NATIXIS BEYOND BANKING

MODEL DOCUMENTATION

January 8, 2022

Object	**Fixed-Coupon and Floating Coupon Defaultable Bond**
Direction	Global Market
Author(s)	Omar ABDELMOULA
Reference	Pricer
Diffusion	Restricted
Version	1.1.1
Complexity	Meduim
Proposed tier	

Contents

1 General information

Model Development Information	
Model ID	CRD_BOND_CF
Model Name	DETERMINISTIC MODEL
Business unit	Credit Solutions Trading
Model purpose	Pricing and Hedging of Bond
Name of the model development team	LD-M-FIC6 R&D GLOBAL
Model development start date	January 2020
Model development completion date	June 2020
Date of the most recent revision	June 2020
Model Implementation Information	
Implementation plateform	ARM/Summit
Name of model implementation team	IT Department / Fixed Income / R&D
Date of implementation	
Model Governance Information	
Name of the model owner	Samuel Cornut
Date model deployment approved for initial Business use	N.A
Most recent date model approved for continued Busines use	
Approved uses	Pricing, Hedging for Bond
Model restrictions	
Approved users	
Name of the model validator lead for the most recent validation	
Date of the most recent prior validation/review	
Type of the most recent prior validation/review	
Model Key Documentation	
Model documentation names	Omar Abdelmoula
Previous validation report names	N.A
Business requirement documentation name	N.A
Specification documentation name	Samuel Cornut
IT implementation sign off documentation name	
User acceptance report name	Samuel Cornut
Regulatory references	Antoine Kremer, SR11.07
Calibration files name	
Back testing documentation name	
Stress testing documentation name	

2 Model objectives

The model objective is to price a defaultable bond given a default, interest rate and basis cash CDS curves.

3 Stakeholders

The defaultable Bond pricer is used by the following stakeholder

- ▷ Model Owner : Credit Solutions Trading team - Samuel Cornut
- ▷ Model Developers : Omar ABDELMOULA
- ▷ Model Implementers : LD-M-ARM
- ▷ Model Users : LD-M-CreditTrading-Solutions

4 Tiering

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5 In house and Vendor

The model and the numerical method is developed in house by the Credit Quant team in ARM pricing library.

6 Defaultable fixed or floating coupon Bond Payoff

6.1 Defaultable fixed-coupon Bond

A Defaultable fixed-coupon Bond has the following payment flows:

- ▷ If no default occurs
  - ▷ Until maturity: Coupons payment of c at coupon payment dates.
  - ▷ At maturity: Payment of the principal.
- ▷ If a default event occurs
  - ▷ Before default: coupons payment of c at coupon payment dates.
  - ▷ Upon default:
    - ▷ Pays recovery on notional at default date + settlement lag.
    - ▷ Pays accrued coupon at default date + accrued coupon payment lag.

The contract entails the definition of:

- ▷ The payment currency
- ▷ The reference entity name.
- ▷ The credit event.
- ▷ The contract coupons (constant).
- ▷ The notional (constant, amortizing, step-up ...).
- ▷ The start date of the bond.
- ▷ The maturity of the bond.
- ▷ The coupon payment schedule.
- ▷ The recovery rate.
- ▷ The grace period between default event and accrued coupon and notional payment.
- ▷ The settlement lag between default event and defaulted instrument delivery.

6.2 Defaultable fixed-coupon Bond Characterization

Data	**Description**
Purchase/Sale	whether the Bank purchased or sold the bond
CCY_R	payment currency
E	Credit reference entity
$\mathcal{T}_N = \{t_1^{(n)}, \ldots, t_p^{(n)}\}$	nominal knots dates
$(N_1, \ldots, N_p)$	nominal amounts (piecewise constant function of time)
δ_0	step in lag
T_s	start date
T	maturity
R	recovery (fixed: $R \in [0,1]$, or market recovery: $R = \emptyset$)
c	coupon values
$(t_1^{(s)}, \ldots, t_n^{(s)})$	start accrual dates for coupon paiement
$(t_1^{(e)}, \ldots, t_n^{(e)})$	end accrual dates for coupon paiement
$(t_0^{(p)}, \ldots, t_n^{(p)})$	schedule of payment dates
δ_C	payment lag for coupon upon default (in *bd*)
δ_B	settlement lag (in *bd*)
α	day count convention for the premium payments
Nominal Exchange Type	NONE, BOTH, START or END
coupon on default settlement	ACC or ACC_NOT_SET

6.3 Defaultable floating-coupon Bond

Here the coupon payments equal Libor plus a spread.

6.4 Defaultable floating-coupon Bond Characterization

Data	**Description**
Purchase/Sale	whether the Bank purchased or sold the bond
CCY_R	payment currency
E	Credit reference entity
$\mathcal{T}_N = \{t_1^{(n)}, \ldots, t_p^{(n)}\}$	nominal knots dates
$(N_1, \ldots, N_p)$	nominal amounts (piecewise constant function of time)
δ_0	step in lag
T_s	start date
T	maturity
R	recovery (fixed: $R \in [0,1]$, or market recovery: $R = \emptyset$)
I	Interest Rate Index
s	spread
$(t_1^{(s)}, \ldots, t_n^{(s)})$	start accrual dates for coupon paiement
$(t_1^{(e)}, \ldots, t_n^{(e)})$	end accrual dates for coupon paiement
$(t_0^{(p)}, \ldots, t_n^{(p)})$	schedule of payment dates
δ_C	payment lag for coupon upon default (in *bd*)
δ_B	settlement lag (in *bd*)
α	day count convention for the premium payments
Nominal Exchange Type	NONE, BOTH, START or END
coupon on default settlement	ACC or ACC_NOT_SET

7 Mathematical Background

All processes and random variables that we introduce are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G} = (\mathcal{G}_t)_{t\geq 0})$, where Ω is the set of possible states of nature, the filtration $(\mathcal{F}_t)_{t\geq 0}$ represents the information structure of the setup, and $\mathbb{Q}$ is the probability measure that attaches probabilities to the events in Ω.

Definition 7.1. Let τ a $\mathbb{G}-$ stopping time and consider the process $N_t = \mathbb{1}_{\{\tau \leq t\}}$. The non negative and $\mathbb{G}-$ adapted process $(\lambda_t)_{t\geq 0}$ is the intensity of τ if

$$N_t - \int_0^t \mathbb{1}_{]0,\tau]}(s)\lambda_s \mathrm{d}s = N_t - \int_0^{\tau\wedge t} \lambda_s \mathrm{d}s$$

is a $(\mathbb{G}, \mathbb{P})-$ martingale

Intuitively, this definition means that, conditionally to the realization of the event $\{\tau > t\}$ we have

$$\mathbb{P}\left(\tau \in]t, t+dt] | \mathcal{G}_t\right) = \lambda_t \mathrm{d}t$$

We are now interested in the situation where the random variable τ is a stopping time of a greater filtration than one generated by the hazard rate. So we assume that $\mathbb{F} = (\mathcal{F})_{t\geq 0}$ is a filtration such as $\mathcal{F}_t \subset \mathcal{G}_t$.

Definition 7.2. Let τ a $\mathbb{G}-$ stopping time with intensity λ. τ is doubly stochastic relatively to the filtration $\mathbb{F}$ is λ is a $\mathbb{F}-$adapted process and $\forall t \leq s$, we have

$$\mathbb{P}\left(\tau > s | \mathcal{F}_s \vee \mathcal{G}_t\right) = \exp\left(-\int_t^s \mathrm{d}u\lambda_u\right) \mathbb{1}_{\{\tau > t\}}$$

The precedent definition means that, conditionally to $\mathcal{F}_\infty$, τ is the first jump time of Poisson process $\tilde{N}$ w.r.t $\mathbb{G}$ with intensity $(\lambda_t)_{t\geq 0}$.

In practice, we start from a filtration $\tilde{\mathbb{F}}$ which is, in general, the filtration generated by a Markov process (say, an affine process) representing market information outside the default instant and we construct a random variable $\tilde{\tau}$ and a filtration $\tilde{\mathbb{G}}$ as:

Consider a $\tilde{\mathbb{F}}-$ adapted process λ and U an uniform random variable on $[0,1]$ independent from $\tilde{\mathcal{F}}_\infty$. If we define

$$\tilde{\tau} = \inf\left\{t \geq 0; \quad \exp\left(-\int_0^t \mathrm{d}s\lambda_s\right) \leq U\right\}$$

as the $\sigma-$algebra

where $\tilde{\mathcal{H}}_t = \sigma(\tau \wedge t)$.

Lemma 1. If $\mathcal{F}$ is sub $\sigma-$algebra of $\mathcal{G}_\infty$. For all t we have

$$\mathcal{H}_t \cap \mathcal{F} \subset \{A \in \mathcal{G}_\infty; \exists B \in \mathcal{F}, A \cap \{\tau > t\} = B \cap \{\tau > t\}\}$$

7.1 Conditional Expectation

We shall first focus on the conditional expectation $\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} Y | H_t \vee \mathcal{F}\right]$, where Y is a $\mathcal{G}_\infty-$integrable random variable. We start by the following result:

For any $\mathcal{G}_\infty-$ measurable random variable Y and any $t \in \mathbb{R}_+$ we have

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} Y | \mathcal{H}_t \vee \mathcal{F}\right] = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} Y | \mathcal{F}\right]}{\mathbb{P}\{\tau > t | \mathcal{F}\}} \tag{1}$$

We will demonstrate that the Cox process defined in the previous subsection is doubly stochastic. We resume for a moment the notations previously defined. In other words

Proposition 2.

$$\forall t \leq s, \quad \mathbb{P}\left[\tilde{\tau} > s | \tilde{\mathcal{F}}_s \vee \tilde{\mathcal{G}}_t\right] = \exp\left(-\int_t^s \mathrm{d}u\lambda_u\right) \mathbb{1}_{\{\tilde{\tau} > t\}}$$

Proof. The assumptions of the Lemma (1) are satisfied ($\mathbb{G} = \mathbb{F} \vee \mathbb{H}$), we have

$$\begin{aligned}
\forall t \leq s, \quad \mathbb{P}\left[\tilde{\tau} > s \middle| \tilde{\mathcal{F}}_s \vee \tilde{\mathcal{G}}_t\right] &= \mathbb{P}\left[\tilde{\tau} > s \middle| \tilde{\mathcal{F}}_s \vee \tilde{\mathcal{H}}_t\right] \\
&= \mathbb{1}_{\{\tilde{\tau}>t\}} \frac{\mathbb{P}[\tilde{\tau} > s | \tilde{\mathcal{F}}_s]}{\mathbb{P}[\tilde{\tau} > t | \tilde{\mathcal{F}}_s]}
\end{aligned}$$

On the other side, because of independence of U and $\mathcal{G}_\infty$ and then of $\tilde{\mathcal{F}}_s$, one can write

$$\forall t \leq s, \quad \mathbb{P}\left[\tilde{\tau} > t \middle| \tilde{\mathcal{F}}_s\right] = \exp\left(-\int_0^t \mathrm{d}u\lambda_u\right)$$

□

7.2 Default Contingent Valuation

Theorem 3. Let be τ a doubly stochastic $\mathbb{G}-$ stopping time relatively to filtration $\mathbb{F}$ and denote λ its intensity. If Z is a $\mathbb{F}-$ predictable and measurable process. Then

$$\mathbb{E}\left[\mathbb{1}_{\{t<\tau\leq s\}} Z_\tau \middle| \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} \mathbb{E}\left[\int_{]t,s]} \mathrm{d}u Z_u \lambda_u \exp\left(-\int_0^u \mathrm{d}s \lambda_s\right) \middle| \mathcal{F}_t\right]$$

8 Defaultabe Bond Pricing

In this section, we are interested in the valuation of Defaultable Bond. We place ourselves in the context of the lack of arbitration opportunity. We assume that all the assets considered are defined on a filtered probability space $\left(\Omega, \mathbb{Q}, \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\right)$. We assume the reference entity default time τ is $\mathbb{G}-$ stopping time with intensity λ, doubly stochastic relatively to filtration $\mathbb{F} \subset \mathbb{G}$, such as $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathcal{H}_t = \sigma(\tau \wedge t)$.

The probability $\mathbb{Q}$ is the classic risk-neutral probability relatively to interest rate process $(r_t)_{t\geq 0)}$ adapted to the filtration $\mathbb{F}$.

8.1 Recovery Payment valuation

The recovery payment consists of a payment of the recovery part $R \times N(\tau)$ at $\tau + \delta_B$.

The price of the recovery part is at evaluation date t

$$\begin{aligned}
V^{(\mathrm{rec})}(t, T; N, R, \delta_B) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{t<\tau\leq T\}} e^{-\int_t^{\tau+\delta_B} \mathrm{d}s\, r_s} R\, N(\tau) \middle| \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} R\, \mathbb{E}^{\mathbb{Q}}\left[\int_{]t\vee T_s, T]} \mathrm{d}u N(u) \lambda_u \exp\left(-\int_0^u \mathrm{d}s \lambda_s\right) \exp\left(-\int_t^{u+\delta_B} \mathrm{d}s\, r_s\right) \middle| \mathcal{F}_t\right]
\end{aligned}$$

8.2 Principal Payment valuation

The Principal payment occurs at the maturity of the bond is there is no default event.

$$\begin{aligned}
V^{(\mathrm{nml})}(t, T; N) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau\geq T\}} e^{-\int_t^T \mathrm{d}s\, r_s} N(T) \middle| \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} N(T) \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \mathrm{d}s \lambda_s\right) \exp\left(-\int_t^T \mathrm{d}s\, r_s\right) \middle| \mathcal{F}_t\right]
\end{aligned}$$

8.3 Premium leg valuation

The coupon leg consists of two parts: Regular premium payments until the default or the expiry of the Bond, denoted by $V^{(\mathrm{prem})}$, and a single payment of accrued premium in the event of default, denoted by $V^{(\mathrm{rkyacc})}$.

8.3.1 Fixed-Coupon Cash Flows

$$\begin{aligned}
V^{(\mathrm{prem})}(t, \mathcal{S}; c, N) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} c\alpha\left(t_i^{(s)}, t_i^{(e)}\right) \tilde{N}_i\, e^{-\int_t^{t_i^{(p)}} \mathrm{d}s\, r_s} \mathbb{1}_{\left\{\tau>t_i^{(e)}\right\}} \middle| \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} c\alpha\left(t_i^{(s)}, t_i^{(e)}\right) \tilde{N}_i\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_i^{(e)}} \mathrm{d}s \lambda_s} e^{-\int_t^{t_i^{(p)}} \mathrm{d}s\, r_s} \middle| \mathcal{F}_t\right]
\end{aligned}$$

where

$$\tilde{N}_i = \frac{1}{\alpha\left(t_i^{(s)}, t_i^{(e)}\right)} \int_{t_i^{(s)}}^{t_i^{(e)}} \mathrm{d}u N(u)$$

The second part of the premium leg is the accrued interest paid on default, its price is given by:

$$\begin{aligned} V^{(\mathrm{rkyacc})}(t, \mathcal{S}; c, N, \delta_C) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} c\alpha\left(t_i^{(s)}, \tau\right) N(\tau)\, e^{-\int_t^{\tau+\delta_C} \mathrm{d}s\, r_s} \mathbb{1}_{\left\{t \vee t_i^{(s)} \leq \tau < t_i^{(e)}\right\}} \middle| \mathcal{G}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \eta_i c \mathbb{E}^{\mathbb{Q}}\left[\int_{t\vee t_i^{(s)}}^{t_i^{(e)}} \mathrm{d}u\,(u - t_i^{(s)}) N(u) \lambda_u e^{-\int_0^u \mathrm{d}s \lambda_s} e^{-\int_t^{u+\delta_C} \mathrm{d}s\, r_s} \middle| \mathcal{F}_t \right] \end{aligned}$$

where

$$\eta_i = \frac{\alpha(t_i^{(s)}, t_i^{(e)})}{t_i^{(e)} - t_i^{(s)}}$$

Hence, the dirty price of a fixed-coupon defaultable bond is:

$$\begin{aligned} V^{(\mathrm{bond})}(t, \mathcal{S}; c, N, \delta_C, \delta_B, R) &= V^{(\mathrm{prem})}(t, \mathcal{S}; c, N) + V^{(\mathrm{rkyacc})}(t, \mathcal{S}; c, N, \delta_C) \\ &+ V^{(\mathrm{nml})}(t, T; N) \\ &+ V^{(\mathrm{rec})}(t, T; N, R, \delta_B) \end{aligned} \tag{2}$$

8.3.2 Floating-Coupon Cash Flows

We denote $L^{(I)}_{T,T+\delta}$ the Libor rate relative to the index I fixing at T and paying at $T+\delta$ and $F^{(I)}_{t,T,T+\delta}$ its forward rate seen from $t \leq T$

$$\begin{aligned} V^{(\mathrm{prem})}(t, \mathcal{S}; I, s, N) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \left(L^{(I)}_{t_i^{(s)}, t_i^{(p)}} + s\right) \alpha\left(t_i^{(s)}, t_i^{(e)}\right) \tilde{N}_i\, e^{-\int_t^{t_i^{(p)}} \mathrm{d}s\, r_s} \mathbb{1}_{\left\{\tau > t_i^{(e)}\right\}} \middle| \mathcal{G}_t \right] \\ &= V^{(\mathrm{prem})}(t, \mathcal{S}; s, N) \\ &+ \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \alpha\left(t_i^{(s)}, t_i^{(e)}\right) \tilde{N}_i\, \mathbb{E}^{\mathbb{Q}}\left[L^{(I)}_{t_i^{(s)}, t_i^{(p)}} e^{-\int_0^{t_i^{(e)}} \mathrm{d}s \lambda_s} e^{-\int_t^{t_i^{(p)}} \mathrm{d}s\, r_s} \middle| \mathcal{F}_t \right] \end{aligned}$$

By supposing the independence of rates and default intensity, we could write:

$$V^{(\mathrm{prem})}(t, \mathcal{S}; I, s, N) = \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \mathrm{d}u \lambda_u} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \alpha\left(t_i^{(s)}, t_i^{(e)}\right) \left(F^{(I)}_{t, t_i^{(s)}, t_i^{(p)}} + s\right) \tilde{N}_i\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_i^{(e)}} \mathrm{d}s \lambda_s} e^{-\int_t^{t_i^{(p)}} \mathrm{d}s\, r_s} \middle| \mathcal{F}_t \right]$$

8.3.3 Accrued

Accrued coupon of a fIxed-coupon defaultable bond at date $t \in [t_i^{(s)}, t_i^{(e)}[$ is given by

$$V^{(\mathrm{acc})}(t, \mathcal{S}; s, N) = \alpha\big(t_i^{(s)}, t\big)\, c\, \tilde{N}_i$$

For the floating coupon defaultable bond, there is two cases when calculating the accrued:

▷ the last Libor rate fixing $\hat{L}$ is available then

$$V^{(\mathrm{acc})}(t, \mathcal{S}; I, s, N) = \alpha\big(t_i^{(s)}, t\big)\, (\hat{L} + s)\, \tilde{N}_i$$

▷ the fixing is not available, we use the forward as of the valuation date

$$V^{(\mathrm{acc})}(t,\mathcal{S};I,s,N) = \alpha\big(t_i^{(s)},t\big)\left(\frac{1}{\alpha_I(t,t_i^{(p)})}\Big(1/P^{(I)}(t,t_i^{(p)})-1\Big)+s\right)\tilde{N}_i$$

where α_I is the day count corresponding to the rate index I.

8.4 Jump to Default (DTR)

The gross JtD formula at time t for the Defaultable bond is calculated as

$$\begin{aligned}
\mathrm{DTR} &= \omega\times\Big(R\times N(t)-V^{(\mathrm{prem})}(t,R)-V^{(\mathrm{rkyacc})}(t,R)-V^{(\mathrm{nml})}(t,R)-V^{(\mathrm{rec})}(t,R)\Big)\\
\forall R\in[0,1]\quad \mathrm{DTR}_0(R) &= \omega\times\Big(R\times N(t)-V^{(\mathrm{prem})}(t,R)-V^{(\mathrm{rkyacc})}(t,R)-V^{(\mathrm{nml})}(t,R)-V^{(\mathrm{rec})}(t,R)\Big),\quad \text{if}\quad R=\varnothing\\
&= \mathrm{DTR},\quad \text{else}
\end{aligned}$$

where

▷ $\omega=\pm1$ equal to $+1$ if protection seller and -1 else,

▷ R is the recovery of the credit reference entity (forced recovery $R\neq\varnothing$ or market recovery $R=\varnothing$),

▷ $N(t)$ the nominal value at time t.

9 Deterministic framework

9.1 Cash CDS basis

If the bonds of a given issuer were perfectly priced according to our framework, and if the term structure of credit risk as well as the recovery value was in agreement between the bond and CDS markets, the bond prices would satisfy the formula given by (2). The default intensity in this equation would be identical to the one calibrated from the CDS market.

When the two markets, bonds and CDS, show different levels of implied default probability, there is an ambiguity as to which of these is correct, if any. It is a common assumption that if there exists a liquid CDS market with full quoted term structure, then it is this market which is less biased as far as credit risk is concerned.

In this case, the natural definition of the CDS-Bond basis would result from the comparison of the bond market price with its fair value in the survival-based framework where the hazard rate is taken from the CDS market. Thus, the definition of the Basis Spread curve $(b_t)_{t\geq0}$ is:

$$\begin{aligned}
V^{(\mathrm{bond})}_{\mathrm{mkt}}(t,\mathcal{S};c,N,\delta_C,\delta_B,R) &= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \mathrm{d}u\lambda_u}\sum_{\substack{i=1\\ t_i^{(e)}>t}}^{n} c\alpha\left(t_i^{(s)},t_i^{(e)}\right)\tilde{N}_i\,\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_i^{(e)}}\mathrm{d}s\lambda_s}e^{-\int_t^{t_i^{(p)}}\mathrm{d}s\,(r_s+b_s)}\middle|\mathcal{F}_t\right]\\
&+ \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \mathrm{d}u\lambda_u}\sum_{\substack{i=1\\ t_i^{(e)}>t}}^{n}\eta_i c\mathbb{E}^{\mathbb{Q}}\left[\int_{t\vee t_i^{(s)}}^{t_i^{(e)}}\mathrm{d}u\,(u-t_i^{(s)})N(u)\lambda_u e^{-\int_0^u\mathrm{d}s\lambda_s}e^{-\int_t^{u+\delta_C}\mathrm{d}s\,(r_s+b_s)}\middle|\mathcal{F}_t\right]\\
&+ \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \mathrm{d}u\lambda_u}N(T)\mathbb{E}^{\mathbb{Q}}\left[\int_{]t\vee T_s,T]}\mathrm{d}u\lambda_u e^{-\int_0^u\mathrm{d}s\lambda_s}e^{-\int_t^{T}\mathrm{d}s\,(r_s+b_s)}\middle|\mathcal{F}_t\right]\\
&+ \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \mathrm{d}u\lambda_u}R\,\mathbb{E}^{\mathbb{Q}}\left[\int_{]t\vee T_s,T]}\mathrm{d}uN(u)\lambda_u e^{-\int_0^u\mathrm{d}s\lambda_s}e^{-\int_t^{u+\delta_B}\mathrm{d}s\,(r_s+b_s)}\middle|\mathcal{F}_t\right]
\end{aligned}$$

9.2 Interest Rate, Cash-Cds Basis and Default Intensity Curves

9.2.1 cPATZ Interpolation

The interest rate curve is interpolated log linearly on bond prices. *i.e.*

$$\forall i \in [\![0, n_r - 1]\!], \quad \forall t \in [t_i^{(r)}, t_{i+1}^{(r)}[, \quad P(0,t) \quad = \quad P(0,t_i)e^{-f_i \times (t-t_i)}$$

where $P(0,t) = e^{-\int_0^t \mathrm{d}u\, r(u)}$ is the t-zero coupon bond price at 0 and $\mathcal{T}_r = \{t_0^{(r)}, \ldots, t_{n_r}^{(r)}\}$ are the nodes of the interest rate curve.

9.3 Default Intensity Curve

9.3.1 Piecewise Constant Interpolation

The instantaneous forward intensity curve is piecewise constant on $\mathcal{T}_c = \left\{t_0^{(c)}, \ldots, t_{n_c}^{(c)}\right\}$. *i.e.*

$$\forall i \in [\![0, n_c - 1]\!], \quad \forall t \in [t_i^{(c)}, t_{i+1}^{(c)}[, \quad Q(0,t) \quad = \quad Q(0,t_i)e^{-h_i \times (t-t_i)}$$

where $Q(0,t) = e^{-\int_0^t \mathrm{d}u\, \lambda(u)}$ is the survival probability up to t. (*cf.* [1])

9.4 Cash-CDS Basis Curve

9.4.1 Piecewise Constant Interpolation

The Cash-CDS basis curve is interpolated log linearly on bond prices. *i.e.*

$$\forall i \in [\![0, n_b - 1]\!], \quad \forall t \in [t_i^{(b)}, t_{i+1}^{(b)}[, \quad B(0,t) \quad = \quad B(0,t_i)e^{-b_i \times (t-t_i)}$$

where $B(0,t) = e^{-\int_0^t \mathrm{d}u\, b(u)}$ is the equivalent "t-zero coupon" Basis price at 0 and $\mathcal{T}_b = \{t_0^{(r)}, \ldots, t_{n_b}^{(b)}\}$ are the nodes of the Cash-CDS basis curve.

9.5 Building Blocks Pricing

9.5.1 Laged Defautable zero-coupon bonds

We have to evaluate the following expression:

$$\bar{P}_{t,T}^{\delta} \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \mathrm{d}s \lambda_s\right)\exp\left(-\int_t^{T+\delta} \mathrm{d}s\, r_s\right)\middle|\mathcal{F}_t\right]$$

We denote the combined set of nodes by

$$\mathcal{T} \quad = \quad \{T_s, T\} \cup \mathcal{T}_c \cup \mathcal{T}_N \cup \mathcal{T}_r \cup \mathcal{T}_r^{-\delta_B} \cup \mathcal{T}_b \cup \mathcal{T}_b^{-\delta_B} \cup \mathcal{T}_r^{-\delta_C} = \{t_1, \ldots, t_n\}$$

and by $\beta(u)$ is the first date among the t_i s that follows u
where $\mathcal{T}_r^{-\delta} = \{t_0^{(r)} - \delta, \ldots, t_{n_r}^{(r)} - \delta\}$

Using the time grid $\mathcal{T}$ ensure that on each interval $[t_k, t_{k+1}[$ the forward intensity, rates and CDS-Cash basis rate are constant *i.e.* $\lambda(u) = \lambda_k, r(u) = r_k, b(u) = b_k$

Proposition 4. $\exists\, p \in [\![0, n_r - 1]\!], \quad [t_k, t_{k+1} + \delta_B[\subset [t_p^{(r)}, t_{p+1}^{(r)}]$

Proof. It is obvious that $p = \max\left\{t_p^{(r)} \leq t_k\right\}$ exists, since $t_0^{(r)} = t_0^{(c)} = 0$. By *reductio ad absurdum* we assume that $t_{p+1}^{(r)} - \delta_B < t_{k+1}$, however $t_{p+1}^{(r)} - \delta_B \in \mathcal{T}$ then necessarily $t_{k+1} + \delta_B \leq t_{p+1}^{(r)}$ □

9.5.2 Expression of the principal part

$$V^{(\mathrm{nml})}(t,T;N) \quad = \quad \mathbb{1}_{\{\tau > t\}} P(t,T)Q(t,T)B(t,T)N(T)$$

9.5.3 Expression of the protection part

The price of the recovery part is

$$
\begin{aligned}
V^{(\mathrm{rec})}(t,T;N,\delta_B) &= \mathbb{1}_{\{\tau>t\}} R \int_{]t\vee T_s,T]} \mathrm{d}u\, N(u)\lambda(u)\exp\left(-\int_t^u \mathrm{d}s\lambda(s)\right) e^{-\int_t^{u+\delta_B}\mathrm{d}s\,(r(s)+b(s))} \\
&= \mathbb{1}_{\{\tau>t\}} R \sum_{k=\beta(t\vee T_s)}^{\beta(T)-1} N_k \int_{t_k}^{t_{k+1}} \mathrm{d}u\,\lambda(u)\exp\left(-\int_t^u \mathrm{d}s\lambda(s)\right) e^{-\int_t^{u+\delta_B}\mathrm{d}s\,(r(s)+b(s))} \\
&= \mathbb{1}_{\{\tau>t\}} R \sum_{k=\beta(t\vee T_s)}^{\beta(T)-1} N_k P(t,t_k)Q(t,t_k)B(t,t_k)\lambda_k e^{-\delta_B(r_k+b_k)} \int_{t_k}^{t_{k+1}} \mathrm{d}u\, e^{-(\lambda_k+r_k+b_k)(u-t_k)} \\
&\overset{\text{proposition (4)}}{=} \mathbb{1}_{\{\tau>t\}} R \sum_{k=\beta(t\vee T_s)}^{\beta(T)-1} I_k(\delta_B)
\end{aligned}
$$

where

$$
I_k(\delta) = N_k P(t,t_k)Q(t,t_k)B(t,t_k)\frac{\lambda_k e^{-\delta(r_k+b_k)}}{\lambda_k+r_k+b_k}\left(1-e^{-(\lambda_k+r_k+b_k)(t_{k+1}-t_k)}\right)
$$

If $\lambda_k+r_k+b_k$ is very small ($\lambda_k+r_k+b_k\to 0$), we use the Taylor expansion form of the function $\epsilon : x \mapsto \frac{1-e^x}{x}$ in the fourth order, $\epsilon(x) \sim -1-\frac{1}{2}x-\frac{1}{6}x^2-\frac{1}{24}x^3-\frac{1}{120}x^4$, which is well defined for $x\to 0$[1], the expression of I_k becomes then

$$
I_k(\delta) = N_k P(t,t_k)Q(t,t_k)B(t,t_k)\lambda_k(t_{k+1}-t_k)e^{-\delta(r_k+b_k)}\epsilon(-(\lambda_k+r_k+b_k)(t_{k+1}-t_k))
$$

9.5.4 Fixed-Coupon Cash Flows

Under deterministic default intensity, rate and cash-CDS basis the expression of the present value of the survival contingent cash flows is straightforward

$$
V^{(\mathrm{prem})}(t,\mathcal{S};c,N) = \mathbb{1}_{\{\tau>t\}} \sum_{\substack{i=1\\ t_i^{(e)}>t}}^{n} c\alpha\left(t_i^{(s)},t_i^{(e)}\right)\tilde{N}_i P(t,t_i^{(p)})B(t,t_i^{(p)})Q(t,t_i^{(e)})
$$

Therefore, we just focus on the coupon upon default over the $i-$th payment period.

$$
\begin{aligned}
J_i &= \mathbb{E}^{\mathbb{Q}}\left[\int_{t\vee t_i^{(s)}}^{t_i^{(e)}} \mathrm{d}u\,(u-t_i^{(s)})N(u)\lambda_u e^{-\int_t^u \mathrm{d}s\lambda_s} e^{-\int_t^{u+\delta_C}\mathrm{d}s\,(r_s+b_s)}\middle|\mathcal{F}_t\right] \\
&= \int_{t\vee t_i^{(s)}}^{t_i^{(e)}} \mathrm{d}u\,(u-t_i^{(s)})N(u)\lambda(u)e^{-\int_t^u \mathrm{d}s\lambda(s)} e^{-\int_t^{u+\delta_C}\mathrm{d}s\,(r(s)+b(s))}
\end{aligned}
$$

If we now truncate $\mathcal{T}$, so it only contains the n_i nodes between $t\vee t_i^{(s)}$ and $t_i^{(e)}$ exclusively, then add $t\vee t_i^{(s)}$ and $t_i^{(e)}$ as the first and last node we have (*i.e.* $\tilde{t}_0 = t\vee t_i^{(s)}, \tilde{t}_{n_i} = t_i^{(e)}$)

By denoting $h_k = \lambda_k + r_k + b_k$ and $\tilde{\Delta}_k = \tilde{t}_k - \tilde{t}_{k-1}$

[1] The threshold is $|x| < 10^{-04}$

$$
\begin{aligned}
J_i &= \sum_{k=1}^{n_i} N_k P(t,\tilde{t}_{k-1}) Q(t,\tilde{t}_{k-1}) B(t,\tilde{t}_{k-1}) \lambda_k e^{-(r_k+b_k)\delta_C} \int_{\tilde{t}_{k-1}}^{\tilde{t}_k} \mathrm{d}u \, (u - t_i^{(s)}) e^{-h_k(u-\tilde{t}_{k-1})} \\
&= \sum_{k=1}^{n_i} N_k P(t,\tilde{t}_{k-1}) Q(t,\tilde{t}_{k-1}) B(t,\tilde{t}_{k-1}) e^{-(r_k+b_k)\delta_C} \frac{\lambda_k}{h_k} \left((\tilde{t}_{k-1} - t_i^{(s)}) - (\tilde{t}_k - t_i^{(s)}) e^{-h_k \tilde{\Delta}_k} + \frac{1-e^{-h_k\tilde{\Delta}_k}}{h_k} \right) \\
&= \sum_{k=1}^{n_i} N_k P(t,\tilde{t}_{k-1}) Q(t,\tilde{t}_{k-1}) B(t,\tilde{t}_{k-1}) e^{-(r_k+b_k)\delta_C} \frac{\lambda_k}{h_k} \left(\tilde{\Delta}_k \left(\frac{1-e^{-h_k\tilde{\Delta}_k}}{h_k \tilde{\Delta}_k} - e^{-h_k\tilde{\Delta}_k} \right) + (\tilde{t}_{k-1} - t_i^{(s)})(1-e^{-h_k\tilde{\Delta}_k}) \right)
\end{aligned}
$$

We must consider again the case of $h_k \to 0$ can be handled using the Taylor Expansion form.

$$
V^{(\mathrm{rkyacc})}(t,\mathcal{S};c,N,\delta_C) = \mathbb{1}_{\{\tau>t\}} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \eta_i c J_i
$$

9.5.5 Floating-Coupon Cash Flows

In the case of floating-coupon

$$
V^{(\mathrm{prem})}(t,\mathcal{S};I,s,N) = \mathbb{1}_{\{\tau>t\}} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \alpha\left(t_i^{(s)},t_i^{(e)}\right) \left(F^{(I)}_{t,t_i^{(s)},t_i^{(p)}} + s\right) \tilde{N}_i P(t,t_i^{(p)}) B(t,t_i^{(p)}) Q(t,t_i^{(e)})
$$

We can deduce the same formula as for the fixed-coupon case

$$
V^{(\mathrm{rkyacc})}(t,\mathcal{S};I,s,N,\delta_C) = \mathbb{1}_{\{\tau>t\}} \sum_{\substack{i=1 \\ t_i^{(e)}>t}}^{n} \eta_i \frac{1}{\alpha_I(t_i^{(s)},t_i^{(p)})} \left(\frac{P^{(I)}(t,t_i^{(s)})}{P^{(I)}(t,t_i^{(p)})} - 1 \right) J_i
$$

10 Tests

For our tests, we use the following risky bonds:

Table 1: 7Y Fixed-coupon defaultable bond on TELECOM-MB

Issuer	
Name	TELECOMMB
Seniority	SRUNSEC
Doc Clause	MR
Credit Lag	30
Fixed-Coupon Bond	
Notional	1 000 000
Start Date	08-AUG-16
Maturity	03-JUL-23
Coupon	2.9%
Day Count	A360
Currency	USD
Payment Frequency	Q
Coupon on default	Yes
Nominal Exchange	NXBOTH

Table 2: 7Y Floating-coupon defaultable bond on TELECOM-MB

Issuer	
Name	TELECOMMB
Seniority	SRUNSEC
Doc Clause	MR
Credit Lag	30
Fixed-Coupon Bond	
Notional	1 000 000
Start Date	lun 08-AUG-16
Maturity	lun 03-JUL-23
Index Tenor	12M
Spread	20 bp
Day Count	A360
Currency	USD
Payment Frequency	A
coupon on default	ACC
Nominal Exchange	NXBOTH

Here, we present the test steps:

- ▷ price the contracts (Table 1, Table 2) with real market data without using the CDS-basis curve between 08-Aug-2016 and 09-Apr-2021,
- ▷ price the same contact with a synthetic stressed market data,
- ▷ compare the two pricers on some selected dates.

10.1 Pricing Example

10.1.1 Market Data

Table 3

As Of Date	LIBOR 3M Tenor	LIBOR 3M Rate	LIBOR 12M Tenor	LIBOR 12M Rate	Tenor	Spread
16-mars-16	2D	0.92578	2D	0.92578	3M	12.8612
	3M	1.15178	3M	2.60178	6M	12.8612
	JUN17	98.68442	JUN17	99.37035	1Y	19.1221
	SEP17	98.54349	SEP17	98.85801	3Y	55.993
	DEC17	98.41638	DEC17	97.48747	5Y	98.0785
	MAR18	98.30513	MAR18	97.62837	7Y	124.9597
	JUN18	98.1746	JUN18	97.69753	10Y	141.1707
	SEP18	98.05975	SEP18	98.04424		
	DEC18	97.93562	DEC18	97.56314	Recovery Rate	
	MAR19	97.86238	MAR19	97.06327	40%	
	3Y	1.8545	3Y	2.226048		
	4Y	2.0065	4Y	2.319791		
	5Y	2.1215	5Y	2.428675		
	6Y	2.2185	6Y	2.520445		
	7Y	2.3005	7Y	2.597352		
	8Y	2.369	8Y	2.659566		
	9Y	2.42875	9Y	2.723044		
	10Y	2.48225	10Y	2.79028		
	11Y	2.529382	11Y	2.831562		
	12Y	2.568	12Y	2.874268		
	13Y	2.601077	13Y	2.901441		
	14Y	2.627568	14Y	2.932035		
	15Y	2.65	15Y	2.958612		
	20Y	2.71925	20Y	3.028823		
	25Y	2.7415	25Y	3.047028		
	30Y	2.745	30Y	3.046495		
	40Y	2.73125	40Y	3.032766		
	50Y	2.71125	50Y	2.990077		

Dirty Price	978 444.20
Clean Price	972 644.20
Accrued Coupon	5 800.00
Risky Accrued Coupon	420.81
Recovery Price	45 572.42

Table 4: All pricing measures of the Fixed-Coupon Bond as of 16-mar-2016

Dirty Price	1 050 842.07
Clean Price	1 031 737.04
Accrued Coupon	19 105.03
Risky Accrued Coupon	2 500.46
Recovery Price	45 572.42

Table 5: All pricing measures of the Floating-Coupon Bond as of 16-mar-2016

10.2 Backtest Pricing

In order to test the Bond pricer, we ran a pricing between 08-Aug-2016 and 09-Apr-2021 using the market data (rates and CDS curves) at each pricing date. The idea is to ensure that there is no issues when using historical data and that there is no price jump that could not be explained by changes in the market data.

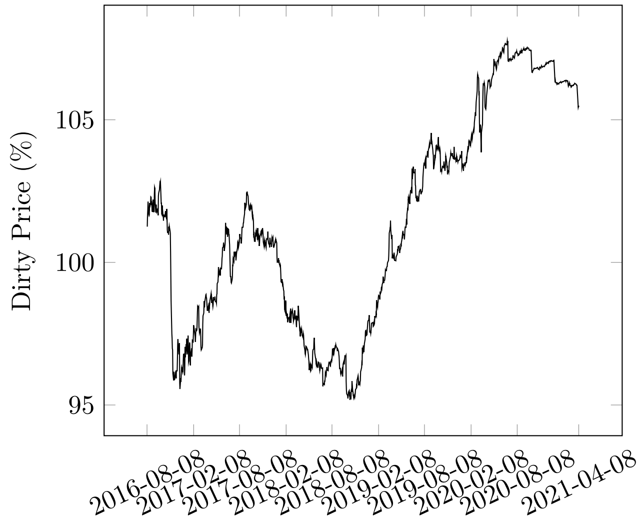


Figure 1: Fixed-Coupon bond dirty prices

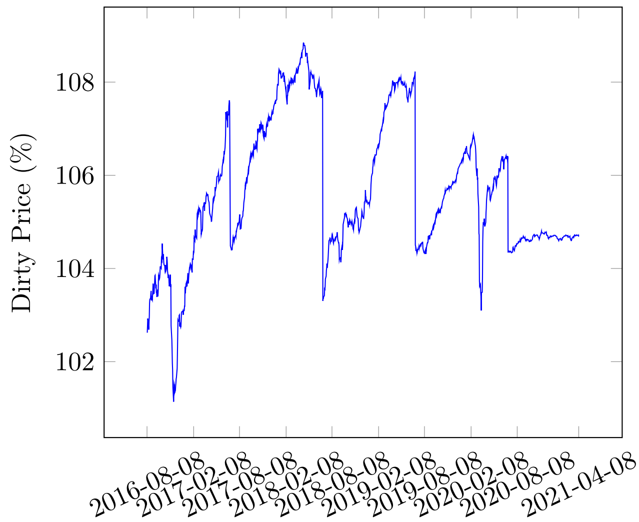


Figure 2: Floating-Coupon bond dirty prices

10.2.1 Stressed Market Data

Table 6: Add caption

As Of Date	LIBOR 3M Tenor	LIBOR 3M Rate	LIBOR 12M Tenor	LIBOR 12M Rate
16-mars-16	2D	0.92578	2D	0.92578
	3M	1.15178	3M	2.60178
	JUN17	98.68442	JUN17	99.37035
	SEP17	98.54349	SEP17	98.85801
	DEC17	98.41638	DEC17	97.48747
	MAR18	98.30513	MAR18	97.62837
	JUN18	98.1746	JUN18	97.69753
	SEP18	98.05975	SEP18	98.04424
	DEC18	97.93562	DEC18	97.56314
	MAR19	97.86238	MAR19	97.06327
	3Y	1.8545	3Y	2.226048
	4Y	2.0065	4Y	2.319791
	5Y	2.1215	5Y	2.428675
	6Y	2.2185	6Y	2.520445
	7Y	2.3005	7Y	2.597352
	8Y	2.369	8Y	2.659566
	9Y	2.42875	9Y	2.723044
	10Y	2.48225	10Y	2.79028
	11Y	2.529382	11Y	2.831562
	12Y	2.568	12Y	2.874268
	13Y	2.601077	13Y	2.901441
	14Y	2.627568	14Y	2.932035
	15Y	2.65	15Y	2.958612
	20Y	2.71925	20Y	3.028823
	25Y	2.7415	25Y	3.047028
	30Y	2.745	30Y	3.046495
	40Y	2.73125	40Y	3.032766
	50Y	2.71125	50Y	2.990077

Tenor	Stressed Spread (x100)
3M	1286.12
6M	1286.12
1Y	1912.21
3Y	5599.3
5Y	9807.85
7Y	12495.97
10Y	14117.07

Recovery Rate
40%

Dirty Price	434 451.28
Clean Price	428 651.28
Accrued Coupon	5 800.00
Risky Accrued Coupon	3 596.97
Recovery Price	392 443.66

Table 7: All pricing measures of the Fixed-Coupon Bond as of 16-mar-2016 with stressed CDS spreads

Dirty Price	452 874.87
Clean Price	433 769.84
Accrued Coupon	19 105.03
Risky Accrued Coupon	18 081.89
Recovery Price	392 443.66

Table 8: All pricing measures of the Floating-Coupon Bond as of 16-mar-2016 with stressed CDS spreads

10.3 Benchmark

To check that the new pricer (*Pricer* 2) addresses the issues observed when pricing the default contingent cash-flows (protection Npv and accrued on default) within the current pricer (*Pricer* 1). We perform a benchmark between the two pricers as of 17-mar-2017

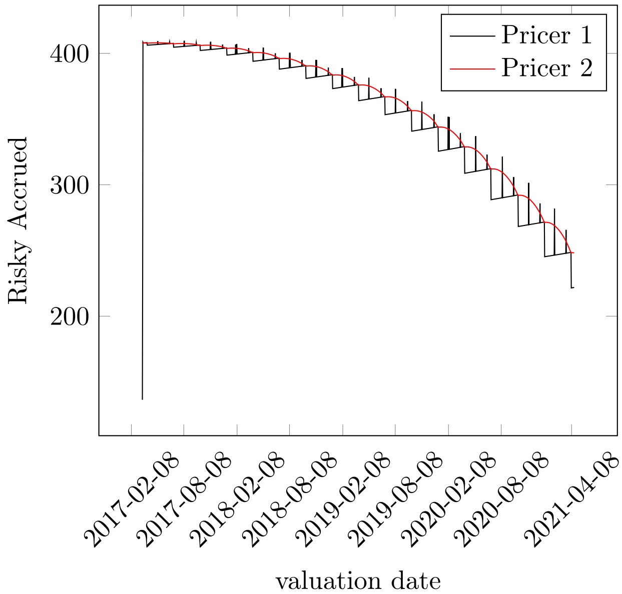
Table 9

	Pricer 2	Pricer 1
Dirty Price	981 216.38	517 079.54
Clean Price	975 335.82	511 198.99
Accrued Coupon	5 880.56	5 880.56
Risky Accrued Coupon	408.03	4 889.17
Recovery Price	44 184.95	331 198.97

We could remark that price given by the Pricer 1 is very far from the one given by the new pricer (Pricer 2).

10.3.1 Risky Accrued Profile

Here we compare the accrued on default (risky accrued) calculated by both pricer 1 and 2, while varying the evaluation date.



We can observe that the pricer 1 presents some issues. Indeed the risky accrued profile by definition is a continuous polynomial function. However, the risky accrued within the pricer 1 present discontinuities at each accrued period and even a jump on the first date (17 march 2017).

10.3.2 JTD

Here we calculate the DTR for both fixed and floating coupon bond as of 16-MAR-2017:

	Fixed Coupon Bond	Floating Coupon Bond
DTR	- 578 444.20	- 650 842.07

If we force the recovery rate to be equal to 40%, we get the expected results from the pricer
If we set the recovery rate on recovery market we get also the expected results

Fixed Coupon Bond r	DTR_0	Floating Coupon Bond r	DTR_0
0%	- 578 444.20	0%	- 650 842.07
5%	- 578 444.20	5%	- 650 842.07
10%	- 578 444.20	10%	- 650 842.07
15%	- 578 444.20	15%	- 650 842.07
20%	- 578 444.20	20%	- 650 842.07
25%	- 578 444.20	25%	- 650 842.07
30%	- 578 444.20	30%	- 650 842.07
35%	- 578 444.20	35%	- 650 842.07
40%	- 578 444.20	40%	- 650 842.07
45%	- 578 444.20	45%	- 650 842.07
50%	- 578 444.20	50%	- 650 842.07
55%	- 578 444.20	55%	- 650 842.07
60%	- 578 444.20	60%	- 650 842.07
65%	- 578 444.20	65%	- 650 842.07
70%	- 578 444.20	70%	- 650 842.07
75%	- 578 444.20	75%	- 650 842.07
80%	- 578 444.20	80%	- 650 842.07
85%	- 578 444.20	85%	- 650 842.07
90%	- 578 444.20	90%	- 650 842.07
95%	- 578 444.20	95%	- 650 842.07
100%	- 578 444.20	100%	- 650 842.07

Fixed Coupon Bond r	DTR_0	Floating Coupon Bond r	DTR_0
0%	- 978 444.20	0%	- 1 050 842.07
5%	- 928 444.20	5%	- 1 000 842.07
10%	- 878 444.20	10%	- 950 842.07
15%	- 828 444.20	15%	- 900 842.07
20%	- 778 444.20	20%	- 850 842.07
25%	- 728 444.20	25%	- 800 842.07
30%	- 678 444.20	30%	- 750 842.07
35%	- 628 444.20	35%	- 700 842.07
40%	- 578 444.20	40%	- 650 842.07
45%	- 528 444.20	45%	- 600 842.07
50%	- 478 444.20	50%	- 550 842.07
55%	- 428 444.20	55%	- 500 842.07
60%	- 378 444.20	60%	- 450 842.07
65%	- 328 444.20	65%	- 400 842.07
70%	- 278 444.20	70%	- 350 842.07
75%	- 228 444.20	75%	- 300 842.07
80%	- 178 444.20	80%	- 250 842.07
85%	- 128 444.20	85%	- 200 842.07
90%	- 78 444.20	90%	- 150 842.07
95%	- 28 444.20	95%	- 100 842.07
100%	21 555.80	100%	- 50 842.07

10.4 Conclusion

The price jumps observed within the (*Pricer* 1) disappeared using the new pricer, moreover the (*Pricer* 2 ensure continuity of the risky accrued price over time.

References

[1] Jean-Marc PRIE. Yield curve stripping with forecast and discount curves. *FI QR Document*, 2012.