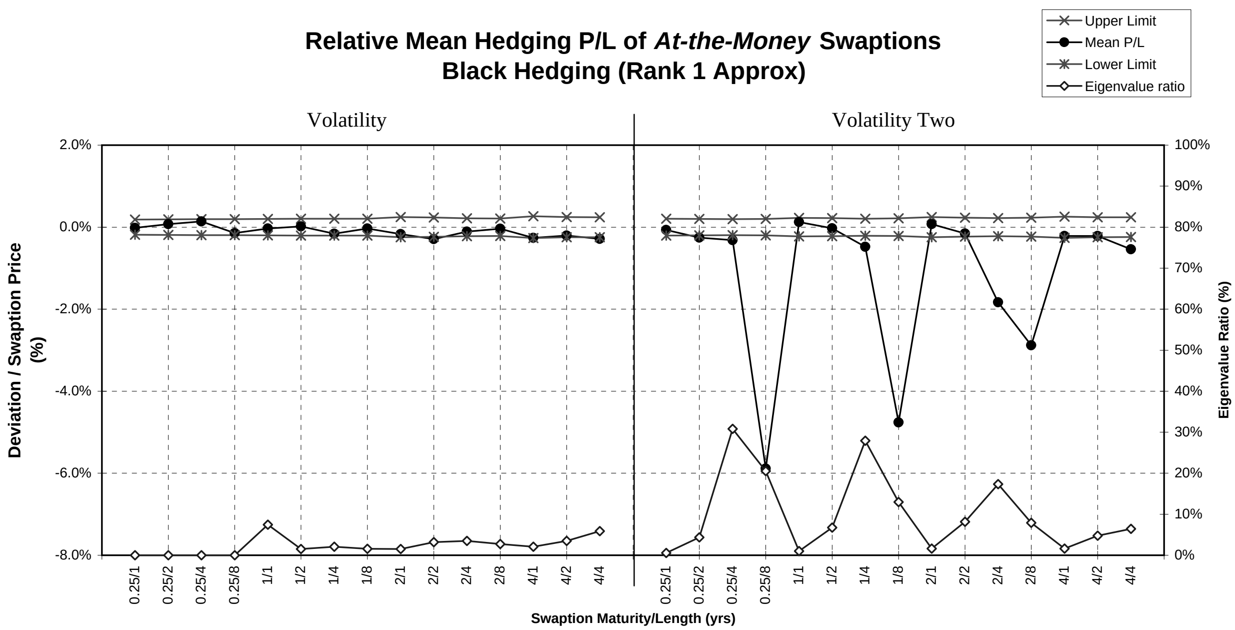



Figure 4: Rank 1 Black hedging P/L biases (and error bounds) relative to swaption price for both volatility structures.

covariance matrix. However, the size of the bias is not necessarily directly implied by the size of this ratio, with some very extreme ratios inducing only small biases. Figure 5 shows the case when hedging based on the initial rank 2 price is employed — here the deviations can be seen to have been essentially eliminated.

These results clearly suggest that, in extreme cases, hedges based on rank 1 approximations can lead to biases, while these can be essentially avoided using rank 2 formulae. Even in these extreme cases, Black–style hedging appears just as effective as LIBOR model

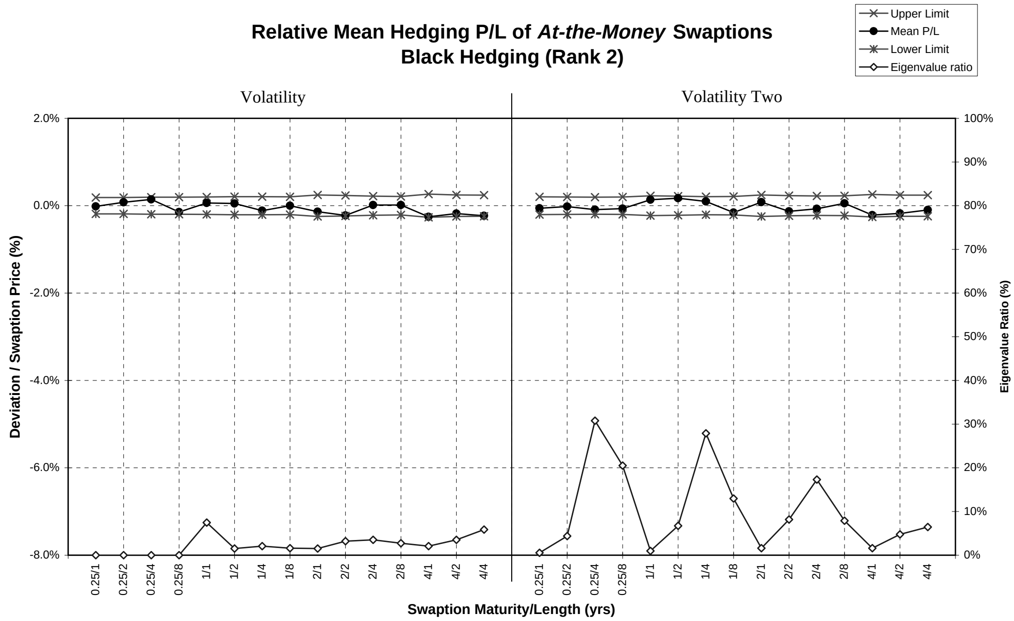



Figure 5: Rank 2 Black hedging P/L biases (and error bounds) relative to swaption price for both volatility structures.

hedging. The next section tests these conclusions at higher hedging frequencies.

## 4.4. *High Frequency Hedging*

The results presented up to this point all seem to indicate that Black and LIBOR model hedging techniques produce effectively identical results. However, at a hedging frequency of $f = 20$ the error due to discrete hedging is still quite substantial. It is worthwhile, therefore, to examine how these results hold up at greater hedging frequencies.

Such tests were carried out on a range of five *at–the–money* swaptions under the (extreme) second volatility structure, employing the Black and LIBOR rank 2 hedging techniques. Starting with a value of $f = 20$, simulations were performed with the hedging frequency successively doubled to an upper limit of $f = 640$. The results are summarized in figure 6.

The top line in this figure represents the ratio of the standard deviation of the Black hedge relative to that of the LIBOR hedge. On the whole, the ratio is close to unity, but for each swaption considered, the ratio shows an increasing trend with increasing hedging frequency. This means that as the hedging frequency increases, the gap between the two approaches widens. The effect is very small, however, with the greatest deviation (in the case of the highly rank two 1yr/8yr swaption rehedged 640 times) less than 1%.

The bottom portion of figure 6 shows the average P/L (with 95% error bounds) for both hedging methods. It is again apparent that the P/L distributions are very close, even at high hedging frequencies, and although the 95% bounds are touched on several occasions, both methods appear to be free of any hedging bias, as they are now based on the initial rank 2 price.

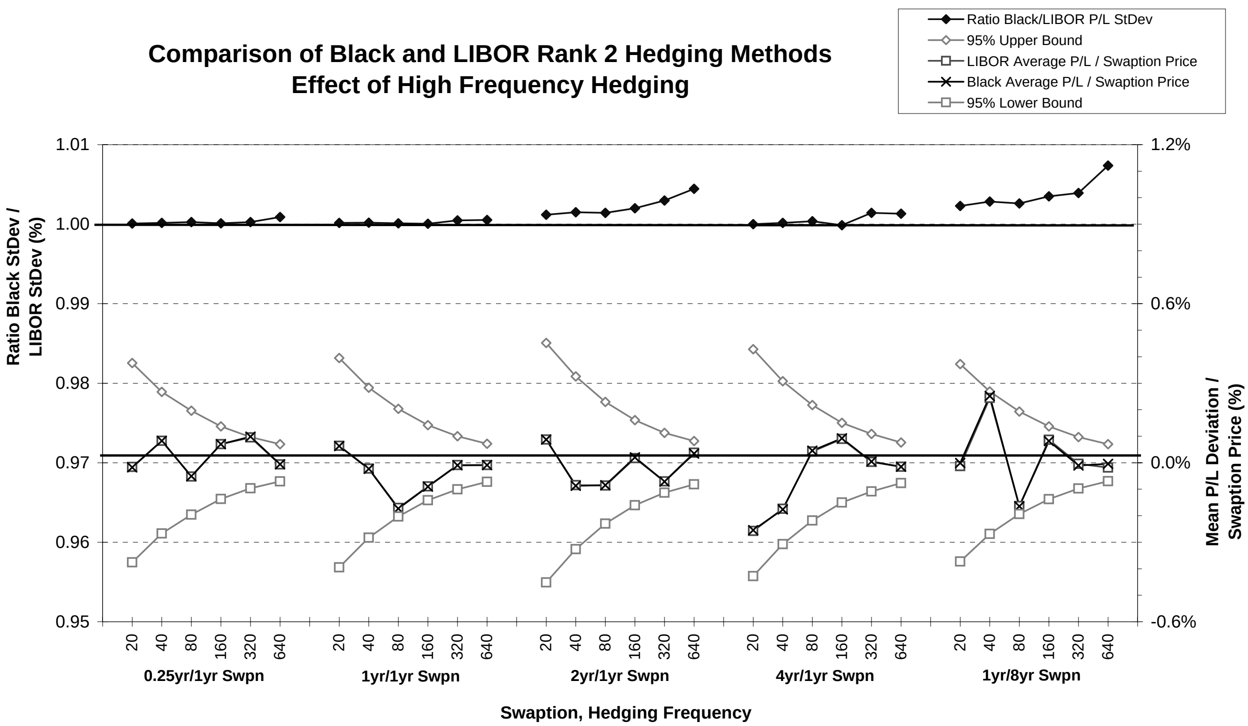



Figure 6: Effect of hedging frequency on Black and LIBOR hedging P/L.

## 5. Conclusion

The simulation study shows that the theoretical inconsistency of pricing and hedging swaptions as well as caps and floors using adaptations of the Black/Scholes formula is of no practical consequence. Any hedging error attributable to the approximations involved in Black–type hedging of swaptions in the context of a lognormal forward LIBOR model is orders of magnitude smaller than the error due to the necessity of implementing the hedging strategy in discrete time. The fact that the lognormal forward LIBOR framework is chosen to represent the "true" model in the study only reinforces this result, as hedging consistent with the assumptions of this model will add the most value in this case. In real–world applications, model misspecification is another source of hedging error which will further reduce the significance of the theoretical inconsistency.

However, particularly in more extreme volatility situations, it is important the volatility for pricing and hedging the swaption is correctly extracted from the forward LIBOR covariance matrix. As demonstrated in the simulations, using a rank 2 approximation to the covariance matrix effectively eliminates any bias in the profit and loss distribution. The practical consequences of this observation depend on one's perspective of the market; i.e. whether, when pricing swaptions, the covariance matrix is taken as given or whether one wishes to calibrate the covariance matrix to cap/floor *and* swaption prices. In the latter case, given the present results it may be convenient to restrict the covariance matrix to rank 1 or 2, as a higher rank may result in over–fitting.

**Acknowledgements**
The authors would like to thank Alan Brace and Marek Musiela for helpful discussions and claim responsibility for any remaining errors.

## Appendix A: Approximate Swaption Formulae in the LFM

### Appendix A.1. *Pricing*

In order to obtain a closed form solution for swaption prices in the lognormal forward LIBOR model, Brace et al. [2] make two simplifying assumptions. First, note that the forward LIBORs have a level dependent drift under all measures except the one to the end of the corresponding accumulation period (cf. eqns. (18) and (20)). If one performs a Wiener chaos expansion of order zero, i.e. replacing the level dependent drift by its (deterministic) initial value, the forward LIBORs are jointly lognormal under all forward measures. As shown in [15], this permits the approximate swaption price to be expressed as

$$\mathrm{Pswpn}(t) = \delta \sum_{j=1}^{n} P(t,T_j) \left\{ K(t,T_{j-1}) \mathbf{N}\left(h_j^{(k)}\right) - \kappa \mathbf{N}\left(\bar{h}_j^{(k)}\right) \right\} \tag{A.1}$$

where $k$ is the rank of the forward LIBOR covariance matrix $\lambda$ (as defined in eq. (14)) and

$$\begin{aligned} h_j^{(k)} &= -\frac{s_1 + d_1^{(j)} - \Gamma_{j,1} - \sum_{l=2}^{k} s_l \left(d_l^{(j)} - \Gamma_{j,l}\right)}{\sqrt{1 + \sum_{l=2}^{k} s_l^2}} \\ \bar{h}_j^{(k)} &= -\frac{s_1 + d_1^{(j)} - \sum_{l=2}^{k} s_l d_l^{(j)}}{\sqrt{1 + \sum_{l=2}^{k} s_l^2}} \end{aligned}$$

The $n \times k$ matrix $\Gamma$ is obtained by orthonormal decomposition of the forward LIBOR covariance matrix; i.e. let $V$ denote the $n \times k$ matrix of eigenvectors of $\lambda$ and $D$ the $k \times k$ diagonal matrix of the associated eigenvalues, then

$$\Gamma = V\sqrt{D}$$

Furthermore,

$$d_l^{(j)} = \sum_{i=1}^{j} \frac{\delta K(t,T_{i-1})}{1 + \delta K(t,T_{i-1})} \Gamma_{i,j}$$

and the $k$-dimensional vector $s$ determined in such a way that

$$\mathrm{Pswap}(T) = \delta \sum_{j=1}^{n} \frac{K(t,T_{j-1}) \exp\left\{\Gamma_{j,\cdot}(s + d^{(j)}) - \frac{1}{2}\|\Gamma_{j,\cdot}\|^2\right\} - \kappa}{\prod_{i=1}^{j} \left(1 + \delta K(t,T_{j-1}) \exp\left\{\Gamma_{j,\cdot}(s + d^{(j)}) - \frac{1}{2}\|\Gamma_{j,\cdot}\|^2\right\}\right)} = 0 \tag{A.2}$$

where $T$ is the option expiry.

The second simplifying assumption made in [2] is based on the observation that in most real–world situations, the largest eigenvalue of the covariance matrix dominates all further eigenvalues and thus a rank one approximation is sufficient. In particular, this reduces the fixed point problem (A.2) to one dimension. In those situations where a rank one approximation is not sufficiently accurate, such as the more extreme volatility scenario considered in the simulations, a rank two approximation should be used. As Brace [15] argues, the need to consider covariance matrices of higher rank should not arise.

## Appendix A.2. *Hedging*

The hedge is constructed in zero coupon bonds. Note, however, that it can be converted into a hedge based on forward instruments if at least one forward contract is available at two different levels, e.g. two forward swaps differing only in the contracted swap levels. If one selects the proportion of $P(t,T_j)$ in the hedge portfolio to match the first derivative

$$\frac{\partial}{\partial P(t,T_j)}\mathrm{Pswpn}(t)$$

then by Ito's Lemma the diffusion terms of Pswpn$(t)$ and the value process of the hedge portfolio coincide.

Let $\mathbb{E}_{T_j}$ denote the expectation operator under the time $T_j$ forward measure. Taking up an idea in [5], if Pswpn$(t)$ is written in terms of the relevant expectations under the forward measures, one needs to evaluate, for $0 < i < n$,[21]

$$\begin{aligned}
&\frac{\partial}{\partial P(t,T_i)}\left(\delta\sum_{j=1}^{n}P(t,T_j)\mathbb{E}_{T_j}[(K(T,T_{j-1})-\kappa)\mathbb{I}(A)|\mathcal{F}_t]\right)\\
=\;&\delta\mathbb{E}_{T_i}[(K(T,T_{i-1})-\kappa)\mathbb{I}(A)|\mathcal{F}_t]\\
&+\delta P(t,T_i)\mathbb{E}_{T_i}\left[\left.\frac{\partial K(T,T_{i-1})}{\partial P(t,T_i)}\mathbb{I}(A)\right|\mathcal{F}_t\right]\\
&+\delta P(t,T_{i+1})\mathbb{E}_{T_{i+1}}\left[\left.\frac{\partial K(T,T_i)}{\partial P(t,T_i)}\mathbb{I}(A)\right|\mathcal{F}_t\right]\\
&+\delta\sum_{j=1}^{n}P(t,T_j)\mathbb{E}_{T_j}\left[\left.(K(T,T_{j-1})-\kappa)\frac{\partial\mathbb{I}(A)}{\partial P(t,T_i)}\right|\mathcal{F}_t\right]\\
=\;&\delta\mathbb{E}_{T_i}[(K(T,T_{i-1})-\kappa)\mathbb{I}(A)|\mathcal{F}_t]\\
&-\frac{P(t,T_{i-1})}{P(t,T_i)}\mathbb{E}_{T_i}\left[\left.\frac{\partial K(T,T_{i-1})}{\partial K(t,T_{i-1})}\mathbb{I}(A)\right|\mathcal{F}_t\right]\\
&+\mathbb{E}_{T_{i+1}}\left[\left.\frac{\partial K(T,T_i)}{\partial K(t,T_i)}\mathbb{I}(A)\right|\mathcal{F}_t\right]\\
&+\delta\sum_{j=1}^{n}P(t,T_j)\mathbb{E}_{T}\left[\left.\frac{P(T,T_j)}{P(t,T_j)}P(t,T)(K(T,T_{j-1})-\kappa)\frac{\partial\mathbb{I}(A)}{\partial P(t,T_i)}\right|\mathcal{F}_t\right]
\end{aligned}$$

In turn, consider each of the four terms in the above sum:

$$\begin{aligned}
&\delta P(t,T)\sum_{j=1}^{n}\mathbb{E}_{T}\left[\left.P(T,T_j)(K(T,T_{j-1})-\kappa)\frac{\partial\mathbb{I}(A)}{\partial P(t,T_i)}\right|\mathcal{F}_t\right]\\
=\;&P(t,T)\mathbb{E}_{T}\left[\left.\mathrm{Pswap}(T)\frac{\partial\mathbb{I}(A)}{\partial P(t,T_i)}\right|\mathcal{F}_t\right]=0
\end{aligned}$$

since

$$\frac{\partial\mathbb{I}(A)}{\partial P(t,T_i)}=0\qquad\text{if }\mathrm{Pswap}(T)\neq 0$$

[21] Cases $i=0$ and $i=n$ see below.

as in [5], as well as

$$\begin{aligned}
&\mathbb{E}_{T_{i+1}}\left[\frac{\partial K(T,T_i)}{\partial K(t,T_i)}\mathbb{I}(A)\bigg|\mathcal{F}_t\right] \\
= \; &\mathbb{E}_{T_{i+1}}\left[\exp\left\{\int_t^T \gamma(s,T_i)dW_{T_{i+1}}(s) - \frac{1}{2}\int_t^T \|\gamma(s,T_i)\|^2 ds\right\}\mathbb{I}(A)\bigg|\mathcal{F}_t\right] \\
= \; &\mathbf{N}(h_{i+1}^{(k)})
\end{aligned}$$

Similarly,

$$\mathbb{E}_{T_i}\left[\frac{\partial K(T,T_{i-1})}{\partial K(t,T_{i-1})}\mathbb{I}(A)\bigg|\mathcal{F}_t\right] = \mathbf{N}(h_i^{(k)})$$

Lastly,

$$\begin{aligned}
&\delta\mathbb{E}_{T_i}[(K(T,T_{i-1})-\kappa)\mathbb{I}(A)|\mathcal{F}_t] \\
= \; &\delta(\mathbb{E}_{T_i}[K(T,T_{i-1})\mathbb{I}(A)|\mathcal{F}_t] - \kappa\mathbb{E}_{T_i}[\mathbb{I}(A)|\mathcal{F}_t]) \\
= \; &\delta(K(t,T_{i-1})\mathbf{N}(h_i^{(k)}) - \kappa\mathbf{N}(\bar{h}_i^{(k)}))
\end{aligned}$$

Thus for $0 < i < n$,

$$\begin{aligned}
\frac{\partial}{\partial P(t,T_i)}\mathrm{Pswpn}(t) &= \delta(K(t,T_{i-1})\mathbf{N}(h_i^{(k)}) - \kappa\mathbf{N}(\bar{h}_i^{(k)})) - \frac{P(t,T_{i-1})}{P(t,T_i)}\mathbf{N}(h_i^{(k)}) + \mathbf{N}(h_{i+1}^{(k)}) \\
&= \mathbf{N}(h_{i+1}^{(k)}) - \mathbf{N}(h_i^{(k)}) - \delta\kappa\mathbf{N}(\bar{h}_i^{(k)})
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial}{\partial P(t,T_0)}\mathrm{Pswpn}(t) &= \delta P(t,T_1)\mathbb{E}_{T_1}\left[\frac{\partial K(T,T_0)}{\partial P(t,T_0)}\mathbb{I}(A)\bigg|\mathcal{F}_t\right] \\
&= \mathbf{N}(h_1^{(k)}) \\
\frac{\partial}{\partial P(t,T_n)}\mathrm{Pswpn}(t) &= \delta\mathbb{E}_{T_n}[(K(T,T_{n-1})-\kappa)\mathbb{I}(A)|\mathcal{F}_t] \\
&\quad + \delta P(t,T_n)\mathbb{E}_{T_n}\left[\frac{\partial K(T,T_{n-1})}{\partial P(t,T_n)}\mathbb{I}(A)\bigg|\mathcal{F}_t\right] \\
&= \delta(K(t,T_{n-1})\mathbf{N}(h_n^{(k)}) - \kappa\mathbf{N}(\bar{h}_n^{(k)}) - \frac{P(t,T_{n-1})}{P(t,T_n)}\mathbf{N}(h_n^{(k)}) \\
&= -\mathbf{N}(h_n^{(k)}) - \delta\kappa\mathbf{N}(\bar{h}_n^{(k)})
\end{aligned}$$

Therefore, the sum of zero coupon bond hedges is

$$\begin{aligned}
\sum_{j=0}^{n} P(t,T_j)\frac{\partial}{\partial P(t,T_j)}\mathrm{Pswpn}(t) &= P(t,T_0)\mathbf{N}(h_1^{(k)}) \\
&\quad + \sum_{j=1}^{n-1} P(t,T_j)(\mathbf{N}(h_{j+1}^{(k)}) - \mathbf{N}(h_j^{(k)}) - \delta\kappa\mathbf{N}(\bar{h}_j^{(k)})) \\
&\quad + P(t,T_n)(-\mathbf{N}(h_n^{(k)}) - \delta\kappa\mathbf{N}(\bar{h}_n^{(k)})) \\
&= \delta\sum_{j=1}^{n} P(t,T_j)(K(t,T_{j-1})\mathbf{N}(h_j^{(k)}) - \kappa\mathbf{N}(\bar{h}_j^{(k)})) \\
&= \mathrm{Pswpn}(t)
\end{aligned}$$

Thus the value of the hedge portfolio at any point in time equals the value of the approximate swaption price and one has a hedge which replicates the desired price process.

Note that no additional assumptions to those made in deriving the price approximation are needed. In fact, the only approximation lies in the calculation of the exercise probability under different forward measures, as the calculations above are equally valid for swaption covariance matrices of any rank.

The hedge is, of course, imperfect in the sense that the strategy is not self-financing; i.e. one is paying for the inaccuracy of the pricing formula as one goes along. This is purely academic, however, since in practice inaccuracies from other sources will be several orders of magnitude greater.

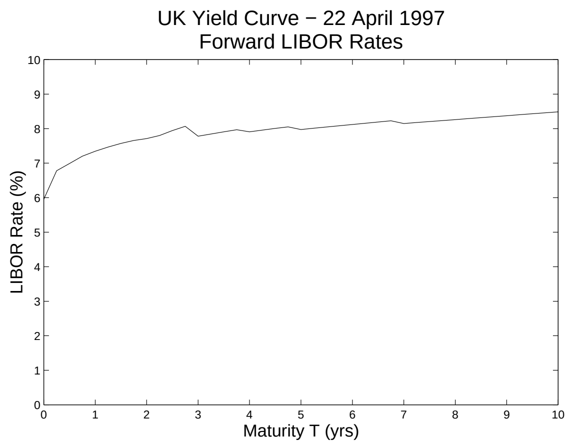
Note also the difference to the delta hedge derived in [5]: Taking the difference between that hedge and the approximate swaption price yields

$$\begin{aligned} & \delta\sum_{j=1}^{n} P(t,T_j)(K(t,T_j)\mathbf{N}(h_j) - \kappa\mathbf{N}(\bar{h}_j)) - \sum_{j=1}^{n} \delta P(t,T_j)\mathbf{N}(h_j)K(t,T_{j-1}) \\ = \quad & \delta\kappa\sum_{j=1}^{n} P(t,T_j)\mathbf{N}(\bar{h}_j) \end{aligned} \tag{A.3}$$

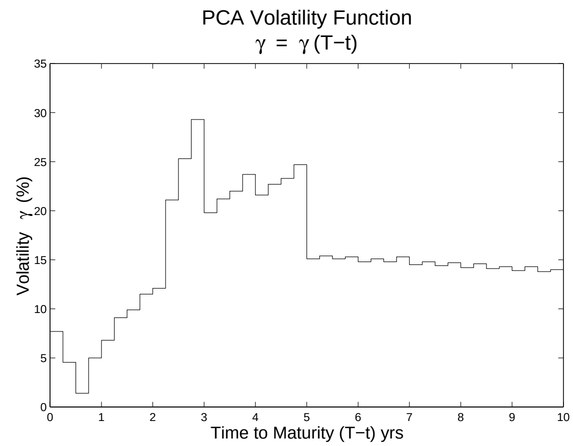
This term is not a multiple of the PVBP and thus [5] in effect proposes to approximate (A.3) by a position in the PVBP, thereby introducing an additional source of inaccuracy. This inaccuracy is quantifiable by the difference between the partial derivatives with respect to the zero coupon bond prices of the PVBP and the term (A.3). Again, this distinction is rather academic.

| Quote | Number | Maturity |
|---|---|---|
| LIBOR rates | 6 | 1 – 6 months |
| Eurodollar futures | 9 | ∼ 0 – 27months |
| Swap rates | 7 | 1, 2, 3, 4, 5, 7, 10 years |

Table B.1: Summary of the UK historical data obtained for the period July 1994 to September 1997.




(a) Yield curve




(b) Volatility function

Figure B.1: First volatility structure.

## Appendix B: Volatility Structures

### Appendix B.1. *First Volatility Structure*

The first volatility structure is a single-factor volatility function portraying typical (or *mild*) market conditions. The choice of a single-factor volatility is justified on the basis of the simplicity and apparent popularity of such functions, while authenticity is ensured by basing the volatility on historical market data.

Dun [16] analysed UK market data for the period of July 1994 to September 1997. The data are summarised in table B.1. Following [17], LIBOR volatility functions are extracted from an estimate of the joint quadratic variation of the yields. The volatility function used in the present paper was chosen to be equated to the vector associated with the dominant eigenvalue resulting from a PCA analysis on five months of UK data (from April 22 to September 29, 1997), representing a fairly typical period in the absence of major market upheavals. The initial yield curve for the simulations of this volatility scenario was selected as the one pertaining to the beginning of this data period and is shown in Figure B.1(a), while the volatility function appears in Figure B.1(b). Both the yield curve and volatility values are also summarised in Table B.2.

| Tenor (yrs) | LIBOR Rates (%) | PCA Volatility | Tenor (yrs) | LIBOR Rates (%) | PCA Volatility | Tenor (yrs) | LIBOR Rates (%) | PCA Volatility |
|---|---|---|---|---|---|---|---|---|
| 0 | 5.95% | - | 3.5 | 7.91% | 0.212 | 7 | 8.15% | 0.153 |
| 0.25 | 6.78% | 0.077 | 3.75 | 7.97% | 0.220 | 7.25 | 8.18% | 0.145 |
| 0.5 | 6.99% | 0.046 | 4 | 7.91% | 0.237 | 7.5 | 8.21% | 0.148 |
| 0.75 | 7.20% | 0.014 | 4.25 | 7.96% | 0.216 | 7.75 | 8.23% | 0.144 |
| 1 | 7.35% | 0.050 | 4.5 | 8.01% | 0.227 | 8 | 8.26% | 0.147 |
| 1.25 | 7.47% | 0.068 | 4.75 | 8.05% | 0.233 | 8.25 | 8.29% | 0.142 |
| 1.5 | 7.57% | 0.091 | 5 | 7.97% | 0.247 | 8.5 | 8.32% | 0.146 |
| 1.75 | 7.66% | 0.099 | 5.25 | 8.01% | 0.151 | 8.75 | 8.35% | 0.141 |
| 2 | 7.71% | 0.115 | 5.5 | 8.05% | 0.154 | 9 | 8.38% | 0.143 |
| 2.25 | 7.80% | 0.121 | 5.75 | 8.08% | 0.151 | 9.25 | 8.40% | 0.139 |
| 2.5 | 7.94% | 0.211 | 6 | 8.12% | 0.153 | 9.5 | 8.43% | 0.143 |
| 2.75 | 8.07% | 0.253 | 6.25 | 8.16% | 0.148 | 9.75 | 8.46% | 0.138 |
| 3 | 7.78% | 0.293 | 6.5 | 8.19% | 0.151 | 10 | 8.49% | 0.140 |
| 3.25 | 7.84% | 0.198 | 6.75 | 8.23% | 0.148 | | | |

Table B.2: Tabulated values of the initial yield curve in terms of LIBOR rates and the volatility function used for the first volatility structure.

## Appendix B.2. *Second Volatility Structure*

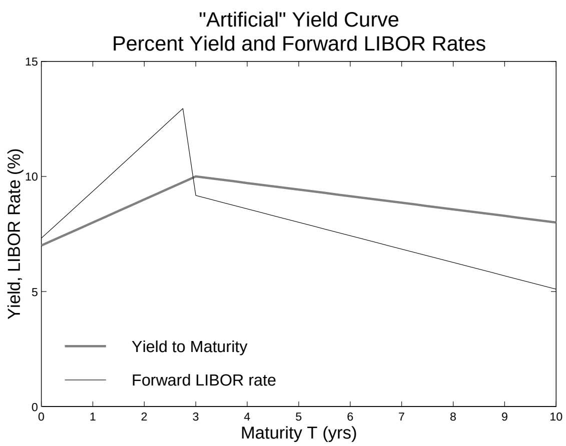
The second volatility is a contrived two-factor structure designed specifically to produce swaptions with highly rank 2 behaviour. The volatility selected is described by the vector function $\gamma(t,T)$ defined as

$$\gamma(t,T) = \begin{bmatrix} \gamma_1(T-t) \\ \gamma_2(T-t) \end{bmatrix}$$

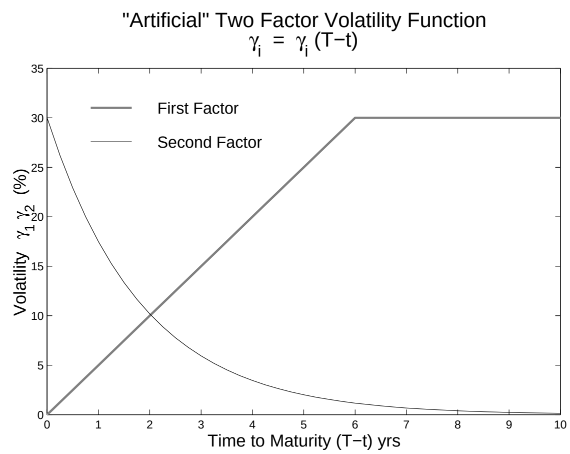
$$\gamma_1(T-t) = \begin{cases} 0.05(T-t), & (T-t) < 6 \text{ years} \\ 0.30, & (T-t) \geq 6 \text{ years} \end{cases}$$

$$\gamma_2(T-t) = 0.30e^{-0.54(T-t)}$$

and is shown graphically in Figure B.2(b).




(a) Yield curve




(b) Volatility function

Figure B.2: Second volatility structure.

The yield curve associated with this volatility structure was also intended to be fairly extreme, rising steeply from 7 to 10% in three years, and then slowly dropping for longer maturities, as described by the piece-wise linear relation

$$Y(T) = \begin{cases} 0.07 + \frac{0.03}{3}T, & T < 3 \text{ years} \\ 0.10 - \frac{0.02}{7}T, & T \geq 3 \text{ years} \end{cases}$$

where $Y(T)$ describes the continuously compounded yield on an investment maturing at time $T$. This yield to maturity (and its associated forward LIBOR curve) is plotted in Figure B.2(a), where the piece-wise linear nature of the yields can be seen to introduce a discontinuity in the LIBORs. This last feature is most likely a contributing factor to the extreme behaviour of some of the swaptions considered under this volatility structure.

## Appendix C: Additional Swaption Hedging Results

For completeness, this appendix contains additional simulation results, in particular for *in–the–money* and *out–of–the–money* swaptions. *In–the–money* and *out–of–the–money* are defined in terms of the swaption *delta*, i.e. the strikes for the swaptions are chosen such that the position in the underlying swap in the Black–type hedge is 25% (of the option's nominal) for *out–of–the–money* and 75% for *in–the–money*. The *at–the–money* strike rate matches the initial swap rate.

| Rebalancing Frequency (Times Until Maturity) | In-the-Money Hedging Profit and Loss (bp) | | Out-of-the-Money Hedging Profit and Loss (bp) | |
|---|---|---|---|---|
| | Mean (95% Error) | Standard Deviation (95% Error) | Mean (95% Error) | Standard Deviation (95% Error) |
| None | -5.5 (9.5) | 476.9 (6.7) | -2.8 (4.8) | 238.6 (3.4) |
| 1 | -3.2 (3.0) | 147.8 (2.1) | -4.2 (3.4) | 168.6 (2.4) |
| 5 | -2.9 (1.5) | 73.6 (1.0) | -3.5 (1.6) | 79.7 (1.1) |
| 10 | -3.1 (1.1) | 52.7 (0.7) | -3.6 (1.2) | 58.2 (0.8) |
| 20 | -3.1 (0.8) | 37.7 (0.5) | -3.9 (0.8) | 42.5 (0.6) |
| 100 | -3.2 (0.3) | 17.4 (0.2) | -3.8 (0.4) | 19.6 (0.3) |

Table C.1: Hedging P/L mean and standard deviations (with error bounds) as a function of hedging frequency, for *away–from–the–money* 2yr/4yr swaptions simulated under the second volatility structure.

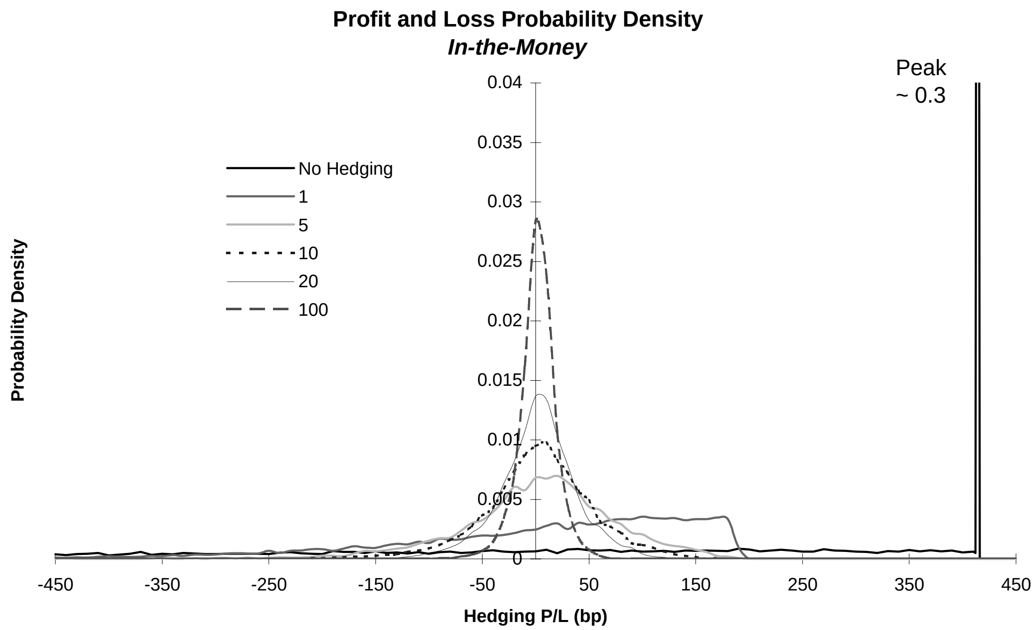



Figure C.1: Estimated hedging P/L probability densities as a function of hedging frequency for an *in–the–money* 2yr/4yr swaption simulated under the second volatility structure, hedged using the approximate Black (rank 1) method.

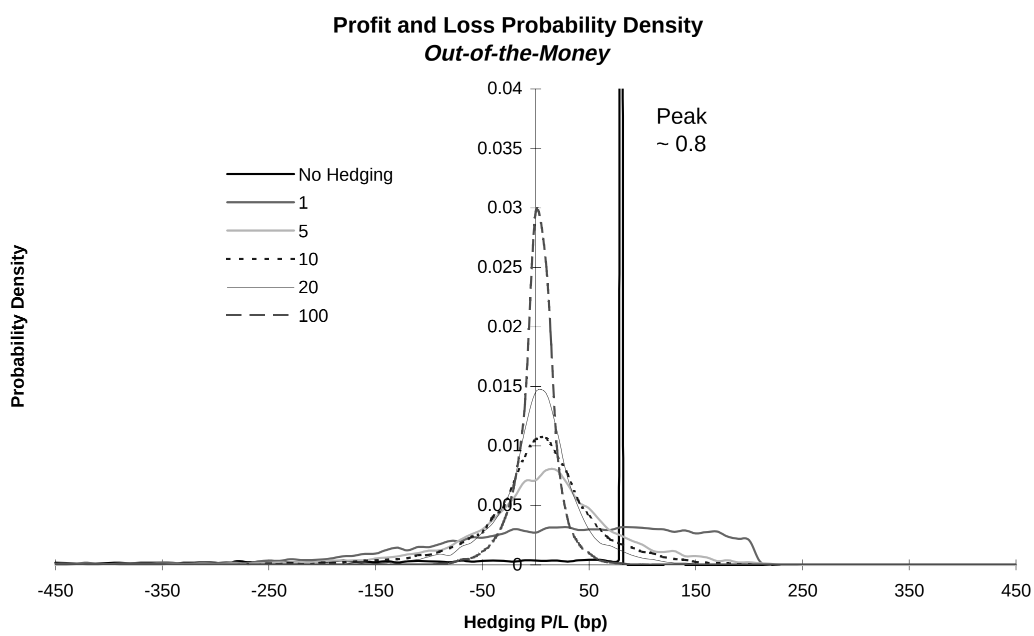



Figure C.2: Estimated hedging P/L probability densities as a function of hedging frequency for an *out–of–the–money* 2yr/4yr swaption simulated under the second volatility structure, hedged using the approximate Black (rank 1) method.

| Swaption Mat (yrs) | Swaption Strike | Description | Swaption Length (yrs) | | | |
|---|---|---|---|---|---|---|
| | | | 1 | 2 | 4 | 8 |
| 0.25 | IN | Black (Rank 1 Approx) | -0.01 (0.02) / 7.91% | 0.00 (0.06) / 8.14% | 0.14 (0.23) / 8.07% | -0.02 (0.45) / 8.26% |
| | | LIBOR (Rank 1 Approx) | -0.01 (0.02) / 7.91% | 0.00 (0.06) / 8.14% | 0.14 (0.23) / 8.07% | -0.02 (0.45) / 8.26% |
| | | Black (Rank 2 Calib$^n$) | -0.01 (0.02) / 7.91% | 0.00 (0.06) / 8.14% | 0.14 (0.23) / 8.08% | -0.02 (0.45) / 8.26% |
| | | LIBOR (Rank 2 FD) | -0.01 (0.02) / 7.91% | 0.00 (0.06) / 8.14% | 0.14 (0.23) / 8.07% | -0.02 (0.45) / 8.26% |
| | | Benchmark Swaption | -0.01 (0.02) / 7.93% | 0.00 (0.06) / 8.16% | 0.14 (0.23) / 8.14% | 0.00 (0.46) / 8.32% |
| | AT | Black (Rank 1 Approx) | 0.00 (0.02) / 18.81% | 0.01 (0.07) / 18.96% | 0.10 (0.28) / 19.22% | -0.20 (0.53) / 18.94% |
| | | LIBOR (Rank 1 Approx) | 0.00 (0.02) / 18.81% | 0.01 (0.07) / 18.96% | 0.10 (0.28) / 19.21% | -0.20 (0.53) / 18.93% |
| | | Black (Rank 2 Calib$^n$) | 0.00 (0.02) / 18.81% | 0.01 (0.07) / 18.96% | 0.10 (0.28) / 19.23% | -0.20 (0.53) / 18.95% |
| | | LIBOR (Rank 2 FD) | 0.00 (0.02) / 18.81% | 0.01 (0.07) / 18.96% | 0.10 (0.28) / 19.21% | -0.20 (0.53) / 18.94% |
| | | Benchmark Swaption | 0.00 (0.02) / 18.81% | 0.01 (0.07) / 18.96% | 0.10 (0.28) / 19.22% | -0.22 (0.53) / 18.96% |
| | OUT | Black (Rank 1 Approx) | 0.00 (0.02) / 44.23% | 0.02 (0.06) / 44.13% | 0.01 (0.24) / 46.17% | -0.37 (0.46) / 45.84% |
| | | LIBOR (Rank 1 Approx) | 0.00 (0.02) / 44.23% | 0.02 (0.06) / 44.13% | 0.01 (0.24) / 46.15% | -0.37 (0.46) / 45.82% |
| | | Black (Rank 2 Calib$^n$) | 0.00 (0.02) / 44.24% | 0.02 (0.06) / 44.13% | 0.01 (0.24) / 46.19% | -0.37 (0.46) / 45.86% |
| | | LIBOR (Rank 2 FD) | 0.00 (0.02) / 44.23% | 0.02 (0.06) / 44.13% | 0.02 (0.24) / 46.15% | -0.37 (0.46) / 45.82% |
| | | Benchmark Swaption | 0.00 (0.02) / 44.27% | 0.02 (0.06) / 44.14% | 0.00 (0.24) / 46.28% | -0.42 (0.47) / 45.88% |
| 1 | IN | Black (Rank 1 Approx) | 0.00 (0.05) / 8.46% | 0.02 (0.16) / 8.54% | -0.14 (0.48) / 8.52% | -0.10 (0.86) / 8.52% |
| | | LIBOR (Rank 1 Approx) | 0.00 (0.05) / 8.46% | 0.02 (0.16) / 8.53% | -0.14 (0.48) / 8.51% | -0.10 (0.86) / 8.51% |
| | | Black (Rank 2 Calib$^n$) | 0.01 (0.05) / 8.46% | 0.04 (0.16) / 8.55% | -0.09 (0.48) / 8.53% | -0.04 (0.86) / 8.53% |
| | | LIBOR (Rank 2 FD) | 0.01 (0.05) / 8.46% | 0.04 (0.16) / 8.53% | -0.09 (0.48) / 8.51% | -0.06 (0.86) / 8.51% |
| | | Benchmark Swaption | 0.01 (0.05) / 8.47% | 0.04 (0.16) / 8.60% | -0.10 (0.49) / 8.57% | 0.04 (0.87) / 8.55% |
| | AT | Black (Rank 1 Approx) | 0.00 (0.06) / 19.76% | 0.01 (0.20) / 19.77% | -0.24 (0.58) / 18.91% | -0.08 (1.03) / 18.66% |
| | | LIBOR (Rank 1 Approx) | 0.00 (0.06) / 19.75% | 0.01 (0.20) / 19.75% | -0.24 (0.58) / 18.89% | -0.08 (1.03) / 18.62% |
| | | Black (Rank 2 Calib$^n$) | 0.01 (0.06) / 19.76% | 0.03 (0.20) / 19.79% | -0.18 (0.58) / 18.93% | 0.00 (1.03) / 18.67% |
| | | LIBOR (Rank 2 FD) | 0.01 (0.06) / 19.76% | 0.03 (0.20) / 19.75% | -0.18 (0.58) / 18.89% | -0.03 (1.03) / 18.62% |
| | | Benchmark Swaption | 0.01 (0.06) / 19.76% | 0.02 (0.20) / 19.77% | -0.18 (0.58) / 18.92% | -0.04 (1.04) / 18.66% |
| | OUT | Black (Rank 1 Approx) | 0.02 (0.05) / 46.82% | -0.06 (0.17) / 48.33% | -0.03 (0.53) / 48.74% | 0.30 (0.92) / 48.05% |
| | | LIBOR (Rank 1 Approx) | 0.02 (0.05) / 46.82% | -0.06 (0.17) / 48.29% | -0.03 (0.53) / 48.67% | 0.30 (0.92) / 47.96% |
| | | Black (Rank 2 Calib$^n$) | 0.03 (0.05) / 46.83% | -0.04 (0.17) / 48.39% | 0.02 (0.53) / 48.73% | 0.36 (0.92) / 48.08% |
| | | LIBOR (Rank 2 FD) | 0.03 (0.05) / 46.82% | -0.04 (0.17) / 48.29% | 0.01 (0.53) / 48.66% | 0.35 (0.92) / 47.96% |
| | | Benchmark Swaption | 0.04 (0.05) / 46.85% | -0.05 (0.17) / 48.46% | -0.04 (0.53) / 49.12% | 0.09 (0.94) / 48.70% |
| 2 | IN | Black (Rank 1 Approx) | -0.04 (0.14) / 10.24% | -0.36 (0.34) / 9.66% | -0.61 (0.72) / 9.05% | 0.16 (1.19) / 8.94% |
| | | LIBOR (Rank 1 Approx) | -0.04 (0.14) / 10.23% | -0.36 (0.34) / 9.65% | -0.62 (0.72) / 9.04% | 0.17 (1.19) / 8.92% |
| | | Black (Rank 2 Calib$^n$) | -0.03 (0.14) / 10.25% | -0.30 (0.34) / 9.67% | -0.35 (0.72) / 9.05% | 0.33 (1.19) / 8.95% |
| | | LIBOR (Rank 2 FD) | -0.03 (0.14) / 10.23% | -0.30 (0.34) / 9.64% | -0.34 (0.72) / 9.04% | 0.33 (1.19) / 8.91% |
| | | Benchmark Swaption | 0.02 (0.14) / 10.27% | -0.17 (0.35) / 9.66% | -0.07 (0.72) / 9.01% | 0.68 (1.19) / 8.88% |
| | AT | Black (Rank 1 Approx) | -0.06 (0.17) / 22.93% | -0.29 (0.42) / 20.78% | -0.26 (0.87) / 18.71% | -0.14 (1.41) / 18.47% |
| | | LIBOR (Rank 1 Approx) | -0.06 (0.17) / 22.91% | -0.29 (0.42) / 20.77% | -0.27 (0.87) / 18.69% | -0.13 (1.41) / 18.42% |
| | | Black (Rank 2 Calib$^n$) | -0.05 (0.17) / 22.94% | -0.22 (0.42) / 20.81% | 0.03 (0.87) / 18.70% | 0.06 (1.42) / 18.49% |
| | | LIBOR (Rank 2 FD) | -0.05 (0.17) / 22.90% | -0.23 (0.42) / 20.76% | 0.04 (0.86) / 18.68% | 0.06 (1.41) / 18.41% |
| | | Benchmark Swaption | 0.00 (0.17) / 22.91% | -0.04 (0.42) / 20.74% | 0.42 (0.87) / 18.69% | 0.35 (1.43) / 18.49% |
| | OUT | Black (Rank 1 Approx) | -0.06 (0.15) / 57.63% | -0.32 (0.36) / 52.43% | 0.04 (0.80) / 51.62% | -0.62 (1.34) / 51.98% |
| | | LIBOR (Rank 1 Approx) | -0.06 (0.15) / 57.58% | -0.31 (0.36) / 52.41% | 0.01 (0.80) / 51.57% | -0.61 (1.34) / 51.81% |
| | | Black (Rank 2 Calib$^n$) | -0.05 (0.15) / 57.67% | -0.27 (0.36) / 52.51% | 0.21 (0.80) / 51.62% | -0.49 (1.34) / 52.06% |
| | | LIBOR (Rank 2 FD) | -0.05 (0.15) / 57.57% | -0.27 (0.36) / 52.39% | 0.21 (0.80) / 51.55% | -0.50 (1.34) / 51.84% |
| | | Benchmark Swaption | -0.01 (0.15) / 57.61% | -0.15 (0.36) / 52.79% | 0.26 (0.81) / 52.18% | -0.38 (1.38) / 53.34% |
| 4 | IN | Black (Rank 1 Approx) | -0.12 (0.28) / 10.85% | -0.24 (0.51) / 10.08% | -1.11 (0.94) / 9.87% | |
| | | LIBOR (Rank 1 Approx) | -0.12 (0.28) / 10.85% | -0.23 (0.51) / 10.09% | -1.10 (0.94) / 9.87% | |
| | | Black (Rank 2 Calib$^n$) | -0.12 (0.28) / 10.85% | -0.20 (0.51) / 10.09% | -1.01 (0.94) / 9.87% | |
| | | LIBOR (Rank 2 FD) | -0.12 (0.28) / 10.85% | -0.20 (0.51) / 10.09% | -0.99 (0.94) / 9.86% | |
| | | Benchmark Swaption | 0.03 (0.27) / 10.80% | 0.17 (0.51) / 10.03% | -0.24 (0.93) / 9.77% | |
| | AT | Black (Rank 1 Approx) | -0.20 (0.34) / 21.40% | -0.33 (0.62) / 19.41% | -0.85 (1.15) / 19.28% | |
| | | LIBOR (Rank 1 Approx) | -0.21 (0.34) / 21.41% | -0.32 (0.62) / 19.44% | -0.82 (1.16) / 19.30% | |
| | | Black (Rank 2 Calib$^n$) | -0.20 (0.34) / 21.41% | -0.29 (0.62) / 19.43% | -0.69 (1.16) / 19.30% | |
| | | LIBOR (Rank 2 FD) | -0.20 (0.34) / 21.41% | -0.28 (0.62) / 19.43% | -0.69 (1.15) / 19.28% | |
| | | Benchmark Swaption | -0.01 (0.34) / 21.38% | 0.12 (0.62) / 19.43% | 0.14 (1.15) / 19.23% | |
| | OUT | Black (Rank 1 Approx) | -0.21 (0.32) / 64.19% | -0.40 (0.61) / 60.42% | -1.41 (1.08) / 57.37% | |
| | | LIBOR (Rank 1 Approx) | -0.21 (0.32) / 64.23% | -0.40 (0.61) / 60.50% | -1.38 (1.09) / 57.44% | |
| | | Black (Rank 2 Calib$^n$) | -0.21 (0.32) / 64.23% | -0.38 (0.61) / 60.47% | -1.25 (1.09) / 57.42% | |
| | | LIBOR (Rank 2 FD) | -0.21 (0.32) / 64.19% | -0.37 (0.61) / 60.47% | -1.25 (1.08) / 57.38% | |
| | | Benchmark Swaption | -0.07 (0.32) / 64.45% | -0.03 (0.62) / 60.91% | -0.83 (1.10) / 58.26% | |

Table C.2: Hedging P/L means (with 95% error bounds, both in basis points $bp$) and standard deviations relative to the swaption price for a range of swaption maturities, lengths and strike values simulated under the first volatility structure.

| Swaption Mat (yrs) | Swaption Strike | Description | Swaption Length (yrs) 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| 0.25 | IN | Black (Rank 1 Approx) | 0.02 (0.12) / 8.54% | -0.12 (0.21) / 8.44% | -0.15 (0.32) / 8.23% | -8.32 (0.58) / 8.63% |
| | | LIBOR (Rank 1 Approx) | 0.02 (0.12) / 8.54% | -0.11 (0.20) / 8.44% | -0.15 (0.32) / 8.22% | -8.31 (0.58) / 8.61% |
| | | Black (Rank 2 Calib$^n$) | 0.03 (0.12) / 8.54% | 0.01 (0.21) / 8.45% | 0.07 (0.32) / 8.23% | 0.23 (0.57) / 8.50% |
| | | LIBOR (Rank 2 FD) | 0.03 (0.12) / 8.54% | 0.02 (0.20) / 8.44% | 0.07 (0.32) / 8.22% | 0.23 (0.57) / 8.51% |
| | | Benchmark Swaption | 0.03 (0.12) / 8.55% | 0.02 (0.21) / 8.49% | 0.09 (0.32) / 8.25% | 0.11 (0.55) / 8.26% |
| | AT | Black (Rank 1 Approx) | -0.02 (0.14) / 19.31% | -0.16 (0.24) / 19.31% | -0.32 (0.38) / 18.97% | -10.47 (0.68) / 19.15% |
| | | LIBOR (Rank 1 Approx) | -0.02 (0.14) / 19.31% | -0.16 (0.24) / 19.30% | -0.32 (0.38) / 18.97% | -10.47 (0.68) / 19.17% |
| | | Black (Rank 2 Calib$^n$) | -0.02 (0.14) / 19.31% | -0.01 (0.24) / 19.32% | -0.09 (0.38) / 18.97% | -0.12 (0.68) / 18.99% |
| | | LIBOR (Rank 2 FD) | -0.02 (0.14) / 19.31% | -0.01 (0.24) / 19.30% | -0.09 (0.38) / 18.96% | -0.13 (0.68) / 18.99% |
| | | Benchmark Swaption | -0.02 (0.14) / 19.31% | -0.01 (0.24) / 19.30% | -0.08 (0.38) / 18.97% | -0.08 (0.66) / 19.01% |
| | OUT | Black (Rank 1 Approx) | -0.04 (0.13) / 49.30% | -0.19 (0.21) / 47.32% | -0.42 (0.34) / 47.16% | -8.28 (0.61) / 46.34% |
| | | LIBOR (Rank 1 Approx) | -0.04 (0.13) / 49.29% | -0.19 (0.21) / 47.28% | -0.43 (0.34) / 47.12% | -8.28 (0.61) / 46.52% |
| | | Black (Rank 2 Calib$^n$) | -0.04 (0.13) / 49.30% | -0.09 (0.21) / 47.32% | -0.28 (0.34) / 47.15% | -0.35 (0.60) / 45.69% |
| | | LIBOR (Rank 2 FD) | -0.04 (0.13) / 49.29% | -0.08 (0.21) / 47.28% | -0.29 (0.34) / 47.11% | -0.35 (0.60) / 45.69% |
| | | Benchmark Swaption | -0.04 (0.13) / 49.36% | -0.10 (0.21) / 47.26% | -0.30 (0.34) / 47.16% | -0.16 (0.58) / 47.11% |
| 1 | IN | Black (Rank 1 Approx) | 0.05 (0.24) / 9.62% | -0.28 (0.41) / 9.20% | -0.51 (0.60) / 8.77% | -12.57 (1.12) / 9.31% |
| | | LIBOR (Rank 1 Approx) | 0.05 (0.24) / 9.62% | -0.29 (0.40) / 9.19% | -0.54 (0.60) / 8.76% | -12.56 (1.12) / 9.29% |
| | | Black (Rank 2 Calib$^n$) | 0.06 (0.24) / 9.63% | -0.05 (0.41) / 9.20% | 0.27 (0.60) / 8.77% | 0.80 (1.10) / 9.18% |
| | | LIBOR (Rank 2 FD) | 0.06 (0.24) / 9.62% | -0.06 (0.40) / 9.19% | 0.24 (0.60) / 8.76% | 0.80 (1.10) / 9.18% |
| | | Benchmark Swaption | 0.06 (0.24) / 9.63% | -0.02 (0.41) / 9.22% | 0.24 (0.59) / 8.66% | 0.20 (1.05) / 8.87% |
| | AT | Black (Rank 1 Approx) | 0.09 (0.28) / 20.19% | -0.03 (0.48) / 20.05% | -0.90 (0.71) / 18.98% | -16.22 (1.29) / 18.99% |
| | | LIBOR (Rank 1 Approx) | 0.09 (0.28) / 20.17% | -0.04 (0.48) / 20.01% | -0.94 (0.71) / 18.95% | -16.21 (1.30) / 19.02% |
| | | Black (Rank 2 Calib$^n$) | 0.10 (0.28) / 20.19% | 0.21 (0.48) / 20.05% | 0.19 (0.71) / 18.98% | -0.53 (1.28) / 18.77% |
| | | LIBOR (Rank 2 FD) | 0.09 (0.28) / 20.17% | 0.19 (0.48) / 20.01% | 0.13 (0.71) / 18.94% | -0.53 (1.28) / 18.76% |
| | | Benchmark Swaption | 0.09 (0.28) / 20.18% | 0.18 (0.49) / 20.02% | 0.21 (0.71) / 18.92% | -0.44 (1.24) / 18.84% |
| | OUT | Black (Rank 1 Approx) | 0.14 (0.26) / 54.82% | -0.30 (0.44) / 52.89% | -1.16 (0.65) / 48.98% | -12.90 (1.20) / 49.00% |
| | | LIBOR (Rank 1 Approx) | 0.14 (0.26) / 54.78% | -0.31 (0.44) / 52.81% | -1.21 (0.65) / 48.89% | -12.90 (1.20) / 49.28% |
| | | Black (Rank 2 Calib$^n$) | 0.14 (0.26) / 54.82% | -0.15 (0.44) / 52.91% | -0.17 (0.65) / 48.99% | -1.17 (1.18) / 48.40% |
| | | LIBOR (Rank 2 FD) | 0.14 (0.26) / 54.77% | -0.16 (0.44) / 52.81% | -0.23 (0.65) / 48.85% | -1.17 (1.18) / 48.39% |
| | | Benchmark Swaption | 0.14 (0.26) / 54.97% | -0.23 (0.44) / 52.97% | -0.10 (0.64) / 49.68% | -0.47 (1.15) / 51.14% |
| 2 | IN | Black (Rank 1 Approx) | 0.09 (0.32) / 10.12% | -0.22 (0.48) / 9.66% | -3.37 (0.77) / 9.19% | -9.83 (1.40) / 9.55% |
| | | LIBOR (Rank 1 Approx) | 0.09 (0.32) / 10.12% | -0.22 (0.48) / 9.64% | -3.34 (0.77) / 9.18% | -9.80 (1.39) / 9.54% |
| | | Black (Rank 2 Calib$^n$) | 0.10 (0.32) / 10.12% | -0.20 (0.48) / 9.66% | 0.08 (0.77) / 9.19% | 1.68 (1.38) / 9.45% |
| | | LIBOR (Rank 2 FD) | 0.10 (0.32) / 10.12% | -0.20 (0.48) / 9.64% | 0.11 (0.76) / 9.18% | 1.71 (1.38) / 9.43% |
| | | Benchmark Swaption | 0.10 (0.32) / 10.12% | -0.25 (0.48) / 9.56% | 0.02 (0.75) / 9.01% | 0.46 (1.30) / 9.10% |
| | AT | Black (Rank 1 Approx) | 0.07 (0.39) / 21.19% | -0.22 (0.58) / 20.09% | -4.42 (0.93) / 19.16% | -12.74 (1.67) / 18.89% |
| | | LIBOR (Rank 1 Approx) | 0.07 (0.39) / 21.18% | -0.22 (0.58) / 20.06% | -4.40 (0.92) / 19.13% | -12.73 (1.67) / 18.93% |
| | | Black (Rank 2 Calib$^n$) | 0.08 (0.39) / 21.19% | -0.19 (0.58) / 20.09% | -0.17 (0.93) / 19.15% | 0.25 (1.66) / 18.75% |
| | | LIBOR (Rank 2 FD) | 0.08 (0.39) / 21.18% | -0.18 (0.58) / 20.05% | -0.14 (0.92) / 19.11% | 0.30 (1.65) / 18.69% |
| | | Benchmark Swaption | 0.07 (0.39) / 21.22% | -0.22 (0.58) / 20.09% | -0.18 (0.91) / 19.11% | 0.28 (1.59) / 18.75% |
| | OUT | Black (Rank 1 Approx) | -0.20 (0.38) / 61.33% | -0.21 (0.54) / 55.27% | -3.40 (0.84) / 51.07% | -10.55 (1.59) / 51.66% |
| | | LIBOR (Rank 1 Approx) | -0.19 (0.38) / 61.28% | -0.20 (0.54) / 55.17% | -3.38 (0.84) / 50.98% | -10.56 (1.60) / 51.96% |
| | | Black (Rank 2 Calib$^n$) | -0.19 (0.38) / 61.34% | -0.16 (0.54) / 55.28% | -0.04 (0.84) / 51.02% | -1.16 (1.58) / 51.33% |
| | | LIBOR (Rank 2 FD) | -0.19 (0.38) / 61.27% | -0.15 (0.54) / 55.17% | -0.01 (0.84) / 50.89% | -1.08 (1.57) / 51.15% |
| | | Benchmark Swaption | -0.21 (0.38) / 61.75% | -0.09 (0.54) / 55.94% | 0.23 (0.83) / 52.07% | 0.04 (1.53) / 54.78% |
| 4 | IN | Black (Rank 1 Approx) | -0.07 (0.26) / 10.59% | -0.11 (0.46) / 9.94% | -1.14 (0.87) / 9.96% | |
| | | LIBOR (Rank 1 Approx) | -0.06 (0.26) / 10.59% | -0.10 (0.46) / 9.94% | -1.15 (0.87) / 9.97% | |
| | | Black (Rank 2 Calib$^n$) | -0.07 (0.26) / 10.59% | -0.06 (0.46) / 9.94% | -0.03 (0.87) / 9.97% | |
| | | LIBOR (Rank 2 FD) | -0.07 (0.26) / 10.59% | -0.06 (0.46) / 9.93% | -0.03 (0.87) / 9.97% | |
| | | Benchmark Swaption | -0.08 (0.26) / 10.57% | -0.14 (0.45) / 9.88% | -0.28 (0.85) / 9.88% | |
| | AT | Black (Rank 1 Approx) | -0.16 (0.31) / 20.72% | -0.31 (0.56) / 19.44% | -1.51 (1.05) / 18.78% | |
| | | LIBOR (Rank 1 Approx) | -0.16 (0.31) / 20.72% | -0.31 (0.56) / 19.43% | -1.53 (1.05) / 18.79% | |
| | | Black (Rank 2 Calib$^n$) | -0.16 (0.31) / 20.72% | -0.25 (0.56) / 19.45% | -0.27 (1.05) / 18.80% | |
| | | LIBOR (Rank 2 FD) | -0.16 (0.31) / 20.72% | -0.25 (0.56) / 19.43% | -0.27 (1.05) / 18.78% | |
| | | Benchmark Swaption | -0.18 (0.32) / 20.74% | -0.36 (0.56) / 19.49% | -0.45 (1.05) / 18.92% | |
| | OUT | Black (Rank 1 Approx) | -0.04 (0.30) / 62.68% | 0.03 (0.54) / 59.07% | -1.00 (1.02) / 58.00% | |
| | | LIBOR (Rank 1 Approx) | -0.04 (0.30) / 62.67% | 0.03 (0.54) / 59.06% | -1.02 (1.03) / 58.03% | |
| | | Black (Rank 2 Calib$^n$) | -0.04 (0.30) / 62.69% | 0.08 (0.54) / 59.09% | -0.16 (1.02) / 58.01% | |
| | | LIBOR (Rank 2 FD) | -0.04 (0.30) / 62.70% | 0.07 (0.54) / 59.05% | -0.15 (1.02) / 57.93% | |
| | | Benchmark Swaption | -0.05 (0.31) / 63.22% | -0.01 (0.55) / 60.16% | -0.01 (1.04) / 60.01% | |

Table C.3: Hedging P/L means (with 95% error bounds, both in basis points $bp$) and standard deviations relative to the swaption price for a range of swaption maturities, lengths and strike values simulated under the second volatility structure.

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