

**AMATH 586 SPRING 2020**  
**HOMEWORK 2 — DUE APRIL 24 ON GITHUB BY 11PM**

Be sure to do a `git pull` to update your local version of the `amath-586-2020` repository.

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**Problem 1:** Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3}v'(t) = 0,$$

where  $\beta_1 < \beta_2 < \beta_3$ . It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where  $\operatorname{cn}(x, k)$  is the Jacobi cosine function and  $k$  is the elliptic modulus. Some notations use  $\operatorname{cn}(x, m)$  where  $m = k^2$ . The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 7 — produce a plot and a table.

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**Problem 2:** Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

- (a)  $U^{n+3} = U^{n+1} + 2kf(U^n)$ ,
  - (b)  $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$ ,
  - (c)  $U^{n+1} = U^n$ ,
  - (d)  $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$ ,
  - (e)  $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$ .
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**Problem 3:** For the one-step method (6.17), with  $\Psi$  given in (6.18), show that the Lipschitz constant is  $L' = L + \frac{k}{2}L^2$  where  $L$  is the Lipschitz constant for  $f$ .

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**Problem 4:** The Fibonacci numbers

- (a) Determine the general solution to the linear difference equation  $U^{n+2} = U^{n+1} + U^n$ .
- (b) Determine the solution to this difference equation with the starting values  $U^0 = 1, U^1 = 1$ . Use this to determine  $U^{30}$ . (Note, these are the *Fibonacci numbers*, which of course should all be integers.)

- (c) Show that for large  $n$  the ratio of successive Fibonacci numbers  $U^n/U^{n-1}$  approaches the “golden ratio”  $\phi \approx 1.618034$ .

**Problem 5:** Any  $r$ -stage Runge-Kutta method applied to  $u' = \lambda u$  will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where  $z = \lambda k$  and  $R(z)$  is a rational function, a ratio of polynomials in  $z$  each having degree at most  $r$ . For an explicit method  $R(z)$  will simply be a polynomial of degree  $r$  and for an implicit method it will be a more general rational function.

Since  $u(t_{n+1}) = e^z u(t_n)$  for this problem, we expect that a  $p$ th order accurate method will give a function  $R(z)$  satisfying

$$R(z) = e^z + O(z^{p+1}) \quad \text{as } z \rightarrow 0.$$

This indicates that the one-step error is  $O(z^{p+1})$  on this problem, as expected for a  $p$ th order accurate method.

The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to  $u'(t) = \lambda u(t)$ . Show that in fact when applied to this problem the method becomes  $U^{n+1} = R(z)U^n$  where  $R(z)$  is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of  $e^z$  through  $O(z^4)$  terms.

We will see that this function  $R(z)$  is also important in the study of absolute stability of a one-step method.

**Problem 6:** Determine the function  $R(z)$  described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for  $u'(t) = \lambda u(t)$ . (You might want to convince yourself these are the same method).

Confirm that  $R(z)$  agrees with  $e^z$  to the expected order.

Note that for this implicit method  $R(z)$  will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1 - \epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots.$$