Advanced Dynamics - Assignment Report

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Contents

1 Assignment 1

1 Wheel rolling without slipping on a 2D track

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.

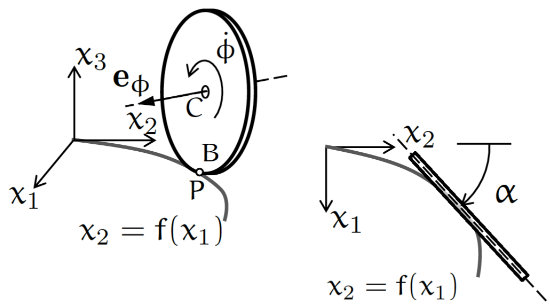

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3
2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.
3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic
4. Rolling without slipping:
   $v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance
5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability: To Do

$$v_B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C \boldsymbol{e_1} + \dot{x}_2^C \boldsymbol{e_2} - R\dot{\phi}\cos\alpha\boldsymbol{e_2} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

Now if we reformulate this in the linear velocity form:

$$\sum_{i=1}^{n} a_i(\boldsymbol{q}, t)\dot{q}_i + b(\boldsymbol{q}, t) \text{ with } \mathbf{q} = [x_1^C, x_2^C, x_3^C, \alpha, \phi] \tag{4}$$

We get

$$
\begin{aligned}
&a_1 = 1, \quad a_5 = -R\sin(\alpha), b = 0 \\
&\Rightarrow \frac{\partial(Cb)}{\partial q_1} = \frac{\partial(Ca_1)}{\partial t} \Rightarrow \frac{\partial C}{\partial t} = 0 \Rightarrow C \neq C(t) \\
&\Rightarrow \frac{\partial(Ca_1)}{\partial q_5} = \frac{\partial(Ca_5)}{\partial q_1} \Rightarrow \frac{\partial(C)}{\partial \phi} = \frac{\partial(-CR\sin(\alpha))}{\partial x_1^C} \\
&\Rightarrow \frac{\partial(C)}{\partial \phi} = -R\sin(\alpha)\frac{\partial(C)}{\partial x_1^C}
\end{aligned}
\tag{5}
$$

Perhaps we can show directly that the constraint can be written in the solution form:

$$
\dot{x}_1^C \boldsymbol{e_1} - R\dot{\phi}\sin\alpha\boldsymbol{e_1} = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{q}, t)}{x_1^C}\dot{x}_1^C + \frac{\partial f(\boldsymbol{q}, t)}{\phi}\dot{\phi} \tag{6}
$$

1.2

See section 1.1

1.3

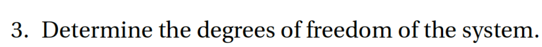
3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e_\phi}$ while all the other generalized coordinates follow accordingly.

2 Assignment 2

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2M\ddot{\phi}}{3}+\frac{L^2m\ddot{\phi}}{3}-\frac{L^2m\ddot{\phi}\cos(\beta)^2}{3}+\frac{LMg\cos(\beta)\sin(\phi)}{2}+\frac{L^2M\Omega^2\sin(\beta)\sin(\phi)}{2}+\frac{L^2M\Omega^2\cos(\beta)\sin(\beta)\sin(\phi)}{4}-\frac{L^2M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6}=0 \tag{7}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi}\left(\frac{L^2(M+m)}{3}-\frac{L^2m\cos(\beta)^2}{3}\right)+\sin(\phi)\left(\frac{LMg\cos(\beta)}{2}+\frac{L^2M\Omega^2\sin(\beta)}{2}+\frac{L^2M\Omega^2\cos(\beta)\sin(\beta)}{4}\right)-\frac{L^2M\Omega^2\cos(\beta)^2\cos(\phi)\sin(\phi)}{6}=0 \tag{8}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2\ddot{\theta}\left(16M+16m+12M\cos(\beta)+12m\cos(\beta)+2M\cos(\beta)^2+4m\cos(\beta)^2-2M\cos(\beta)^2\cos(\phi)^2\right)}{12}+\frac{L^2\ddot{\theta}\left(12M\cos(\phi)\sin(\beta)+6M\cos(\beta)\cos(\phi)\sin(\beta)\right)}{12} \tag{9}$$

3.4

$$-c\left(L\dot{\theta}+\dot{\theta}x_2\cos(\beta)-\dot{\phi}x_1\cos(\phi)+L\dot{\theta}\cos(\beta)-\dot{\theta}x_1\cos(\phi)\sin(\beta)\right)^2-cx_1{}^2\sin(\phi)^2\left(\dot{\theta}+\dot{\phi}\sin(\beta)\right)^2-c\dot{\phi}^2x_1{}^2\cos(\beta)^2\sin(\phi)^2 \tag{10}$$

To find the whole work done by this force distribution we have to integrate it:

x_1 goes from -L to 0 and x_2 goes from -L/2 to L/2.

$$-c\int_{-L}^{0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\underbrace{\left[\left(L\dot{\theta}+\dot{\theta}x_2\cos(\beta)-\dot{\phi}x_1\cos(\phi)+L\dot{\theta}\cos(\beta)-\dot{\theta}x_1\cos(\phi)\sin(\beta)\right)^2\right]}_{\text{A}}dx_2dx_1$$
$$-c\int_{-L}^{0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\underbrace{\left[{x_1}^2\left(\sin(\phi)^2\left(\dot{\theta}+\dot{\phi}\sin(\beta)\right)^2+\dot{\phi}^2\cos(\beta)^2\sin(\phi)^2\right)\right]}_{\text{B}}dx_2dx_1 \tag{11}$$

Starting with the integration of A:

$$-c\int_{-L}^{0}\left[\frac{1}{3\dot{\theta}\cos(\beta)}\left(L\dot{\theta}+\dot{\theta}x_2\cos(\beta)-\dot{\phi}x_1\cos(\phi)+L\dot{\theta}\cos(\beta)-\dot{\theta}x_1\cos(\phi)\sin(\beta)\right)^3\right]_{-\frac{L}{2}}^{\frac{L}{2}}dx_1= \tag{12}$$

$$-c\int_{-L}^{0}\left[\frac{1}{3\dot{\theta}\cos(\beta)}\left(L\dot{\theta}+\dot{\theta}\frac{L}{2}\cos(\beta)-\dot{\phi}x_1\cos(\phi)+L\dot{\theta}\cos(\beta)-\dot{\theta}x_1\cos(\phi)\sin(\beta)\right)^3\right]dx_1$$
$$+c\int_{-L}^{0}\left[\frac{1}{3\dot{\theta}\cos(\beta)}\left(L\dot{\theta}-\dot{\theta}\frac{L}{2}\cos(\beta)-\dot{\phi}x_1\cos(\phi)+L\dot{\theta}\cos(\beta)-\dot{\theta}x_1\cos(\phi)\sin(\beta)\right)^3\right]dx_1 \tag{13}$$

Which can be summarized as

$$\begin{aligned}&-\frac{c}{3\dot{\theta}\cos(\beta)}\int_{-L}^{0}\left(L\dot{\theta}-x_1\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{L\dot{\theta}\cos(\beta)}{2}\right)^3\\&-\left(L\dot{\theta}-x_1\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{3L\dot{\theta}\cos(\beta)}{2}\right)^3dx_1=\end{aligned} \tag{14}$$

$$\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left[\left(L\dot{\theta}-x_1\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{L\dot{\theta}\cos(\beta)}{2}\right)^4\right]_{-L}^{0}$$
$$-\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left[\left(L\dot{\theta}-x_1\cos(\phi)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{3L\dot{\theta}\cos(\beta)}{2}\right)^4\right]_{-L}^{0}= \tag{15}$$

Which finally yields:

$$
\begin{aligned}
&\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left(L\dot{\theta}+\frac{L\dot{\theta}\cos\left(\beta\right)}{2}\right)^{4}\\
&-\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left(L\dot{\theta}+\frac{3L\dot{\theta}\cos\left(\beta\right)}{2}\right)^{4}-\\
&-\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left(L\dot{\theta}+L\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{L\dot{\theta}\cos\left(\beta\right)}{2}\right)^{4}\\
&+\frac{c}{3\dot{\theta}\cos(\beta)}\frac{1}{\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)}\left(L\dot{\theta}+L\cos\left(\phi\right)\left(\dot{\phi}+\dot{\theta}\sin(\beta)\right)+\frac{3L\dot{\theta}\cos\left(\beta\right)}{2}\right)^{4}
\end{aligned}
\tag{16}
$$

4 Assignment 4