Advanced Dynamics - Assignment Report

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1 Assignment 1

1 Wheel rolling without slipping on a 2D track

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.

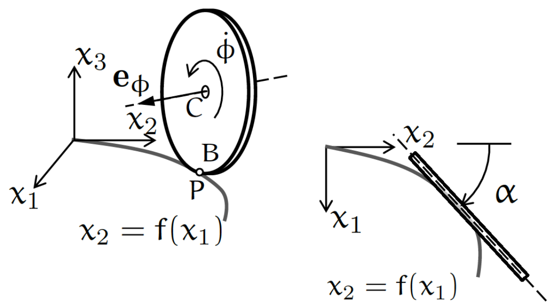

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

1.1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C\boldsymbol{e_1} + \dot{x}_2^C\boldsymbol{e_2} - R\dot{\phi}\cos\alpha\boldsymbol{e_2} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

As α is not dependent on time it can be easily seen that 3 is integrable:

$$\begin{aligned} &\int \dot{x}_1^C dt = \int R\dot{\phi}\sin\alpha dt \\ &\Rightarrow x_1^C = R\phi\sin\alpha \end{aligned} \tag{4}$$

1.1.2

See subsubsection 1.1.1

1.1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e}_{\phi}$ while all the other generalized coordinates follow accordingly.

2 Two bars linkage

Two bars AB and BC of equal length L are hinged at point B, and move in the plane spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The velocity $\mathbf{v}_C$ of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use x_1^B, x_2^B, θ_1 and θ_2 as generalized coordinates.

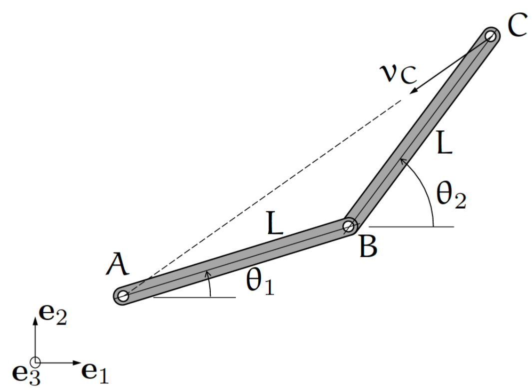

Figure 2.1: A two bar linkage in 2D. The velocity v_C must be directed towards A at all times.

Figure 3: Task 1.1

1.2

I want to show that the constraint that v_c always points in the direction of AD is not integrable. I would like to express this constraint as:

$$\begin{aligned} &v_C * AC_p = 0 \\ &\text{where } AC_p \text{ is a vector perpendicular to } AC \end{aligned} \tag{5}$$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} \tag{6}$$

And AC is simply:

$$AC = \begin{pmatrix} L\,(\cos\theta_1 + \cos\theta_2) \\ L\,(\sin\theta_1 + \sin\theta_2) \end{pmatrix} \tag{7}$$

A simple vector that is perpendicular to AC can be gotten by switching the e_1 and e_2 entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L(\sin\theta_1 + \sin\theta_2) \\ -L(\cos\theta_1 + \cos\theta_2) \end{pmatrix} \tag{8}$$

$$\begin{aligned} &\Rightarrow v_C * AC_p = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L(\sin\theta_1 + \sin\theta_2) \\ -L(\cos\theta_1 + \cos\theta_2) \end{pmatrix} = \\ &\quad (L(\sin(\theta_1) + \sin(\theta_2)))\,\dot{\mathrm{x}}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2)))\,\dot{x}_2^B - \\ &\quad L^2\dot{\theta}_2\cos(\theta_2)(\cos(\theta_1) + \cos(\theta_2)) - L^2\dot{\theta}_2\sin(\theta_2)(\sin(\theta_1) + \sin(\theta_2)) = 0 \end{aligned} \tag{9}$$

After applying trigonometry:

$$\begin{aligned} &(L(\sin(\theta_1) + \sin(\theta_2)))\,\dot{\mathrm{x}}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2)))\,\dot{x}_2^B - \\ &L^2\dot{\theta}_2(\cos(\theta_2)\cos(\theta_1) + \sin(\theta_2)\sin(\theta_1) + 1) = 0 \end{aligned} \tag{10}$$

To simplify the notation I will refer to $\sin\theta_1$ as s_1 and $\sin\theta_2$ as s_2:

$$L\dot{x}_1^B(s_1 + s_2) - L\dot{x}_2^B(c_1 + c_2) - L^2\dot{\theta}_2^2(c_2c_1 + s_2s_1 + 1) = 0 \tag{11}$$

Dividing by L:

$$\dot{x}_1^B(s_1 + s_2) - \dot{x}_2^B(c_1 + c_2) - L\dot{\theta}_2^2(c_2c_1 + s_2s_1 + 1) = 0 \tag{12}$$

With the generalized coordinates $q = [x_1^B, x_2^B, \theta_1, \theta_2]$
Writing down the coefficients of the non-holonomic constraint:

$$\begin{aligned} a_1 &= s_1 + s_2 \\ a_2 &= -(c_1 + c_2) \\ a_3 &= 0 \\ a_4 &= -L(1 + s_1s_2 + c_1c_2) \\ b &= 0 \end{aligned} \tag{13}$$

To check for the exact velocity form we want to proof that there can't exist a $C(q) \neq 0$ for which holds:

$$\frac{\partial(Cb)}{\partial q_i} = \frac{\partial(Ca_1)}{\partial t} \quad \text{and} \quad \frac{\partial(Ca_i)}{\partial q_k} = \frac{\partial(Ca_k)}{\partial q_i} \quad \text{for all the gen. coord.} \tag{14}$$

q1-q3:

$$\begin{aligned}
&\frac{\partial(Ca_1)}{\partial\theta_1} = 0 \Rightarrow C * c_1 + \frac{\partial C}{\partial\theta_1}(s1+s2) = 0\\
&\Rightarrow \frac{c_1}{s1+s2} + \frac{1}{C}\frac{dC}{d\theta_1} = 0 \Rightarrow \int \frac{1}{C}dC = -\int \frac{c_1}{s_1+s_2}d\theta_1\\
&\Rightarrow lnC = \frac{D}{s_1+s_2} + D \Rightarrow C = \frac{D}{s_1+s_2} \quad \text{where the integration const. D was updated}\\
&\Rightarrow C = \frac{D(x_1^B, x_2^B, \theta_2)}{s1+s2}
\end{aligned} \tag{15}$$

q1 - q4:

$$\frac{\partial Ca_1}{\partial\theta_2} = \underbrace{\frac{\partial Ca_4}{\partial x_1^B}}_{0} = 0 \Rightarrow \frac{\partial D}{\partial\theta_2} = 0 \Rightarrow C = \frac{D(x_1^B, x_2^B)}{s1+s2} \tag{16}$$

q1 - q2:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = \frac{\partial Ca_2}{\partial x_1^B} \Rightarrow \frac{\partial C}{\partial x_2^B}(s_1+s_2) = \frac{\partial C}{\partial x_1^B}(c_1+c_2) \Rightarrow \frac{\partial C}{\partial x_2^B} = \frac{\partial C}{\partial x_1^B}\frac{c_1+c_2}{s_1+s_2}\\
&\text{Remembering that } D = D(x_1^B, x_2^B) \Rightarrow D = const.\\
&\Rightarrow C = \frac{D}{s_1+s_2} \text{ where D const.}
\end{aligned} \tag{17}$$

q2 - q3:

$$\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = 0 \Rightarrow D\frac{\partial \frac{(c_1+c_2)}{s_1+s_2}}{\partial\theta_1} = 0\\
&D\frac{\partial \frac{c_1}{s_1+s_2}}{\partial\theta_1} = -\frac{Ds_1}{s_1+s_2} - \frac{Dc_1^2}{(s_1+s_2)^2} = -D\frac{s_1(s_1+s_2)-c_1^2}{(s_1+s_2)^2}
\end{aligned} \tag{18}$$

$$D\frac{\partial \frac{c_2}{s_1+s_2}}{\partial\theta_1} = -D\frac{c_1c_2}{(s_1+s_2)^2} \tag{19}$$

Which leads to:

$$-D\frac{c_1^2 + s_1^2 + s_1 s_2 + c_1 c_2}{(s_1 + s_2)^2} \overset{!}{=} 0 \tag{20}$$

Which is a contradiction. Therefore this is a non-holonomic constraint.

2 Assignment 2

This assignment asks to get the equations of motion using the Hamilton Principle:

$$\int_{t_1}^{t_2} \delta(T - V + W)dt = 0 \tag{21}$$

Using the theory discussed in the lecture we can split the term into different contributions::

Remembering the result of the Hamilton principle from the lecture:

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right) + f(x,t) + \sum F_i D(x - x_i) = 0, \quad x \in [0; l]$$

Boundary conditions:

$$\left[\frac{\partial L}{\partial v'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right]\delta v = 0, \quad x = 0, l$$

$$\frac{\partial L}{\partial v''}\delta v' = 0, \quad x = 0, l$$

Figure 4: Hamilton Formulas for a continuous system

We consider w our variable instead of v.

$$\frac{\partial L}{\partial w} = 0, \qquad \frac{\partial}{\partial x}\frac{\partial L}{\partial w'} = 0, \tag{22}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial w''} = -E\frac{\partial^2}{\partial x^2}I(x), \qquad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 2\delta A(x)\ddot{w}, \tag{23}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}'}\right)\right) = 2\rho I(x)\ddot{w}'' + 2\rho I(x)'\ddot{w}' \tag{24}$$

$$f(x,t) = p(x,t) \tag{25}$$

Therefore the green part comes to:
While for the red we get:
And for lastly for the blue:

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2 M\ddot{\phi}}{3} + \frac{L^2 m\ddot{\phi}}{3} - \frac{L^2 m\ddot{\phi}\cos(\beta)^2}{3} + \frac{LMg\cos(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2\sin(\beta)\sin(\phi)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)\sin(\phi)}{4} - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{26}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi}\left(\frac{L^2(M+m)}{3} - \frac{L^2 m\cos(\beta)^2}{3}\right) + \sin(\phi)\left(\frac{LMg\cos(\beta)}{2} + \frac{L^2 M\Omega^2\sin(\beta)}{2} + \frac{L^2 M\Omega^2\cos(\beta)\sin(\beta)}{4}\right) - \frac{L^2 M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6} = 0 \tag{27}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2\ddot{\theta}\left(16M + 16m + 12M\cos(\beta) + 12m\cos(\beta) + 2M\cos(\beta)^2 + 4m\cos(\beta)^2 - 2M\cos(\beta)^2\cos(\phi)^2\right)}{12} + \frac{L^2\ddot{\theta}\,(12M\cos(\phi)\sin(\beta) + 6M\cos(\beta)\cos(\phi)\sin(\beta))}{12} \tag{28}$$

3.4

After matlab integration of the work per area we get for the work

$$W = \frac{L^4\left(c\left(\dot{\theta} + \dot{\phi}\sin(\beta)\right)^2\left(\cos(\phi)^2 - 1\right) - c\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right)^2 + c\dot{\phi}^2\cos(\beta)^2\left(\cos(\phi)^2 - 1\right)\right)}{3} - L\left(Lc\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)^2 + \frac{L^3 c\dot{\theta}^2\cos(\beta)^2}{12}\right) - L^3 c\left(L\dot{\theta} + L\dot{\theta}\cos(\beta)\right)\left(\dot{\phi}\cos(\phi) + \dot{\theta}\cos(\phi)\sin(\beta)\right) \tag{29}$$

Which results in the equations of motion:

$$\frac{L^2M\ddot{\phi}}{3}+\frac{2L^4c\left(\dot{\phi}+\dot{\theta}\sin\left(\beta\right)\right)}{3}+\frac{L^2m\ddot{\phi}}{3}-\frac{L^2m\ddot{\phi}\cos\left(\beta\right)^2}{3}+\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2}+$$
$$L^4c\dot{\theta}\cos\left(\phi\right)\left(\cos\left(\beta\right)+1\right)+\frac{L^2M\dot{\theta}^2\sin\left(\beta\right)\sin\left(\phi\right)}{2}-\frac{L^2M\dot{\theta}^2\cos\left(\beta\right)^2\cos\left(\phi\right)\sin\left(\phi\right)}{6}+$$
$$\frac{L^2M\dot{\theta}^2\cos\left(\beta\right)\sin\left(\beta\right)\sin\left(\phi\right)}{4}\qquad\text{for }\phi \tag{30}$$

And

$$\frac{cL^4}{6}\left(16\dot{\theta}+24\dot{\theta}\cos\left(\beta\right)+6\dot{\phi}\cos\left(\phi\right)+4\dot{\phi}\sin\left(\beta\right)+13\dot{\theta}\cos\left(\beta\right)^2-4\dot{\theta}\cos\left(\beta\right)^2\cos\left(\phi\right)^2+6\dot{\phi}\cos\left(\beta\right)\cos\left(\phi\right)+12\dot{\theta}\cos\right.$$
$$\left.12\dot{\theta}\cos\left(\beta\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)+$$
$$\frac{\ddot{\theta}}{12}\left(16M+16m+12M\cos\left(\beta\right)+12m\cos\left(\beta\right)+2M\cos\left(\beta\right)^2+4m\cos\left(\beta\right)^2-2M\cos\left(\beta\right)^2\cos\left(\phi\right)^2+\right.$$
$$\left.12M\cos\left(\phi\right)\sin\left(\beta\right)+6M\cos\left(\beta\right)\cos\left(\phi\right)\sin\left(\beta\right)\right)L^2\qquad\text{for }\theta \tag{31}$$

3.5 Plugging in the values:

Let $L=0.25[m], \beta=\frac{\pi}{6}, M=0.5[kg], m=0.2[kg], g=9.81[\frac{m}{s^2}]$ and $c=0.1[\frac{Ns}{m^3}]$.

The equations become:

$$\frac{\dot{\phi}}{3840}+\frac{11\ddot{\phi}}{960}+\frac{\dot{\theta}}{7680}+\frac{\dot{\theta}^2\sin\left(\phi\right)}{128}+\frac{981\sqrt{3}\sin\left(\phi\right)}{3200}+\frac{\sqrt{3}\dot{\theta}^2\sin\left(\phi\right)}{512}+\frac{\dot{\theta}\cos\left(\phi\right)\left(\frac{\sqrt{3}}{2}+1\right)}{2560}$$
$$-\frac{\dot{\theta}^2\cos\left(\phi\right)\sin\left(\phi\right)}{256}\qquad\text{for }\phi \tag{32}$$

And

$$\frac{\dot{\phi}}{7680}+\frac{103\dot{\theta}}{61440}-\frac{\dot{\theta}\cos\left(\phi\right)^2}{5120}+\frac{\sqrt{3}\dot{\theta}}{1280}+$$
$$\frac{\ddot{\theta}}{192}\left(3\cos\left(\phi\right)-\frac{3\cos\left(\phi\right)^2}{4}+\frac{21\sqrt{3}}{5}+\frac{3\sqrt{3}\cos\left(\phi\right)}{4}+\frac{251}{20}\right)+$$
$$\frac{\dot{\phi}\cos\left(\phi\right)}{2560}+\frac{\dot{\theta}\cos\left(\phi\right)}{2560}+\frac{\sqrt{3}\dot{\phi}\cos\left(\phi\right)}{5120}+\frac{\sqrt{3}\dot{\theta}\cos\left(\phi\right)}{5120}\qquad\text{for }\theta \tag{33}$$

3.6 Equilibrium State

Reminder- The potential energy looks as follows:

$$\frac{LMg\left(2\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2} \tag{34}$$

Case 1: Ω fixed:

The derivative of the potential energy w.r.t. θ is always 0.

$$\frac{\partial V}{\partial\theta}=0 \tag{35}$$

The derivative of the potential energy w.r.t. ϕ not however:

$$\frac{\partial V}{\partial\phi}=\frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2} \tag{36}$$

Equation 12 is only 0 for $\boldsymbol{\phi = k * \pi}$ which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

Case 2: $\dot{\theta}$ can change:

As V is independent of θ altogether the result is the same for this case.

The equation of motions are after plugging in equilibrium state:

$$\begin{aligned}&\phi:\\&0=0\end{aligned} \tag{37}$$

$$\begin{aligned}&\theta:\\&\frac{L^2\ddot{\theta}}{12}\Big(16M+16m+12M\cos\left(\beta\right)+12M\sin\left(\beta\right)+12m\cos\left(\beta\right)+\\&4m{\cos\left(\beta\right)}^2+6M\cos\left(\beta\right)\sin\left(\beta\right)\Big)=0\end{aligned} \tag{38}$$

As the coefficient of $\ddot{\theta}$ is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant Ω.

4 Assignment 4