Advanced Dynamics - Assignment Report

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1 Assignment 1

1 WHEEL ROLLING WITHOUT SLIPPING ON A 2D TRACK

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.

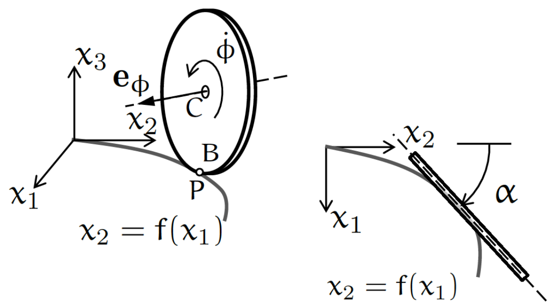

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability: To Do

$$v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C \boldsymbol{e_1} + \dot{x}_2^C \boldsymbol{e_2} - R\dot{\phi}\cos\alpha\boldsymbol{e_2} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

Now if we reformulate this in the linear velocity form:

$$\sum_{i=1}^{n} a_i(\boldsymbol{q}, t)\dot{q}_i + b(\boldsymbol{q}, t) \text{ with } \mathbf{q} = [x_1^C, x_2^C, x_3^C, \alpha, \phi] \tag{4}$$

We get

$$
\begin{aligned}
& a_1 = 1, \quad a_5 = -R\sin(\alpha), b = 0 \\
& \Rightarrow \frac{\partial(Cb)}{\partial q_1} = \frac{\partial(Ca_1)}{\partial t} \Rightarrow \frac{\partial C}{\partial t} = 0 \Rightarrow C \neq C(t) \\
& \Rightarrow \frac{\partial(Ca_1)}{\partial q_5} = \frac{\partial(Ca_5)}{\partial q_1} \Rightarrow \frac{\partial(C)}{\partial \phi} = \frac{\partial(-CR\sin(\alpha))}{\partial x_1^C} \\
& \Rightarrow \frac{\partial(C)}{\partial \phi} = -R\sin(\alpha)\frac{\partial(C)}{\partial x_1^C}
\end{aligned} \tag{5}
$$

Perhaps we can show directly that the constraint can be written in the solution form:

$$
\dot{x}_1^C \boldsymbol{e_1} - R\dot{\phi}\sin\alpha \boldsymbol{e_1} = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{q}, t)}{x_1^C}\dot{x}_1^C + \frac{\partial f(\boldsymbol{q}, t)}{\phi}\dot{\phi} \tag{6}
$$

1.2

See section 1.1

1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e_\phi}$ while all the other generalized coordinates follow accordingly.

2 Assignment 2

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2 M\ddot{\phi}}{3} + \frac{L^2 m\ddot{\phi}}{3} - \frac{L^2 m\ddot{\phi}\cos(\beta)^2}{3} + \frac{LMg\cos(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2 \sin(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)\sin(\phi)}{4} - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{7}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi}\left(\frac{L^2(M+m)}{3} - \frac{L^2 m\cos(\beta)^2}{3}\right) + \sin(\phi)\left(\frac{LMg\cos(\beta)}{2} + \frac{L^2 M\Omega^2\sin(\beta)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)}{4}\right) - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{8}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2\ddot{\theta}\left(16M + 16m + 12M\cos(\beta) + 12m\cos(\beta) + 2M\cos(\beta)^2 + 4m\cos(\beta)^2 - 2M\cos(\beta)^2\cos(\phi)^2\right)}{12} + \frac{L^2\ddot{\theta}\left(12M\cos(\phi)\sin(\beta) + 6M\cos(\beta)\cos(\phi)\sin(\beta)\right)}{12} \tag{9}$$

3.4

After matlab integration of the work per area we get for the work

$$W = \frac{L^4\left(c\left(\dot{\theta} + \dot{\phi}\sin(\beta)\right)^2\left(\cos(\phi)^2 - 1\right) - c\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right)^2 + c\dot{\phi}^2\cos(\beta)^2\left(\cos(\phi)^2 - 1\right)\right)}{3} - L\left(Lc\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)^2 + \frac{L^3 c\dot{\theta}^2\cos(\beta)^2}{12}\right) - L^3 c\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right) \tag{10}$$

Which results in the equations of motion:

$$\frac{L^2 M\ddot{\phi}}{3} + \frac{2L^4 c\left(\dot{\phi} + \dot{\theta}\sin(\beta)\right)}{3} + \frac{L^2 m\ddot{\phi}}{3} - \frac{L^2 m\ddot{\phi}\cos(\beta)^2}{3} + \frac{LMg\cos(\beta)\sin(\phi)}{2} +$$
$$L^4 c\dot{\theta}\cos(\phi)(\cos(\beta)+1) + \frac{L^2 M\dot{\theta}^2\sin(\beta)\sin(\phi)}{2} - \frac{L^2 M\dot{\theta}^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} +$$
$$\frac{L^2 M\dot{\theta}^2\cos(\beta)\sin(\beta)\sin(\phi)}{4} \qquad \text{for } \phi \tag{11}$$

And

$$\frac{cL^4}{6}\Big(16\dot{\theta} + 24\dot{\theta}\cos(\beta) + 6\dot{\phi}\cos(\phi) + 4\dot{\phi}\sin(\beta) + 13\dot{\theta}\cos(\beta)^2 - 4\dot{\theta}\cos(\beta)^2\cos(\phi)^2 + 6\dot{\phi}\cos(\beta)\cos(\phi) + 12\dot{\theta}\cos$$
$$12\dot{\theta}\cos(\beta)\cos(\phi)\sin(\beta)\Big) +$$
$$\frac{\ddot{\theta}}{12}\Big(16M + 16m + 12M\cos(\beta) + 12m\cos(\beta) + 2M\cos(\beta)^2 + 4m\cos(\beta)^2 - 2M\cos(\beta)^2\cos(\phi)^2 +$$
$$12M\cos(\phi)\sin(\beta) + 6M\cos(\beta)\cos(\phi)\sin(\beta)\Big)L^2 \qquad \text{for } \theta \tag{12}$$

3.5 Plugging in the values:

Let $L = 0.25[m], \beta = \frac{\pi}{6}, M = 0.5[kg], m = 0.2[kg], g = 9.81[\frac{m}{s^2}]$ and $c = 0.1[\frac{Ns}{m^3}]$.

The equations become:

$$\frac{\dot{\phi}}{3840} + \frac{11\ddot{\phi}}{960} + \frac{\dot{\theta}}{7680} + \frac{\dot{\theta}^2\sin(\phi)}{128} + \frac{981\sqrt{3}\sin(\phi)}{3200} + \frac{\sqrt{3}\dot{\theta}^2\sin(\phi)}{512} + \frac{\dot{\theta}\cos(\phi)\left(\frac{\sqrt{3}}{2}+1\right)}{2560}$$
$$- \frac{\dot{\theta}^2\cos(\phi)\sin(\phi)}{256} \qquad \text{for } \phi \tag{13}$$

And

$$\frac{\dot{\phi}}{7680} + \frac{103\dot{\theta}}{61440} - \frac{\dot{\theta}\cos(\phi)^2}{5120} + \frac{\sqrt{3}\dot{\theta}}{1280} +$$
$$\frac{\ddot{\theta}}{192}\left(3\cos(\phi) - \frac{3\cos(\phi)^2}{4} + \frac{21\sqrt{3}}{5} + \frac{3\sqrt{3}\cos(\phi)}{4} + \frac{251}{20}\right) +$$
$$\frac{\dot{\phi}\cos(\phi)}{2560} + \frac{\dot{\theta}\cos(\phi)}{2560} + \frac{\sqrt{3}\dot{\phi}\cos(\phi)}{5120} + \frac{\sqrt{3}\dot{\theta}\cos(\phi)}{5120} \qquad \text{for } \theta \tag{14}$$

3.6 Equilibrium State

Reminder- The potential energy looks as follows:

$$\frac{LMg\left(2\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2} \tag{15}$$

Case 1: Ω fixed:

The derivative of the potential energy w.r.t. θ is always 0.

$$\frac{\partial V}{\partial \theta}=0 \tag{16}$$

The derivative of the potential energy w.r.t. ϕ not however:

$$\frac{\partial V}{\partial \phi}=\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2} \tag{17}$$

Equation 12 is only 0 for $\boldsymbol{\phi = k * \pi}$ which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

Case 2: $\dot{\theta}$ can change:

As V is independent of θ altogether the result is the same for this case.

The equation of motions are after plugging in equilibrium state:

$$\begin{gathered}\phi:\\ 0=0\end{gathered} \tag{18}$$

$$\begin{aligned}&\theta:\\ &\frac{L^2\ddot{\theta}}{12}\left(16M+16m+12M\cos\left(\beta\right)+12M\sin\left(\beta\right)+12m\cos\left(\beta\right)+\right.\\ &\left.4m\cos\left(\beta\right)^2+6M\cos\left(\beta\right)\sin\left(\beta\right)\right)=0\end{aligned} \tag{19}$$

As the coefficient of $\ddot{\theta}$ is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant Ω.

4 Assignment 4