

# Optimal Broadcast Auctions with Costly Actions

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## 1. INTRODUCTION

## 2. COST MODEL AND SETTINGS

In this section, we define our cost model and explain why it makes sense. Our cost model is inspired from online second-hand item transactions, including examples like eBay, craigslist and universities' mailing-lists.

**DEFINITION 1.** *In our settings, there's one seller selling one item to  $n$  buyers (bidders) whose valuations  $v_i$  ( $1 \leq i \leq n$ ) are independently and identically distributed (i.i.d.) over  $[0, 1]$  with PDF  $f(x)$  and CDF  $F(x)$ . The seller can broadcast a message to all bidders which costs  $b$  (the broadcast cost) for the seller. A bidder may reply to that broadcast with cost  $c$  (bidding cost) or remain silent with no cost. Such bidding cost  $c$  may contain two parts,  $\beta_1, \beta_2$  where  $\beta_1 + \beta_2 = c$ . The first part  $\beta_1$  is charged to the seller while the second is charged to the corresponding bidder. The bidder's reply should be deterministic with respect to the seller's broadcast (for simplicity, we only consider pure strategy equilibrium).*

(shall we insert a graph to show our cost model?)

Most settings in definition 1 are quite standard except for the cost and broadcast capability.

The bidding costs  $c$  may be caused by communication and other verification actions required to put a bidding. For example, the bidder may have to input his credit card number and prepay an amount of money. Without such verification for the bidder, a bidder may bid very high and refuse to pay in the end. A verification for the seller might also be needed. For example, a bidder may want to set up an appointment with the seller to check the item. Setting up such an appointment might be costly because they need to discuss time and place via emails or phone calls. Attending that appointment may also cost travel fees and time.

We introduce broadcast capability because it's shared by many auctions. For example, a Vickrey auction or a first-price auction with reserve price can be described as a broadcast auction with only one broadcast: telling every bidder the reserve price. The bisection auction [cite bisection auctions] is another example which has many rounds of broadcasts. In each round, it will broadcast a price and ask

bidders to reply whether his valuation is beyond or below that price. In real world, sellers make such broadcasts via sending emails to a bunch of receivers (typically a mailing-list), listing items on a platform such as craigslist and eBay, or even showing ads on Internet/TV. Such broadcast activities costs either money (e.g. a list or ads fee) or time and effort (e.g. writing and sending an email).

Note that in our model, we only give sellers broadcast capability so they cannot find or communicate with each bidder one by one. The first reason to have this constraint is there are too many potential bidders on the Internet (our model focus on online item transactions) and it's hard to explicitly find them one by one. On the other hand, in offline cases where the set of bidders are small and explicit (e.g. the government want to sell a land to one of three companies), it might be helpful to let the seller communicate with bidders one by one [cite search mechanisms]. The second reason to have this constraint is that we want to focus on mechanisms that avoids time consuming bargaining. Such feature is very important as one of the most vital advantages of online transactions are their convenience and the time consuming bargaining can ruin it.

Finally, we define optimal mechanisms to be the ones that maximize seller's utility since when facing many different auction mechanisms, a rational seller will choose the one that gives him maximum utility.

**DEFINITION 2.** *We say a mechanism is optimal if it gives the seller maximum utility (revenue) which equals to all the value payed to this seller minus all the cost charged to this seller. A class of mechanisms are optimal if they contain one optimal mechanism.*

## 3. OPTIMAL MECHANISMS WITH EFFICIENCY AND ONLY SELLER'S COST

In this section, we consider a simplified optimizing problem with efficiency constraint and only seller's cost. Though these two constraints simplify our problem a lot, they are reasonable in real cases such as craigslist or moving sales in mailing-lists:

1. In many cases, sellers only have 0 valuation for the item and they cannot commit to withhold the item or prevent re-sales between buyers. For example, some second-hand items will be tossed if they cannot be sold by a particular day, e.g. the day the seller moves out the house. We encounter many free items during second-hand sales as well, which is another demonstration of zero valuation. Under such circumstances, an

**Appears in:** *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2012)*, Conitzer, Winikoff, Padgham, and van der Hoek (eds.), June, 4–8, 2012, Valencia, Spain.

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efficient mechanism not only maximizes the social welfare but also maximizes the seller's revenue [cite The optimality of being efficient].

2. The bidding cost for each bidder is sometimes negligible compared to bidding cost charged to the seller. For example, if 100 bidders replied to the seller by a 1-minute call, each bidder only has a tiny 1 minute cost. But for the seller, it's a big 100 minutes cost which is very annoying. It's also necessary to remove bidder's bidding cost to achieve efficiency. Otherwise, the item may not be able to allocate to the highest bidder when that highest valuation is below the bidder's bidding cost.

The rest of this section is organized as follows. First of all, we introduce a mechanism called Multi-round Vickrey Auction (MVA) based on what's been used in realworld online second-hand item transactions. Then we prove that MVAs are optimal (so there exists a MVA that's optimal). After that we'll try to find the specific MVA that achieves the optimality. Finally, we conduct some experiments to compare the optimal MVA with other mechanisms.

### 3.1 Multi-round Vickrey Auctions

A Multi-round Vickrey Auction (MVA) has multiple rounds of Vickrey auctions with progressively decreasing reserve prices. This kind of auction effectively occurs on eBay. The seller may set up a reserve price and let buyers bid for this item. The proxy bidding functionality makes such an auction equivalent to a Vickrey auction with a reserve price. If no buyers bid for a given reserve price, the seller may lower the reserve price, which makes the whole process equivalent to an MVA.

**DEFINITION 3.** (*Multi-round Vickrey Auction, MVA*)

*In a Multi-round Vickrey Auction (MVA), there's a sequence of reserve prices  $r_k, r_{k-1}, \dots, r_0$  where  $r_k > r_{k-1}$ . The seller creates a Vickrey auction with a reserve price  $r_{k-i}$  at time  $i$  (or round  $k-i$ ). In each Vickrey auction, if only one buyer bids, he/she gets the item and pays reserve price. Otherwise, the buyer with the highest bidding gets the item and pays the second highest bidding.*

MVAs require Vickrey auctions (or equivalent English auctions) as basic steps. In reality, however, such functionality won't always be provided by online platforms such as craigslist. Thus a simplified version of MVA occur very often in those platforms. People call it first-come first served which means for every reserve price  $r_i$ , the first one who accept that price wins the item and pays  $r_i$  directly. This mechanism may loose revenue and social efficiency as the person with lower valuation  $p$  may get the item for  $r_i$  while there's someone else who is willing to pay a higher amount of  $q$  where  $r_i \leq p < q < r_{i+1}$ . We won't focus on this first-come first served mechanism because it's harder to analyze analytically and it's inferior than MVAs in terms of both sellers' utility and social welfare.

Since there's no cost charged to buyers, it's obvious to see that whenever a bidder decides to bid, he/she must bid truthfully. Thus the Bayesian Nash Equilibria (BNE) for MVAs can be described as  $k$  thresholds  $a_k, a_{k-1}, \dots, a_0$  where  $a_i > a_{i-1}$ . Whenever a bidder's valuation for the item is greater than  $a_i$ , he/she is going to bid in round  $i$  whose

reserve price is  $r_i$ . Because of efficiency constraint we also have  $r_0 = a_0 = 0$ .

In later analysis, we will think about the equilibrium from another perspective. We firstly decide thresholds  $a_i$  since it's more meaningful for bidders to make decisions and for us to make analysis. For a set of thresholds, we then determine the right reserve prices  $r_i$  that make bidders incentive compatible to bid according to  $a_i$ . The following equations connects  $a_i$  and  $r_i$ :

$$r_0 = a_0 = 0 \text{ and } \forall i (1 \leq i \leq k),$$

$$P(a_i)(a_i - r_i) = \int_{a_{i-1}}^{a_i} (a_i - x)p(x)dx + P(a_{i-1})(a_i - r_{i-1}) \quad (1)$$

assuming

$$\begin{aligned} P(x) &:= F(x)^{n-1} \\ p(x) &:= P'(x) = (n-1)F(x)^{n-2} \end{aligned}$$

The equation 1 says that the bidder with valuation  $a_i$  should be indifferent from bidding in round  $i$  (the left hand side) and bidding in round  $i-1$  (the right hand side). The following theorem describes the equilibrium of MVAs determined by equations above.

**THEOREM 1.** *In a MVA with a pure strategy Bayesian Nash Equilibrium characterized by thresholds  $a_k, a_{k-1}, \dots, a_0$  where the bidder with valuation greater than  $a_i$  (but not greater than  $a_{i+1}$ ) will bid in round  $i$  (recall that round  $k$  is the first round and round 0 is the last round), the reserve prices  $r_i$  shall be:*

$$\begin{aligned} r_0 &= a_0 = 0 \\ r_i &= \left( \int_0^{a_i} x p(x) dx \right) / P(a_i) \quad (i > 0) \end{aligned} \quad (2)$$

**PROOF.** By equation 2, we have  $r_i P(a_i) = \int_0^{a_i} x p(x) dx$  for all  $i$ . Thus the right hand side of equation 1 is:

$$\begin{aligned} & \int_{a_{i-1}}^{a_i} a_i p(x) dx - \int_{a_{i-1}}^{a_i} x p(x) dx + P(a_{i-1})(a_i - r_{i-1}) \\ &= a_i P(a_i) - \frac{a_i P(a_{i-1})}{P(a_i)} - r_i P(a_i) + \frac{r_{i-1} P(a_{i-1})}{P(a_i)} \\ & \quad + \frac{P(a_{i-1}) a_i}{P(a_i)} - \frac{P(a_{i-1}) r_{i-1}}{P(a_i)} \\ &= \text{left hand side of equation 1} \end{aligned}$$

□

This tells us that a bidder will bid in a round of MVA if and only if the expected second highest bidding conditional on this bidder's valuation is the highest is greater than the reserve price of that round. For example, if the distribution is uniform, i.e.  $F(x) = x$ ,  $r_i = \frac{n-1}{n} a_i$  for  $i > 0$ .

### 3.2 Optimality of MVAs

**DEFINITION 4.** *Assume there are  $n$  values  $v_i$  ( $1 \leq i \leq n$ ) independently and identically distributed over  $[0, 1]$  with PDF  $f(x)$  and CDF  $F(x)$ . A query strategy is to find the maximum value by asking queries  $Q_1, Q_2, \dots$  sequentially where  $Q_i \subset [0, 1]$  and  $\forall i \neq j, Q_i \cap Q_j = \emptyset$ . After a query  $Q_i$ , all numbers within  $Q_i$  will be reported. Note that  $Q_i$  may depend on results of  $Q_1, Q_2, \dots, Q_{i-1}$ . Thus a strategy can be denoted as a function:*

$$S(f, m, V, \mathcal{Q} = \{Q_1, Q_2, \dots, Q_{i-1}\}) = Q_i$$

which means, given  $m$  i.i.d. unknown values whose PDF is  $f(x)$ , the reported values set  $V$  and the set of queries asked  $\mathcal{Q}$ , the strategy  $S$  will make  $Q_i$  as the next query.

The cost of each query is equal to  $b + j \cdot c$  where  $j$  is the number of reported values from that query. The cost of a strategy is equal to the sum of all queries' costs it has to ask before identifying the maximum value. The optimal query strategy is the one that has minimum expected cost. We will write such minimum expected cost as  $C^*(f, n)$ , a function of PDF  $f(x)$  and number of values  $n$ .

LEMMA 1. Assume uniform PDF  $u(x) = 1$  and define  $C^*(n) = C^*(u, n)$ . For any other PDF  $f(x)$ , we have

$$C^*(f, n) = C^*(u, n) = C^*(n)$$

PROOF. Define  $F^{-1}(x) = \sup\{y \mid F(y) = x\}$ . For any strategy  $S_f$  that works for PDF  $f$ , we can come up with a strategy  $S_{u|f}$  for uniform PDF  $u$ :

$$S_{u|f}(u, m, V, \mathcal{Q}) = F(S_f(f, m, F^{-1}(V), F^{-1}(\mathcal{Q})))$$

Thus  $C^*(u, n) \leq C^*(f, n)$  since we can adopt any strategy for  $f$  to run under  $u$  with the same cost. Similarly, for any strategy  $S_u$  that works for PDF  $u$ , we can make strategy  $S_{f|u}$  for PDF  $f$ :

$$S_{f|u}(f, m, V, \mathcal{Q}) = F^{-1}(S_u(u, m, F(V), F(\mathcal{Q})))$$

Therefore  $C^*(f, n) \leq C^*(u, n)$ . Combining with  $C^*(u, n) \leq C^*(f, n)$  we have  $C^*(f, n) = C^*(u, n) = C^*(n)$ . Figure [make an illustration] illustrates those two constructions we used for this proof.

□

LEMMA 2. There exists an optimal query strategy that always asks descending queries  $Q_1 = [q_1, 1], Q_2 = [q_2, q_1], \dots$

PROOF. If not, there must be a strategy with non-descending queries and non of its non-descending queries can be changed to descending queries without increasing the cost. In that strategy  $S$ , there must be a first non-descending query  $Q_{i+1} = S(F, m, V, \mathcal{Q} = \{Q_1, Q_2, \dots, Q_i\})$  where  $Q_1$  to  $Q_i$  are all descending. We can make another descending query  $Q'_{i+1} = [q'_{i+1}, q_i]$  (or  $Q'_{i+1} = [q'_{i+1}, 1]$  if  $i = 0$ ) such that

$$\Pr(v_i \in Q_{i+1}) = \int_{Q_{i+1}} f(x)dx = \int_{Q'_{i+1}} f(x)dx = \Pr(v_i \in Q'_{i+1})$$

After  $Q'_{i+1}$  or  $Q_{i+1}$ , we'll use as optimal query as possible.

Since  $Q_1$  to  $Q_i$  are all descending, we have  $m = n$  and  $V = \emptyset$  (otherwise the strategy should terminate without asking  $Q_{i+1}$ ). Thus the expected cost of using  $Q_{i+1}$  and later queries is  $C = b + \sum_{j=0}^n p_j(j \cdot c + C_j)$  where: 1)  $p_j$  is the probability that there are  $j$  reported values within  $Q_{i+1}$ ; 2)  $C_j$  is the expected cost of later queries given that  $j$  values have been found in  $Q_{i+1}$ . Similarly the expected cost of using  $Q'_{i+1}$  and later queries is  $C' = b + \sum_{j=0}^n p'_j(j \cdot c + C'_j)$

As  $\int_{x \in Q'_{i+1}} f(x)dx = \int_{x \in Q_{i+1}} f(x)dx$ , we have  $p'_j = p_j$ . Since  $Q'_{i+1}$  is a descending query,  $\forall j > 0, C'_j = 0 \leq C_j$ . And by lemma 1,  $C'_0 = C_0 = C^*(n)$  because knowing no value is in  $Q_1, Q_2, \dots, Q_{i+1}$  is equivalent to revise PDF  $f(x)$  to a refined PDF

$$f_{i+1}(x) = \begin{cases} \lambda f(x), & x \notin Q_1 \cup Q_2 \cup \dots \cup Q_{i+1} \\ 0, & x \in Q_1 \cup Q_2 \cup \dots \cup Q_{i+1} \end{cases}$$

where  $\lambda$  is a constant to make  $\int_0^1 f_{i+1}(x)dx = 1$ .

Thus  $C' \leq C$  which contradicts to that no non-descending query can be changed to descending query without increasing the cost.

□

THEOREM 2. Among all mechanisms that can include multiple rounds of broadcasts and are required to be efficient (allocate the item to the bidder with highest valuation), Multi-round Vickrey Auctions (MVAs) are of minimum cost.

PROOF. The mechanism is required to be efficient so the mechanism should at least find out the bidder with highest valuation. The best case is that every reply contains the exact and truthful valuation of the corresponding bidder since every non-silent reply has a cost  $c$ . By doing that, we never need someone to reply twice. We then prove that if this best case can be achieved, the descending query mechanisms are optimal, which gives us a lower bound of minimum cost. After that, it's easy to show that MVAs can achieve this lower bound so this is the optimal cost and MVAs are optimal mechanisms.

Firstly, let's prove that descending query mechanisms are optimal if all replies consists the exact valuation of corresponding bidders.

□

COROLLARY 1. If all broadcast costs and bidding costs are charged to sellers, MVAs are optimal if efficiency is required. Such optimal MVA is the one that minimizes the overall cost.

### 3.3 Cost Minimized $\alpha$ -MVA

### 3.4 Approximation of $\alpha$ and Experiments

It's difficult to get an exact closed formula for optimal  $\alpha$ . Thus we are going to use a closed formula to approximate this  $\alpha$ . We'll conduct experiments to compare our approximation with the optimal  $\alpha$  that's computed numerically. We are also going to show comparisons between optimal MVA, approximate optimal MVA and other conventional mechanisms such as Vickrey auctions.

## 4. OPTIMAL MECHANISMS WITH BOTH SELLER'S AND BIDDER'S COST

In this section, we take out the constraint and prove that MVAs are optimal in general. We will also try to find the specific MVA to achieve such optimality and it turns out to be significantly more complicated than our previous simplified case.

### 4.1 Spending Equivalence Theorem and Revenue Optimization Strategy

DEFINITION 5. We say mechanisms satisfy relaxed efficiency constraint with lowest type  $l$  if:

1. They only allocate the item to bidders whose valuation are at least  $l$  (the lowest type is  $l$ )
2. If they will allocate the item, they will always allocate the item to the bidder with highest valuation.

*When we say a mechanism with a lowest type  $l$ , we imply that this mechanism satisfy relaxed efficiency constraint with lowest  $l$ .*

**THEOREM 3.** *For all mechanisms with the same lowest type  $l$ , they will have the same overall spending from all bidders (including bidders' bidding costs and payments to sellers).*

**COROLLARY 2.** *For mechanisms with a fixed lowest type  $l$ , the maximum utility for sellers is achieved when the mechanism minimizes the cost.*

**THEOREM 4.** *MVAs have the minimum cost among all mechanisms with a lowest type  $l$*

**THEOREM 5.** *MVAs are optimal. (watch out for relaxed efficiency constraint)*

**COROLLARY 3.** *Shall we increase lowest type  $l$  a little above Myerson's optimal lowest type to trade payment with cost? Or is it optimal to use Myerson's  $l$  and then minimize cost according to that?*

## **4.2 Experiments to Discover Optimal MVA with a Given Lowest Type**

## **4.3 Analysis of Optimal MVA with Lowest Type**

## **4.4 Using Piecewise Linear MVA to Approximate Optimal MVA**

## **4.5 Choosing Lowest Type?**

## **4.6 Experiments**

Now we are going to compare revenue in general cases. We not only compare our approximate optimal MVAs to optimal MVAs computed numerically, but also compare MVAs to other conventional mechanisms.

## **5. CONCLUSION**