Optimal Broadcast Auctions with a new Cost Model

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1. INTRODUCTION

Auctions constitute a favored method for allocating scarce resources in multiagent systems. However, communication requirements can pose a bottleneck. Motivated by the revelation principle, much of the theory of mechanism design considers *direct-revelation* mechanisms, in which each agent declares its entire valuation function to the auctioneer. The corresponding overhead, not only in terms of communication per se but also in terms of the agent having to completely *determine* its valuation function, can be prohibitive. All of this is well understood (for further discussion, see, e.g., [2]), and a significant amount of research has been devoted to the design of *iterative* auction mechanisms (e.g., [8]) and (roughly equivalently) auctions with explicit *elicitation* of agents' valuations (e.g., [9]). Such auctions aim to avoid the communication of unnecessary information. For example, in a Vickrey auction, if it is known that bidders 1 and 2 have valuations above $100 and bidder 3 has one below $100, then there is no need to query bidder 3 any further.

How do we evaluate how effective a particular iterative auction mechanism is in reducing communication? Perhaps the simplest measure is the number of bits communicated (see, e.g., []). Within a particular query model, it may also make sense to minimize the total number of queries (see, e.g., []). However, in this paper, we argue that such existing models fail to capture important aspects of the cost of communication in certain types of Internet auctions. In particular, in such auctions often the most costly communication that takes place is the *first* time that a bidder gives a positive response, indicating having a nonzero value for an item. This can be the case for several reasons. One possibility is that the auction website at this point may insist that the bidder provides payment (say, credit card) information or places money in escrow. Otherwise, a malicious user may steer the auction in a particular direction and in the end refuse to pay. Alternatively, in other settings (such as an item having been posted for sale on craigslist or a similar list), at this point the bidder may wish to set up an appointment with the seller to check the item. In both cases, these actions come at costs for both the potential buyer and the seller, in terms of effort and time, loss of privacy, and various risks. While participation costs have been studied before in auctions [12, 13], a distinguishing feature of our model is that a bidder can observe the proceedings of the auction at no cost, until the bidder decides to actively participate by at some indicating a nonzero valuation and thereby changing the course of the auction. This appears to us to be a more natural model of Internet (or other highly anonymous) auctions of the type discussed above.

The rest of this paper is organized as follows. In Section 2, we provide a formal model of elicitation cost motivated by the observations just discussed. In Section 3, we study the special case in which efficient allocation is a constraint and bidders experience no cost from bidding. The characterization of the optimal mechanism in this special case turns out to be closely related to earlier work on finding an optimal agent by iteratively relaxing the parameters of the search []. In Section 4, we drop the assumptions of efficiency and no bidder cost for elicitation.

2. COST MODEL AND SETTINGS

In this section, we formally define our cost model and explain its motivation.

DEFINITION 1 (SETTING). *One seller is selling one item to n buyers (bidders) whose valuations v_i $(1 \leq i \leq n)$ are independently and identically distributed (i.i.d.) over $[0,1)$ with PDF $f(x)$ and CDF $F(x)$. (We assume that for all $\epsilon > 0$, we have $F(\epsilon) > 0$.) The seller can broadcast a message to all bidders, at cost b to the seller. A bidder can reply to that broadcast, or remain silent. If a bidder replies (does not stay silent), this comes at a cost β_1 to the seller and a cost β_2 to that bidder, for a total cost of $c = \beta_1 + \beta_2$.*

Two key aspects of this model are that (1) staying silent comes at no cost and (2) replying comes at a positive cost, *and this positive cost is the same no matter how complex the query and answer are*. This is motivated by the settings discussed in the introduction, where a bidder can observe the process of the auction (or messages posted on a board) silently at no cost, but once the bidder acts in the auction, costs occur—e.g., the bidder has to submit credit card information, the bidder and the seller have to arrange an in-person meeting, etc.). A key aspect of such costs is that they tend to be the same regardless of the level of detail in the bidder's answer: for example, if the bidder just reports having a valuation greater than $10 without specifying what it is exactly (rather than reporting a valuation of exactly $14), this is not likely to reduce any of the above costs. In particular, the seller is likely to want to verify the bidder's authenticity at any point where the bidder's reply affects the course of the auction from then on. This leads to the following easy proposition:

PROPOSITION 1. *In the model defined in 1, without loss of optimality, we can restrict our attention to (broadcast) queries that result in each bidder either staying silent, or immediately revealing his exact valuation to the seller.*

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Another notable point is that we restrict communication from the seller to broadcast queries. This is a common restriction: any sealed-bid auction can be considered a broadcast auction with only one broadcast: the reserve price. The bisection auction [4] is an example auction with many rounds of broadcasts. In each round, it broadcasts a price and asks bidders to reply whether their valuation is above or below that price. Besides the broadcast model being simple, natural, and common in existing auction mechanisms, it is also naturally motivated in the Internet domains we consider: the bidders are entirely anonymous until their first reply, so before this point querying such bidders individually is not feasible and they can only be reached by, say, posting on a public website; and after they have replied, we will know their valuation exactly (by Proposition 1) and we no longer need to query them. Of course, there are offline cases where the set of bidders is small and explicit (e.g., the government wants to sell land or spectrum to one of three known companies); in such settings, it can indeed be helpful for the seller to communicate with bidders individually [6, 10]. Such settings do not fit our model; we explicitly focus on highly anonymous settings, and the costs that the seller incurs from the broadcast query correspond to the time, effort, and third-party charges associated with posting a public message.

DEFINITION 2. *A* mechanism *in our setting consists of (1) a full contingency plan for which query to broadcast at each point, depending on answers given so far, and a termination condition; and (2) an allocation and pricing rule that is defined on each terminal state. A mechanism is* individually rational *if losing bidders never pay and winning bidders never pay more than their valuations. We say an individually rational mechanism is* optimal *if it has a Bayes-Nash equilibrium for the bidders that maximizes the seller's profit (among all Bayes-Nash equilibria of all individually rational auction mechanisms). Here, seller profit is revenue minus seller elicitation costs. A class of mechanisms is optimal if it contains at least one optimal mechanism.*

3. OPTIMAL MECHANISMS WITH EFFICIENCY AND ONLY SELLER'S COST

In this section, we make two simplifying assumptions: (1) we restrict attention to mechanisms that allocate the item efficiently and (2) we assume that the bidder cost for replying (β_2) is zero.[1] For example, someone who is moving and selling furniture on craigslist is likely to have 0 valuation for the item and cannot commit to withhold the item or prevent re-sale between bidders. Under such circumstances, efficient mechanisms not only maximizes the social welfare but also maximizes the seller's revenue [1] (though this does not consider elicitation costs). In Section 4 we drop these assumptions.

The rest of this section is organized as follows. First, we introduce a class of mechanisms called multi-round Vickrey auctions (MVA). Then, we prove that we can restrict attention to MVAs without loss of optimality. After that, we find the specific MVA that is optimal. Finally, we experimentally compare this optimal MVA to some other natural mechanisms.

3.1 Multi-round Vickrey Auctions

In an MVA, the seller runs a Vickrey auction with a reserve price; if nobody bids (above the reserve price), the seller runs another Vickrey auction with a lower reserve price, etc., until the item is sold. (In Section 4, we will also consider MVAs that can terminate without having sold the item.) For example, consider a sequence of eBay auctions (with proxy bidding) in which the seller is repeatedly lowering the reserve price.

DEFINITION 3 (MULTI-ROUND VICKREY AUCTION (MVA)). *An MVA is defined by a sequence of reserve prices $r_1, r_2, \ldots$ (which may be finite or infinite) where $r_i > r_{i+1}$. In round i, a Vickrey auction with reserve price r_i is run (if the item has not been sold yet).*

In an MVA, *if* a bidder decides to bid (above the reserve price), it is optimal to bid truthfully since doing so is dominant in a Vickrey auction. However, a bidder may choose strategically to stay silent even with a valuation above the reserve price, in the belief that nobody else will bid this round so that the price will decrease in the next round. Thus, in our (symmetric) setting, Bayes-Nash Equilibria (BNE) for MVAs can be described by a sequence of thresholds $a_1, a_2, \ldots$ where $a_i > a_{i+1}$; in round i, a bidder bids if and only if his valuation is at least a_i.

We will be interested in the following question: given desired thresholds a_i, which reserve prices r_i result in these thresholds? Then, by the revelation principle, we can convert this to a mechanism in which we query agents whether their valuations are above a_i and it is optimal for them to respond truthfully.

LEMMA 1. *Consider a symmetric strategy profile in an MVA where in the ith round, valuation a_i is the threshold for bidding (so that bidders with lower valuations stay silent and those with higher valuations bid). Then this constitutes a Bayes-Nash equilibrium if:*

$$P(a_i)(a_i - r_i) = \int_{a_{i+1}}^{a_i} (a_i - x)p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \quad (1)$$

(where $P(x) := F(x)^{n-1}$ and $p(x) := P'(x) = (n-1)F(x)^{n-2}f(x)$) when i is not the last round, and either $a_i = r_i$ when i is the last round or $\lim_{i\to\infty} a_i = \lim_{i\to\infty} r_i$.

PROOF. Consider a bidder with valuation a_i; we first show that in round i, such a bidder is indifferent between bidding and staying silent if the condition holds. If i is the last round, clearly he is indifferent between bidding and not iff $a_i = r_i$. We now consider the case where i is not the last round. If another bidder bids in round i, that bidder will bid at least a_i, and our bidder will have zero utility. Therefore, the left-hand side of (1) represents the expected utility to our bidder for bidding now. Corresponding to the right-hand side, if our bidder stays silent, and there is a next round and our bidder bids in that next round, then he will win in that round, either with another bidder bidding (first term) or not (second term).

If we now consider a bidder with valuation above a_i, a similar analysis shows that this bidder strictly prefers bidding in round i to waiting one more round; inductively, he will prefer it to waiting any number $k > 0$ more rounds; and he will also prefer this to never bidding because either $a_j = r_j$ when j is the last round or $\lim_{j\to\infty} a_j = \lim_{j\to\infty} r_j$. Finally, let us consider a bidder whose valuation lies below all the a_i. Because either $a_j = r_j$ when j is the last round or $\lim_{j\to\infty} a_j = \lim_{j\to\infty} r_j$, this bidder is best off never bidding. □

For a mechanism that allocates efficiently, we either have $r_k = a_k = 0$ for some k, or $\lim_{i\to\infty} r_i = 0$.

THEOREM 1. *Given a decreasing sequence of $a_i \in [0, 1)$ that ends at or converges to 0, let*

$$r_i = \left(\int_0^{a_i} x\, p(x)dx \right) / P(a_i) \qquad (\textit{if } a_i > 0) \quad (2)$$

[1]Note that if (2) does not hold, then allocating efficiently may not be possible without the seller compensating the bidders for their bidding costs.

and $r_i = 0$ if $a_i = 0$. The corresponding MVA has a pure strategy Bayes-Nash equilibrium characterized by bidding thresholds $a_1, a_2, \ldots$.

PROOF. We show that the conditions of Lemma 1 hold. If i is the last round, then $a_i = 0 = r_i$. Also, $\lim_i r_i \le \lim_i \left(\int_0^{a_i} a_i\, p(x)dx\right) / P(a_i) = \lim_i a_i = 0$. If i is not the last round, then by (2), we have $r_i P(a_i) = \int_0^{a_i} x\, p(x)dx$ for all i (this clearly also holds if $a_i = 0$). Thus the right-hand side of equation 1 is:

$$\begin{aligned}
&\int_{a_{i+1}}^{a_i} a_i p(x)dx - \int_{a_{i+1}}^{a_i} x\, p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \\
&= a_i P(a_i) - \cancel{a_i P(a_{i+1})} - r_i P(a_i) + \cancel{r_{i+1} P(a_{i+1})} \\
&\quad + \cancel{P(a_{i+1}) a_i} - \cancel{P(a_{i+1}) r_{i+1}} \\
&= \text{left hand side of (1)}
\end{aligned}$$

□

This tells us that a bidder will bid in a round of an MVA if and only if the expected second-highest bid conditional on this bidder's valuation being the highest is greater than the reserve price of that round. For example, if the distribution is uniform, we have $r_i = \frac{n-1}{n} a_i$.

A similar analysis to the one in this subsection (in a model that includes discounting) appears in [5]. However, that paper does not consider communication costs, and so we diverge from that work in what follows.

3.2 Optimality of MVAs

Since the mechanism is required to be efficient and bidders' communications costs (β_2) are zero, any mechanism that gives utility zero to an agent with valuation zero results in the same revenue for the seller, by the revenue equivalence theorem [7].[2] Hence, maximizing profit is equivalent to minimizing the seller's query costs.

By the revelation principle, we can restrict our attention to mechanisms in which agents always answer queries truthfully in equilibrium.[3] By Proposition **??**, we can assume that an agent reveals his entire valuation when not staying silent. By the efficiency constraint, the mechanism must at least discover a bidder with the highest valuation. This optimization problem corresponds to Definition 4. We will then prove that MVAs can achieve this lower bound and are hence optimal.

DEFINITION 4. *A* (direct-revelation) query *is given by a subset $Q \subseteq [0, 1)$, such that if the agent's valuation is in Q, he replies with his exact valuation, and otherwise stays silent. The cost of a query is $b + j \cdot c$ where j is the number of bidders who do not stay silent. A strategy for asking queries can be represented by a function*

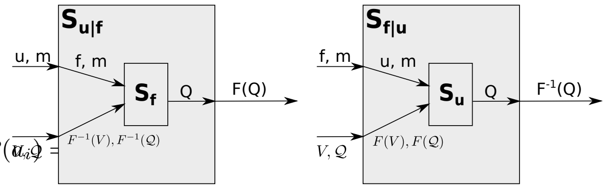
$$S\big(f, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_{i-1}\}\big) = Q_i$$

which means: suppose that the set of queries asked previously is $\mathcal{Q}$, the set of values already reported is V, and there are m bidders left who have not responded (whose valuations were drawn i.i.d. from f); then the strategy S will next ask Q_i.

$C_{f,n}(S)$ is the expected cost of strategy S, and we wish to find $C^*_{f,n} = \inf_S C_{f,n}(S)$.

[2]Moreover, by individual rationality we cannot give an agent with valuation zero less than zero utility; we could give such an agent more than zero utility (as in redistribution mechanisms []), but this would only hurt revenue.

[3]For example, for MVAs, we can directly query the agents whether their valuations are above the a_i while still charging according to the r_i.



(a) Construct $S_{u|f}$ from S_f (b) Construct $S_{f|u}$ from S_u

Figure 1: These two figures illustrates how to construct a strategy for uniform PDF u from another strategy for another arbitrary PDF f and vice versa. Here we depict a strategy as a box which takes four inputs $f, m, V, \mathcal{Q}$ (PDF, number of unknown values, reported values set, set of asked queries) and make an output Q (the next query)

To find out the optimal query strategy, we first introduce a lemma saying that the minimum cost is independent of PDF $f(x)$.

LEMMA 2. *Assume uniform PDF $u(x) = 1$ and define $C^*(n) = C^*(u, n)$. For any other PDF $f(x)$, we have*

$$C^*(f, n) = C^*(u, n) = C^*(n)$$

PROOF. Define $F^{-1}(x) = \sup\{y \mid F(y) = x\}$. For any strategy S_f that works for PDF f, we can come up with a strategy $S_{u|f}$ for uniform PDF u:

$S_{u|f}\big(u, m, V, \mathcal{Q}\big) = F\Big(S_f\big(f, m, F^{-1}(V), F^{-1}(\mathcal{Q})\big)\Big)$

Thus $C^*(u, n) \le C^*(f, n)$ since we can adopt any strategy for f to run under u with the same cost. Similarly, for any strategy S_u that works for PDF u, we can make strategy $S_{f|u}$ for PDF f:

$S_{f|u}\big(f, m, V, \mathcal{Q}\big) = F^{-1}\Big(S_u\big(u, m, F(V), F(\mathcal{Q})\big)\Big)$

Therefore $C^*(f, n) \le C^*(u, n)$. Combining with $C^*(u, n) \le C^*(f, n)$ we have $C^*(f, n) = C^*(u, n) = C^*(n)$. Figure 1 illustrates those two constructions we used for this proof.

□

Then we prove that descending query strategies are optimal:

LEMMA 3. *There exists an optimal strategy with only descending queries $Q_1 = [a_1, 1],\ Q_2 = [a_2, a_1),\ Q_3 = [a_3, a_2) \ldots$*

PROOF. If not, there must be an optimal strategy where none of its non-descending queries can be changed to descending queries without increasing the cost. In that strategy S, there must be a first non-descending query $Q_{i+1} = S\big(F, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_i\}\big)$ where Q_1 to Q_i are all descending. We can make another descending query $Q'_{i+1} = [a'_{i+1}, a_i)$ (or $Q'_{i+1} = [a'_{i+1}, 1]$ if $i = 0$) such that

$$\Pr(v \in Q_{i+1}) = \int\limits_{Q_{i+1}} f(x)\mathrm{d}x = \int\limits_{Q'_{i+1}} f(x)\mathrm{d}x = \Pr(v \in Q'_{i+1})$$

After Q'_{i+1}, we will use as optimal query as possible.

Since Q_1 to Q_i are all descending, we have $m = n$ and $V = \emptyset$ (otherwise the strategy should terminate without asking Q_{i+1}). Define C to be the expected cost of using Q_{i+1} and later queries. Similarly we define C' for Q'_{i+1}:

$C = b + \sum_{j=0}^{n} p_j (j \cdot c + C_j)$ and $C' = b + \sum_{j=0}^{n} p'_j (j \cdot c + C'_j)$ where: p_j (or p'_j) is the probability that there are j reported values within Q_{i+1} (or Q'_{i+1}); C_j (or C'_j) is the expected cost of later queries given that j values have been found in Q_{i+1} (or Q'_{i+1}).

As $\Pr(v \in Q_{i+1}) = \Pr(v \in Q'_{i+1})$, we have $p'_j = p_j$. Since Q'_{i+1} is a descending query, $\forall j > 0$, $C'_j = 0 \le C_j$. And by lemma 2, $C'_0 = C_0 = C^*(n)$ because knowing no value is in $Q_1, Q_2, \ldots, Q_{i+1}$ is equivalent to revise PDF $f(x)$ to a refined PDF

$$f_{i+1}(x) = \begin{cases} \lambda f(x), & x \notin Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \\ 0, & x \in Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \end{cases}$$

where λ is a constant to make $\int_0^1 f_{i+1}(x)dx = 1$.

Thus $C' \le C$ which contradicts to that no non-descending query can be changed to descending query without increasing the cost.

□

Finally we have

THEOREM 2. *Among all mechanisms that can include multiple rounds of broadcasts and are required to be efficient (allocate the item to the bidder with highest valuation), Multi-round Vickrey Auctions (MVAs) are of minimum cost.*

PROOF. The best case optimizing problem defined in definition 4 provides us a lower bound of minimum cost we can achieve by any mechanisms. By lemma 3, such lower bound minimum cost can be achieved by descending query strategy $Q_1 = [a_1, 1], Q_2 = [a_2, a_1), Q_3 = [a_3, a_2) \ldots$. And by theorem 2 we are able to design such an MVA with reserve prices $r_1, r_2, \ldots$ whose Bayesian Nash Equilibrium achieves this best case descending query strategy. Thus, MVAs are of minimum cost.

□

And by revenue equivalence theorem, MVAs are optimal:

COROLLARY 1. *If all broadcast costs and bidding costs are charged to sellers, MVAs are optimal if efficiency is required. Such optimal MVA is the one that minimizes the overall cost.*

3.3 Cost Minimized α-MVA

Now let us try to calculate the parameters (thresholds a_i or equivalently reserve prices r_i) of the optimal MVA for a given settings F, n, b, c (valuation CDF, number of bidders, broadcast cost, bidding cost). By lemma 2, the cost is indifferent with F and we can always derive an optimal mechanism for any F from a uniform distribution. Thus we will focus on uniform cases below. We will also introduce $\rho = \frac{b}{c}$ to simplify our analysis by normalize bidding cost c to 1 and thus broadcast cost b to ρ.

An optimal MVA must be an α-MVA where each round only $(1-\alpha)n$ bidders are expected to bid, i.e. $a_1 = \alpha, a_{i+1} = \alpha \cdot a_i$ for uniform cases. Then the expected overall cost C satisfies: $C = \rho + (1-\alpha)n + \alpha^n C$. In the right hand side, the first term ρ is the broadcast we have to use in the first round, the second term $(1-\alpha)n$ is the expected bidding cost for the first round, the third term $\alpha^n C$ is a recursive term, the probability that no one bids times if that happens the same cost C should be expected in later rounds.

From that equation, we get $C = \frac{\rho + (1-a)n}{1-\alpha^n}$. To minimize cost C, we either choose boundary cases $\alpha = 0, 1$ or we have:

$$\frac{\partial C}{\partial \alpha} = \frac{\alpha^{n-1}\, n\ (\rho + (1-\alpha)\ n)}{(1-\alpha^n)^2} - \frac{n}{1-\alpha^n} = 0$$
$$\Updownarrow$$
$$\alpha^{n-1}(\rho + (1-\alpha)n) - (1-\alpha^n) = 0 \qquad (3)$$

The boundary case $\alpha = 0$ can be ruled out because $\frac{\partial C}{\partial \alpha}|_{\alpha=0} < 0$. When $\alpha = 1$, α-MVA becomes Dutch Auction thus ρ must be 0 (otherwise total broadcast cost would be infinity). And this is indeed a solution of equation 3 when $\rho = 0$. Thus equation 3 characterize the optimal α in all cases. [11] has a proof for why α-MVA is optimal among MVAs and how to determine α as well. Thus if they not intuitive please reference more details there. Our simplified cost model (with efficiency constraint and no buyer's cost) is equivalent to their cost model where learning cost is linear to the number of replied agents. That paper makes descending query as a contraint [4] and proves that α-MVA is optimal among descending query mechanisms. Our focus in this section, however, is to have a preliminary introduction for MVA and proves optimality of descending queries and eventually MVAs. Thus we omit the proof about α-MVA. Even for the α, we will be more interested in its relation with larger n so we will give approximations for α.

3.4 Approximation of α

It is difficult to get an exact closed formula for optimal α by equation 3. Thus we are going to use some simpler formulas to approximate α. We will conduct experiments to compare our approximation with the optimal α that is computed numerically.

Firstly, $\alpha = 1 - 1/n$ is a natural guess which means each round the expected number of biddings is equal to 1. It turns out to be quite good:

THEOREM 3. *$\alpha = 1 - 1/n$ is a $1/(1-e^{-1})$ approximation of optimal α-MVA. That means, by simply choosing $\alpha = 1 - 1/n$, we would at most get about 1.582 times of optimal cost. Another obersavation of this approximation is that no matter how large n is, the cost of this simple approximation is at most $(\rho+1)/(1-e^{-1}) = O(1)$. Thus the optimal cost is bounded by constant $O(1)$ no matter how large n is.*

PROOF. $C(\alpha = 1-1/n) = (\rho+1)/(1-(1-1/n)^n)$. Because $(1-1/n)^n \le e^{-1}$, we have $C(\alpha = 1-1/n) \le (\rho+1)/(1-e^{-1})$. It is obvious that at least one broadcast and one bidding is required to terminate so $C \ge \rho + 1$. This completes the proof. □

A better approximation when n is large is to observe that $\alpha \to 1$ when n grows large. Thus we guess that $(1-a)n \approx A$ (for some constant A) and $\alpha^n \approx \alpha^{n-1}$. Then we have:

$$n(\rho + A) \cdot \alpha^n - n(1 - \alpha^n) = 0$$

which gives us $\alpha = (1+\rho+A)^{-1/n}$. Put this back to $\lim_{n\to\infty}(1-a)n = A$ we have $\ln(1+\rho+A) = A$ which gives us

$$A = -1 - \rho - W(-1-\rho)$$
$$\alpha = (-W(-1-\rho))^{-1/n} \qquad (4)$$

where $W(x)$ is the Lambert W function [cite wikipedia?] defined by $W(x)e^{W(x)} = x$. Actually, $w\,e^w = x$ has two solutions for w when $-1 < x < 0$. Here our $W(x)$ refers to the lower[5] branch $W_{-1}(x) < -1$. This second approximation that converges to the optimal one when n is large:

THEOREM 4. *Suppose that the optimal α is α^* which satisfies equation 3. Then $\alpha = (-W(-1-\rho))^{-1/n}$ satisfies*

$$\lim_{n\to\infty} C(\alpha^*) = \lim_{n\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n})$$

That is, our approximation's cost will converge to optimal cost when n grows to infinity.

[4] In their model they want to find the minimum value so increasing threshold search is equivalent to descending query

[5] The upper branch is $W_0(x) > -1$ when $-1 < x < 0$

PROOF. Define sequence α_n^*, C_n^* where $n = 1, 2, 3, \ldots$ to be sequences of optimal α^* and corresponding optimal cost C^* when there are n bidders. We first show that C_n^* is increasing: if we make $(\alpha_{n-1})^{n-1} = (\alpha_n^*)^n$, then we have 1) The expected broadcast cost of α_{n-1}-MVA with $n-1$ bidders is equal to that of α_n^*-MVA with n bidders as the probability that one round will terminate is the same; 2) $\alpha_{n-1} < \alpha_n^*$ thus the expected bidding cost of α_{n-1}-MVA with $n-1$ bidders should be less than that of α_n^*-MVA with n bidders. Therefore, $C_{n-1}^* \leq C_{n-1}(\alpha_{n-1}) < C_n^*$. Thus sequence C^* is indeed strictly increasing.

Secondly, theorem 4 says C_n^* is bounded. Therefore $(1-\alpha^*)n$ must also be bounded otherwise $C = \frac{\rho+(1-\alpha)n}{1-\alpha^n}$ cannot be bounded. Thus according to Bolzano-Weierstrass theorem [cite wikipedia?], there must be a subsequence $\alpha_{n_i}^*$ such that $(1-\alpha_{n_i}^*)n$ converges to some constant A. Recall that α^* satisfies equation 3 and obviously $\lim_{n\to\infty} \alpha^* = 1$, we could use calculations similar to what we used for equation 4 to derive

$$\lim_{n_i\to\infty}(1-\alpha_{n_i}^*)n_i = A = -1-\rho-W(-1-\rho)$$
$$\lim_{n_i\to\infty}(\alpha_{n_i}^*)^{n_i} = \lim_{n_i\to\infty}(\alpha_{n_i}^*)^{n_i-1} = (-W(-1-\rho))^{-1}$$

This proves that

$$\lim_{n_i\to\infty} C(\alpha^*) = \lim_{n_i\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n_i})$$

Then using the fact that C_n^* is strictly increasing and bounded completes the proof. $\square$

Experiments in figure 2 compare the optimal α, our first approximation of $\alpha = 1 - 1/n$ and our second approximation $\alpha = (-W(-1-\rho))^{-1/n}$ together with their corresponding cost under settings $\rho = 0.2, 1, 5$.

As figure 2 shows, the first approximation $1 - 1/n$ is bounded to be a constant time of optimal cost while the second approximation converges to optimal cost when n grows large. When ρ is close to 1, both two approximations are very close to the optimal one. But the second approximation is much better when ρ is much smaller or greater than 1. Anyway, the second approximation is not always better than the first approximation, as the case $\rho = 1, n = 2$ shows.

3.5 Experiments

Now let us compare optimal α-MVA with other kinds of MVA. In all following experiments, the valuation distribution is always uniform over $[0, 1]$. According to lemma 2, other distributions can always be adapted to uniform distribution so this will not be a problem. Also recall that under this simpler model, the revenue is fixed thus the profit is completely determined by cost. Thus we will only compare cost.

α-MVA can potentially have infinite many rounds. But in reality, it is more naturaly to come up with an MVA that has finite many, say k rounds at most. Let us call them k-MVA where k is some positive integers. One particualr k- MVA is uniform k-MVA where the k thresholds is uniformly distributed over $[0, 1]$. It is also known as fixed-step search strategy in [11, 3]. An optimal uniform MVA is the uniform k-MVA that minimize the cost by choosing the best k.

In practice, such k might also be very limited. For example, if someone need to sell something in 3 days before moving out, the k might be limited to 3 since it is too annoying to send out two broadcast messages per day (e.g. it might be labeled as spam by selling platform). With this limitation, one can still use uniform thresholds as a baseline. Or we may formulate this as another optimzing problem and solve the best k thresholds under this constraint (this problem is defined and solved in section 4.4 and equation 6). Between the baseline k uniform thresholds and the optimal k thresholds, another heurestic way to get k thresholds is to use $\alpha, \alpha^2, \ldots, \alpha^{k-1}$ as thresholds. That means, in first $k-1$ rounds, we query as if we have infinite many rounds, and we query all the left in last round. We call this mechanism as α-cutoff-k-MVA.

The comparison results achieved from simulation experiments are shown in figure 3. Here are some observations:

- The optimal uniform-MVA's cost is very close to optimal α-MVA's, especially when n is large. That is probably because
  1. α approaching 1 when n grows large, which makes the first k thresholds $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^k$ close to uniform thresholds $1-(1-\alpha), 1-2(1-\alpha), \ldots, 1-k(1-\alpha)$;
  2. the probability that the highest value falls out of first k thresholds, α^{nk}, becomes negligible for large n.
- The optimal k-MVA's cost decreases and approaches to optimal cost quickly when k grows (check optimal 2-MVA and 3-MVA).
- When k is small, uniform thresholds has significant higher cost than optimal k-MVA, and the heuristic α-cutoff-k-MVA.
- The heuristic α-cutoff-k-MVA works pretty well especially when ρ is large as shown in figure 3(a). But it is not as good as optimal 2-MVA when ρ is small and n is large as shown in figure 3(b). Thus find the right thresholds is even more important than adding one more round in those cases.

4. OPTIMAL MECHANISMS WITH BOTH SELLER'S AND BIDDER'S COST

In this section, we take out the constraints and prove that MVAs are optimal in general. We will also try to find the specific MVA to achieve such optimality, which turns out to be significantly more complex than previous simplified case.

Let us first look at bidder's bidding cost. Sending emails, making phone calls, enterring credit card nubmers, depositing money and clicking buttons are all costly for bidders, though sometimes very tiny. Bidders may not bid when this cost is greater than their expected utility. Note that even if the valuation is very high, the expected utility can be very small because of tense competition, which is very common on the Internet as n, the number of potential bidders, is very large.

This behaviour (bidders will not bid because of competitions) is very different compared to that in previous model [5] of sequential auctions. In that model, there is a time discount which makes bidders eager to bid in early rounds with high reserve prices to avoid waiting lost. Suppose that the seller posts an auction with a very low reserve price in the first round, most bidders with high valuation must be happy to bid according to that time discount cost model. But in our model, a lot bidders may be reluctant to bid because of competence, which can hurt seller's revenue.

The bidder's bidding cost will also make revenue equivalence theorem no longer applicable. That is not strange as the revenue equivalence theorem assumes that the utility of a bidder winning the item is equal to the valuation minus the payment. This assumption is no longer true as now the utility is also influenced by the cost charged to this bidder. And even if the bidder doesn't win, the cost still exists so we can't simply revise the valuation.

Therefore, the first issue we are going to solve is to make a very similar theorem applicable to our model again.

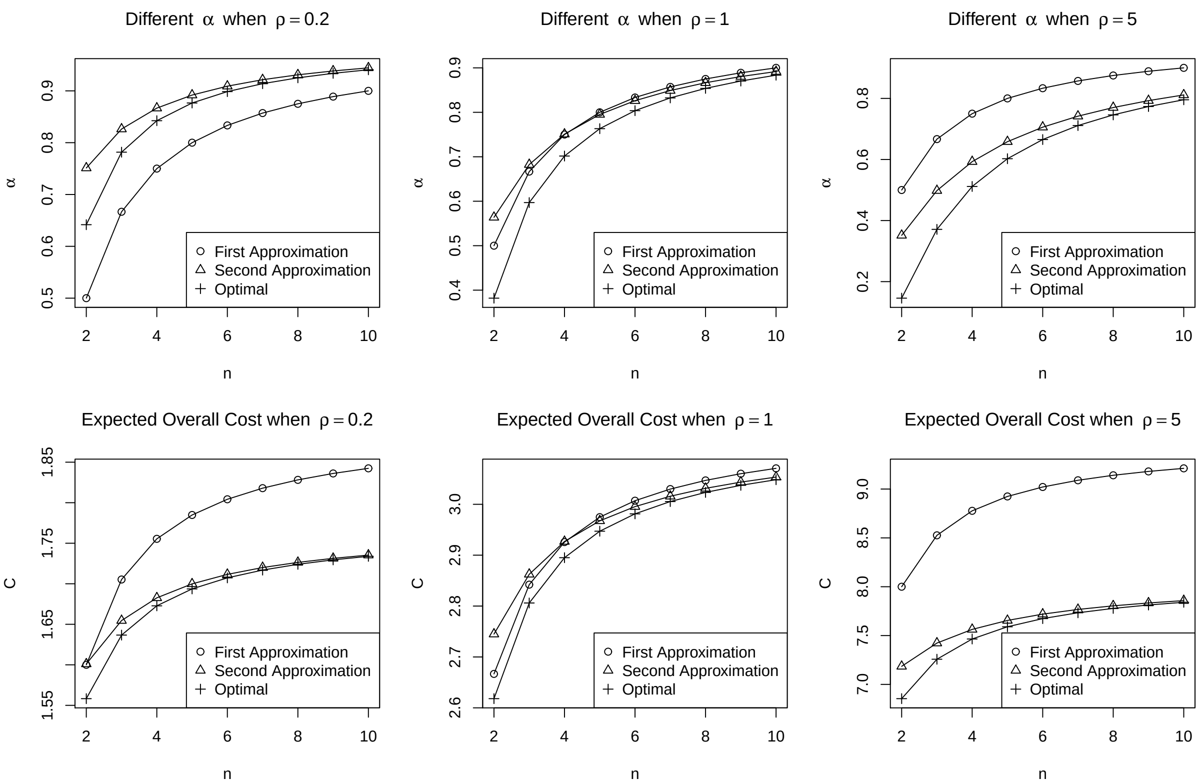
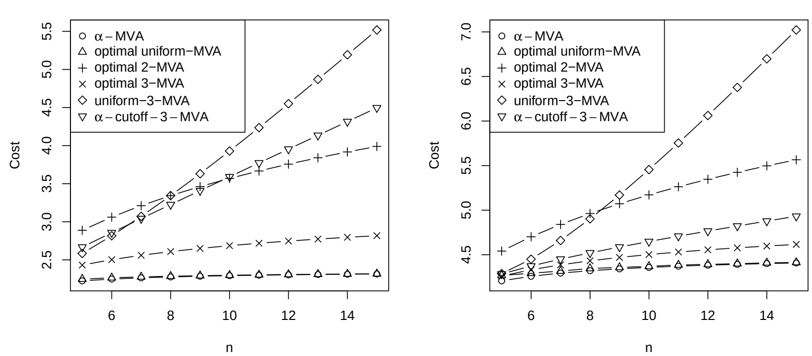


Figure 2: Comparisons for optimal α and its approximations. The first approximation is $\alpha = 1 - 1/n$, the second is $\alpha = (-W(-1-\rho))^{-1/n}$. The second row is the corresponding cost for different α



(a) $\rho = 0.5$ (b) $\rho = 2$

Figure 3: Cost comparison for different MVAs over n, number of bidders, and ρ, broadcast/bidding cost ratio.

4.1 Spending Equivalence Theorem and Revenue Optimization Strategy

THEOREM 5. *The expected overall spendings from bidders (including their bidding costs and payments to the seller) is completely determined by the expected utility of lowest type bidders and allocation probability function*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

where p_i is the probability that bidder i will get the item.

PROOF. To prove it, let us construct another mechanism M' without bidding cost from our mechanism M with bidding cost so M' fits into the original revenue equivalence theorem's model. Suppose there is a virtual seller in M', who collects valuations from all bidders at no cost (a direct revelation mechanism). Then this virtual seller will make n virtual bidders delegating bidders to communicate with the true seller in our mechanism M. When our mechanism ends by allocating the item to virtual bidder i, the virtual seller also allocate the item to the real bidder i. The payment from each bidder i to this virtual seller will be equal to the payment that virtual bidder i pays to our real seller plus all the bidding costs charged to virtual bidder i. M' will satisfy revenue equivalence theorem and the payment from real bidder i to the virtual seller has two parts, one payed to the real seller, another consumed by bidding costs, which sum up to total spending. $\square$

Thanks to theorem 5, our profit maximization problem is now greatly simplified:

COROLLARY 2. *To maximize profit for a given allocation rule $p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$, we only need find the minimum total cost (including both seller's cost and bidders' cost).*

PROOF. The total spending, substracts cost charged to bidders, will be the revenue that the seller receives. This revenue, substracts cost charged to the seller, will be profit. Thus profit is total spending minus total cost. As total spending is fixed by allocation rule, we only need to find the minimum cost to maximze profit. $\square$

The highlight here is that we will not have to differentiate cost charged to bidders and cost charged to sellers if we just want to maximize seller's profit. The difference of them may make revenue different, but as long as their sum does not change, the profit will not change. This not only helps us simplify our analysis, but also helps us simplify the optimal mechanism:

PROPOSITION 2. *In original MVA, to make a set of thresholds a_i works (the equilibrium holds), we may have to make r_i negative. That says, we need to compensate bidders if only one bids at reserve price r_i rather than charge him something, in order to maintain bidders incentive to bid in spite of bidding cost. That's not intuitive and sellers/bidders may not be able to easily understand it. A better way to achieve a_i might be compensating the bidder with the bidding cost immediately after the bidding. Equivalently, the seller buys the bidding cost for bidders and make it as a part of seller's bidding cost. This won't change his profit but this will make everything looks more intuitive (r_i is non-negative again).*

Since profit is total spending minus overall cost where total spending is decided by allocation rule which is studied in Myerson's work, our optimizing problem is greately related to Myerson's if we were able to minimize the overall cost. However, the optimal allocation rule here is not as simple as the one that is discovered by Myerson [7]: allocate the item to the bidder with highest positive virtural value. Theorem 5 tells us that this rule will maximize the total spending. But we must substract the cost from the spending to get the profit. Therefore, there might be another weird allocation rule that has less total spending but even much less minimum cost.

Thus, the profit optimal mechanism will depend on how minimum cost is defined given an allocation rule.

DEFINITION 5. *A mechanism with allocation rule*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

does not allocate blindly if it satisfies: if $p_i > 0$ for some valuation profile $(v_1, v_2, \ldots, v_n)$, there must be a broadcast query that the i-th bidder (v_i) reply to the seller under that profile setting.

The no-blind-allocation property defined above will ensure the optimality (minimum cost) of the following mechanisms.

DEFINITION 6. *Mechanisms satisfy relaxed efficiency constraint with low value l if they always allocate the item to the bidder with highest valuation that's at least l (no allocation if everyone is below l). When we say a mechanism has a low value l, we imply that it satisfies relatex efficiency constraint with low value l.*

THEOREM 6. *In regular cases (the virtual valuation is monotone strictly increasing []), mechanisms satisfying relaxed efficiency constraint with some low value l are optimal among all mechanisms that won't allocate blindly.*

PROOF. Suppose that there's an arbitrary optimal mechanism. We can describe it by a query strategy $S(f, m, V, \mathcal{Q})$ (see definition 4) and an allocation function based on V, the set of reported values set, since we can only allocate item to reported bidders. We then construct another mechanism with query strategy $S'(f, m, V', \mathcal{Q}')$ where $\mathcal{Q}'$ are decsending and

$$|S'(f, m, \emptyset, \{Q'_1, \ldots, Q'_{i-1}\})| = |S(f, m, \emptyset, \{Q_1, \ldots, Q_{i-1}\})|$$

That says, if there's no reported bidders yet, we will always ask the descending query which has the same length as the query of S. And we pretend that we asked the same query as S so we can continue to ask S what's the next query length. Once we get reply in S', we terminate the mechanism immediately and allocate the item to the replied bidder with highest valuation with probability 1. Note that we may have $|Q'_i| = |Q_i| = 0$ which means both mechanisms terminate after i queries without any reply and allocation. It's clear that our new mechanism won't cause more cost than original mechanism. And our new mechanism's allocation rule will get at least as high total spending as original mechanism because it makes allocation probability of high value bidders as much as possible. Therefore our new mechanism, which satisfies relaxed efficiency constraint with some low value l, is also optimal. $\square$

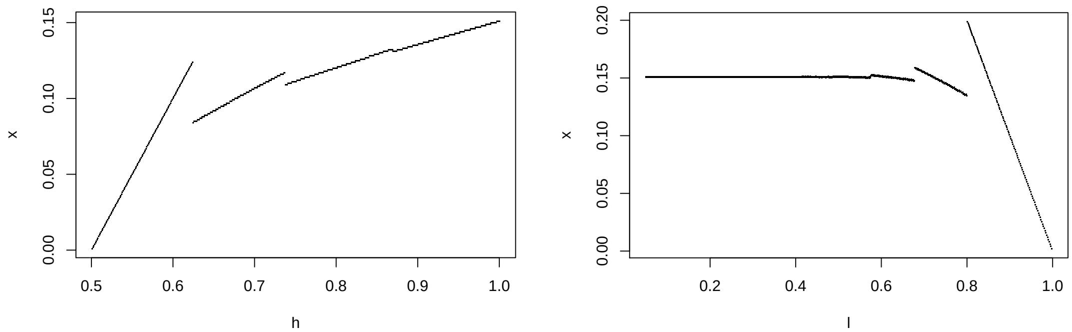
For simplicity, we will always assume regularly and will not mention virtual valuation below. It's easy to extend our result to general cases by mapping all values to virtual values and ask broadcast queries according to virtual values.

4.2 MVAs' Optimality in General

We have already narrowed down optimal mechanism to relaxed efficient mechanisms and by theorem 5, it is straightforward to see

COROLLARY 3. *For mechanisms with a fixed low value l, the maximum profit is achieved when the mechanism minizes the cost.*

Our next question is naturally: what is the cost minimized mechanism given a low value l. A similar lemma can be proved using almost identical technique to lemma 2. Thus for space limit, we will not elaborate it again.



(a) Best query length x over h with fixed l

(b) Normalized best query length x over low value l

Figure 4: In the first subfigure, we plot the best query length $x = \hat{x}/D$ over $h = \hat{h}/D$ (the highest undiscovered value) with discretized low value $\hat{l} = 500$, broadcast bidding cost ratio $\rho = 2$, number of bidders $n = 10$ and maximum discretized valuation $D = 1000$. In the second subfigure, we normalize x to be $x = \hat{x}/\hat{h}$ and l to be $l = \hat{l}/\hat{h}$, as if h is always 1.

LEMMA 4. *Suppose that there are two cases n, F_1, f_1, l_1 and n, F_2, f_2, l_2 where n is the number of values for both cases, F_i, f_i are CDF and PDF of the n i.i.d. values in case i, l_i is the low value for case i. If $F_1(l_1) = F_2(l_2)$, then these two cases have the same minimum cost to find the maximum value above the low value l_i.*

Finally, we conclude

THEOREM 7. *MVAs have the minimum cost among all mechanisms with a low value l. Thus they are optimal.*

PROOF. A special case of this theorem when $l = 0$ is theorem 2. We proved that special case by introducing lemma 2 and 3. To prove the general cases with arbitrary l, we just need to revise lemma 2 a little to lemma 4. All other part of the proof remains similar. For space limit, the detailed proof is omitted. $\square$

Then the only parameters we are going to determine for the specific optimal MVA are 1) the low value l; 2) the descending query thresholds $a_1, a_2, a_3, \ldots$. When we later investigate such parameters that minimizes the cost, we will also assume uniform distribution $F(x) = x$ in default because distribution will not change this minimum cost and we can always adapt an optimal MVA for uniform distribution to an optimal MVA for any distribution easily.

4.3 Experiments to Discover Optimal MVA with a Given Low Value

To discover the specific MVA that is optimal, we first try to identify the optimal thresholds a_i given a low value l (recall that in i-th round, MVA will ask all bidders whose valuation is within $[a_i, a_{i-1})$ to bid). If we can write the minimum cost C^* as a function of n, ρ, l (recall that $\rho = b/c$ is the ratio between broadcast and bidding cost), we may then determine the optimal l by that.

As it is not immediately clear what optimal thresholds should be like, we present a simple algorithm to calculate such thresholds numerically. We firstly discretize the continuous valuation $[0, 1)$ to D discrete values $\{0, 1, \ldots, D-1\}$. That means, the original valuation v will be transformed to integer value $\hat{v} = \lfloor vD \rfloor$. Then we use dynamic programming to inductively calculate $\hat{x}_{\hat{h}}$, the length of next optimal descending query $[\hat{h} - \hat{x}_{\hat{h}}, \hat{h})$ to ask, conditional on that we have already queried $[\hat{h}, D)$ and no one replies.

Algorithm 1 runs in time $O(D^2)$. Having $\hat{x}$, we can then infer the best strategy x for original continuous problem by converting $\hat{h}, \hat{x}$ back to h, x and using them to interpolate continuous strategy.

Algorithm 1 Calculate discretized best query lengths

Require: $\hat{l}$ is the discretized low value, ρ is the ratio between broadcast and bidding cost, n is the number of bidders, D is the maximum discretized valuation

function BESTQUERYLENGTHS($\hat{l}$, ρ, n, D)
  $\hat{x}_{\hat{l}} \leftarrow 0$
  $C_{\hat{l}} \leftarrow 0$
  for $\hat{h} = \hat{l} + 1$ to D **do**
    $\hat{x}_{\hat{h}} \leftarrow \underset{1 \le \hat{x} \le \hat{h} - l}{\arg\min}\, \rho + n\frac{\hat{x}}{\hat{h}} + (\frac{\hat{h} - x}{\hat{h}})^n C_{\hat{h} - \hat{x}}$
    $C_{\hat{h}} \leftarrow \rho + n\frac{\hat{x}_{\hat{h}}}{\hat{h}} + (\frac{\hat{h} - \hat{x}_{\hat{h}}}{\hat{h}})^n C_{\hat{h} - \hat{x}_{\hat{h}}}$
  end for
  return $\hat{x}$
end function

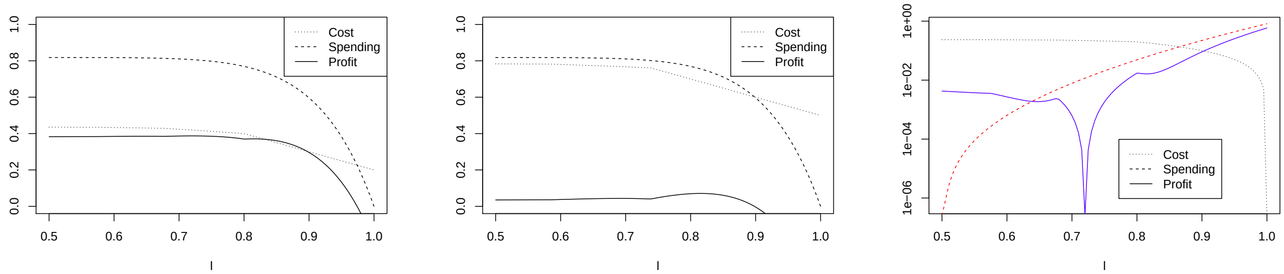
The larger D is the more accurate it will be. But it will also require more running time. Running this for case $l = .5, \rho = 2, n = 10, D = 1000$ we get x showed in figure 4(a). It seems that x is a piecewise linear function over h. For those h which is close to l, obviously that the optimal strategy should be $x = h - l$, which means using only one query to explore all potential bidders. But it is unclear why x is linear when the best strategy is using multiple queries to explore the valuation range.

Another way to plot the graph is to normalize x and l so that h becomes 1 since our model is $v_i \in [0, 1)$. It is showed in figure 4(b). This again looks like a piecewise linear function. A more interesting observation is that the left-most piece is quite a long straight line $x = 1 - \alpha$. Thus it seems that $x = 1 - \alpha$ is optimal for quite a lot l which is not far from 0. Here the α is the the optimal α of α-MVA if we set $l = 0$.

4.4 Analysis of Optimal MVA with a Positive Low Value

Figure 4(a) shows that the optimal MVA with a low value $l > 0$ is much more complicated but there might still be hope to get a nice analytical result: piecewise linear function. That is, we cannot use a single α to describe such optimal MVA, but perhaps we can use a sequence of α to describe it. Now let us take an analytical treatment.

The first key to analyze the optimal MVA with $l > 0$ is to utilize



(a) $n = 10, b = 0.2, c = 0.1$ (b) $n = 10, b = 0.5, c = 0.1$ (c) Suboptimality, $n = 10, b = 0.2, c = 0.1$

Figure 5: Optimal MVA's cost, spending, profit over l, n i.i.d. uniform distributed bidders, broadcast b and bidding cost c

the fact that there is a maximal number of rounds to exploit the whole reportable valuation range $[l, 1)$. Let us call that number k.[6]

For convenience, define $\vec{a} = (a_0, a_1, a_2, \ldots, a_k)$, the vector of k thresholds in such optimal MVA. We make $a_0 = h, a_k = l$ so in i-th round the query would be $[a_i, a_{i-1})$. Now define cost $C(\vec{a}, k, \rho, n)$ to be the expected cost for MVA defined by $k, \vec{a}$ when there are n i.i.d. $[0, 1)$-uniform bidders (note that l, h are implicitly defined by a_k, a_0). If we can get a neat form of C, we can use $\frac{\partial C}{\partial a_i} = 0$ to characterize optimal MVA, as we did in section 3.3.[7]

Thus the second key is to represent this C. Rather than considering one round after another recursively as we did before, we now consider all rounds together. Let CR_i denote the cost that occur in round i (the broadcast of that round and the bidding cost charged in that round) and $v^* = \max_i v_i$. Then expected cost would be sum of expected cost of each round:

$$\begin{aligned} C(\vec{a}, k, \rho, n) &= \sum_{i=1}^{k} E[CR_i] \\ &= \sum_{i=1}^{k} \Pr\left(v^* < a_{i-1}\right) E[CR_i \mid v^* < a_{i-1}] \\ &= \sum_{i=1}^{k} \frac{a_{i-1}^n}{a_0^n} \left(\rho + \frac{a_{i-1} - a_i}{a_{i-1}} n\right) \end{aligned} \tag{5}$$

Taking derivative we get

$$\frac{\partial C}{\partial a_i} = \frac{n(n+\rho)a_i^{n-1} - n(n-1)a_{i+1}a_i^{n-2} - na_{i-1}^{n-1}}{a_0^n} \tag{6}$$

Unfortunately, equation 6 is not neat enough to get a piecewise linear query length. Recall that in previous subsection, the experiment seems to show that query length x is piecewise linear over h, which means $a_1 = a_0 - x = h - x$ must also be piecewise linear over $a_0 = h$. One counter example is the simple case $k = 2, n = 3$. We have: $a_1 = (\sqrt{{a_0}^2\,\rho + {a_2}^2 + 3\,{a_0}^2} + a_2)/(\rho + 3)$. Anyway, by definition we have $a_1 \geq a_2$. And the former equation indeed looks very linear when $a_1 \geq a_2$.

Though we failed characterizing the optimal MVA using piecewise linear functions, equation 5 gives us a better way to calculate thresholds a_i. We use an R package called BB [14] to solve these non-linear equation systems. Before throw those equations to that package, we have to first decide k and an initial guess of $\vec{a}$.

We found a lemma that's very useful for deciding k:

LEMMA 5. *Two sets of optimal thresholds $\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_m\}$ can't have $p_2 \leq q_2 < q_1 < p_1$ where p_2, p_1 and q_2, q_1 are two consecutive thresholds, say $a_i, a_{i-1}, b_j, b_{j-1}$, for these two sets of thresholds.*

PROOF. Denote $C(l)$ to be the optimal cost for finding the maximum value above low value l (suppose h, n, ρ and everything else are constants now). Note that thresholds $a_0, \ldots, a_i$ must be optimal thresholds for $l = a_i$, otherwise $a_1, a_2, \ldots, a_k$ can't be optimal for $l = a_k$. Similarly, $b_0, b_1, \ldots, b_j$ must be optimal thresholds for $l = b_j$. Then

$$C(q_1) + q_1^n[\rho + n(q_1 - q_2)] \leq C(p_1) + p_1^n[\rho + n(p_1 - q_2)]$$
$$C(p_1) + p_1^n[\rho + n(p_1 - p_2)] \leq C(q_1) + q_1^n[\rho + n(q_1 - p_2)]$$

Adding these two inequations together and do some cancellations we get $p_1^n \leq q_1^n$, which contradicts $q_1 < p_1$. ☐

Then we can find an upper bound for k:

LEMMA 6. *$k_\alpha = \lceil \log_\alpha (h/l) \rceil$ is an upper bound for the optimal k. Here α is the optimal α for α-MVA when $l = 0$.*

PROOF. We will prove by contradiction. Assume $k > k_\alpha$ for optimal thresholds $(b_0, b_1, b_2, \ldots, b_k)$ where $b_0 = h, b_k = l$ as defined. Points $a_0 = \alpha^0, a_1 = \alpha^1, \ldots, a_{k_\alpha} = \alpha^{k_\alpha}$ split interval $[l, h)$ into k_α subintervals $[a_i, a_{i-1})$, which together contain k thresholds $b_1, \ldots, b_k$. Since $k > k_\alpha$, at least one subinterval $[a_i, a_{i-1})$ contain two thresholds b_j, b_{j-1}. Equivalently, there exists $a_i \leq b_j < b_{j-1} < a_{i-1}$. This contradicts with Lemma 5. ☐

Having this upper bound, we can eigther bruteforcely search all k between 1 and k_α, or use ternary search [cite wiki?] to get optimal k in $O(\log k_\alpha)$ time if we can prove it is unimodal.[8] But in light of the proof above, we actually could have something much better:

THEOREM 8. *$\lceil \log_\alpha (h/l) \rceil$ and $\lfloor \log_\alpha (h/l) \rfloor$ are upper and lower bounds for the optimal k.*

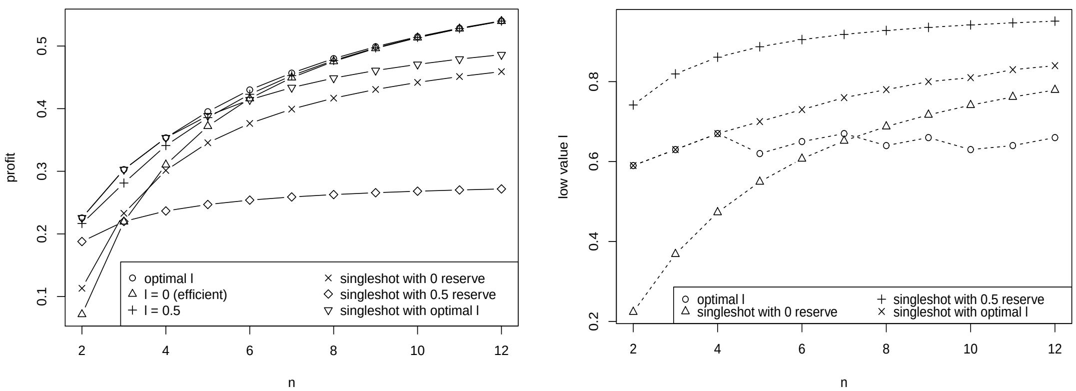
PROOF. We have already proved the upper bound in lemma 6. Similarly, if the lower bound does not hold, we can construct $b_j = p_2 \leq \alpha^i = q_2 < \alpha^{i-1} = q_1 < b_{j-1} = p_1$, which violates the Lemma 5. ☐

By these bounds, we only need to try at most two values to get the optimal k. The next important thing is the initial thresholds $\vec{a}$. A bad choice may lead to much more computation and even non-convergence. For example, the trivial uniform $a_i = (1 - i/k)a_0 - (i/k)a_k$ is a bad initial guess. We find the initial guess $a_i = \alpha^i$ (the

[6] We have argued this before: if such k does not exist, or equivalently the maximum number of rounds is unbounded, the cost will be infinite as the possibility that no values lie in $[l, h)$ is positive.

[7] It is easy to see that boundary cases $a_i = a_{i-1}, a_{i+1}$ are not optimal too.

[8] We did not prove that it is unimodal but experiments seems to support this property.



(a) profit (b) low value

Figure 7: Compare profit and corresponding low value l over n. The broadcast cost $b = 0.1$. Bidding cost for seller and buyer are $\beta_1 = \beta_2 = 0.05$ (thus $c = \beta_1 + \beta_2 = 0.1$)

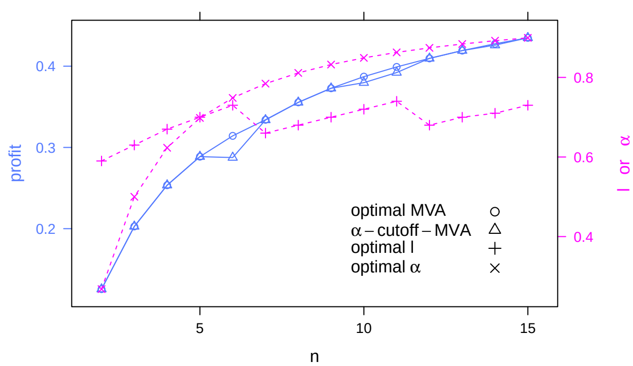


Figure 6: The profit of optimal MVA and α-cutoff-MVA are plotted in solid line. The low value l is chosen to maximze optimal MVA's profit. We also plot l and α in dashed lines to see how they affect the relative profit difference. Note that α does not depend on l.

same α of k_α) to be very efficient. Let us call this MVA with thresholds $\alpha, \alpha^1, \ldots, \alpha^{k-1}$ where $\alpha^{k-1} \geq l, \alpha^k < l$ an α-cutoff-MVA, similar to what we did in previous section 3.5. One explanation about its good performance is that these initial thresholds doesn't violate proposition **??**. Thus it's more likely to be close to actual optimal thresholds. In experiments, this α-cutoff-MVA's profit is pretty good as shown in figure 6. In most cases its profit is very close to optimal. The significant difference only occur when low value l is a little below α and this is not likely to occur when n is large where α is much closer to 1 than l.

4.5 Choosing Low Value

Having k and a_i, we are still one step away from optimal MVA: choosing the low value l. It is obvious to see that optimal cost C is non-increasing as l increases. By Myerson's optimal auction theory and theorem 5, setting virtual value of l to be 0 will yield the maximum total spending. Name such l as l_0. We then conclude that optimal l must satisfy $l \geq l_0$. The question is, how to search or determine optimal l. Let us first check the simple case when the value distribution is uniform.

Figure 5(a) and 5(b) plot cost, spending and profit over l. It is clear in figure 5(b) that increasing l from 0.5 to about 0.8 will get a significant profit increase. However, it is not so clear in these two plots whether profit is unimodal and whether we can do ternary search.[9] To see it more clear, we plot figure 5(c), which transforms the cost, spending and profit to their corresponding suboptimality, i.e. the distance between the specific value and the optimal value (to make the log-scale work for distance 0, we add some small constant to that suboptimality). Thus, the new plot will preserve the peaks and global optimal point as the lowest point.

From figure 5(c), the cost is not unimodal over l and it is quite wierd. Thus in later experiments, we are just going to brute forcely search over all possible l from 0.5 to 1, with a search increment of 0.01.

4.6 Experiments

Finally we are going to compare profit of different mechanisms in general settings. To keep it simple, we still use a uniform valuation distribution for all experiments. As theorem 5 shows, the profit is simply spending minus cost where spending is solely determined by l. In fact, the total spending will not be very sensitive to this l when n is large (except that l is very close to 1) and we will see this in later comparisons. Thus most experimental comparisons that we have done for cost in section 3.5 would also give us a lot information for profit comparison. As a result, we will compare mechanisms very different from those in section 3.5. Specifically, we will not be interested in k-MVA where k is very limited. Instead, we are going to compare how l is going to affect the profit and how much profit we lose if we ignore the cost.

If we ignore the cost, Myerson has already proved that a singleshot Vickrey auction with reserve price 0.5 would be optimal for uniform i.i.d. bidders. Thus we will compare this singleshot mechanism, as well as its variants, the singleshot Vickrey auction with 0 reserve price. But be aware of that reserve price is not equivalent with low value l when cost exists. To make it more fair, we add one more singleshot Vickrey auction whose reserve is set to be optimal among all singleshot Vickrey auctions (i.e. we optimize l for this particular singleshot mechanism). One may also suspect whether it is good enough to set low value l to be 0.5 in such singleshot Vickrey auction, since it will bring us maximal total spending. The

[9] Unlike k which is an integer with a fairly small upper bound, l is continuous over $[0.5, 1)$, thus a ternary search is more needed.

answer is no, because its bidding cost is $0.5n \times c$, which grows way too large when n is large.

The experimental result is shown in figure 7. As shown in profit comparison figure 7(a), the profit of optimal l, $l = 0$ and $l = 0.5$ become close very quickly when n grows and they are almost identical for $n \geq 8$ in this experiment setting. Thus choosing l will not be a critical issue for large n. However, if we use singleshot mechanism (which is optimal if cost does not exist), the gap is significant. This is because the bidding cost will drive low value l very close to 1 thus lower the total spending significantly. For example, even if we set reserve price to 0, when $n = 10$, l will be between 0.6 to 0.8. The optimal reserve price 0.5 becomes the worst as its l is too high. Even if we adapt our reserve price to optimal one, the singleshot mechanism is still not so good because it cannot balance the total spending and bidding cost, i.e. it either makes l close to 1 to lose a lot total spending, or makes l close to 0.5 to cause a big bidding cost.

5. CONCLUSION

6. ACKNOWLEDGEMENT

7. REFERENCES

[1] L. M. Ausubel and P. Cramton. The optimality of being efficient. 1999.

[2] V. Conitzer and T. Sandholm. Computational criticisms of the revelation principle. In *The Conference on Logic and the Foundations of Game and Decision Theory (LOFT-04)*, Leipzig, Germany, 2004. Earlier versions appeared as a short paper at ACM-EC-04, and in the workshop on Agent-Mediated Electronic Commerce (AMEC-03).

[3] J. Hassan and S. Jha. On the optimization trade-offs of expanding ring search. In N. Das, A. Sen, S. K. Das, and B. P. Sinha, editors, *IWDC*, volume 3326 of *Lecture Notes in Computer Science*, pages 489–494. Springer, 2004.

[4] P. J.-J. Herings, R. Müller, and D. Vermeulen. Bisection auctions. *SIGecom Exch.*, 8(1):6:1–6:5, July 2009.

[5] R. McAfee and D. Vincent. Sequentially optimal auctions. *Games and Economic Behavior*, 18(2):246 – 276, 1997.

[6] R. P. McAfee and J. McMillan. Search mechanisms. *Journal of Economic Theory*, 44(1):99 – 123, 1988.

[7] R. B. Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6:58–73, 1981.

[8] D. Parkes. Iterative combinatorial auctions. In P. Cramton, Y. Shoham, and R. Steinberg, editors, *Combinatorial Auctions*, chapter 2, pages 41–77. MIT Press, 2006.

[9] T. Sandholm and C. Boutilier. Preference elicitation in combinatorial auctions. In P. Cramton, Y. Shoham, and R. Steinberg, editors, *Combinatorial Auctions*, chapter 10, pages 233–263. MIT Press, 2006.

[10] T. Sandholm and A. Gilpin. Sequences of take-it-or-leave-it offers: Near-optimal auctions without full valuation revelation. In *Proceedings of the International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1127–1134, Hakodate, Japan, 2006.

[11] D. Sarne, S. Shamoun, and E. Rata. Increasing threshold search for best-valued agents. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, Atlanta, GA, USA, 2010.

[12] M. Stegeman. Participation costs and efficient auctions. *Journal of Economic Theory*, 71(1):228–259, October 1996.

[13] G. Tan and O. Yilankaya. Equilibria in second price auctions with participation costs. *Journal of Economic Theory*, 130(1):205 – 219, 2006.

[14] R. Varadhan and P. Gilbert. Bb: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. *Journal of Statistical Software*, 32(4):1–26, 10 2009.