Advanced Dynamics - Assignment Report

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Contents

1 Assignment 1

1 WHEEL ROLLING WITHOUT SLIPPING ON A 2D TRACK

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C \; x_2^C \; x_3^C \; \alpha \; \phi]$.
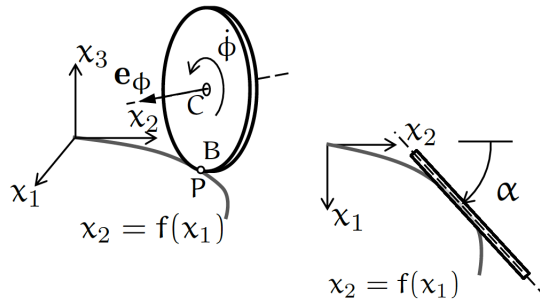

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.

2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1

1.1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C \boldsymbol{e_1} + \dot{x}_2^C \boldsymbol{e_2} - R\dot{\phi}\cos\alpha \boldsymbol{e_2} - R\dot{\phi}\sin\alpha \boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

As α is not dependent on time it can be easily seen that 3 is integrable:

$$\int \dot{x}_1^C dt = \int R\dot{\phi}\sin\alpha \, dt$$
$$\Rightarrow x_1^C = R\phi\sin\alpha \tag{4}$$

1.1.2

See subsubsection 1.1.1

1.1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around e_ϕ while all the other generalized coordinates follow accordingly.

2 TWO BARS LINKAGE

Two bars AB and BC of equal length L are hinged at point B, and move in the plane spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The velocity $\mathbf{v}_C$ of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use x_1^B, x_2^B, θ_1 and θ_2 as generalized coordinates.
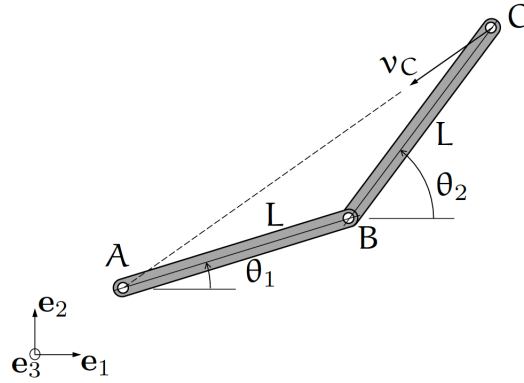

Figure 2.1: A two bar linkage in 2D. The velocity v_C must be directed towards A at all times.

Figure 3: Task 1.1

1.2

I want to show that the constraint that v_c always points in the direction of AD is not integrable. I would like to express this constraint as:

$$v_C * AC_p = 0$$
$$\text{where } AC_p \text{ is a vector perpendicular to } AC \tag{5}$$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} \tag{6}$$

And AC is simply:

$$AC = \begin{pmatrix} L\left(\cos\theta_1 + \cos\theta_2\right) \\ L\left(\sin\theta_1 + \sin\theta_2\right) \end{pmatrix} \tag{7}$$

A simple vector that is perpendicular to AC can be gotten by switching the e_1 and e_2 entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} \tag{8}$$

$$\Rightarrow v_C * AC_p = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} =$$
$$\left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{x}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B -$$
$$L^2\dot{\theta}_2\cos\left(\theta_2\right)\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right) - L^2\dot{\theta}_2\sin\left(\theta_2\right)\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right) = 0 \tag{9}$$

After applying trigonometry:

$$\left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{x}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B -$$
$$L^2\dot{\theta}_2\left(\cos\left(\theta_2\right)\cos\left(\theta_1\right) + \sin\left(\theta_2\right)\sin\left(\theta_1\right) + 1\right) = 0 \tag{10}$$

To simplify the notation I will refer to $\sin\theta_1$ as s_1 and $\sin\theta_2$ as s_2:

$$L\dot{x}_1^B\left(s_1 + s_2\right) - L\dot{x}_2^B\left(c_1 + c_2\right) - L^2\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{11}$$

Dividing by L:

$$\dot{x}_1^B\left(s_1 + s_2\right) - \dot{x}_2^B\left(c_1 + c_2\right) - L\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{12}$$

With the generalized coordinates $q = [x_1^B, x_2^B, \theta_1, \theta_2]$
Writing down the coefficients of the non-holonomic constraint:

$$\begin{aligned} a_1 &= s_1 + s_2 \\ a_2 &= -\left(c_1 + c_2\right) \\ a_3 &= 0 \\ a_4 &= -L\left(1 + s_1s_2 + c_1c_2\right) \\ b &= 0 \end{aligned} \tag{13}$$

7

To check for the exact velocity form we want to proof that there can't exist a $C(q) \neq 0$ for which holds:

$$\frac{\partial(Cb)}{\partial q_i} = \frac{\partial(Ca_1)}{\partial t} \quad \text{and} \quad \frac{\partial(Ca_i)}{\partial q_k} = \frac{\partial(Ca_k)}{\partial q_i} \quad \text{for all the gen. coord.} \quad (14)$$

q1-q3:

$$\frac{\partial(Ca_1)}{\partial \theta_1} = 0 \Rightarrow C * c_1 + \frac{\partial C}{\partial \theta_1}(s1 + s2) = 0$$

$$\Rightarrow \frac{c_1}{s1 + s2} + \frac{1}{C}\frac{dC}{d\theta_1} = 0 \Rightarrow \int \frac{1}{C}dC = -\int \frac{c_1}{s_1 + s_2}d\theta_1$$

$$\Rightarrow lnC = \frac{D}{s_1 + s_2} + D \Rightarrow C = \frac{D}{s_1 + s_2} \quad \text{where the integration const. D was updated}$$

$$\Rightarrow C = \frac{D(x_1^B, x_2^B, \theta_2)}{s1 + s2}$$

$$(15)$$

q1 - q4:

$$\frac{\partial Ca_1}{\partial \theta_2} = \underbrace{\frac{\partial Ca_4}{\partial x_1^B}}_{0} = 0 \Rightarrow \frac{\partial D}{\partial \theta_2} = 0 \Rightarrow C = \frac{D(x_1^B, x_2^B)}{s1 + s2} \quad (16)$$

q1 - q2:

$$\frac{\partial Ca_1}{\partial x_2^B} = \frac{\partial Ca_2}{\partial x_1^B} \Rightarrow \frac{\partial C}{\partial x_2^B}(s_1 + s_2) = \frac{\partial C}{\partial x_1^B}(c_1 + c_2) \Rightarrow \frac{\partial C}{\partial x_2^B} = \frac{\partial C}{\partial x_1^B}\frac{c_1 + c_2}{s_1 + s_2}$$

Remembering that $D = D(x_1^B, x_2^B) \Rightarrow D = const.$

$$\Rightarrow C = \frac{D}{s_1 + s_2} \quad \text{where D const.}$$

$$(17)$$

q2 - q3:

$$\frac{\partial Ca_1}{\partial x_2^B} = 0 \Rightarrow D\frac{\partial \frac{(c_1 + c_2)}{s_1 + s_2}}{\partial \theta_1} = 0$$

$$(18)$$

$$D\frac{\partial \frac{c_1}{s_1 + s_2}}{\partial \theta_1} = -\frac{Ds_1}{s_1 + s_2} - \frac{Dc_1^2}{(s_1 + s_2)^2} = -D\frac{s_1(s_1 + s_2) - c_1^2}{(s_1 + s_2)^2}$$

$$D\frac{\partial \frac{c_2}{s_1 + s_2}}{\partial \theta_1} = -D\frac{c_1 c_2}{(s_1 + s_2)^2} \quad (19)$$

8

Which leads to:

$$-D\frac{c_1^2 + s_1^2 + s_1 s_2 + c_1 c_2}{(s_1 + s_2)^2} \overset{!}{=} 0 \tag{20}$$

Which is a contradiction. Therefore this is a non-holonomic constraint.

2 Assignment 2

2.1 Hamilton's Principle

This assignment asks to get the equations of motion using the Hamilton Principle:

$$\int_{t_1}^{t_2} \delta(T - V + W)dt = 0 \tag{21}$$

Using the theory discussed in the lecture we can split the term into different contributions::

Remembering the result of the Hamilton principle from the lecture:

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right) + f(x, t) + \sum F_i D(x - x_i) = 0, \quad x \in [0; l]$$

Boundary conditions:

$$\left[\frac{\partial L}{\partial v'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right]\delta v = 0, \quad x = 0, l$$

$$\frac{\partial L}{\partial v''}\delta v' = 0, \quad x = 0, l$$

Figure 4: Hamilton Formulas for a continuous system

We consider w our variable instead of v.

$$\frac{\partial L}{\partial w} = 0, \quad \frac{\partial}{\partial x}\frac{\partial L}{\partial w'} = 0, \tag{22}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial w''} = -E\frac{\partial^2}{\partial x^2}I(x), \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 2\delta A(x)\ddot{w}, \tag{23}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}'}\right)\right) = 2\rho I(x)\ddot{w}'' + 2\rho I(x)'\ddot{w}' \tag{24}$$

$$f(x, t) = p(x, t) \tag{25}$$

$$\frac{\partial L}{\partial w'} = 0, \quad \frac{\partial}{\partial x}(\frac{\partial L}{\partial w''}) = -EI(x)' \tag{26}$$

10

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \ddot{w}'}) = 2\rho I(x)\ddot{w}' \tag{27}$$

Therefore the green part comes to:

$$-E\frac{\partial^2}{\partial x^2}I(x) - 2\rho A(x)\ddot{w} + 2\rho I(x)\ddot{w}'' + 2\rho I(x)'\ddot{w}' + p(x,t) = 0 \tag{28}$$

$$-E\frac{\partial^2}{\partial x^2}I(x) - 2\rho A(x)\ddot{w} + 2\rho I(x)\ddot{w}'' + 2\rho I(x)'\ddot{w}' + p(x,t) = 0 \tag{29}$$

While for the red we get:

$$\left[EI(x)' - 2\rho I(x)\ddot{w}'\right]\delta w = 0, \quad x = 0, L \tag{30}$$

And for lastly for the blue:

$$EI(x)\delta w' = 0, \quad x = 0, L \tag{31}$$

2.2 Analysis

3 Assignment 3

3.1

Lagrange equations:

$$\frac{L^2 M \ddot{\phi}}{3} + \frac{L^2 m \ddot{\phi}}{3} - \frac{L^2 m \ddot{\phi} \cos{(\beta)}^2}{3} + \frac{LMg \cos{(\beta)} \sin{(\phi)}}{2} + \frac{L^2 M \Omega^2 \sin{(\beta)} \sin{(\phi)}}{2} +$$
$$\frac{L^2 M \Omega^2 \cos{(\beta)} \sin{(\beta)} \sin{(\phi)}}{4} - \frac{L^2 M \Omega^2 \cos{(\beta)}^2 \cos{(\phi)} \sin{(\phi)}}{6} = 0$$
$$\tag{32}$$

3.2

This can be reformulated for the differential equation of $\ddot{\phi}$:

$$\ddot{\phi} \left(\frac{L^2 (M+m)}{3} - \frac{L^2 m \cos(\beta)^2}{3} \right) + \sin(\phi) \left(\frac{LMg \cos(\beta)}{2} + \frac{L^2 M \Omega^2 \sin(\beta)}{2} + \frac{L^2 M \Omega^2 \cos(\beta) \sin(\beta)}{4} \right) -$$
$$\frac{L^2 M \Omega^2 \cos(\beta)^2 \cos(\phi) \sin(\phi)}{6} = 0$$
$$\tag{33}$$

3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\frac{L^2 \ddot{\theta} \left(16M + 16m + 12M \cos{(\beta)} + 12m \cos{(\beta)} + 2M \cos{(\beta)}^2 + 4m \cos{(\beta)}^2 - 2M \cos{(\beta)}^2 \cos{(\phi)}^2 \right)}{12} +$$
$$\frac{L^2 \ddot{\theta} \left(12M \cos{(\phi)} \sin{(\beta)} + 6M \cos{(\beta)} \cos{(\phi)} \sin{(\beta)} \right)}{12}$$
$$\tag{34}$$

3.4

After matlab integration of the work per area we get for the work

$$W =$$
$$\frac{L^4 \left(c \left(\dot{\theta} + \dot{\phi} \sin{(\beta)} \right)^2 \left(\cos{(\phi)}^2 - 1 \right) - c \left(\dot{\phi} \cos{(\phi)} + \dot{\theta} \cos{(\phi)} \sin{(\beta)} \right)^2 + c \dot{\phi}^2 \cos{(\beta)}^2 \left(\cos{(\phi)}^2 - 1 \right) \right)}{3}$$
$$- L \left(Lc \left(L\dot{\theta} + L\dot{\theta} \cos{(\beta)} \right)^2 + \frac{L^3 c \dot{\theta}^2 \cos{(\beta)}^2}{12} \right) - L^3 c \left(L\dot{\theta} + L\dot{\theta} \cos{(\beta)} \right) \left(\dot{\phi} \cos{(\phi)} + \dot{\theta} \cos{(\phi)} \sin{(\beta)} \right)$$
$$\tag{35}$$

Which results in the equations of motion:

$$\frac{L^2 M \ddot{\phi}}{3} + \frac{2L^4 c \left(\dot{\phi} + \dot{\theta}\sin(\beta)\right)}{3} + \frac{L^2 m \ddot{\phi}}{3} - \frac{L^2 m \ddot{\phi}\cos(\beta)^2}{3} + \frac{L M g \cos(\beta)\sin(\phi)}{2} +$$

$$L^4 c \dot{\theta}\cos(\phi)\left(\cos(\beta)+1\right) + \frac{L^2 M \dot{\theta}^2 \sin(\beta)\sin(\phi)}{2} - \frac{L^2 M \dot{\theta}^2 \cos(\beta)^2 \cos(\phi)\sin(\phi)}{6} +$$

$$\frac{L^2 M \dot{\theta}^2 \cos(\beta)\sin(\beta)\sin(\phi)}{4} \qquad \text{for } \phi$$

$$(36)$$

And

$$\frac{cL^4}{6}\left(16\dot{\theta} + 24\dot{\theta}\cos(\beta) + 6\dot{\phi}\cos(\phi) + 4\dot{\phi}\sin(\beta) + 13\dot{\theta}\cos(\beta)^2 - 4\dot{\theta}\cos(\beta)^2\cos(\phi)^2 + 6\dot{\phi}\cos(\beta)\cos(\phi) + 12\dot{\theta}\cos\right.$$

$$\left. 12\dot{\theta}\cos(\beta)\cos(\phi)\sin(\beta)\right) +$$

$$\frac{\ddot{\theta}}{12}\left(16M + 16m + 12M\cos(\beta) + 12m\cos(\beta) + 2M\cos(\beta)^2 + 4m\cos(\beta)^2 - 2M\cos(\beta)^2\cos(\phi)^2 +\right.$$

$$\left. 12M\cos(\phi)\sin(\beta) + 6M\cos(\beta)\cos(\phi)\sin(\beta)\right)L^2 \qquad \text{for } \theta$$

$$(37)$$

3.5 Plugging in the values:

Let $L = 0.25[m], \beta = \frac{\pi}{6}, M = 0.5[kg], m = 0.2[kg], g = 9.81[\frac{m}{s^2}]$ and $c = 0.1[\frac{Ns}{m^3}]$.

The equations become:

$$\frac{\dot{\phi}}{3840} + \frac{11\ddot{\phi}}{960} + \frac{\dot{\theta}}{7680} + \frac{\dot{\theta}^2 \sin(\phi)}{128} + \frac{981\sqrt{3}\sin(\phi)}{3200} + \frac{\sqrt{3}\dot{\theta}^2 \sin(\phi)}{512} + \frac{\dot{\theta}\cos(\phi)\left(\frac{\sqrt{3}}{2}+1\right)}{2560}$$

$$- \frac{\dot{\theta}^2 \cos(\phi)\sin(\phi)}{256} \qquad \text{for } \phi$$

$$(38)$$

And

$$\frac{\dot{\phi}}{7680} + \frac{103\dot{\theta}}{61440} - \frac{\dot{\theta}\cos(\phi)^2}{5120} + \frac{\sqrt{3}\dot{\theta}}{1280} +$$

$$\frac{\ddot{\theta}}{192}\left(3\cos(\phi) - \frac{3\cos(\phi)^2}{4} + \frac{21\sqrt{3}}{5} + \frac{3\sqrt{3}\cos(\phi)}{4} + \frac{251}{20}\right) +$$

$$\frac{\dot{\phi}\cos(\phi)}{2560} + \frac{\dot{\theta}\cos(\phi)}{2560} + \frac{\sqrt{3}\dot{\phi}\cos(\phi)}{5120} + \frac{\sqrt{3}\dot{\theta}\cos(\phi)}{5120} \qquad \text{for } \theta$$

$$(39)$$

13

3.6 Equilibrium State

Reminder- The potential energy looks as follows:

$$\frac{LMg\left(2\sin\left(\beta\right)-\cos\left(\beta\right)\cos\left(\phi\right)\right)}{2} \tag{40}$$

Case 1: Ω fixed:

The derivative of the potential energy w.r.t. θ is always 0.

$$\frac{\partial V}{\partial \theta} = 0 \tag{41}$$

The derivative of the potential energy w.r.t. ϕ not however:

$$\frac{\partial V}{\partial \phi} = \frac{LMg\cos\left(\beta\right)\sin\left(\phi\right)}{2} \tag{42}$$

Equation 12 is only 0 for $\phi = k * \pi$ which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

Case 2: $\dot{\theta}$ can change:

As V is independent of θ altogether the result is the same for this case.

The equation of motions are after plugging in equilibrium state:

$$\phi : \\ 0 = 0 \tag{43}$$

θ :
$$\frac{L^2\ddot{\theta}}{12}\Big(16M + 16m + 12M\cos\left(\beta\right) + 12M\sin\left(\beta\right) + 12m\cos\left(\beta\right) + \\ 4m\cos\left(\beta\right)^2 + 6M\cos\left(\beta\right)\sin\left(\beta\right)\Big) = 0 \tag{44}$$

As the coefficient of $\ddot{\theta}$ is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant Ω.

4   Assignment 4