Cross currency swap valuation

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Abstract

Cross currency swaps are powerful instruments to transfer assets or liabilities from one currency into another. The market charges for this a liquidity premium, the cross currency basis spread, which should be taken into account by the valuation methodology. We describe and compare two valuation methods for cross currency swaps which are based upon using two different discounting curves. The first method is very popular in practice but inconsistent with singe currency swap valuation methods. The second method is consistent for all swap valuations but leads to mark to market values for single currency off market swaps which can be quite different to standard valuation results.

1 Single currency swap valuation

Denote by $\mathbf{DF}(T)$ the discount factor from the swap curve for a cash flow at time T.

Consider a fixed-floating standard interest rate swap with reference dates $0 = \bar{T}_0 < \bar{T}_1 < \cdots < \bar{T}_n$ on the fixed leg and reference dates $0 = T_0 < T_1 < \cdots < T_m$ for the floating leg, $\bar{T}_n = T_m$. Denote by $\bar{\Delta}_i$ (resp. Δ_i) the length (day count fraction) of the period $[\bar{T}_{i-1}, \bar{T}_i]$ (resp. $[T_{i-1}, T_i]$) according to the specified fixed (resp. floating) leg day count convention. For the period $[T_{i-1}, T_i]$ the variable rate (Libor) L_i is set (fixed) in the market at time T_{i-1} and the amount $\Delta_i \cdot L_i$ is paid at time T_i. The *forward rate* L_i^0 for the period $[T_{i-1}, T_i]$ is defined as

$$L_i^0 = \frac{\frac{\mathbf{DF}(T_{i-1})}{\mathbf{DF}(T_i)} - 1}{\Delta_i}. \tag{1}$$

One says, the forward rate L_i^0 is *projected or forecasted from the discount curve*. By well-known replication and no-arbitrage arguments the value today of the floating interest rate (Libor) cash flow $\Delta_i \cdot L_i$ for period $[T_{i-1}, T_i]$ is its discounted forward rate

$$\Delta_i \cdot L_i^0 \cdot \mathbf{DF}(T_i) = \mathbf{DF}(T_{i-1}) - \mathbf{DF}(T_i). \tag{2}$$

Therefore the value of the whole floating leg is simply

$$\mathbf{DF}(T_0) - \mathbf{DF}(T_m) = 1 - \mathbf{DF}(T_m) = 1 - \mathbf{DF}(\bar{T}_n). \tag{3}$$

As a consequence *the value of a floating rate bond is always at par*

$$1 = \sum_{i=1}^{m} \Delta_i \cdot L_i^0 \cdot \mathbf{DF}(T_i) + \mathbf{DF}(T_m).$$

The value of the fixed leg with fixed rate C is obviously

$$\sum_{i=1}^{n} C\, \bar{\Delta}_i \mathbf{DF}(\bar{T}_i). \tag{4}$$

If C_n is the fair swap rate for maturity $\bar{T}_n$, i.e., (3) = (4), we get the following equation

$$1 = \sum_{i=1}^{n} C\, \bar{\Delta}_i \mathbf{DF}(\bar{T}_i) + \mathbf{DF}(\bar{T}_n),$$

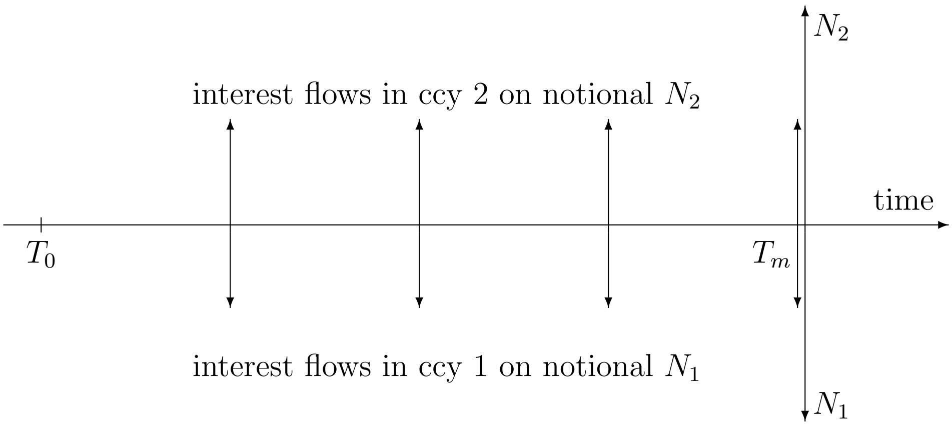
i.e., *a fixed coupon bond with maturity $\bar{T}_n$ and coupon C_n admits a price of par*. This is the basis for the recursive bootstrapping relationship for the discount factors

$$\mathbf{DF}(\bar{T}_n) = \frac{1 - C_n \sum_{i=1}^{n-1} \bar{\Delta}_i \mathbf{DF}(\bar{T}_i)}{1 + \bar{\Delta}_n C_n}, \quad n = 2, \ldots \tag{5}$$

from market quoted fair swap rates C_n.

2 Cross currency basis swaps

Cross currency swaps differ from single currency swaps by the fact that the interests rate payments on the two legs are in different currencies. So on one leg interest rate payments are in currency 1 on a notional amount N_1 and on the other leg interest rate payments are in currency 2 calculated on a notional amount N_2 in that currency. At inception of the trade the notional principal amounts in the two currencies are usually set to be fair given the spot foreign exchange rate X, i.e. $N_1 = X \cdot N_2$. Contrary to single currency swaps there is always an exchange of principals at maturity. So a cross currency swap can be seen as exchange of the payments of two bonds, one in currency 1 with principal N_1, the other in currency 2 with principal N_2.



If a leg is floating the variable reference rate refers to the payment currency of that leg – otherwise this would be a so-called quanto swap.

From the possible types of cross currency swaps: fixed versus fixed, fixed versus floating and floating versus floating, the latter type is particularly important and called a *basis swap*. Combining a basis swap with a single currency swap the other types can be generated synthetically. It therefore suffices to investigate basis swaps and this also explains why the market quotes only basis swaps.

A basis swap is basically an exchange of two floating rate bonds. Following the arguments of the previous section the price of a floater is always[1] par. For a cross currency basis swap this means that the two legs should have a value of N_1 and N_2, respectively. Consequently, if the two principal amounts are linked by today's foreign exchange rate X: $N_1 = X \cdot N_2$, the basis swap is fair. This is theoretically true, but in practice the market quotes basis swaps to be fair if there is a certain spread, called *cross currency basis spread*, on top of the floating rate of one leg of the basis swap. Theoretically this would imply an arbitrage opportunity. However, cross currency swaps are powerful instruments to transfer assets or liabilities from one currency into another one and the market is charging liquidity premiums of one currency over the other. The market quotes cross currency basis spreads usually relative to a

[1] At the beginning of each interest period.

liquidity benchmark, e.g. USD or EUR Libor. Here is an example of cross currency basis swap quotes against the liquidity benchmark USD:

13:47 18DEC03 GARBAN-INTERCAPITAL UK04138 ICAB1
Basis Swaps - All currencies vs. 3m USD LIBOR - Also see <ICAB2>

	REC/PAY EUR	REC/PAY JPY	REC/PAY GBP	REC/PAY CHF	REC/PAY SEK
1 Yr	+3.125/+1.125	-02.00/-05.00	+02.50/-01.50	+1.50/-1.50	-3.75/-6.75
2 Yr	+3.000/+1.000	-02.00/-05.00	+02.50/-01.50	+1.00/-2.00	-2.50/-5.50
3 Yr	+3.000/+1.000	-01.75/-04.75	+02.25/-01.75	+0.25/-3.00	-1.00/-4.00
4 Yr	+3.000/+1.000	-01.75/-04.75	+02.00/-02.00	-0.75/-3.75	-0.25/-3.25
5 Yr	+2.750/+0.750	-02.00/-05.00	+01.75/-02.25	-1.25/-4.25	+0.25/-3.25
7 Yr	+2.750/+0.750	-02.50/-05.50	+00.75/-03.25	-1.50/-4.50	+0.25/-2.75
10Yr	+2.750/+0.750	-04.50/-07.50	-00.75/-04.75	-1.50/-4.50	+0.25/-2.75
15Yr	+4.125/+0.125	-10.50/-13.50	-02.25/-06.25	-0.75/-4.75	+2.50/-2.50
20Yr	+4.125/+0.125	-15.25/-18.25	-02.50/-06.50	-0.25/-4.25	+2.50/-2.50
30Yr	+4.125/+0.125	-23.25/-26.25	-02.50/-06.50	*FOR 3M V 6M EUR/EUR <ICAB4>**	

	DKK	NOK	CAD	CZK	PLN
1 Yr	-1.00/-5.00	-4.00/-8.00	+12.50/+08.50	+02.00/-07.00	+05.00/-12.00
2 Yr	-1.50/-4.50	-4.00/-8.00	+13.50/+09.50	+01.50/-06.50	+05.00/-12.00
3 Yr	-1.25/-4.25	-4.00/-8.00	+14.25/+10.25	+01.50/-06.50	+05.00/-12.00
4 Yr	-1.00/-4.00	-3.75/-7.75	+15.25/+11.25	+01.50/-06.50	+03.00/-10.00
5 Yr	-0.25/-3.25	-3.75/-7.75	+16.25/+12.25	+01.50/-06.50	+03.00/-10.00
7 Yr	+0.25/-3.00	-3.75/-7.75	+17.00/+13.00	+01.50/-06.50	+03.00/-10.00
10Yr	+0.25/-3.00	-3.75/-7.75	+17.00/+13.00	+01.50/-06.50	+03.00/-10.00
15Yr	+1.25/-3.75	-3.50/-8.50	+17.25/+13.25		
20Yr	+1.25/-3.75	-3.50/-8.50	+17.25/+13.25		

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For example, a 10 years cross currency basis swap of 3 months USD Libor flat against JPY Libor is fair with a spread if -4.5 basis points if USD Libor is received and with a spread of -7.5 basis points if USD Libor is paid.

Evaluating cross currency swaps requires discounting the cash flows with the discount factors for the respective currency of the flow. But clearly, a valuation of those instruments along the lines of Section 1 would show a profit or loss which is not existent. It is therefore necessary to incorporate the cross currency basis spread into the valuation methodology to be consistent with the market.

First of all one has to agree on a liquidity reference currency (benchmark) which is usually chosen to be USD or EUR. Swap cash flows in the liquidity reference currency are valued exactly as described in Section 1 since there is no need for liquidity adjustments there.

For all currencies different from the liquidity benchmark the idea is to use two different discount factor curves depending on whether to forecast or value variable cash flows or to discount cash flows.

Denote by s_m the market quoted fair cross currency basis spread[2] for maturity T_m on top of the floating rate for the given currency relative to the chosen liquidity reference. The fact that s_m is the fair spread is equivalent to saying that a floating rate bond with maturity T_m in the given currency which pays Libor plus spread s_m values to par.

[2] In practice we would take the mid-market spread from the market quotes.

2.1 Valuation based on a modified discount curve

We start by describing a valuation methodology for cross currency swaps which is quite popular among practitioners but unfortunately inconsistent with the standard single currency swap valuation method.

In this approach we use two discount factor curves: one for projecting forward rates according to (1) and the other for finally discounting all cash flows.

From single currency swap quotes the standard curve of discount factors $\mathbf{DF}(t)$ is extracted following formula (5). These standard discount factors are solely used to project forward rates according to (1):

$$L_i^0 = \frac{\frac{\mathbf{DF}(T_{i-1})}{\mathbf{DF}(T_i)} - 1}{\Delta_i}.$$

Now in order to assure that a floater which pays Libor plus spread s_n is at par one has to introduce another curve of discount factors $\mathbf{DF}^\star(t)$ to be used exclusively for discounting cash flows. The defining condition for this curve $\mathbf{DF}^\star(t)$ is then

$$1 = \sum_{i=1}^{m} \Delta_i (L_i^0 + s_m)\, \mathbf{DF}^\star(T_i) + \mathbf{DF}^\star(T_m),\ m = 1, \ldots$$

and we obtain the recursive bootstrapping relation

$$\mathbf{DF}^\star(T_m) = \frac{1 - \sum_{i=1}^{m-1} \Delta_i (L_i^0 + s_m)\, \mathbf{DF}^\star(T_i)}{1 + \Delta_m (L_m^0 + s_m)},\ m = 1, \ldots \tag{6}$$

The discount factors from the curve $\mathbf{DF}^\star(t)$ are used for discounting any fixed or floating cash flows in a cross currency swap. Cross currency swap valuations are thus consistent with the cross currency swap market quotes.

However, applying the same methodology to single currency swaps obviously leads to a mispricing of those instruments. That is single currency swaps have to be valued differently and according to the standard methodology of Section 1. On one hand, this is clearly unsatisfactory since the valuation methodology for one and the same cash flow should not depend on the type of originating trade for that cash flow. Applying different and non-consistent valuation methodologies to single currency and cross currency swaps implies that there are theoretically opportunities for arbitrage within these products.

On the other hand, since the market charges different premiums for liquidity for single currency and cross currency swaps this can also be seen to justify the use different discounting curves for the two types of trades.

Here is an illustrating example for the two discount factor curves assuming for simplicity everywhere an annual payment frequency and a 30/360 day count convention, i.e. $\Delta_i = \bar{\Delta}_i = 1$.

T_n	C_n	s_n	$\mathbf{DF}(T_n)$ from (5)	$\mathbf{DF}^\star(T_n)$ from (6)
1	5,00%	-0,10%	0,952381	0,953289
2	5,10%	-0,12%	0,905260	0,907339
3	5,20%	-0,14%	0,858748	0,862218
4	5,30%	-0,16%	0,812945	0,817985
5	5,40%	-0,18%	0,767947	0,774694
6	5,50%	-0,20%	0,723838	0,732392
7	5,60%	-0,22%	0,680698	0,691121
8	5,70%	-0,24%	0,638596	0,650917
9	5,80%	-0,26%	0,597595	0,611810
10	5,90%	-0,28%	0,557750	0,573823

2.2 Valuation based on modified fixed and floating discount curve

We will use two discount factors curves:

(i) the first one, $\mathbf{DF}(t)$, will be used to discount all fixed cash flows,

(ii) the second one, $\mathbf{DF}^\star(t)$, we be applied to completely value floating cash flows, see equation (9) below.

The power of this approach is that both, cross currency swaps and single currency swaps, are valued consistently in one and the same framework.

The two conditions on the two discount factor curves are

- the value of a coupon bond with coupon equal to the swap rate C_n is identical to the value of a floating rate bond,
- a floating rate bond which pays Libor plus cross currency basis spread s_n values to par.

Combining these conditions a fixed coupon bond paying the coupon C_n plus the cross currency basis spread s_n should have a value of par:

$$1 = \sum_{i=1}^{n} \bar{\Delta}_i C_n \mathbf{DF}(\bar{T}_i) + \sum_{j=1}^{m} \Delta_j s_m \mathbf{DF}(T_j) + \mathbf{DF}(\bar{T}_n). \tag{7}$$

From this equation the curve $\mathbf{DF}(t)$ can be extracted. If in particular, the floating and fixed legs admit the same frequency, i.e. $\bar{T}_i = T_i$, then we have the following simple bootstrapping equation

$$\mathbf{DF}(\bar{T}_n) = \frac{1 - \sum_{i=1}^{n-1}(\bar{\Delta}_i C_n + \Delta_i s_n)\mathbf{DF}(\bar{T}_i)}{1 + \bar{\Delta}_n C_n + \Delta_n s_n}. \tag{8}$$

Now, in order to determine the second curve of discount factors, analogously to equation (2) we *define the value today of the floating interest rate (Libor) cash flow for period* $[T_{i-1}, T_i]$ *as given by*

$$\mathbf{DF}^\star(T_{i-1}) - \mathbf{DF}^\star(T_i). \tag{9}$$

This also implies the desirable property that the value of a series of subsequent floating rate cash flows over the time interval $[T_0, T_m]$ is just $\mathbf{DF}^\star(T_0) - \mathbf{DF}^\star(T_m)$ and is thus independent of the payment frequency. The second condition above now implies the following requirement

$$1 = \mathbf{DF}^\star(T_0) - \mathbf{DF}^\star(T_m) + \sum_{j=1}^{m} \Delta_j s_m \mathbf{DF}(T_j) + \mathbf{DF}(T_m).$$

Setting, $\mathbf{DF}^\star(T_0) = 1, T_0 = 0$, this gives

$$\mathbf{DF}^\star(T_m) = \mathbf{DF}(T_m) + s_m \sum_{j=1}^{m} \Delta_j \mathbf{DF}(T_j). \tag{10}$$

This defines the second curve of discount factors $\mathbf{DF}^\star(t)$.

The principal idea of this approach is closely related to the approach proposed by Fruchard, Zammouri and Williams [1]. Somehow their idea did get not much attraction in practice - one possible reason for that might be that their paper [1] is hard to understand. In their paper [1] Fruchard, Zammouri & Willems use a so-called margin function $F(t)$ to adjust forward rates. First discount factors $\mathbf{DF}(t)$ are extracted from (7) and then a floating rate cash flow for period $[T_{i-1}, T_i]$ is evaluated according to the following formula

$$\left(\frac{\frac{\mathbf{DF}(T_{i-1})}{\mathbf{DF}(T_i)} - 1}{\Delta_i} + \frac{F(T_i) - F(T_{i-1})}{\Delta_i \mathbf{DF}(T_i)} \right) \Delta_i \mathbf{DF}(T_i). \tag{11}$$

Here

$$L_i^{0,adj} = \frac{\frac{\mathbf{DF}(T_{i-1})}{\mathbf{DF}(T_i)} - 1}{\Delta_i} + \frac{F(T_i) - F(T_{i-1})}{\Delta_i \mathbf{DF}(T_i)} \tag{12}$$

is an adjusted forward rate consisting of the standard forward rate from the discount curve $\mathbf{DF}(t)$ (cf. (1)) plus an adjustment for liquidity defined by the margin function $F(t)$. Simplifying (11) we end up with

$$(11) = \mathbf{DF}(T_{i-1}) - \mathbf{DF}(T_i) + F(T_{i-1}) - F(T_i)$$

and comparing with (9) the relationship of the margin function $F(t)$ to our discount curve $\mathbf{DF}^{\star}(t)$ above is simply

$$F(t) = \mathbf{DF}^{\star}(t) - \mathbf{DF}(t).$$

Now consider the same example as in the previous approach to illustrate the calculation of the two curves.

T_n	C_n	s_n	$\mathbf{DF}(T_n)$ from (8)	$\mathbf{DF}^{\star}(T_n)$ from (10)
1	5,00%	-0,10%	0,953289	0,952336
2	5,10%	-0,12%	0,907384	0,905151
3	5,20%	-0,14%	0,862368	0,858555
4	5,30%	-0,16%	0,818316	0,812650
5	5,40%	-0,18%	0,775298	0,767528
6	5,50%	-0,20%	0,733375	0,723275
7	5,60%	-0,22%	0,692602	0,679968
8	5,70%	-0,24%	0,653026	0,637676
9	5,80%	-0,26%	0,614687	0,596460
10	5,90%	-0,28%	0,577622	0,556375

For easy comparison we also show the standard discount factors form (5), the standard forward rates and the adjusted forward rates from (12).

T_n	$\mathbf{DF}(T_n)$ from (5)	L_n^0 from (1)	$L_n^{0,adj}$ from (12)
1	0,952381	5,000%	5,000%
2	0,905260	5,205%	5,200%
3	0,858748	5,416%	5,403%
4	0,812945	5,634%	5,610%
5	0,767947	5,860%	5,820%
6	0,723838	6,094%	6,034%
7	0,680698	6,338%	6,253%
8	0,638596	6,593%	6,476%
9	0,597595	6,861%	6,705%
10	0,557750	7,144%	6,940%

Although the standard forward rates and the adjusted forward rates are quite different in most cases, fortunately, as already indicated by the example, the forward rate from (1) and the adjusted forward rate (12) for the first period $[T_0, T_1]$ are identical. In fact, let the floating and fixed legs admit the same frequency and day count convention, i.e. $\bar{T}_i = T_i$, $\bar{\Delta}_i = \Delta_i$. Then the standard forward rate L_1^0 for the period $[T_0, T_1]$ (cf (1)) is just $L_1^0 = C_1$. On the other hand, the adjusted forward rate $L_1^{0,adj}$ is

$$L_1^{0,adj} = \frac{\mathbf{DF}^\star(T_0) - \mathbf{DF}^\star(T_1)}{\Delta_1 \mathbf{DF}(T_1)}$$

with

$$\begin{aligned} \mathbf{DF}(T_1) &= \frac{1}{1 + \Delta_1 C_1 + \Delta_1 s_1} \text{ see (8)} \\ \mathbf{DF}^\star(T_1) &= \mathbf{DF}(T_1)(1 + \Delta_1 s_1) \text{ see (10).} \end{aligned}$$

Substituting we end up again with

$$L_1^{0,adj} = C_1.$$

This is important to ensure that projected cash flows turn continuously into their fixings.

So far the current approach seems appropriate since it really captures simultaneously both types of market information, the single currency fair swap rates C_n and the cross currency basis spreads s_n. But what are the drawbacks?

Generally speaking in the current approach also cash flows in standard single currency swaps are discounted differently compared to the standard approach in Section 1. For example, the present value of the upcoming next cash flow on the floating side is $5\% \cdot \mathbf{DF}(T_1) = 4,76644\%$ in the current approach compared to $5\% \cdot \mathbf{DF}(T_1) = 4,76190\%$ in the standard valuation.

This gets even more pronounced when it comes to mark-to-market valuation of off market swaps. Consider a 10 years single currency swap with off market fixed rate of $C = 7\%$. In the standard approach of Section 1 its net present value is 824,53 basis points compared to 875,33 basis points in the current approach. Obviously this has dramatic consequences, for example, on the fair values for unwinding a swap position with a counter party. The only way to overcome this problem seems that all market participants have to move over to the new methodology.

3 Conclusion

We describe and discuss two valuation approaches for cross currency swaps. The challenging element of cross currency swap valuation is that the market quotes a certain liquidity premium of one currency over the other which has to be taken care of in the pricing methodology.

The first approach, being widely used in practice yields inconsistencies and thus arbitrage opportunities between single currency and cross currency swaps.

The second approach is able to handle both types of swaps consistently in one and the same framework. The major drawback of this approach is that mark-to-market valuations of single currency swaps can be quite different from the results of the current standard valuation method. Future developments have to show if market participants adopt this methodology.

References

[1] Fruchard, E., Zammouri, C. and Willems, E.: Basis for change, *RISK*, Vol. 8, No. 10, October 1995, 70-75

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