

# Advanced Dynamics - Assignment Report

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# 1 Assignment 1

## 1 WHEEL ROLLING WITHOUT SLIPPING ON A 2D TRACK

A thin wheel of radius  $R$  rolls without slipping on a track on the  $x_1 - x_2$  plane, defined by  $x_2 = f(x_1)$ . The wheel plane stays vertical and tangent to such track at the contact point  $P$ . Denote with  $\alpha$  the angle the disk plane forms with the  $x_2$  axis, and with  $\phi$  the rotation of the disk about its axis  $\mathbf{e}_\phi$ . The position of the center of the disk  $C$  is indicated by  $x_1^C$ ,  $x_2^C$  and  $x_3^C$ . Assume a set of generalized coordinate  $\mathbf{q} = [x_1^C \ x_2^C \ x_3^C \ \alpha \ \phi]$ .

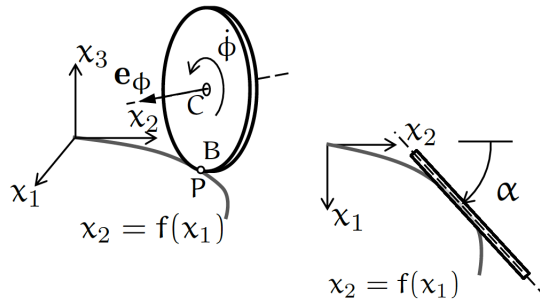


Figure 1.1: Wheel rolling without slipping on a track.

1. State all the constraints acting on the disk.
2. Determine whether the constraints are holonomic or non-holonomic.

Figure 1: Task 1.1

## 1.1

### 1.1.1

The list of constraints looks as follows:

1. The wheel always stays vertical (plane parallel to  $x_3$ )  
This is a holonomic constraint:  $f = \theta = 0$  where  $\theta$  denotes the angle between the disk and  $x_3$
2.  $x_2$  follows a fixed trajectory, given  $x_1$  (and vice versa):  
 $f_2 : x_2 = f(x_1) \Rightarrow x_2 - f(x_1) = 0 \Rightarrow$  holonomic.
3.  $\alpha$  is the angle between the trajectory and the  $x_2$  axis:  
 $\alpha = \frac{\pi}{2} - \frac{\partial f(x_1)}{\partial x_1}$  or written differently:  
 $f(\alpha, x_1) = \alpha - \frac{\pi}{2} + \frac{\partial f(x_1)}{\partial x_1} = 0 \Rightarrow$  holonomic
4. Rolling without slipping:  
 $v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0$  with  $\omega = \dot{\alpha}\mathbf{e}_3 + \dot{\phi}\mathbf{e}_\phi$   
Seems to be non-holonomic at first glance
5. The disk does not leave the ground:  
 $x_3^C - R = 0$  aka the  $x_3$  component of the center of mass is R.  
This is holonomic as well

So far we have a 3D system (6 DoF) and 4 holonomic constraints and 1 non-holonomic constraint.

4. Check for integrability:

$$v^B = 0 \Rightarrow v_C + \omega \times R_{CB} = 0 \quad (1)$$

Plugging in  $\dot{x}_1^C, \dot{x}_2^C, \omega = \dot{\alpha}\mathbf{e}_3 + \dot{\phi}\mathbf{e}_\phi$  and  $R_{CB} = [0, 0, -R]^T$ :

$$\dot{x}_1^C \mathbf{e}_1 + \dot{x}_2^C \mathbf{e}_2 - R\dot{\phi} \cos \alpha \mathbf{e}_2 - R\dot{\phi} \sin \alpha \mathbf{e}_1 = 0 \quad (2)$$

Considering the part in  $\mathbf{e}_1$  direction:

$$\dot{x}_1^C - R\dot{\phi} \sin \alpha = 0 \quad (3)$$

As  $\alpha$  is not dependent on time it can be easily seen that 3 is integrable:

$$\begin{aligned} \int \dot{x}_1^C dt &= \int R\dot{\phi} \sin \alpha dt \\ \Rightarrow x_1^C &= R\phi \sin \alpha \end{aligned} \quad (4)$$

### 1.1.2

See subsubsection 1.1.1

### 1.1.3

3. Determine the degrees of freedom of the system.

Figure 2: Task 1.1.3

As we have a 3D body with 6 generalized coordinates (here 5 are given, already considering constraint 1) and 5 holonomic constraints. We get a total of  $6 - 5 = 1$  degree of freedom. That could for instance be the rotation of the wheel around  $\mathbf{e}_\phi$  while all the other generalized coordinates follow accordingly.

## 2 TWO BARS LINKAGE

Two bars AB and BC of equal length  $L$  are hinged at point B, and move in the plane spanned by the unit vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . The velocity  $\mathbf{v}_C$  of point C is required to be directed towards point A at all times, as shown. Show that such constraint is non-holonomic. Use  $x_1^B, x_2^B, \theta_1$  and  $\theta_2$  as generalized coordinates.

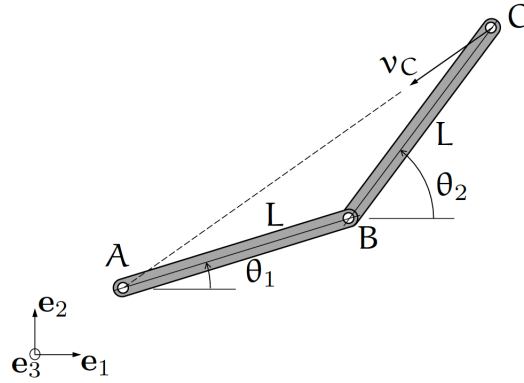


Figure 2.1: A two bar linkage in 2D. The velocity  $\mathbf{v}_C$  must be directed towards A at all times.

Figure 3: Task 1.1

### 1.2

I want to show that the constraint that  $v_c$  always points in the direction of  $AD$  is not integrable. I would like to express this constraint as:

$$v_C * AC_p = 0 \quad (5)$$

where  $AC_p$  is a vector perpendicular to  $AC$

The velocity of point C can be found either by expressing the position of C and derivating or using the velocity transfer formula from B to C which yields:

$$v_C = \begin{pmatrix} \dot{x}_1^B - L \sin(\theta_2) \dot{\theta}_2 \\ \dot{x}_2^B + L \cos(\theta_2) \dot{\theta}_2 \end{pmatrix} \quad (6)$$

And AC is simply:

$$AC = \begin{pmatrix} L (\cos \theta_1 + \cos \theta_2) \\ L (\sin \theta_1 + \sin \theta_2) \end{pmatrix} \quad (7)$$

A simple vector that is perpendicular to AC can be gotten by switching the  $e_1$  and  $e_2$  entries and switching the sign of one of them:

$$AC_p = \begin{pmatrix} L(\sin \theta_1 + \sin \theta_2) \\ -L(\cos \theta_1 + \cos \theta_2) \end{pmatrix} \quad (8)$$

$$\begin{aligned} \Rightarrow v_C * AC_p &= \begin{pmatrix} \dot{x}_1^B - L \sin(\theta_2) \dot{\theta}_2 \\ \dot{x}_2^B + L \cos(\theta_2) \dot{\theta}_2 \end{pmatrix} * \begin{pmatrix} L(\sin \theta_1 + \sin \theta_2) \\ -L(\cos \theta_1 + \cos \theta_2) \end{pmatrix} = \\ & (L(\sin(\theta_1) + \sin(\theta_2))) \dot{x}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2))) \dot{x}_2^B - L^2 \dot{\theta}_2 \cos(\dot{\theta}_2) (\cos(\theta_1) + \cos(\theta_2)) - \\ & L^2 \dot{\theta}_2 \sin(\dot{\theta}_2) (\sin(\theta_1) + \sin(\theta_2)) \end{aligned} \quad (9)$$

After applying trigonometry:

$$\begin{aligned} & (L(\sin(\theta_1) + \sin(\theta_2))) \dot{x}_1^B + (-L(\cos(\theta_1) + \cos(\theta_2))) \dot{x}_2^B - L^2 \dot{\theta}_2 \cos(\dot{\theta}_2) (\cos(\theta_1) + \cos(\theta_2)) - \\ & L^2 \dot{\theta}_2 \sin(\dot{\theta}_2) (\sin(\theta_1) + \sin(\theta_2)) \end{aligned} \quad (10)$$

## 2 Assignment 2



### 3 Assignment 3

#### 3.1

Lagrange equations:

$$\begin{aligned} & \frac{L^2 M \ddot{\phi}}{3} + \frac{L^2 m \ddot{\phi}}{3} - \frac{L^2 m \ddot{\phi} \cos(\beta)^2}{3} + \frac{LMg \cos(\beta) \sin(\phi)}{2} + \frac{L^2 M \Omega^2 \sin(\beta) \sin(\phi)}{2} + \\ & \frac{L^2 M \Omega^2 \cos(\beta) \sin(\beta) \sin(\phi)}{4} - \frac{L^2 M \Omega^2 \cos(\beta)^2 \cos(\phi) \sin(\phi)}{6} = 0 \end{aligned} \quad (11)$$

#### 3.2

This can be reformulated for the differential equation of  $\ddot{\phi}$ :

$$\begin{aligned} & \ddot{\phi} \left( \frac{L^2(M+m)}{3} - \frac{L^2 m \cos(\beta)^2}{3} \right) + \sin(\phi) \left( \frac{LMg \cos(\beta)}{2} + \frac{L^2 M \Omega^2 \sin(\beta)}{2} + \frac{L^2 M \Omega^2 \cos(\beta) \sin(\beta)}{4} \right) - \\ & \frac{L^2 M \Omega^2 \cos(\beta)^2 \cos(\phi) \sin(\phi)}{6} = 0 \end{aligned} \quad (12)$$

#### 3.3

Equation of motion for the case that the rotation around the vertical bar is not constant:

$$\begin{aligned} & \frac{L^2 \ddot{\theta} \left( 16M + 16m + 12M \cos(\beta) + 12m \cos(\beta) + 2M \cos(\beta)^2 + 4m \cos(\beta)^2 - 2M \cos(\beta)^2 \cos(\phi)^2 \right)}{12} + \\ & \frac{L^2 \ddot{\theta} (12M \cos(\phi) \sin(\beta) + 6M \cos(\beta) \cos(\phi) \sin(\beta))}{12} \end{aligned} \quad (13)$$

#### 3.4

After matlab integration of the work per area we get for the work

$W =$

$$\begin{aligned} & \frac{L^4 \left( c \left( \dot{\theta} + \dot{\phi} \sin(\beta) \right)^2 \left( \cos(\phi)^2 - 1 \right) - c \left( \dot{\phi} \cos(\phi) + \dot{\theta} \cos(\phi) \sin(\beta) \right)^2 + c \dot{\phi}^2 \cos(\beta)^2 \left( \cos(\phi)^2 - 1 \right) \right)}{3} \\ & - L \left( Lc \left( L\dot{\theta} + L\dot{\theta} \cos(\beta) \right)^2 + \frac{L^3 c \dot{\theta}^2 \cos(\beta)^2}{12} \right) - L^3 c \left( L\dot{\theta} + L\dot{\theta} \cos(\beta) \right) \left( \dot{\phi} \cos(\phi) + \dot{\theta} \cos(\phi) \sin(\beta) \right) \end{aligned} \quad (14)$$

Which results in the equations of motion:

$$\begin{aligned} & \frac{L^2 M \ddot{\phi}}{3} + \frac{2L^4 c (\dot{\phi} + \dot{\theta} \sin(\beta))}{3} + \frac{L^2 m \ddot{\phi}}{3} - \frac{L^2 m \ddot{\phi} \cos(\beta)^2}{3} + \frac{LMg \cos(\beta) \sin(\phi)}{2} + \\ & L^4 c \dot{\theta} \cos(\phi) (\cos(\beta) + 1) + \frac{L^2 M \dot{\theta}^2 \sin(\beta) \sin(\phi)}{2} - \frac{L^2 M \dot{\theta}^2 \cos(\beta)^2 \cos(\phi) \sin(\phi)}{6} + \\ & \frac{L^2 M \dot{\theta}^2 \cos(\beta) \sin(\beta) \sin(\phi)}{4} \end{aligned} \quad \text{for } \phi \quad (15)$$

And

$$\begin{aligned} & \frac{cL^4}{6} \left( 16\dot{\theta} + 24\dot{\theta} \cos(\beta) + 6\dot{\phi} \cos(\phi) + 4\dot{\phi} \sin(\beta) + 13\dot{\theta} \cos(\beta)^2 - 4\dot{\theta} \cos(\beta)^2 \cos(\phi)^2 + 6\dot{\phi} \cos(\beta) \cos(\phi) + 12\dot{\theta} \cos(\beta) \cos(\phi) \sin(\beta) \right) + \\ & \frac{\ddot{\theta}}{12} \left( 16M + 16m + 12M \cos(\beta) + 12m \cos(\beta) + 2M \cos(\beta)^2 + 4m \cos(\beta)^2 - 2M \cos(\beta)^2 \cos(\phi)^2 + \right. \\ & \left. 12M \cos(\phi) \sin(\beta) + 6M \cos(\beta) \cos(\phi) \sin(\beta) \right) L^2 \end{aligned} \quad \text{for } \theta \quad (16)$$

### 3.5 Plugging in the values:

Let  $L = 0.25[m]$ ,  $\beta = \frac{\pi}{6}$ ,  $M = 0.5[kg]$ ,  $m = 0.2[kg]$ ,  $g = 9.81[\frac{m}{s^2}]$  and  $c = 0.1[\frac{Ns}{m^3}]$ .

The equations become:

$$\begin{aligned} & \frac{\dot{\phi}}{3840} + \frac{11\ddot{\phi}}{960} + \frac{\dot{\theta}}{7680} + \frac{\dot{\theta}^2 \sin(\phi)}{128} + \frac{981\sqrt{3} \sin(\phi)}{3200} + \frac{\sqrt{3}\dot{\theta}^2 \sin(\phi)}{512} + \frac{\dot{\theta} \cos(\phi) \left( \frac{\sqrt{3}}{2} + 1 \right)}{2560} \\ & - \frac{\dot{\theta}^2 \cos(\phi) \sin(\phi)}{256} \end{aligned} \quad \text{for } \phi \quad (17)$$

And

$$\begin{aligned} & \frac{\dot{\phi}}{7680} + \frac{103\dot{\theta}}{61440} - \frac{\dot{\theta} \cos(\phi)^2}{5120} + \frac{\sqrt{3}\dot{\theta}}{1280} + \\ & \frac{\ddot{\theta}}{192} \left( 3 \cos(\phi) - \frac{3 \cos(\phi)^2}{4} + \frac{21\sqrt{3}}{5} + \frac{3\sqrt{3} \cos(\phi)}{4} + \frac{251}{20} \right) + \\ & \frac{\dot{\phi} \cos(\phi)}{2560} + \frac{\dot{\theta} \cos(\phi)}{2560} + \frac{\sqrt{3}\dot{\phi} \cos(\phi)}{5120} + \frac{\sqrt{3}\dot{\theta} \cos(\phi)}{5120} \end{aligned} \quad \text{for } \theta \quad (18)$$

### 3.6 Equilibrium State

Reminder- The potential energy looks as follows:

$$\frac{LMg (2 \sin (\beta) - \cos (\beta) \cos (\phi))}{2} \quad (19)$$

**Case 1:  $\Omega$  fixed:**

The derivative of the potential energy w.r.t.  $\theta$  is always 0.

$$\frac{\partial V}{\partial \theta} = 0 \quad (20)$$

The derivative of the potential energy w.r.t.  $\phi$  not however:

$$\frac{\partial V}{\partial \phi} = \frac{LMg \cos (\beta) \sin (\phi)}{2} \quad (21)$$

Equation 12 is only 0 for  $\phi = k * \pi$  which makes sense as in this configuration the square has a momentary velocity in horizontal direction which doesn't change the altitude of any body.

**Case 2:  $\dot{\theta}$  can change:**

As V is independent of  $\theta$  altogether the result is the same for this case.

**The equation of motions are after plugging in equilibrium state:**

$$\begin{aligned} \phi : \\ 0 &= 0 \end{aligned} \quad (22)$$

$$\begin{aligned} \theta : \\ \frac{L^2 \ddot{\theta}}{12} (16M + 16m + 12M \cos (\beta) + 12M \sin (\beta) + 12m \cos (\beta) + \\ 4m \cos (\beta)^2 + 6M \cos (\beta) \sin (\beta)) &= 0 \end{aligned} \quad (23)$$

As the coefficient of  $\ddot{\theta}$  is constant, the angular acceleration of the vertical shaft has to be 0. This results in a constant angular velocity which recovers the case of a constant  $\Omega$ .

## 4 Assignment 4