# Optimal Broadcast Auctions with a new Cost Model

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## 1. INTRODUCTION

## 2. COST MODEL AND SETTINGS

In this section, we define our cost model and explain why it makes sense. Our cost model is inspired from online second-hand item transactions, including examples like eBay, craigslist and universities' mailing-lists.

DEFINITION 1. *In our settings, there's one seller selling one item to n buyers (bidders) whose valuations* $v_i$ $(1 \leq i \leq n)$ *are independently and identically distributed (i.i.d.) over* $[0, 1]$ *with PDF* $f(x)$ *and CDF* $F(x)$*. The seller can broadcast a message to all bidders which costs b (the broadcast cost) for the seller. A bidder may reply to that broadcast with cost c (bidding cost) or remain silent with no cost. Such bidding cost c may contain two parts,* $\beta_1, \beta_2$ *where* $\beta_1 + \beta_2 = c$*. The first part* $\beta_1$ *is charged to the seller while the second is charged to the corresponding bidder. The bidder's reply should be deterministic with respect to the seller's broadcast (for simplicity, we only consider pure strategy equilibrium).*

Most settings in definition 1 are quite standard except for the cost and broadcast capability.

The bidding costs $c$ may be caused by communication and other verification actions required to put a bidding. For example, the bidder may have to input his credit card number and prepay an amount of money. Without such verification for the bidder, a bidder may bid very high and refuse to pay in the end. A verification for the seller might also be needed. For example, a bidder may want to set up an appointment with the seller to check the item. Setting up such an appointment might be costly because they need to dicuss time and place via emails or phone calls. Attending that appointment may also cost travel fees and time. Note that our bidding cost is different from conventional participation cost studied before [cite Participation Costs and Efficient Auctions, Equilibria in second price auctions with participation costs]. In our model, it's free for bidders to participate without bidding, which is more close to online cases where receiving information is almost free. In another word, you don't have to buy a ticket to walk in and observe.

We introduce broadcast capability because it's shared by many auctions. For example, a Vickrey auction or a first-price auction with reserve price can be described as a broadcast auction with only one broadcast: telling every bidder the reserve price. The bisection auction [cite bisection auctions] is another example which has many rounds of broadcasts. In each round , it will broadcast a price and ask bidders to reply whether his valuation is beyond or below that price. In real world, sellers make such broadcasts via sending emails to a bunch of receivers (typically a mailing-list), listing items on a platform such as craigslist and eBay, or even showing ads on Internet/TV. Such broadcast activities costs either money (e.g. a list or ads fee) or time and effort (e.g. writing and sending an email).

Note that in our model, we only give sellers broadcast capability so they cannot find or communicate with each bidder one by one. The first reason to have this constraint is there are too many potential bidders on the Internet (our model focus on online item transactions) and it's hard to explicitly find them one by one. On the other hand, in offline cases where the set of bidders are small and explicit (e.g. the government want to sell a land to one of three companies), it might be helpful to let the seller communicate with bidders one by one [cite search mechanisms]. The second reason to have this constraint is that we want to focus on mechanisms that avoids time consuming bargaining. Such feature is very important as one of the most vital advantages of online transactions are their convenience and the time consuming bargainning can ruin it.

Finally, we define optimal mechanisms to be the ones that maximze seller's utility since when facing many different auction mechanisms, a rational seller will choose the one that gives him maximum utility.

DEFINITION 2. *We say a mechanism is optimal if it gives the seller maximum utility (revenue) which equals to all the value payed to this seller minus all the cost charged to this seller. A class of mechanisms are optimal if they contain one optimal mechanism.*

## 3. OPTIMAL MECHANISMS WITH EFFICIENCY AND ONLY SELLER'S COST

In this section, we consider a simplified optimizing problem with efficiency constraint and only seller's cost. Though these two constraints simplify our problem a lot, they are reasonable in real cases such as craigslist or moving sales in mailing-lists:

1. In many cases, sellers only have 0 valuation for the item and they cannot commit to withhold the item or

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prevent re-sales between buyers. For example, some second-hand items will be tossed if they cannot be sold by a particular day, e.g. the day the seller moves out the house. We encounter many free items during second-hand sales as well, which is another demonstration of zero valuation. Under such circumstances, an efficient mechanism not only maximzes the social welfare but also maximizes the seller's revenue [cite The optimality of being efficient].

2. The bidding cost for each bidder is sometimes negligible compared to bidding cost charged to the seller. For example, if 100 bidders replied to the seller by a 1-minue call, each bidder only has a tiny 1 minue cost. But for the seller, it's a big 100 minutes cost which is very annoying. It's also necessary to remove bidder's bidding cost to achieve efficiency. Otherwise, the item may not be able to allocate to the highest bidder when that highest valuation is below the bidder's bidding cost.

The rest of this section is organized as follows. First of all, we introduce a mechanism called Multi-round Vickrey Auction (MVA) based on what's been used in realworld online second-hand item transactions. Then we prove that MVAs are optimal (so there exists a MVA that's optimal). After that we'll try to find the specific MVA that achieves the optimality. Finally, we conduct some experiments to compare the optimal MVA with other mechanisms.

## 3.1 Multi-round Vickrey Auctions

A Multi-round Vickrey Auction (MVA) has multiple rounds of Vickrey auctions with progressively decreasing reserve prices. This kind of auction effectively occurs on eBay. The seller may set up a reserve price and let buyers bid for this item. The proxy bidding functionality makes such an auction equivalent to a Vickrey auction with a reserve price. If no buyers bid for a given reserve price, the seller may lower the reserve price, which makes the whole process equivalent to an MVA.

Definition 3. *(Multi-round Vickrey Auction, MVA)* *In a Multi-round Vickrey Auction (MVA), there's a sequence of reserve prices $r_1, r_2, \ldots, r_k$ where $r_k > r_{k+1}$. The seller creates a Vickrey auction with a reserve price $r_i$ at time i (or round i). In each Vickrey auction, if only one buyer bids, he/she gets the item and pays reserve price. Otherwise, the buyer with the highest bidding gets the item and pays the second highest bidding.*

MVAs require Vickrey auctions (or equivalent English auctions) as basic steps. In reality, however, such functionality won't always be provided by online platforms such as craigslist. Thus a simplified version of MVA occur very often in those platforms. People call it first-come first served which means for every reserve price $r_i$, the first one who accept that price wins the item and pays $r_i$ directly. This mechanism may loose revenue and social efficiency as the person with lower valuation $p$ may get the item for $r_i$ while there's someone else who is willing to pay a higher amount of $q$ where $r_i \le p < q < r_{i-1}$. We won't focus on this first-come first served mechanism because it's harder to analyze analytically and it's inferior than MVAs in terms of both sellers' utility and social welfare.

Since there's no cost charged to buyers, it's obvious to see that whenever a bidder decides to bid, he/she must bid truthfully. Thus the Bayesian Nash Equilibria (BNE) for MVAs can be described as $k$ thresholds $a_1, a_2, \ldots, a_k$ where $a_i > a_{i+1}$. Whenever a bidder's valuation for the item is greater than $a_i$, he/she is going to bid in round $i$ whose reserve price is $r_i$. Because of efficiency constraint we also have $r_k = a_k = 0$.

In later analysis, we will think about the equilibrium from another perspective. We firstly decide thresholds $a_i$ since they are more meaningful for bidders to make decisions and for us to make analysis. For a set of thresholds, we then determine the right reserve prices $r_i$ that make bidders incentive compatible to bid according to $a_i$. The following equations connects $a_i$ and $r_i$:

$r_k = a_k = 0$ and $\forall i\ (1 \le i < k)$,

$$P(a_i)(a_i - r_i) = \int_{a_{i+1}}^{a_i} (a_i - x)p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \tag{1}$$

assuming

$$\begin{aligned} P(x) &:= F(x)^{n-1} \\ p(x) &:= P'(x) = (n-1)F(x)^{n-2}f(x) \end{aligned}$$

The equation 1 says that the bidder with valuation $a_i$ should be indifferent from bidding in round $i$ (the left hand side) and bidding in round $i-1$(the right hand side). The following theorem describes the equilibrium of MVAs determined by equations above.

Theorem 1. *If we make reserve prices $r_i$ to be:*

$$\begin{aligned} r_k &= a_k = 0 \\ r_i &= \left(\int_0^{a_i} x\, p(x)dx\right) / P(a_i) \qquad (i < k) \end{aligned} \tag{2}$$

*Such MVA will have a pure strategy Bayesian Nash Equibibrium characterized by thresholds $a_1, a_2, \ldots, a_k$ where the bidder with valuation greater than $a_i$ (but not greater than $a_{i-1}$) will bid in round $i$*

Proof. By equation 2, we have $r_i P(a_i) = \int_0^{a_i} x\, p(x)dx$ for all $i$. Thus the right hand side of equation 1 is:

$$\begin{aligned} &\int_{a_{i+1}}^{a_i} a_i p(x)dx - \int_{a_{i+1}}^{a_i} x\, p(x)dx + P(a_{i+1})(a_i - r_{i+1}) \\ &= a_i P(a_i) - \cancel{a_i P(a_{i+1})} - r_i P(a_i) + \cancel{r_{i+1} P(a_{i+1})} \\ &\quad + \cancel{P(a_{i+1}) a_i} - \cancel{P(a_{i+1}) r_{i+1}} \\ &= \text{left hand side of equation 1} \end{aligned}$$

□

This tells us that a bidder will bid in a round of MVA if and only if the expected second highest bidding conditional on this bidder's valuation is the highest is greater than the reserve price of that round. For example, if the distribution is uniform, i.e. $F(x) = x$, $r_i = \frac{n-1}{n} a_i$ for $i > 0$. [cite sequentailly optimal auctions] has given some more discussions and proof (e.g. once a bidder choose to bid, bid truthfully is a unique weekly dominant strategy) about the equilibrium of this kind of sequential auctions. That paper, however, aims on the optimal auctions for a different cost model with time discount but without broadcast or bidding cost. We will discuss this difference later when we introduces bidding cost for bidders.

## 3.2 Optimality of MVAs

Since the mechanism is required to be efficient, by revenue equivalence theorem [cite] we know that the seller's gross revenue without substracting costs is fixed. Thus to maximize his utility is equivalent to minimize the cost. To satisfy efficiency constraint, the mechanism should at least find out the bidder with highest valuation. The best case is that every reply contains the exact and truthful valuation of the corresponding bidder since every non-silent reply has a cost $c$. By doing that, we never need someone to reply twice. The optimizing problem in this best case is defined as definition 4. The minimum cost of this best case optimizing problem provides us a lower bound for minimum cost of our mechanisms. We will then prove that MVAs can achieve this lower bound so MVAs are optimal.

DEFINITION 4. *Assume there are $n$ values $v_i$ ($1 \le i \le n$) independently and identically distributed over $[0,1]$ with PDF $f(x)$ and CDF $F(x)$. A query strategy is to find the maximum value by asking queries $Q_1, Q_2, \ldots$ sequentially where $Q_i \subset [0,1]$ and $\forall i \neq j,\ Q_i \cap Q_j = \emptyset$. After a query $Q_i$, all numbers within $Q_i$ will be reported. Note that $Q_i$ may depend on results of $Q_1, Q_2, \ldots, Q_{i-1}$. Thus a strategy can be denoted as a function:*

$$S\big(f, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_{i-1}\}\big) = Q_i$$

*which means, given $m$ i.i.d. unknown values whose PDF is $f(x)$, the reported values set $V$ and the set of queries asked $\mathcal{Q}$, the strategy $S$ will make $Q_i$ as the next query.*

*The cost of each query is equal to $b + j \cdot c$ where $j$ is the number of reported values from that query. The cost of a strategy is equal to the sum of all queries' costs it has to ask before identifying the maximum value. The optimal query strategy is the one that has minimum expected cost. We will write such minimum expected cost as $C^*(f,n)$, a function of PDF $f(x)$ and number of values $n$.*

We are then going to find out the optimal query strategy. Firstly, we need a lemma that tells us that the minimum cost is independent of PDF $f(x)$.

LEMMA 1. *Assume uniform PDF $u(x) = 1$ and define $C^*(n) = C^*(u,n)$. For any other PDF $f(x)$, we have*

$$C^*(f,n) = C^*(u,n) = C^*(n)$$

PROOF. Define $F^{-1}(x) = \sup\{y \mid F(y) = x\}$. For any strategy $S_f$ that works for PDF $f$, we can come up with a strategy $S_{u|f}$ for uniform PDF $u$:

$$S_{u|f}\big(u, m, V, \mathcal{Q}\big) = F\Big(S_f\big(f, m, F^{-1}(V), F^{-1}(\mathcal{Q})\big)\Big)$$

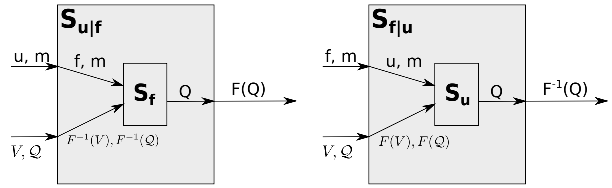
Thus $C^*(u,n) \le C^*(f,n)$ since we can adopt any strategy for $f$ to run under $u$ with the same cost. Similarly, for any strategy $S_u$ that works for PDF $u$, we can make strategy $S_{f|u}$ for PDF $f$:

$$S_{f|u}\big(f, m, V, \mathcal{Q}\big) = F^{-1}\Big(S_u\big(u, m, F(V), F(\mathcal{Q})\big)\Big)$$

Therefore $C^*(f,n) \le C^*(u,n)$. Combining with $C^*(u,n) \le C^*(f,n)$ we have $C^*(f,n) = C^*(u,n) = C^*(n)$. Figure [make an illustration] illustrates those two constructions we used for this proof.

□

After that, we prove that descending queries are optimal for this best case optimizing problem.




(a) Construct $S_{u|f}$ from $S_f$ (b) Construct $S_{f|u}$ from $S_u$

**Figure 1: These two figures illustrates how to construct a strategy for uniform PDF $u$ from another strategy for another arbitrary PDF $f$ and vice versa. Here we depict a strategy as a box which takes four inputs $f, m, V, \mathcal{Q}$ (PDF, number of unknown values, reported values set, set of asked queries) and make an output $Q$ (the next query)**

LEMMA 2. *There exists an optimal strategy with only descending queries $Q_1 = [q_1, 1],\ Q_2 = [q_2, q_1),\ Q_3 = [q_3, q_2) \ldots$*

PROOF. If not, there must be an optimal strategy where none of its non-descending queries can be changed to descending queries without increasing the cost. In that strategy $S$, there must be a first non-descending query $Q_{i+1} = S\big(F, m, V, \mathcal{Q} = \{Q_1, Q_2, \ldots, Q_i\}\big)$ where $Q_1$ to $Q_i$ are all descending. We can make another descending query $Q'_{i+1} = [q'_{i+1}, q_i)$ (or $Q'_{i+1} = [q'_{i+1}, 1]$ if $i = 0$) such that

$$\Pr(v \in Q_{i+1}) = \int_{Q'_{i+1}} f(x)\mathrm{d}x = \int_{Q_{i+1}} f(x)\mathrm{d}x = \Pr(v \in Q'_{i+1})$$

After $Q'_{i+1}$, we'll use as optimal query as possible.

Since $Q_1$ to $Q_i$ are all descending, we have $m = n$ and $V = \emptyset$ (otherwise the strategy should terminate without asking $Q_{i+1}$). Define $C$ to be the expected cost of using $Q_{i+1}$ and later queries. Similarly we define $C'$ for $Q'_{i+1}$: $C = b + \sum_{j=0}^{n} p_j(j \cdot c + C_j)$ and $C' = b + \sum_{j=0}^{n} p'_j(j \cdot c + C'_j)$ where: $p_j$ (or $p'_j$) is the probability that there are $j$ reported values within $Q_{i+1}$ (or $Q'_{i+1}$); $C_j$ (or $C'_j$) is the expected cost of later queries given that $j$ values have been found in $Q_{i+1}$ (or $Q'_{i+1}$).

As $\Pr(v \in Q_{i+1}) = \Pr(v \in Q'_{i+1})$, we have $p'_j = p_j$. Since $Q'_{i+1}$ is a descending query, $\forall j > 0,\ C'_j = 0 \le C_j$. And by lemma 1, $C'_0 = C_0 = C^*(n)$ because knowing no value is in $Q_1, Q_2, \ldots, Q_{i+1}$ is equivalent to revise PDF $f(x)$ to a refined PDF

$$f_{i+1}(x) = \begin{cases} \lambda f(x), & x \notin Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \\ 0, & x \in Q_1 \cup Q_2 \cup \ldots \cup Q_{i+1} \end{cases}$$

where $\lambda$ is a constant to make $\int_0^1 f_{i+1}(x)dx = 1$.

Thus $C' \le C$ which contradicts to that no non-descending query can be changed to descending query without increasing the cost.

□

Finally, we conclude the optimality of MVAs.

THEOREM 2. *Among all mechanisms that can include multiple rounds of broadcasts and are required to be efficient (allocate the item to the bidder with highest valuation), Multi-round Vickrey Auctions (MVAs) are of minimum cost.*

PROOF. The best case optimizing problem defined in definition 4 provides us a lower bound of minimum cost we can achieve by any mechanisms. By lemma 2, such lower bound minimum cost can be achieved by descending query strategy $Q_1 = [q_1, 1], Q_2 = [q_2, q_1), Q_3 = [q_3, q_2) \ldots$ Making $a_1 = q_1, a_2 = q_2, a_3 = q_3, \ldots$, by theorem 2 we are able to design such an MVA with reserve prices $r_1, r_2, \ldots$ whose Bayesian Nash Equilibrium achieves this best case descending query strategy. Thus, MVAs are optimal.

□

And by revenue equivalence theorem, MVAs maximize the seller utility.

COROLLARY 1. *If all broadcast costs and bidding costs are charged to sellers, MVAs are optimal if efficiency is required. Such optimal MVA is the one that minimizes the overall cost.*

## 3.3 Cost Minimized $\alpha$-MVA

Now let's try to calculate the parameters (thresholds $a_i$ or equivalently reserve prices $r_i$) of the optimal MVA for a given settings $F, n, b, c$ (valuation CDF, number of bidders, broadcast cost, bidding cost). By lemma 1, the cost is indifferent with $F$ and we can always derive an optimal mechanism for any $F$ from a uniform distribution. Thus we will focus on uniform cases below. We will also introduce $\rho = \frac{b}{c}$ to simplify our analysis by normalize bidding cost $c$ to 1 and thus broadcast cost $b$ to $\rho$.

An optimal MVA must be an $\alpha$-MVA where each round only $(1-\alpha)n$ bidders are expected to bid, i.e. $a_1 = \alpha, a_{i+1} = \alpha \cdot a_i$ for uniform cases. Then the expected overall cost $C$ satisfies: $C = \rho + (1-\alpha)n + \alpha^n C$. In the right hand side, the first term $\rho$ is the broadcast we have to use in the first round, the second term $(1-\alpha)n$ is the expected bidding cost for the first round, the third term $\alpha^n C$ lis a recursive term, the probability that no one bids $\alpha^n$ and if that happens the same cost $C$ should be expected in later rounds. Thus $C = \frac{\rho+(1-a)n}{1-\alpha^n}$. To minimize cost $C$, we have:

$$\frac{\partial C}{\partial \alpha} = \frac{\alpha^{n-1}\, n\ (\rho + (1-\alpha)\ n)}{(1-\alpha^n)^2} - \frac{n}{1-\alpha^n} = 0$$

$$\Updownarrow$$

$$\alpha^{n-1}(\rho + (1-\alpha)n) - (1-\alpha^n) = 0 \qquad (3)$$

[Increasing Threshold Search for Best-Valued Agents] has a proof for why $\alpha$-MVA is optimal among MVAs and how to determine $\alpha$. Thus if you find them not intuitive you may reference more details there. Our simplified cost model (with efficiency constraint and no buyer's cost) is equivalent to their cost model where learning cost is linear to the number of replied agents. That paper makes descending query as a contraint [1] and proves that $\alpha$-MVA is optimal among descending query mechanisms. Our focus in this section, however, is to have a preliminary introduction for MVA and proves optimality of descending queries and eventually MVAs. Thus we omit the proof about $\alpha$-MVA. Even for the $\alpha$, we will be more interested in its relation with larger $n$ so we will give approximations for $\alpha$.

[1]In their model they want to find the minimum value so increasing threshold search is equivalent to descending query

## 3.4 Approximation of $\alpha$ and Experiments

It's difficult to get an exact closed formula for optimal $\alpha$. Thus we are going to use some simpler formulas to approximate $\alpha$. We'll conduct experiments to compare our approximation with the optimal $\alpha$ that's computed numerically. We are also going to show comparisons between optimal MVA, approximate optimal MVA and other conventional mechanisms such as Vickrey auctions.

Firstly, $\alpha = 1 - 1/n$ is a natural guess which means each round the expected number of biddings is equal to 1. Thus we have $C(\alpha = 1-1/n) = (\rho+1)/(1-(1-1/n)^n)$. Because $(1-1/n)^n \leq e^{-1}$, we have $C(\alpha = 1-1/n) \leq (\rho+1)(1-e^{-1})$. It's obvious that at least one broadcast and one bidding is required to terminate so $C \geq \rho + 1$. Thus we have

THEOREM 3. *$\alpha = 1-1/n$ is a $1/(1-e^{-1})$ approximation of optimal $\alpha$-MVA. That means, by simply choosing $\alpha = 1 - 1/n$, we would at most get about $1.582$ times of optimal cost. Another obersavation of this approximation is that no matter how large $n$ is, the cost of this simple approximation is at most $(\rho+1)/(1-e^{-1}) = O(1)$. Thus the optimal cost is bounded by constant $O(1)$ no matter how large $n$ is.*

A better approximation when $n$ is large is to observe that $\lim_{n\to\infty} \alpha = 1$. Thus we guess [2] that $(1-a)n \approx A$ (for some constant $A$) and $\alpha^n \approx \alpha^{n-1}$. Then we have:

$$n(\rho + A) \cdot \alpha^n - n(1-\alpha^n) = 0$$

which gives us $\alpha = (1 + \rho + A)^{-1/n}$. Put this back to $\lim_{n\to\infty}(1-a)n = A$ we have $\ln(1+\rho+A) = A$ which gives us

$$A = -1 - \rho - W(-1-\rho)$$
$$\alpha = (-W(-1-\rho))^{-1/n} \qquad (4)$$

where $W(x)$ is the Lambert W function [cite wikipedia?] defined by $W(x)e^{W(x)} = x$. Actually, $w\,e^w = x$ has two solutions for $w$ when $-1 < x < 0$. Here our $W(x)$ refers to the lower[3] branch $W_{-1}(x) < -1$. So this is our second approximation that converges to the optimal one:

THEOREM 4. *Suppose that the optimal $\alpha$ is $\alpha^*$ which satisfies equation 3. Then $\alpha = (-W(-1-\rho))^{-1/n}$ satisfies*

$$\lim_{n\to\infty} C(\alpha^*) = \lim_{n\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n})$$

*That is, our approximation's cost will converge to optimal cost when $n$ grows to infinity.*

PROOF. Define sequence $\alpha_n^*, C_n^*$ where $n = 1, 2, 3, \ldots$ to be sequences of optimal $\alpha^*$ and corresponding optimal cost $C^*$ when there are $n$ bidders. We first show that $C_n^*$ is increasing: if we make $(\alpha_{n-1})^{n-1} = (\alpha_n^*)^n$, then we have 1) The expected broadcast cost of $\alpha_{n-1}$-MVA with $n-1$ bidders is equal to that of $\alpha_n^*$-MVA with $n$ bidders as the probability that one round will terminate is the same; 2) $\alpha_{n-1} < \alpha_n^*$ thus the expected bidding cost of $\alpha_{n-1}$-MVA with $n-1$ bidders should be less than that of $\alpha_n^*$-MVA with

[2]We tried several guesses and this is the one that finally turns out to work. Note that $(1-\alpha)n$ must be bounded (otherwise optimal cost won't be bounded by $O(1)$ as well) thus it's either a constant or a weird perturbation. It's natural to try constant first

[3]The upper branch is $W_0(x) > -1$ when $-1 < x < 0$

$n$ bidders. Therefore, $C^*_{n-1} \leq C_{n-1}(\alpha_{n-1}) < C^*_n$. Thus sequence $C^*$ is indeed strictly increasing.

Secondly, theorem 4 sais $C^*_n$ is bounded. Therefore $(1-\alpha^*)n$ must also be bounded otherwise $C = \frac{\rho+(1-\alpha)n}{1-\alpha^n}$ cannot be bounded. Thus according to Bolzano-Weierstrass theorem [cite wikipedia?], there must be a subsequence $\alpha^*_{n_i}$ such that $(1-\alpha^*_{n_i})n$ converges to $A$. And since $\alpha^*$ satisfies equation 3 and obviously $\lim_{n\to\infty} \alpha^* = 1$, we could use calculations similar to what we used for equation 4 to derive

$$\lim_{n_i\to\infty}(1-\alpha^*_{n_i})n_i = A = -1-\rho-W(-1-\rho)$$

$$\lim_{n_i\to\infty}(\alpha^*_{n_i})^{n_i} = (-W(-1-\rho))^{-1}$$

This proves that

$$\lim_{n_i\to\infty} C(\alpha^*) = \lim_{n_i\to\infty} C(\alpha = (-W(-1-\rho))^{-1/n_i})$$

Then using the fact that $C^*_n$ is strictly increasing and bounded completes the proof. □

Experiments in figure 2 compare the optimal $\alpha$, our first approximation of $\alpha = 1-1/n$ and our second approximation $\alpha = (-W(-1-\rho))^{-1/n}$ together with their corresponding cost under settings $\rho = 0.2, 1, 5$. Experiments [ref:figures] further compare these cost with two-step, fixed-step California-split strategy [cite and see Increasing Threshold Search for these strategies]. Recall that under this simpler model, the gross revenue is fixed thus the utility is completely determined by this cost.

As you can see from figure 2, the first approximation $1-1/n$ is bounded to be a constant time of optimal cost while the second approximation converges to optimal cost when $n$ grows large. When $\rho$ is close to 1, both two approximations are very close to the optimal one. But the second approximation is much better when $\rho$ is much smaller or greater than 1. Anyway, the second approximation isn't always better than the first approximation, as the case $\rho = 1, n = 2$ shows.

# 4. OPTIMAL MECHANISMS WITH BOTH SELLER'S AND BIDDER'S COST

In this section, we take out the constraints and prove that MVAs are optimal in general. We will also try to find the specific MVA to achieve such optimality, which turns out to be significantly more complex than previous simplified case.

The first constraint we are going to remove is seller's cost only. It's exciting to introduce bidder's bidding cost since it occurs very often in real cases and it plays an important role. Sending emails, making phone calls, enterring credit card nubmers, depositing money and clicking buttons are all costly for bidders, though sometimes very tiny. Bidders may not bid when this cost is greater than their expected utility. Note that even if the valuation is very high, the expected utility can be very small because of tense competition, which is very common in online cases as $n$, the number of potential bidders, is very large.

This behaviour (bidders won't bid because of competitions) is very different compared to that in previous model [cite Sequential Optimal Auctions] of sequential auctions. In that model, there's a time discount which makes bidders eager to bid in early rounds with high reserve prices to avoid waiting lost. That makes a lot sense in some cases but sometimes it may not. For example, if the seller posts an auction with a very low reserve price in the first round, most bidders with high valuation must be happy to bid according to that time discount cost model. But this may not be true. For example, when I encounter such an auction online [4], I might be very reluctant to bid because there's a big probability that my bidding will be over taken by someone else's so it's just a waste of effort. Our cost model can describe this behaviour very well.

Assume an extreme case where the broadcast cost is 0, the bidder's bidding cost is 0.1 and there are $n\to\infty$ many $[0,1]$-uniform distributed bidders. In a Dutch auction (a Dutch auction has infinite many rounds of broadcasts so we have to set braodcast cost to 0), only one bidder is expected to bid (no competition), thus every bidder with a valuation $v_i > 0.1$ should be benifitable to bid when the reserve price drops to a little > bit below $v_i - 0.1$ (recall that $n\to\infty$). In a Vickrey auction, however, the competition is very tense. Only bidders with valuation greater than $t$ can accept such intense competition where $t$ satisfies $t^{n-1}t - 0.1 = 0$ (the expected utility for a bidder with valuation $t$ is 0). Thus $t = \sqrt[n]{0.1}$ which is arbitrary close to 1 as $n$ grows to infinity. Thus almost all bidders can't bare this competition when $n$ is really large.

So a bad mechanism (e.g. a Vickrey auction) with too much cost will have less participations and therefore decreases seller's utility significantly. To see this, look at previous extreme case again. The revenue of Dutch auction will converge to 0.9 (someone with valuation very close to 1 will bid for price very close to 0.9) when $n$ becomes infinity. The revenue of a Vickrey auction, however, is only $\int_t^1 x\,(n-1)\,n\,(1-x)\,x^{n-2}dx$ which converges to about 0.67 when $n$ grows to infinity.

Another problem caused by bidder's cost is that revenue equivalence theorem seems to be no longer applicable. That's not strange as the revenue equivalence theorem assumes that the utility of a bidder is equal to the valuation minus the payment. This assumption is no longer true as now the utility is also influenced by the cost charged to this bidder.

Finally, the byproduct of removing the first constraint (or equivalently allowing bidder's bidding cost) is that we have to remove our second constraint, the efficiency of the mechanism, as you may have already observed. Since there's cost for bidders to bid, we could no longer enforce the mechanism to always allocate the item to the bidder with highest valuation. If we do so, it will become infeasible when that highest valuation is less than the bidding cost, i.e. the expected utility will be negative for some bidders to participate this mechanism.

In summary, we now introduce bidder's bidding cost and drops efficiency constraint for our mechanisms. The first issue we are going to solve is to make revenue equivalence theorem, or a very similar theorem, applicable to our model again. That's vital for seller's utility maximization.

## 4.1 Spending Equivalence Theorem and Revenue Optimization Strategy

THEOREM 5. *The expected overall spendings from all bidders (including their bidding costs and payments to the seller) in a feasible mechanism (with our cost model) is completely determined by the expected utility of lowest type bidders and*

[4] For example, when I see a very good item in the Auction House of Diablo III with a very low current bidding

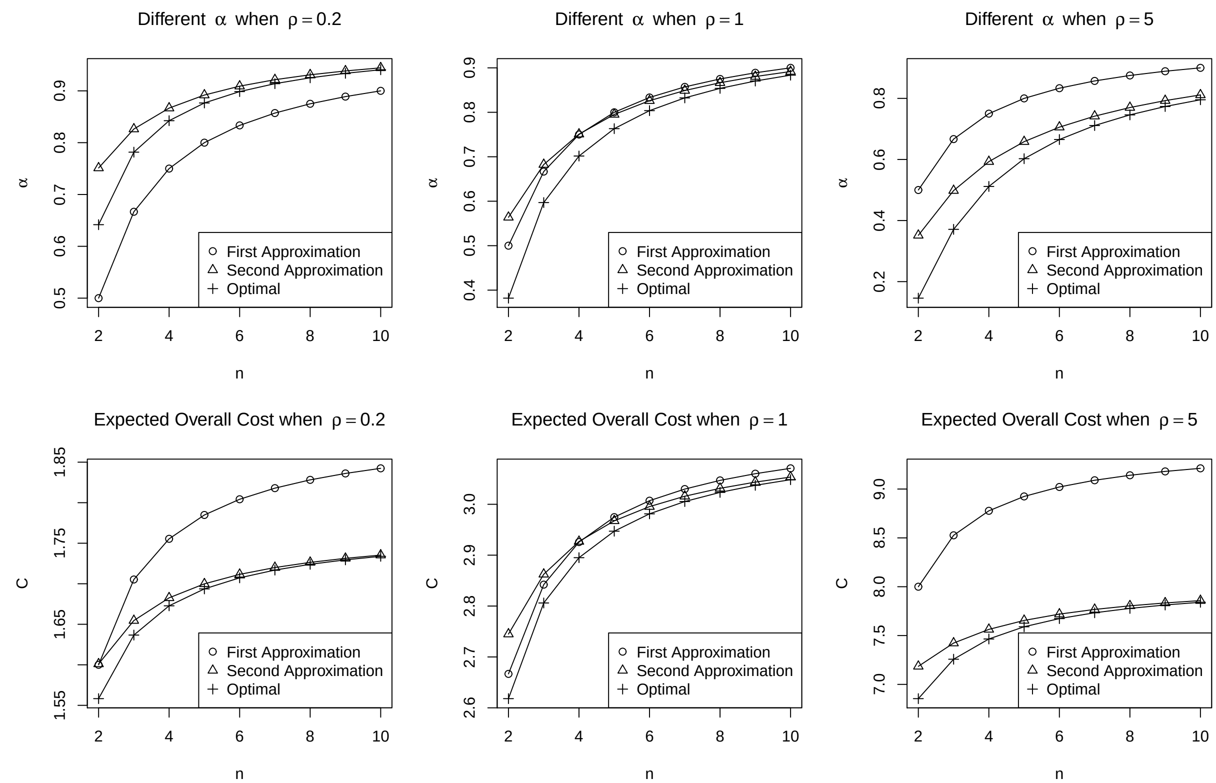



**Figure 2: Comparisons for optimal $\alpha$ and its approximations. The first approximation is $\alpha = 1 - 1/n$, the second is $\alpha = (-W(-1-\rho))^{-1/n}$. The second row is the corresponding cost for different $\alpha$**

*allocation probability function*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

*where $p_i$ is the probability that bidder $i$ will get the item.*

Proof. This theorem is exactly the same as revenue equivalence theorem except that we exchange revenue with spending. To prove it, let's construct another mechanism $M'$ (from our mechanism $M$) that fits into the original revenue equivalence theorem's model. Suppose there's a virtual seller in $M'$, who collects valuations from all bidders at no cost (a direct revelation mechanism). Then this virtual seller will make $n$ virtual bidders delegating all bidders to communicate with the true seller in our mechanism $M$. When our mechanism ends by allocating the item to virtual bidder $i$, the virtual seller also allocate the item to the real bidder $i$. The payment from each bidder $i$ to this virtual seller will be equal to the payment that virtual bidder $i$ pays to our real seller plus all the bidding costs charged to virtual bidder $i$. Thus, from the real bidders' aspects, this mechanism $M'$ is just a direct revelation mechanism which will satisfy revenue equivalence theorem. The only difference is that the payment from real bidder $i$ to the virtual seller actually has two parts, one is payed to the real seller, another is payed to bidding costs, which sum up to the total spending. $\square$

Thanks to theorem 5, our revenue maximization problem is now greatly simplified: we only need to determine optimal allocation rule $p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$ and the minimum cost given that allocation rule. However, the optimal allocation rule here isn't as simple as the one that's discovered by Myerson [ref:Optimal Auction]: allocate the item to the bidder with highest virtural value if it's positive. This rule, though optimal in Myerson's setting, may not be optimal here. Theorem 5 tells us that this rule will maximize the total spending. But we must substract the cost from the spending to get the revenue. Therefore, can there be another weird allocation rule that has less total spending but even much less minimum cost, so a greater revenue is achieved by that allocation rule.

It's not hard to find one example to support our suspicion. We know that for bidders with uniform valuation over $[0, 1]$, the spending maximizing allocation rule is allocate the item to the bidder with highest valuation that's greater than $1/2$. However, if the broadcast is too large, for example 1 (which is equal to the highest possible valuation), the cost to find out whether there's any bidder with valuation greater than $1/2$ is at least 1. Thus if we use this allocation rule, the final revenue would be negative since the total spending must be less than the minimum cost. However, the allocation rule that never allocate the item will have 0 cost and 0 spending, which achieves 0 revenue, better than previous allocation rule.

Thus, the revenue optimal mechianism will depend on how minimum cost is defined. We previously defined how cost is charged and proved that a specific mechanism (MVA) has minimum cost when allocation rule is allocate efficiently. But unfortunately, that's not enough to give us a well defined minimum cost for any allocation rule. For example, one allocation rule might be always allocate the item to each bidder with the same probability $1/n$, i.e. $p(v_1, v_2, \ldots) =$

$(1/n, 1/n, \ldots)$. You might think that the minimum cost for this is 0 since we don't have to know anyone's valuation. But that cost isn't realistic: how can you ever allocate the item to someone that you have never communicated with? Thus the minimum cost seems to be at least $b$, the cost for one round of broadcast, if we ever allocate the item to some bidder. In order to make a realistic constraint and get a well defined minimum cost for any allocation rule, we define

Definition 5. *For an allocation rule*

$$p : (v_1, v_2, \ldots, v_n) \to (p_1, p_2, \ldots, p_n)$$

*if $p_i > 0$ for some valuation profile $(v_1, v_2, \ldots, v_n)$, there must be a broadcast query that the $i$-th bidder with valuation $v_i$ reply to the seller under that profile setting.*

With this definition, we have

Theorem 6. *The optimal mechanism should always allocate the item to the bidder with highest virtual valuation $v_i - \frac{1-F(v_i)}{f(v_i)}$ if it decides to allocate the item. In regular cases when the virtual valuation is monotone strictly increasing, the optimal mechanism should always allocate the item to the bidder with highest valuation if it decides to allocate the item.*

To see why this theorem holds, firstly notice that in our cost model, the minimum cost to let any bidder reply is equal to the minimum cost to let the max-valuation bidder (or max-virtual-valuation bidder, we won't mention virtual valuation below because they are indifferent in our cost context so you can always substitude one by another, adding or removing regularity constraint) reply. This is because that the mechanism is only allowed to ask broadcast queries and it's equivalent to specify a range $Q \subseteq [0, 1]$ and ask all bidders whose valuations are within that range to reply. That means, by properly mappings of valuations and query ranges, we can always adapt a strategy that lets any bidder reply to another strategy that lets the max-valuation bidder reply and vise versa (just like what we did to prove lemma 2). This means that finding the max-valuation bidder won't cost more than any other cases except doing nothing (which costs 0). Thus the optimal mechanism either do nothing (no one replies and thus $p_i = 0$ by definition 5) which has minimum cost or ask queries to find the max-valuation bidder and allocate the item to that bidder which has maximum total spending. The flexibility of an optimal mechanism is that it can choose the cases in which it won't allocate the item (in any other cases it will allocate the item to the max-valuation bidder). It's also intuitive to see that the no allocation cases can be described by a single parameter $l$: no allocation if every bidder's valuation is below $l$. For limit of space, we won't list the detailed proof for the above theorem and arguments.

Now we can narrow our optimal mechanisms with relaxed efficiency constraint:

Definition 6. *We say mechanisms satisfy relaxed efficiency constraint with lowest type $l$ if:*

1. *They only allocate the item to bidders whose valuation are at least $l$ (the lowest type is $l$)*

2. *If they will allocate the item, they will always allocate the item to the bidder with highest valuation.*

*When we say a mechanism with a lowest type $l$, we imply that this mechanism satisfy relatex efficiency constraint with lowest $l$.*

## 4.2 Cost Minimization with Lowest Type and MVAs' Optimality in General

We have already narrowed down optimal mechanism to relatex efficient mechanisms and by theorem 5, it's straightforward to see

Corollary 2. *For mechanisms with a fixed lowest type $l$, the maximum utility for sellers is achieved when the mechanism minizes the cost.*

Our next question is naturally: what's the cost minimized mechanism given a lowest type $l$. As expected:

Theorem 7. *MVAs have the minimum cost among all mechanisms with a lowest type $l$*

Proof. A special case of this theorem when $l = 0$ is theorem 2. We proved that special case by introducing lemma 1 and 2. To prove the general cases with arbitrary $l$, we just need to revise lemma 1 a little as following lemma 3. All other part of the proof remains similar. For space limit, the detailed proof is omitted. □

Lemma 3. *Suppose that there are two cases $n, F_1, f_1, l_1$ and $n, F_2, f_2, l_2$ where $n$ is the number of values for both cases, $F_i, f_i$ are CDF and PDF of the $n$ i.i.d. values in case $i$, $l_i$ is the lowest type for case $i$. If $F_1(l_1) = F_2(l_2)$, then these two cases have the same minimum cost to find the maximum value above the lowest type $l_i$.*

The proof of this revised lemma is almost identical to the one for original lemma. Thus for space limit, we won't elaborate it again. Finally, we conclude that MVAs are optimal in general.

Corollary 3. *MVAs are optimal. The only parameters we are going to determine for the specific optimal MVA are 1) the lowest type $l$; 2) the descending query thresholds $q_1, q_2, q_3, \ldots$.*

*Shall we increase lowest type $l$ a little above Myerson's optimal lowest type to trade payment with cost? Or is it optimal to use Myerson's $l$ and then minimize cost according to that?*

## 4.3 Experiments to Discover Optimal MVA with a Given Lowest Type

To discover the specific MVA that's optimal, we first try to identify the optimal thresholds $a_i$ given a lowest type $l$ (recall that in $i$-th round, MVA will ask all bidders whose valuation is within $[a_i, a_{i-1})$ to bid). If we can write the minimum cost $C^*$ as a function of $n, \rho, l$ (recall that $rho = b/c$ is the ratio between broadcast and bidding cost), we may then determine the optimal $l$ using this function.

In the special case that we studied in previous section where $l = 0$ (efficiency is enforced), the optimal thresholds can be easily described as a single parameter $\alpha$. This strategy won't work when $l > 0$ (more precisely, $F(l) > 0$, but because of lemma 3, we only focus on uniform distribution in the following). Obviously that $a_i \geq l$, thus we can't let $a_i = \alpha a_{i-1}$ since $\lim_{i\to\infty} a_i = 0 < l$. Additionally, if we revise the equation by letting $a_i - l = \alpha(a_{i-1} - l)$, we will get a positive possibility $F(l)$ that we would ask infinite many broadcast queries, which is even worse.

As it's not immediately clear what optimal thresholds should be like, we present a simple algorithm to calculate such thresholds numerically. We firstly discretize the continuous valuation to $D$ discrete values $\{0, 1, \ldots, D-1\}$. That means, the original valuation $v$ will be transformed to integer value $\hat{v} = \lfloor vD \rfloor$. Then we use dynamic programming to inductively calculate $x_r$, the length of next optimal descending query to ask (therefore the query would be $[r - x_r, r)$), conditional on that we have already queried $[r, D)$ and no one replies.

**Algorithm 1** Calculate discretized best query lengths

**Require:** $\hat{l}$ is the discretized lowest type, $\rho$ is the ratio between broadcast and bidding cost, $n$ is the number of bidders, $D$ is the maximum discretized valuation

**function** BESTQUERYLENGTHS($\hat{l}$, $\rho$, $n$, $D$)
  $x_{\hat{l}} \leftarrow 0$
  $C_{\hat{l}} \leftarrow 0$
  **for** $r = \hat{l} + 1$ to $D$ **do**
    $x_r \leftarrow \arg\min_{1 \le x \le r-l} \rho + n\frac{x}{r} + (\frac{r-x}{r})^n C_{r-x}$
    $C_r \leftarrow \rho + n\frac{x_r}{r} + (\frac{r-x_r}{r})^n C_{r-x_r}$
  **end for**
  **return** $x$
**end function**

Algorithm 1 runs in time $O(D^2)$. Having $x$, we can then infer the best strategy for original continuous problem by first discretize it and then convert the discretized optimal strategy back to continuous strategy. The larger $D$ is the more accurate it will be. But it will also require more running time. Running this for case $\hat{l} = 500, \rho = 2, n = 10, D = 1000$ we get $x$ showed in figure 3. It seems that $x$ is a piecewise linear function over $r$. For those $r$ which is close to $\hat{l}$, obviously that the optimal strategy should be $x = r - \hat{l}$, which means using only one query to explore all potential bidders. But it's unclear why $x$ is linear when the best strategy is using multiple queries to explore the valuation range.

## 4.4 Analysis of Optimal MVA with Lowest Type

Figure 3 shows that the optimal MVA with a lowest type $l$ is much more complicated but there might still be hope to get a nice analytical result: piecewise linear function. That is, we can't use a single $\alpha$ to describe such optimal MVA, but perhaps we can use a sequence of $\alpha$ to describe it. Now let's take an analytical treatment.

The first key point to analyze the optimal MVA with lowest type is to utilize the fact that there's a fixed maximal number of rounds to exploit the whole reportable valuation range $[l, 1)$. Let's call that number $k$. We have argued this before: if not, because the possibility that no values lie in $[l, 1)$ is possitive, the cost will be infinte.

For convenience, define $\vec{a} = (a_0, a_1, a_2, \ldots, a_k)$, the vector of $k$ thresholds in such optimal MVA. We make $a_0 = 1, a_k = l$ so in $i$-th round the query would be $[a_i, a_{i-1})$. Now define cost $C(\vec{a}, k, \rho, n)$ to be the expected cost for MVA defined by $k, \vec{a}$ when there are $n$ i.i.d. $[0, 1)$-uniform bidders. If we can get a neat form of $C$, we can use $\frac{\partial C}{\partial a_i} = 0$ to characterize optimal MVA, as we did in section 3.3.

Thus the second key point is to how to represent this $C$.

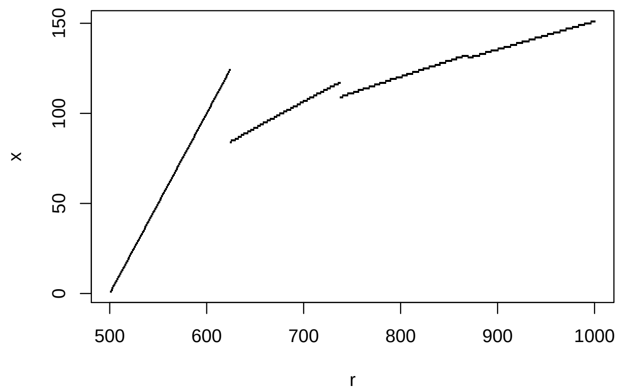



**Figure 3: The best query lengths over $r$ (the upper bound of undiscovered value) when discretized lowest type $\hat{l} = 500$, ratio between broadcast and bidding cost $\rho = 2$, number of bidders $n = 10$ and maximum discretized valuation $D = 1000$**

Rather than considering one round after another as we did before, now we consider all rounds together:

$$C(\vec{a}, k, \rho, n) = \sum_{i=0}^{k} P\big(H_i = 1\big) C\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big)$$

where $H_k$ is the indicate random variable for the event that the highest valuation is below $a_k$ and $H_i (0 \le i < k)$ is for the event that the highest valuation is in $[a_{i+1}, a_i)$.

Since bidders are uniformly distributed,

$$P(H_k = 1) = \frac{a_k^n}{a_0^n} = a_k^n = l^n$$
$$P(H_i = 1) = \frac{a_i^n - a_{i+1}^n}{a_0^n} = a_i^n - a_{i+1}^n$$

It's also clear that $C(\vec{a}, k, \rho, n \bigm| H_k = 1) = k\rho$ because there will be only $k$ broadcasts and no biddings.

For $C\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big)$ where $i < k$, we partition it into two parts $C\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big) = C_b\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big) + C_c\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big)$. The broadcast part $C_b\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big)$ is simply $(i+1)\rho$ as the MVA finishes in $(i+1)$-th round. The not-so-obvious part is the overall bidding cost $C_c\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big)$. It can be expressed as:

$$\begin{aligned} P(H_i = 1) C_c\big(\vec{a}, k, \rho, n \bigm| H_i = 1\big) &= E\left(H_i \cdot \sum_{j=1}^{n} X_{ij}\right) \\ &= E\left(\left(\prod_{j=1}^{n} B_{ij}\right) \cdot \left(\sum_{j=1}^{n} X_{ij}\right)\right) \end{aligned}$$

Here $X_{ij}, A_{ij}$ are both indicate random variables of bidder $j$'s valuation $v_j$:

$$X_{ij} = \begin{cases} 0, & \text{if } v_j \notin [a_{i+1}, a_i) \\ 1, & \text{if } v_j \in [a_{i+1}, a_i) \end{cases} \qquad B_{ij} = \begin{cases} 0, & \text{if } v_j \ge a_i \\ 1, & \text{if } v_j < a_i \end{cases}$$

You may notice that our original random variable $H_i$ is not exactly the same as our substitute $\prod_{j=1}^{n} B_{ij}$. The only difference between them, however, is the case that every $v_j$

is below $a_{i+1}$, which is the case that every $X_{ij} = 0$, so the equation still holds.

Now we use the properties of expectation to finish our calculation:

$$E\left(\left(\prod_{j=1}^{n} B_{ij}\right) \cdot \left(\sum_{j=1}^{n} X_{ij}\right)\right) = \sum_{j=1}^{n} E\left(X_{ij} \prod_{k=1}^{n} B_{ik}\right)$$
$$= \sum_{j=1}^{n}\left(E(X_{ij}B_{ij}) \prod_{k \neq j} E(B_{ik})\right) = n a_i^{n-1}(a_i - a_{i+1})$$

Combine all those above, finally we get (note that lowest type $l = a_k$):

$$C(\vec{a}, k, \rho, n) = a_k^n k\rho + \sum_{i=0}^{k-1}\left((a_i^n - a_{i+1}^n)(i+1)\rho + n a_i^{n-1}(a_i - a_{i+1})\right) \tag{5}$$

### 4.5 Using Piecewise Linear MVA to Approximate Optimal MVA

### 4.6 Choosing Lowest Type?

### 4.7 Experiments

Now we are going to compare revenue in general cases. We not only compare our approximate optimal MVAs to optimal MVAs computed numerically, but also compare MVAs to other conventional mechanisms.

## 5. CONCLUSION