Advanced Dynamics - Assignment Report

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Contents

1 Assignment 1

1.1 Wheel Rolling Without Slipping

1 WHEEL ROLLING WITHOUT SLIPPING ON A 2D TRACK

A thin wheel of radius R rolls without slipping on a track on the $x_1 - x_2$ plane, defined by $x_2 = f(x_1)$. The wheel plane stays vertical and tangent to such track at the contact point P. Denote with α the angle the disk plane forms with the x_2 axis, and with ϕ the rotation of the disk about its axis $\mathbf{e}_\phi$. The position of the center of the disk C is indicated by x_1^C, x_2^C and x_3^C. Assume a set of generalized coordinate $\mathbf{q} = [x_1^C\ x_2^C\ x_3^C\ \alpha\ \phi]$.

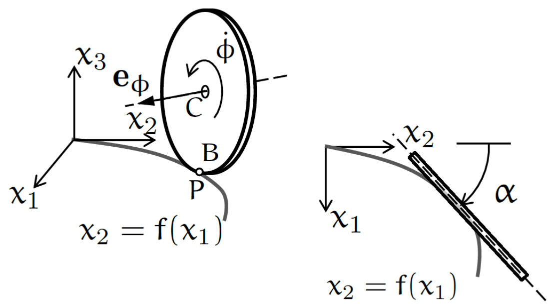

Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

1.1.1 Constraints

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to x_3)
   This is a holonomic constraint: $f = \theta = 0$ where θ denotes the angle between the disk and x_3

2. x_2 follows a fixed trajectory, given x_1 (and vice versa):
   $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$ holonomic.

3. α is the angle between the trajectory and the x_2 axis:
   $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$ or written differently:
   $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$ holonomic

4. Rolling without slipping:
   $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$ with $\omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$
   Seems to be non-holonomic at first glance

5. The disk does not leave the ground:
   $x_3^C - R = 0$ aka the x_3 component of the center of mass is R.
   This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \tag{1}$$

Plugging in $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\boldsymbol{e_3} + \dot{\phi}\boldsymbol{e_\phi}$ and $R_{CB} = [0, 0, -R]^T$:

$$\dot{x}_1^C \boldsymbol{e_1} + \dot{x}_2^C \boldsymbol{e_2} - R\dot{\phi}\cos\alpha \boldsymbol{e_2} - R\dot{\phi}\sin\alpha \boldsymbol{e_1} = 0 \tag{2}$$

Considering the part in $\boldsymbol{e_1}$ direction:

$$\dot{x}_1^C - R\dot{\phi}\sin\alpha = 0 \tag{3}$$

As α is not dependent on time it can be easily seen that (3) is integrable:

$$\begin{aligned} &\int \dot{x}_1^C dt = \int R\dot{\phi}\sin\alpha dt \\ &\Rightarrow x_1^C = R\phi\sin\alpha \end{aligned} \tag{4}$$

1.1.2 Constraint Explanation

See subsubsection (1.1.1)

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

1.1.3 Degrees of freedom

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of $6-5=1$ degree of freedom. That could for instance be the rotation of the wheel around $\boldsymbol{e}_\phi$ while all the other generalized coordinates follow accordingly.

1.2 Two Bars Linkage

2 Two bars linkage

Two bars AB and BC of equal length L are hinged at point B, and move in the plane spanned by the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The velocity $\mathbf{v}_C$ of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use x_1^B, x_2^B, θ_1 and θ_2 as generalized coordinates.

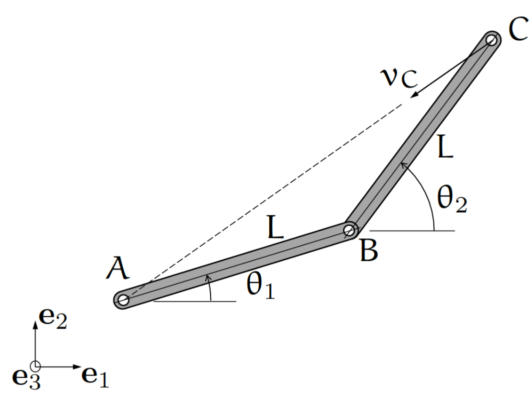

Figure 2.1: A two bar linkage in 2D. The velocity v_C must be directed towards A at all times.

Figure 3: Task 1.1

I want to show that the constraint that v_c always points in the direction of AD

is not integrable. I would like to express this constraint as:

$$\begin{aligned} &v_C * AC_p = 0 \\ &\text{where } AC_p \text{ is a vector perpendicular to } AC \end{aligned} \tag{5}$$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} \tag{6}$$

And AC is simply:

$$AC = \begin{pmatrix} L\left(\cos\theta_1 + \cos\theta_2\right) \\ L\left(\sin\theta_1 + \sin\theta_2\right) \end{pmatrix} \tag{7}$$

A simple vector that is perpendicular to AC can be gotten by switching the e_1 and e_2 entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} \tag{8}$$

$$\begin{aligned} \Rightarrow & v_C * AC_p = \begin{pmatrix} \dot{x}_1^B - L\sin(\theta_2)\dot{\theta}_2 \\ \dot{x}_2^B + L\cos(\theta_2)\dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L\left(\sin\theta_1 + \sin\theta_2\right) \\ -L\left(\cos\theta_1 + \cos\theta_2\right) \end{pmatrix} = \\ & \left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{\mathrm{x}}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B - \\ & L^2\dot{\theta}_2\cos\left(\theta_2\right)\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right) - L^2\dot{\theta}_2\sin\left(\theta_2\right)\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right) = 0 \end{aligned} \tag{9}$$

After applying trigonometry:

$$\begin{aligned} & \left(L\left(\sin\left(\theta_1\right) + \sin\left(\theta_2\right)\right)\right)\dot{\mathrm{x}}_1^B + \left(-L\left(\cos\left(\theta_1\right) + \cos\left(\theta_2\right)\right)\right)\dot{x}_2^B - \\ & L^2\dot{\theta}_2\left(\cos\left(\theta_2\right)\cos\left(\theta_1\right) + \sin\left(\theta_2\right)\sin\left(\theta_1\right) + 1\right) = 0 \end{aligned} \tag{10}$$

To simplify the notation I will refer to $\sin\theta_1$ as s_1 and $\sin\theta_2$ as s_2:

$$L\dot{x}_1^B\left(s_1 + s_2\right) - L\dot{x}_2^B\left(c_1 + c_2\right) - L^2\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{11}$$

Dividing by L:

$$\dot{x}_1^B\left(s_1 + s_2\right) - \dot{x}_2^B\left(c_1 + c_2\right) - L\dot{\theta}_2^2\left(c_2c_1 + s_2s_1 + 1\right) = 0 \tag{12}$$

With the generalized coordinates $q = [x_1^B, x_2^B, \theta_1, \theta_2]$
Writing down the coefficients of the non-holonomic constraint:

$$
\begin{aligned}
a_1 &= s_1 + s_2 \\
a_2 &= -\left(c_1 + c_2\right) \\
a_3 &= 0 \\
a_4 &= -L\left(1 + s_1 s_2 + c_1 c_2\right) \\
b &= 0
\end{aligned}
\tag{13}
$$

To check for the exact velocity form we want to proof that there can't exist a $C(q) \neq 0$ for which holds:

$$
\frac{\partial(Cb)}{\partial q_i} = \frac{\partial(Ca_1)}{\partial t} \quad \text{and} \quad \frac{\partial(Ca_i)}{\partial q_k} = \frac{\partial(Ca_k)}{\partial q_i} \quad \text{for all the gen. coord.} \tag{14}
$$

q1-q3:

$$
\begin{aligned}
&\frac{\partial(Ca_1)}{\partial\theta_1} = 0 \Rightarrow C * c_1 + \frac{\partial C}{\partial\theta_1}(s1 + s2) = 0 \\
&\Rightarrow \frac{c_1}{s1 + s2} + \frac{1}{C}\frac{dC}{d\theta_1} = 0 \Rightarrow \int \frac{1}{C} dC = -\int \frac{c_1}{s_1 + s_2} d\theta_1 \\
&\Rightarrow lnC = \frac{D}{s_1 + s_2} + D \Rightarrow C = \frac{D}{s_1 + s_2} \qquad \text{where the integration const. D was updated} \\
&\Rightarrow C = \frac{D(x_1^B, x_2^B, \theta_2)}{s1 + s2}
\end{aligned}
\tag{15}
$$

q1 - q4:

$$
\frac{\partial Ca_1}{\partial\theta_2} = \underbrace{\frac{\partial Ca_4}{\partial x_1^B}}_{0} = 0 \Rightarrow \frac{\partial D}{\partial\theta_2} = 0 \Rightarrow C = \frac{D(x_1^B, x_2^B)}{s1 + s2} \tag{16}
$$

q1 - q2:

$$
\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = \frac{\partial Ca_2}{\partial x_1^B} \Rightarrow \frac{\partial C}{\partial x_2^B}\left(s_1 + s_2\right) = \frac{\partial C}{\partial x_1^B}\left(c_1 + c_2\right) \Rightarrow \frac{\partial C}{\partial x_2^B} = \frac{\partial C}{\partial x_1^B}\frac{c_1 + c_2}{s_1 + s_2} \\
&\text{Remembering that } D = D(x_1^B, x_2^B) \Rightarrow D = const. \\
&\Rightarrow C = \frac{D}{s_1 + s_2} \text{ where D const.}
\end{aligned}
\tag{17}
$$

q2 - q3:

$$
\begin{aligned}
&\frac{\partial Ca_1}{\partial x_2^B} = 0 \Rightarrow D\frac{\partial \frac{(c_1+c_2)}{s_1+s_2}}{\partial \theta_1} = 0 \\
&D\frac{\partial \frac{c_1}{s_1+s_2}}{\partial \theta_1} = -\frac{Ds_1}{s_1+s_2} - \frac{Dc_1^2}{(s_1+s_2)^2} = -D\frac{s_1(s_1+s_2)-c_1^2}{(s_1+s_2)^2}
\end{aligned}
\tag{18}
$$

$$
D\frac{\partial \frac{c_2}{s_1+s_2}}{\partial \theta_1} = -D\frac{c_1c_2}{(s_1+s_2)^2} \tag{19}
$$

Which leads to:

$$
-D\frac{c_1^2+s_1^2+s_1s_2+c_1c_2}{(s_1+s_2)^2} \stackrel{!}{=} 0 \tag{20}
$$

Which is a contradiction. Therefore this is a non-holonomic constraint.

2 Assignment 2

2.1 Hamilton's Principle

This assignment asks to get the equations of motion using the Hamilton Principle:

$$\int_{t_1}^{t_2} delta(T - V + W)dt = 0 \tag{21}$$

Using the theory discussed in the lecture we can split the term into different contributions::
Remembering the result of the Hamilton principle from the lecture:

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right) + f(x,t) + \sum F_i D(x - x_i) = 0, \quad x \in [0; l]$$

Boundary conditions:

$$\left[\frac{\partial L}{\partial v'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}'}\right)\right]\delta v = 0, \quad x = 0, l$$

$$\frac{\partial L}{\partial v''}\delta v' = 0, \quad x = 0, l$$

Figure 4: Hamilton Formulas for a continuous system

As the axial displacement u depends on the first derviative of w w.r.t. x we consider w our variable.

$$\frac{\partial L}{\partial w} = 0 \tag{22}$$

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial w'} = 0 \tag{23}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial w''} = -E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) \tag{24}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = A(x)\rho\ddot{w} \tag{25}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}'}\right)\right) = \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) \tag{26}$$

$$f(x,t) = p(x,t) \tag{27}$$

$$\frac{\partial L}{\partial w'} = 0 \tag{28}$$

$$\frac{\partial}{\partial x}(\frac{\partial L}{\partial w''}) = -E\left(I(x)w^{(3)} + I'(x)w''\right) \tag{29}$$

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{w}'}) = I(x)\rho\ddot{w}' \tag{30}$$

Where a (i) paranthesis in the exponent denotes the ith derivative of that variable w.r.t. x.

2.1.1 Differential Equation aka green part

Therefore the green part comes to:

$$\begin{aligned} &p(x,t) - E\left(I(x)w^{(4)} + 2I'(x)w^{(3)} + I'(x)'w''\right) + \rho\left(I(x)\ddot{w}'' + I'(x)\ddot{w}'\right) \\ &- A(x)\rho\ddot{w} = 0 \end{aligned} \tag{31}$$

Comparing this solution to the Euler-Bernoulli Equation:

$$p(x,t) = EIw^{(4)} + A(x)\rho\ddot{w} \tag{32}$$

What we can see is that the additional terms from the green expression come from the derivative of I. This comes from the changing cross section and the thus changing moment of Inertia I. Secondly we had a term with $\dot{w}'$ in the kinetic energy arising through the change in thickness which is considered here. (not the case for the euler bernoulli beam).

2.1.2 Boundary conditions with virtual displacement aka Red Part

$$\left[\delta\mathrm{w(x)}\left(E\left(I(x)w^{(3)} + I'(x)w''\right) - I(x)\rho\ddot{w}'\right)\right]_0^L = 0 \tag{33}$$

2.1.3 Blue part

$$\left[-\delta\mathrm{w}'\mathrm{(x)}EI(x)w''\right]_0^L = 0 \tag{34}$$

2.2 Analysis

$$I(0) = \frac{1}{12}h_0^3 \text{ and } I(L) = \frac{1}{12}h_L^3 \tag{35}$$

as well as

$$I'(0) = \frac{1}{4}h(0)^2 * \frac{h_L - h_0}{L} = \frac{1}{4}h_0^2 * \frac{h_L - h_0}{L} \tag{36}$$

and

$$I'(L) = \frac{1}{4}h_L^2 * \frac{h_L - h_0}{L} \tag{37}$$

Beginning with the blue part and pluggin in the boundary values:

$$-\delta \mathrm{w}'(\mathrm{L}) E \mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0) E \mathrm{h}_0^3 w''(0) = 0 \tag{38}$$

As we have a clamped end (w, δw and their first three derivatives are 0) and a supported end (w, δw and their first derivative is 0) we arrive at:

$$0 = 0 \tag{39}$$

For the red part:

$$\begin{aligned} &\delta w(L)\left(E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)\right) - \\ &\delta w(0)\left(E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)\right) = 0 \end{aligned} \tag{40}$$

Note that both $\delta w(0) = \delta w(L) = 0$. The coefficients

$$E\left(I(L)w^{(3)}(L) + I'(L)w''(L)\right) - I(L)\rho\ddot{w}'(L)$$

and

$$E\left(I(0)w^{(3)}(0) + I'(0)w''(0)\right) - I(0)\rho\ddot{w}'(0)$$

represent the forces at the boundaries. To determine these forces we would have to solve the differential equation to get a solution for w which we can use to get the forces.

2.3 Analysis with open second end

Nothing changes in the application of the Hamilton Principle. However when analysing the boundaries we see some changes:
Starting again with the blue:

$$-\delta \mathrm{w}'(\mathrm{L}) E \mathrm{h}_L^3 w''(L) + \delta \mathrm{w}'(0) E \mathrm{h}_0^3 w''(0) = 0 \tag{41}$$

As before the w terms at $x = 0$ vanish. However now $\delta w'(L) \neq 0$ thus we get

$$E \mathrm{h}_L^3 w''(L) = 0 \tag{42}$$

So the slope at the vertical displacement of the free end has to stay constant.
For the red part we get:

$$\begin{aligned} &\delta w(L) \Big(E \Big(I(L) w^{(3)}(L) + I'(L) w''(L) \Big) - I(L) \rho \ddot{w}'(L) \Big) - \\ &\delta w(0) \Big(E \Big(I(0) w^{(3)}(0) + I'(0) w''(0) \Big) - I(0) \rho \ddot{w}'(0) \Big) = 0 \end{aligned} \tag{43}$$

As before $\delta w(0) = 0$ and the force of the boundary constraint is:

$$E \Big(I(0) w^{(3)}(0) + I'(0) w''(0) \Big) - I(0) \rho \ddot{w}'(0) \tag{44}$$

However for the second end we have $\delta w(L) \neq 0$ which leads to:

$$E \Big(I(L) w^{(3)}(L) + I'(L) w''(L) \Big) - I(L) \rho \ddot{w}'(L) = 0 \tag{45}$$

Pluggin in the values for I and I':

$$E \left(h_L w^{(3)}(L) + 3 * \frac{h_L - h_0}{L} w''(L) \right) - h_L \rho \ddot{w}'(L) = 0 \tag{46}$$

3 Assignment 3

A few notes before starting this assignment:
I'm considering four frames.
The first one is the inertial frame. It is in A with the first axis straight up, the second right and the third into the plane of the image.

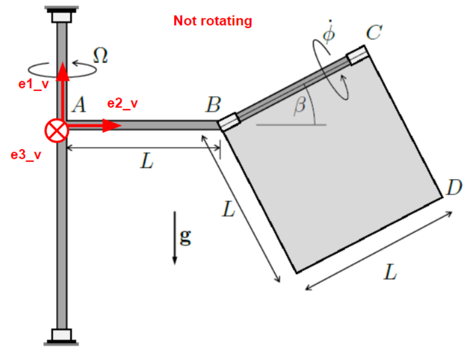

Figure 5: Inertial Frame

The second one is located at the same position but rotating with Omega. I will call it the vertical frame.

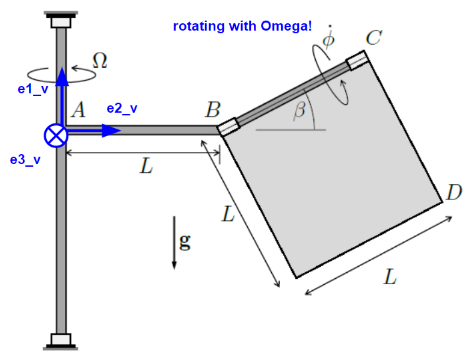

Figure 6: Vertical Frame

The third one, that is called rod frame is located in the point B and relative to the vertical frame it is tilted by β around the positive e_3^v axis.

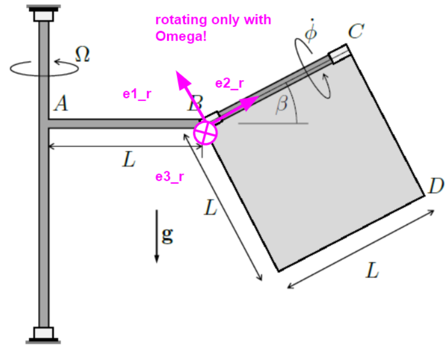

Figure 7: Rod Frame

The last frame is the square frame which relative to the rod frame is also rotation with ϕ:

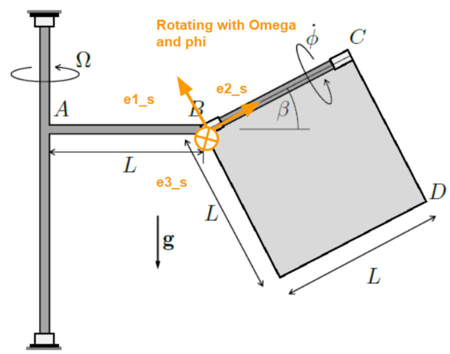

Figure 8: Square Frame

3.1 Kinetic and Potential Energy

3.1.1 Vertical Shaft

As the vertical bar rotates around the thin axis and is static regarding the height of it's CoM the contribution to the energy is 0.

$$T_{\text{vertical bar}} = V_{\text{vertical bar}} = 0 \tag{47}$$

3.1.2 Horizontal Bar

Note: I will refer to the horizontal bar as bar and to the tilted bar as rod.
As point A which is part of the bar is not moving we can simply consider the rotational part w.r.t. to this fixed point A.:

$$T_{\text{bar}} = \frac{L^2\Omega^2 m}{6} \tag{48}$$

As the bar is horizontal on the height of A we have no contribution of the potential energy.

$$V_{\text{bar}} = 0 \tag{49}$$

3.1.3 Rod

As the rod has no static point we will use a superposition of the rotational and translational kinetic energy.

$$T_{\text{Rod transl}} = \frac{L^2\Omega^2 m(\cos(\beta) + 2)^2}{8} \tag{50}$$

For the rotational part considering the moment of inertia of the center of mass we get:

$$T_{\text{Rod rot}} = \frac{L^2\Omega^2 m \cos(\beta)^2}{24} \tag{51}$$

Which leads to:

$$\Rightarrow T_{\text{Rod}} = \frac{L^2\Omega^2 m \left(\cos(\beta)^2 + 3\cos(\beta) + 3\right)}{6} \tag{52}$$

For the potential energy we have the verical part of the incline of the rod:

$$V_{\text{Rod}} = \frac{L}{2} mg \sin\beta \tag{53}$$

3.1.4 Square Plate

As for the rod, first the translational kinetic energy:

$$
\begin{aligned}
T_{\text{square transl}} =& \frac{M}{2}\left(\left(\left(\Omega+\dot{\phi}\sin(\beta)\right)\left(\frac{L\left(\sin(\Omega t)\sin(\phi)-\cos(\Omega t)\cos(\phi)\sin(\beta)\right)}{2}-\frac{L\cos(\Omega t)\cos(\beta)}{2}\right)\right.\right. \\
& \left. -L\Omega\cos(\Omega t)+\frac{L\dot{\phi}\cos(\Omega t)\cos(\beta)\left(\sin(\beta)-\cos(\beta)\cos(\phi)\right)}{2}\right)^2 \\
& +\left(\left(\Omega+\dot{\phi}\sin(\beta)\right)\left(\frac{L\left(\cos(\Omega t)\sin(\phi)+\sin(\Omega t)\cos(\phi)\sin(\beta)\right)}{2}+\frac{L\sin(\Omega t)\cos(\beta)}{2}\right)\right. \\
& \left.\left. +L\Omega\sin(\Omega t)-\frac{L\dot{\phi}\sin(\Omega t)\cos(\beta)\left(\sin(\beta)-\cos(\beta)\cos(\phi)\right)}{2}\right)^2+\frac{L^2\dot{\phi}^2\cos(\beta)^2\sin(\phi)^2}{4}\right)
\end{aligned}
\tag{54}
$$

For the rotational part we get, considering the moment of inertia w.r.t. the c.o.m. in the s frame:

$$
\begin{aligned}
T_{\text{square rot}} =& \frac{L^2M\left(\dot{\phi}+\Omega\sin(\beta)\right)\left(\frac{\dot{\phi}}{2}+\frac{\Omega\sin(\beta)}{2}\right)}{12} \\
& +\frac{L^2M\Omega^2\cos(\beta)^2\cos(\phi)^2}{24}
\end{aligned}
\tag{55}
$$

Which leads to:

$$
\begin{aligned}
T_{\text{square}} =& \frac{L^2M}{12}\left(-\Omega^2\cos(\beta)^2\cos(\phi)^2+\Omega^2\cos(\beta)^2\right. \\
& +3\sin(\beta)\,\Omega^2\cos(\beta)\cos(\phi)+6\Omega^2\cos(\beta) \\
& +6\sin(\beta)\,\Omega^2\cos(\phi)+8\Omega^2+3\Omega\dot{\phi}\cos(\beta)\cos(\phi) \\
& \left.+6\Omega\dot{\phi}\cos(\phi)+4\sin(\beta)\,\Omega\dot{\phi}+2\dot{\phi}^2\right)
\end{aligned}
\tag{56}
$$

For the potential energy we get:

$$
\frac{L}{2}Mg\left(\sin(\beta)-\cos(\beta)\cos(\phi)\right) \tag{57}
$$

Which can be seen as the contribution of the axis-parallel parts of the vector to the center of mass of the square in the s frame. The sin part points up (to c.o.m of the rod) and the cos part points down.

3.1.5 Total Energy

The total kinetic energy comes to:

$$
\begin{aligned}
&T = \underbrace{T_{\text{vertical bar}}}_{0} + T_{\text{bar}} + T_{\text{Rod}} + T_{\text{Square}} = \\
&\frac{L^2}{12}\Big(8M\Omega^2 + 8\Omega^2 m + 2M\dot{\phi}^2 + 6M\Omega^2\cos(\beta) + 6\Omega^2 m\cos(\beta) \\
&+ M\Omega^2\cos(\beta)^2 + 2\Omega^2 m\cos(\beta)^2 + 6M\Omega^2\cos(\phi)\sin(\beta) \\
&+ 6M\Omega\dot{\phi}\cos(\phi) + 4M\Omega\dot{\phi}\sin(\beta) - M\Omega^2\cos(\beta)^2\cos(\phi)^2 \\
&+ 3M\Omega\dot{\phi}\cos(\beta)\cos(\phi) + 3M\Omega^2\cos(\beta)\cos(\phi)\sin(\beta)\Big)
\end{aligned}
\tag{58}
$$

Analogous the potential energy:

$$
\begin{aligned}
&V = \underbrace{V_{\text{vertical bar}}}_{0} + V_{\text{bar}} + V_{\text{Rod}} + V_{\text{Square}} = \\
&\frac{L}{2}Mg\left(2\sin(\beta) - \cos(\beta)\cos(\phi)\right)
\end{aligned}
\tag{59}
$$

Where one can see nicely the double contribution of the c.o.m of the rod for the rod and the square and the negative part of the square.

3.2 Lagrange Equations

Having an expression for the kinetic and potential energy we can use them to denote our lagrange equations:

$$
\frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0 \tag{60}
$$

Which yields:

$$
\begin{aligned}
&\frac{LM}{12}\Big(-2L\cos(\phi)\sin(\phi)\,\Omega^2\cos(\beta)^2 + 3L\sin(\beta)\sin(\phi)\,\Omega^2\cos(\beta) \\
&+ 6L\sin(\beta)\sin(\phi)\,\Omega^2 + 6g\sin(\phi)\cos(\beta) + 4L\ddot{\phi}\Big) = 0
\end{aligned}
\tag{61}
$$

Which is a differential equation for ϕ.

3.3 Non constant angular velocity around vertical Shaft

3.3.1 Vertical shaft

Unchanged

3.3.2 Bar

$$T_{\text{bar}} = \frac{L^2\Omega^2 m}{6} \tag{62}$$

$$V_{\text{bar}} = 0 \tag{63}$$

3.3.3 Rod

$$T_{\text{rod}} = L^2 m\dot{\theta}^2\left(\frac{\cos(\beta)^2}{2} + \cos(\beta) + 1\right) \tag{64}$$

$$V_{\text{rod}} = \frac{L}{2} gm\sin(\beta) \tag{65}$$

3.3.4 Square

$$\begin{aligned} T_{\text{square}} =& \frac{L^2 M}{12}\left(2\dot{\phi}^2 + 3\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi) + 6\dot{\phi}\dot{\theta}\cos(\phi) + 4\sin(\beta)\,\dot{\phi}\dot{\theta}\right.\\ &- \dot{\theta}^2\cos(\beta)^2\cos(\phi)^2 + \dot{\theta}^2\cos(\beta)^2 + 3\sin(\beta)\,\dot{\theta}^2\cos(\beta)\cos(\phi)\\ &\left.+ 6\dot{\theta}^2\cos(\beta) + 6\sin(\beta)\,\dot{\theta}^2\cos(\phi) + 8\dot{\theta}^2\right) \end{aligned} \tag{66}$$

$$V_{\text{square}} = \frac{L}{2} Mg\left(\sin(\beta) - \cos(\beta)\cos(\phi)\right) \tag{67}$$

3.3.5 Total Energy

$$\begin{aligned} T =& \frac{L^2}{12}\left(2M\dot{\phi}^2 + 8M\dot{\theta}^2 + 8m\dot{\theta}^2 + 6M\dot{\theta}^2\cos(\beta) + 6m\dot{\theta}^2\cos(\beta)\right.\\ &+ M\dot{\theta}^2\cos(\beta)^2 + 2m\dot{\theta}^2\cos(\beta)^2 + 6M\dot{\theta}^2\cos(\phi)\sin(\beta)\\ &+ 6M\dot{\phi}\dot{\theta}\cos(\phi) + 4M\dot{\phi}\dot{\theta}\sin(\beta) - M\dot{\theta}^2\cos(\beta)^2\cos(\phi)^2\\ &\left.+ 3M\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi) + 3M\dot{\theta}^2\cos(\beta)\cos(\phi)\sin(\beta)\right) \end{aligned} \tag{68}$$

$$V = \frac{L}{2} gm\sin(\beta) + \frac{L}{2} Mg\left(\sin(\beta) - \cos(\beta)\cos(\phi)\right) \tag{69}$$

3.3.6 Lagrange Equations

As we have now two generalized coordinates the lagrange equations look like this:

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial \mathbf{q}} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0 \tag{70}$$

Where $\mathbf{q} = \begin{pmatrix} \phi \\ \theta \end{pmatrix}$

This leads to two equations. For visualization purposes I will write down the entries one after the other:

$$\frac{L^2\, M\, \ddot{\phi}}{3} + \frac{L^2\, M\, \ddot{\theta}\, \left(6\, \cos\left(\phi\right) + 4\, \sin\left(\beta\right) + 3\, \cos\left(\beta\right)\, \cos\left(\phi\right)\right)}{12} + \frac{L\, M\, g\, \cos\left(\beta\right)\, \sin\left(\phi\right)}{2}$$
$$+ \frac{L^2\, M\, \dot{\theta}\, \sin\left(\phi\right)\, \left(6\, \dot{\phi} + 3\, \dot{\phi}\, \cos\left(\beta\right) + 6\, \dot{\theta}\, \sin\left(\beta\right) + 3\, \dot{\theta}\, \cos\left(\beta\right)\, \sin\left(\beta\right) - 2\, \dot{\theta} \cos\left(\beta\right)^2\, \cos\left(\phi\right)\right)}{12}$$
$$- \frac{L^2\, M\, \dot{\phi}\, \dot{\theta}\, \sin\left(\phi\right)\, \left(\cos\left(\beta\right) + 2\right)}{4} \tag{71}$$

And

$$\frac{L^2\, \ddot{\theta}}{12} \left(16\, M + 16\, m + 12\, M\, \cos\left(\beta\right) + 12\, m\, \cos\left(\beta\right) + 2\, M \cos\left(\beta\right)^2\right.$$
$$\left. + 4\, m \cos\left(\beta\right)^2 - 2\, M \cos\left(\beta\right)^2 \cos\left(\phi\right)^2 + 12\, M\, \cos\left(\phi\right)\, \sin\left(\beta\right) + 6\, M\, \cos\left(\beta\right)\, \cos\left(\phi\right)\, \sin\left(\beta\right)\right)$$
$$+ \frac{L^2\, M\, \ddot{\phi}\, \left(6\, \cos\left(\phi\right) + 4\, \sin\left(\beta\right) + 3\, \cos\left(\beta\right)\, \cos\left(\phi\right)\right)}{12}$$
$$- \frac{L^2\, M\, \dot{\phi}\, \sin\left(\phi\right)\, \left(6\, \dot{\phi} + 3\, \dot{\phi}\, \cos\left(\beta\right) + 12\, \dot{\theta}\, \sin\left(\beta\right) + 6\, \dot{\theta}\, \cos\left(\beta\right)\, \sin\left(\beta\right) - 4\, \dot{\theta} \cos\left(\beta\right)^2\, \cos\left(\phi\right)\right)}{12} \tag{72}$$

3.4 Non Conservative Forces

Now we have a non conservative force attacking on the square.
We have to derive the velocity of a random point on the square. In the following I will parametrize said point with x_1 and x_2 which are the coordinates of the point on the square in the s frame from B.

$$v_{Pi} = v_{Bi} + \omega_i \times BP_i \tag{73}$$

Therefore the work of the force contribution at a single point P is:

$$
\begin{aligned}
dW =& - cvP_i \bullet vP_i = \\
&= -c \Bigg(\Big(x_1 \left(\sin\left(\phi\right)\sin\left(\theta\right) - \cos\left(\phi\right)\sin\left(\beta\right)\cos\left(\theta\right)\right) \\
&+ x_2 \cos\left(\beta\right)\cos\left(\theta\right) \Big) \Big(\dot{\theta} + \dot{\phi}\sin\left(\beta\right) \Big) + L\dot{\theta}\cos\left(\theta\right) \\
&- \dot{\phi}\cos\left(\beta\right)\cos\left(\theta\right)\left(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\right) \Bigg)^2 \\
&- c \Bigg(\Big(x_1 \left(\cos\left(\theta\right)\sin\left(\phi\right) + \cos\left(\phi\right)\sin\left(\beta\right)\sin\left(\theta\right)\right) - x_2\cos\left(\beta\right)\sin\left(\theta\right) \Big) \\
&\Big(\dot{\theta} + \dot{\phi}\sin\left(\beta\right) \Big) - L\dot{\theta}\sin\left(\theta\right) \\
&+ \dot{\phi}\cos\left(\beta\right)\sin\left(\theta\right) \Big(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right) \Big) \Bigg)^2 \\
&- c\dot{\phi}^2 {x_1}^2 {\cos\left(\beta\right)}^2 {\sin\left(\phi\right)}^2
\end{aligned}
\tag{74}
$$

As we want to derive the whole work done by the area distributed force we integrate the expression of dW over the whole square:

$$
\begin{aligned}
W =& \int_{x_1=-L}^{0} \int_{x_2=0}^{L} -c \Bigg(\Big(x_1 \left(\sin\left(\phi\right)\sin\left(\theta\right) - \cos\left(\phi\right)\sin\left(\beta\right)\cos\left(\theta\right)\right) \\
&+ x_2 \cos\left(\beta\right)\cos\left(\theta\right) \Big) \Big(\dot{\theta} + \dot{\phi}\sin\left(\beta\right) \Big) + L\dot{\theta}\cos\left(\theta\right) \\
&- \dot{\phi}\cos\left(\beta\right)\cos\left(\theta\right)\left(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right)\right) \Bigg)^2 \\
&- c \Bigg(\Big(x_1 \left(\cos\left(\theta\right)\sin\left(\phi\right) + \cos\left(\phi\right)\sin\left(\beta\right)\sin\left(\theta\right)\right) - x_2\cos\left(\beta\right)\sin\left(\theta\right) \Big) \\
&\Big(\dot{\theta} + \dot{\phi}\sin\left(\beta\right) \Big) - L\dot{\theta}\sin\left(\theta\right) \\
&+ \dot{\phi}\cos\left(\beta\right)\sin\left(\theta\right) \Big(x_2\sin\left(\beta\right) + x_1\cos\left(\beta\right)\cos\left(\phi\right) \Big) \Bigg)^2 \\
&- c\dot{\phi}^2 {x_1}^2 {\cos\left(\beta\right)}^2 {\sin\left(\phi\right)}^2 \, dx_2 dx_1
\end{aligned}
\tag{75}
$$

Which comes to:

$$
\begin{aligned}
W =& -\frac{cL^4\dot{\phi}^2}{3} - \frac{cL^4\dot{\phi}\dot{\theta}\cos(\beta)\cos(\phi)}{2} - cL^4\dot{\phi}\dot{\theta}\cos(\phi) - \\
&\frac{2c\sin(\beta)\,L^4\dot{\phi}\dot{\theta}}{3} + \frac{cL^4\dot{\theta}^2\cos(\beta)^2\cos(\phi)^2}{3} - \frac{cL^4\dot{\theta}^2\cos(\beta)^2}{3} - \\
&\frac{c\sin(\beta)\,L^4\dot{\theta}^2\cos(\beta)\cos(\phi)}{2} - cL^4\dot{\theta}^2\cos(\beta) - c\sin(\beta)\,L^4\dot{\theta}^2\cos(\phi) \\
&-\frac{4cL^4\dot{\theta}^2}{3}
\end{aligned}
\tag{76}
$$

And finally the generalized forces are:

$$
Q_\phi = \frac{\partial W}{\partial \phi} = -\frac{L^4c\left(4\dot{\phi} + 6\dot{\theta}\cos(\phi) + 4\dot{\theta}\sin(\beta) + 3\dot{\theta}\cos(\beta)\cos(\phi)\right)}{6} \tag{77}
$$

And

$$
\begin{aligned}
Q_\theta =& -\frac{L^4c}{6}\Big(16\dot{\theta} + 12\dot{\theta}\cos(\beta) + 6\dot{\phi}\cos(\phi) + 4\dot{\phi}\sin(\beta) + 4\dot{\theta}\cos(\beta)^2 \\
&- 4\dot{\theta}\cos(\beta)^2\cos(\phi)^2 + 3\dot{\phi}\cos(\beta)\cos(\phi) + 12\dot{\theta}\cos(\phi)\sin(\beta) \\
&+ 6\dot{\theta}\cos(\beta)\cos(\phi)\sin(\beta)\Big)
\end{aligned}
\tag{78}
$$

3.5 Values plugged in

We plug in the values:

$$
L = 0.25, \beta = \pi/6, M = 0.5, m = 0.2, g = 9.81, c = 0.1 \tag{79}
$$

3.6 Equilibrium

In the equilibrium configuration we have

$$
\ddot{\phi} = \dot{\phi} = 0, \quad \phi = \phi_{\mathrm{eq}} \tag{80}
$$

And

$$
\ddot{\theta} = \dot{\theta} = 0, \quad \theta = \theta_{\mathrm{eq}} \tag{81}
$$

For the rheonomic system with a constant Omega and the plugged in values we get:

$$\frac{981\sqrt{3}\sin(\phi)}{3200} + \frac{\Omega^2\sin(\phi)}{128} + \frac{\sqrt{3}\Omega^2\sin(\phi)}{512} - \frac{\Omega^2\cos(\phi)\sin(\phi)}{256} = 0 \tag{82}$$

Or for the case of a free rotation around A:

$$\left(\begin{array}{c} \frac{981\sqrt{3}\sin(\phi_{\mathrm{eq}})}{3200} \\ 0 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \tag{83}$$

And as can be seen the above equation (83) can be read as:

$$\begin{pmatrix} A\sin(\phi_{\mathrm{eq}}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{84}$$

With a constant A. Which yields

$$\phi_{\mathrm{eq}} = n * \pi, \quad \text{for } n = 0, 1, 2... \tag{85}$$

This makes sense as $\phi = n * \pi$ describes states where the square plate stays vertical. As we are not considering aerial drag (non-conservative forces) it is imaginable that this is an equilibrium. θ can be chosen arbitrarily which makes sense as the system is symmetric w.r.t. theta.
Note: as for this case we have a scleronomic system we could have also just used the following:

$$\frac{\partial V}{\partial \mathbf{q}} = \mathbf{0} \tag{86}$$

Which when plugging it in gives us:

$$\left(\begin{array}{cc} \frac{981\sqrt{3}\sin(\phi_{\mathrm{eq}})}{3200} & 0 \end{array} \right) = \mathbf{0} \tag{87}$$

Which is the same as equation (83)
Lastly when considering the non-conservative forces as well we get:

$$\left(\begin{array}{c} \frac{981\sqrt{3}\sin(\phi_{\mathrm{eq}})}{3200} \\ 0 \end{array} \right) \tag{88}$$

Which is the same equilibrium configuration as for the case with the free rotation of the vertical shaft.

Looking at the equations of motion for the case of constant rotation around the vertical shaft whilst considering the non-conservative forces:

$$
\begin{aligned}
&\frac{\Omega}{7680}+\frac{\Omega\,\cos\left(\phi_{\mathrm{eq}}\right)}{2560}+\frac{981\,\sqrt{3}\,\sin\left(\phi_{\mathrm{eq}}\right)}{3200}+\frac{\Omega^{2}\,\sin\left(\phi_{\mathrm{eq}}\right)}{128}+\frac{\sqrt{3}\,\Omega^{2}\,\sin\left(\phi_{\mathrm{eq}}\right)}{512}\\
&-\frac{\Omega^{2}\,\cos\left(\phi_{\mathrm{eq}}\right)\,\sin\left(\phi_{\mathrm{eq}}\right)}{256}+\frac{\sqrt{3}\,\Omega\,\cos\left(\phi_{\mathrm{eq}}\right)}{10240}=0
\end{aligned}
\tag{89}
$$

From the first view we can see that $\phi = 0$ is not the solution anymore which makes sense because the non-conservative forces simulate aerial drag which would induce a tilt in the square when rotating at a constant angular velocity.

Trying to solve this equation yielded no solution for ϕ_{eq}. This can be interpreted as follows: We have a constant angular velocity around the vertical shaft. This rotation induces a velocity in the bar, the rod and the square plate. This generates a drag force on the square plate which will remain there as long as there is an angular velocity and therefore no equilibrium can be reached.

3.7 Integration

To perform the time integration of the state space representation we use the same approach as given in the car exercise. Aka we say that:

$$
\begin{bmatrix} M(q) & 0 \\ 0I & \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix}=\begin{bmatrix} f \\ \dot{q} \end{bmatrix}
\tag{90}
$$

Where $f = -(\text{equations of motion} - M(q)\ddot{q})$

The result of this time integration for iniitial conditions of:

$$
\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \phi \\ \theta \end{pmatrix}=\begin{pmatrix} \pi/s \\ 0 \\ 0 \\ 0 \end{pmatrix}
\tag{91}
$$

is the following for phi:

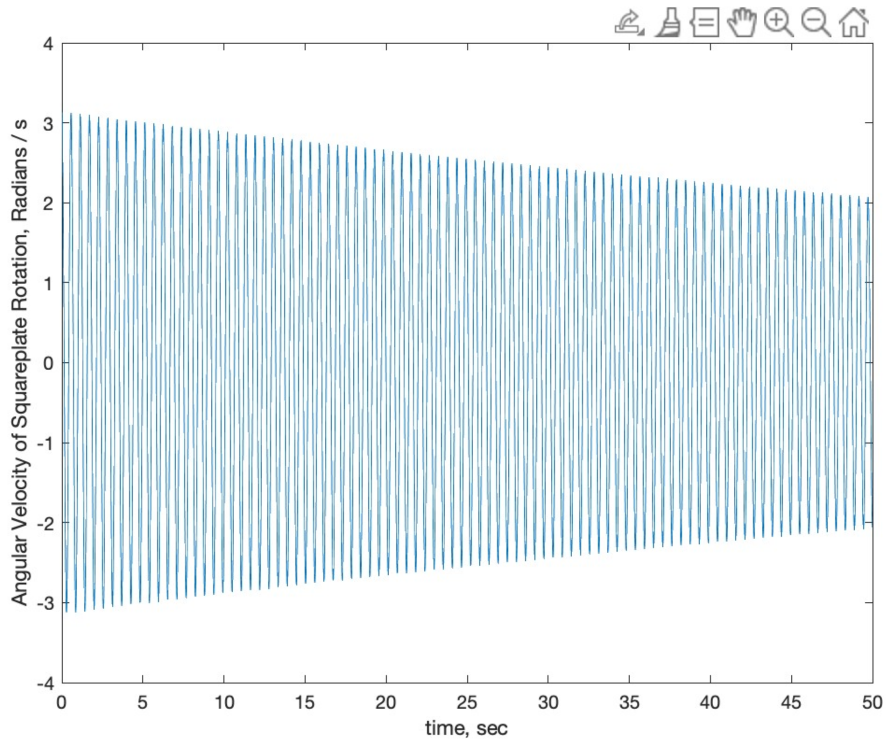

Figure 9: $\dot{\phi}$ for initial rotation of the square plate

and this for theta:

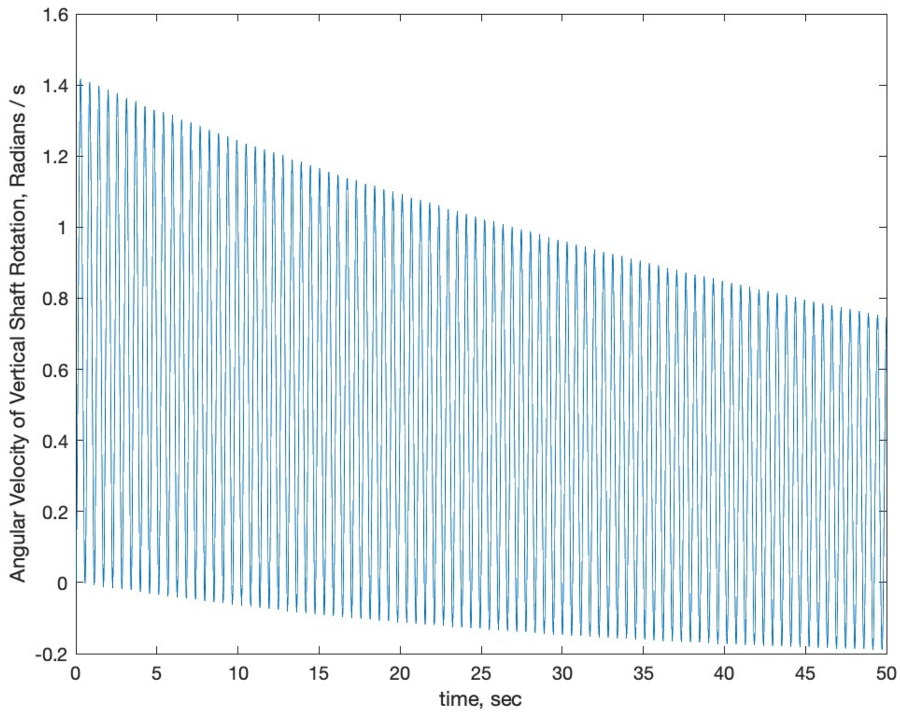

Figure 10: $\dot{\theta}$ for initial rotation of the square plate

Interpretation:
With initial conditions of only a phi rotation we get an induced angular momentum in the system which has to be compensated by the angular momentum around the vertical shaft. Therefore we get a precession around the vertical shaft. The induced precession in turn leads to a rotation of the square plate in the opposite direction and so on. This highly oscillatory movement can be seen in figures 9 and 10. Also it can be noted that the amplitude of the oscillation diminishes for both rotations. This is due to the drag forces on the square plate.

And for the second initial conditions of exciting the theta around the vertical shaft:

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ \pi/s \\ 0 \\ 0 \end{pmatrix} \tag{92}$$

we get for phi:

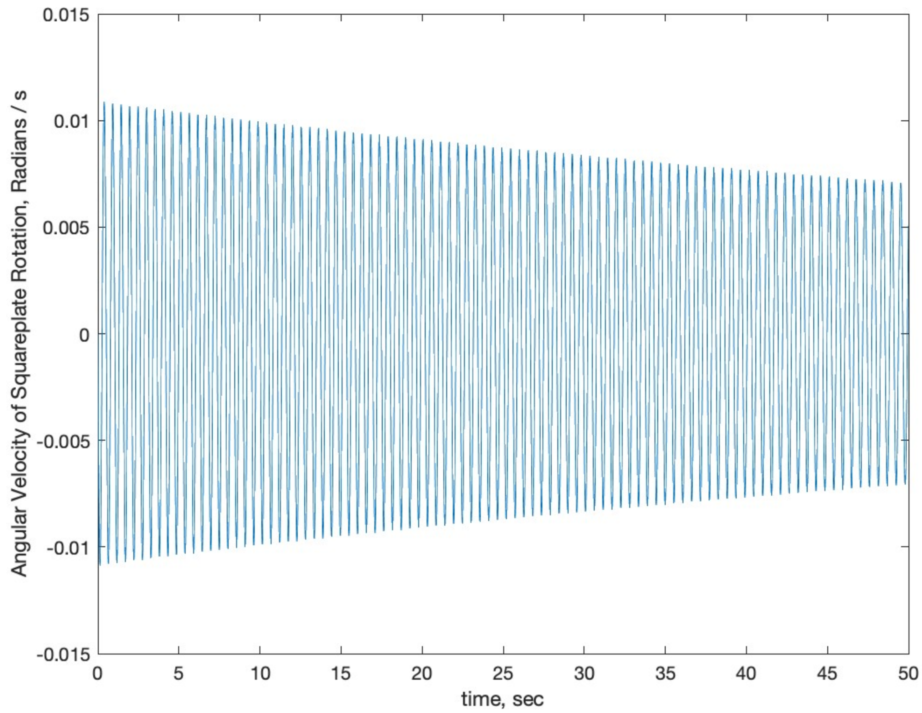

Figure 11: $\dot{\phi}$ for initial rotation of the vertical shaft

and for theta:

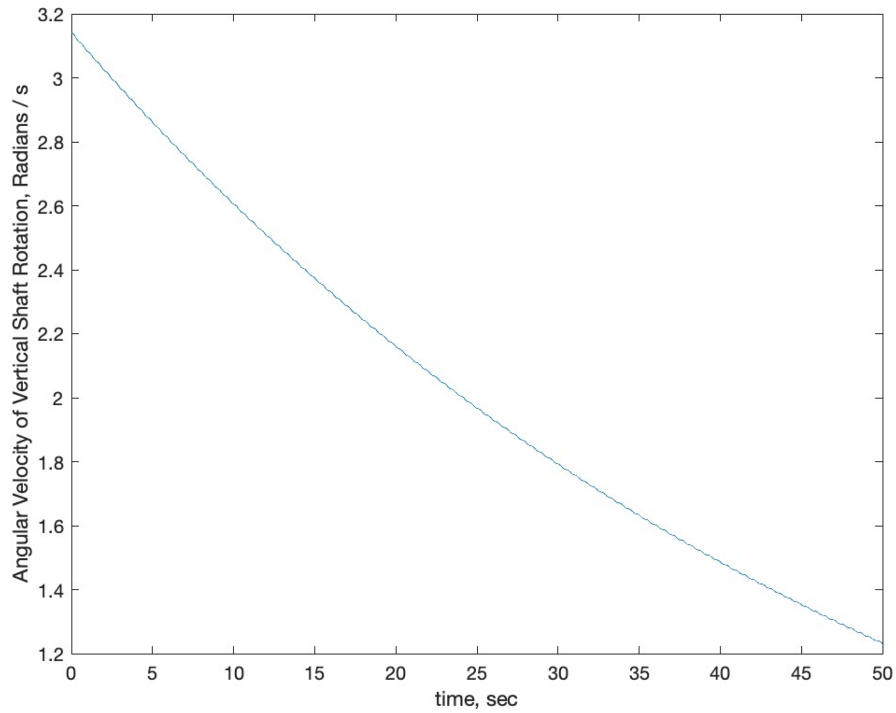

Figure 12: $\dot{\theta}$ for initial rotation of the vertical shaft

Interpretation
Here we have a smooth rotation of the system around the vertical shaft. The angular velocity of theta goes towards 0 as can be seen in figure 12 this is of course again due to the drag force acting on the smoothly rotating system. The angular velocity of the square plate is again oscillatory but with very small magnitudes. This is the case because the square plate tries to find it's equilibrium position during the rotation around the vertical shaft but can't as the drag force pusheds the plate up and gravity pulls it down and $\dot{\theta}$ is diminishing so the equilibrium always changes aka there is no equilibrium. In the end this looks like a rotation around the vertical shaft with a tilted square plate that wobbles in the wind.
To proof this we can also plot the angle of the square plate over time:

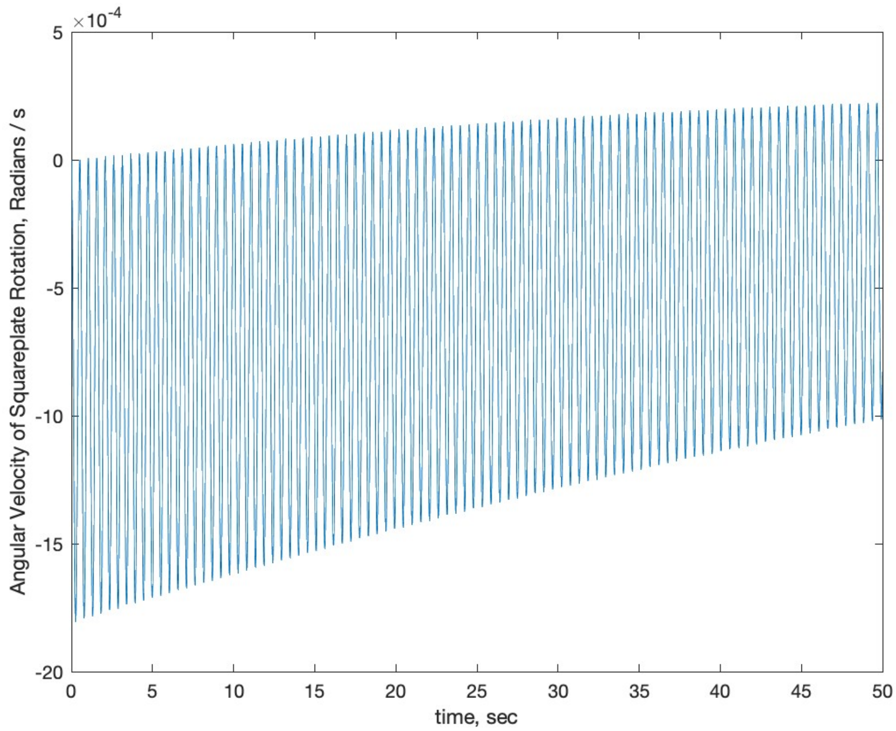

Figure 13: ϕ for initial rotation of the vertical shaft

And as expected we have very tiny fluctuations (range 1e-4) of the ϕ angle.

4 Assignment 4

4.1 DONE

4.2 DONE

4.3 DONE

4.4 DONE

4.5

4.5.1 DONE

We fixed the front wheel to remove the singularity of K. q_{init} was given as:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 0 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 0.22 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 0.21 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = \pi \\ \beta_{\text{Link Front}} = 0 \end{pmatrix} \tag{93}$$

Which represents this position:

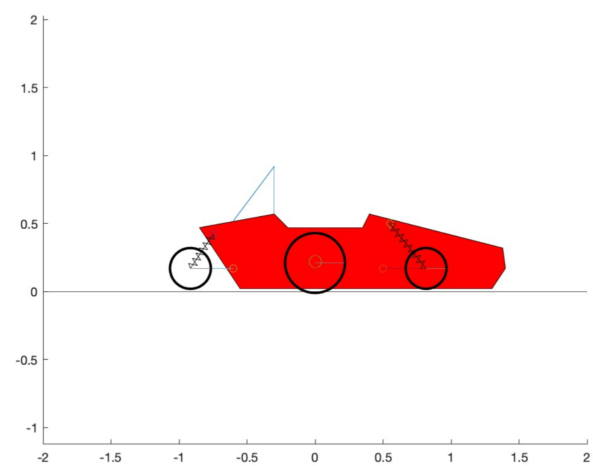

Figure 14: Initial Position 1

This resulted in the equilibrium:

$$q_{\text{equilibrium 1}} = \begin{pmatrix} \theta_{\text{Frame}} = -1.3896e-02 = -0.014 \\ x_{\text{Frame}} = -8.1104e-01 = -0.811 \\ y_{\text{Frame}} = 2.1926e-01 = -0.219 \\ \theta_{\text{Wheel Back}} = 7.8437e+00 = 7.844 \\ \theta_{\text{Tire Front}} = 2.2254e-21 \approx 0 \\ \theta_{\text{Tire Back}} = 7.8437e+00 = 7.844 \\ y_{\text{Tire Front}} = 2.1883e-01 = 0.219 \\ y_{\text{Tire Back}} = 2.1895e-01 = 0.219 \\ \beta_{\text{Link Back}} = 3.0209e+00 = 3.021 \\ \beta_{\text{Link Front}} = 1.6787e-01 = 0.168 \end{pmatrix} \tag{94}$$

Which is visualized by this figure:

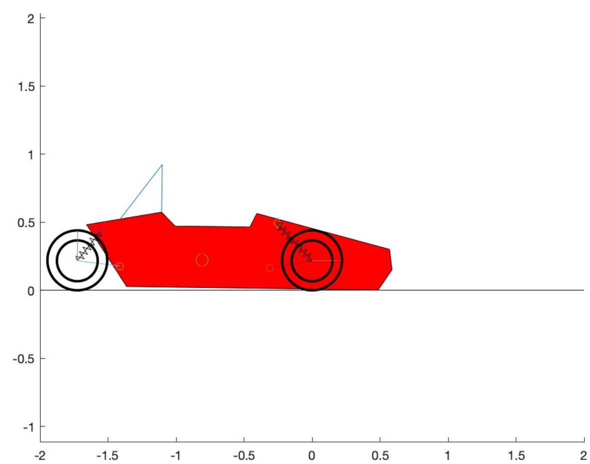

Figure 15: Equilibrium Position 1

In this example the choice of initial generalized coordinates was obviously very good. For the first task of this assignment the goal is to find two more (probably bad) equilibria.
The eigenvalues of the stiffness matrix are:

4.5.2 Different Equilibrium states

The first alternative is luckily already given in the code. Again we fix the front wheel's rotation and start with the initial state:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi/2 \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{95}$$

Which represents this position:

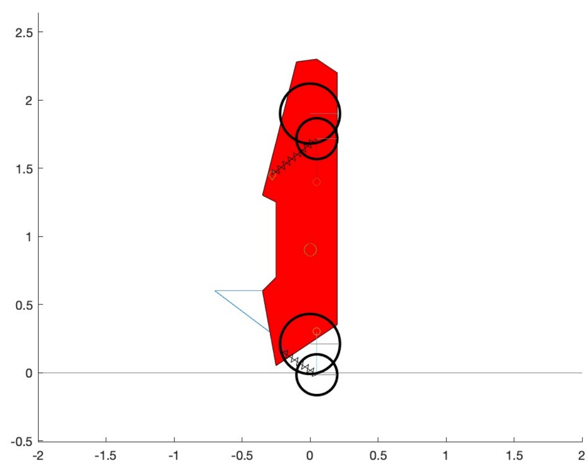

Figure 16: Initial Position 2

Note: in the code provided θ_{Frame} was π which was a different initial set of gen. coord. but converged to the same solution

As can be seen, this is obviously not a good choice of initial coordinates. This setting converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 1.5248e+00 \\ x_{\text{Frame}} = -6.9703e-02 \\ y_{\text{Frame}} = 1.1280e+00 \\ \theta_{\text{Wheel Back}} = 3.0427e-01 \\ \theta_{\text{Tire Front}} = 2.9622e-24 \\ \theta_{\text{Tire Back}} = 3.0427e-01 \\ y_{\text{Tire Front}} = 1.9414e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -1.6327e+00 \\ \beta_{\text{Link Front}} = 1.5811e+00 \end{pmatrix} \tag{96}$$

Which looks as follows:

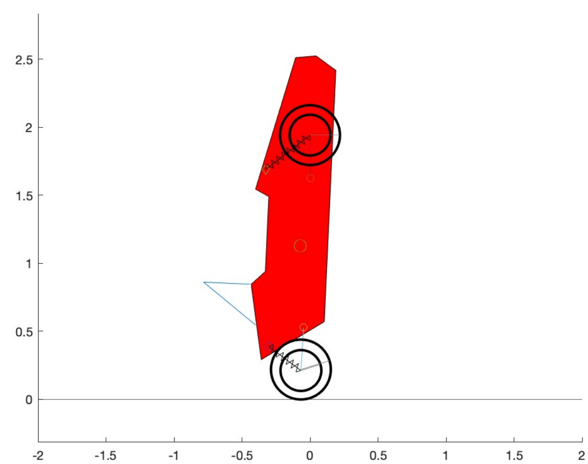

Figure 17: Equilibrium Position 2

The code example which has even worse initial conditions:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = \pi \\ x_{\text{Frame}} = 0 \\ y_{\text{Frame}} = 90 \\ \theta_{\text{Wheel Back}} = 0 \\ \theta_{\text{Tire Front}} = 0 \\ \theta_{\text{Tire Back}} = 0 \\ y_{\text{Tire Front}} = 1.90 \\ y_{\text{Tire Back}} = 0.21 \\ \beta_{\text{Link Back}} = -\pi/2 \\ \beta_{\text{Link Front}} = \pi/2 \end{pmatrix} \tag{97}$$

And looks like this:

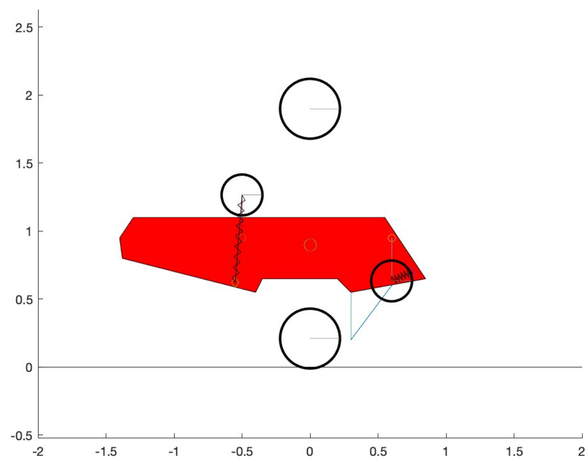

Figure 18: Initial Position 3

Converges to:

$$q_{\text{init}} = \begin{pmatrix} \theta_{\text{Frame}} = 2.0225e+00 \\ x_{\text{Frame}} = -1.2734e-01 \\ y_{\text{Frame}} = 6.3440e-01 \\ \theta_{\text{Wheel Back}} = 5.5591e-01 \\ \theta_{\text{Tire Front}} = 2.0275e-25 \\ \theta_{\text{Tire Back}} = 5.5591e-01 \\ y_{\text{Tire Front}} = 1.2043e+00 \\ y_{\text{Tire Back}} = 2.1777e-01 \\ \beta_{\text{Link Back}} = -3.4446e+00 \\ \beta_{\text{Link Front}} = 3.1586e-01 \end{pmatrix} \tag{98}$$

Which represents:

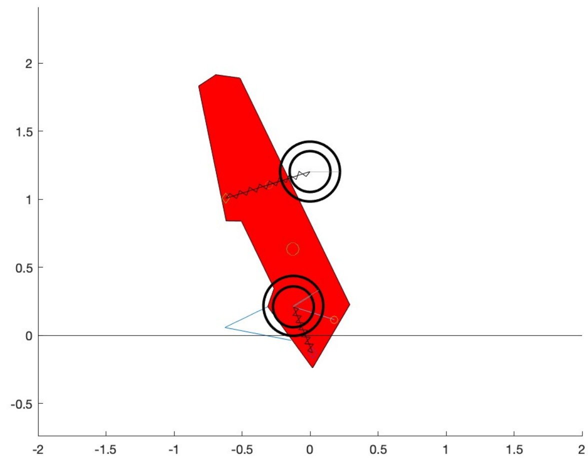

Figure 19: Equilibrium Position 3

4.6 Analysis

As we are not considering drag yet we have a conservative, scleronomic system. For an equilibrium in such a system to be stable we need that K_{eq} must be positive definite. Aka the eigenvalues of K_{eq} must be real and positive.
Looking at the eigenvalues for the first case:

$$\begin{pmatrix} 1.6913e-07 \\ 2.1774e+00 \\ 2.2340e+00 \\ 3.0009e+01 \\ 3.0821e+01 \\ 3.4894e+01 \\ 3.5642e+01 \\ 6.8555e+01 \\ 6.8673e+01 \\ 1.3035e+02 \\ 1.3036e+02 \end{pmatrix} \tag{99}$$

We can see that they are indeed all positive and real. Thus 94 is a stable equilibrium. This represents logical assumptions as the car staying in horizontal position is as we know a stable equilibrium.

For the second equilibrium we get the following eigenvalues:

$$\begin{pmatrix} 0.0000e+00+4.9405e-01i \\ 0.0000e+00+1.2122e-07i \\ 4.4151e+00+0.0000e+00i \\ 7.1552e+00+0.0000e+00i \\ 7.4830e+00+0.0000e+00i \\ 4.3728e+01+0.0000e+00i \\ 4.3983e+01+0.0000e+00i \\ 6.3302e+01+0.0000e+00i \\ 7.2017e+01+0.0000e+00i \\ 7.2124e+01+0.0000e+00i \\ 1.2904e+02+0.0000e+00i \end{pmatrix} \tag{100}$$

Where we can see that the eigenvalues ar of complex nature. Mainly the first one

Lastly for the third setting we have the following eigenvalues:

$$\begin{pmatrix} 0.0000e+00+4.9405e-01i \\ 1.9956e-07+0.0000e+00i \\ 4.4151e+00+0.0000e+00i \\ 7.1552e+00+0.0000e+00i \\ 7.4830e+00+0.0000e+00i \\ 4.3728e+01+0.0000e+00i \\ 4.3983e+01+0.0000e+00i \\ 6.3302e+01+0.0000e+00i \\ 7.2017e+01+0.0000e+00i \\ 7.2124e+01+0.0000e+00i \\ 1.2904e+02+0.0000e+00i \end{pmatrix} \tag{101}$$

Where we see again, that we have complex eigenvalues denoting an unstable equilibrium.

Another interesting example is the one seen in fig 20:

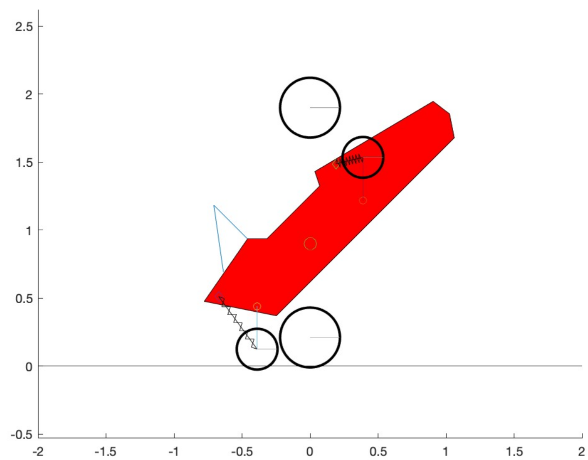

Figure 20: Initial Position 4

When we start from the tilted starting position of 16 but changing merely the frame angle to a smaller $\pi/4$ we can see that it converges to something quite reasonable looking.
And indeed when looking at the eigenvalues:

$$\begin{pmatrix} 2.5694e-07 \\ 2.1774e+00 \\ 2.2340e+00 \\ 3.0009e+01 \\ 3.0821e+01 \\ 3.4894e+01 \\ 3.5642e+01 \\ 6.8555e+01 \\ 6.8673e+01 \\ 1.3035e+02 \\ 1.3036e+02 \end{pmatrix} \tag{102}$$

We see that it's a stable equilibrium. From this we can take away that a perturbation from a stable equilibrium that is small enough will converge to the stable equilibrium.

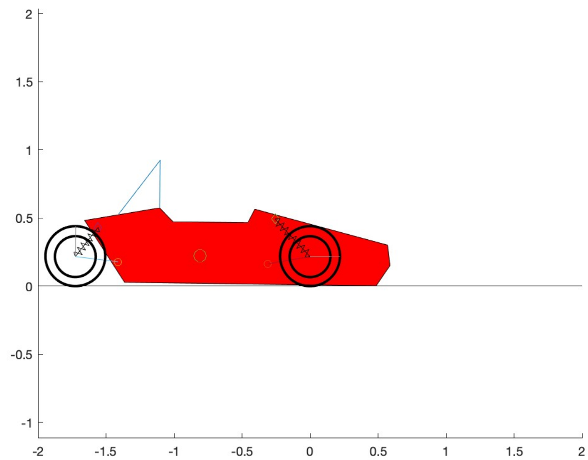

Figure 21: Equilibrium Position 4

4.7 Linearized Equations - DONE

4.8 Eigenmodes and Eigenfrequencies

Looking at the eigenfrequencies in equation (99) we can see that the dominant ones with the large motion impact seem to be the first as well as the second and third.
The eigenmodes associated with these frequencies are:

$$\begin{pmatrix} 6.0858e-01 \\ 9.4719e-02 \\ -9.7389e-02 \\ 0 \\ 0 \\ 1.2034e-02 \\ 8.8040e-03 \\ -4.7960e-01 \\ 5.7699e-01 \\ 1.7042e-01 \\ 1.3773e-01 \end{pmatrix} \tag{103}$$

,

$$\begin{pmatrix} 4.8632e-02 \\ 9.8644e-01 \\ 0 \\ 0 \\ 0 \\ 1.0851e-01 \\ 1.0851e-01 \\ 0 \\ 0 \\ -1.8772e-02 \\ -2.6059e-02 \end{pmatrix} \tag{104}$$

and

$$\begin{pmatrix} -4.0695e-02 \\ 0 \\ 8.0283e-01 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4.0141e-01 \\ -4.0141e-01 \\ -1.2602e-01 \\ 1.2515e-01 \end{pmatrix} \tag{105}$$

Plotted out we get the following configurations:
This result I am not sure about at the current time.

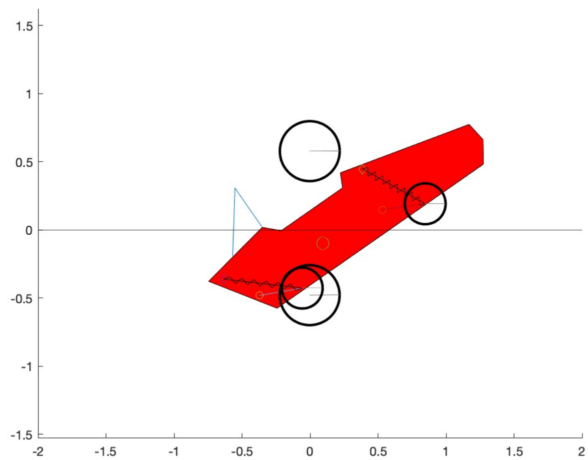

Figure 22: Eigenmode Configuration 1

4.8.1 DONE

4.8.2 Mode Shapes

4.9 Drag and Damping

4.9.1 Damping

The contributions to the damping force are:

$$Q_{\text{suspension back}} = \begin{pmatrix} Q_{\text{susb}(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_{\text{susb}(2)} \\ 0 \end{pmatrix} \tag{106}$$

Which has a contribution towards the rotation of the frame and the rotation of the back link. Both of these contributions have a negative sign which makes sense as the suspension of the back tire will lift the back wheel resulting in a rotation of the bar and the frame in negative Z direction.

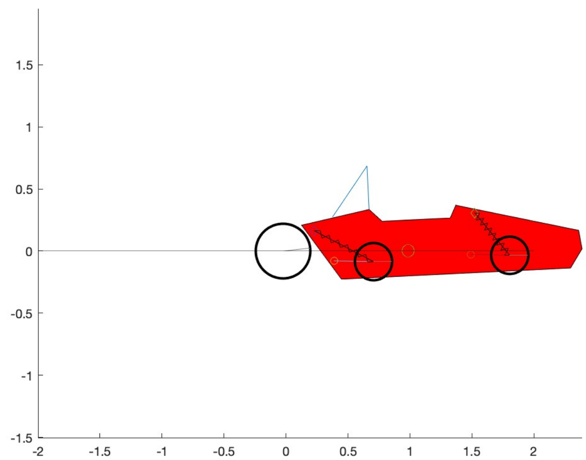

Figure 23: Eigenmode Configuration 2

Analogous for the front suspension:

$$Q_{\text{suspension front}} = \begin{pmatrix} Q_{\text{susf}(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_{\text{susf}(2)} \end{pmatrix} \tag{107}$$

Therefore the front suspension force leads to an impact on the frame angle and the front link angle.
Note: these two suspension forces come from the spring approximation of the reality.

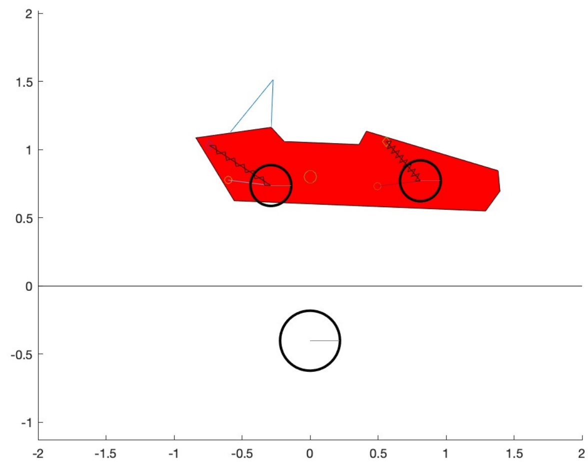

Figure 24: Eigenmode Configuration 3

$$Q_{\text{tire back}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_{\text{tB}(9)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{108}$$

The back tire contact force has an impact on the elevation of the back tire.

$$Q_{\text{tire front}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_{\text{tF}(9)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{109}$$

This has an impact on the elevation of the front tire component.

4.9.2 Drag Forces

$$Q_{\text{drag}} = \begin{pmatrix} 0 \\ -(63 * \dot{x}_{\text{fr}}^2)/100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{110}$$

This is the aerodynamic drag on the system. This can be nicely seen as there is only a negative component in the second row (depending on the velocity in this direction scquared) which corresponds to the x movement of the frame. Thus we have aerodynamic drag in that direction.

$$Q_{\text{friction wheels}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dot{\theta}_{\text{wF}}^3/50000 \\ -\dot{\theta}_{\text{wB}}^3/50000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{111}$$

Here we can nicely see friciton drag impacting the rotation of the front and the back wheel.
Lastly we have for the torque:

$$Q_{\text{wheel}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -300 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{112}$$

Representing the load applied in negative Z direction on the back wheel to make the car drive to the right.

4.10 Nonlinear Time Integration

Speed of the Car:

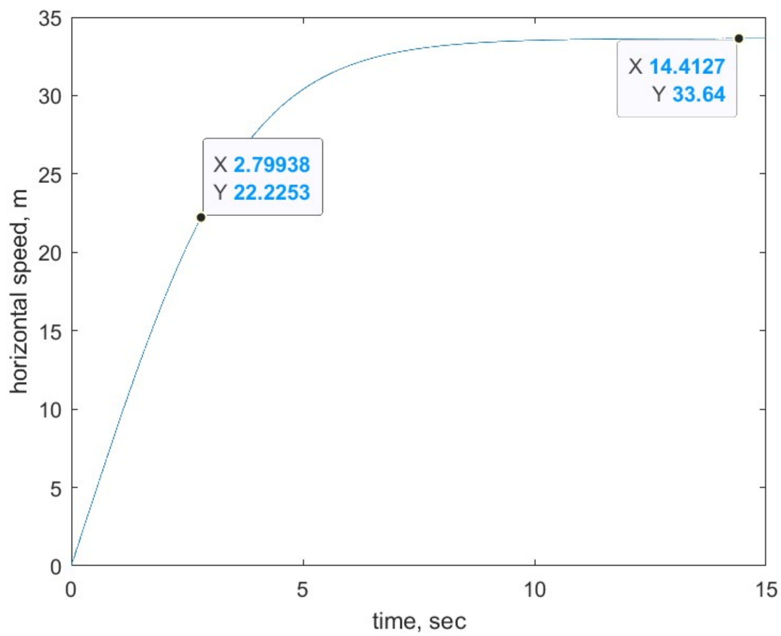

Figure 25: Speed of Car with convergence speed indicated

As we can see in figure 25 the horizontal velocity converges towards 33.64 m/s which is equivalent to 120.6 km/h. (When performing the integration for a longer time interval the more exact convergence value is 33.6466) Also for an acceleration to 80 km/h (= 22.22 m/s) it takes about 2.8s.

4.10.1 Horizontal displacement of the frame

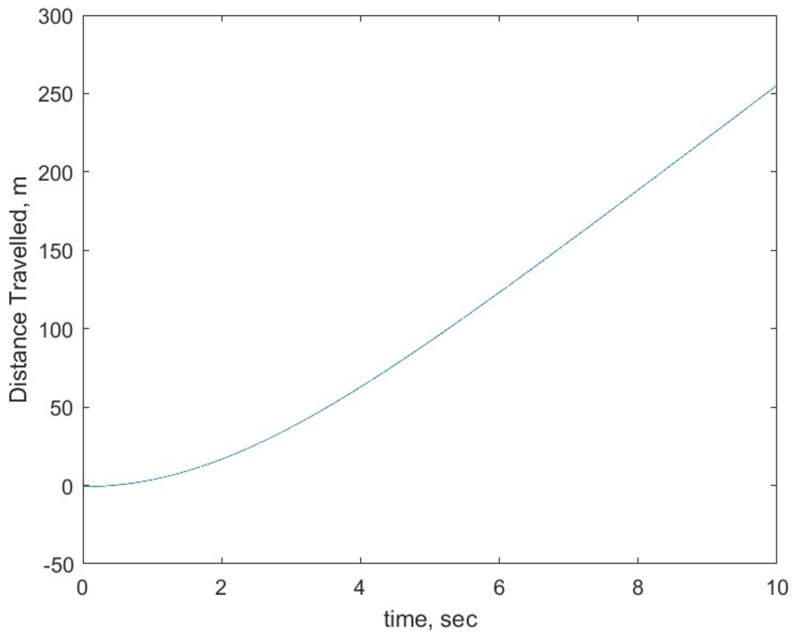

Figure 26: Horizontal Displacement

Here above we can nicely the horizontal displacement with time. As already assumed the displacement starts at zero and at one point behaves linear in time due to the constant velocity which can be seen in figure 25.

4.10.2 Vertical displacement of the frame

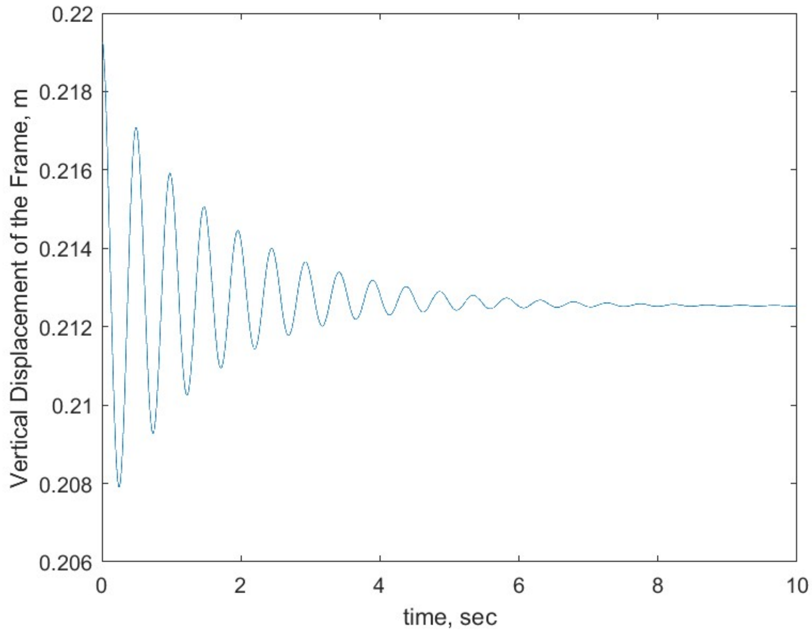

Figure 27: Vertical Displacement

Here we can see a very small oscillating displacement which converges towards 0.

4.10.3 Wheel-ground elastic forces

As the wheel-ground elastic forces are modelled with linear springs we have:

$$\begin{aligned} f_{\text{Contact}} &= k_{\text{Contact}} * \Delta y_{\text{tF}} \\ \text{with } \Delta y_{\text{tF}} &= |y_{\text{tF}} - r_{\text{tire}}| \end{aligned} \tag{113}$$

And thus for the elastic forces of the tire we get the following plot:

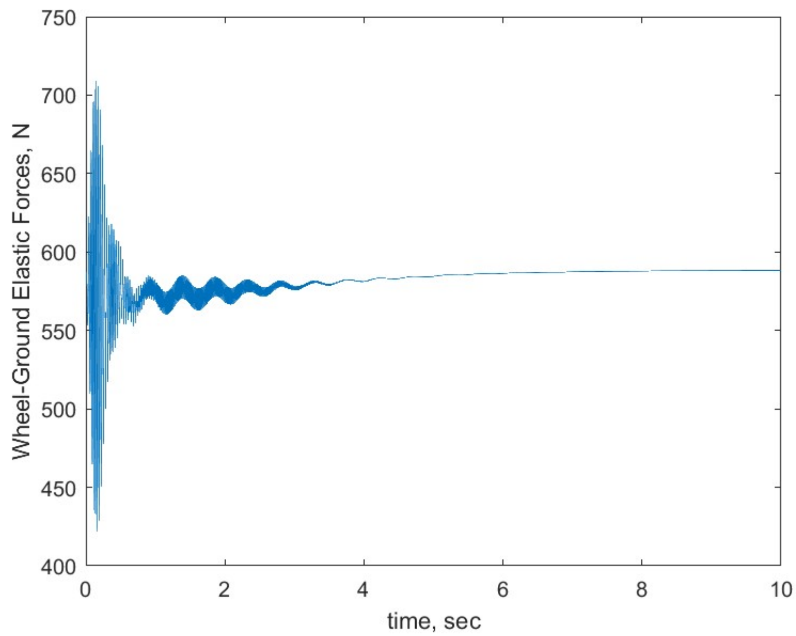

Figure 28: Elastic Contact Forces

4.10.4 orientation of the rear linkage

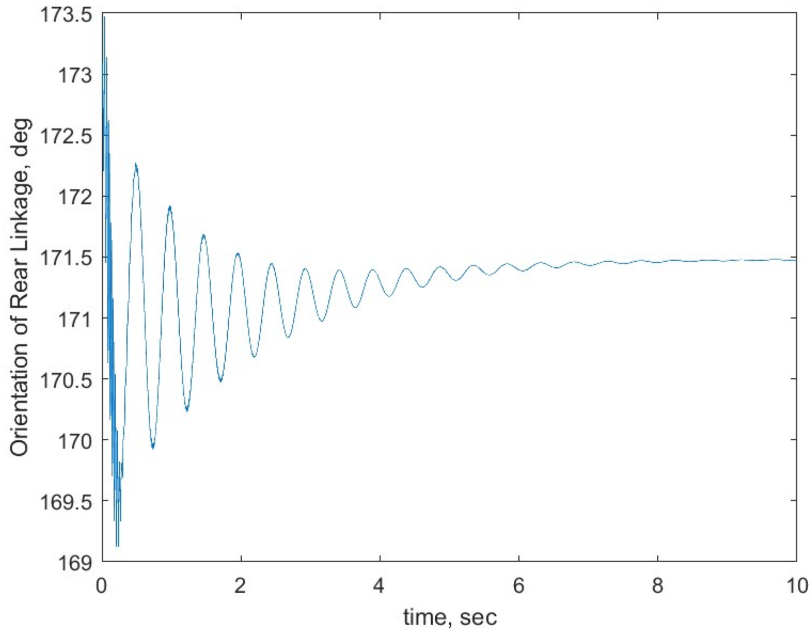

Figure 29: Rear Linkage angle

4.11 Adding Wings

To add the effect of the wings to the car we have to derive the contribution of the generalized forces. For this we calculate the work done at either wing and consider the derivate w.r.t. q.

$$Q_{\text{L Front}} = \frac{\partial}{\partial q} W_{\text{L Front}} \tag{114}$$

For the work done we multiply the force with the respective velocity component. The velocity component consists of both ends of the rigid body translation and the contribution from the frame rotation.

$$v_F = v_{\text{Frame}} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \times \begin{pmatrix} x_F \\ y_F \\ 0 \end{pmatrix} \tag{115}$$

$$\Rightarrow W_{\text{L Front}} = F_{\text{L Front}} * v_F \tag{116}$$

4.11.1 Horizontal Displacement of the Frame

Implementing the above contribution of the wings in the code yields the following plots:

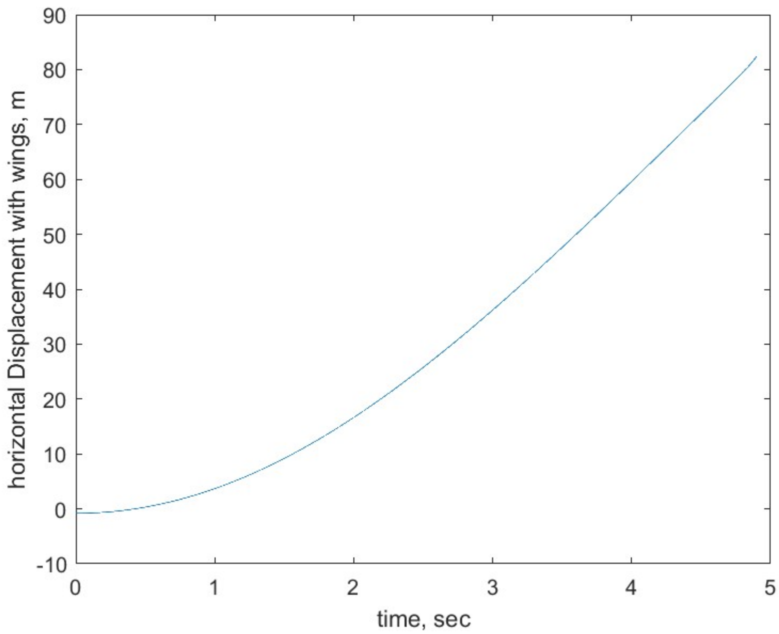

Figure 30: Horizontal Displacement with Wings

The distance covered as can be seen above is significantly less than for the case without wings. This is due to the drag induced by the wings.

4.11.2 Vertical Displacement of the Frame

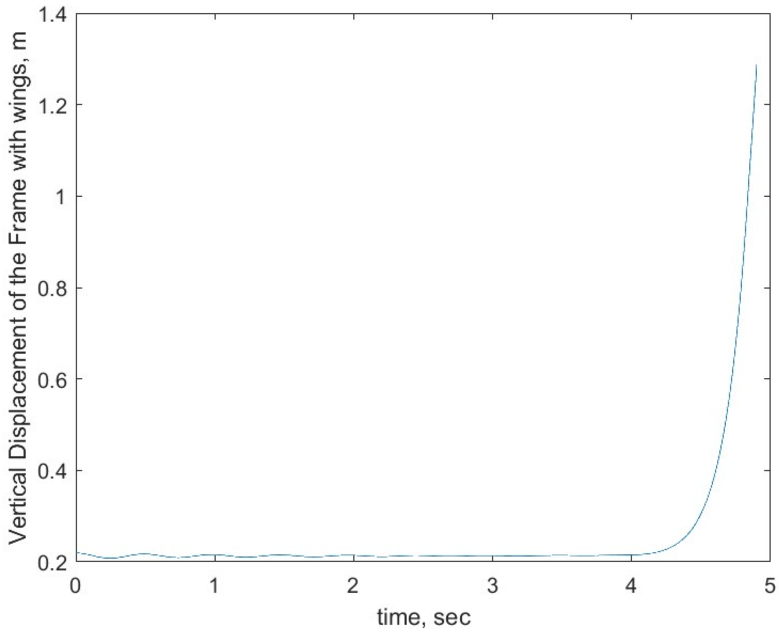

Figure 31: Vertical Displacement with Wings

The vertical displacement is stable at first and then the car gains altitude rapidly. As the wings induce drag forces the car seems to tip over at the front resulting in an elevation of the center of mass.

4.11.3 Wheel-Ground Elastic Forces

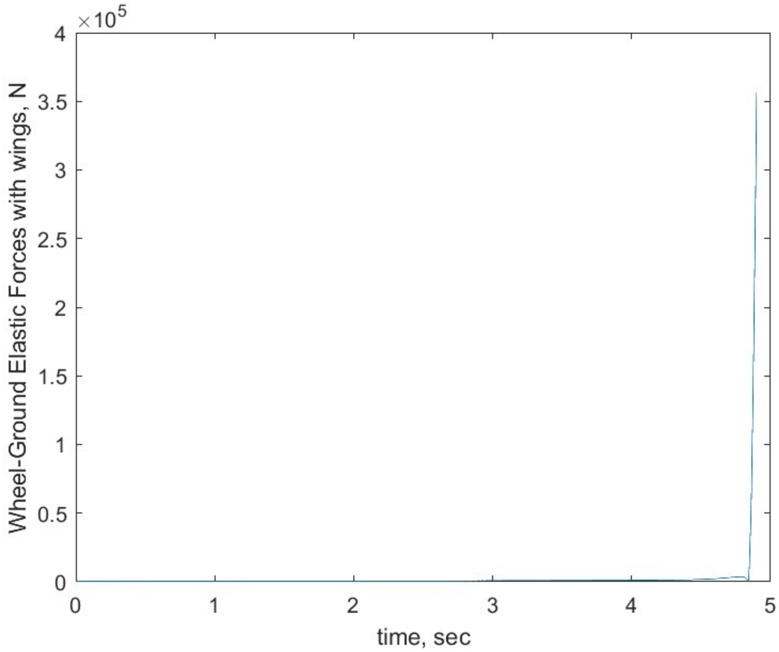

Figure 32: Elastic Contact Forces with Wings

Starting off we see the elastic force in the same range as without the wings. Around the 5 second mark the elastic force increases by a lot. This is the same point in time as the aforementioned supposed tipping of the car.

4.11.4 Orientation of the Rear Linkage

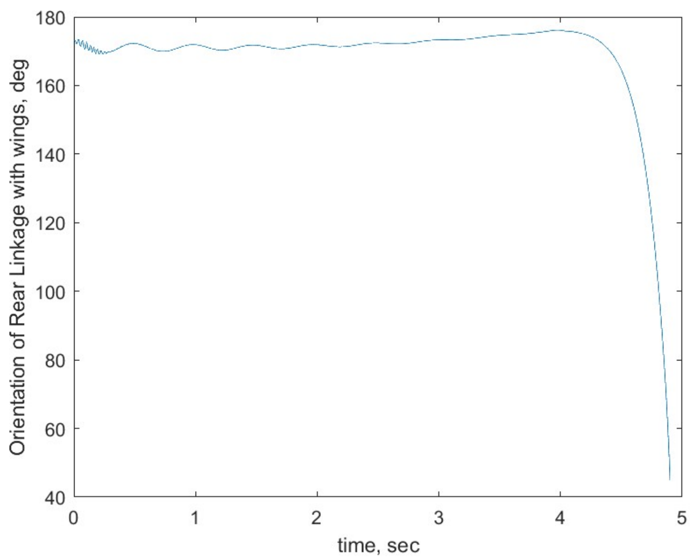

Figure 33: Rear Linkage Orientation with Wings

The rear linkage angle starts off at 180° as before but then turns clockwise.

Wings Conclusion Looking at the last state of the generalized coordinates we

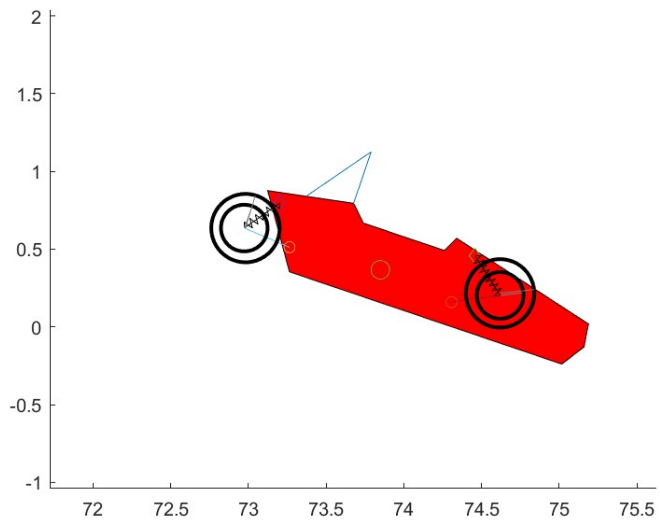

Figure 34: End Pose Car with Wings

can see the car in a tilted state. From the additional drag force from the wings the car seems to turn in a clockwise motion.

4.12 Harmonic Response

Not completed.