Advanced Dynamics - Assignment Report

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1 Assignment 1

1 Wheel rolling without slipping on a 2D track

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.

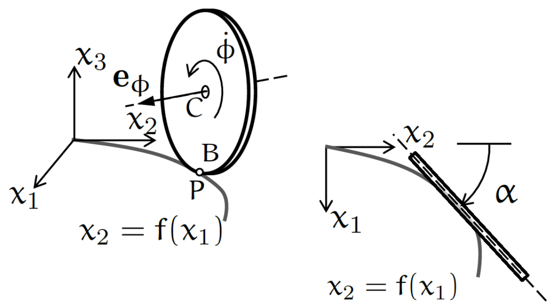

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

1.1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C\boldsymbol{e_1} + \dot{x}_2^C\boldsymbol{e_2} - R\dot{\phi}\cos\alpha\boldsymbol{e_2} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

As α is not dependent on time it can be easily seen that 3 is integrable:

$$\begin{aligned} &\int \dot{x}_1^C dt = \int R\dot{\phi}\sin\alpha dt \\ &\Rightarrow x_1^C = R\phi\sin\alpha \end{aligned} \tag{4}$$

1.1.2

See subsubsection 1.1.1

1.1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e}_{\phi}$ while all the other generalized coordinates follow accordingly.

2 Two bars linkage

Two bars AB and BC of equal length L are hinged at point B, and move in the plane spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The velocity $\mathbf{v}_C$ of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use x_1^B, x_2^B, θ_1 and θ_2 as generalized coordinates.

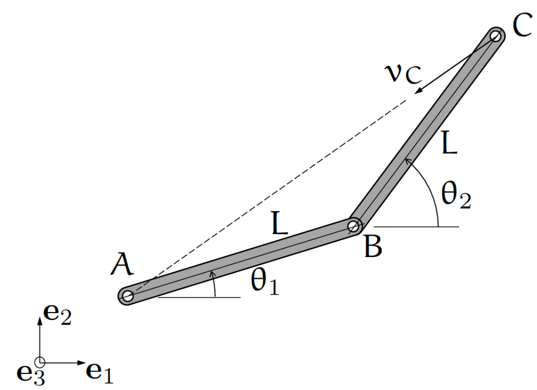

Figure 2.1: A two bar linkage in 2D. The velocity v_C must be directed towards A at all times.

Figure 3: Task 1.1

1.2

I want to show that the constraint that v_c always points in the direction of AD is not integrable. I would like to express this constraint as:

$$\begin{aligned} & v_C * AC_p = 0 \\ & \text{where } AC_p \text{ is a vector perpendicular to } AC \end{aligned} \tag{5}$$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} \tag{6}$$

And AC is simply:

$$AC = \begin{pmatrix} L\left(\cos\theta_1 + \cos\theta_2\right) \\ L\left(\sin\theta_1 + \sin\theta_2\right) \end{pmatrix} \tag{7}$$

A simple vector that is perpendicular to AC can be gotten by switching the e_1 and e_2 entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L(\sin\theta_1 + \sin\theta_2) \\ -L(\cos\theta_1 + \cos\theta_2) \end{pmatrix} \tag{8}$$

$$\begin{aligned} &\Rightarrow v_C * AC_p = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L(\sin\theta_1 + \sin\theta_2) \\ -L(\cos\theta_1 + \cos\theta_2) \end{pmatrix} = \\ &\quad (L(\sin(\theta_1) + \sin(\theta_2)))\,\dot{\mathrm{x}}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2)))\,\dot{x}_2^B - \\ &\quad L^2\dot{\theta}_2\cos(\theta_2)(\cos(\theta_1) + \cos(\theta_2)) - L^2\dot{\theta}_2\sin(\theta_2)(\sin(\theta_1) + \sin(\theta_2)) = 0 \end{aligned} \tag{9}$$

After applying trigonometry:

$$\begin{aligned} &(L(\sin(\theta_1) + \sin(\theta_2)))\,\dot{\mathrm{x}}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2)))\,\dot{x}_2^B - \\ &L^2\dot{\theta}_2(\cos(\theta_2)\cos(\theta_1) + \sin(\theta_2)\sin(\theta_1) + 1) = 0 \end{aligned} \tag{10}$$

To simplify the notation I will refer to $\sin\theta_1$ as s_1 and $\sin\theta_2$ as s_2:

$$L\dot{x}_1^B(s_1 + s_2) - L\dot{x}_2^B(c_1 + c_2) - L^2\dot{\theta}_2^2(c_2c_1 + s_2s_1 + 1) = 0 \tag{11}$$

Dividing by L:

$$\dot{x}_1^B(s_1 + s_2) - \dot{x}_2^B(c_1 + c_2) - L\dot{\theta}_2^2(c_2c_1 + s_2s_1 + 1) = 0 \tag{12}$$

With the generalized coordinates $q = [x_1^B, x_2^B, \theta_1, \theta_2]$
Writing down the coefficients of the non-holonomic constraint:

$$\begin{aligned} a_1 &= s_1 + s_2 \\ a_2 &= -(c_1 + c_2) \\ a_3 &= 0 \\ a_4 &= -L(1 + s_1s_2 + c_1c_2) \\ b &= 0 \end{aligned} \tag{13}$$

To check for the exact velocity form we want to proof that there can't exist a $C(q) \neq 0$ for which holds:

$$\frac{\partial(Cb)}{\partial q_i} = \frac{\partial(Ca_1)}{\partial t} \quad \text{and} \quad \frac{\partial(Ca_i)}{\partial q_k} = \frac{\partial(Ca_k)}{\partial q_i} \quad \text{for all the gen. coord.} \tag{14}$$

q1-q3:

$$\begin{aligned}
&\frac{\partial(Ca_1)}{\partial \theta_1} = 0 \Rightarrow C * c_1 + \frac{\partial C}{\partial \theta_1}(s1+s2) = 0 \\
&\Rightarrow \frac{c_1}{s1+s2} + \frac{1}{C}\frac{dC}{d\theta_1} = 0 \Rightarrow \int \frac{1}{C}dC = -\int \frac{c_1}{s_1+s_2}d\theta_1 \\
&\Rightarrow lnC = \frac{D}{s_1+s_2} + D \Rightarrow C = \frac{D}{s_1+s_2} \quad \text{where the integration const. D was updated} \\
&\Rightarrow C = \frac{D(x_1^B, x_2^B, \theta_2)}{s1+s2}
\end{aligned} \tag{15}$$

q1 - q4:

$$\frac{\partial Ca_1}{\partial \theta_2} = \underbrace{\frac{\partial Ca_4}{\partial x_1^B}}_{0} = 0 \Rightarrow \frac{\partial D}{\partial \theta_2} = 0 \Rightarrow C = \frac{D(x_1^B, x_2^B)}{s1+s2} \tag{16}$$

q1 - q2:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = \frac{\partial Ca_2}{\partial x_1^B} \Rightarrow \frac{\partial C}{\partial x_2^B}(s_1+s_2) = \frac{\partial C}{\partial x_1^B}(c_1+c_2) \Rightarrow \frac{\partial C}{\partial x_2^B} = \frac{\partial C}{\partial x_1^B}\frac{c_1+c_2}{s_1+s_2} \\
&\text{Remembering that } D = D(x_1^B, x_2^B) \Rightarrow D = const. \\
&\Rightarrow C = \frac{D}{s_1+s_2} \text{ where D const.}
\end{aligned} \tag{17}$$

q2 - q3:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = 0 \Rightarrow D\frac{\partial \frac{(c_1+c_2)}{s_1+s_2}}{\partial \theta_1} = 0 \\
&D\frac{\partial \frac{c_1}{s_1+s_2}}{\partial \theta_1} = -\frac{Ds_1}{s_1+s_2} - \frac{Dc_1^2}{(s_1+s_2)^2} = -D\frac{s_1(s_1+s_2)-c_1^2}{(s_1+s_2)^2}
\end{aligned} \tag{18}$$

$$D\frac{\partial \frac{c_2}{s_1+s_2}}{\partial \theta_1} = -D\frac{c_1c_2}{(s_1+s_2)^2} \tag{19}$$

Which leads to:

$$-D\frac{c_1^2 + s_1^2 + s_1 s_2 + c_1 c_2}{(s_1 + s_2)^2} \stackrel{!}{=} 0 \tag{20}$$

Which is a contradiction. Therefore this is a non-holonomic constraint.

2 Assignment 2

2.1 Hamilton's Principle

This assignment asks to get the equations of motion using the Hamilton Principle:

$$\int_{t_1}^{t_2} delta(T - V + W)dt = 0 \tag{21}$$

Using the theory discussed in the lecture we can split the term into different contributions::

Remembering the result of the Hamilton principle from the lecture:

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right) + f(x,t) + \sum F_i D(x - x_i) = 0, \quad x \in [0; l]$$

Boundary conditions:

$$\left[\frac{\partial L}{\partial v'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right]\delta v = 0, \quad x = 0, l$$

$$\frac{\partial L}{\partial v''}\delta v' = 0, \quad x = 0, l$$

Figure 4: Hamilton Formulas for a continuous system

We consider w our variable instead of v.

$$\frac{\partial L}{\partial w} = 0 \tag{22}$$

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial w'} = 0 \tag{23}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial w''} = -E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) \tag{24}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = A(x)\rho\ddot{w} \tag{25}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}'}\right)\right) = \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) \tag{26}$$

$$f(x,t) = p(x,t) \tag{27}$$

$$\frac{\partial L}{\partial w'} = 0 \tag{28}$$

$$\frac{\partial}{\partial x}(\frac{\partial L}{\partial w''}) = -E\left(I(x)w^{(3)} + I'(x)w''\right) \tag{29}$$

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{w}'}) = I(x)\rho\ddot{w}' \tag{30}$$

2.1.1 Differential Equation aka green part

Therefore the green part comes to:

$$p(x,t) - E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) + \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) - A(x)\rho\ddot{w} = 0 \tag{31}$$

Comparing this solution to the Euler-Bernoulli Equation:

$$p(x,t) = EIw^{(4)} + A(x)\rho\ddot{w} \tag{32}$$

What we can see is that the additional terms from the green expression come from the derivative of I. This comes from the changing cross section and the thus changing moment of Inertia I. Secondly we had a term with $\dot{w}'$ in the kinetic energy arising through the change in thickness which is considered here. (not the case for the euler bernoulli beam).

2.1.2 Boundary conditions with virtual displacement aka Red Part

$$\left[\delta \mathrm{w(x)}\left(E\left(I(x)w^{(3)} + I'(x)w''\right) - I(x)\rho\ddot{w}'\right)\right]_0^L = 0 \tag{33}$$

2.1.3 Blue part

$$\left[-\delta \mathrm{w'(x)}EI(x)w''\right]_0^L = 0 \tag{34}$$

2.2 Analysis

$$I(0) = \frac{1}{12}h_0^3 \text{ and } I(L) = \frac{1}{12}h_L^3 \tag{35}$$

as well as

$$I'(0) = \frac{1}{4}h(0)^2 * \frac{h_L - h_0}{L} = \frac{1}{4}h_0^2 * \frac{h_L - h_0}{L} \tag{36}$$

and

$$I'(L) = \frac{1}{4}h_L^2 * \frac{h_L - h_0}{L} \tag{37}$$

Beginning with the blue part and pluggin in the boundary values:

$$-\delta \mathrm{w}'(\mathrm{L})E\mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0)E\mathrm{h}_0^3 w''(0) = 0 \tag{38}$$

As we have a clamped end (w, δw and their first three derivatives are 0) and a supported end (w, δw and their first derivative is 0) we arrive at:

$$0 = 0 \tag{39}$$

For the red part:

$$\begin{aligned} &\delta w(L)\left(E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)\right) - \\ &\delta w(0)\left(E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)\right) = 0 \end{aligned} \tag{40}$$

Note that both $\delta w(0) = \delta w(L) = 0$. The coefficients

$$E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)$$

and

$$E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)$$

represent the forces at the boundaries. To determine these forces we would have to solve the differential equation to get a solution for w which we can use to get the forces.

2.3 Analysis with open second end

Nothing changes in the application of the Hamilton Principle. However when analysing the boundaries we see some changes:
Starting again with the blue:

$$-\delta \mathrm{w}'(\mathrm{L}) E \mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0) E \mathrm{h}_0^3 w''(0) = 0 \tag{41}$$

As before the w terms at $x = 0$ vanish. However now $\delta w'(L) \neq 0$ thus we get

$$E \mathrm{h}_L^3 w''(L) = 0 \tag{42}$$

So the slope at the vertical displacement of the free end has to stay constant.
For the red part we get:

$$\begin{aligned} &\delta w(L) \left(E \left(I(L) w^{(3)}(L) + I'(L) w''(L) \right) - I(L) \rho \ddot{w}'(L) \right) - \\ &\delta w(0) \left(E \left(I(0) w^{(3)}(0) + I'(0) w''(0) \right) - I(0) \rho \ddot{w}'(0) \right) = 0 \end{aligned} \tag{43}$$

As before $\delta w(0) = 0$ and the force of the boundary constraint is:

$$E \left(I(0) w^{(3)}(0) + I'(0) w''(0) \right) - I(0) \rho \ddot{w}'(0) \tag{44}$$

However for the second end we have $\delta w(L) \neq 0$ which leads to:

$$E \left(I(L) w^{(3)}(L) + I'(L) w''(L) \right) - I(L) \rho \ddot{w}'(L) = 0 \tag{45}$$

Pluggin in the values for I and I':

$$E \left(h_L w^{(3)}(L) + 3 * \frac{h_L - h_0}{L} w''(L) \right) - h_L \rho \ddot{w}'(L) = 0 \tag{46}$$

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2 M\ddot{\phi}}{3} + \frac{L^2 m\ddot{\phi}}{3} - \frac{L^2 m\ddot{\phi}\cos(\beta)^2}{3} + \frac{LMg\cos(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2\sin(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)\sin(\phi)}{4} - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{47}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi}\left(\frac{L^2(M+m)}{3} - \frac{L^2 m\cos(\beta)^2}{3}\right) + \sin(\phi)\left(\frac{LMg\cos(\beta)}{2} + \frac{L^2 M\Omega^2\sin(\beta)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)}{4}\right) - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{48}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2\ddot{\theta}\left(16M + 16m + 12M\cos(\beta) + 12m\cos(\beta) + 2M\cos(\beta)^2 + 4m\cos(\beta)^2 - 2M\cos(\beta)^2\cos(\phi)^2\right)}{12} + \frac{L^2\ddot{\theta}\left(12M\cos(\phi)\sin(\beta) + 6M\cos(\beta)\cos(\phi)\sin(\beta)\right)}{12} \tag{49}$$

3.4

After matlab integration of the work per area we get for the work

$$W = \frac{L^4\left(c\left(\dot{\theta} + \dot{\phi}\sin(\beta)\right)^2\left(\cos(\phi)^2 - 1\right) - c\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right)^2 + c\dot{\phi}^2\cos(\beta)^2\left(\cos(\phi)^2 - 1\right)\right)}{3} - L\left(Lc\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)^2 + \frac{L^3 c\dot{\theta}^2\cos(\beta)^2}{12}\right) - L^3 c\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right) \tag{50}$$

Which results in the equations of motion:

$$\frac{L^2M\ddot{\phi}}{3}+\frac{2L^4c\left(\dot{\phi}+\dot{\theta}\sin\left(\beta\right)\right)}{3}+\frac{L^2m\ddot{\phi}}{3}-\frac{L^2m\ddot{\phi}\cos\left(\beta\right)^2}{3}+\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2}+$$
$$L^4c\dot{\theta}\cos\left(\phi\right)\left(\cos\left(\beta\right)+1\right)+\frac{L^2M\dot{\theta}^2\sin\left(\beta\right)\sin\left(\phi\right)}{2}-\frac{L^2M\dot{\theta}^2\cos\left(\beta\right)^2\cos\left(\phi\right)\sin\left(\phi\right)}{6}+$$
$$\frac{L^2M\dot{\theta}^2\cos\left(\beta\right)\sin\left(\beta\right)\sin\left(\phi\right)}{4}\qquad\text{for } \phi \tag{51}$$

And

$$\frac{cL^4}{6}\left(16\dot{\theta}+24\dot{\theta}\cos\left(\beta\right)+6\dot{\phi}\cos\left(\phi\right)+4\dot{\phi}\sin\left(\beta\right)+13\dot{\theta}\cos\left(\beta\right)^2-4\dot{\theta}\cos\left(\beta\right)^2\cos\left(\phi\right)^2+6\dot{\phi}\cos\left(\beta\right)\cos\left(\phi\right)+12\dot{\theta}\cos\right.$$
$$\left.12\dot{\theta}\cos\left(\beta\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)+$$
$$\frac{\ddot{\theta}}{12}\left(16M+16m+12M\cos\left(\beta\right)+12m\cos\left(\beta\right)+2M\cos\left(\beta\right)^2+4m\cos\left(\beta\right)^2-2M\cos\left(\beta\right)^2\cos\left(\phi\right)^2+\right.$$
$$\left.12M\cos\left(\phi\right)\sin\left(\beta\right)+6M\cos\left(\beta\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)L^2\qquad\text{for } \theta \tag{52}$$

3.5 Plugging in the values:

Let $L=0.25[m],\beta=\frac{\pi}{6},M=0.5[kg],m=0.2[kg],g=9.81[\frac{m}{s^2}]$ and $c=0.1[\frac{Ns}{m^3}]$.

The equations become:

$$\frac{\dot{\phi}}{3840}+\frac{11\ddot{\phi}}{960}+\frac{\dot{\theta}}{7680}+\frac{\dot{\theta}^2\sin\left(\phi\right)}{128}+\frac{981\sqrt{3}\sin\left(\phi\right)}{3200}+\frac{\sqrt{3}\dot{\theta}^2\sin\left(\phi\right)}{512}+\frac{\dot{\theta}\cos\left(\phi\right)\left(\frac{\sqrt{3}}{2}+1\right)}{2560}$$
$$-\frac{\dot{\theta}^2\cos\left(\phi\right)\sin\left(\phi\right)}{256}\qquad\text{for } \phi \tag{53}$$

And

$$\frac{\dot{\phi}}{7680}+\frac{103\dot{\theta}}{61440}-\frac{\dot{\theta}\cos\left(\phi\right)^2}{5120}+\frac{\sqrt{3}\dot{\theta}}{1280}+$$
$$\frac{\ddot{\theta}}{192}\left(3\cos\left(\phi\right)-\frac{3\cos\left(\phi\right)^2}{4}+\frac{21\sqrt{3}}{5}+\frac{3\sqrt{3}\cos\left(\phi\right)}{4}+\frac{251}{20}\right)+$$
$$\frac{\dot{\phi}\cos\left(\phi\right)}{2560}+\frac{\dot{\theta}\cos\left(\phi\right)}{2560}+\frac{\sqrt{3}\dot{\phi}\cos\left(\phi\right)}{5120}+\frac{\sqrt{3}\dot{\theta}\cos\left(\phi\right)}{5120}\qquad\text{for } \theta \tag{54}$$

3.6 Equilibrium State

Reminder- The potential energy looks as follows:

$$\frac{LMg\left(2\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2} \tag{55}$$

Case 1: Ω fixed:

The derivative of the potential energy w.r.t. θ is always 0.

$$\frac{\partial V}{\partial \theta}=0 \tag{56}$$

The derivative of the potential energy w.r.t. ϕ not however:

$$\frac{\partial V}{\partial \phi}=\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2} \tag{57}$$

Equation 12 is only 0 for $\boldsymbol{\phi = k * \pi}$ which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

Case 2: $\dot{\theta}$ can change:

As V is independent of θ altogether the result is the same for this case.

The equation of motions are after plugging in equilibrium state:

$$\begin{aligned}&\phi:\\&0=0\end{aligned} \tag{58}$$

$$\begin{aligned}&\theta:\\&\frac{L^2\ddot{\theta}}{12}\left(16M+16m+12M\cos\left(\beta\right)+12M\sin\left(\beta\right)+12m\cos\left(\beta\right)+\right.\\&\left.4m\cos\left(\beta\right)^2+6M\cos\left(\beta\right)\sin\left(\beta\right)\right)=0\end{aligned} \tag{59}$$

As the coefficient of $\ddot{\theta}$ is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant Ω.

$$\frac{L^2\,M\,\left(-2\,\cos\left(\phi\right)\,\sin\left(\phi\right)\,\dot{\theta}^2\cos\left(\beta\right)^2+3\,\sin\left(\beta\right)\,\sin\left(\phi\right)\,\dot{\theta}^2\,\cos\left(\beta\right)+6\,\sin\left(\beta\right)\,\sin\left(\phi\right)\,\dot{\theta}^2+4\,\ddot{\phi}\right)}{12} \tag{60}$$

4 Assignment 4

4.1 DONE

4.2 DONE

4.3 DONE

4.4 DONE

4.5

4.5.1 DONE

We fixed the front wheel to remove the singularity of K. q_{init} was given as:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 0 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 0.22 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 0.21 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = \pi \\ \beta_{\text{Link Front}} = 0 \end{pmatrix} \tag{61}$$

Which represents this position:

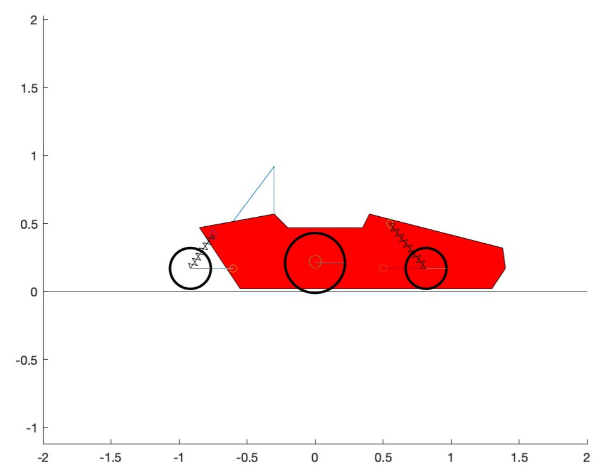

Figure 5: Initial Position 1

This resulted in the equilibrium:

$$q_{\text{equilibrium 1}} = \begin{pmatrix} \theta_{\text{Frame}} = -1.3896e-02 = -0.014 \\ x_{\text{Frame}} = -8.1104e-01 = -0.811 \\ y_{\text{Frame}} = 2.1926e-01 = -0.219 \\ \theta_{\text{Wheel Back}} = 7.8437e+00 = 7.844 \\ \theta_{\text{Tire Front}} = 2.2254e-21 \approx 0 \\ \theta_{\text{Tire Back}} = 7.8437e+00 = 7.844 \\ y_{\text{Tire Front}} = 2.1883e-01 = 0.219 \\ y_{\text{Tire Back}} = 2.1895e-01 = 0.219 \\ \beta_{\text{Link Back}} = 3.0209e+00 = 3.021 \\ \beta_{\text{Link Front}} = 1.6787e-01 = 0.168 \end{pmatrix} \tag{62}$$

Which is visualized by this figure:

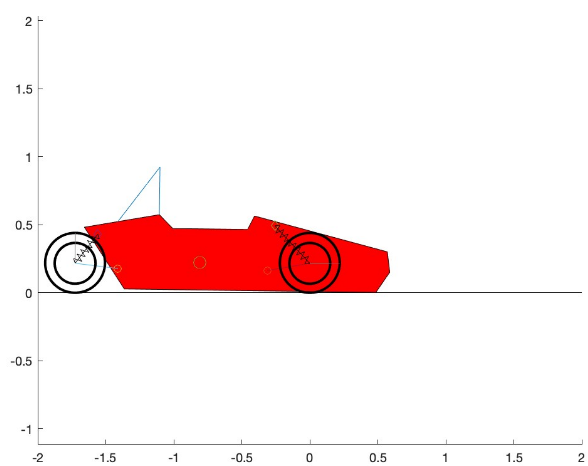

Figure 6: Equilibrium Position 1

In this example the choice of initial generalized coordinates was obviously very good. For the first task of this assignment the goal is to find two more (probably bad) equilibria.

4.5.2 Different Equilibrium states

The first alternative is luckily already given in the code. Again we fix the front wheel's rotation and start with the initial state:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi/2 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{63}$$

Which represents this position:

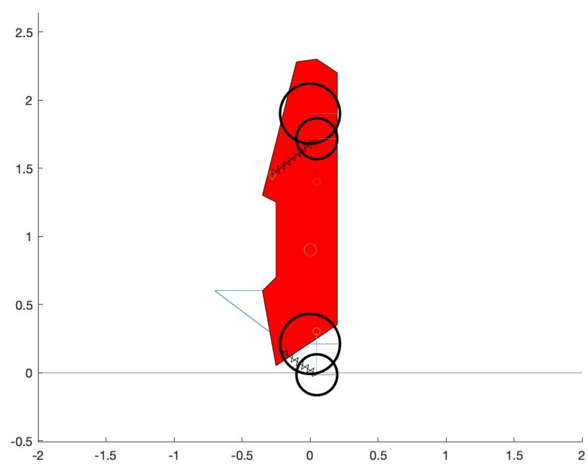

Figure 7: Initial Position 2

Note: in the code provided θ_{Frame} was π which was a different initial set of gen. coord. but converged to the same solution

As can be seen, this is obviously not a good choice of initial coordinates.

This setting converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 1.5248e+00 \\ x_{\text{Frame}} = -6.9703e-02 \\ y_{\text{Frame}} = 1.1280e+00 \\ \theta_{\text{Wheel Back}} = 3.0427e-01 \\ \theta_{\text{Tire Front}} = 2.9622e-24 \\ \theta_{\text{Tire Back}} = 3.0427e-01 \\ y_{\text{Tire Front}} = 1.9414e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -1.6327e+00 \\ \beta_{\text{Link Front}} = 1.5811e+00 \end{pmatrix} \tag{64}$$

Which looks as follows:

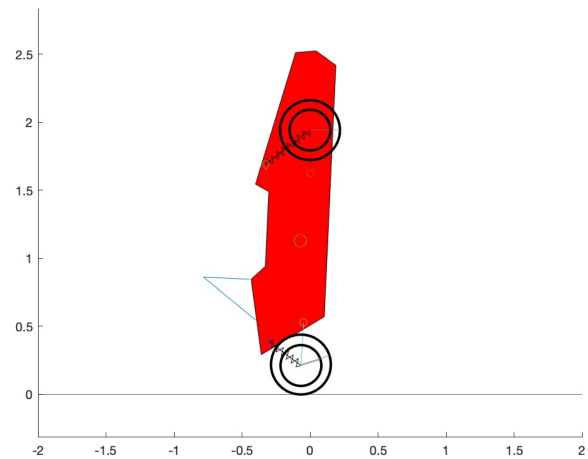

Figure 8: Equilibrium Position 2

The code example which has even worse initial conditions:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{65}$$

And looks like this:

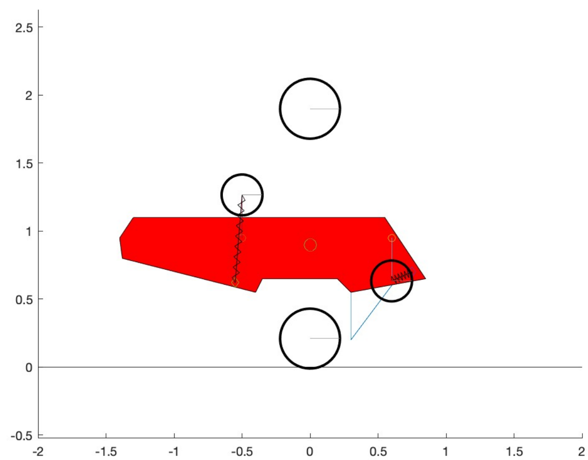

Figure 9: Initial Position 3

Converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 2.0225e+00 \\ x_{\text{Frame}} = -1.2734e-01 \\ y_{\text{Frame}} = 6.3440e-01 \\ \theta_{\text{Wheel Back}} = 5.5591e-01 \\ \theta_{\text{Tire Front}} = 2.0275e-25 \\ \theta_{\text{Tire Back}} = 5.5591e-01 \\ y_{\text{Tire Front}} = 1.2043e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -3.4446e+00 \\ \beta_{\text{Link Front}} = 3.1586e-01 \end{pmatrix} \tag{66}$$

Which represents:

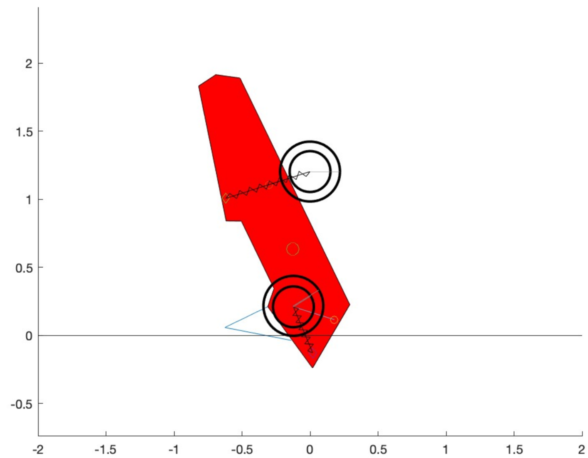

Figure 10: Equilibrium Position 3

4.6 Analysis

As we are working with a first order linearization of the dynamics of the system we have to stay in the proximity of the desired state of convergence. So a horizontal car with the initial conditions as seen in equation 61 will converge to a reasonable solution.

Initial conditions like a 90° or 180° rotation of the frame can converge to a local solution as seen in Figure 10

The equilibrium from figure 6 surely is stable as we all know it from reality. The two alternatives however are not stable. This can be seen from the example in fig 11 when the angle of the frame is between 0 and 90°:

Which converges to

So as we can see even with nonsensical other parameters when rotating the frame a bit less than 90° we converge to the stable solution.

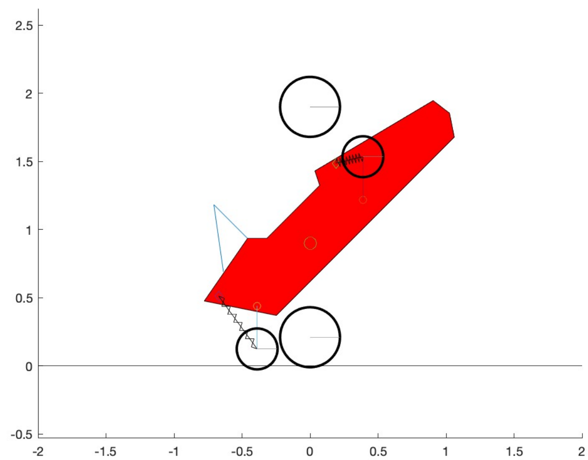

Figure 11: Initial Position 4

4.7 DONE

4.8 Eigenmodes and Eigenfrequencies

4.8.1 DONE

4.8.2 Mode Shapes

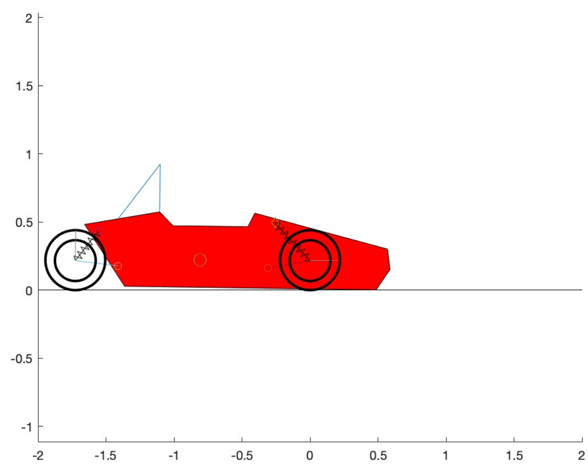

Figure 12: Equilibrium Position 4