AMATH 586 SPRING 2020
HOMEWORK 2 — DUE APRIL 24 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2020` repository.

Problem 1: Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\frac{\sqrt{\beta_3 - \beta_1}}{12} t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\operatorname{cn}(x, k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\operatorname{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 7 — produce a plot and and a table.

Problem 2: Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+3} = U^{n+1} + 2kf(U^n)$,
(b) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$,
(c) $U^{n+1} = U^n$,
(d) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$,
(e) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

Problem 3: For the one-step method (6.17), with Ψ given in (6.18), show that the Lipschitz constant is $L' = L + \frac{k}{2}L^2$ where L is the Lipschitz constant for f.

Problem 4: The Fibonacci numbers

(a) Determine the general solution to the linear difference equation $U^{n+2} = U^{n+1} + U^n$.
(b) Determine the solution to this difference equation with the starting values $U^0 = 1$, $U^1 = 1$. Use this to determine U^{30}. (Note, these are the *Fibonacci numbers*, which of course should all be integers.)

(c) Show that for large n the ratio of successive Fibonacci numbers U^n/U^{n-1} approaches the "golden ratio" $\phi \approx 1.618034$.

Problem 5: Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r. For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathrm{O}(z^{p+1}) \quad \text{as } z \to 0.$$

This indicates that the one-step error is $\mathrm{O}(z^{p+1})$ on this problem, as expected for a pth order accurate method.

The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to $u'(t) = \lambda u(t)$. Show that in fact when applied to this problem the method becomes $U^{n+1} = R(z)U^n$ where $R(z)$ is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of e^z through $O(z^4)$ terms.

We will see that this function $R(z)$ is also important in the study of absolute stability of a one-step method.

Problem 6: Determine the function $R(z)$ described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for $u'(t) = \lambda u(t)$. (You might want to convince yourself these are the same method).

Confirm that $R(z)$ agrees with e^z to the expected order.

Note that for this implicit method $R(z)$ will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1-\epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots .$$