Advanced Dynamics - Assignment Report

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1 Assignment 1

1 Wheel rolling without slipping on a 2D track

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.

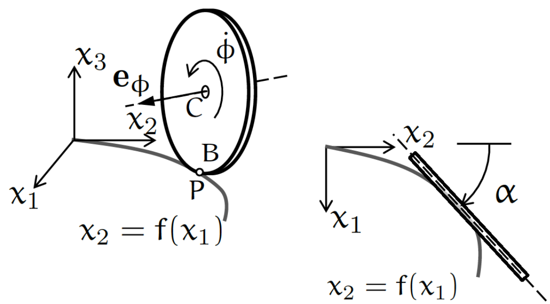

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3
2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.
3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic
4. Rolling without slipping:
   $v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance
5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability: To Do

$$v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C \boldsymbol{e_1} + \dot{x}_2^C \boldsymbol{e_2} - R\dot{\phi}\cos\alpha \boldsymbol{e_2} - R\dot{\phi}\sin\alpha \boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

Now if we reformulate this in the linear velocity form:

$$\sum_{i=1}^{n} a_i(\boldsymbol{q}, t)\dot{q}_i + b(\boldsymbol{q}, t) \text{ with } \mathbf{q} = [x_1^C, x_2^C, x_3^C, \alpha, \phi] \tag{4}$$

We get

$$
\begin{aligned}
& a_1 = 1, \quad a_5 = -R\sin(\alpha), b = 0 \\
& \Rightarrow \frac{\partial(Cb)}{\partial q_1} = \frac{\partial(Ca_1)}{\partial t} \Rightarrow \frac{\partial C}{\partial t} = 0 \Rightarrow C \neq C(t) \\
& \Rightarrow \frac{\partial(Ca_1)}{\partial q_5} = \frac{\partial(Ca_5)}{\partial q_1} \Rightarrow \frac{\partial(C)}{\partial \phi} = \frac{\partial(-CR\sin(\alpha))}{\partial x_1^C} \\
& \Rightarrow \frac{\partial(C)}{\partial \phi} = -R\sin(\alpha)\frac{\partial(C)}{\partial x_1^C}
\end{aligned}
\tag{5}
$$

Perhaps we can show directly that the constraint can be written in the solution form:

$$
\dot{x}_1^C \boldsymbol{e_1} - R\dot{\phi}\sin\alpha \boldsymbol{e_1} = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{q}, t)}{x_1^C}\dot{x}_1^C + \frac{\partial f(\boldsymbol{q}, t)}{\phi}\dot{\phi} \tag{6}
$$

1.2

See section 1.1

1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5 = 1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e_\phi}$ while all the other generalized coordinates follow accordingly.

2 Assignment 2

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2M\ddot{\phi}}{3}+\frac{L^2m\ddot{\phi}}{3}-\frac{L^2m\ddot{\phi}\cos(\beta)^2}{3}+\frac{LMg\cos(\beta)\sin(\phi)}{2}+\frac{L^2M\Omega^2\sin(\beta)\sin(\phi)}{2}+\frac{L^2M\Omega^2\cos(\beta)\sin(\beta)\sin(\phi)}{4}-\frac{L^2M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6}=0 \tag{7}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi}\left(\frac{L^2(M+m)}{3}-\frac{L^2m\cos(\beta)^2}{3}\right)+\sin(\phi)\left(\frac{LMg\cos(\beta)}{2}+\frac{L^2M\Omega^2\sin(\beta)}{2}+\frac{L^2M\Omega^2\cos(\beta)\sin(\beta)}{4}\right)-\frac{L^2M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6}=0 \tag{8}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2\ddot{\theta}\left(16M+16m+12M\cos(\beta)+12m\cos(\beta)+2M\cos(\beta)^2+4m\cos(\beta)^2-2M\cos(\beta)^2\cos(\phi)^2\right)}{12}+\frac{L^2\ddot{\theta}\left(12M\cos(\phi)\sin(\beta)+6M\cos(\beta)\cos(\phi)\sin(\beta)\right)}{12} \tag{9}$$

3.4

After matlab integration of the work per area we get for the work

$$W=\frac{L^4\left(c\left(\dot{\theta}+\dot{\phi}\sin(\beta)\right)^2\left(\cos(\phi)^2-1\right)-c\left(\dot{\phi}\cos(\phi)+\dot{\theta}\cos(\phi)\sin(\beta)\right)^2+c\dot{\phi}^2\cos(\beta)^2\left(\cos(\phi)^2-1\right)\right)}{3}-L\left(Lc\left(L\dot{\theta}+L\dot{\theta}\cos(\beta)\right)^2+\frac{L^3c\dot{\theta}^2\cos(\beta)^2}{12}\right)-L^3c\left(L\dot{\theta}+L\dot{\theta}\cos(\beta)\right)\left(\dot{\phi}\cos(\phi)+\dot{\theta}\cos(\phi)\sin(\beta)\right) \tag{10}$$

3.5

Reminder- The potential energy looks as follows:

$$\frac{LMg\left(2\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2} \tag{11}$$

Case 1: Ω fixed:

The derivative of the potential energy w.r.t. θ is always 0.

$$\frac{\partial V}{\partial\theta}=0 \tag{12}$$

The derivative of the potential energy w.r.t. ϕ not however:

$$\frac{\partial V}{\partial\phi}=\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2} \tag{13}$$

Equation 12 is only 0 for $\phi = k*\pi$ which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

Case 2: $\dot{\theta}$ can change:

As V is independent of θ altogether the result is the same for this case.

The equation of motions are:

$$\frac{L^2M\ddot{\phi}}{3}+\frac{L^2m\ddot{\phi}}{3}-\frac{L^2m\ddot{\phi}\cos\left(\beta\right)^2}{3}+\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2}+\frac{L^2M\dot{\theta}^2\sin\left(\beta\right)\sin\left(\phi\right)}{2}- \frac{L^2M\dot{\theta}^2\cos\left(\beta\right)^2\cos\left(\phi\right)\sin\left(\phi\right)}{6}+\frac{L^2M\dot{\theta}^2\cos\left(\beta\right)\sin\left(\beta\right)\sin\left(\phi\right)}{4} \quad \text{for } \phi \tag{14}$$

$$\frac{L^2\ddot{\theta}}{12}\Big(16M+16m+12M\cos\left(\beta\right)+12m\cos\left(\beta\right)+2M\cos\left(\beta\right)^2+4m\cos\left(\beta\right)^2- 2M\cos\left(\beta\right)^2\cos\left(\phi\right)^2+12M\cos\left(\phi\right)\sin\left(\beta\right)+6M\cos\left(\beta\right)\cos\left(\phi\right)\sin\left(\beta\right)\Big) \quad \text{for } \theta \tag{15}$$

Plugging in $\phi=\dot{\phi}=\ddot{\phi}=0$:

$$\begin{aligned}&\phi:\\&0=0\end{aligned} \tag{16}$$

$$\begin{aligned}&\theta:\\&\frac{L^2\ddot{\theta}}{12}\left(16M+16m+12M\cos(\beta)+12M\sin(\beta)+12m\cos(\beta)+\right.\\&\left.4m\cos(\beta)^2+6M\cos(\beta)\sin(\beta)\right)=0\Rightarrow\ddot{\theta}=0\end{aligned} \tag{17}$$

As the coefficient of $\ddot{\theta}$ is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant Ω.

3.6

4 Assignment 4