Optimal Internet Auctions with Costly Communication

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ABSTRACT

Iterative auctions can reach an outcome before all bidders have revealed all their preference information. This can decrease costs associated with communication, deliberation, and loss of privacy. We propose an explicit cost model that is inspired by single-item Internet auctions, such as those taking place on auction sites (eBay) or via informal communication (craigslist, mailing lists). A nonzero bid comes at a cost to both the seller and the bidder, and the seller can send broadcast queries at a cost. Under this model, we study auctions that maximize the seller's profit (revenue minus seller cost). We consider multi-round Vickrey auctions (MVAs), in which the seller runs multiple Vickrey auctions, with decreasing reserve prices. We prove that restricting attention to this class is without loss of optimality, show how to compute an optimal MVA, and compare experimentally to some other natural MVAs. Among our findings are that (1) the expected total cost is bounded by a constant for arbitrarily many bidders, and (2) the optimal MVA and profit remain the same as long as the total bid cost is fixed, regardless of which portion of it belongs to the seller and which to the buyer.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multi-agent systems*; J.4 [**Social and Behavioral Sciences**]: Economics

General Terms

Algorithms, Economics, Theory

Keywords

Auctions, mechanism design, communication costs

1. INTRODUCTION

Auctions constitute a favored method for allocating scarce resources in multiagent systems. However, communication requirements can pose a bottleneck. Motivated by the revelation principle, much of the theory of mechanism design considers *direct-revelation* mechanisms, in which each agent declares its entire valuation function to the auctioneer. The corresponding overhead, not only in terms of communication per se but also in terms of the agent having to completely *determine* its valuation function, can be prohibitive. All of this is well understood (for further discussion, see, e.g., [2]), and a significant amount of research has been devoted to the design of *iterative* auction mechanisms (e.g., [8]) and (roughly equivalently) auctions with explicit *elicitation* of agents' valuations (e.g., [9]). Such auctions aim to avoid the communication of unnecessary information. For example, in a Vickrey auction, if it is known that bidders 1 and 2 have valuations above $100 and bidder 3 has one below $100, then there is no need to query bidder 3 any further.

How do we evaluate how effective a particular iterative auction mechanism is in reducing communication? Perhaps the simplest measure is the number of bits communicated (see, e.g., []). Within a particular query model, it may also make sense to minimize the total number of queries (see, e.g., []). However, in this paper, we argue that such existing models fail to capture important aspects of the cost of communication in certain types of Internet auctions. In particular, in such auctions often the most costly communication that takes place is the *first* time that a bidder gives a positive response, indicating having a nonzero value for an item. This can be the case for several reasons. One possibility is that the auction website at this point may insist that the bidder provides payment (say, credit card) information or places money in escrow. Otherwise, a malicious user may steer the auction in a particular direction and in the end refuse to pay. Alternatively, in other settings (such as an item having been posted for sale on craigslist or a similar list), at this point the bidder may wish to set up an appointment with the seller to check the item. In both cases, these actions come at costs for both the potential buyer and the seller, in terms of effort and time, loss of privacy, and various risks. While participation costs have been studied before in auctions [12, 13], a distinguishing feature of our model is that a bidder can observe the proceedings of the auction at no cost, until the bidder decides to actively participate by at some indicating a nonzero valuation and thereby changing the course of the auction. This appears to us to be a more natural model of Internet (or other highly anonymous) auctions of the type discussed above.

The rest of this paper is organized as follows. In Section 2, we provide a formal model of elicitation cost motivated by the observations just discussed. In Section 3, we study the special case in which efficient allocation is a constraint and bidders experience no cost from bidding. The characterization of the optimal mechanism in this special case turns out to be closely related to earlier work on finding an optimal agent by iteratively relaxing the parameters of the search []. In Section 4, we drop the assumptions of efficiency and no bidder cost for elicitation.

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2. COST MODEL AND SETTINGS

In this section, we formally define our cost model and explain its motivation.

DEFINITION 1 (SETTING). *One seller is selling one item to n buyers (bidders) whose valuations v_i $(1 \leq i \leq n)$ are independently and identically distributed (i.i.d.) over $[0, 1)$ with PDF $f(x)$ and CDF $F(x)$ which are assumed to have no gaps ($F(x)$ is strictly increasing everywhere). The seller can broadcast a message to all bidders, at cost b to the seller. A bidder can reply to that broadcast, or remain silent. If a bidder replies (does not stay silent), this comes at a cost β_1 to the seller and a cost β_2 to that bidder, for a total cost of $c = \beta_1 + \beta_2$.*

Two key aspects of this model are that (1) staying silent comes at no cost and (2) replying comes at a positive cost, *and this positive cost is the same no matter how complex the query and answer are*. This is motivated by the settings discussed in the introduction, where a bidder can observe the process of the auction (or messages posted on a board) silently at no cost, but once the bidder acts in the auction, costs occur—e.g., the bidder has to submit credit card information, the bidder and the seller have to arrange an in-person meeting, etc.). A key aspect of such costs is that they tend to be the same regardless of the level of detail in the bidder's answer: for example, if the bidder just reports having a valuation greater than \$10 without specifying what it is exactly (rather than reporting a valuation of exactly \$14), this is not likely to reduce any of the above costs. In particular, the seller is likely to want to verify the bidder's authenticity at any point where the bidder's reply affects the course of the auction from then on. This leads to the following easy proposition:

PROPOSITION 1. *In the model defined in 1, without loss of optimality, we can restrict our attention to (broadcast) queries that result in each bidder either staying silent, or immediately revealing his exact valuation to the seller.*

Another notable point is that we restrict communication from the seller to broadcast queries. This is a common restriction: any sealed-bid auction can be considered a broadcast auction with only one broadcast: the reserve price. The bisection auction [4] is an example auction with many rounds of broadcasts. In each round, it broadcasts a price and asks bidders to reply whether their valuation is above or below that price. Besides the broadcast model being simple, natural, and common in existing auction mechanisms, it is also naturally motivated in the Internet domains we consider: the bidders are entirely anonymous until their first reply, so before this point querying such bidders individually is not feasible and they can only be reached by, say, posting on a public website; and after they have replied, we will know their valuation exactly (by Proposition 1) and we no longer need to query them. Of course, there are offline cases where the set of bidders is small and explicit (e.g., the government wants to sell land or spectrum to one of three known companies); in such settings, it can indeed be helpful for the seller to communicate with bidders individually [6, 10]. Such settings do not fit our model; we explicitly focus on highly anonymous settings, and the costs that the seller incurs from the broadcast query correspond to the time, effort, and third-party charges associated with posting a public message.

DEFINITION 2. *A* mechanism *in our setting consists of (1) a full contingency plan for which query to broadcast at each point, depending on answers given so far, and a termination condition; and (2) an allocation and pricing rule that is defined on each terminal state. A mechanism is* individually rational *if losing bidders never pay and winning bidders never pay more than their valuations. We say an individually rational mechanism is* optimal *if it has a Bayes-Nash equilibrium for the bidders that maximizes the seller's profit (among all Bayes-Nash equilibria of all individually rational auction mechanisms). Here, seller profit is revenue minus seller elicitation costs. A class of mechanisms is optimal if it contains at least one optimal mechanism.*

3. OPTIMAL MECHANISMS WITH EFFICIENCY AND ONLY SELLER'S COST

In this section, we make two simplifying assumptions: (1) we restrict attention to mechanisms that allocate the item efficiently and (2) we assume that the bidder cost for replying (β_2) is zero.[1] For example, someone who is moving and selling furniture on craigslist is likely to have 0 valuation for the item and cannot commit to withhold the item or prevent re-sale between bidders. Under such circumstances, efficient mechanisms not only maximizes the social welfare but also maximizes the seller's revenue [1] (though this does not consider elicitation costs). In Section 4 we drop these assumptions.

The rest of this section is organized as follows. First, we introduce a class of mechanisms called multi-round Vickrey auctions (MVA). Then, we prove that we can restrict attention to MVAs without loss of optimality. After that, we find the specific MVA that is optimal. Finally, we experimentally compare this optimal MVA to some other natural mechanisms.

3.1 Multi-round Vickrey Auctions

In an MVA, the seller runs a Vickrey auction with a reserve price; if nobody bids (above the reserve price), the seller runs another Vickrey auction with a lower reserve price, etc., until the item is sold. (In Section 4, we will also consider MVAs that can terminate without having sold the item.) For example, consider a sequence of eBay auctions (with proxy bidding) in which the seller is repeatedly lowering the reserve price.

DEFINITION 3 (MULTI-ROUND VICKREY AUCTION (MVA)). *An MVA is defined by a sequence of reserve prices $r_1, r_2, \ldots$ (which may be finite or infinite) where $r_i > r_{i+1}$. In round i, a Vickrey auction with reserve price r_i is run (if the item has not been sold yet).*

In an MVA, *if* a bidder decides to bid (above the reserve price), it is optimal to bid truthfully since doing so is dominant in a Vickrey auction. However, a bidder may choose strategically to stay silent even with a valuation above the reserve price, in the belief that nobody else will bid this round so that the price will decrease in the next round. Thus, in our (symmetric) setting, Bayes-Nash Equilibria (BNE) for MVAs can be described by a sequence of thresholds $a_1, a_2, \ldots$ where $a_i > a_{i+1}$; in round i, a bidder bids if and only if his valuation is at least a_i.

We will be interested in the following question: given desired thresholds a_i, which reserve prices r_i result in these thresholds? Then, by the revelation principle, we can convert this to a mechanism in which we query agents whether their valuations are above a_i and it is optimal for them to respond truthfully.

LEMMA 1. *Consider a symmetric strategy profile in an MVA where in the ith round, valuation a_i is the threshold for bidding (so*

[1] Note that if (2) does not hold, then allocating efficiently may not be possible without the seller compensating the bidders for their bid costs.

that bidders with lower valuations stay silent and those with higher valuations bid). Then this constitutes a Bayes-Nash equilibrium if:

$$P(a_i)(a_i - r_i) = \int_{a_{i+1}}^{a_i} (a_i - x)p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \tag{1}$$

(where $P(x) := F(x)^{n-1}$ and $p(x) := P'(x) = (n-1)F(x)^{n-2}f(x)$) when i is not the last round, and either $a_i = r_i$ when i is the last round or $\lim_{i\to\infty} a_i = \lim_{i\to\infty} r_i$.

PROOF. Consider a bidder with valuation a_i; we first show that in round i, such a bidder is indifferent between bidding and staying silent if the condition holds. If i is the last round, clearly he is indifferent between bidding and not iff $a_i = r_i$. We now consider the case where i is not the last round. If another bidder bids in round i, that bidder will bid at least a_i, and our bidder will have zero utility. Therefore, the left-hand side of (1) represents the expected utility to our bidder for bidding now. Corresponding to the right-hand side, if our bidder stays silent, and there is a next round and our bidder bids in that next round, then he will win in that round, either with another bidder bidding (first term) or not (second term).

If we now consider a bidder with valuation above a_i, a similar analysis shows that this bidder strictly prefers bidding in round i to waiting one more round; inductively, he will prefer it to waiting any number $k > 0$ more rounds; and he will also prefer this to never bidding because either $a_j = r_j$ when j is the last round or $\lim_{j\to\infty} a_j = \lim_{j\to\infty} r_j$. Finally, let us consider a bidder whose valuation lies below all the a_i. Because either $a_j = r_j$ when j is the last round or $\lim_{j\to\infty} a_j = \lim_{j\to\infty} r_j$, this bidder is best off never bidding. □

For a mechanism that allocates efficiently, we either have $r_k = a_k = 0$ for some k, or $\lim_{i\to\infty} r_i = 0$.

THEOREM 1. *Given a decreasing sequence of $a_i \in [0, 1)$ that ends at or converges to 0, let*

$$r_i = \left(\int_0^{a_i} x\, p(x)dx\right) / P(a_i) \qquad (\text{if } a_i > 0) \tag{2}$$

and $r_i = 0$ if $a_i = 0$. The corresponding MVA has a pure strategy Bayes-Nash equilibrium characterized by bidding thresholds $a_1, a_2, \ldots$.

PROOF. We show that the conditions of Lemma 1 hold. If i is the last round, then $a_i = 0 = r_i$. Also,

$$\lim_i r_i \le \lim_i \left(\int_0^{a_i} a_i\, p(x)dx\right) / P(a_i) = \lim_i a_i = 0$$

If i is not the last round, by (2) we have $r_i P(a_i) = \int_0^{a_i} x\, p(x)dx$ for all i (this clearly also holds if $a_i = 0$). Thus the right-hand side of equation 1 is:

$$\begin{aligned}
&\int_{a_{i+1}}^{a_i} a_i p(x)dx - \int_{a_{i+1}}^{a_i} x\, p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \\
&= a_i P(a_i) - \cancel{a_i P(a_{i+1})} - r_i P(a_i) + \cancel{r_{i+1} P(a_{i+1})} \\
&\quad + \cancel{P(a_{i+1})a_i} - \cancel{P(a_{i+1})r_{i+1}} \\
&= \text{left hand side of (1)}
\end{aligned}$$

□

This tells us that a bidder will bid in a round of an MVA if and only if the expected second-highest bid conditional on this bidder's valuation being the highest is greater than the reserve price of that round. For example, if the distribution is uniform, we have $r_i = \frac{n-1}{n} a_i$.

A similar analysis to the one in this subsection (in a model that includes discounting) appears in [5]. However, that paper does not consider communication costs, and so we diverge from that work in what follows.

3.2 Optimality of MVAs

Since the mechanism is required to be efficient and bidders' communications costs (β_2) are zero, any mechanism that gives utility zero to an agent with valuation zero results in the same revenue for the seller, by the revenue equivalence theorem [7].[2] Hence, maximizing profit is equivalent to minimizing the seller's query costs.

By the revelation principle, we can restrict our attention to mechanisms in which agents always answer queries truthfully in equilibrium.[3] By Proposition 1, we can assume that an agent reveals his entire valuation when not staying silent. By the efficiency constraint, the mechanism must at least discover a bidder with the highest valuation. This optimization problem corresponds to Definition 4. We will then prove that MVAs can achieve this lower bound and are hence optimal.

DEFINITION 4. *A* (direct-revelation) query *is given by a subset $Q \subseteq [0, 1)$, such that if the agent's valuation is in Q, he replies with his exact valuation, and otherwise stays silent. The cost of a query is $b + j \cdot c$ where j is the number of bidders who do not stay silent. A strategy for asking queries can be represented by a function*

$$S\big(f, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_{i-1}\}\big) = Q_i$$

which means: suppose that the set of queries asked previously is $\mathcal{Q}$, the set of values already reported is V, and there are m bidders left who have not responded (whose valuations were drawn i.i.d. from f); then the strategy S will next ask Q_i. $C_{f,n}(S)$ is the expected cost of strategy S, and we wish to find $C^*_{f,n} = \inf_S C_{f,n}(S)$.

To find out the optimal query strategy, we first show that the minimum cost is independent of the PDF $f(x)$.

LEMMA 2. *Consider the uniform PDF $f_u(x) = 1$ and let $C^*(n) = C^*(f_u, n)$. For any other PDF $f(x)$, we have $C^*(f, n) = C^*(n)$.*

PROOF. For any strategy S_{f_u} for the uniform distribution, we can transform it into a strategy S_f for the distribution with PDF f by querying at the equivalent percentiles. Specifically,

$$S_f\big(f, m, V, \mathcal{Q}\big) = F^{-1}\Big(S_{f_u}\big(f_u, m, F(V), F(\mathcal{Q})\big)\Big)$$

The performance of S_f on f is the same as that of S_{f_u} on f_u. Therefore, $C^*(f, n) \le C^*(f_u, n)$. Conversely and similarly, for any strategy S_f for PDF f, we can transform it into a strategy S_{f_u} for uniform PDF f_u. Specifically,

$$S_{f_u}\big(f_u, m, V, \mathcal{Q}\big) = F\Big(S_f\big(f, m, F^{-1}(V), F^{-1}(\mathcal{Q})\big)\Big)$$

Thus, $C^*(f_u, n) \le C^*(f, n)$. Hence, we have $C^*(f, n) = C^*(u, n) = C^*(n)$. Figure 1 illustrates those two constructions we used for this proof. □

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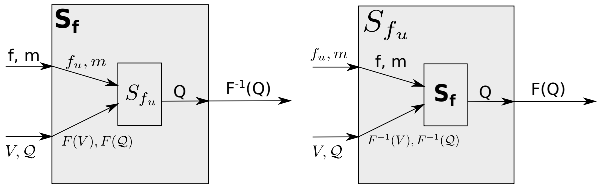
[2]Moreover, by individual rationality we cannot give an agent with valuation zero less than zero utility; we could give such an agent more than zero utility (as in redistribution mechanisms []), but this would only hurt revenue.

[3]For example, for MVAs, we can directly query the agents whether their valuations are above the a_i while still charging according to the r_i.



(a) Construct S_f from S_{f_u} (b) Constructing S_{f_u} from S_f.

Figure 1: These two figures illustrate how to construct a strategy for an arbitrary PDF f from one for the uniform distribution f_u, and vice versa. Here, we depict a strategy as a box that takes four inputs $f, m, V, \mathcal{Q}$ (PDF, number of unknown values, reported values set, and set of asked queries) and returns Q (the next query).

We next prove that descending query strategies, in which all bidders are asked whether their valuations are above a_i for a decreasing sequence of a_i, are optimal. (For a descending query strategy, as soon as we get a positive reply, we are done, so the contingency plan does not need to branch.)

LEMMA 3. *Every optimal strategy uses only descending queries $Q_1 = [a_1, 1),\ Q_2 = [a_2, a_1),\ Q_3 = [a_3, a_2) \ldots$ (with the possible exception of strategies that differ only on a measure zero set and are therefore equivalent to a descending-query strategy with probability 1).*

PROOF. Let us assume, for the sake of contradiction, that there exists an optimal strategy that uses a non-descending query (meaning, one that differs from a descending query on a non-measure-zero set). In that strategy S, there must be a first non-descending query $Q_{i+1} = S\big(F, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_i\}\big)$ Consider an alternative descending query $Q'_{i+1} = [a'_{i+1}, a_i)$ (or $Q'_{i+1} = [a'_{i+1}, 1)$ if $i = 0$) such that

$$\Pr(v \in Q_{i+1}) = \int_{Q_{i+1}} f(x)\mathrm{d}x = \int_{Q'_{i+1}} f(x)\mathrm{d}x = \Pr(v \in Q'_{i+1})$$

Because Q_1 to Q_i are all descending, we have $m = n$ and $V = \emptyset$ (otherwise an optimal strategy should terminate without asking Q_{i+1}). Let C be the expected cost of using S from this point on (starting with Q_{i+1}), and let C' be the expected cost of of using Q'_{i+1} and querying optimally after that. We have $C = b + \sum_{j=0}^{n} p_j(j \cdot c + C_j)$ and $C' = b + \sum_{j=0}^{n} p'_j(j \cdot c + C'_j)$ where p_j (or p'_j) is the probability that there are j reported values within Q_{i+1} (or Q'_{i+1}), and C_j (or C'_j) is the expected cost of later queries given that j values have been found in Q_{i+1} (or Q'_{i+1}).

Because $\Pr(v \in Q_{i+1}) = \Pr(v \in Q'_{i+1})$, we have $p'_j = p_j$ for all j. We have $C_n = C'_n = 0$. By Lemma 2, $C_0 = C'_0 = C^*(n)$ because if we know that no value lies in $Q_1, Q_2, \ldots, Q_{i+1}$, the remaining problem is equivalent to that with the revised PDF

$$f_{i+1}(x) = \begin{cases} \lambda f(x), & x \notin Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \\ 0, & x \in Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \end{cases}$$

where λ is a constant that makes $\int_0^1 f_{i+1}(x)dx = 1$. Finally, since Q'_{i+1} is a descending query, for all $0 < j < n$, $C'_j = 0$ but $C_j > 0$ (because for these values of j, it is possible that all reported values in Q_{i+1} lie below some value that has not been queried yet). Moreover, there exists some j with $0 < j < n$ and $p_j > 0$ unless (1) the probability of each agent's reply to Q_{i+1} is 1—but in this case it differs only on a measure zero set from the descending query that asks for all the remaining values, contradicting our assumption; (2) the probability of each agent's reply to Q_{i+1} is 0—but such a measure zero query is clearly suboptimal; or (3) $n = 1$—but in this case any query that differs by more than a measure zero set from the query that asks for all the remaining values is clearly suboptimal. Hence, $C' < C$, contradicting our initial assumption. □

It follows that:

THEOREM 2. *Among all mechanisms that are required to allocate efficiently, Multi-round Vickrey Auctions (MVAs) are of minimum cost.*

PROOF. By Lemma 3, we can restrict our attention to mechanisms with descending query strategies $Q_1 = [a_1, 1], Q_2 = [a_2, a_1), Q_3 = [a_3, a_2) \ldots$. For any such mechanism, Theorem 2 tells us how to find reserve prices $r_1, r_2, \ldots$ such that the corresponding MVA has a Bayes-Nash equilibrium that is equivalent to this descending-query mechanism. □

By the revenue equivalence theorem, we obtain that MVAs are optimal:

COROLLARY 1. *If the bidders have no cost for bidding ($\beta_2 = 0$), MVAs that minimize cost for the seller are optimal among mechanisms that allocate efficiently.*

3.3 MVAs with Minimum Cost

The above analysis still leaves open what the optimal parameters (thresholds a_i, or equivalently, reserve prices r_i) of the optimal MVA are for a given setting f, n, b, c (valuation PDF, number of bidders, broadcast cost, bid cost). According to Lemma 2, the cost does not depend on f and it suffices to restrict our attention to the uniform distribution. Let $\rho = \frac{b}{c}$ (for cases where $c > 0$).

DEFINITION 5. *Given f, we define the α-MVA to be the MVA in which in each round, the expected number of nonsilent bids in that round (conditional on having reached that round) is $(1-\alpha)n$. In the case where f is uniform, the α-MVA is characterized by $a_i = \alpha^i$.*

PROPOSITION 2. *For the purpose of minimizing total cost when $\beta_2 = 0$ under the constraint of efficient allocation: If $c = 0$, the Vickrey auction (the 0-MVA) is optimal. If $\rho = b = 0$, the Dutch auction (which is approximated by the $(1-\epsilon)$-MVA) is optimal. Otherwise, it is optimal to use an α-MVA where α satisfies*

$$\alpha^{n-1}(\rho + (1-\alpha)n) - (1-\alpha^n) = 0$$

PROOF. If $c = 0$, the Vickrey auction has cost b, which is optimal. If $\rho = b = 0$, the expected total cost of the Dutch auction is c (since with probability 1 only one agent will reply), which is optimal. For the remaining case, we first argue that there must be some α-MVA that is optimal. By Lemma 2, we can assume f is uniform. By Theorem 2, some MVA must be optimal. For this MVA, consider a_1. If $a_i = 0$, this is the 0-MVA. If $a_i > 0$, then if at least one bidder is above a_1, we finish after the first query; otherwise, the resulting conditional distribution for each bidder is $f|_{[0,a_1)}$. According to Lemma 2, we can rescale this conditional distribution to the uniform distribution over $[0, 1)$ to arrive back at our original problem; and an optimal mechanism for that is the same MVA that starts with query a_1 (which translates to a_1^2 without rescaling). Repeated application of this reasoning results in the α-MVA with $\alpha = a_1$.

All that remains to show is the characterization of this α. If C_α is the expected overall cost for the α-MVA, then we have $C_\alpha =$

$b + (1-\alpha)nc + \alpha^n C_\alpha$, or equivalently $C_\alpha = \frac{b+(1-\alpha)nc}{1-\alpha^n}$. If we optimize this with respect to α, we must either have

$$\frac{\partial C}{\partial \alpha} = \frac{\alpha^{n-1}\, n\, (b + (1-\alpha)\, nc)}{(1-\alpha^n)^2} - \frac{nc}{1-\alpha^n} = 0$$
$$\Updownarrow$$
$$\alpha^{n-1}(\rho + (1-\alpha)n) - (1-\alpha^n) = 0 \qquad (3)$$

or have α at a boundary value (0 or 1). However, $\alpha = 1$ (or approaching 1) is clearly suboptimal unless $b = 0$, a case that we have already covered; $\alpha = 0$ can be ruled out because $\frac{\partial C}{\partial \alpha}|_{\alpha=0} < 0$. □

This result is analogous to a result by Sarne et al. [11]; translating[4] their result into our setting also gives a proof that α-MVAs are optimal among MVAs and also characterizes the optimal value of α. However, they do not provide a proof that MVAs are optimal among all mechanisms; their setup implicitly restricts attention to MVAs (when translated to our setting).

3.4 Approximation of α

It is difficult to get an exact closed formula for optimal α by equation 3. Thus we are going to use some simpler formulas to approximate α. We will conduct experiments to compare our approximation with the optimal α that is computed numerically.

Firstly, $\alpha = 1 - 1/n$ is a natural guess which means each round the expected number of biddings is equal to 1. It turns out to be quite good:

THEOREM 3. *$\alpha = 1 - 1/n$ is a $1/(1-e^{-1})$ approximation of optimal α-MVA. That means, by simply choosing $\alpha = 1 - 1/n$, we would at most get about 1.582 times of optimal cost. Another obersavation of this approximation is that no matter how large n is, the cost of this simple approximation is at most $(\rho+1)/(1-e^{-1}) = O(1)$. Thus the optimal cost is bounded by constant $O(1)$ no matter how large n is.*

PROOF. $C(\alpha = 1-1/n) = (\rho+1)/(1-(1-1/n)^n)$. Because $(1-1/n)^n \le e^{-1}$, we have $C(\alpha = 1-1/n) \le (\rho+1)/(1-e^{-1})$. It is obvious that at least one broadcast and one bidding is required to terminate so $C \ge \rho + 1$. This completes the proof. □

A better approximation when n is large is to observe that $\alpha \to 1$ when n grows large. Thus we guess that $(1-a)n \approx A$ (for some constant A) and $\alpha^n \approx \alpha^{n-1}$. Then we have:

$$n(\rho + A) \cdot \alpha^n - n(1-\alpha^n) = 0$$

which gives us $\alpha = (1+\rho+A)^{-1/n}$. Put this back to $\lim_{n\to\infty}(1-a)n = A$ we have $\ln(1+\rho+A) = A$ which gives us

$$A = -1 - \rho - W(-1-\rho)$$
$$\alpha = (-W(-1-\rho))^{-1/n} \qquad (4)$$

where $W(x)$ is the Lambert W function [cite wikipedia?] defined by $W(x)e^{W(x)} = x$. Actually, $w\, e^w = x$ has two solutions for w when $-1 < x < 0$. Here our $W(x)$ refers to the lower[5] branch $W_{-1}(x) < -1$. This second approximation that converges to the optimal one when n is large:

THEOREM 4. *Suppose that the optimal α is α^* which satisfies equation 3. Then $\alpha = (-W(-1-\rho))^{-1/n}$ satisfies*

$$\lim_{n\to\infty} C(\alpha^*) = \lim_{n\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n})$$

That is, our approximation's cost will converge to optimal cost when n grows to infinity.

PROOF. Define sequence α_n^*, C_n^* where $n = 1, 2, 3, \dots$ to be sequences of optimal α^* and corresponding optimal cost C^* when there are n bidders. We first show that C_n^* is increasing: if we make $(\alpha_{n-1})^{n-1} = (\alpha_n^*)^n$, then we have 1) The expected broadcast cost of α_{n-1}-MVA with $n-1$ bidders is equal to that of α_n^*-MVA with n bidders as the probability that one round will terminate is the same; 2) $\alpha_{n-1} < \alpha_n^*$ thus the expected bidding cost of α_{n-1}-MVA with $n-1$ bidders should be less than that of α_n^*-MVA with n bidders. Therefore, $C_{n-1}^* \le C_{n-1}(\alpha_{n-1}) < C_n^*$. Thus sequence C^* is indeed strictly increasing.

Secondly, theorem 4 says C_n^* is bounded. Therefore $(1-\alpha^*)n$ must also be bounded otherwise $C = \frac{\rho+(1-\alpha)n}{1-\alpha^n}$ cannot be bounded. Thus according to Bolzano-Weierstrass theorem [cite wikipedia?], there must be a subsequence $\alpha_{n_i}^*$ such that $(1-\alpha_{n_i}^*)n$ converges to some constant A. Recall that α^* satisfies equation 3 and obviously $\lim_{n\to\infty}\alpha^* = 1$, we could use calculations similar to what we used for equation 4 to derive

$$\lim_{n_i\to\infty}(1-\alpha_{n_i}^*)n_i = A = -1 - \rho - W(-1-\rho)$$
$$\lim_{n_i\to\infty}(\alpha_{n_i}^*)^{n_i} = \lim_{n_i\to\infty}(\alpha_{n_i}^*)^{n_i-1} = (-W(-1-\rho))^{-1}$$

This proves that

$$\lim_{n_i\to\infty} C(\alpha^*) = \lim_{n_i\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n_i})$$

Then using the fact that C_n^* is strictly increasing and bounded completes the proof. □

Experiments in figure 2 compare the optimal α, our first approximation of $\alpha = 1 - 1/n$ and our second approximation $\alpha = (-W(-1-\rho))^{-1/n}$ together with their corresponding cost under settings $\rho = 0.2, 1, 5$.

As figure 2 shows, the first approximation $1 - 1/n$ is bounded to be a constant time of optimal cost while the second approximation converges to optimal cost when n grows large. When ρ is close to 1, both two approximations are very close to the optimal one. But the second approximation is much better when ρ is much smaller or greater than 1. Anyway, the second approximation is not always better than the first approximation, as the case $\rho = 1, n = 2$ shows.

3.5 Experiments

Now let us compare optimal α-MVA with other kinds of MVA. In all following experiments, the valuation distribution is always uniform over $[0, 1]$. According to lemma 2, other distributions can always be adapted to uniform distribution so this will not be a problem. Also recall that under this simpler model, the revenue is fixed thus the profit is completely determined by cost. Thus we will only compare cost.

α-MVA can potentially have infinite many rounds. But in reality, it is more naturaly to come up with an MVA that has finite many, say k rounds at most. Let us call them k-MVA where k is some positive integers. One particualr k- MVA is uniform k-MVA where the k thresholds is uniformly distributed over $[0, 1]$. It is also known as fixed-step search strategy in [11, 3]. An optimal uniform MVA is the uniform k-MVA that minimize the cost by choosing the best k.

[4] Sarne et al. motivate their model as performing increasingly broad searches for an optimal agent and thereby focus on increasing threshold search to find a minimum, rather than decreasing queries to find a maximum.

[5] The upper branch is $W_0(x) > -1$ when $-1 < x < 0$

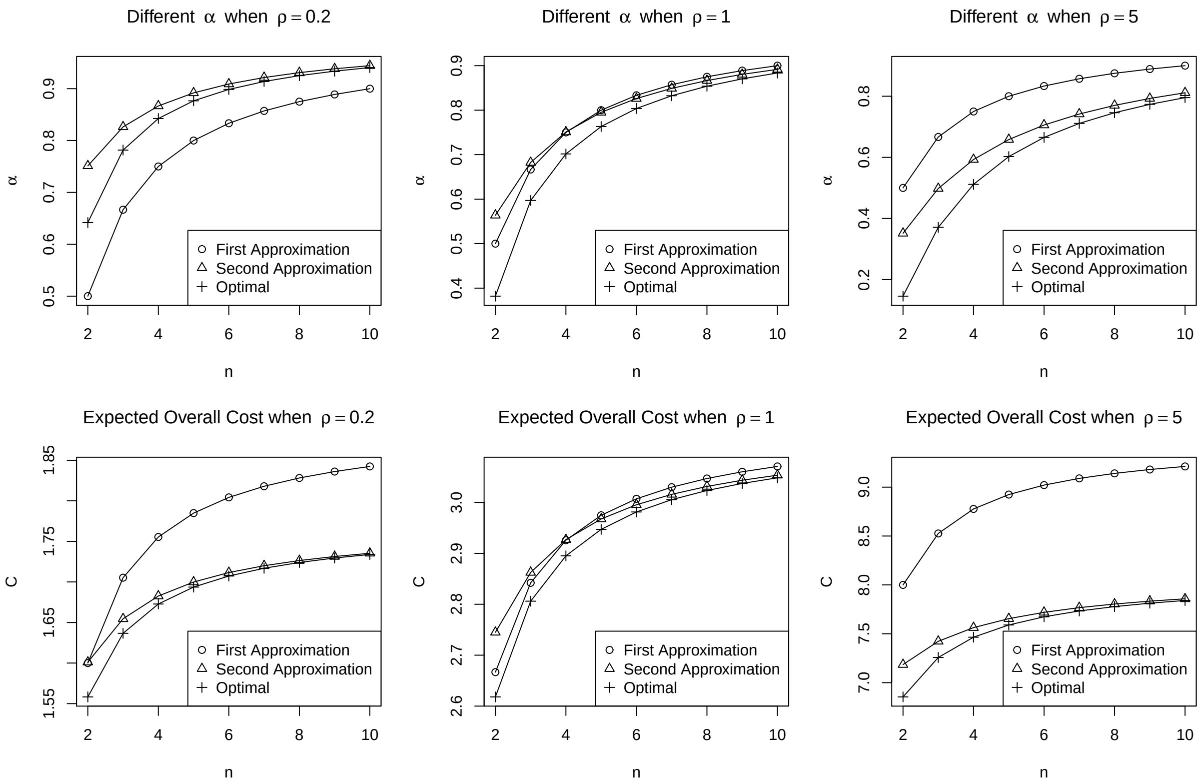


Figure 2: Comparisons for optimal α and its approximations. The first approximation is $\alpha = 1 - 1/n$, the second is $\alpha = (-W(-1-\rho))^{-1/n}$. The second row is the corresponding cost for different α

In practice, such k might also be very limited. For example, if someone need to sell something in 3 days before moving out, the k might be limited to 3 since it is too annoying to send out two broadcast messages per day (e.g. it might be labeled as spam by selling platform). With this limitation, one can still use uniform thresholds as a baseline. Or we may formulate this as another optimzing problem and solve the best k thresholds under this constraint (this problem is defined and solved in section 4.3 and equation 6). Between the baseline k uniform thresholds and the optimal k thresholds, another heurestic way to get k thresholds is to use $\alpha, \alpha^2, \ldots, \alpha^{k-1}$ as thresholds. That means, in first $k-1$ rounds, we query as if we have infinite many rounds, and we query all the left in last round. We call this mechanism as α-cutoff-k-MVA.

The comparison results achieved from simulation experiments are shown in figure 3. Here are some observations:

- The optimal uniform-MVA's cost is very close to optimal α-MVA's, especially when n is large. That is probably because
  1. α approaching 1 when n grows large, which makes the first k thresholds $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^k$ close to uniform thresholds $1-(1-\alpha), 1-2(1-\alpha), \ldots, 1-k(1-\alpha)$;
  2. the probability that the highest value falls out of first k thresholds, α^{nk}, becomes negligible for large n.
- The optimal k-MVA's cost decreases and approaches to optimal cost quickly when k grows (check optimal 2-MVA and 3-MVA).
- When k is small, uniform thresholds has significant higher cost than optimal k-MVA, and the heuristic α-cutoff-k-MVA.
- The heuristic α-cutoff-k-MVA works pretty well especially when ρ is large as shown in figure 3(a). But it is not as good as optimal 2-MVA when ρ is small and n is large as shown in figure 3(b). Thus find the right thresholds is even more important than adding one more round in those cases.

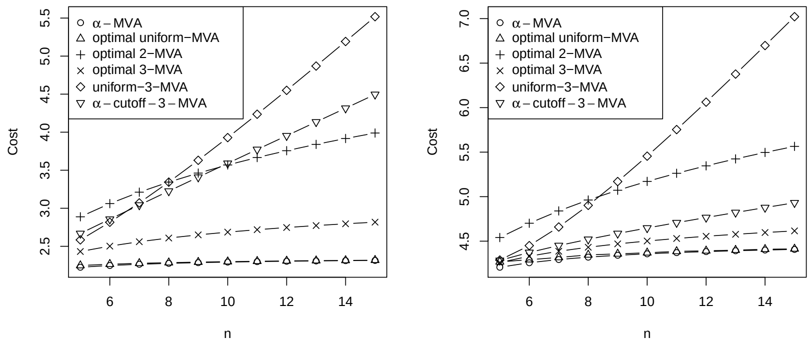
4. THE GENERAL CASE

In this section, we drop the constraints that (1) the mechanism must allocate efficiently and (2) bidders have no cost for bidding ($\beta_2 = 0$), and prove that MVAs are optimal even without these constraints (i.e., in the fully general model). We also study methods for finding the specific optimal MVA, though we do not manage to obtain a characterization that is as elegant as in the restricted case.

An interesting effect of $\beta_2 > 0$ is that it gives bidders an incentive *not* to bid when they expect many others to bid, because if they do not win they still pay β_2 for bidding. (Also, we note that their probability of winning with a fixed value is exponentially small in the number of bidders, so that even small bidding costs, such as the effort needed to send an e-mail, can be significant.) This effect is rather opposite to the effect observed in earlier parts of this paper that bidders are less inclined to wait for later rounds when there are many bidders.

4.1 Spending Equivalence Theorem and Revenue Optimization Strategy

Another effect of $\beta_2 > 0$ is that the revenue equivalence theorem (straightforwardly interpreted) no longer applicable, because this theorem assumes that the utility is equal to valuation minus payment, which is no longer true because cost also plays a role.



(a) $\rho = 0.5$ (b) $\rho = 2$

Figure 3: Cost comparison for different MVAs over n, number of bidders, and ρ, broadcast/bid cost ratio.

Intuitively, we can fold this cost into the payment to make the theorem applicable again, though we have to keep in mind that this cost does not benefit the seller, unlike (true) payments. This is how we proceed.

THEOREM 5 (SPENDING EQUIVALENCE). *The bidders' expected total spending (payment + bidding costs) is completely determined by (1) the expected utility of lowest-type bidders and (2) the allocation probability function*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

where p_i is the probability that bidder i will get the item.

PROOF. Any mechanism in our domain (with bidding costs) can also be used in a domain without costs, letting the seller collect the bidding costs (as well as the payments) instead. The bidders' incentives will be identical in this mechanism, and so the result follows from applying the standard revenue equivalence theorem to these transformed mechanisms. $\square$

Thanks to theorem 5, our profit maximization problem is now greatly simplified:

COROLLARY 2. *To maximize profit for a given allocation rule $p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$, we only need find the minimum total cost (including both seller's cost and bidders' cost).*

PROOF. The total spending, substracts cost charged to bidders, will be the revenue that the seller receives. This revenue, substracts cost charged to the seller, will be profit. Thus profit is total spending minus total cost. As total spending is fixed by allocation rule, we only need to find the minimum cost to maximze profit. $\square$

The highlight here is that we will not have to differentiate cost charged to bidders and cost charged to sellers if we just want to maximize seller's profit. The difference of them may make revenue different, but as long as their sum does not change, the profit will not change. This not only helps us simplify our analysis, but also helps us simplify the optimal mechanism:

PROPOSITION 3. *In original MVA, to make a set of thresholds a_i works (the equilibrium holds), we may have to make r_i negative. That says, we need to compensate bidders if only one bids at reserve price r_i rather than charge him something, in order to maintain bidders incentive to bid in spite of bid cost. That's not intuitive and sellers/bidders may not be able to easily understand it. A better way to achieve a_i might be compensating the bidder with the bid*

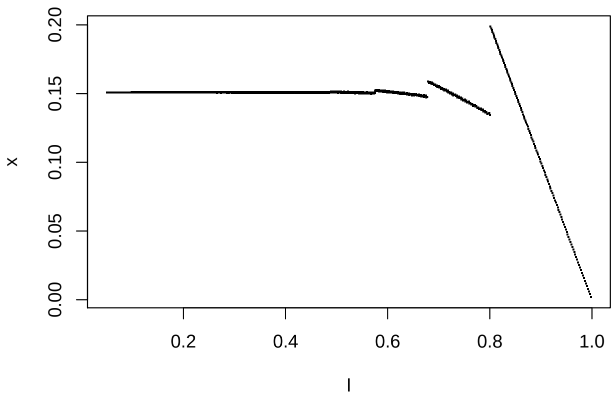


Figure 4: Best first-query-length x over low value l. In the experiment, we set $\rho = 2$ and discretize $[0, 1)$ into 1000 grids.

cost immediately after the bidding. Equivalently, the seller buys the bid cost for bidders and make it as a part of seller's bid cost. This won't change his profit but this will make everything looks more intuitive (r_i is non-negative again).

Since profit is total spending minus overall cost where total spending is decided by allocation rule which is studied in Myerson's work, our optimizing problem is greately related to Myerson's if we were able to minimize the overall cost. However, the optimal allocation rule here is not as simple as the one that is discovered by Myerson [7]: allocate the item to the bidder with highest positive virtural value. Theorem 5 tells us that this rule will maximize the total spending. But we must substract the cost from the spending to get the profit. Therefore, there might be another weird allocation rule that has less total spending but even much less minimum cost.

Thus, the profit optimal mechianism will depend on how minimum cost is defined given an allocation rule.

DEFINITION 6. *A mechanism with allocation rule*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

does not allocate blindly if it satisfies: if $p_i > 0$ for some valuation profile $(v_1, v_2, \ldots, v_n)$, there must be a broadcast query that the i-th bidder (v_i) reply to the seller under that profile setting.

The no-blind-allocation property defined above will ensure the optimality (minimum cost) of the following mechanisms.

DEFINITION 7. *Mechanisms satisfy relaxed efficiency constraint with low value l if they always allocate the item to the bidder with highest valuation that's at least l (no allocation if everyone is below l). When we say a mechanism has a low value l, we imply that it satisfies relatex efficiency constraint with low value l.*

THEOREM 6. *In regular cases (the virtual valuation is monotone strictly increasing []), mechanisms satisfying relaxed efficiency constraint with some low value l are optimal among all mechanisms that won't allocate blindly.*

PROOF. Suppose that there's an arbitrary optimal mechanism. We can describe it by a query strategy $S(f, m, V, \mathcal{Q})$ (see definition 4) and an allocation function based on V, the set of reported values set, since we can only allocate item to reported bidders. We then construct another mechanism with query strategy $S'(f, m, V', \mathcal{Q}')$ where $\mathcal{Q}'$ are decsending and

$$|S'(f, m, \emptyset, \{Q'_1, \ldots, Q'_{i-1}\})| = |S(f, m, \emptyset, \{Q_1, \ldots, Q_{i-1}\})|$$

That says, if there's no reported bidders yet, we will always ask the descending query which has the same length as the query of S. And we pretend that we asked the same query as S so we can continue to ask S what's the next query length. Once we get reply in S', we terminate the mechanism immediately and allocate the item to the replied bidder with highest valuation with probability 1. Note that we may have $|Q'_i| = |Q_i| = 0$ which means both mechanisms terminate after i queries without any reply and allocation. It's clear that our new mechanism won't cause more cost than original mechanism. And our new mechanism's allocation rule will get at least as high total spending as original mechanism because it makes allocation probability of high value bidders as much as possible. Therefore our new mechanism, which satisfies relaxed efficiency constraint with some low value l, is also optimal. $\square$

For simplicity, we will always assume regularily and will not mention virtual valuation below. It's easy to extend our result to general cases by mapping all values to virtual values and ask broadcast queries according to virtual values.

4.2 MVAs' Optimality in General

We have already narrowed down optimal mechanism to relaxed efficient mechanisms and by theorem 5, it is straightforward to see

COROLLARY 3. *For mechanisms with a fixed low value l, the maximum profit is achieved when the mechanism minizes the cost.*

Our next question is naturally: what is the cost minimized mechanism given a low value l. A similar lemma can be proved using almost identical technique to lemma 2. Thus for space limit, we will not elaborate it again.

LEMMA 4. *Suppose that there are two cases n, F_1, f_1, l_1 and n, F_2, f_2, l_2 where n is the number of values for both cases, F_i, f_i are CDF and PDF of the n i.i.d. values in case i, l_i is the low value for case i. If $F_1(l_1) = F_2(l_2)$, then these two cases have the same minimum cost to find the maximum value above the low value l_i.*

Finally, we conclude

THEOREM 7. *MVAs have the minimum cost among all mechanisms with a low value l. Thus they are optimal.*

PROOF. A special case of this theorem when $l = 0$ is theorem 2. We proved that special case by introducing lemma 2 and 3. To prove the general cases with arbitrary l, we just need to revise lemma 2 a little to lemma 4. All other part of the proof remains similar. For space limit, the detailed proof is omitted. $\square$

Then the only parameters we are going to determine for the specific optimal MVA are 1) the low value l; 2) the descending query thresholds $a_1, a_2, a_3, \ldots$. When we later investigate such parameters that minimizes the cost, we will also assume uniform distribution $F(x) = x$ in default because distribution will not change this minimum cost and we can always adapt an optimal MVA for uniform distribution to an optimal MVA for any distribution easily.

4.3 Analysis of Optimal MVA with a Positive Low Value

Before an analytical treatment, we first conduct an experiment to compute the optimal MVA approximately for getting some insights. In that experiment, we discretize the value into fine grids and then use dynamic programming to compute the optimal first-query-length over different l (this is sufficient to derive the whole optimal MVA because a failed query is equivalent to increase the l after renormalizing). The result is plotted in Figure 4 and it seems to be much more complicated than α-MVA but there might still be hope to get a nice analytical result: it seems like a piecewise linear function. Specifically, we cannot use a single α to describe such optimal MVA, but perhaps we can use a sequence of α to describe it. Now let us take an analytical treatment.

The first key to analyze the optimal MVA with $l > 0$ is to utilize the fact that there is a maximal number of rounds to exploit the whole reportable valuation range $[l, 1)$. Let us call that number k.[6]

For convenience, define $\boldsymbol{a} = (a_0, a_1, a_2, \ldots, a_k)$, the vector of thresholds in such optimal MVA. We make $a_0 = 1, a_k = l$ so in i-th round the query would be $[a_i, a_{i-1})$. Now define cost $C(\boldsymbol{a}, k, \rho, n)$ to be the expected cost for MVA defined by $k, \boldsymbol{a}$ when there are n i.i.d. $[0, 1)$-uniform bidders (note that l, h are implicitly defined by a_k, a_0). If we can get a neat form of C, we can use $\frac{\partial C}{\partial a_i} = 0$ to characterize optimal MVA, as we did in 3.3.[7]

Thus the second key is to represent this C. Rather than considering one round after another recursively as we did before, we now consider all rounds together. Let CR_i denote the cost that occur in round i (the broadcast of that round and the bid cost charged in that round) and $v^* = \max_i v_i$. Then expected cost would be sum of expected cost of each round:

$$\begin{aligned} C(\boldsymbol{a}, k, \rho, n) &= \sum_{i=1}^{k} E[CR_i] \\ &= \sum_{i=1}^{k} \Pr\left(v^* < a_{i-1}\right) E[CR_i \mid v^* < a_{i-1}] \\ &= \sum_{i=1}^{k} \frac{a_{i-1}^n}{a_0^n} \left(\rho + \frac{a_{i-1} - a_i}{a_{i-1}} n\right) \end{aligned} \tag{5}$$

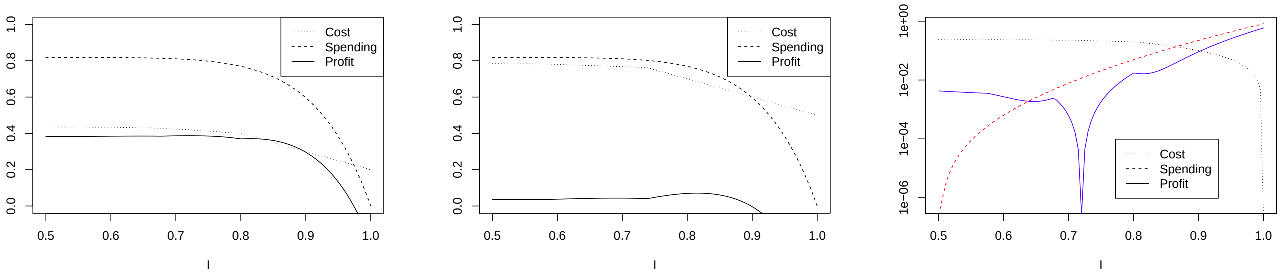
Taking derivative we get

$$\frac{\partial C}{\partial a_i} = \frac{n(n+\rho)a_i^{n-1} - n(n-1)a_{i+1}a_i^{n-2} - na_{i-1}^{n-1}}{a_0^n} \tag{6}$$

Unfortunately, equation 6 is not neat enough to get a piecewise linear query length. Recall that in previous subsection, the experiment seems to show that query length x is piecewise linear over l, which means $a_1 = a_2 + x = l + x$ must also be piecewise linear over $a_2 = l$. One counter example is the simple case $k = 2, n = 3$. We have: $a_1 = (\sqrt{{a_0}^2\,\rho + {a_2}^2 + 3\,{a_0}^2} + a_2)/(\rho + 3)$. Anyway,

[6]We have argued this before: if such k does not exist, or equivalently the maximum number of rounds is unbounded, the cost will be infinite as the possibility that no values lie in $[l, 1)$ is positive.

[7]It is easy to see that boundary cases $a_i = a_{i-1}, a_{i+1}$ are not optimal too.



(a) $n = 10, b = 0.2, c = 0.1$ (b) $n = 10, b = 0.5, c = 0.1$ (c) Suboptimality, $n = 10, b = 0.2, c = 0.1$

Figure 5: Optimal MVA's cost, spending, profit over l, n i.i.d. uniform distributed bidders, broadcast b and bid cost c

by definition we have $a_1 \geq a_2$. And the former equation indeed looks very linear when $a_1 \geq a_2$.

Though we failed characterizing the optimal MVA using piecewise linear functions, equation 5 gives us a better way to calculate thresholds a_i. We use an R package called BB [14] to solve these non-linear equation systems. Before throw those equations to that package, we have to first decide k and an initial guess of $\boldsymbol{a}$.

We found a lemma that's very useful for deciding k:

LEMMA 5. *Two vectors of optimal thresholds $\boldsymbol{a} = (a_0, \ldots, a_k)$ and $\boldsymbol{a}' = (a'_0, \ldots, a'_m)$ can't have $p_2 \leq q_2 < q_1 < p_1$ where p_2, p_1 and q_2, q_1 are two consecutive thresholds $a_i, a_{i-1}, a'_j, a'_{j-1}$.*

PROOF. Denote $C(l)$ to be the optimal cost for finding the maximum value above low value l (suppose h, n, ρ and everything else are constants now). Note that thresholds $a_0, \ldots, a_i$ must be optimal thresholds for $l = a_i$, otherwise $a_0, \ldots, a_k$ can't be optimal for $l = a_k$. Similarly, $a'_0, \ldots, a'_j$ must be optimal thresholds for $l = a'_j$. Then

$$C(q_1) + q_1^n[\rho + n(q_1 - q_2)] \leq C(p_1) + p_1^n[\rho + n(p_1 - q_2)]$$
$$C(p_1) + p_1^n[\rho + n(p_1 - p_2)] \leq C(q_1) + q_1^n[\rho + n(q_1 - p_2)]$$

Adding these two inequations together and do some cancellations we get $p_1^n \leq q_1^n$, which contradicts $q_1 < p_1$. □

From that we derive an upper bound for k. Intuitively, it says we will need less rounds if we have less reportable range (a greater l).

LEMMA 6. *$k_\alpha = \lceil \log_\alpha (1/l) \rceil$ is an upper bound for the optimal k. Here α is the optimal α for α-MVA when $l = 0$.*

PROOF. We will prove by contradiction. Assume $k > k_\alpha$ for optimal thresholds $(a_0^*, a_1^*, a_2^*, \ldots, a_k^*)$ where $a_0^* = 1, a_k^* = l$ as defined. Points $a_0 = \alpha^0, a_1 = \alpha^1, \ldots, a_{k_\alpha} = \alpha^{k_\alpha}$ split interval $[l, 1)$ into k_α subintervals $[a_i, a_{i-1})$, which together contain k thresholds $a_1^*, \ldots, a_k^*$. Since $k > k_\alpha$, at least one subinterval $[a_i, a_{i-1})$ contain two thresholds a_j^*, a_{j-1}^*. Equivalently, there exists $a_i \leq a_j^* < a_{j-1}^* < a_{i-1}$. This contradicts with Lemma 5. □

Having this upper bound, we can eigther bruteforcely search all k between 1 and k_α, or use ternary search [cite wiki?] to get optimal k in $O(\log k_\alpha)$ time if we can prove it is unimodal.[8] But in light of the proof above, we actually could have something much better:

THEOREM 8. *$\lceil \log_\alpha (1/l) \rceil$ and $\lfloor \log_\alpha (1/l) \rfloor$ are upper and lower bounds for the optimal k.*

[8]We did not prove that it is unimodal but experiments seems to support this property.

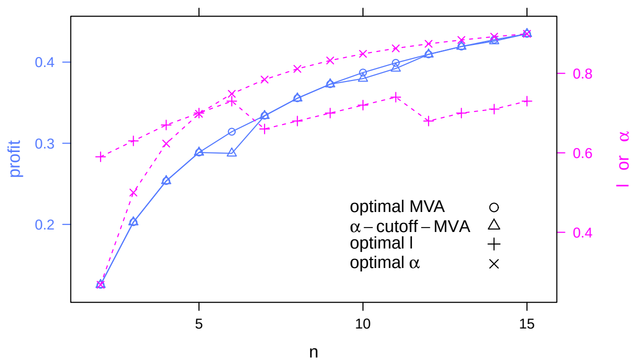


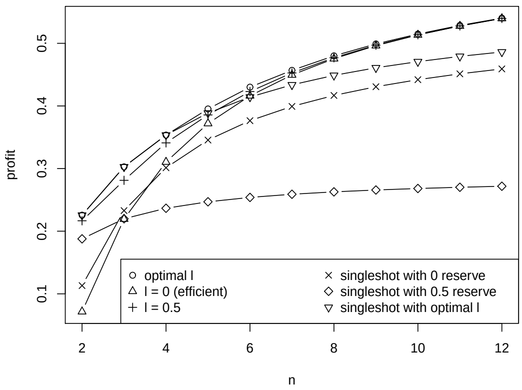
Figure 6: The profit of optimal MVA and α-cutoff-MVA are plotted in solid line. The low value l is chosen to maximze optimal MVA's profit. We also plot l and α in dashed lines to see how they affect the relative profit difference. Note that α does not depend on l.

PROOF. We have proved the upper bound in Lemma 6. Similarly, if the lower bound does not hold, we have $a_j^* = p_2 \leq \alpha^i = q_2 < \alpha^{i-1} = q_1 < a_{j-1}^* = p_1$ which violates Lemma 5. □

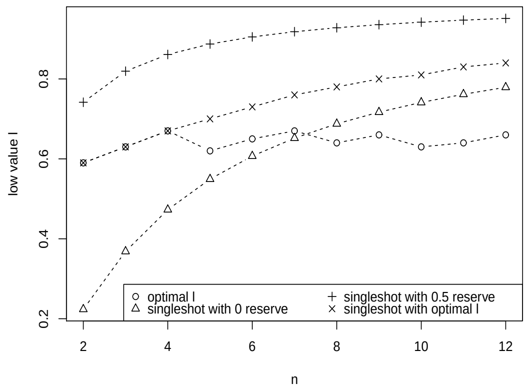
By these bounds, we only need to try at most two values to get the optimal k. The next important thing is the initial thresholds $\boldsymbol{a}$. A bad choice may lead to much more computation and even non-convergence. For example, the trivial uniform $a_i = (1 - i/k)a_0 - (i/k)a_k$ is a bad initial guess. We find the initial guess $a_i = \alpha^i$ (the same α of k_α) to be very efficient. Let us call this MVA with thresholds $\alpha, \alpha^1, \ldots, \alpha^{k-1}$ where $\alpha^{k-1} \geq l, \alpha^k < l$ an α-cutoff-MVA, similar to what we did in previous section 3.5. One explanation about its good performance is that these initial thresholds doesn't violate Lemma 5 (while the uniform thresholds do). Thus it's more likely to be close to actual optimal thresholds. In experiments, this α-cutoff-MVA's profit is pretty good as shown in figure 6. In most cases its profit is very close to optimal. The significant difference only occur when low value l is a little below α and this is not likely to occur when n is large where α is much closer to 1 than l.

4.4 Choosing Low Value

Having k and a_i, we are still one step away from optimal MVA: choosing the low value l. It is obvious to see that optimal cost C is non-increasing as l increases. By Myerson's optimal auction theory and theorem 5, setting virtual value of l to be 0 will yield the maximum total spending. Name such l as l_0. We then conclude



(a) profit



(b) low value

Figure 7: Compare profit and corresponding low value l over n. The broadcast cost $b = 0.1$. Bidding cost for seller and buyer are $\beta_1 = \beta_2 = 0.05$ (thus $c = \beta_1 + \beta_2 = 0.1$)

that optimal l must satisfy $l \geq l_0$. The question is, how to search or determine optimal l. Let us first check the simple case when the value distribution is uniform.

Figure 5(a) and 5(b) plot cost, spending and profit over l. It is clear in figure 5(b) that increasing l from 0.5 to about 0.8 will get a significant profit increase. However, it is not so clear in these two plots whether profit is unimodal and whether we can do ternary search.[9] To see it more clear, we plot figure 5(c), which transforms the cost, spending and profit to their corresponding suboptimality, i.e. the distance between the specific value and the optimal value (to make the log-scale work for distance 0, we add some small constant to that suboptimality). Thus, the new plot will preserve the peaks and global optimal point as the lowest point.

From figure 5(c), the cost is not unimodal over l and it is quite wierd. Thus in later experiments, we are just going to brute forcely search over all possible l from 0.5 to 1, with a search increment of 0.01.

4.5 Experiments

Finally we are going to compare profit of different mechanisms in general settings. To keep it simple, we still use a uniform valuation distribution for all experiments. As theorem 5 shows, the profit is simply spending minus cost where spending is solely determined by l. In fact, the total spending will not be very sensitive to this l when n is large (except that l is very close to 1) and we will see this in later comparisons. Thus most experimental comparisons that we have done for cost in section 3.5 would also give us a lot information for profit comparison. As a result, we will compare mechanisms very different from those in section 3.5. Specifically, we will not be interested in k-MVA where k is very limited. Instead, we are going to compare how l is going to affect the profit and how much profit we lose if we ignore the cost.

If we ignore the cost, Myerson has already proved that a singleshot Vickrey auction with reserve price 0.5 would be optimal for uniform i.i.d. bidders. Thus we will compare this singleshot mechanism, as well as its variants, the singleshot Vickrey auction with 0 reserve price. But be aware of that reserve price is not equivalent with low value l when cost exists. To make it more fair, we add one more singleshot Vickrey auction whose reserve is set to be optimal among all singleshot Vickrey auctions (i.e. we optimize l for this particular singleshot mechanism). One may also suspect whether it is good enough to set low value l to be 0.5 in such singleshot Vickrey auction, since it will bring us maximal total spending. The answer is no, because its bid cost is $0.5n \times c$, which grows way too large when n is large.

The experimental result is shown in figure 7. As shown in profit comparison figure 7(a), the profit of optimal l, $l = 0$ and $l = 0.5$ become close very quickly when n grows and they are almost identical for $n \geq 8$ in this experiment setting. Thus choosing l will not be a critical issue for large n. However, if we use singleshot mechanism (which is optimal if cost does not exist), the gap is significant. This is because the bid cost will drive low value l very close to 1 thus lower the total spending significantly. For example, even if we set reserve price to 0, when $n = 10$, l will be between 0.6 to 0.8. The optimal reserve price 0.5 becomes the worst as its l is too high. Even if we adapt our reserve price to optimal one, the singleshot mechanism is still not so good because it cannot balance the total spending and bid cost, i.e. it either makes l close to 1 to lose a lot total spending, or makes l close to 0.5 to cause a big bid cost.

5. CONCLUSION

6. ACKNOWLEDGEMENT

7. REFERENCES

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