Optimal Broadcast Auctions with Costly Actions

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1. INTRODUCTION

2. COST MODEL AND SETTINGS

In this section, we define our cost model and explain why it makes sense. Our cost model is inspired from online second-hand item transactions, including examples like eBay, craigslist and universities' mailing-lists.

DEFINITION 1. *In our settings, there's one seller selling one item to n buyers (bidders) whose valuations* v_i $(1 \leq i \leq n)$ *are independently and identically distributed (i.i.d.) over* $[0, 1]$ *with PDF* $f(x)$ *and CDF* $F(x)$*. The seller can broadcast a message to all bidders which costs b (the broadcast cost) for the seller. A bidder may reply to that broadcast with cost c (bidding cost) or remain silent with no cost. Such bidding cost c may contain two parts,* β_1, β_2 *where* $\beta_1 + \beta_2 = c$*. The first part* β_1 *is charged to the seller while the second is charged to the corresponding bidder. The bidder's reply should be deterministic with respect to the seller's broadcast (for simplicity, we only consider pure strategy equilibrium).*

Most settings in definition 1 are quite standard except for the cost and broadcast capability.

The bidding costs c may be caused by communication and other verification actions required to put a bidding. For example, the bidder may have to input his credit card number and prepay an amount of money. Without such verification for the bidder, a bidder may bid very high and refuse to pay in the end. A verification for the seller might also be needed. For example, a bidder may want to set up an appointment with the seller to check the item. Setting up such an appointment might be costly because they need to dicuss time and place via emails or phone calls. Attending that appointment may also cost travel fees and time.

We introduce broadcast capability because it's shared by many auctions. For example, a Vickrey auction or a first-price auction with reserve price can be described as a broadcast auction with only one broadcast: telling every bidder the reserve price. The bisection auction [cite bisection auctions] is another example which has many rounds of broadcasts. In each round , it will broadcast a price and ask bidders to reply whether his valuation is beyond or below that price. In real world, sellers make such broadcasts via sending emails to a bunch of receivers (typically a mailing-list), listing items on a platform such as craigslist and eBay, or even showing ads on Internet/TV. Such broadcast activities costs either money (e.g. a list or ads fee) or time and effort (e.g. writing and sending an email).

Appears in: *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2012),* Conitzer, Winikoff, Padgham, and van der Hoek (eds.), June, 4–8, 2012, Valencia, Spain.

Note that in our model, we only give sellers broadcast capability so they cannot find or communicate with each bidder one by one. The first reason to have this constraint is there are too many potential bidders on the Internet (our model focus on online item transactions) and it's hard to explicitly find them one by one. On the other hand, in offline cases where the set of bidders are small and explicit (e.g. the government want to sell a land to one of three companies), it might be helpful to let the seller communicate with bidders one by one [cite search mechanisms]. The second reason to have this constraint is that we want to focus on mechanisms that avoids time consuming bargaining. Such feature is very important as one of the most vital advantages of online transactions are their convenience and the time consuming bargainning can ruin it.

Finally, we define optimal mechanisms to be the ones that maximze seller's utility since when facing many different auction mechanisms, a rational seller will choose the one that gives him maximum utility.

DEFINITION 2. *We say a mechanism is optimal if it gives the seller maximum utility (revenue) which equals to all the value payed to this seller minus all the cost charged to this seller. A class of mechanisms are optimal if they contain one optimal mechanism.*

3. OPTIMAL MECHANISMS WITH EFFICIENCY AND ONLY SELLER'S COST

In this section, we consider a simplified optimizing problem with efficiency constraint and only seller's cost. Though these two constraints simplify our problem a lot, they are reasonable in real cases such as craigslist or moving sales in mailing-lists:

1. In many cases, sellers only have 0 valuation for the item and they cannot commit to withhold the item or prevent re-sales between buyers. For example, some second-hand items will be tossed if they cannot be sold by a particular day, e.g. the day the seller moves out the house. We encounter many free items during second-hand sales as well, which is another demonstration of zero valuation. Under such circumstances, an

efficient mechanism not only maximzes the social welfare but also maximizes the seller's revenue [cite The optimality of being efficient].

2. The bidding cost for each bidder is sometimes negligible compared to bidding cost charged to the seller. For example, if 100 bidders replied to the seller by a 1-minue call, each bidder only has a tiny 1 minue cost. But for the seller, it's a big 100 minutes cost which is very annoying. It's also necessary to remove bidder's bidding cost to achieve efficiency. Otherwise, the item may not be able to allocate to the highest bidder when that highest valuation is below the bidder's bidding cost.

The rest of this section is organized as follows. First of all, we introduce a mechanism called Multi-round Vickrey Auction (MVA) based on what's been used in realworld online second-hand item transactions. Then we prove that MVAs are optimal (so there exists a MVA that's optimal). After that we'll try to find the specific MVA that achieves the optimality. Finally, we conduct some experiments to compare the optimal MVA with other mechanisms.

3.1 Multi-round Vickrey Auctions

A Multi-round Vickrey Auction (MVA) has multiple rounds of Vickrey auctions with progressively decreasing reserve prices. This kind of auction effectively occurs on eBay. The seller may set up a reserve price and let buyers bid for this item. The proxy bidding functionality makes such an auction equivalent to a Vickrey auction with a reserve price. If no buyers bid for a given reserve price, the seller may lower the reserve price, which makes the whole process equivalent to an MVA.

Definition 3. *(Multi-round Vickrey Auction, MVA)*
In a Multi-round Vickrey Auction (MVA), there's a sequence of reserve prices $r_1, r_2, \ldots, r_k$ where $r_k > r_{k+1}$. The seller creates a Vickrey auction with a reserve price r_i at time i (or round i). In each Vickrey auction, if only one buyer bids, he/she gets the item and pays reserve price. Otherwise, the buyer with the highest bidding gets the item and pays the second highest bidding.

MVAs require Vickrey auctions (or equivalent English auctions) as basic steps. In reality, however, such functionality won't always be provided by online platforms such as craigslist. Thus a simplified version of MVA occur very often in those platforms. People call it first-come first served which means for every reserve price r_i, the first one who accept that price wins the item and pays r_i directly. This mechanism may loose revenue and social efficiency as the person with lower valuation p may get the item for r_i while there's someone else who is willing to pay a higher amount of q where $r_i \le p < q < r_{i-1}$. We won't focus on this first-come first served mechanism because it's harder to analyze analytically and it's inferior than MVAs in terms of both sellers' utility and social welfare.

Since there's no cost charged to buyers, it's obvious to see that whenever a bidder decides to bid, he/she must bid truthfully. Thus the Bayesian Nash Equilibria (BNE) for MVAs can be described as k thresholds $a_1, a_2, \ldots, a_k$ where $a_i > a_{i+1}$. Whenever a bidder's valuation for the item is greater than a_i, he/she is going to bid in round i whose reserve price is r_i. Because of efficiency constraint we also have $r_k = a_k = 0$.

In later analysis, we will think about the equilibrium from another perspective. We firstly decide thresholds a_i since they are more meaningful for bidders to make decisions and for us to make analysis. For a set of thresholds, we then determine the right reserve prices r_i that make bidders incentive compatible to bid according to a_i. The following equations connects a_i and r_i:

$r_k = a_k = 0$ and $\forall i\ (1 \le i < k)$,

$$P(a_i)(a_i - r_i) = \int_{a_{i+1}}^{a_i} (a_i - x)p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \tag{1}$$

assuming

$$P(x) := F(x)^{n-1}$$
$$p(x) := P'(x) = (n-1)F(x)^{n-2}f(x)$$

The equation 1 says that the bidder with valuation a_i should be indifferent from bidding in round i (the left hand side) and bidding in round $i-1$(the right hand side). The following theorem describes the equilibrium of MVAs determined by equations above.

Theorem 1. *If we make reserve prices r_i to be:*

$$r_k = a_k = 0$$
$$r_i = \left(\int_0^{a_i} x\,p(x)dx\right)/P(a_i) \qquad (i < k) \tag{2}$$

Such MVA will have a pure strategy Bayesian Nash Equibibrium characterized by thresholds $a_1, a_2, \ldots, a_k$ where the bidder with valuation greater than a_i (but not greater than a_{i-1}) will bid in round i

Proof. By equation 2, we have $r_i P(a_i) = \int_0^{a_i} x\,p(x)dx$ for all i. Thus the right hand side of equation 1 is:

$$\int_{a_{i+1}}^{a_i} a_i p(x)dx - \int_{a_{i+1}}^{a_i} x\,p(x)dx + P(a_{i+1})(a_i - r_{i+1})$$
$$= a_i P(a_i) - \cancel{a_i P(a_{i+1})} - r_i P(a_i) + \cancel{r_{i+1} P(a_{i+1})}$$
$$+ \cancel{P(a_{i+1}) a_i} - \cancel{P(a_{i+1}) r_{i+1}}$$
$$= \text{left hand side of equation 1}$$

□

This tells us that a bidder will bid in a round of MVA if and only if the expected second highest bidding conditional on this bidder's valuation is the highest is greater than the reserve price of that round. For example, if the distribution is uniform, i.e. $F(x) = x$, $r_i = \frac{n-1}{n} a_i$ for $i > 0$. [cite sequentailly optimal auctions] has given some more discussions and proof (e.g. once a bidder choose to bid, bid truthfully is a unique weekly dominant strategy) about the equilibrium of this kind of sequential auctions. That paper, however, aims on the optimal auctions for a different cost model with time discount but without broadcast or bidding cost. We will discuss this difference later when we introduces bidding cost for bidders.

3.2 Optimality of MVAs

Since the mechanism is required to be efficient, by revenue equivalence theorem [cite] we know that the seller's gross revenue without substracting costs is fixed. Thus to maximize his utility is equivalent to minimize the cost. To satisfy efficiency constraint, the mechanism should at least find out the bidder with highest valuation. The best case is that every reply contains the exact and truthful valuation of the corresponding bidder since every non-silent reply has a cost c. By doing that, we never need someone to reply twice. The optimizing problem in this best case is defined as definition 4. The minimum cost of this best case optimizing problem provides us a lower bound for minimum cost of our mechanisms. We will then prove that MVAs can achieve this lower bound so MVAs are optimal.

DEFINITION 4. *Assume there are n values v_i $(1 \leq i \leq n)$ independently and identically distributed over $[0,1]$ with PDF $f(x)$ and CDF $F(x)$. A query strategy is to find the maximum value by asking queries $Q_1, Q_2, \ldots$ sequentially where $Q_i \subset [0,1]$ and $\forall i \neq j,\ Q_i \cap Q_j = \emptyset$. After a query Q_i, all numbers within Q_i will be reported. Note that Q_i may depend on results of $Q_1, Q_2, \ldots, Q_{i-1}$. Thus a strategy can be denoted as a function:*

$$S\big(f, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_{i-1}\}\big) = Q_i$$

which means, given m i.i.d. unknown values whose PDF is $f(x)$, the reported values set V and the set of queries asked $\mathcal{Q}$, the strategy S will make Q_i as the next query.

The cost of each query is equal to $b + j \cdot c$ where j is the number of reported values from that query. The cost of a strategy is equal to the sum of all queries' costs it has to ask before identifying the maximum value. The optimal query strategy is the one that has minimum expected cost. We will write such minimum expected cost as $C^*(f,n)$, a function of PDF $f(x)$ and number of values n.

We are then going to find out the optimal query strategy. Firstly, we need a lemma that tells us that the minimum cost is independent of PDF $f(x)$.

LEMMA 1. *Assume uniform PDF $u(x) = 1$ and define $C^*(n) = C^*(u,n)$. For any other PDF $f(x)$, we have*

$$C^*(f,n) = C^*(u,n) = C^*(n)$$

PROOF. Define $F^{-1}(x) = \sup\{y \mid F(y) = x\}$. For any strategy S_f that works for PDF f, we can come up with a strategy $S_{u|f}$ for uniform PDF u:

$$S_{u|f}\big(u, m, V, \mathcal{Q}\big) = F\Big(S_f\big(f, m, F^{-1}(V), F^{-1}(\mathcal{Q})\big)\Big)$$

Thus $C^*(u,n) \leq C^*(f,n)$ since we can adopt any strategy for f to run under u with the same cost. Similarly, for any strategy S_u that works for PDF u, we can make strategy $S_{f|u}$ for PDF f:

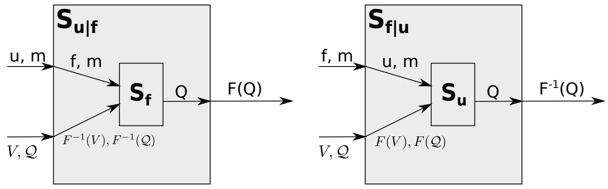
$$S_{f|u}\big(f, m, V, \mathcal{Q}\big) = F^{-1}\Big(S_u\big(u, m, F(V), F(\mathcal{Q})\big)\Big)$$

Therefore $C^*(f,n) \leq C^*(u,n)$. Combining with $C^*(u,n) \leq C^*(f,n)$ we have $C^*(f,n) = C^*(u,n) = C^*(n)$. Figure [make an illustration] illustrates those two constructions we used for this proof.

□

After that, we prove that descending queries are optimal for this best case optimizing problem.

LEMMA 2. *There exists an optimal strategy with only descending queries $Q_1 = [q_1, 1],\ Q_2 = [q_2, q_1),\ Q_3 = [q_3, q_2) \ldots$*



(a) Construct $S_{u|f}$ from S_f (b) Construct $S_{f|u}$ from S_u

Figure 1: These two figures illustrates how to construct a strategy for uniform PDF u from another strategy for another arbitrary PDF f and vice versa. Here we depict a strategy as a box which takes four inputs $f, m, V, \mathcal{Q}$ (PDF, number of unknown values, reported values set, set of asked queries) and make an output Q (the next query)

PROOF. If not, there must be an optimal strategy where none of its non-descending queries can be changed to descending queries without increasing the cost. In that strategy S, there must be a first non-descending query $Q_{i+1} = S\big(F, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_i\}\big)$ where Q_1 to Q_i are all descending. We can make another descending query $Q'_{i+1} = [q'_{i+1}, q_i)$ (or $Q'_{i+1} = [q'_{i+1}, 1]$ if $i = 0$) such that

$$\Pr(v \in Q_{i+1}) = \int\limits_{Q'_{i+1}} f(x)\mathrm{d}x = \int\limits_{Q_{i+1}} f(x)\mathrm{d}x = \Pr(v \in Q'_{i+1})$$

After Q'_{i+1}, we'll use as optimal query as possible.

Since Q_1 to Q_i are all descending, we have $m = n$ and $V = \emptyset$ (otherwise the strategy should terminate without asking Q_{i+1}). Define C to be the expected cost of using Q_{i+1} and later queries. Similarly we define C' for Q'_{i+1}: $C = b + \sum_{j=0}^{n} p_j (j \cdot c + C_j)$ and $C' = b + \sum_{j=0}^{n} p'_j (j \cdot c + C'_j)$ where: p_j (or p'_j) is the probability that there are j reported values within Q_{i+1} (or Q'_{i+1}); C_j (or C'_j) is the expected cost of later queries given that j values have been found in Q_{i+1} (or Q'_{i+1}).

As $\Pr(v \in Q_{i+1}) = \Pr(v \in Q'_{i+1})$, we have $p'_j = p_j$. Since Q'_{i+1} is a descending query, $\forall j > 0,\ C'_j = 0 \leq C_j$. And by lemma 1, $C'_0 = C_0 = C^*(n)$ because knowing no value is in $Q_1, Q_2, \ldots, Q_{i+1}$ is equivalent to revise PDF $f(x)$ to a refined PDF

$$f_{i+1}(x) = \begin{cases} \lambda f(x), & x \notin Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \\ 0, & x \in Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \end{cases}$$

where λ is a constant to make $\int_0^1 f_{i+1}(x)dx = 1$.

Thus $C' \leq C$ which contradicts to that no non-descending query can be changed to descending query without increasing the cost.

□

Finally, we conclude the optimality of MVAs.

THEOREM 2. *Among all mechanisms that can include multiple rounds of broadcasts and are required to be efficient (allocate the item to the bidder with highest valuation), Multiround Vickrey Auctions (MVAs) are of minimum cost.*

PROOF. The best case optimizing problem defined in definition 4 provides us a lower bound of minimum cost we can achieve by any mechanisms. By lemma 2, such lower bound minimum cost can be achieved by descending query strategy $Q_1 = [q_1, 1], Q_2 = [q_2, q_1), Q_3 = [q_3, q_2) \ldots$. Making $a_1 = q_1, a_2 = q_2, a_3 = q_3, \ldots$, by theorem 2 we are able to design such an MVA with reserve prices $r_1, r_2, \ldots$ whose Bayesian Nash Equilibrium achieves this best case descending query strategy. Thus, MVAs are optimal.

□

And by revenue equivalence theorem, MVAs maximize the seller utility.

COROLLARY 1. *If all broadcast costs and bidding costs are charged to sellers, MVAs are optimal if efficiency is required. Such optimal MVA is the one that minimizes the overall cost.*

3.3 Cost Minimized α-MVA

Now let's try to calculate the parameters (thresholds a_i or equivalently reserve prices r_i) of the optimal MVA for a given settings F, n, b, c (valuation CDF, number of bidders, broadcast cost, bidding cost). By lemma 1, the cost is indifferent with F and we can always derive an optimal mechanism for any F from a uniform distribution. Thus we will focus on uniform cases below. We will also introduce $\rho = \frac{b}{c}$ to simplify our analysis by normalize bidding cost c to 1 and thus broadcast cost b to ρ.

An optimal MVA must be an α-MVA where each round only $(1-\alpha)n$ bidders are expected to bid, i.e. $a_1 = \alpha, a_{i+1} = \alpha \cdot a_i$ for uniform cases. Then the expected overall cost C satisfies: $C = \rho + (1-\alpha)n + \alpha^n C$. In the right hand side, the first term ρ is the broadcast we have to use in the first round, the second term $(1-\alpha)n$ is the expected bidding cost for the first round, the third term $\alpha^n C$ lis a recursive term, the probability that no one bids α^n and if that happens the same cost C should be expected in later rounds. Thus $C = \frac{\rho + (1-a)n}{1-\alpha^n}$. To minimize cost C, we have:

$$\frac{\partial C}{\partial \alpha} = \frac{\alpha^{n-1}\, n\ (\rho + (1-\alpha)\ n)}{(1-\alpha^n)^2} - \frac{n}{1-\alpha^n} = 0 \qquad (3)$$

[Increasing Threshold Search for Best-Valued Agents] has a proof for why α-MVA is optimal among MVAs and how to determine α. Thus if you find them not intuitive you may reference more details there. Our simplified cost model (with efficiency constraint and no buyer's cost) is equivalent to their cost model where learning cost is linear to the number of replied agents. That paper makes descending query as a contraint (in their model they want to find the minimum value so descending query becomes increasing threshold search) and proves that α-MVA is optimal among descending query mechanisms. Our focus in this section, however, is to have a preliminary introduction for MVA and proves optimality of descending queries and eventually MVAs. Thus we omit the proof about α-MVA. Even for the α, we will be more interested in its relation with larger n so we will give approximations for α.

3.4 Approximation of α and Experiments

Equation 3 can be solved numerically but it's hard to get a closed form for α. Thus we want to approximate α using some simpler formulas.

Firstly, $\alpha = \frac{n-1}{n}$ is a natural guess which means each round the expected number of biddings is equal to 1. Thus we have $C(\alpha = \frac{n-1}{n}) = \frac{\rho+1}{1-(\frac{n-1}{n})^n}$. Because $(\frac{n-1}{n})^n \le e^{-1}$, we have $C(\alpha = \frac{n-1}{n}) \le \frac{\rho+1}{1-e^{-1}}$. It's obvious that at least one broadcast and one bidding is required to terminate so $C \ge \rho + 1$. Thus $\alpha = \frac{n-1}{n}$ is a $\frac{1}{1-e^{-1}}$ approximation. That means, by simply choosing $\alpha = \frac{n-1}{n}$, we would at most get about 1.582 times of optimal cost. Another obersavation of this approximation is that no matter how large n is, the cost is at most $\frac{1}{1-e^{-1}}(\rho+1)$. Thus the optimal cost is bounded by constant $O(1)$ no matter how large n is.

To solve it approximately, we assume that $(1-a)n \ll \rho$ and $(1-\alpha^n) \approx (1-\alpha^{n-1})$ since intuitively α will be very close to 1 (more than $1 - \frac{1}{n}$) when n becomes large. Then we have:

$$n\rho \cdot \alpha^n - n(1-\alpha^n) = 0$$

So we get $\alpha = (1+\rho)^{-\frac{1}{n}}$ and

$$\lim_{n\to\infty} C(\alpha = (1+\rho)^{-\frac{1}{n}}) = \frac{\rho^2 + (\log(\rho+1)+1)\ \rho + \log(\rho+1)}{\rho} + O(\frac{1}{n})$$

This is amazing as the cost is bounded to about ρ no matter how large n is.

However, this α doesn't satisfy our assumption that $(1-a)n \ll \rho$ since $\lim_{n\to\infty}(1-\alpha)n = \ln(1+\rho)$. Therefore, we fix our assumption by $(1-a)n \approx \ln(1+\rho)$. Then we have:

$$\alpha = \left(1 + \rho + \ln(1+\rho)\right)^{-\frac{1}{n}}$$

.

It's difficult to get an exact closed formula for optimal α. Thus we are going to use a closed formula to approximate this α. We'll conduct experiments to compare our approximation with the optimal α that's computed numerically. We are also going to show comparisons between optimal MVA, approximate optimal MVA and other conventional mechanisms such as Vickrey auctions.

4. OPTIMAL MECHANISMS WITH BOTH SELLER'S AND BIDDER'S COST

In this section, we take out the constraint and prove that MVAs are optimal in general. We will also try to find the specific MVA to achieve such optimality and it turns out to be significantly more complicated than our previous simplified case.

4.1 Spending Equivalence Theorem and Revenue Optimization Strategy

DEFINITION 5. *We say mechanisms satisfy relaxed efficiency constraint with lowest type l if:*

1. *They only allocate the item to bidders whose valuation are at least l (the lowest type is l)*

2. *If they will allocate the item, they will always allocate the item to the bidder with highest valuation.*

When we say a mechanism with a lowest type l, we imply that this mechanism satisfy relatex efficiency constraint with lowest l.

THEOREM 3. *For all mechanisms with the same lowest type l, they will have the same overall spending from all bidders (including bidders' bidding costs and payments to sellers).*

COROLLARY 2. *For mechanisms with a fixed lowest type l, the maximum utility for sellers is achieved when the mechanism minizes the cost.*

THEOREM 4. *MVAs have the minimum cost among all mechanisms with a lowest type l*

THEOREM 5. *MVAs are optimal. (watch out for relaxed efficiency constraint)*

COROLLARY 3. *Shall we increase lowest type l a little above Myerson's optimal lowest type to trade payment with cost? Or is it optimal to use Myerson's l and then minimize cost according to that?*

4.2 Experiments to Discover Optimal MVA with a Given Lowest Type

4.3 Analysis of Optimal MVA with Lowest Type

4.4 Using Piecewise Linear MVA to Approximate Optimal MVA

4.5 Choosing Lowest Type?

4.6 Experiments

Now we are going to compare revenue in general cases. We not only compare our approximate optimal MVAs to optimal MVAs computed numerically, but also compare MVAs to other conventional mechanisms.

5. CONCLUSION