ESERCIZIO 1.1

i) $\{x\in\mathbb{R}^3 : d(x,A)=3d(x,B)\}$ dove $A=(1,1/2,1)$ e $B=(1,-2,1)$

- $d(x,A)^2=9d(x,B)^2$
- $(x_1-1)^2+(x_2-1/2)^2+(x_3-1)^2=9(x_1-1)^2+9(x_2+2)^2+9(x_3-1)^2$
- $8(x_1^2-2x_1+1)+9(x_2^2+4x_2+4)+8(x_3^2-2x_3+1)-x_2^2+x_2-1/4=0$
- $8x_1^2+8x_2^2+8x_3^2-16x_1+37x_2-16x_3+199/4=0$
- $x_1^2+x_2^2+x_3^2-2x_1+37/8\,x_2-2x_3+199/32=0$

$d(x,A)=\sqrt{(x_1-a_1)^2+\dots+(x_n-a_n)^2}$

$d(x,A)=\alpha d(x,B) \Rightarrow d(x,A)^2=\alpha^2(x,B)^2$

$(x_1-a_1)^2+\dots+(x_n-a_n)^2=(x_1-b_1)^2+\dots$

ii) $\{x\in\mathbb{R}^3: d(x,A)=2d(x,\pi)\}$ dove $A=(0,1,1)$ e $\pi: x-y+1=0$

$d(x,\pi)=\frac{|ax_1+\dots+zx_n+d|}{\sqrt{a^2+\dots+z^2}}$

- $d(x,A)^2=4d(x,\pi)^2$
- $(x_1-0)^2+(x_2-1)^2+(x_3-1)^2=4\left(\frac{|x_1-x_2+1|}{\sqrt{1^2+(-1)^2}}\right)^2$
- $x_1^2+x_2^2-2x_2+1+x_3^2-2x_3+1=4\frac{(x_1-x_2+1)^2}{2}$
- $x_1^2+x_2^2+x_3^2-2x_2-2x_3+2=2(x_1^2-x_1x_2+x_1-x_1x_2+x_2^2-x_2+x_1-x_2+1)$
- $x_1^2+x_2^2+x_3^2-2x_2-2x_3+\cancel{2}=2x_1^2+2x_2^2-4x_1x_2+4x_1-4x_2+\cancel{2}$
- $x_1^2+x_2^2-x_3^2-4x_1x_2+4x_1-2x_2+2x_3=0$

iii) $\{x\in\mathbb{R}^3: 2d(x,\pi)=2d(x,r)\}$ dove $\pi: x-y=0$ e $r:(1,0,0)+t(0,1,0)$

- $4d(x,\pi)^2=4d(x,r)^2$
- $4\left(\frac{|x_1-x_2|}{\sqrt{1^2+(-1)^2}}\right)^2=4(x_1^2+x_3^2-2x_1+1)$
- $2(x_1-x_2)^2=4x_1^2+4x_3^2-8x_1+4$
- $2x_1^2+2x_2^2-4x_1x_2=4x_1^2+4x_3^2-8x_1+4$
- $2x_1^2-2x_2^2+4x_3^2+4x_1x_2-8x_1+4=0$
- $x_1^2-x_2^2+2x_3^2+2x_1x_2-4x_1+2=0$

$d(x,r)$:

- $Q=(1,0,0)$ e $\vec v=(0,1,0)$
- $\vec{PQ}=(x_1-1,x_2-0,x_3-0)=(x_1-1,x_2,x_3)$
- $\vec{PQ}\cdot\vec v=\begin{vmatrix}\hat i&\hat j&\hat z\\ x_1-1&x_2&x_3\\ 0&1&0\end{vmatrix}\to$ determinante

  $=\hat i(-x_3)-\hat j(0)+\hat z(x_1-1)\to(-x_3,0,x_1-1)$
- $\|\vec{PQ}\cdot\vec v\|=\sqrt{(x_3)^2+(x_1-1)^2}$
- $\|\vec v\|=\sqrt{(1)^2}=1$
- $d(x,r)^2=\left(\frac{\|\vec{PQ}\cdot\vec v\|}{\|\vec v\|}\right)^2=x_3^2+x_1^2-2x_1+1$

ESERCIZIO 1.2

i) $\{x\in\mathbb{R}^2: 3x^2+2xy+3y^2+2x+2y+1=0\}$

- $A=\begin{pmatrix}3&1\\1&3\end{pmatrix}\to A\underline{v}=-\underline{b}\to\begin{pmatrix}3&1\\1&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}\to\begin{pmatrix}3x+y\\x+3y\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}\to\underline{v}_0=(1/4,1/4)$
- Metrica:
  - $\chi_A(t)=\det\begin{pmatrix}3-t&1\\1&3-t\end{pmatrix}=0\to(3-t)^2-1=0\to(3-t-1)(3-t+1)=0\to(2-t)(4-t)=0$ $\quad\lambda_1=2,\ \lambda_2=4$
  - $2z_1^2+4z_2^2+c'=0\to c'=c+\underline{b}^T\underline{v}_0=1+(-1,-1)\begin{pmatrix}1/4\\1/4\end{pmatrix}=1-2/4=1/2\to2z_1^2+4z_2^2+1/2=0\to$ ellisse
- Affine
  - segnatura $(2,0,0)$
  - dividere per $|c'|\to z_1^2+2z_2^2+1=0\to w_1^2+w_2^2+1=0\to$ circonferenza

ii) $\{x \in \mathbb{R}^2 : x^2+6xy+y^2-3=0\}$

- $A=\begin{pmatrix}1&3\\3&1\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}1&3\\3&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \to \begin{cases} y+3x=0\\ 3x+y=0\end{cases} \to \underline{V}_0=(0,0)$
- Metrica:
  - $\chi_A(t)=\det\begin{pmatrix}1-t&3\\3&1-t\end{pmatrix}=0 \to (1-t)^2-3^2=0 \to (1-t-3)(1-t+3)=0 \to (-2-t)(4-t)=0 \quad \lambda_1=-2,\ \lambda_2=2$
  - $-2z_1^2+4z_2^2+c'=0 \to c'=-3+(0,0)\begin{pmatrix}0\\0\end{pmatrix}=-3 \to -2z_1^2+4z_2^2-3=0 \to$ iperbole
- Affine
  - segnatura $(1,0,1)$
  - $-2/3\, z_1^2+2z_2^2-1=0 \quad \to w_1^2-w_2^2-1=0 \to$ iperbole

iii) $\{x\in\mathbb{R}^3 : 3x^2+2y^2+2xz+3z^2-4=0\}$

- $A=\begin{pmatrix}3&1&0\\1&2&0\\0&0&3\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}3&1&0\\1&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \to \begin{cases}3x+y=0\\x+2y=0\\3z=0\end{cases} \to V_0=(0,0,0)$
- Metrica:
  - $\chi_A(t)=\det\begin{pmatrix}3-t&1&0\\1&2-t&0\\0&0&3-t\end{pmatrix}=0 \to (3-t)(2-t)(3-t)-(3-t)=0 \to (3-t)(6-3t-2t+t^2-1)=0$
    $\to (3-t)(t^2-5t+5)=0 \quad \lambda_1=\frac{5-\sqrt5}{2},\ \lambda_2=\frac{5+\sqrt5}{2},\ \lambda_3=3$
  - $\frac{5-\sqrt5}{2}z_1^2+\frac{5+\sqrt5}{2}z_2^2+3z_3^2-4=0 \to \quad c'=-4+(0,0,0)\begin{pmatrix}0\\0\\0\end{pmatrix}=-4 \to$ ellissoide
- Affine:
  - segnatura: $(3,0,0)$
  - $\frac{5-\sqrt5}{8}z_1^2+\frac{5+\sqrt5}{8}z_2^2+3/4\,z_3^2-1=0 \to w_1^2+w_2^2+w_3^2-1=0 \to$ sfera

iv) $\{x\in\mathbb{R}^3 : x^2+2xy+y^2-z^2+2x+2y+2z=0\}$

- $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&-1\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}1&1&0\\1&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\-1\\-1\end{pmatrix} \to \begin{cases}x+y=-1\\x+y=-1\\z=-1\end{cases}$ ha infiniti centri es: $\underline{V}_0=(0,-1,-1)$
- Metrica:
  - $\chi_A(t)=\det\begin{pmatrix}1-t&1&0\\1&1-t&0\\0&0&-1-t\end{pmatrix}=0 \to (1-t)(1-t)(-1-t)-(-1-t)=0 \to \lambda_1=-1,\ \lambda_2=0,\ \lambda_3=2$
  - $-z_1^2+2z_3^2+2=0 \to c'=0+(-1,-1,-1)\begin{pmatrix}0\\-1\\-1\end{pmatrix}=2 \to$ cilindro iperbolico
- Affine:
  - segnatura $(1,1,1)$
  - $w_1^2-w_2^2+1=0 \to$ cilindro iperbolico

ESERCIZIO 1.3

i) $Q_1:\{x\in\mathbb{R}^2 : x_1^2+2x_1x_2+x_2^2+x_1+2x_2=0\}$

- $A=\begin{pmatrix}1&1\\1&1\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/2\\1\end{pmatrix} \to \begin{cases}x+y=1/2\\x+y=1\end{cases}$ $\nexists$ centro
- $\chi_A(t)=\det\begin{pmatrix}1-t&1\\1&1-t\end{pmatrix}=0 \to (1-t)^2-1=0 \to (1-t-1)(1-t+1)=0 \to (-t)(2-t)=0 \to \lambda_1=0,\ \lambda_2=2$
- segnatura : $(1,0,1)$
- Affine
  - $t^2+2t=0 \to$ parabola

- Metrica
  - autospazi: $\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=0 \to v_1+v_2=0 \quad v_1=\begin{pmatrix}1\\-1\end{pmatrix}$

    $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=0 \to v_1-v_2=0 \quad v_2=\begin{pmatrix}1\\1\end{pmatrix}$
  - $t_1=x_1-x_2,\ t_2=x_1+x_2 \to \lambda_1 t_1^2+\lambda_2 t_2^2=2t_2^2 \to 2t_2^2=0$ retta ?
- Asse: $A^2\underline{v}=A\underline{b} \to \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1/2\\1\end{pmatrix} \to \begin{pmatrix}2&2\\2&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3/2\\3/2\end{pmatrix} \to 2x+2y=3/2$

  $\to x+y=3/4$

ii) $Q_2:\{x\in\mathbb{R}^3: x_1^2-2x_1x_2+x_2^2-4x_1-4x_2-4x_3+4=0$
- $A=\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\-2\\-2\end{pmatrix} \to \begin{cases}x-y=-2\\y-x=-2\\0=-2\end{cases}$ ∄ centro
- $\chi_A(t)=\det\begin{pmatrix}1-t&-1&0\\-1&1-t&0\\0&0&-t\end{pmatrix}=0 \to (1-t)(1-t)(-t)-(-t)=0 \to \lambda_1=0\ \ g_1=2,\ \lambda_2=2$
- segnatura: $(1,0,2)$
- Affine:
  - $t^2+2t=0 \to$ parabola
- Metrica:
  - autospazi: $\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

    $\begin{pmatrix}-1&-1&0\\-1&-1&0\\0&0&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$
  - $t_1=x_1+x_2,\ t_2=x_3,\ t_3=x_1-x_2 \quad \to 2t_2^2$?
- Asse $A^2\underline{v}=A\underline{b} \to \begin{pmatrix}2&-2&0\\-2&2&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \to x-y=0$

ESERCIZIO 1.4

$Q=\{P\in\mathbb{R}^3: d(P,P_0)=d(P,\pi)\}$ dove $P_0=(1,0,1)$ e $\pi: x-y=0$

- $d(P,P_0)^2=d(P,\pi)^2 \to (x_1-1)^2+(x_2-0)^2+(x_3-1)^2=\left(\frac{|x_1-x_2|}{\sqrt{1^2+(-1)^2}}\right)^2 \to x_1^2-2x_1+1+x_2^2+x_3^2-2x_3+1=\frac{(x_1-x_2)^2}{2}$

  $\to 2x_1^2+2x_2^2+2x_3^2-4x_1-4x_3+4=x_1^2-2x_1x_2+x_2^2 \to x_1^2+x_2^2+2x_3^2+2x_1x_2-4x_1-4x_3+4=0$
- $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}x+y\\x+y\\2z\end{pmatrix}=\begin{pmatrix}-2\\0\\-2\end{pmatrix} \to \begin{cases}x+y=-2\\x+y=0\\2z=-2\end{cases}$ ∄ centro
- $\chi_A(t)=\det\begin{pmatrix}1-t&1&0\\1&1-t&0\\0&0&2-t\end{pmatrix}=(1-t)(1-t)(2-t)-(2-t)=0 \to (2-t)(\cancel{1}-t-t+t^2-\cancel{1})=0 \to (2-t)(t(t-2))=0$

  $\to \lambda_1=2,\ \lambda_2=0,\ \lambda_3=2$
- segnatura: $(2,0,1)$
- Affine: $t_1^2+t_2^2+2t_3=0 \to$ paraboloide ellittico
- Metrica:
- e' di rotazione $\alpha=\beta=1$ e non ci sono altri termini

ESERCIZIO 1.5

$Q: 3x_1^2+3x_2^2+2x_3^2-2x_1x_2+8x_1+3=0$

- $A=\begin{pmatrix}3&-1&0\\-1&3&0\\0&0&2\end{pmatrix} \to A\underline{v}=-\underline{b} \to \begin{pmatrix}3x-y\\-x+3y\\2z\end{pmatrix}=\begin{pmatrix}-4\\0\\0\end{pmatrix} \to \begin{cases}3x-y=-4\\-x+3y=0\\2z=0\end{cases} \to \underline{V_0}=(-3/2,-1/2,0)$
- $\chi_A(t)=\det\begin{pmatrix}3-t&-1&0\\-1&3-t&0\\0&0&2-t\end{pmatrix}=(3-t)(3-t)(2-t)-(2-t)=0 \to \lambda_1=2,\ \lambda_2=2,\ \lambda_3=4$
- Metrica: $2z_1^2+2z_2^2+4z_3^2-3=0 \longrightarrow$ ellissoide $\quad c'=3+\underline{b}^T\underline{V_0}=3+(4,0,0)\begin{pmatrix}-3/2\\-1/2\\0\end{pmatrix}=-3$

- Simmetria: $A^2\underline{v} = A\underline{b}$: $\begin{pmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 10x_1 - 6x_2 = 12 \\ -6x_1 + 10x_2 = -4 \\ 4x_3 = 0 \end{cases} \rightarrow \underline{v}_1 = (3/2, 1/2, 0)$
- Rotazione: $\alpha = \beta = 2$ non ci sono altri termini, si!
- $x_3 = 0 \rightarrow 3x_1^2 + 3x_2^2 - 2x_1x_2 + 8x_1 + 3 = 0$
  - $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \rightarrow A\underline{v} = -\underline{b} \rightarrow \begin{cases} 3x_1 - x_2 = -4 \\ -x_1 + 3x_2 = 0 \end{cases}$ $\underline{v}_0 = (-3/2, -1/2, 0)$
  - $\chi_A(t) = \det\begin{pmatrix} 3-t & -1 \\ -1 & 3-t \end{pmatrix} = (3-t)^2 - 1 = (3-t-1)(3-t+1) = (2-t)(4-t) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 4$
  - segnatura: $(2,0,0) \rightarrow$ affine: $x_1^2 + x_2^2 + 1 = 0 \rightarrow$ circonferenza