ETH zürich

Segment: S1.1 (Lecture 1, part 1)

Section: Kinematics

Topic: **Fundamentals (Vectors, Positions, Parameterization)**

Script Sec: 1, 2.1, 2.2

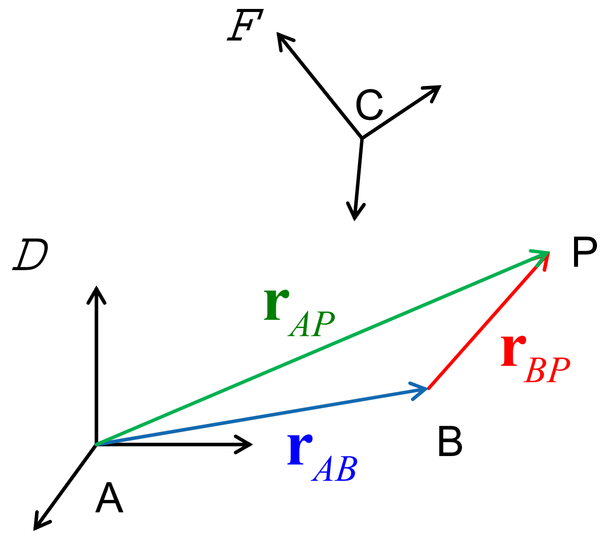
151-0851-00 V: Robot Dynamics

Marco Hutter

Recapitulation: Vectors, Position, and Vector Calculus

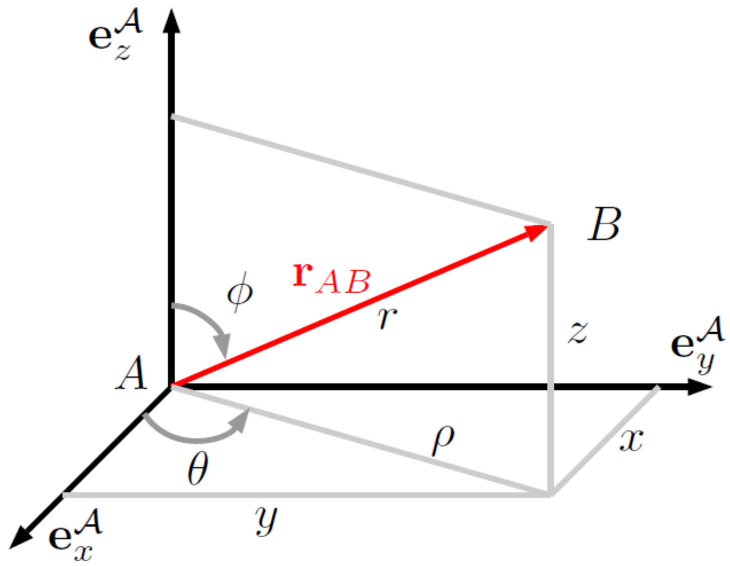
- Builds upon notation of other dynamics classes at ETH and IEEE standards

- Vector: $\mathbf{r}$ (often also $\vec{r}$)
- Vector from point B to P: $\mathbf{r}_{BP}$
- Reference coordinate system $\mathcal{D}$ (calligraphic)
  $\left(\mathbf{e}_x^{\mathcal{A}}, \mathbf{e}_y^{\mathcal{A}}, \mathbf{e}_z^{\mathcal{A}}\right)$:= orthonormal basis of R^3
- Numerical expression of a vector: ${}_{\mathcal{D}}\mathbf{r}_{BP}$
- Addition of vectors: $\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP}$
- Use the same reference frame: ${}_{\mathcal{D}}\mathbf{r}_{AP} = {}_{\mathcal{D}}\mathbf{r}_{AB} + {}_{\mathcal{D}}\mathbf{r}_{BP}$
  ${}_{\mathcal{D}}\mathbf{r}_{AP} \neq {}_{\mathcal{D}}\mathbf{r}_{AB} + {}_{\mathcal{F}}\mathbf{r}_{BP}$



Parameterization of Positions

- Cartesian coordinates $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = x\mathbf{e}_x^{\mathcal{A}} + y\mathbf{e}_y^{\mathcal{A}} + z\mathbf{e}_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$
- Spherical coordinates $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}$

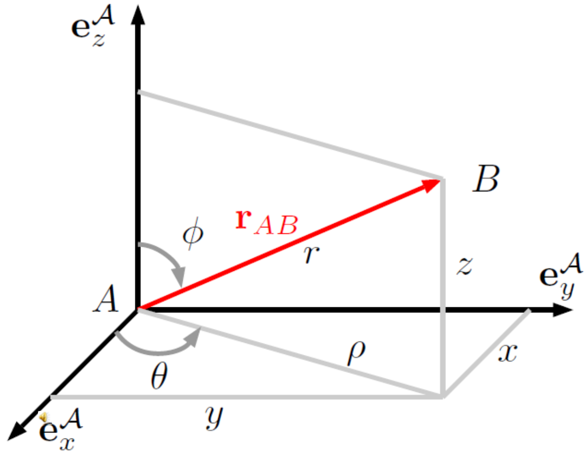


$$\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP} \qquad \chi_{AP} = \chi_{AB} + \chi_{BP}$$

- Only correct for Cartesian coordinates
- NEVER do this for other representations (requires special algebra!!)
  => *we will encounter similar problems for rotations*

Quiz

Parameterization of Positions
Example



$$_D\mathbf{r}_{AP} = {}_D\mathbf{r}_{AB} + {}_D\mathbf{r}_{BP}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} : \begin{cases}\chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases} \qquad \begin{pmatrix}0\\1\\1\end{pmatrix} : \begin{cases}\chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases} \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} : \begin{cases}\chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases}$$

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Segment: S1.2 (Lecture 1, part 2)

Section: Kinematics

Topic: **Linear Velocity**

Script Sec: 2.3

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Differentiation of Representation ⇔ Linear Velocity

- The velocity of point P relative to point B, expressed in frame D is: ${}_D\dot{\mathbf{r}}_{BP}$

- Question: What is the relationship between the velocity $\dot{\mathbf{r}}$ and the time derivative of the representation $\dot{\chi}$

$$\mathbf{r} = \mathbf{r}(\chi)$$

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial \chi}\dot{\chi}$$

$$\Rightarrow$$

$$\dot{\mathbf{r}} = \mathbf{E}_P(\chi)\cdot\dot{\chi}$$

$$\dot{\chi} = \mathbf{E}_P^{-1}(\chi)\cdot\dot{\mathbf{r}}$$

Differentiation of Representation ⇔ Linear Velocity

- Cartesian coordinates: $\mathbf{E}_{Pc}\left(\chi_{Pc}\right) = \mathbf{E}_{Pc}^{-1}\left(\chi_{Pc}\right) = \mathbb{I}$

- Cylindrical coordinates: $_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix} \qquad \dot{\mathbf{r}}\left(\chi_{Pz}\right) = \begin{pmatrix} \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta \\ \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta \\ \dot{z} \end{pmatrix}$

$$\mathbf{E}_{Pz}\left(\chi_{Pz}\right) = \frac{\partial \mathbf{r}\left(\chi_{Pz}\right)}{\partial \chi_{Pz}} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\chi}_{Pz} = \begin{pmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta \\ -\dot{x}\sin\theta/\rho + \dot{y}\cos\theta/\rho \\ \dot{z} \end{pmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}_{Pz}^{-1}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

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Segment: S1.3 (Lecture 1, part 3)
Section: Kinematics
Topic: **Introduction to Rotations, Transformations**
Script Sec: 2.4.1-2.4.4

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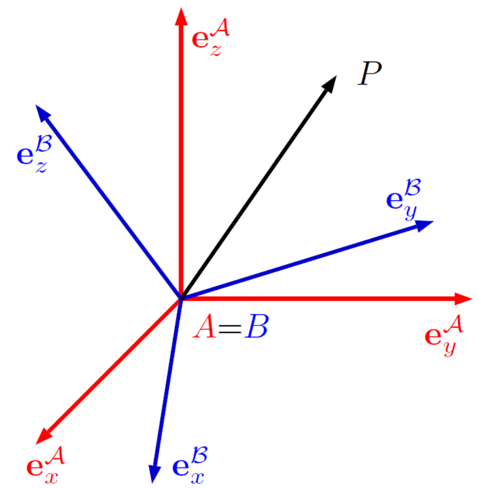
Rotations

- Position of P with respect to A expressed in `D` : $${}_{\mathcal{A}}\mathbf{r}_{AP} = \begin{pmatrix} {}_{\mathcal{A}}r_{AP_x} \\ {}_{\mathcal{A}}r_{AP_y} \\ {}_{\mathcal{A}}r_{AP_z} \end{pmatrix}$$
- Position of P with respect to A expressed in `E`: $${}_{\mathcal{B}}\mathbf{r}_{AP} = \begin{pmatrix} {}_{\mathcal{B}}r_{AP_x} \\ {}_{\mathcal{B}}r_{AP_y} \\ {}_{\mathcal{B}}r_{AP_z} \end{pmatrix}$$
- Write the unit vectors of `E` expressed in `D` as matrix: $\left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right]$
- => $${}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_x} + {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_y} + {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_z}$$
$$\begin{aligned} {}_{\mathcal{A}}\mathbf{r}_{AP} &= \left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right] \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} \\ &= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}. \end{aligned}$$



Rotation Matrix

- The rotation matrix transforms vectors expressed in E to D:

$$\mathbf{C}_{\mathcal{AB}} = \left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right] \qquad\qquad {}_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}$$

- The matrix is orthogonal: $\mathbf{C}_{\mathcal{BA}} = \mathbf{C}_{\mathcal{AB}}^{-1} = \mathbf{C}_{\mathcal{AB}}^{T}$

- Belongs to special orthonormal group SO(3) (and not R^3)
  - This causes difficulties and requires special algebra

- Consecutive rotations: $\begin{aligned} {}_{\mathcal{A}}\mathbf{u} &= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}. \\ {}_{\mathcal{B}}\mathbf{u} &= \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{u}. \end{aligned}$ $\Rightarrow$ $\begin{aligned} {}_{\mathcal{A}}\mathbf{u} &= \mathbf{C}_{\mathcal{AB}} \cdot (\mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{u}) \\ &= \mathbf{C}_{\mathcal{AC}} \cdot {}_{\mathcal{C}}\mathbf{u}. \end{aligned}$ $\boxed{\mathbf{C}_{\mathcal{AC}} = \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}}}$

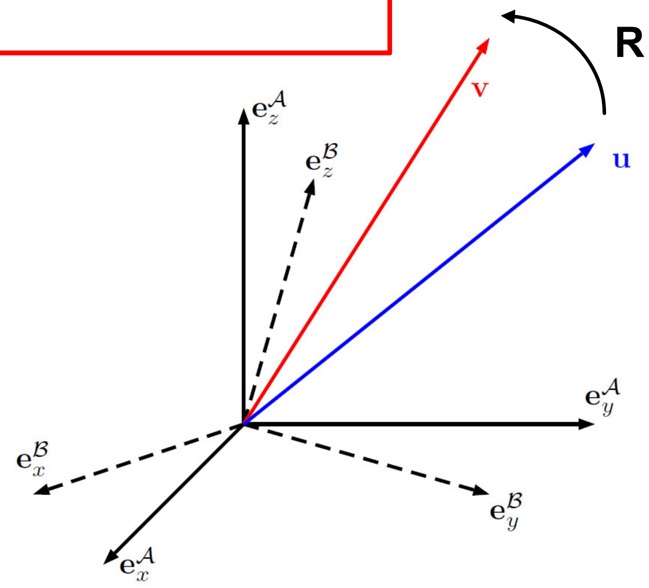
Passive and Active Rotation

- Passive rotation = mapping of the same vector from frame E to D

$$_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}$$

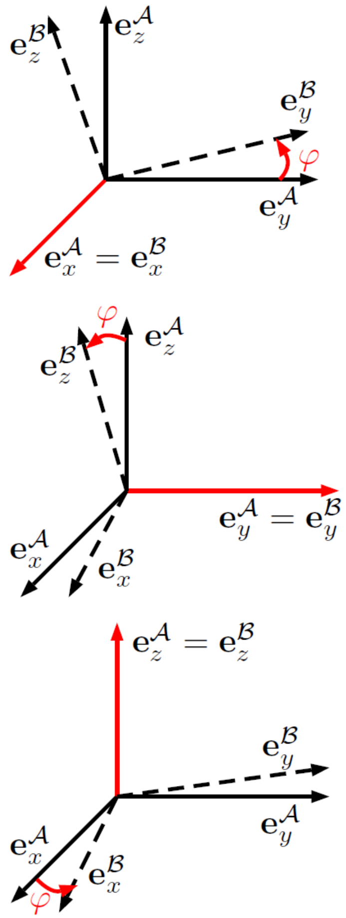
- Active rotation = rotating a vector in the same frame

$$_{\mathcal{A}}\mathbf{v} = \mathbf{R} \cdot_{\mathcal{A}} \mathbf{u}$$



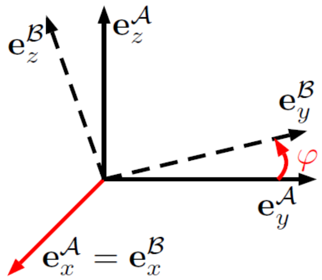
Elementary Rotation

- Find the elementary rotation matrix s.t $\ _{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {_{\mathcal{B}}\mathbf{u}}$

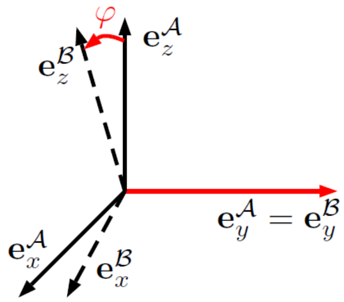


Elementary Rotation

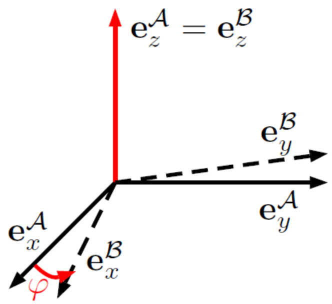
- Find the elementary rotation matrix s.t $\quad {}_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation
Combined Translation and Rotation

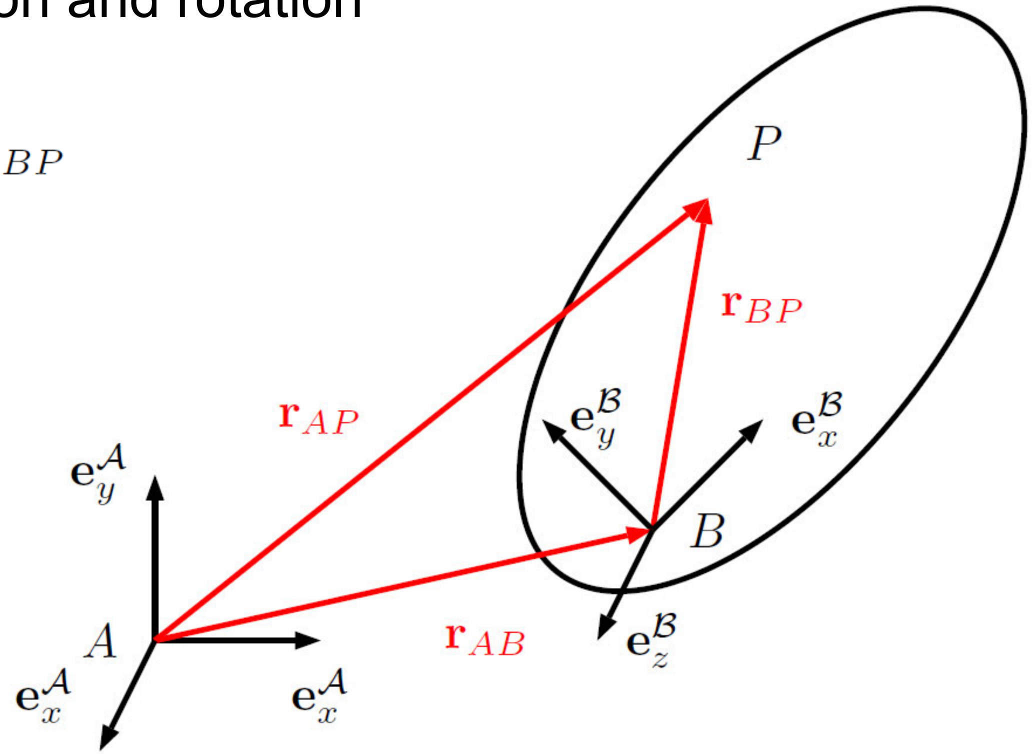
- Homogeneous transformation = translation and rotation

$$\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP}$$

$$_{\mathcal{A}}\mathbf{r}_{AP} = {_{\mathcal{A}}\mathbf{r}_{AB}} + {_{\mathcal{A}}\mathbf{r}_{BP}} = {_{\mathcal{A}}\mathbf{r}_{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}}$$

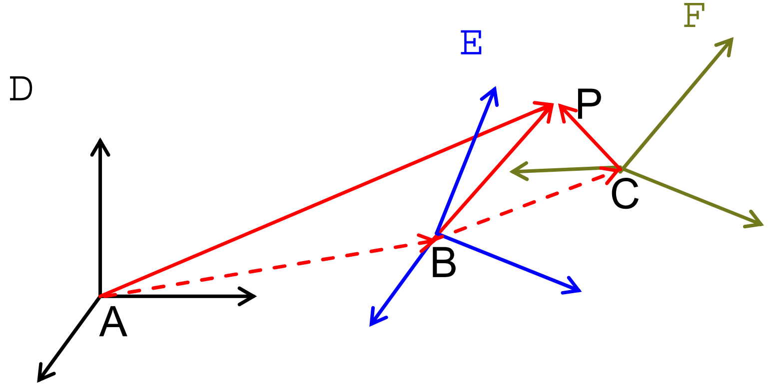
$$\begin{pmatrix} _{\mathcal{A}}\mathbf{r}_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & _{\mathcal{A}}\mathbf{r}_{AB} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}} \begin{pmatrix} _{\mathcal{B}}\mathbf{r}_{BP} \\ 1 \end{pmatrix}$$

- Inverse $\mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^T & \overbrace{-\mathbf{C}_{\mathcal{AB}}^T {_{\mathcal{A}}\mathbf{r}_{AB}}}^{_{\mathcal{B}}\mathbf{r}_{BA}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$



Homogeneous Transformations
Consecutive Transformation

$$\left.\begin{aligned} {}_{\mathrm{D}}\vec{r}_{AP} &= \mathbf{T}_{\mathrm{DE}} \cdot {}_{\mathrm{E}}\vec{r}_{BP} \\ {}_{\mathrm{E}}\vec{r}_{BP} &= \mathbf{T}_{\mathrm{EF}} \cdot {}_{\mathrm{F}}\vec{r}_{CP} \end{aligned}\right\} \mathbf{T}_{\mathrm{DF}} = \mathbf{T}_{\mathrm{DE}} \cdot \mathbf{T}_{\mathrm{EF}}$$



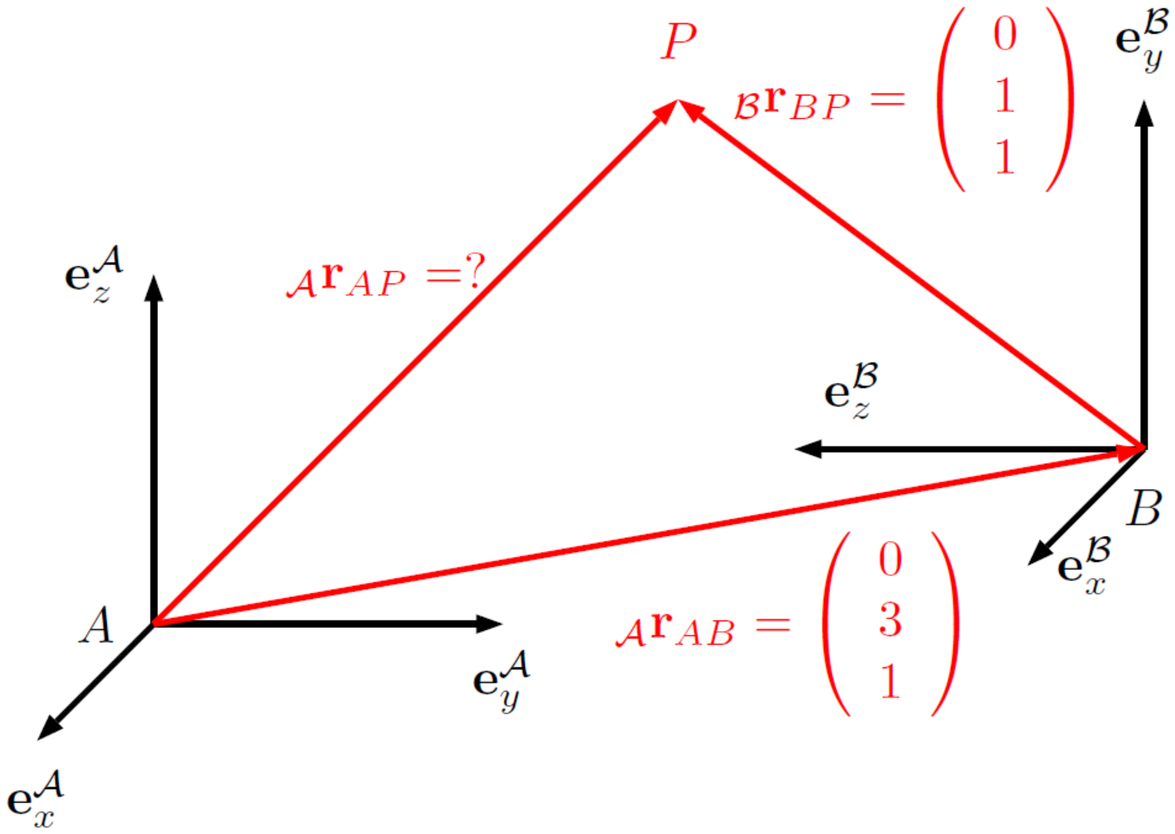
- This allows to transform an arbitrary vector between different reference frames (classical example: mapping of features in camera frame to world frame)

Quiz

Homogeneous Transformation

Simple Example

- Find the position vector ${}_{\mathcal{A}}\mathbf{r}_{AP}$
  - Find the transformation matrix
  - Find the vector



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Segment: S1.4 (Lecture 1, part 4)
Section: Kinematics
Topic: **Angular Velocities**
Script Sec: 2.6

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Angular Velocity

- Angular velocity ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$ describes the relative rotational velocity of E wrt. D expressed in frame D
- The relative velocity of D wrt. E is: $\boldsymbol{\omega}_{\mathrm{ED}} = -\boldsymbol{\omega}_{\mathrm{DE}}$
- Given the rotation matrix $\mathbf{C}_{\mathcal{AB}}(t)$ between two frames, the angular velocity is

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \dot{\mathbf{C}}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{AB}}^T \qquad [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \qquad {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Transformation of angular velocity: ${}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{C}_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}$

- Addition of relative velocities: ${}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AC}} = {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{BC}}$

Quiz

Angular Velocity
Simple Example

- Given the rotation matrix $\mathbf{C}_{\mathcal{A}\mathcal{B}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$

  determine ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}$