ADVANCED DYNAMICS, SPRING SEMESTER 2023
ETH ZÜRICH

Assignment 2

Paolo Tiso

1 EULER-BERNOULLI BEAM

Consider the straight planar beam depicted in Figure 1.1. The beam is aligned along the x axis, while the z axis is perpendicular to the mean line. The beam is made of elastic material of Young modulus E and volumetric density ρ. The cross section is rectangular, of unit depth along the y direction, and the cross sectional area $h(x)$ is linearly tapered from h_0 to h_L. Assume that the bending moment of inertia is $I(x) = \frac{1}{12}h(x)^3$. The beam is full clamped at one end and simply supported at the opposite tip. A arbitrary line load $p(x,t)$ is acting on the beam.

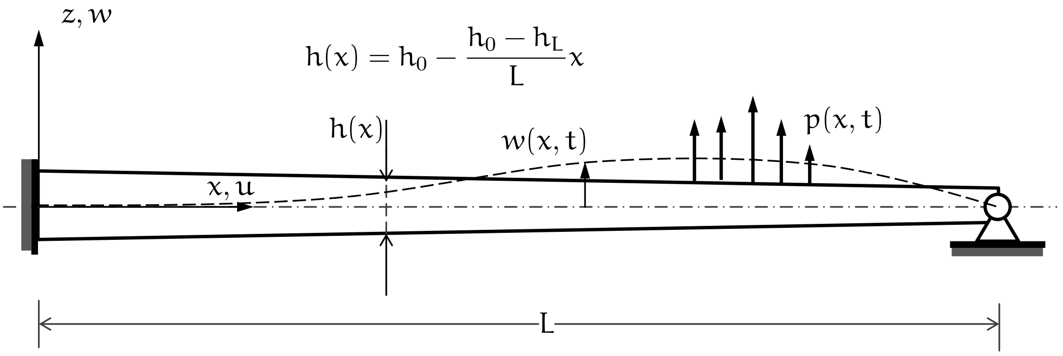

Figure 1.1: Definition of loads, geometry and boundary conditions for the beam under consideration.

The Euler-Bernoulli beam model assumes that[1]:

- the cross sections remain straight;
- the transverse displacement w is uniform along the cross section, i.e. $w = w(x)$;
- the axial displacement $u(x,z)$ results from the rotation of the cross section, i.e.

$$u(x,z) = -z\frac{\partial w}{\partial x}. \tag{1.1}$$

The resulting strains are given by

$$\epsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \tag{1.2}$$

$$\epsilon_z = \frac{\partial w}{\partial z} = 0 \tag{1.3}$$

The Euler-Bernoulli kinematic assumptions are clarified in Figure 1.2.

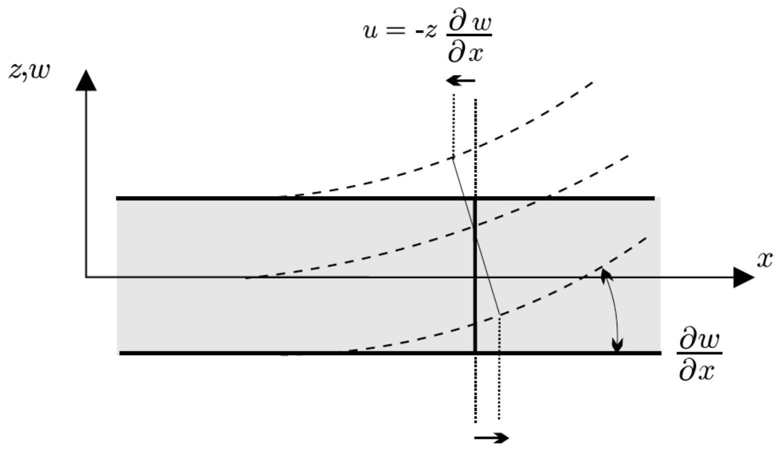

Figure 1.2: Euler-Bernoulli kinematic assumptions.

2 Derivation of equations of motion

We are interested in finding the partial differential equation and the corresponding boundary conditions by using the Hamilton's principle. In order to do this, we need to write the kinetic

[1]This material should be known to you from the course *Mechanics 2*, or equivalent.

and potential energies. Given the kinetic assumptions illustrated above, the kinetic energy is given by

$$T = \frac{1}{2}\int_0^L \int_{A(x)} \rho(\dot{u}^2 + \dot{w}^2)\,dA\,dx = \frac{1}{2}\int_0^L \int_{A(x)} \rho \left[z^2 \left(\frac{\partial \dot{w}}{\partial x}\right)^2 + \dot{w}^2\right] dA\,dx =$$
$$= \frac{1}{2}\int_0^L \left[\rho I(x) \left(\frac{\partial \dot{w}}{\partial x}\right)^2 + \rho A(x)\dot{w}^2\right] dx \tag{2.1}$$

while the elastic potential energy V is given by

$$V = \frac{1}{2}\int_{A(x)} \int_0^L E z^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,dA = \frac{1}{2}\int_0^L EI(x) \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx. \tag{2.2}$$

The work W of the external loads is given by

$$W = \int_0^L p(x,t)w(x,t)\,dx \tag{2.3}$$

3 Assigment tasks

You are asked to:

1. Derive the equations of motion, and the relative boundary conditions (natural and kinematical), using the Hamilton's principle, which writes

$$\int_{t_1}^{t_2} \delta(T - V + W)\,dt = 0 \tag{3.1}$$

2. Discuss how the equation of motion, and the relative boundary conditions, change if the support at $x = L$ is removed.

Note: check your results! The equation of motion of straight beams (at least with constant cross section) should be known material from previous courses. You are then expected to comment each term of the found equations, if asked.