Advanced Dynamics - Assignment Report

Lasse Fierz

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1 Assignment 1

1 Wheel rolling without slipping on a 2D track

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \ x_2^C \ x_3^C \ \alpha \ \phi]$.

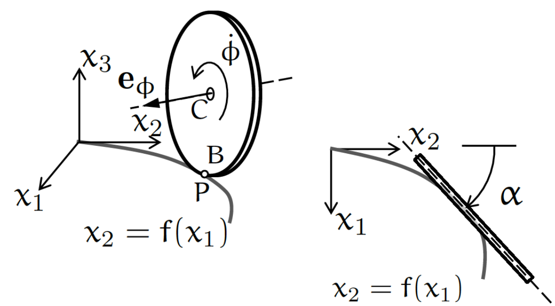

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

1.1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C\boldsymbol{e_1} + \dot{x}_2^C\boldsymbol{e_2} - R\dot{\phi}\cos\alpha\boldsymbol{e_2} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

As α is not dependent on time it can be easily seen that (3) is integrable:

$$\begin{aligned}&\int \dot{x}_1^C dt = \int R\dot{\phi}\sin\alpha dt\\ &\Rightarrow x_1^C = R\phi\sin\alpha\end{aligned} \tag{4}$$

1.1.2

See subsubsection (1.1.1)

1.1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e}_{\phi}$ while all the other generalized coordinates follow accordingly.

2 Two bars linkage

Two bars AB and BC of equal length L are hinged at point B, and move in the plane spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The velocity $\mathbf{v}_C$ of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use x_1^B, x_2^B, θ_1 and θ_2 as generalized coordinates.

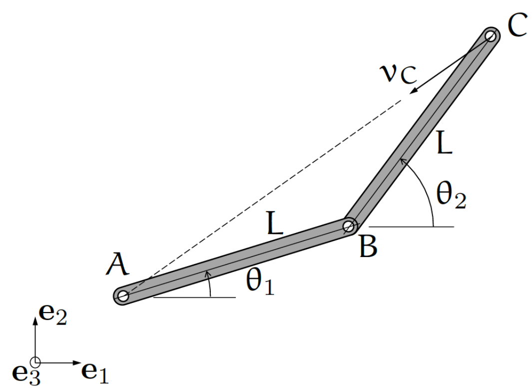

Figure 2.1: A two bar linkage in 2D. The velocity v_C must be directed towards A at all times.

Figure 3: Task 1.1

1.2

I want to show that the constraint that v_c always points in the direction of AD is not integrable. I would like to express this constraint as:

$$\begin{aligned} &v_C * AC_p = 0 \\ &\text{where } AC_p \text{ is a vector perpendicular to } AC \end{aligned} \tag{5}$$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} \tag{6}$$

And AC is simply:

$$AC = \begin{pmatrix} L\,(\cos\theta_1 + \cos\theta_2) \\ L\,(\sin\theta_1 + \sin\theta_2) \end{pmatrix} \tag{7}$$

A simple vector that is perpendicular to AC can be gotten by switching the e_1 and e_2 entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} \tag{8}$$

$$\begin{aligned} \Rightarrow & v_C * AC_p = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} = \\ & \left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{\mathrm{x}}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B - \\ & L^2\dot{\theta}_2\cos\left(\theta_2\right)\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right) - L^2\dot{\theta}_2\sin\left(\theta_2\right)\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right) = 0 \end{aligned} \tag{9}$$

After applying trigonometry:

$$\begin{aligned} & \left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{\mathrm{x}}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B - \\ & L^2\dot{\theta}_2\left(\cos\left(\theta_2\right)\cos\left(\theta_1\right) + \sin\left(\theta_2\right)\sin\left(\theta_1\right) + 1\right) = 0 \end{aligned} \tag{10}$$

To simplify the notation I will refer to $\sin\theta_1$ as s_1 and $\sin\theta_2$ as s_2:

$$L\dot{x}_1^B\left(s_1 + s_2\right) - L\dot{x}_2^B\left(c_1 + c_2\right) - L^2\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{11}$$

Dividing by L:

$$\dot{x}_1^B\left(s_1 + s_2\right) - \dot{x}_2^B\left(c_1 + c_2\right) - L\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{12}$$

With the generalized coordinates $q = [x_1^B, x_2^B, \theta_1, \theta_2]$
Writing down the coefficients of the non-holonomic constraint:

$$\begin{aligned} a_1 &= s_1 + s_2 \\ a_2 &= -\left(c_1 + c_2\right) \\ a_3 &= 0 \\ a_4 &= -L\left(1 + s_1s_2 + c_1c_2\right) \\ b &= 0 \end{aligned} \tag{13}$$

To check for the exact velocity form we want to proof that there can't exist a $C(q) \neq 0$ for which holds:

$$\frac{\partial(Cb)}{\partial q_i} = \frac{\partial(Ca_1)}{\partial t} \quad \text{and} \quad \frac{\partial(Ca_i)}{\partial q_k} = \frac{\partial(Ca_k)}{\partial q_i} \quad \text{for all the gen. coord.} \tag{14}$$

q1-q3:

$$\begin{aligned}
&\frac{\partial(Ca_1)}{\partial\theta_1} = 0 \Rightarrow C * c_1 + \frac{\partial C}{\partial\theta_1}(s1+s2) = 0\\
&\Rightarrow \frac{c_1}{s1+s2} + \frac{1}{C}\frac{dC}{d\theta_1} = 0 \Rightarrow \int \frac{1}{C}dC = -\int \frac{c_1}{s_1+s_2}d\theta_1\\
&\Rightarrow lnC = \frac{D}{s_1+s_2} + D \Rightarrow C = \frac{D}{s_1+s_2} \quad \text{where the integration const. D was updated}\\
&\Rightarrow C = \frac{D(x_1^B, x_2^B, \theta_2)}{s1+s2}
\end{aligned} \tag{15}$$

q1 - q4:

$$\frac{\partial Ca_1}{\partial\theta_2} = \underbrace{\frac{\partial Ca_4}{\partial x_1^B}}_{0} = 0 \Rightarrow \frac{\partial D}{\partial\theta_2} = 0 \Rightarrow C = \frac{D(x_1^B, x_2^B)}{s1+s2} \tag{16}$$

q1 - q2:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = \frac{\partial Ca_2}{\partial x_1^B} \Rightarrow \frac{\partial C}{\partial x_2^B}(s_1+s_2) = \frac{\partial C}{\partial x_1^B}(c_1+c_2) \Rightarrow \frac{\partial C}{\partial x_2^B} = \frac{\partial C}{\partial x_1^B}\frac{c_1+c_2}{s_1+s_2}\\
&\text{Remembering that } D = D(x_1^B, x_2^B) \Rightarrow D = const.\\
&\Rightarrow C = \frac{D}{s_1+s_2} \text{ where D const.}
\end{aligned} \tag{17}$$

q2 - q3:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = 0 \Rightarrow D\frac{\partial \frac{(c_1+c_2)}{s_1+s_2}}{\partial\theta_1} = 0\\
&D\frac{\partial \frac{c_1}{s_1+s_2}}{\partial\theta_1} = -\frac{Ds_1}{s_1+s_2} - \frac{Dc_1^2}{(s_1+s_2)^2} = -D\frac{s_1(s_1+s_2)-c_1^2}{(s_1+s_2)^2}
\end{aligned} \tag{18}$$

$$D\frac{\partial \frac{c_2}{s_1+s_2}}{\partial\theta_1} = -D\frac{c_1 c_2}{(s_1+s_2)^2} \tag{19}$$

Which leads to:

$$-D\frac{c_1^2 + s_1^2 + s_1 s_2 + c_1 c_2}{(s_1 + s_2)^2} \overset{!}{=} 0 \tag{20}$$

Which is a contradiction. Therefore this is a non-holonomic constraint.

2 Assignment 2

2.1 Hamilton's Principle

This assignment asks to get the equations of motion using the Hamilton Principle:

$$\int_{t_1}^{t_2} delta(T-V+W)dt = 0 \tag{21}$$

Using the theory discussed in the lecture we can split the term into different contributions::
Remembering the result of the Hamilton principle from the lecture:

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right) + f(x,t) + \sum F_i D(x-x_i) = 0, \quad x \in [0; l]$$

Boundary conditions:

$$\left[\frac{\partial L}{\partial v'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right]\delta v = 0, \quad x = 0, l$$

$$\frac{\partial L}{\partial v''}\delta v' = 0, \quad x = 0, l$$

Figure 4: Hamilton Formulas for a continuous system

We consider w our variable instead of v.

$$\frac{\partial L}{\partial w} = 0 \tag{22}$$

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial w'} = 0 \tag{23}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial w''} = -E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) \tag{24}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = A(x)\rho\ddot{w} \tag{25}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}'}\right)\right) = \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) \tag{26}$$

$$f(x,t) = p(x,t) \tag{27}$$

$$\frac{\partial L}{\partial w'} = 0 \tag{28}$$

$$\frac{\partial}{\partial x}(\frac{\partial L}{\partial w''}) = -E\left(I(x)w^{(3)} + I'(x)w''\right) \tag{29}$$

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{w}'}) = I(x)\rho\ddot{w}' \tag{30}$$

2.1.1 Differential Equation aka green part

Therefore the green part comes to:

$$p(x,t) - E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) + \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) - A(x)\rho\ddot{w} = 0 \tag{31}$$

Comparing this solution to the Euler-Bernoulli Equation:

$$p(x,t) = EIw^{(4)} + A(x)\rho\ddot{w} \tag{32}$$

What we can see is that the additional terms from the green expression come from the derivative of I. This comes from the changing cross section and the thus changing moment of Inertia I. Secondly we had a term with $\dot{w}'$ in the kinetic energy arising through the change in thickness which is considered here. (not the case for the euler bernoulli beam).

2.1.2 Boundary conditions with virtual displacement aka Red Part

$$\left[\delta\text{w(x)}\left(E\left(I(x)w^{(3)} + I'(x)w''\right) - I(x)\rho\ddot{w}'\right)\right]_0^L = 0 \tag{33}$$

2.1.3 Blue part

$$\left[-\delta\text{w}'\text{(x)}EI(x)w''\right]_0^L = 0 \tag{34}$$

2.2 Analysis

$$I(0) = \frac{1}{12}h_0^3 \text{ and } I(L) = \frac{1}{12}h_L^3 \tag{35}$$

as well as

$$I'(0) = \frac{1}{4}h(0)^2 * \frac{h_L - h_0}{L} = \frac{1}{4}h_0^2 * \frac{h_L - h_0}{L} \tag{36}$$

and

$$I'(L) = \frac{1}{4}h_L^2 * \frac{h_L - h_0}{L} \tag{37}$$

Beginning with the blue part and pluggin in the boundary values:

$$-\delta \mathrm{w}'(\mathrm{L})E\mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0)E\mathrm{h}_0^3 w''(0) = 0 \tag{38}$$

As we have a clamped end (w, δw and their first three derivatives are 0) and a supported end (w, δw and their first derivative is 0) we arrive at:

$$0 = 0 \tag{39}$$

For the red part:

$$\begin{aligned} &\delta w(L)\left(E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)\right) - \\ &\delta w(0)\left(E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)\right) = 0 \end{aligned} \tag{40}$$

Note that both $\delta w(0) = \delta w(L) = 0$. The coefficients

$$E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)$$

and

$$E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)$$

represent the forces at the boundaries. To determine these forces we would have to solve the differential equation to get a solution for w which we can use to get the forces.

2.3 Analysis with open second end

Nothing changes in the application of the Hamilton Principle. However when analysing the boundaries we see some changes:
Starting again with the blue:

$$-\delta \mathrm{w}'(\mathrm{L}) E \mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0) E \mathrm{h}_0^3 w''(0) = 0 \tag{41}$$

As before the w terms at $x = 0$ vanish. However now $\delta w'(L) \neq 0$ thus we get

$$E \mathrm{h}_L^3 w''(L) = 0 \tag{42}$$

So the slope at the vertical displacement of the free end has to stay constant.
For the red part we get:

$$\begin{aligned} &\delta w(L) \left(E \left(I(L) w^{(3)}(L) + I'(L) w''(L) \right) - I(L) \rho \ddot{w}'(L) \right) - \\ &\delta w(0) \left(E \left(I(0) w^{(3)}(0) + I'(0) w''(0) \right) - I(0) \rho \ddot{w}'(0) \right) = 0 \end{aligned} \tag{43}$$

As before $\delta w(0) = 0$ and the force of the boundary constraint is:

$$E \left(I(0) w^{(3)}(0) + I'(0) w''(0) \right) - I(0) \rho \ddot{w}'(0) \tag{44}$$

However for the second end we have $\delta w(L) \neq 0$ which leads to:

$$E \left(I(L) w^{(3)}(L) + I'(L) w''(L) \right) - I(L) \rho \ddot{w}'(L) = 0 \tag{45}$$

Pluggin in the values for I and I':

$$E \left(h_L w^{(3)}(L) + 3 * \frac{h_L - h_0}{L} w''(L) \right) - h_L \rho \ddot{w}'(L) = 0 \tag{46}$$

3 Assignment 3

A few notes before starting this assignment:
I'm considering three frames.
The first one is the inertial frame. It is in A with the first axis straight up, the second right and the third into the plane of the image.

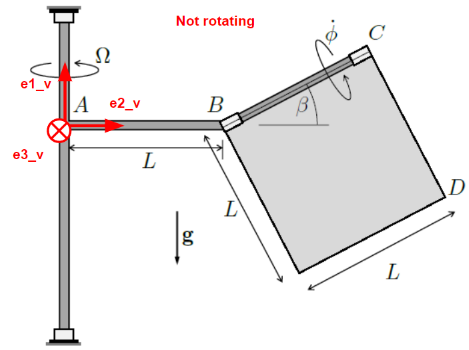

Figure 5: Inertial Frame

The second one is located at the same position but rotating with Omega. I will call it the vertical frame.

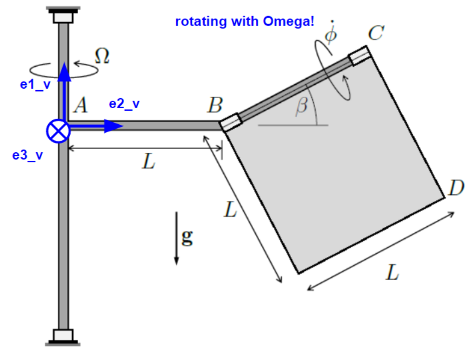

Figure 6: Vertical Frame

The third one that is called rod frame is the same as the vertical frame but with a constant rotation of β around the negative $e3_v$ axis.

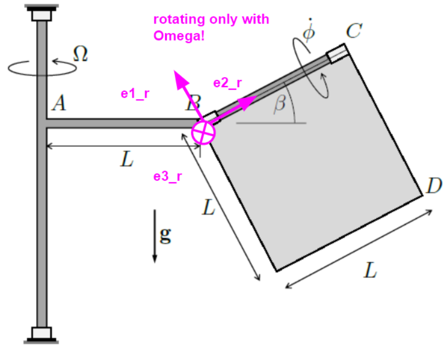

Figure 7: Rod Frame

The last frame is the square frame which relative to the rod frame is also rotation with ϕ:

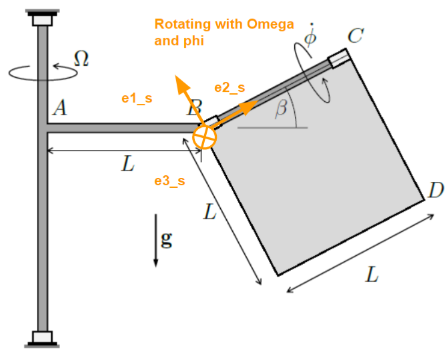

Figure 8: Square Frame

3.1 Kinetic and Potential Energy

3.1.1 Vertical Shaft

As the vertical bar rotates around the thing axis and is static regarding the height of it's CoM the contribution to the energ is 0.

$$T_{\text{vertical bar}} = V_{\text{vertical bar}} = 0 \tag{47}$$

3.1.2 Horizontal Bar

Note: I will refer to the horizontal bar as bar and to the tilted bar as rod. As point A which is part of the bar is not moving we can simply consider the rotational part w.r.t. to this fixed point A.:

$$T_{\text{bar}} = \frac{L^2\Omega^2 m}{6} \tag{48}$$

As the bar is horizontal on the height of A we have no contribution of the potential energy.

$$V_{\text{bar}} = 0 \tag{49}$$

3.1.3 Rod

As the rod has no static point we will use a superposition of the rotational and translational kinetic energy.

$$T_{\text{Rod transl}} = \frac{L^2\Omega^2 m(\cos(\beta) + 2)^2}{8} \tag{50}$$

For the rotational part considering the moment of inertia of the center of mass we get:

$$T_{\text{Rod rot}} = \frac{L^2\Omega^2 m\cos(\beta)^2}{24} \tag{51}$$

Which leads to:

$$\Rightarrow T_{\text{Rod}} = \frac{L^2\Omega^2 m\left(\cos(\beta)^2 + 3\cos(\beta) + 3\right)}{6} \tag{52}$$

For the potential energy we have the verical part of the incline of the rod:

$$V_{\text{Rod}} = \frac{L}{2} mg \sin\beta \tag{53}$$

3.1.4 Square Plate

As for the rod, first the translational kinetic energy:

$$
\begin{aligned}
T_{\text{square transl}} =& \frac{M}{2}\left(\left(\left(\Omega+\dot{\phi}\sin\left(\beta\right)\right)\left(\frac{L\left(\sin\left(\Omega t\right)\sin\left(\phi\right)-\cos\left(\Omega t\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)}{2}-\frac{L\cos\left(\Omega t\right)\cos\left(\beta\right)}{2}\right)\right.\right.\\
&\left.-L\Omega\cos\left(\Omega t\right)+\frac{L\dot{\phi}\cos\left(\Omega t\right)\cos\left(\beta\right)\left(\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2}\right)^{2}\\
&+\left(\left(\Omega+\dot{\phi}\sin\left(\beta\right)\right)\left(\frac{L\left(\cos\left(\Omega t\right)\sin\left(\phi\right)+\sin\left(\Omega t\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)}{2}+\frac{L\sin\left(\Omega t\right)\cos\left(\beta\right)}{2}\right)\right.\\
&\left.\left.+L\Omega\sin\left(\Omega t\right)-\frac{L\dot{\phi}\sin\left(\Omega t\right)\cos\left(\beta\right)\left(\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2}\right)^{2}+\frac{L^{2}\dot{\phi}^{2}{\cos\left(\beta\right)}^{2}{\sin\left(\phi\right)}^{2}}{4}\right)
\end{aligned}
\tag{54}
$$

For the rotational part we get, considering the moment of inertia w.r.t. the c.o.m. in the s frame:

$$
\begin{aligned}
T_{\text{square rot}} =& \frac{L^{2}M\left(\dot{\phi}+\Omega\sin\left(\beta\right)\right)\left(\frac{\dot{\phi}}{2}+\frac{\Omega\sin(\beta)}{2}\right)}{12}\\
&+\frac{L^{2}M\Omega^{2}{\cos\left(\beta\right)}^{2}{\cos\left(\phi\right)}^{2}}{24}
\end{aligned}
\tag{55}
$$

Which leads to:

$$
\begin{aligned}
T_{\text{square}} =& \frac{L^{2}M}{12}\left(-\Omega^{2}{\cos\left(\beta\right)}^{2}{\cos\left(\phi\right)}^{2}+\Omega^{2}{\cos\left(\beta\right)}^{2}\right.\\
&+3\sin\left(\beta\right)\Omega^{2}\cos\left(\beta\right)\cos\left(\phi\right)+6\Omega^{2}\cos\left(\beta\right)\\
&+6\sin\left(\beta\right)\Omega^{2}\cos\left(\phi\right)+8\Omega^{2}+3\Omega\dot{\phi}\cos\left(\beta\right)\cos\left(\phi\right)\\
&\left.+6\Omega\dot{\phi}\cos\left(\phi\right)+4\sin\left(\beta\right)\Omega\dot{\phi}+2\dot{\phi}^{2}\right)
\end{aligned}
\tag{56}
$$

For the potential energy we get:

$$
\frac{L}{2}Mg\left(\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)
\tag{57}
$$

Which are the contributions of the vertical parts of the two vectors of length $\frac{L}{2}$ to the center of mass of the square in the s frame. The sin part points up (to c.o.m of the rod) and the cos part points down.

3.1.5 Total Energy

The total kinetic energy comes to:

$$
\begin{aligned}
&T = \underbrace{T_{\text{vertical bar}}}_{0} + T_{\text{bar}} + T_{\text{Rod}} + T_{\text{Square}} = \\
&\frac{L^2}{12}\Big(8M\Omega^2 + 8\Omega^2 m + 2M\dot{\phi}^2 + 6M\Omega^2\cos(\beta) + 6\Omega^2 m\cos(\beta) \\
&+ M\Omega^2\cos(\beta)^2 + 2\Omega^2 m\cos(\beta)^2 + 6M\Omega^2\cos(\phi)\sin(\beta) \\
&+ 6M\Omega\dot{\phi}\cos(\phi) + 4M\Omega\dot{\phi}\sin(\beta) - M\Omega^2\cos(\beta)^2\cos(\phi)^2 \\
&+ 3M\Omega\dot{\phi}\cos(\beta)\cos(\phi) + 3M\Omega^2\cos(\beta)\cos(\phi)\sin(\beta)\Big)
\end{aligned}
\tag{58}
$$

Analogous the potential energy:

$$
\begin{aligned}
&V = \underbrace{V_{\text{vertical bar}}}_{0} + V_{\text{bar}} + V_{\text{Rod}} + V_{\text{Square}} = \\
&\frac{L}{2}Mg\left(2\sin(\beta) - \cos(\beta)\cos(\phi)\right)
\end{aligned}
\tag{59}
$$

Where one can see nicely the double contribution of the c.o.m of the rod for the rod and the square and the negative part of the square.

3.2 Lagrange Equations

Having an expression for the kinetic and potential energy we can use them to denote our lagrange equations:

$$
\frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0 \tag{60}
$$

Which yields:

$$
\begin{aligned}
&\frac{LM}{12}\Big(-2L\cos(\phi)\sin(\phi)\,\Omega^2\cos(\beta)^2 + 3L\sin(\beta)\sin(\phi)\,\Omega^2\cos(\beta) \\
&+ 6L\sin(\beta)\sin(\phi)\,\Omega^2 + 6g\sin(\phi)\cos(\beta) + 4L\ddot{\phi}\Big) = 0
\end{aligned}
\tag{61}
$$

Which is a differential equation for ϕ.

3.3 Non constant angular velocity around vertical Shaft

3.3.1 Vertical shaft

Unchanged

3.3.2 Bar

$$T_{\text{bar}} = \frac{L^2\Omega^2 m}{6} \tag{62}$$

$$V_{\text{bar}} = 0 \tag{63}$$

3.3.3 Rod

$$T_{\text{rod}} = L^2 m\dot{\theta}^2 \left(\frac{\cos(\beta)^2}{2} + \cos(\beta) + 1\right) \tag{64}$$

$$V_{\text{rod}} = \frac{L}{2} gm \sin(\beta) \tag{65}$$

3.3.4 Square

$$\begin{aligned} T_{\text{square}} =& \frac{L^2 M}{12} \Big(2\dot{\phi}^2 + 3\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi) + 6\dot{\phi}\dot{\theta}\cos(\phi) + 4\sin(\beta)\,\dot{\phi}\dot{\theta} \\ & - \dot{\theta}^2\cos(\beta)^2\cos(\phi)^2 + \dot{\theta}^2\cos(\beta)^2 + 3\sin(\beta)\,\dot{\theta}^2\cos(\beta)\cos(\phi) \\ & + 6\dot{\theta}^2\cos(\beta) + 6\sin(\beta)\,\dot{\theta}^2\cos(\phi) + 8\dot{\theta}^2\Big) \end{aligned} \tag{66}$$

$$V_{\text{square}} = \frac{L}{2} Mg\left(\sin(\beta) - \cos(\beta)\cos(\phi)\right) \tag{67}$$

3.3.5 Total Energy

$$\begin{aligned} T =& \frac{L^2}{12} \Big(2M\dot{\phi}^2 + 8M\dot{\theta}^2 + 8m\dot{\theta}^2 + 6M\dot{\theta}^2\cos(\beta) + 6m\dot{\theta}^2\cos(\beta) \\ & + M\dot{\theta}^2\cos(\beta)^2 + 2m\dot{\theta}^2\cos(\beta)^2 + 6M\dot{\theta}^2\cos(\phi)\sin(\beta) \\ & + 6M\dot{\phi}\dot{\theta}\cos(\phi) + 4M\dot{\phi}\dot{\theta}\sin(\beta) - M\dot{\theta}^2\cos(\beta)^2\cos(\phi)^2 \\ & + 3M\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi) + 3M\dot{\theta}^2\cos(\beta)\cos(\phi)\sin(\beta)\Big) \end{aligned} \tag{68}$$

$$V = \frac{L}{2} gm\sin(\beta) + \frac{L}{2} Mg\left(\sin(\beta) - \cos(\beta)\cos(\phi)\right) \tag{69}$$

3.3.6 Lagrange Equations

As we have now two generalized coordinates the lagrange equations look like this:

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial \mathbf{q}} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0 \tag{70}$$

Where $\mathbf{q} = \begin{pmatrix} \phi \\ \theta \end{pmatrix}$

This leads to two equations. For visualization purposes I will write down the entries one after the other:

$$\frac{L^2\, M\, \ddot{\phi}}{3} + \frac{L^2\, M\, \ddot{\theta}\, \left(6\, \cos\left(\phi\right) + 4\, \sin\left(\beta\right) + 3\, \cos\left(\beta\right)\, \cos\left(\phi\right)\right)}{12} + \frac{L\, M\, g\, \cos\left(\beta\right)\, \sin\left(\phi\right)}{2}$$
$$+ \frac{L^2\, M\, \dot{\theta}\, \sin\left(\phi\right)\, \left(6\, \dot{\phi} + 3\, \dot{\phi}\, \cos\left(\beta\right) + 6\, \dot{\theta}\, \sin\left(\beta\right) + 3\, \dot{\theta}\, \cos\left(\beta\right)\, \sin\left(\beta\right) - 2\, \dot{\theta}\cos\left(\beta\right)^2\, \cos\left(\phi\right)\right)}{12}$$
$$- \frac{L^2\, M\, \dot{\phi}\dot{\theta}\, \sin\left(\phi\right)\, \left(\cos\left(\beta\right) + 2\right)}{4} \tag{71}$$

And

$$\frac{L^2\, \ddot{\theta}}{12} \left(16\, M + 16\, m + 12\, M\, \cos\left(\beta\right) + 12\, m\, \cos\left(\beta\right) + 2\, M \cos\left(\beta\right)^2\right.$$
$$\left. + 4\, m \cos\left(\beta\right)^2 - 2\, M \cos\left(\beta\right)^2 \cos\left(\phi\right)^2 + 12\, M\, \cos\left(\phi\right)\, \sin\left(\beta\right) + 6\, M\, \cos\left(\beta\right)\, \cos\left(\phi\right)\, \sin\left(\beta\right)\right)$$
$$+ \frac{L^2\, M\, \ddot{\phi}\, \left(6\, \cos\left(\phi\right) + 4\, \sin\left(\beta\right) + 3\, \cos\left(\beta\right)\, \cos\left(\phi\right)\right)}{12}$$
$$- \frac{L^2\, M\, \dot{\phi}\, \sin\left(\phi\right)\, \left(6\, \dot{\phi} + 3\, \dot{\phi}\, \cos\left(\beta\right) + 12\, \dot{\theta}\, \sin\left(\beta\right) + 6\, \dot{\theta}\, \cos\left(\beta\right)\, \sin\left(\beta\right) - 4\, \dot{\theta}\cos\left(\beta\right)^2\, \cos\left(\phi\right)\right)}{12} \tag{72}$$

3.4 Non Conservative Forces

Now we have a non conservative force attacking on the square.
We have to derive the velocity of a random point on the square. In the following I will parametrize said point with x_1 and x_2 which are the coordinates of the point on the square in the s frame from B.

$$v_{Pi} = v_{Bi} + \omega_i \times BP_i \tag{73}$$

Therefore the work of the force contribution at a single point P is:

$$
\begin{aligned}
dW =& - cvP_i \bullet vP_i = \\
&= -c\Bigg(\Big(x_1 \left(\sin\left(\phi\right)\sin\left(\theta\right) - \cos\left(\phi\right)\sin\left(\beta\right)\cos\left(\theta\right)\right) \\
&+ x_2\cos\left(\beta\right)\cos\left(\theta\right)\Big)\left(\dot{\theta} + \dot{\phi}\sin\left(\beta\right)\right) + L\dot{\theta}\cos\left(\theta\right) \\
&- \dot{\phi}\cos\left(\beta\right)\cos\left(\theta\right)\left(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\right)\Bigg)^2 \\
&- c\Bigg(\Big(x_1\left(\cos\left(\theta\right)\sin\left(\phi\right) + \cos\left(\phi\right)\sin\left(\beta\right)\sin\left(\theta\right)\right) - x_2\cos\left(\beta\right)\sin\left(\theta\right)\Big) \\
&\left(\dot{\theta} + \dot{\phi}\sin\left(\beta\right)\right) - L\dot{\theta}\sin\left(\theta\right) \\
&+ \dot{\phi}\cos\left(\beta\right)\sin\left(\theta\right)\Big(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\Big)\Bigg)^2 \\
&- c\dot{\phi}^2{x_1}^2{\cos\left(\beta\right)}^2{\sin\left(\phi\right)}^2
\end{aligned} \tag{74}
$$

As we want to derive the whole work done by the area distributed force we integrate the expression of dW over the whole square:

$$
\begin{aligned}
W =& \int_{x_1=-L}^{0}\int_{x_2=0}^{L} -c\Bigg(\Big(x_1 \left(\sin\left(\phi\right)\sin\left(\theta\right) - \cos\left(\phi\right)\sin\left(\beta\right)\cos\left(\theta\right)\right) \\
&+ x_2\cos\left(\beta\right)\cos\left(\theta\right)\Big)\left(\dot{\theta} + \dot{\phi}\sin\left(\beta\right)\right) + L\dot{\theta}\cos\left(\theta\right) \\
&- \dot{\phi}\cos\left(\beta\right)\cos\left(\theta\right)\left(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\right)\Bigg)^2 \\
&- c\Bigg(\Big(x_1\left(\cos\left(\theta\right)\sin\left(\phi\right) + \cos\left(\phi\right)\sin\left(\beta\right)\sin\left(\theta\right)\right) - x_2\cos\left(\beta\right)\sin\left(\theta\right)\Big) \\
&\left(\dot{\theta} + \dot{\phi}\sin\left(\beta\right)\right) - L\dot{\theta}\sin\left(\theta\right) \\
&+ \dot{\phi}\cos\left(\beta\right)\sin\left(\theta\right)\Big(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\Big)\Bigg)^2 \\
&- c\dot{\phi}^2{x_1}^2{\cos\left(\beta\right)}^2{\sin\left(\phi\right)}^2\, dx_2 dx_1
\end{aligned} \tag{75}
$$

Which comes to:

$$
\begin{aligned}
W = & -\frac{cL^4\dot{\phi}^2}{3} - \frac{cL^4\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi)}{2} - cL^4\dot{\phi}\dot{\theta}\cos(\phi) - \\
& \frac{2c\sin(\beta)\,L^4\dot{\phi}\dot{\theta}}{3} + \frac{cL^4\dot{\theta}^2\cos(\beta)^2\cos(\phi)^2}{3} - \frac{cL^4\dot{\theta}^2\cos(\beta)^2}{3} - \\
& \frac{c\sin(\beta)\,L^4\dot{\theta}^2\cos(\beta)\cos(\phi)}{2} - cL^4\dot{\theta}^2\cos(\beta) - c\sin(\beta)\,L^4\dot{\theta}^2\cos(\phi) \\
& -\frac{4cL^4\dot{\theta}^2}{3}
\end{aligned}
\tag{76}
$$

And finally the generalized forces are:

$$
Q_\phi = \frac{\partial W}{\partial \phi} = -\frac{L^4 c\left(4\dot{\phi} + 6\dot{\theta}\cos(\phi) + 4\dot{\theta}\sin(\beta) + 3\dot{\theta}\cos(\beta)\cos(\phi)\right)}{6} \tag{77}
$$

And

$$
\begin{aligned}
Q_\theta = & -\frac{L^4 c}{6}\Big(16\dot{\theta} + 12\dot{\theta}\cos(\beta) + 6\dot{\phi}\cos(\phi) + 4\dot{\phi}\sin(\beta) + 4\dot{\theta}\cos(\beta)^2 \\
& - 4\dot{\theta}\cos(\beta)^2\cos(\phi)^2 + 3\dot{\phi}\cos(\beta)\cos(\phi) + 12\dot{\theta}\cos(\phi)\sin(\beta) \\
& + 6\dot{\theta}\cos(\beta)\cos(\phi)\sin(\beta)\Big)
\end{aligned}
\tag{78}
$$

3.5 Values plugged in

We plug in the values:

$$
L = 0.25, \beta = \pi/6, M = 0.5, m = 0.2, g = 9.81, c = 0.1 \tag{79}
$$

3.6 Equilibrium

In the equilibrium configuration we have

$$
\ddot{\phi} = \dot{\phi} = 0, \quad \phi = \phi_{\mathrm{eq}} \tag{80}
$$

And

$$
\ddot{\theta} = \dot{\theta} = 0, \quad \theta = \theta_{\mathrm{eq}} \tag{81}
$$

For the rheonomic system with a constant Omega and the plugged in values we get:

$$\frac{981\sqrt{3}\sin\left(\phi\right)}{3200}+\frac{\Omega^{2}\sin\left(\phi\right)}{128}+\frac{\sqrt{3}\Omega^{2}\sin\left(\phi\right)}{512}-\frac{\Omega^{2}\cos\left(\phi\right)\sin\left(\phi\right)}{256}=0 \tag{82}$$

Or for the case of a free rotation around A:

$$\left(\begin{array}{c}\frac{981\,\sqrt{3}\,\sin(\phi_{\mathrm{eq}})}{3200}\\ 0\end{array}\right)=\left(\begin{array}{c}0\\ 0\end{array}\right) \tag{83}$$

And as can be seen the above equation (84) can be read as:

$$\begin{pmatrix}A\sin(\phi_{\mathrm{eq}})\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{84}$$

With a constant A. Which yields

$$\phi_{\mathrm{eq}}=n*\pi, \qquad \text{for } n=0,1,2... \tag{85}$$

This makes sense as $\phi = n*\pi$ describes states where the square plate stays vertical. As we are not considering aerial drag (non-conservative forces) it is imaginable that this is an equilibrium. θ can be chosen arbitrarily which makes sense as the system is symmetric w.r.t. theta.

Note: as for this case we have a scleronomic system we could have also just used the following:

$$\frac{\partial V}{\partial \mathbf{q}}=\mathbf{0} \tag{86}$$

Which when plugging it in gives us:

$$\left(\begin{array}{cc}\frac{981\,\sqrt{3}\,\sin(\phi_{\mathrm{eq}})}{3200} & 0\end{array}\right)=\mathbf{0} \tag{87}$$

Which is the same as equation (84)
Lastly when considering the non-conservative forces as well we get:

$$\left(\begin{array}{c}\frac{981\,\sqrt{3}\,\sin(\phi_{\mathrm{eq}})}{3200}\\ 0\end{array}\right) \tag{88}$$

Which in my eyes does not make sense. I believe through the drag force the vertical state of the square plate should not be an equilibrium.

3.7 Integration

4 Assignment 4

4.1 DONE

4.2 DONE

4.3 DONE

4.4 DONE

4.5

4.5.1 DONE

We fixed the front wheel to remove the singularity of K. q_{init} was given as:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 0 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 0.22 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 0.21 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = \pi \\ \beta_{\text{Link Front}} = 0 \end{pmatrix} \tag{89}$$

Which represents this position:

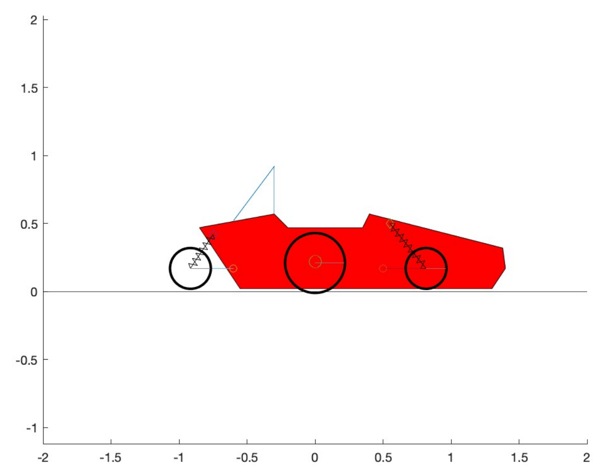

Figure 9: Initial Position 1

This resulted in the equilibrium:

$$q_{\text{equilibrium 1}} = \begin{pmatrix} \theta_{\text{Frame}} = -1.3896e-02 = -0.014 \\ x_{\text{Frame}} = -8.1104e-01 = -0.811 \\ y_{\text{Frame}} = 2.1926e-01 = -0.219 \\ \theta_{\text{Wheel Back}} = 7.8437e+00 = 7.844 \\ \theta_{\text{Tire Front}} = 2.2254e-21 \approx 0 \\ \theta_{\text{Tire Back}} = 7.8437e+00 = 7.844 \\ y_{\text{Tire Front}} = 2.1883e-01 = 0.219 \\ y_{\text{Tire Back}} = 2.1895e-01 = 0.219 \\ \beta_{\text{Link Back}} = 3.0209e+00 = 3.021 \\ \beta_{\text{Link Front}} = 1.6787e-01 = 0.168 \end{pmatrix} \tag{90}$$

Which is visualized by this figure:

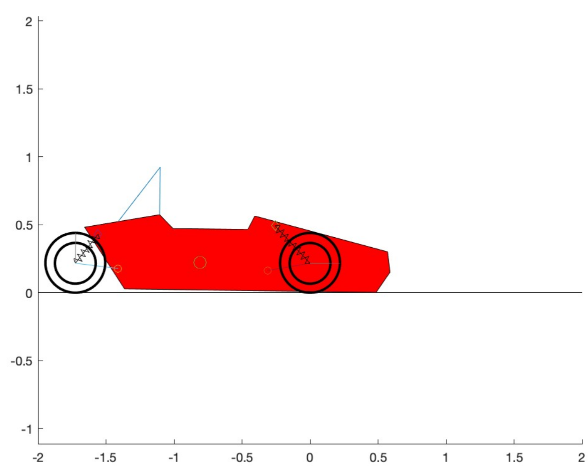

Figure 10: Equilibrium Position 1

In this example the choice of initial generalized coordinates was obviously very good. For the first task of this assignment the goal is to find two more (probably bad) equilibria.
The eigenvalues of the stiffness matrix are:

4.5.2 Different Equilibrium states

The first alternative is luckily already given in the code. Again we fix the front wheel's rotation and start with the initial state:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi/2 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{91}$$

Which represents this position:

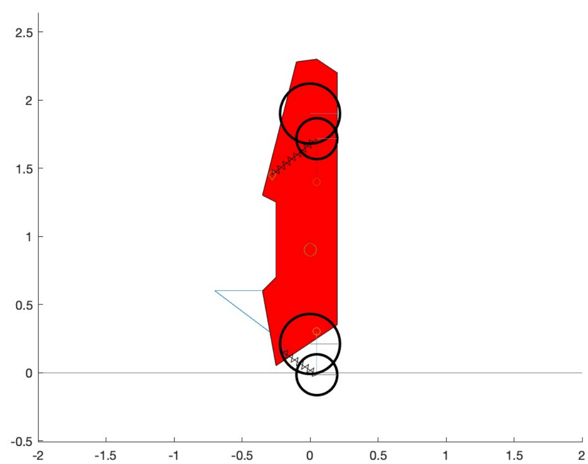

Figure 11: Initial Position 2

Note: in the code provided θ_{Frame} was π which was a different initial set of gen. coord. but converged to the same solution

As can be seen, this is obviously not a good choice of initial coordinates.
This setting converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 1.5248e+00 \\ x_{\text{Frame}} = -6.9703e-02 \\ y_{\text{Frame}} = 1.1280e+00 \\ \theta_{\text{Wheel Back}} = 3.0427e-01 \\ \theta_{\text{Tire Front}} = 2.9622e-24 \\ \theta_{\text{Tire Back}} = 3.0427e-01 \\ y_{\text{Tire Front}} = 1.9414e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -1.6327e+00 \\ \beta_{\text{Link Front}} = 1.5811e+00 \end{pmatrix} \tag{92}$$

Which looks as follows:

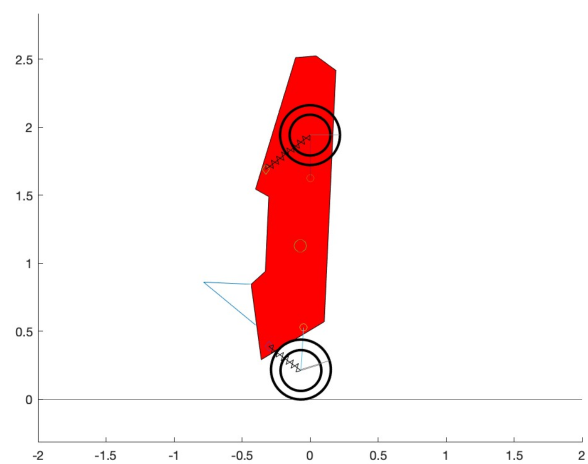

Figure 12: Equilibrium Position 2

The code example which has even worse initial conditions:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{93}$$

And looks like this:

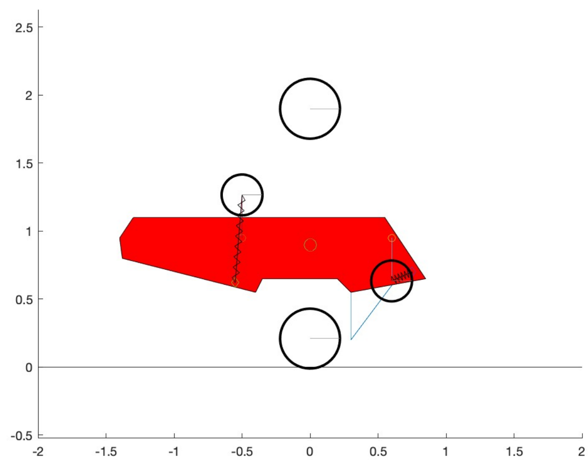

Figure 13: Initial Position 3

Converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 2.0225e+00 \\ x_{\text{Frame}} = -1.2734e-01 \\ y_{\text{Frame}} = 6.3440e-01 \\ \theta_{\text{Wheel Back}} = 5.5591e-01 \\ \theta_{\text{Tire Front}} = 2.0275e-25 \\ \theta_{\text{Tire Back}} = 5.5591e-01 \\ y_{\text{Tire Front}} = 1.2043e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -3.4446e+00 \\ \beta_{\text{Link Front}} = 3.1586e-01 \end{pmatrix} \tag{94}$$

Which represents:

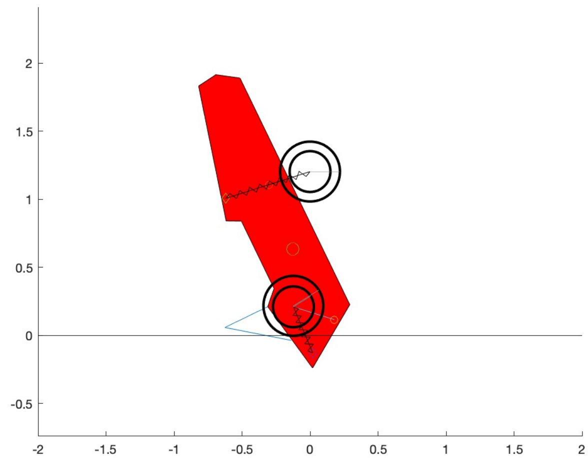

Figure 14: Equilibrium Position 3

4.6 Analysis

As we are not considering drag yet we have a conservative, scleronomic system. For an equilibrium in such a system to be stabel we need that $\mathrm{K_{eq}}$ must be positive definite. Aka the eigenvalues of $\mathrm{K_{eq}}$ must be real and positive.
Looking at the eigenvalues for the first case:

$$\begin{pmatrix} 1.6913e-07 \\ 2.1774e+00 \\ 2.2340e+00 \\ 3.0009e+01 \\ 3.0821e+01 \\ 3.4894e+01 \\ 3.5642e+01 \\ 6.8555e+01 \\ 6.8673e+01 \\ 1.3035e+02 \\ 1.3036e+02 \end{pmatrix} \tag{95}$$

We can see that they are indeed all positive and real. Thus 90 is a stable equilibrium. This represents logical assumptions as the car staying in horizontal position is as we know a stable equilibrium.

For the second equilibrium we get the following eigenvalues:

$$\begin{pmatrix} 0.0000e+00+4.9405e-01i \\ 0.0000e+00+1.2122e-07i \\ 4.4151e+00+0.0000e+00i \\ 7.1552e+00+0.0000e+00i \\ 7.4830e+00+0.0000e+00i \\ 4.3728e+01+0.0000e+00i \\ 4.3983e+01+0.0000e+00i \\ 6.3302e+01+0.0000e+00i \\ 7.2017e+01+0.0000e+00i \\ 7.2124e+01+0.0000e+00i \\ 1.2904e+02+0.0000e+00i \end{pmatrix} \tag{96}$$

Where we can see that the eigenvalues ar of complex nature. Mainly the first one

Lastly for the third setting we have the following eigenvalues:

$$\begin{pmatrix} 0.0000e+00+4.9405e-01i \\ 1.9956e-07+0.0000e+00i \\ 4.4151e+00+0.0000e+00i \\ 7.1552e+00+0.0000e+00i \\ 7.4830e+00+0.0000e+00i \\ 4.3728e+01+0.0000e+00i \\ 4.3983e+01+0.0000e+00i \\ 6.3302e+01+0.0000e+00i \\ 7.2017e+01+0.0000e+00i \\ 7.2124e+01+0.0000e+00i \\ 1.2904e+02+0.0000e+00i \end{pmatrix} \tag{97}$$

Where we see again, that we have complex eigenvalues denoting an unstable equilibrium.

Another interesting example is the one seen in fig 15:

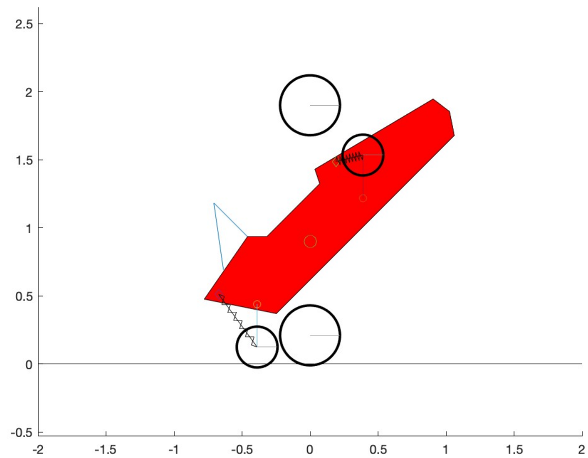

Figure 15: Initial Position 4

When we start from the tilted starting position of 11 but changing merely the frame angle to a smaller $\pi/4$ we can see that it converges to something quite reasonable looking.
And indeed when looking at the eigenvalues:

$$\begin{pmatrix} 2.5694e-07 \\ 2.1774e+00 \\ 2.2340e+00 \\ 3.0009e+01 \\ 3.0821e+01 \\ 3.4894e+01 \\ 3.5642e+01 \\ 6.8555e+01 \\ 6.8673e+01 \\ 1.3035e+02 \\ 1.3036e+02 \end{pmatrix} \tag{98}$$

We see that it's a stable equilibrium. From this we can take away that a perturbation from a stable equilibrium that is small enough will converge to the stable equilibrium.

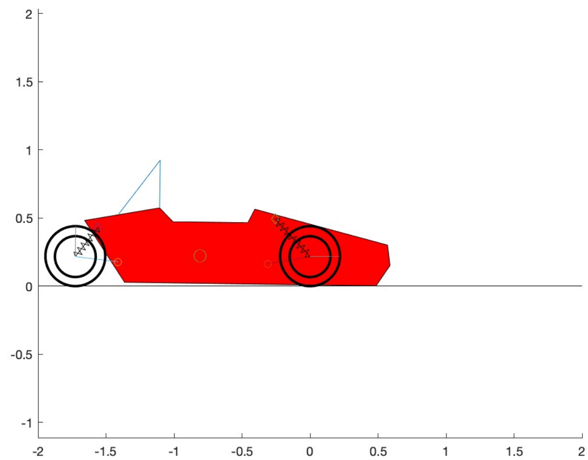

Figure 16: Equilibrium Position 4

4.7 DONE

4.8 Eigenmodes and Eigenfrequencies

Looking at the eigenfrequencies in equation (95) we can see that the dominant ones with the large motion impact seem to be the first as well as the second and third.
The eigenmodes associated with these frequencies are:

$$\begin{pmatrix} 6.0858e-01 \\ 9.4719e-02 \\ -9.7389e-02 \\ 0 \\ 0 \\ 1.2034e-02 \\ 8.8040e-03 \\ -4.7960e-01 \\ 5.7699e-01 \\ 1.7042e-01 \\ 1.3773e-01 \end{pmatrix} \tag{99}$$

,

$$
\begin{pmatrix} 4.8632e-02 \\ 9.8644e-01 \\ 0 \\ 0 \\ 0 \\ 1.0851e-01 \\ 1.0851e-01 \\ 0 \\ 0 \\ -1.8772e-02 \\ -2.6059e-02 \end{pmatrix} \tag{100}
$$

and

$$
\begin{pmatrix} -4.0695e-02 \\ 0 \\ 8.0283e-01 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4.0141e-01 \\ -4.0141e-01 \\ -1.2602e-01 \\ 1.2515e-01 \end{pmatrix} \tag{101}
$$

Plotted out we get the following configurations:
If only these could be explained.

4.8.1 DONE

4.8.2 Mode Shapes

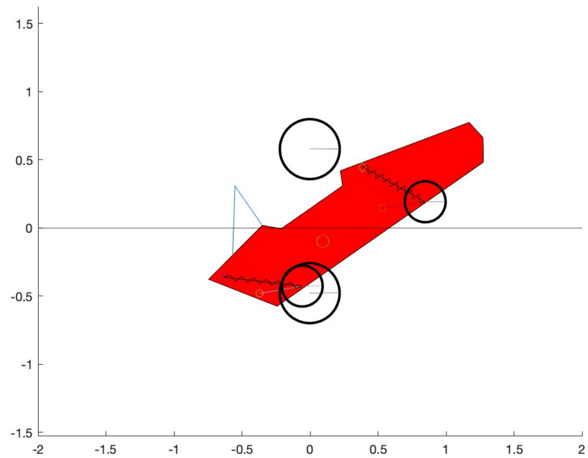

Figure 17: Eigenmode Configuration 1

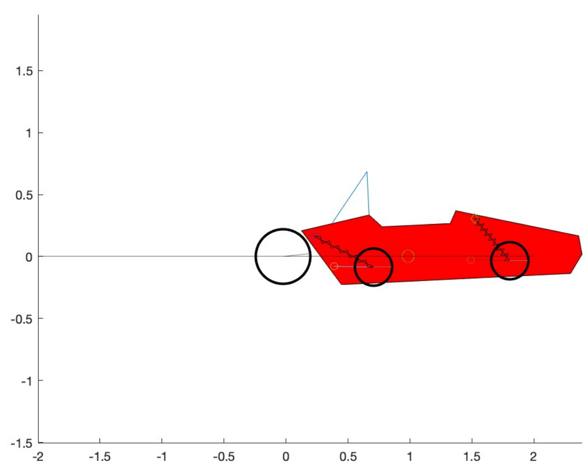

Figure 18: Eigenmode Configuration 2

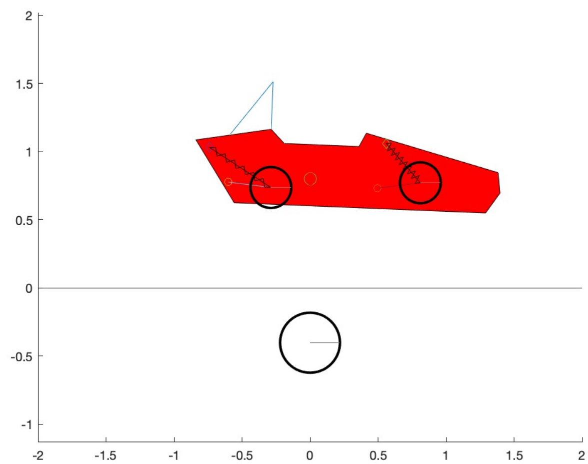

Figure 19: Eigenmode Configuration 3