Scattering Polarimetry & SAR Data Statistical Description

Carlos López-Martínez

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Universitat Politècnica de Catalunya – UPC
Signal Theory and Communications Department – TSC
Institute of Space Studies of Catalonia - IEEC
carlos.lopezmartinez@upc.edu

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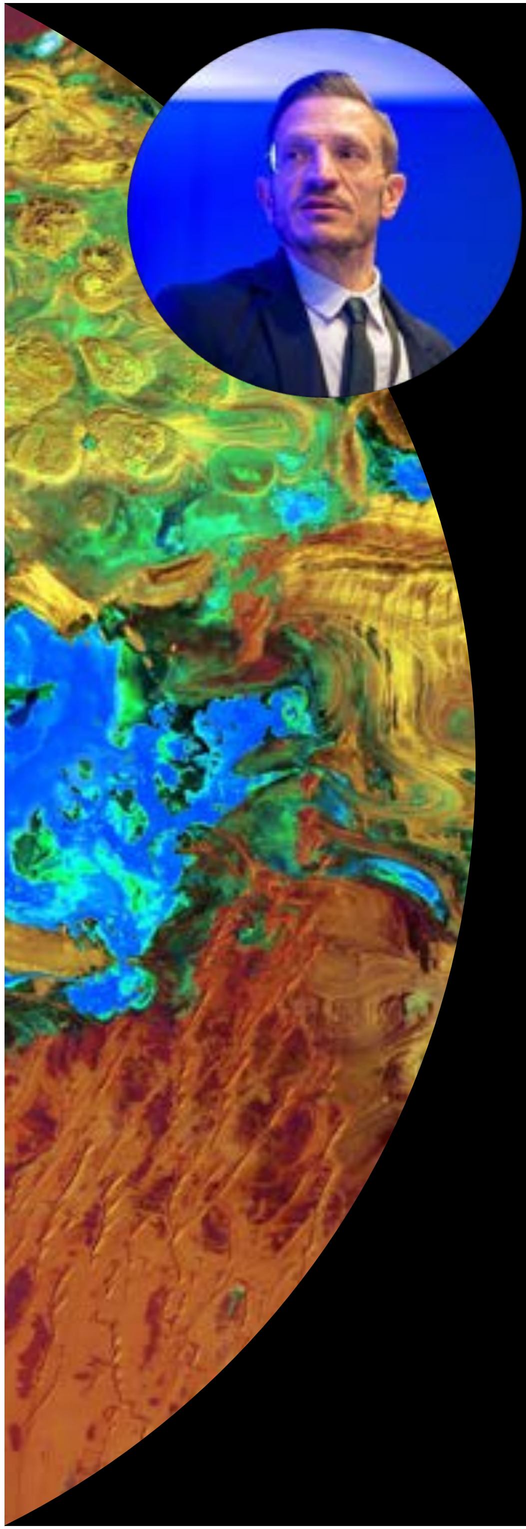


Carlos LÓPEZ-MARTÍNEZ, PhD

Scientist | Engineer | Associate Professor

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Departament de Teoria del Senyal
i Comunicacions

IEEC
Institut d'Estudis
Espacials de Catalunya

Technical University of Catalonia UPC
Signal Theory and Communications Department TSC
Remote Sensing Lab. (RsLAB)

Institute of Space Studies of Catalonia IEEC

+34 93 401 6785
carlos.lopezmartinez@upc.edu
carloslopezmartinez.info
www.linkedin.com/in/clopmar

Outline

- Scattering Polarimetry. Totally & Partially Developed Waves
- SAR Data Statisitical Characterization
- PolSAR Data Statistical Characterization
- Information Estimation/Filtering
- PolSAR Data Speckle Noise Characterization

Wave Covariance Matrix & Stokes Vector

Real representation of the wave polarization state of a monochromatic wave

Wave covariance matrix $$\underline{E} \cdot \underline{E}^{T*} = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix}$$

Any 2x2 matrix can be decomposed in the Pauli set of matrices, since this set is complete

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\underline{E} \cdot \underline{E}^{T*} = \frac{1}{2}\left\{ g_0\sigma_0 + g_1\sigma_1 + g_2\sigma_2 + g_3\sigma_3 \right\} = \frac{1}{2}\begin{bmatrix} g_0 + g_1 & g_2 - jg_3 \\ g_2 + jg_3 & g_0 - g_1 \end{bmatrix}$$

$\{ g_0, g_1, g_2, g_3 \}$ Decomposition coefficients $\Rightarrow$ Stokes parameters

$[g_0, g_1, g_2, g_3]^T$ $\Rightarrow$ Stokes vector

Stokes Vector

Equivalence of the different wave polarization descriptors

$$\underline{E} = \begin{bmatrix} E_x = E_{ox} e^{j\delta_x} \\ E_y = E_{oy} e^{j\delta_y} \end{bmatrix} \Rightarrow \underline{g}_{\underline{E}} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} |E_x|^2 + |E_y|^2 \\ |E_x|^2 - |E_y|^2 \\ 2\Re\left(E_x E_y^*\right) \\ -2\Im\left(E_x E_y^*\right) \end{bmatrix} = \begin{bmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos(\delta) \\ 2E_{0x}E_{0y}\sin(\delta) \end{bmatrix} = \begin{bmatrix} A^2 \\ A^2\cos 2\phi \cos 2\tau \\ A^2 \sin 2\phi \cos 2\tau \\ A^2 \sin 2\tau \end{bmatrix}$$

Jones vector parameters

Polarization ellipse parameters

Orthogonal stokes vectors

Orthogonality $\Rightarrow$ $\phi_2 = \phi_1 + \dfrac{\pi}{2} \qquad \tau_2 = -\tau_1$

$$\underline{g}_{\underline{E}} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} A \\ A\cos 2\phi \cos 2\tau \\ A \sin 2\phi \cos 2\tau \\ A \sin 2\tau \end{bmatrix} \qquad \underline{g}_{\underline{E}_\perp} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} A \\ -A\cos 2\phi \cos 2\tau \\ -A \sin 2\phi \cos 2\tau \\ -A \sin 2\tau \end{bmatrix}$$

Kennaugh Matrix

The Kennaugh Matrix describes the scattering process by relating the Stokes vectors

$$\mathbf{g}_s = \frac{1}{r^2}\mathbf{K}\mathbf{g}_i$$

$$\Downarrow$$

$$\mathbf{K} = \begin{bmatrix} A_0 + B_0 & C & H & F \\ C & A_0 + B & E & G \\ H & E & A_0 - B & D \\ F & G & D & -A_0 + B_0 \end{bmatrix}$$

$$\mathbf{K} = \frac{1}{2}\left(\mathbf{V}^T \left[\mathbf{S} \otimes \mathbf{S}^*\right]\mathbf{V}\right) \qquad \mathbf{V} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & +j \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

- The Kennaugh Matrix is a real symmetric matrix
- Contains the same information as the scattering matrix

Kennaugh Matrix

The parameters of the Kennaugh matrix are employed for man-made target indentification and analysis

- A_0: Generator of target symmety
- B_0+B: Generator of target non-symmetry
- B_0-B: Generator of target irregularity
- C: Generator of target global shape (Linear)
- D: Generator of target local shape (curvature)
- E: generator of target local twist (Torsion)
- F: Generator of target global twist (Helicity)
- G: Generator of target local coupling (Glue)
- H: Generator of target global coupling (Orientation)

These parameters are not mutually independent

Elliptical Basis Transformation

The elliptical basis transformation must be performed expressing the transmitted and scattered stokes vectors in the same coordinate system

First polarization basis

$$\mathbf{g}^{s}_{(A,A_\perp)} = \mathbf{K}_{(A,A_\perp)}\mathbf{g}^{i}_{(A,A_\perp)}$$

Second polarization basis

$$\mathbf{g}^{s}_{(B,B_\perp)} = \mathbf{K}_{(B,B_\perp)}\mathbf{g}^{i}_{(B,B_\perp)}$$

Change of polarization basis

$$\mathbf{g}_{(A,A_\perp)} = \mathbf{O}_{(A,A_\perp)\mapsto(B,B_\perp)}\mathbf{g}_{(B,B_\perp)}$$

- The **O** is a O(4) special unitary elliptical transformation basis

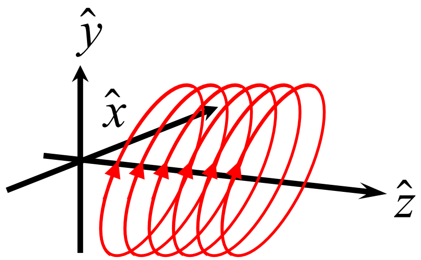
$$\mathbf{K}_{(B,B_\perp)} = \mathbf{O}_{(A,A_\perp)\mapsto(B,B_\perp)}\mathbf{K}_{(A,A_\perp)}\mathbf{O}^{-1}_{(A,A_\perp)\mapsto(B,B_\perp)}$$

Similarity Transformation

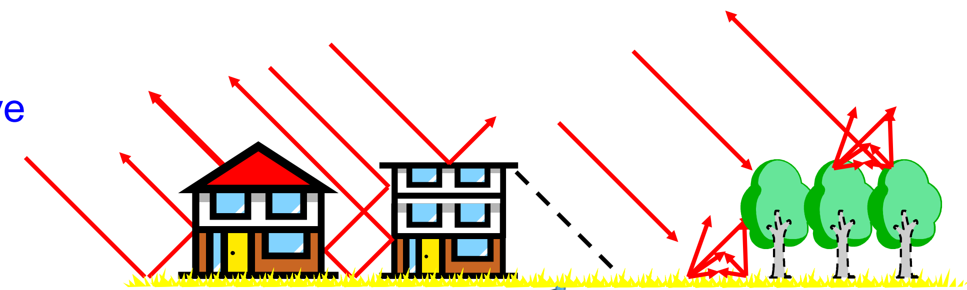
Partially Polarized Waves

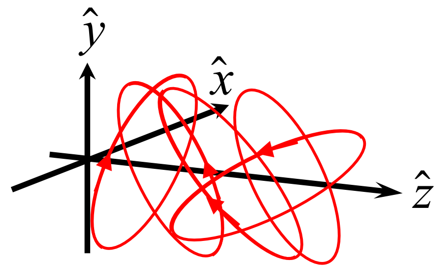
Electromagnetic fields depend on the microscopic arrangement of elementary scatters in the resolution cell, therefore, wave polarization depends also on this effect

- Deterministic scatterers: The electromagnetic field does no vary, in space or time,
  - Wave polarization parameters are constant
  - Completely Polarized Wave
- Distributed scatterers: The electromagnetic field varies, in space or time, due to the complex arrangement of individual scatters
  - Wave polarization parameters vary randomly in time or space
  - Wave polarization descriptors need to accommodate this random nature, i.e., stochastic descriptors are necessary
  - Partially Polarized Wave



Completely Polarized Wave





Partially Polarized Wave

Partially Polarized Waves

Possibility to introduce the expectation operation in the wave polarization descriptors

- Wave covariance matrix

$$E\left\{\underline{E}\cdot\underline{E}^{T*}\right\}=\begin{bmatrix} E\left\{E_xE_x^*\right\} & E\left\{E_xE_y^*\right\} \\ E\left\{E_yE_x^*\right\} & E\left\{E_yE_y^*\right\}\end{bmatrix}$$

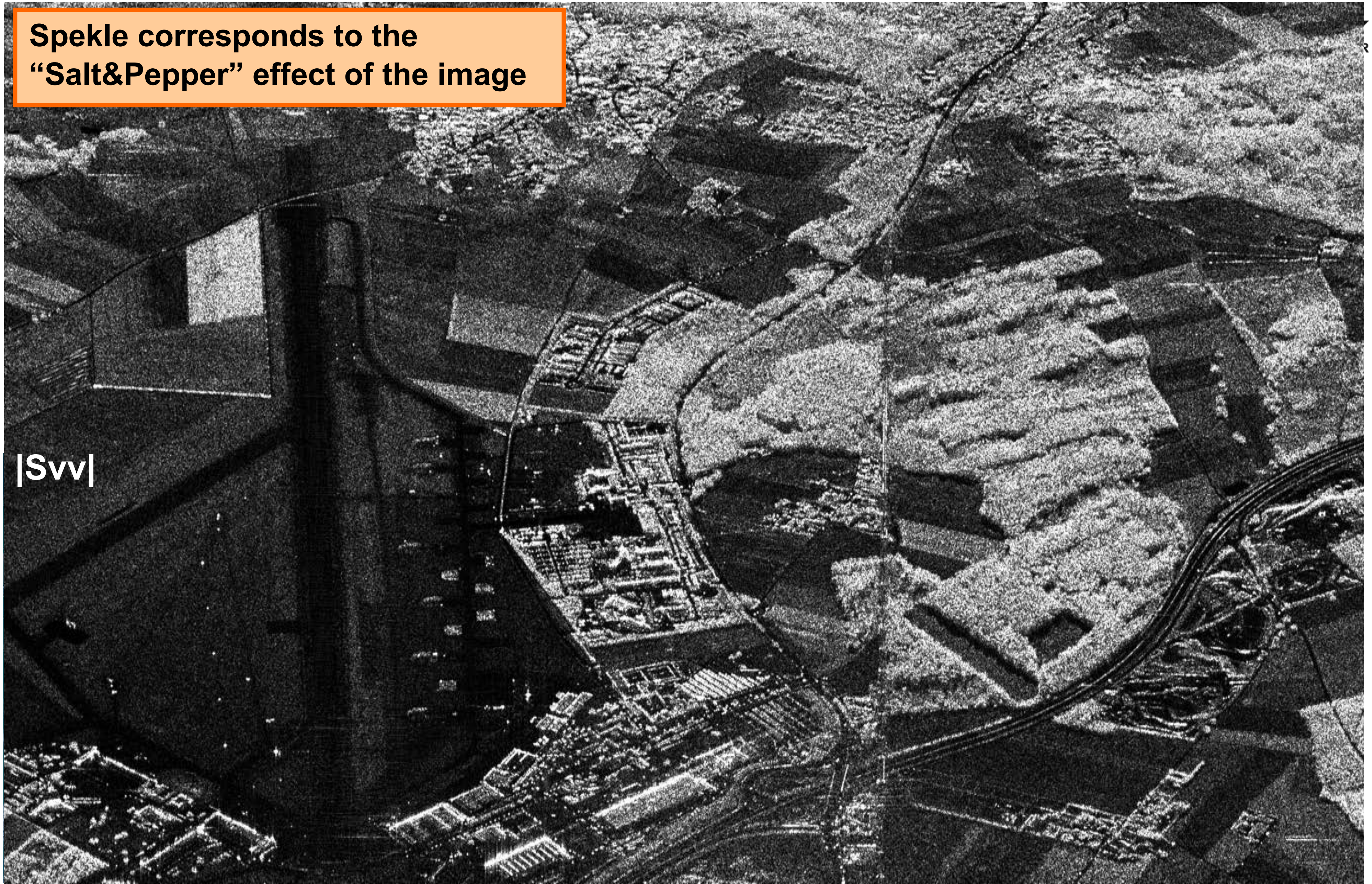
- Stokes vector

$$E\left\{\underline{g}_{\underline{E}}\right\}=\begin{bmatrix} E\left\{g_0\right\} \\ E\left\{g_1\right\} \\ E\left\{g_2\right\} \\ E\left\{g_3\right\}\end{bmatrix}=\begin{bmatrix} E\left\{\left|E_x\right|^2+\left|E_y\right|^2\right\} \\ E\left\{\left|E_x\right|^2-\left|E_y\right|^2\right\} \\ E\left\{2\Re\left(E_xE_y^*\right)\right\} \\ E\left\{-2\Im\left(E_xE_y^*\right)\right\}\end{bmatrix}$$

- Second order moments are not null as they refer to power and correlation

Speckle Noise

Speckle Noise



On the basis of the discrete scatter description

$$S(x,r)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x',r')h(x-x',r-r')\mathrm{d}x'\mathrm{d}r' \Rightarrow S(x,r)=\frac{1}{\sqrt{L}}\sum_{k=1}^{L}\sqrt{\sigma_k}e^{j\theta_k}h(x-x_k,r-r_k)$$

Normalizing factor

L: Number of point scatters embraced by the resolution cell

- L as a deterministic quantity
  - L = 1: or a dominating point scatter: Deterministic scattering
    - Rice/Rician model
  - L >1: Partially developed speckle
    - Not solved model. Even numerical solution difficult
  - L >>1: Fully developed speckle
    - Gaussian model
- L as a stochastic quantity
  - L characterized by a pdf: Image texture
    - K-distribution model

Fully Developed Speckle Noise

Important considerations

- Speckle is a deterministic electromagnetic effect, but due to the complexity of the image formation process, it must be analysed statistically

- Considering completely developed speckle, a SAR image pixel does not give information about the target. Only statistical moments can describe the target or the process

Information

What does it mean information in the presence of Speckle?

- Phase contains no information
- Intensity exponentially distributed

$$p_I(I) = \frac{1}{2\sigma^2}\exp\left(-\frac{I}{2\sigma^2}\right) \quad I \in [0,\infty) \Rightarrow \begin{aligned} E\{I\} &= 2\sigma^2 \\ \sigma_I &= 2\sigma^2 \end{aligned}$$

Exponential pdf — First and second order moments

- Intensity, under the previous hypotheses, is completely determined by the exponential pdf
  - Pdf completely determined by the pdf shape
  - Pdf shape parameterized by σ ⇒ INFORMATION ⇒ RCS σ^0

- Not useful information is considered as NOISE

Fully Developed Speckle Noise Model

Objectives of a Noise Model

- To embed the data distribution into a noise model, that is, a function that allows identifying of the useful information to be retrieved, the noise sources, and how these terms interact
- Optimize the information extraction process, i.e., the noise filtering process

SAR image intensity noise model

SAR image intensity ($I=r^2$) $$p_I(I)=\frac{1}{2\sigma^2}\exp\left(-\frac{I}{2\sigma^2}\right) \quad I\in[0,\infty) \qquad E\{I\}=2\sigma^2 \quad \sigma_I=2\sigma^2$$

$$I=2\sigma^2 n \qquad p_n(n)=\exp(-n) \quad n\in[0,\infty) \qquad E\{I\}=1 \quad \sigma_I=1$$

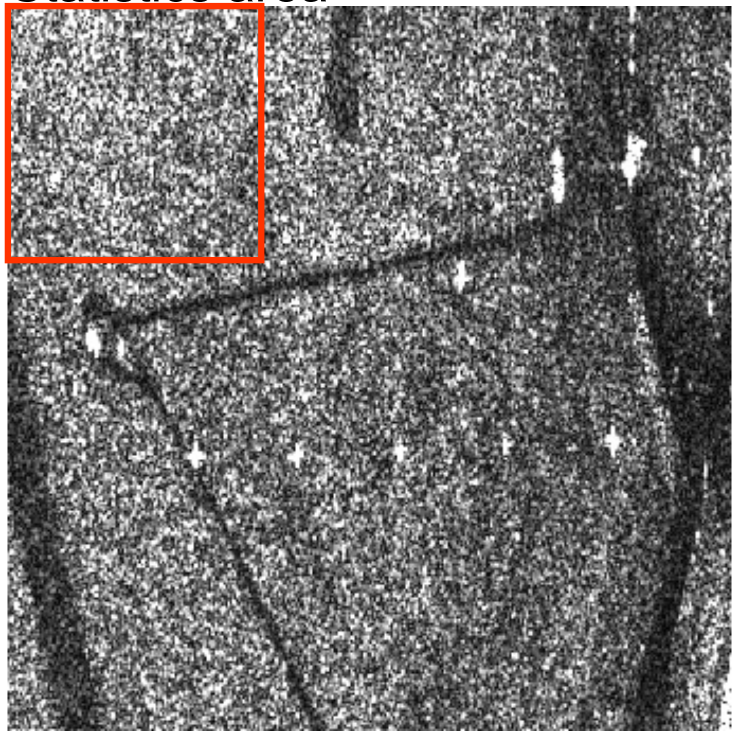
One dimensional speckle noise model (Model over the SAR image intensity - 2nd moment) $\Rightarrow$ $\boxed{I(x,r)=\sigma(x,r)n(x,r)}$

Multiplicative Speckle Noise Model

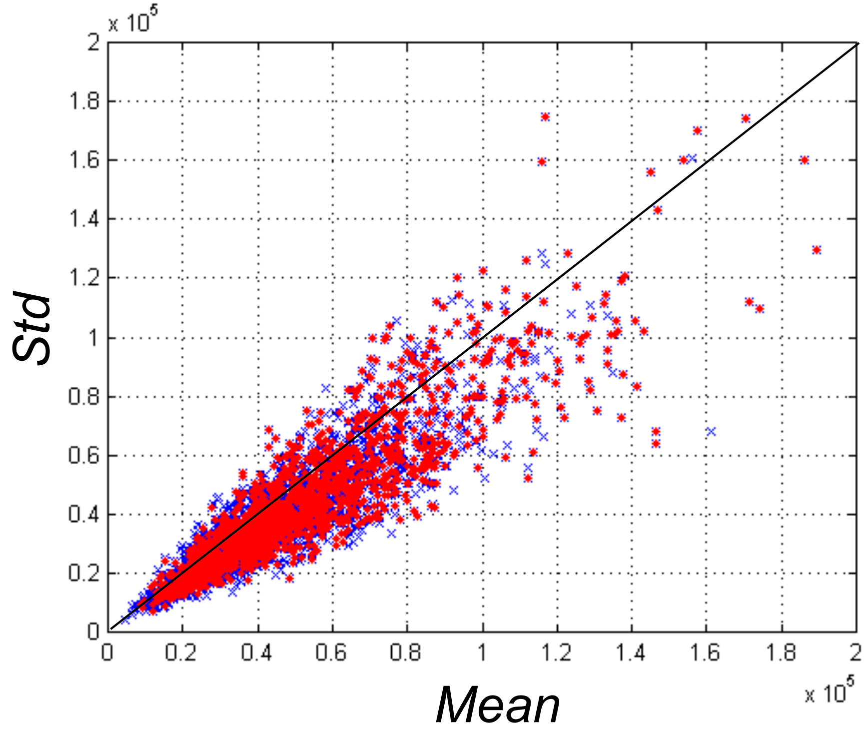
Fully Developed Speckle Noise Model

Moments calculated over local 7x7 local windows

Statistics area



Grass area



Blue: $|S_{hh}|^2$

Red: $|S_{vv}|^2$

S_{hh} amplitude

E-SAR L-band system

Analysis of the Coefficient of Variation CV

$$\boxed{CV = \frac{std}{mean}}$$

$$E\{I\} = 2\sigma^2$$
$$\sigma_I = 2\sigma^2$$

$$CV = \frac{std}{mean} = \frac{2\sigma^2}{2\sigma^2} = 1$$

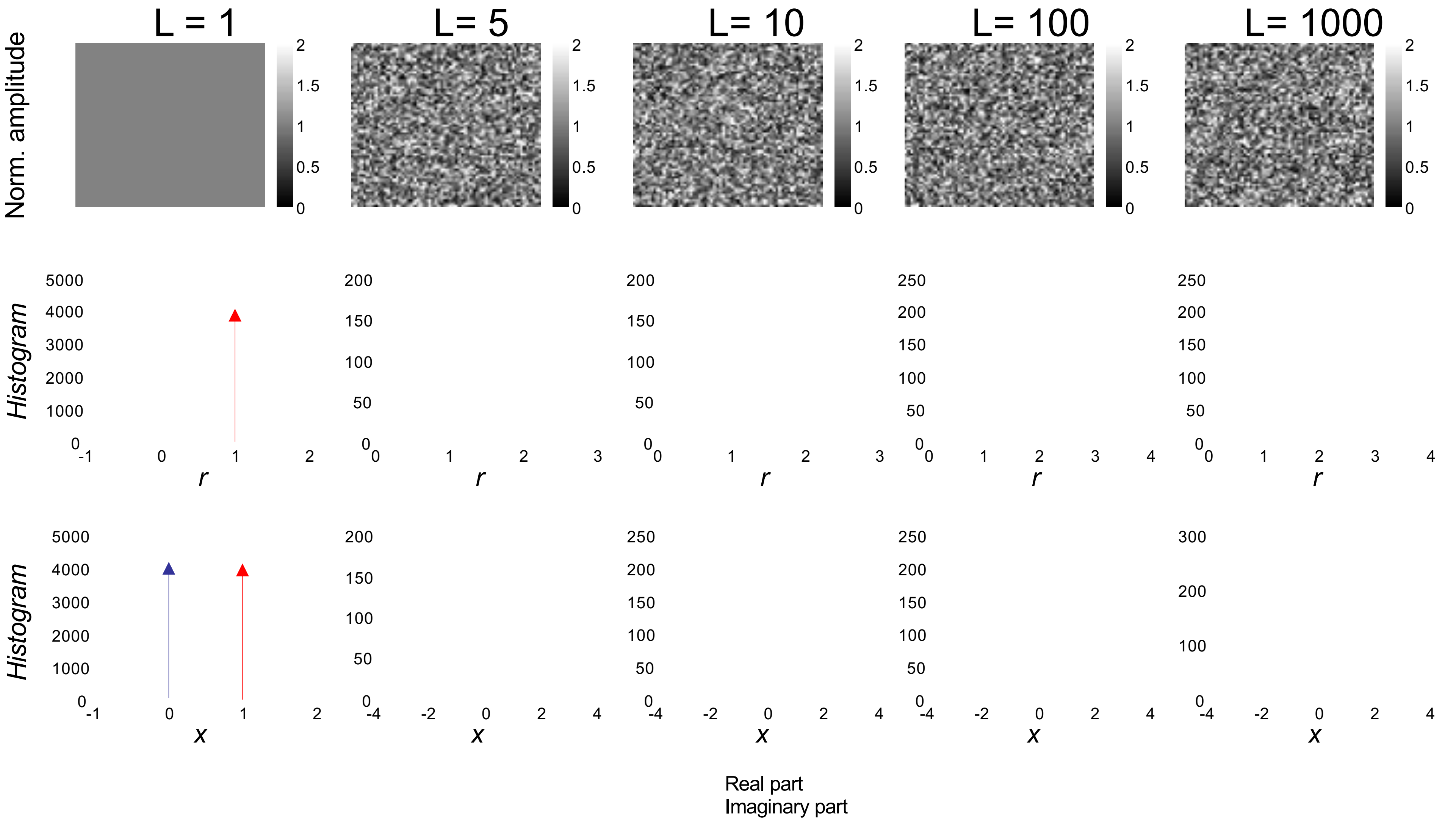
For the exponential PDF *CV*=1

- An increase of the power transmitted by the SAR system does not produce an increase of the Signal to Noise Ratio (SNR)

Analysis of the Equivalent Number of Looks (ENL)

$$\boxed{ENL = CV^{-1} = \frac{mean}{std}}$$

Effect of the number of point scatters L within the resolution cell. All of them with same weight



Statistical Product Model

- Intensity is decomposed into a three term product

$$\boxed{I(x,r)=\sigma(x,r)T(x,r)n(x,r)}$$

σ : Mean value

T : Texture random variable

n : Fading random variable (speckle)

Three scale model

- Coarsest scale : Mean reflectivity, constant value
- Finest scale : Speckle, noise
- Intermediate scale : Texture, spatially correlated fluctuations

As observed, the definition of the three terms is subjected to the notion of scale, or in other words, to where limits between them are placed

- Analysis based in time/frequency tools

SAR Image Texture

How to describe texture in SAR images

- One-point statistics: Mean and Variance
  - K-distribution model

- Two-point statistics: Autocovariance, Autocorrelation function (ACF)
  - Modelization of the autocovariance and autocorrelation functions

One-Point Statistics Texture

Texture can be considered as a fluctuating mean value

$$\boxed{I(x,r)=\sigma(x,r)T(x,r)n(x,r)}$$

$$p_I(I)=\frac{1}{2\sigma^2}\exp\left(-\frac{I}{2\sigma^2}\right) \quad I\in[0,\infty) \quad \Rightarrow \quad p_I(I)=\frac{1}{\sigma}\exp\left(-\frac{I}{\sigma}\right) \quad I\in[0,\infty)$$

Simplification

$$P(I)=\int_0^\infty \underbrace{P(I\,|\,\sigma)}_{\text{Gaussian PDF}}\underbrace{P(\sigma)}_{\text{Fluctuating RCS}}d\sigma=\frac{L^L I^{L-1}}{\Gamma(L)}\int_0^\infty \frac{d\sigma}{\sigma^L}\exp\left[-\frac{LI}{\sigma}\right]P(\sigma)$$

Model results from considering the number of scatters L within the resolution cell as a random quantity

One-Point Statistics Texture

RCS model ➡ Gamma pdf

$$P(\sigma)=\left(\frac{\nu}{\langle\sigma\rangle}\right)^{\nu}\frac{\sigma^{\nu-1}}{\Gamma(\nu)}\exp\left[-\frac{\nu\sigma}{\langle\sigma\rangle}\right]$$

ν : Order parameter

$\langle\sigma\rangle$: Mean RCS $\sigma(x,r)$

Number of scatterers controlled by a bird, death and migration process, the population would be negative binomial

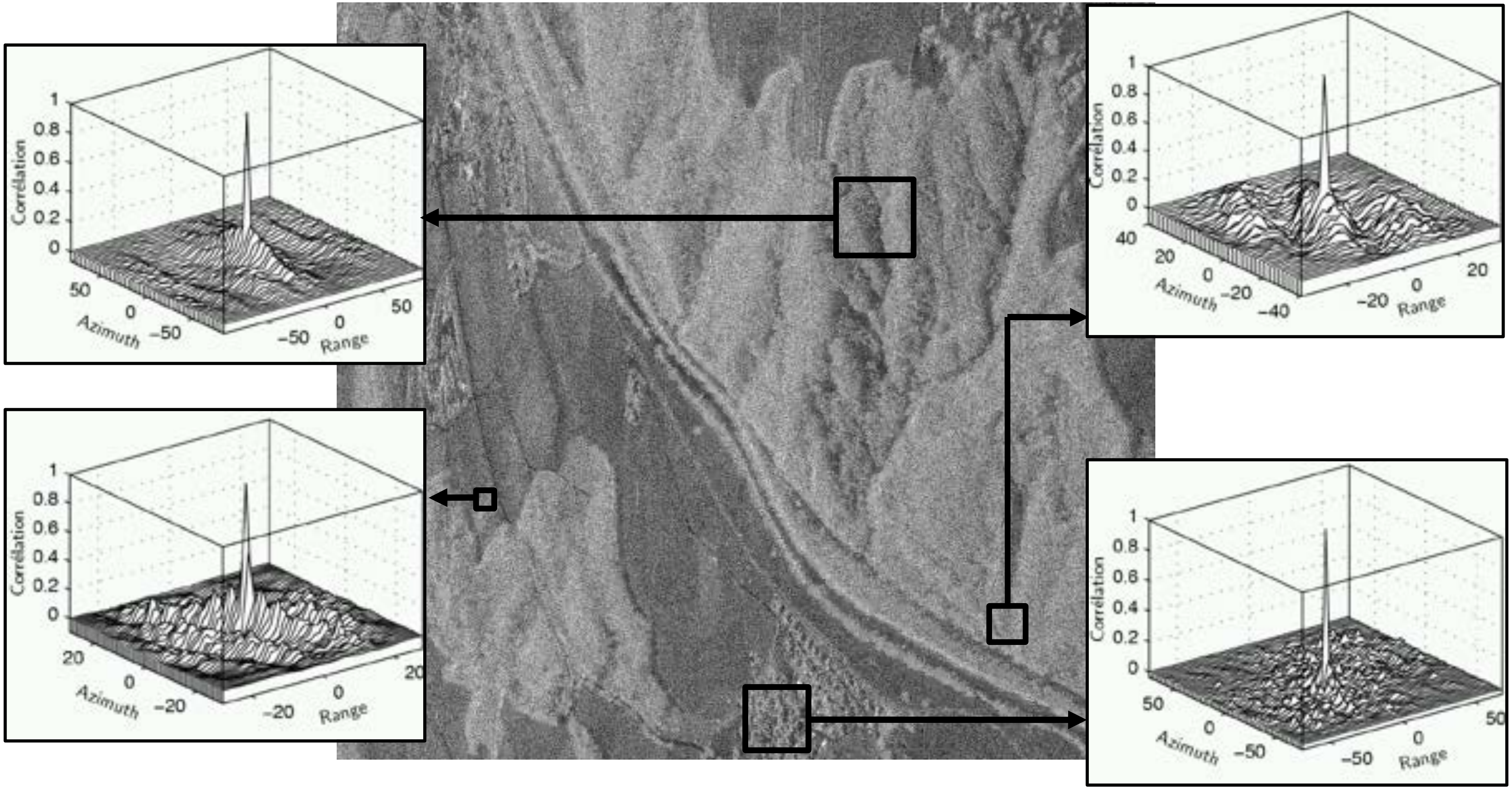
⬇

$$\boxed{P(I)=\frac{2}{\Gamma(L)\Gamma(\nu)}\left(\frac{L\nu}{\langle I\rangle}\right)^{(L+\nu)/2} I^{(L+\nu-2)/2}K_{\nu-L}\left[2\left(\frac{\nu LI}{\langle I\rangle}\right)^{1/2}\right]}$$

Intensity distributed as K-distribution

Texture can be considered as a fluctuating autocovariance function

Trauntstein, ESAR, DLR, L-Band



Nonstationary texture

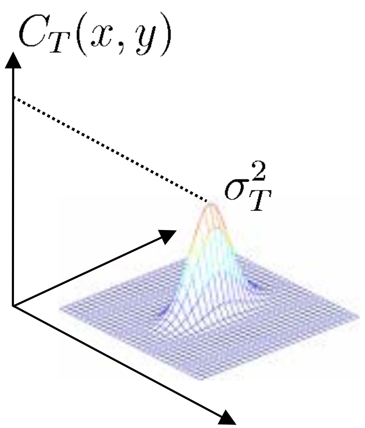
Anisotropic texture

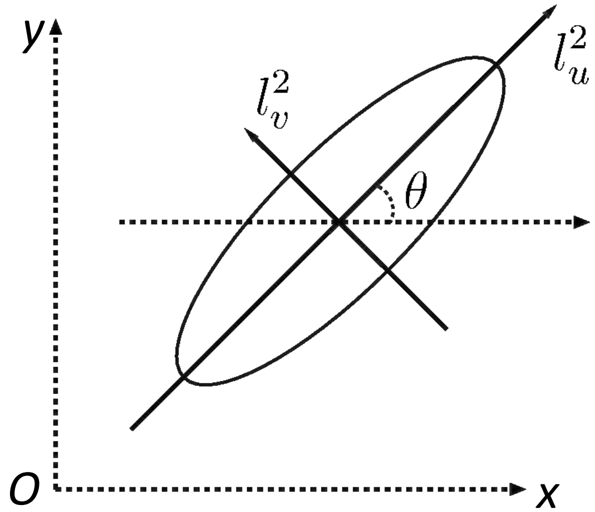
Two-Point Statistics Texture

Modelling of the local autocovariance function considering a Anisotropic Gaussian Kernels AGK

- Consider that locally, the autocovariance function can be approximated by an orientated two-dimensional Gaussian function

$$C_T(\mathbf{d}) = \sigma_T^2 \exp\left(-\mathbf{d}^T \Sigma^{-1} \mathbf{d}\right) \qquad \mathbf{d} = [x, y]^T$$





$$\Sigma = \mathbf{R}_\theta^T \Lambda\, \mathbf{R}_\theta$$

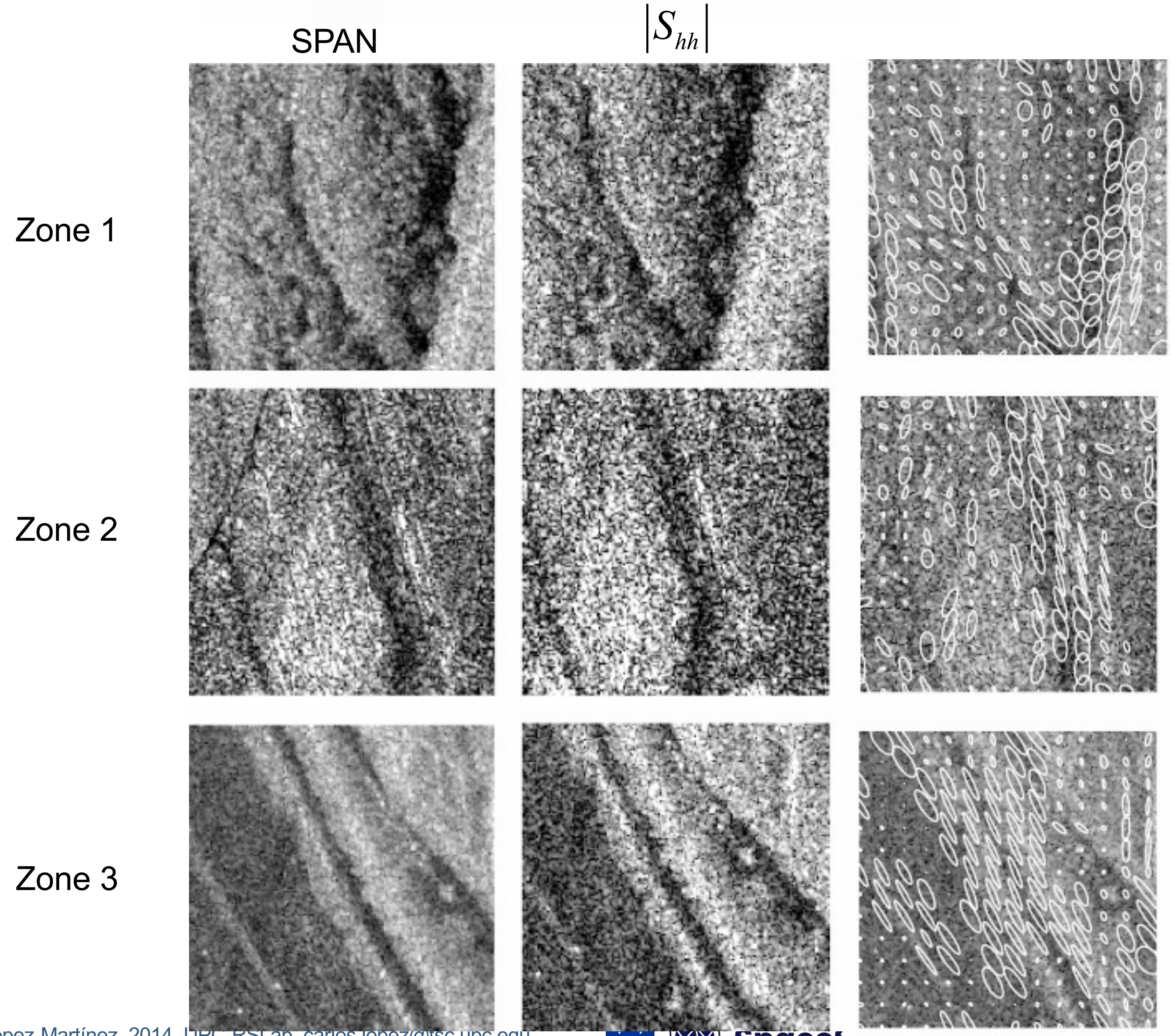
Correlation lengths

$$\Lambda = \begin{bmatrix} l_u^2 & 0 \\ 0 & l_v^2 \end{bmatrix}$$

Orientation Angle

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Two-Point Statistics Texture



Observation

- The Gaussian statistical model is unable to accommodate larger tails, i.e., a higher probability of larger SAR images amplitudes

⬇

Gaussian statistics must be extended

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Consider a family of distributions in which the Gaussian distribution is a member

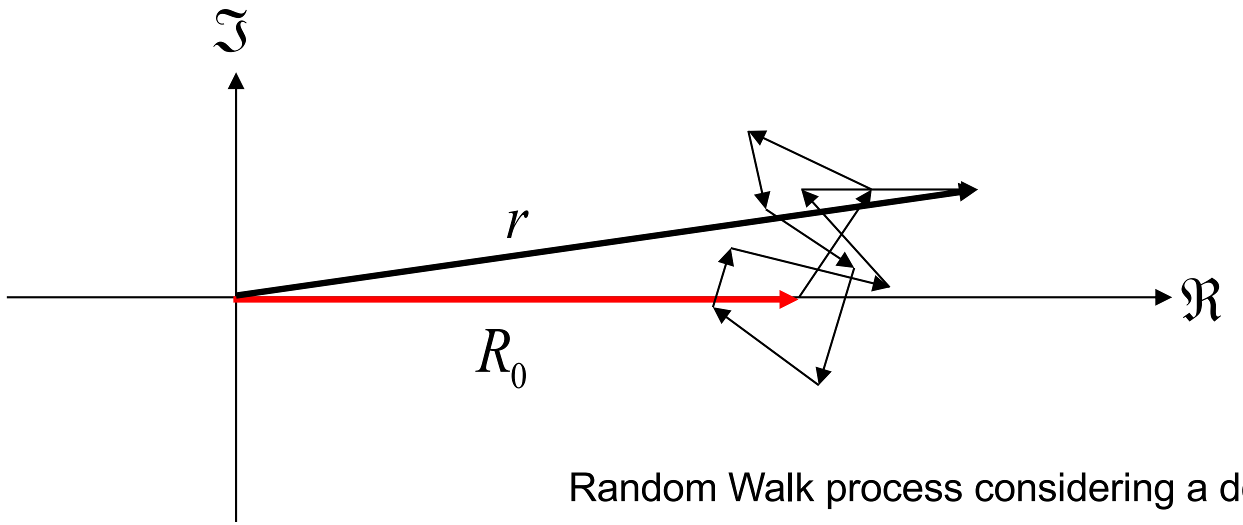
Rice Model

- SAR image formation process

$$S(x,r)=\frac{1}{\sqrt{L}}\sum_{k=1}^{L}\sqrt{\sigma_k}e^{j\theta_k}h(x-x_k,r-r_k)$$

- Now consider that within the resolution cell there is a dominant point scatterer

$$S(x,r)=R_0+\frac{1}{\sqrt{L}}\sum_{k=1}^{L}\sqrt{\sigma_k}e^{j\theta_k}h(x-x_k,r-r_k)$$



Random Walk process considering a dominant scatter

Under the same assumptions for fully developed speckle, but considering the contribution of the dominant point scatterer

- Real and Imaginary Parts

$$p_{\Re\{S\},\Im\{S\}}\left(\Re\{S\},\Im\{S\}\right)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{\left(\Re\{S\}-R_0\right)^2-\Im^2\{S\}}{2\sigma^2}\right)$$

- Amplitude: Rician pdf

$$p_r(r)=\frac{r}{\sigma^2}\exp\left(-\frac{r^2+R_0^2}{2\sigma}\right)I_0\left(\frac{rR_0}{\sigma^2}\right)$$

$I_0(x)$ Bessel function of first kind, order zero

$$E\{r\}=\frac{1}{2}\sqrt{\frac{\pi}{2\sigma^2}}\exp\left(-\frac{R_0^2}{4\sigma^2}\right)\left[\left(R_0^2+2\sigma^2\right)I_0\left(\frac{R_0^2}{4\sigma^2}\right)+R_0^2 I_1\left(\frac{R_0^2}{4\sigma^2}\right)\right]$$

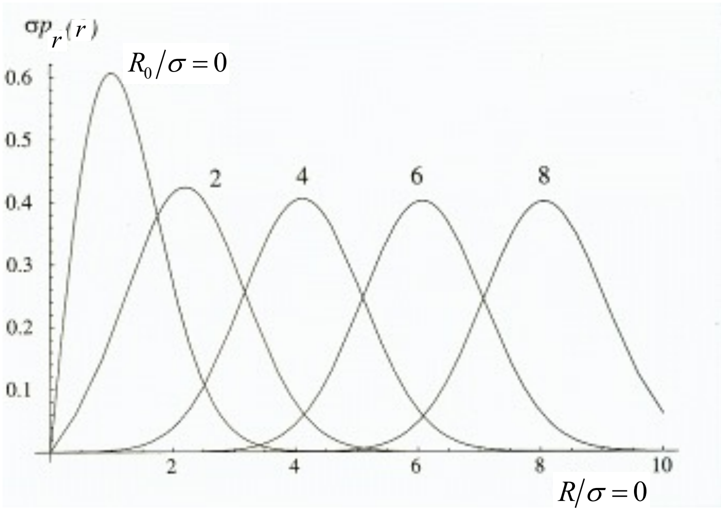
$$E\{r^2\}=R_0^2-2\sigma^2$$

Rice Model

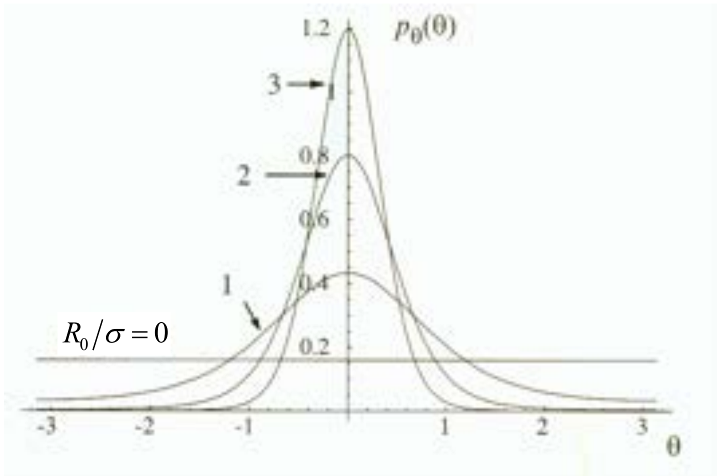
- Phase

$$p_\theta(\theta) = \frac{e^{-\frac{R_0^2}{2\sigma^2}}}{2\pi} + \sqrt{\frac{1}{2\pi}}\,\frac{R_0}{\sigma}\,e^{-\frac{R_0^2}{2\sigma^2}\sin^2\theta}\,\frac{1+\operatorname{erf}\left(\frac{R_0\cos\theta}{\sqrt{2}\sigma}\right)}{2}\cos\theta$$

- Examples of pdfs

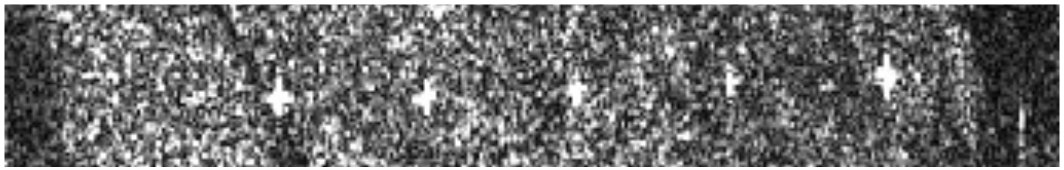


Amplitude pdf



Phase pdf

- SAR image example



Corners reflectors

S_{hh} amplitude
E-SAR L-band system

Models for Extremely Heterogeneous Areas

In extremely heterogeneous areas the Gaussian distribution is unable to predict the data distribution

- The solution is to consider more complex distributions with a larger number of parameters

- Difficulty to estimate these parameters with a reduced number of samples

- These models tend to model the pair Texture/Spekle and not only Speckle. No differences are established between point and distributed scatterers

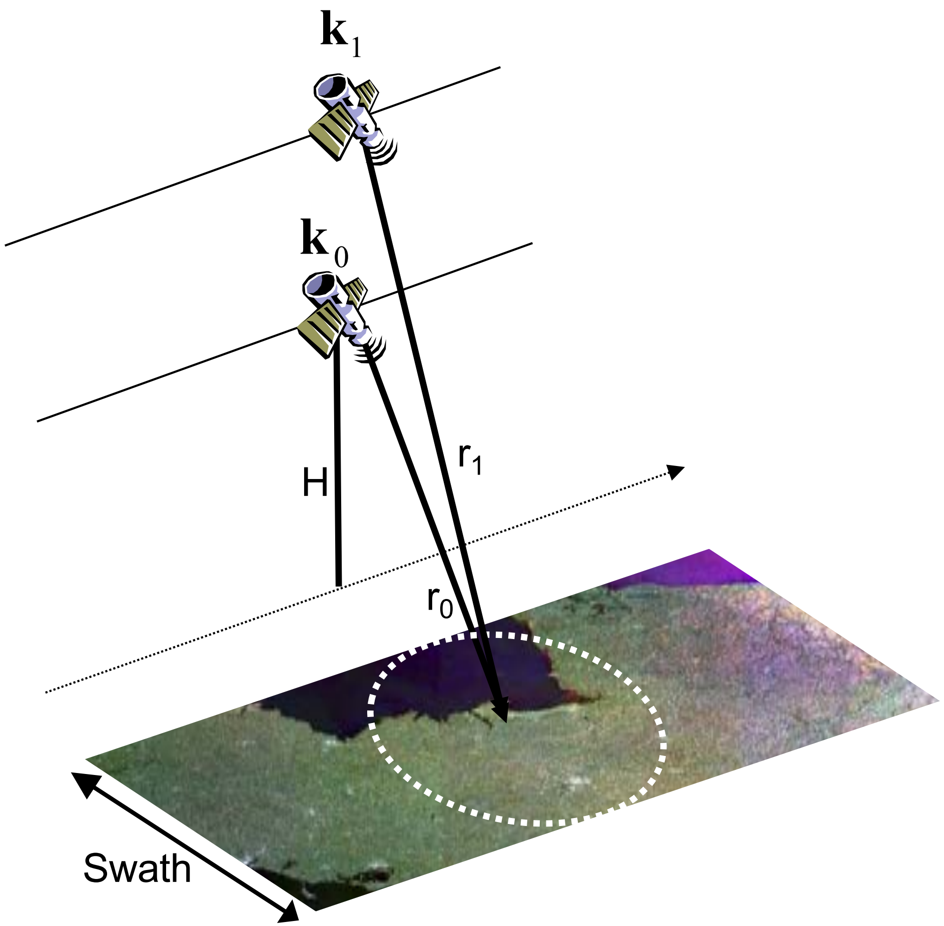
- Extremely heterogeneous areas correspond mainly to urban areas

$$\boxed{I(x,r)=\sigma(x,r)T(x,r)n(x,r)}$$

Polarimetric SAR Systems

The Polarimetric SAR system acquires 3 complex SAR images

Target vector $\mathbf{k} = [S_{hh}, 2S_{hv}, S_{vv}]^T$



The properties of the target vector follow from the properties of a single SAR image

- **k** is deterministic for point scatters. It contains all the necessary information to characterize the scatter
- **k** is a multidimensional random variable for distributed scatters due to speckle. A single sample does not characterize the scatterer

SAR images characterized through second order moments

- Second order moments in multidimensional SAR data are matrix quantities

Fully Developed Speckle Noise

Completely developed Speckle (large L and no dominant scatter)

- Hypotheses
  - The amplitude A_k and the phase θ_{s_k} of the kth scattered wave are statistically independent of each other and from the amplitudes and phases of all other elementary waves (Uncorrelated point scatterers)
  - The phases of the elementary contributions θ_{s_k} are equally likely to lie anywhere in the primary interval *[-π, π)*

Central Limit Theorem $\boxed{S = \mathrm{N}_{C^2}\left(0, \sigma^2/2\right)}$

- Real Part

$$p_{\Re\{S\}}\left(\Re\{S\}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\Re\{S\}}{\sigma}\right)^2\right) \quad \Re\{S\} \in (-\infty, \infty) \quad \text{Gaussian pdf}$$

- Imaginary Part

$$p_{\Im\{S\}}\left(\Im\{S\}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\Im\{S\}}{\sigma}\right)^2\right) \quad \Im\{S\} \in (-\infty, \infty) \quad \text{Gaussian pdf}$$

- Real and imaginary parts are uncorrelated $E\left\{\Re\{S\}\Im\{S\}\right\} = 0$

Mathematical Representation

PDF for non-correlated SAR images

- Zero-mean multidimensional complex (also circular) Gaussian pdf

$$p_{\mathbf{k}}(\mathbf{k}) = \prod_{k=1}^{3} \frac{1}{\pi\sigma^2} \exp\left(-\frac{S_k S_k^H}{\sigma^2}\right) = \frac{1}{\pi^m \sigma^{2m}} \exp\left(-\sum_{k=1}^{m} \frac{S_k S_k^H}{\sigma^2}\right) = \frac{1}{\pi^m \sigma^{2m}} \exp\left(-\frac{1}{\sigma^2} \operatorname{tr}\left(\mathbf{k}\mathbf{k}^H\right)\right)$$

Independent SAR images with the same power $S_k = \mathrm{N}_{C^2}\left(0, \sigma^2/2\right)$

- First order moment

$$E\{\mathbf{k}\} = \mathbf{0}$$

- Second order moment: Covariance matrix

$$\mathbf{C} = E\left\{\mathbf{k}\mathbf{k}^H\right\} = \sigma^2 \mathbf{I}_{m\times m}$$

Multidimensional Gaussian pdf Properties

Characterization of random variables

- Probability Density Function (pdf)
- Moment-generating function
- Statistical moments (mean, power, kurtosis, skewness…)

Zero-mean multidimensional complex Gaussian pdf

$$p_{\mathbf{k}}(\mathbf{k}) = \frac{1}{\pi^3 |\mathbf{C}|} \exp\left(-\mathbf{k}^H \mathbf{C}^{-1} \mathbf{k}\right)$$

- First order moment $E\{\mathbf{k}\} = \mathbf{0}$
- Second order moment: Covariance matrix

$$\mathbf{C} = E\{\mathbf{k}\mathbf{k}^H\} = \begin{bmatrix} E\{|S_{hh}|^2\} & E\{S_{hh}S_{hv}^*\} & E\{S_{hh}S_{vv}^*\} \\ E\{S_{hv}S_{hh}^*\} & E\{|S_{hv}|^2\} & E\{S_{hv}S_{vv}^*\} \\ E\{S_{vv}S_{hh}^*\} & E\{S_{vv}S_{hv}^*\} & E\{|S_{vv}|^2\} \end{bmatrix}$$

$$E\{S_k S_l^*\} \neq 0 \quad k,l \in \{1,\ldots,m\}, k \neq l$$

$$\Downarrow$$

Correlated SAR images

Multidimensional Gaussian pdf Properties

A zero-mean multidimensional complex Gaussian pdf is completely characterized by the second order moments, i.e., the covariance matrix

- Moment theorem for complex Gaussian processes, given Q correlated SAR images
  - For $k \neq l$, where m_k and n_l are integers from $\{1, 2, \ldots, Q\}$

$$E\left\{S_{m_1} S_{m_2} \cdots S_{m_k} S_{n_1}^* S_{n_2}^* \cdots S_{n_l}^*\right\} = 0$$

  - For $k = l$, where π is a permutation of the set of integers $\{1, 2, \ldots, Q\}$

$$E\left\{S_{m_1} S_{m_2} \cdots S_{m_k} S_{n_1}^* S_{n_2}^* \cdots S_{n_l}^*\right\} = \sum_{\pi} E\left\{S_{m_{\pi(1)}} S_{n_1}^*\right\} E\left\{S_{m_{\pi(2)}} S_{n_2}^*\right\} \cdots E\left\{S_{m_{\pi(l)}} S_{n_l}^*\right\}$$

- Considering the covariance matrix
  - Higher order moments are function of the covariance matrix

Mathematical Representation

The covariance matrix contains the correlation structure of the set of m SAR images

$$\mathbf{C} = E\left\{\mathbf{k}\mathbf{k}^H\right\} = \begin{bmatrix} E\left\{\left|S_{hh}\right|^2\right\} & E\left\{S_{hh}S_{hv}^*\right\} & E\left\{S_{hh}S_{vv}^*\right\} \\ E\left\{S_{hv}S_{hh}^*\right\} & E\left\{\left|S_{hv}\right|^2\right\} & E\left\{S_{hv}S_{vv}^*\right\} \\ E\left\{S_{vv}S_{hh}^*\right\} & E\left\{S_{vv}S_{hv}^*\right\} & E\left\{\left|S_{vv}\right|^2\right\} \end{bmatrix}$$

Information

- Diagonal elements: Power information

$$E\left\{S_k S_k^H\right\} = E\left\{\left|S_k\right|^2\right\} \quad k \in \{1, 2, \ldots, m\}$$

- Off-diagonal elements: Correlation information

$$E\left\{S_k S_l^H\right\} \quad k, l \in \{1, 2, \ldots, m\}, k \neq l$$

PDF for correlated SAR images

- Zero-mean multidimensional complex Gaussian pdf

$$p_{\mathbf{k}}(\mathbf{k}) = \frac{1}{\pi^3 |\mathbf{C}|} \exp\left(-\mathbf{k}^H \mathbf{C}^{-1} \mathbf{k}\right)$$

- First order moment

$$E\{\mathbf{k}\} = \mathbf{0}$$

- Second order moment: Covariance matrix

$$\mathbf{C} = E\{\mathbf{k}\mathbf{k}^H\} = \begin{bmatrix} E\{|S_{hh}|^2\} & E\{S_{hh}S_{hv}^*\} & E\{S_{hh}S_{vv}^*\} \\ E\{S_{hv}S_{hh}^*\} & E\{|S_{hv}|^2\} & E\{S_{hv}S_{vv}^*\} \\ E\{S_{vv}S_{hh}^*\} & E\{S_{vv}S_{hv}^*\} & E\{|S_{vv}|^2\} \end{bmatrix}$$

All the information characterizing the set of 3 SAR images is contained in the covariance matrix

Complex Correlation Coefficient

How to consider the correlation information

- Off-diagonal covariance matrix elements

$$E\left\{S_k S_l^H\right\} \quad k,l \in \{1,2,\ldots,m\}, k \neq l$$

  - Absolute correlation information

- Complex correlation coefficient

$$\rho_{k,l} = \frac{E\left\{S_k S_l^*\right\}}{\sqrt{E\left\{\left|S_k\right|^2\right\} \cdot E\left\{\left|S_l\right|^2\right\}}} = \left|\rho_{k,l}\right| e^{j\theta_{k,l}}$$

$$0 \leq \left|\rho_{k,l}\right| \leq 1 \quad \text{Coherence}$$

$$-\pi \leq \theta_{k,l} \leq \pi$$

  - Normalized correlation information

- The complex correlation information represents the most important observable for multidimensional SAR data. Its physical interpretation depends on the multidimensional SAR system configuration

Mathematical Representation

Multidimensional SAR data descriptors for distributed scatterers

First order moment

Second order moment

$$\mathbf{S} = \begin{bmatrix} S_{hh} & S_{hv} \\ S_{hv}^H & S_{vv} \end{bmatrix}$$

Scattering matrix

$$\mathbf{k}\mathbf{k}^H = \begin{bmatrix} S_{hh}S_{hh}^H & \sqrt{2}S_{hh}S_{hv}^H & S_{hh}S_{vv}^H \\ \sqrt{2}S_{hv}S_{hh}^H & 2S_{hv}S_{hv}^H & \sqrt{2}S_{hv}S_{vv}^H \\ S_{vv}S_{hh}^H & \sqrt{2}S_{vv}S_{hv}^H & S_{vv}S_{vv}^H \end{bmatrix}$$

Covariance matrix

Description for PolSAR data

Point scatterers characterization

Description for generalized multidimensional SAR data

$$\mathbf{k} = [S_1, S_2, \ldots, S_m]^T$$

Target vector

$$\mathbf{k}\mathbf{k}^H = \begin{bmatrix} S_1S_1^H & S_1S_2^H & \cdots & S_1S_m^H \\ S_2S_1^H & S_2S_2^H & \cdots & S_2S_m^H \\ \vdots & \vdots & \ddots & \vdots \\ S_mS_1^H & S_mS_2^H & \cdots & S_mS_m^H \end{bmatrix}$$

$$E\{\mathbf{k}\} = \mathbf{0}$$

Characterizes completely the data distribution

Distributed scatterers characterization

$$E\{\mathbf{k}\mathbf{k}^H\} = \mathbf{C}$$

Covariance matrix

$$p_{\mathbf{k}}(\mathbf{k}) = \frac{1}{\pi^m |\mathbf{C}|} \exp\left(-\mathbf{k}^H \mathbf{C}^{-1} \mathbf{k}\right)$$

For multidimensional SAR data, under the hypothesis of Gaussian scattering, all the information is contained in the covariance matrix

$$\mathbf{C} = E\left\{\mathbf{k}\mathbf{k}^H\right\} = \begin{bmatrix} E\left\{\left|S_{hh}\right|^2\right\} & E\left\{S_{hh}S_{hv}^*\right\} & E\left\{S_{hh}S_{vv}^*\right\} \\ E\left\{S_{hv}S_{hh}^*\right\} & E\left\{\left|S_{hv}\right|^2\right\} & E\left\{S_{hv}S_{vv}^*\right\} \\ E\left\{S_{vv}S_{hh}^*\right\} & E\left\{S_{vv}S_{hv}^*\right\} & E\left\{\left|S_{vv}\right|^2\right\} \end{bmatrix}$$

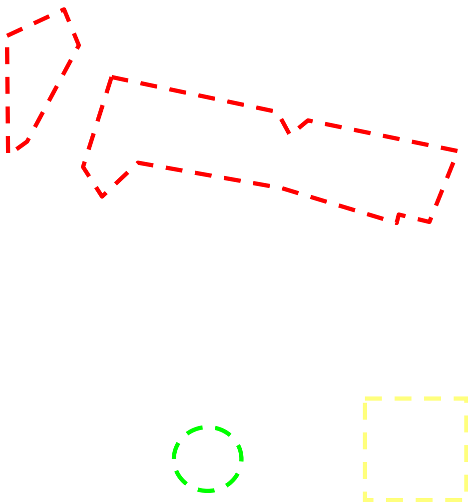
This matrix must be estimated from the available information

- The scattering vector for each pixel/sample of the SAR data

$$\mathbf{k} = \left[S_{hh}, 2S_{hv}, S_{vv}\right]^T$$

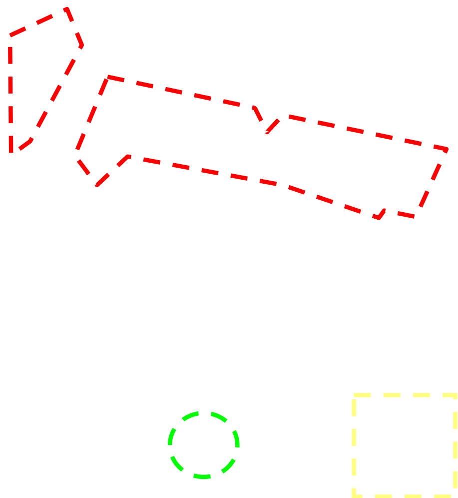
- The estimation process reduces to estimate the ensemble average (expectation operator) $E\{\cdot\}$
- The estimation process also receives the name of data filtering process.

Considerations about speckle noise reduction



Optical image DLR OP

SAR images reflex the Nature's complexity



SAR image DLR OP

Homogeneous areas	Image details	Heterogeneous areas
Maintain useful information (σ)	Maintain spatial details (Shape and value)	Maintain both
RADIOMETRIC RESOLUTION	SPATIAL RESOLUTION	LOCAL ANALYSIS

Image data: S_{hh} amplitude. E-SAR L-band system

Multidimensional SAR data information estimation, i.e., data filtering, based on two main hypotheses

- Ergodicity in mean: The different time/space averages of each process converge to the same limit, i.e., the ensemble average $E\{\}$
  - The statistics in the realizations domain can be calculated in the time/spatial domain
  - Necessary to assume ergodicity since there are not multiple data realizations over the same area
  - Applied to the processes $E\left\{|S_k|^2\right\}$, $E\left\{|S_l|^2\right\}$ and $E\left\{S_k S_l^H\right\}$ $k,l \in \{1,2,\dots,m\}$
- Wide-sense stationary: Given a spatial domain all the samples in this spatial domain belong to the same statistical distribution
  - SAR images can not be considered as wide-sense stationary processes since they are a reflex of the data heterogeneity
  - SAR images can be considered locally wide-sense stationary
  - Applied to the processes $E\left\{|S_k|^2\right\}$, $E\left\{|S_l|^2\right\}$ and $E\left\{S_k S_l^H\right\}$ $k,l \in \{1,2,\dots,m\}$
- Homogeneity: Refers to non-textured data
  - Gaussian distributed data

Sample Covariance Matrix

Covariance matrix estimation by means of a MultiLook (BoxCar)

- Maximum likelihood estimator: Sample covariance matrix

$$\mathbf{Z}_n = \frac{1}{n}\sum_{k=1}^{n}\mathbf{k}\mathbf{k}^H = \begin{bmatrix} \frac{1}{n}\sum_{k=1}^{n} S_1(k)S_1^*(k) & \frac{1}{n}\sum_{k=1}^{n} S_1(k)S_2^*(k) & \cdots & \frac{1}{n}\sum_{k=1}^{n} S_1(k)S_m^*(k) \\ \frac{1}{n}\sum_{k=1}^{n} S_2(k)S_1^*(k) & \frac{1}{n}\sum_{k=1}^{n} S_2(k)S_2^*(k) & \cdots & \frac{1}{n}\sum_{k=1}^{n} S_2(k)S_m^*(k) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n}\sum_{k=1}^{n} S_m(k)S_1^*(k) & \frac{1}{n}\sum_{k=1}^{n} S_m(k)S_2^*(k) & \cdots & \frac{1}{n}\sum_{k=1}^{n} S_m(k)S_m^*(k) \end{bmatrix}$$

  - n represents the total number of samples employed to estimate the covariance matrix, taken a region (square, rectangular, adapted…)
  - $\mathbf{Z}_n$ as estimator of $\mathbf{C}$
    - Does not consider signal morphology/heterogeneity
    - Loss of spatial resolution

The sample covariance matrix Z_n is itself a multidimensional random variable

Sample Covariance Matrix Distribution

The sample covariance matrix $\mathbf{Z}_n$ is characterized by the complex Wishart distribution $\mathbf{Z}_n \sim \mathrm{W}(n, \mathbf{C})$

$$p_{\mathbf{Z}_n}(\mathbf{Z}_n) = \frac{n^{mn} |\mathbf{Z}_n|^{n-m}}{|\mathbf{C}|^n \tilde{\Gamma}_m(n)} \mathrm{etr}\left(-n\mathbf{C}^{-1}\mathbf{Z}_n\right) \qquad \tilde{\Gamma}_m(n) = \pi^{m(m-1)/2} \Pi_{i=1}^m \Gamma(n-i+1)$$

- Multidimensional data distribution
- Valid for $n \geq m$, otherwise $|\mathbf{Z}_n|^{n-m}$ is equal to zero and the Wishart pdf is undetermined
  - Equivalent to $Rank(\mathbf{Z}_n)=m$, i.e., the sample covariance matrix is a full rank matrix
  - The higher the data dimensionality m the higher the number of looks n for the Wishart pdf to be defined

$$S_k = N_{C^2}\left(0, \sigma^2/2\right)$$

Single SAR image

$$\mathbf{k} = \left[S_1, S_2, \ldots, S_m\right]^T$$

Multidimensional SAR dataset

$$p_{\mathbf{k}}\left(\mathbf{k}\right) = \frac{1}{\pi^m \left|\mathbf{C}\right|} \exp\left(-\mathbf{k}^H \mathbf{C}^{-1} \mathbf{k}\right)$$

Multidimensional complex Gaussian pdf

$$\mathbf{Z}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbf{k}_i \mathbf{k}_i^H$$

$$\mathbf{Z}_n \sim W\left(n, \mathbf{C}\right)$$

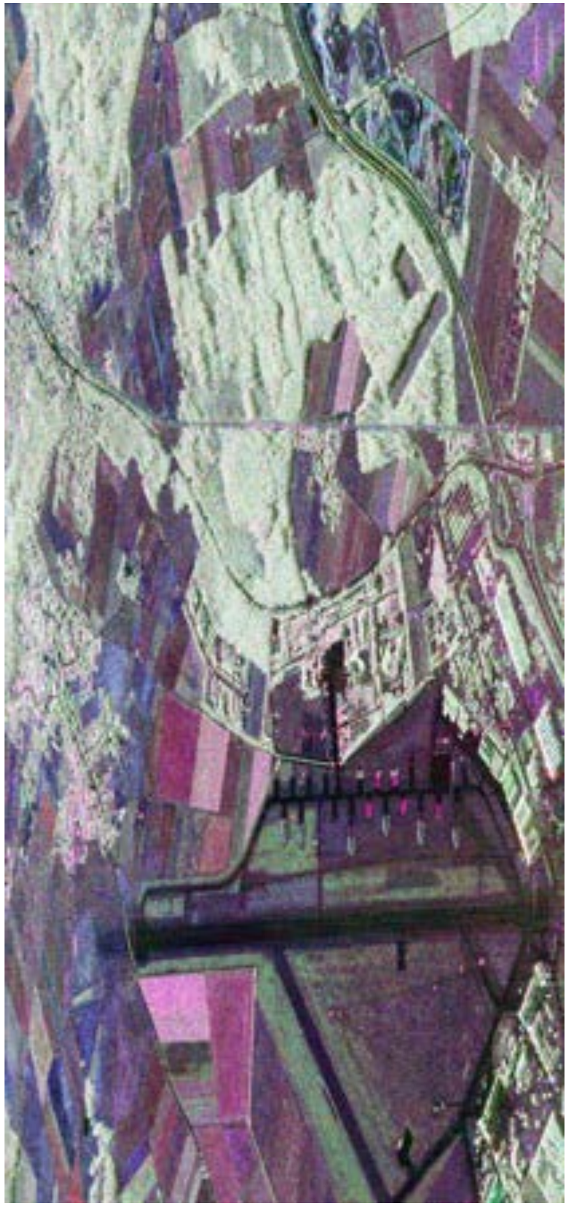
Sample covariance matrix

$$p_{\mathbf{Z}_n}\left(\mathbf{Z}_n\right) = \frac{n^{mn} \left|\mathbf{Z}_n\right|^{n-m}}{\left|\mathbf{C}\right|^n \tilde{\Gamma}_m\left(n\right)} \operatorname{etr}\left(-n \mathbf{C}^{-1} \mathbf{Z}_n\right)$$

Complex Wishart pdf

Multilook/Boxcar Example

|Shh-Svv| 2|Shv| | Shh +Svv|



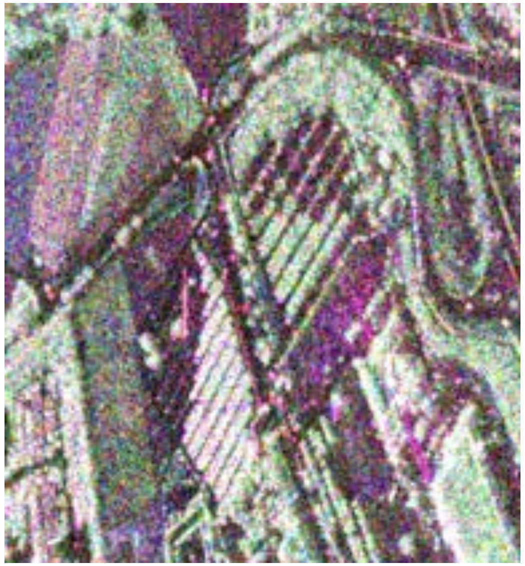
Original data

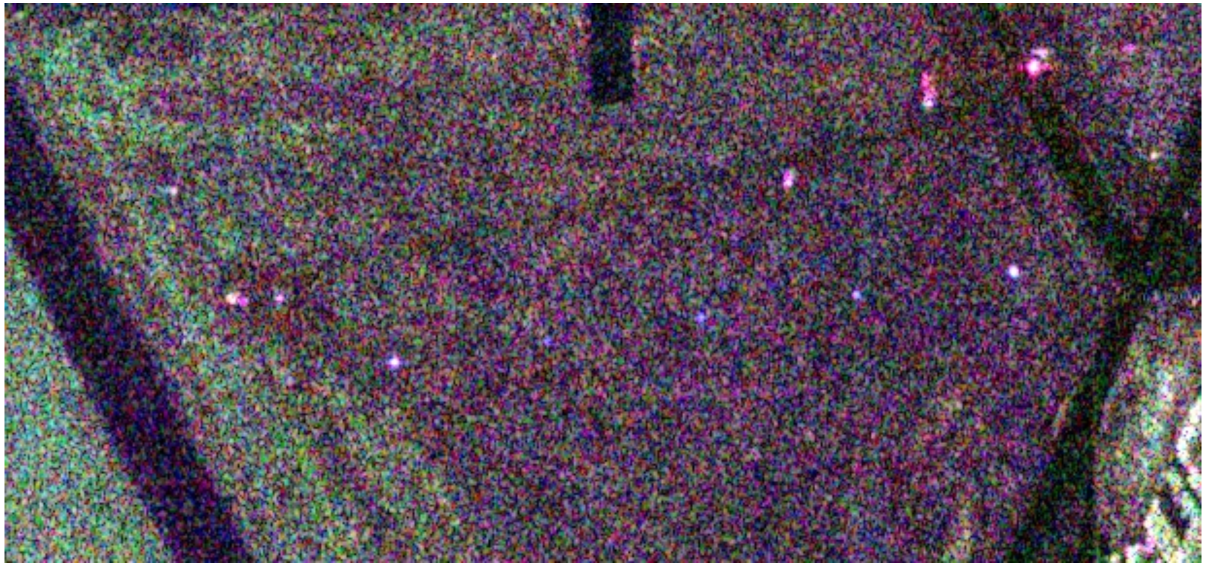
7x7 MLT data

L-band (1.3 GHz) fully PolSAR data
E-SAR system. Oberpfaffenhofen test area (D)

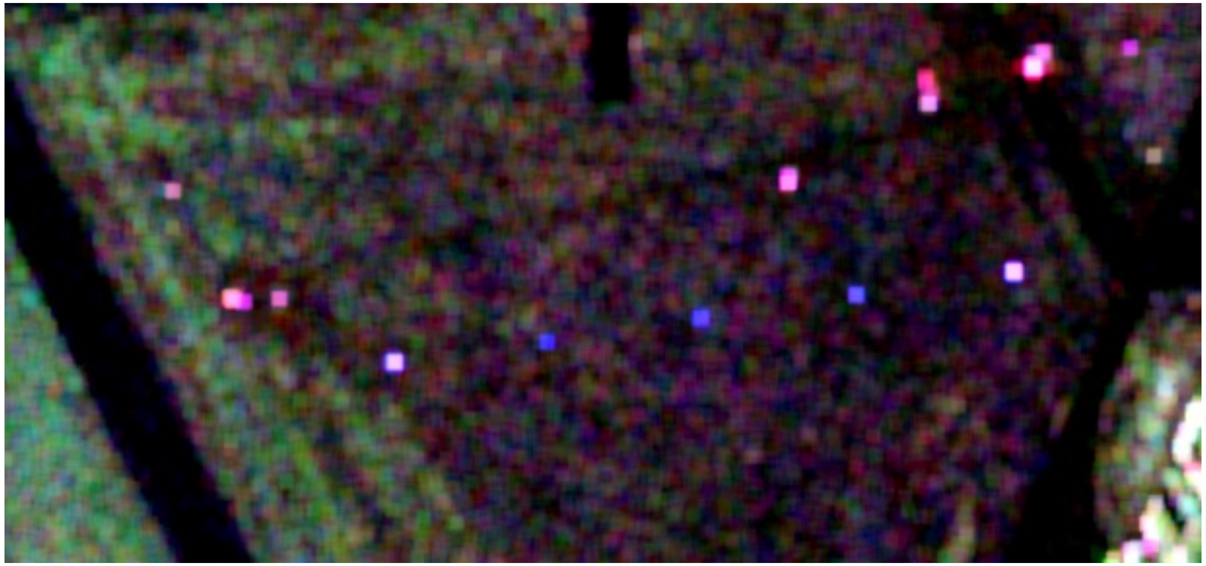
Multilook/Boxcar Example

Original data





7x7 MLT data



|Shh-Svv| 2|Shv| | Shh +Svv|

Local Statistics Linear Filter

Local statistics linear filter (Lee filter)

Filter form $\hat{I}(x,r) = a\,E\{I(x,r)\} + bI(x,r)$

Signal noise model $I(x,r) = \sigma(x,r)n(x,r)$

Minimization criteria (MMSE) $\min\limits_{(a,b)} J = E\{(\hat{I}(x,r) - I(x,r))\}$

MMSE gives $a = \dfrac{1}{E\{n\}} - b$ $\qquad b = E\{n\}\dfrac{\mathrm{var}(\sigma)}{\mathrm{var}(I)}$

$$\hat{I}(x,r) = \frac{E\{I(x,r)\}}{E\{n\}} + b\left(I(x,r) - E\{I(x,r)\}\right)$$

Statistics need to be derived from noisy data

$$a = \frac{1}{E\{n\}} - b = 1 - b$$

$E\{n\} = 1$

$$b = E\{n\}\frac{\mathrm{var}(\sigma)}{\mathrm{var}(I)} = \frac{\mathrm{var}(I) - E^2\{I\}\sigma_n^2}{\mathrm{var}(I)(1+\sigma_n^2)}$$

Information estimated from data

$E\{n\} = 1$

$$\boxed{\hat{I}(x,r) = E\{I(x,r)\} + b\left(I(x,r) - E\{I(x,r)\}\right)}$$

Local statistics $E^2\{I(x,r)\} \qquad \mathrm{var}\{I\}$

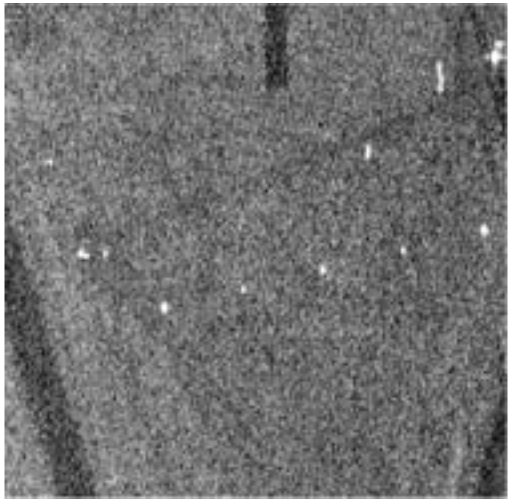
A priori information $\sigma_n^2 = \mathrm{var}(n) = \dfrac{1}{N}$

Local Statistics Linear Filter

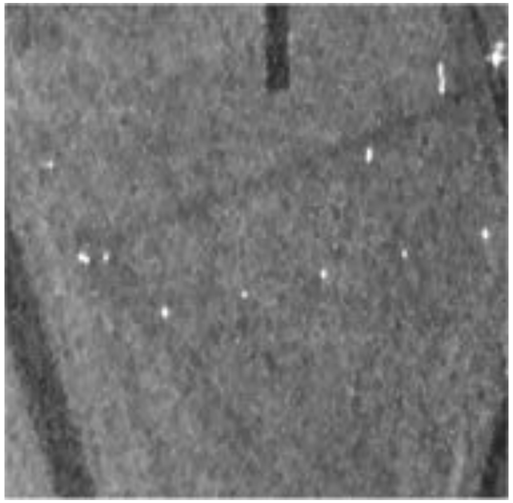
$$\hat{I}(x,r) = E\{I(x,r)\} + b\left(I(x,r) - E\{I(x,r)\}\right)$$

$\mathrm{var}(I) >> E^2\{I\} \Rightarrow b \to 1$ Multiplicative noise model can not explain data variability

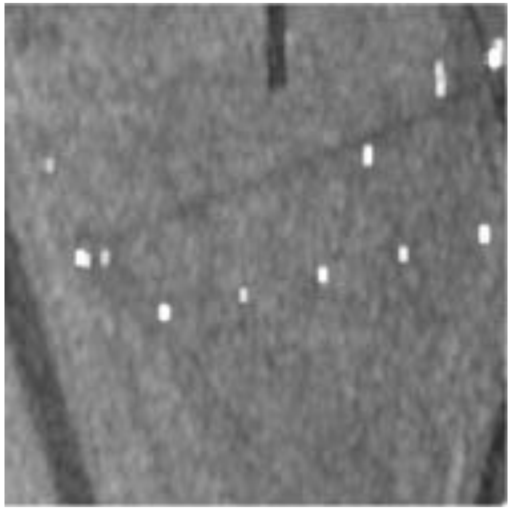
$\mathrm{var}(I) \approx E^2\{I\}\sigma_n^2 \Rightarrow b \to 0$ Multiplicative noise model can explain data variability



Original SAR intensity image



Filtered SAR intensity image
Lee Filter



Filtered SAR intensity image
Boxcar Filter

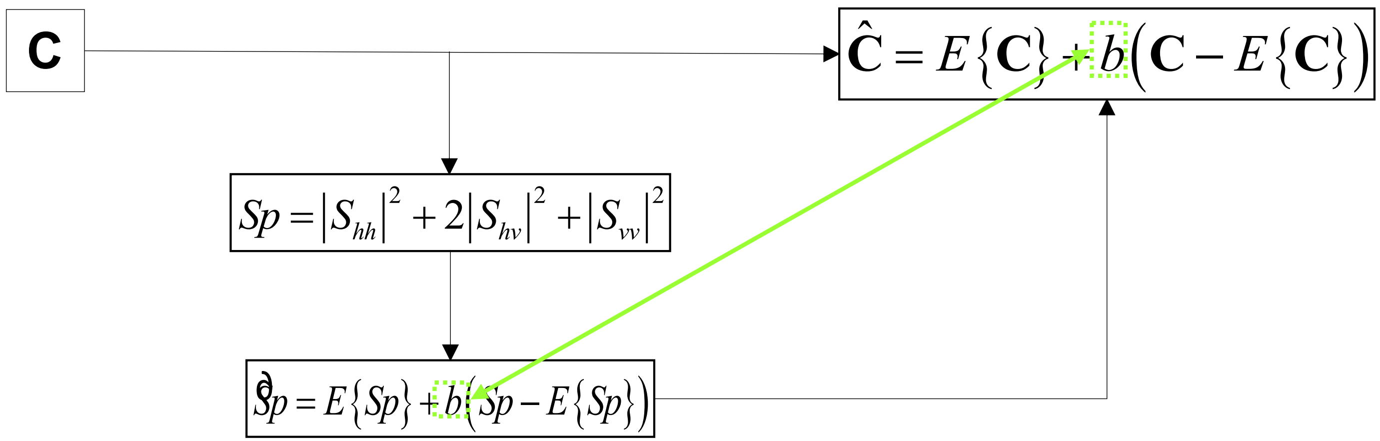
Image data: S_{hh} amplitude. E-SAR L-band system

Polarimetric Lee filter

Nowadays is the most employed polarimetric filtering solution

Extension of the linear scalar Lee filter for SAR images by considering a multiplicative speckle noise model over all the covariance matrix entries

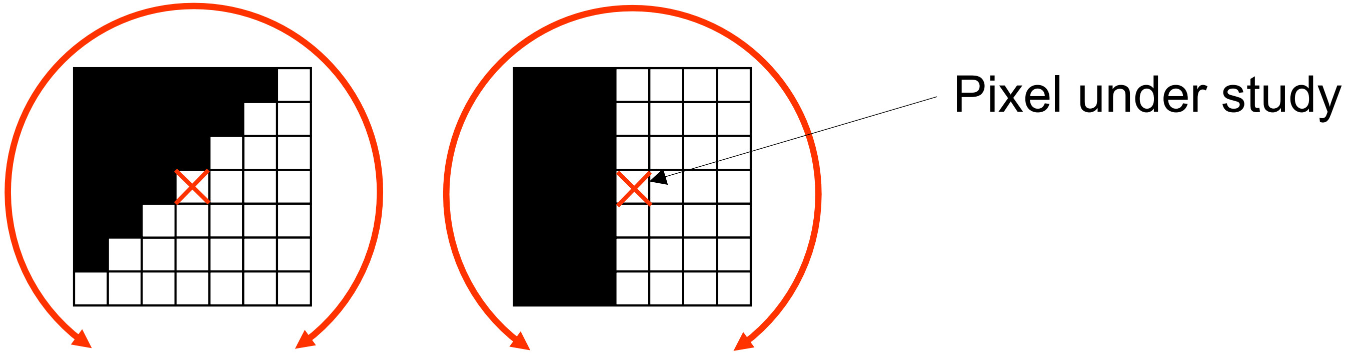
Working principles



Linear scalar Lee filter

Refined Lee filter

Statistics estimation in windows selected according to the signal morphology in order to retain edges, spatial feature and point targets



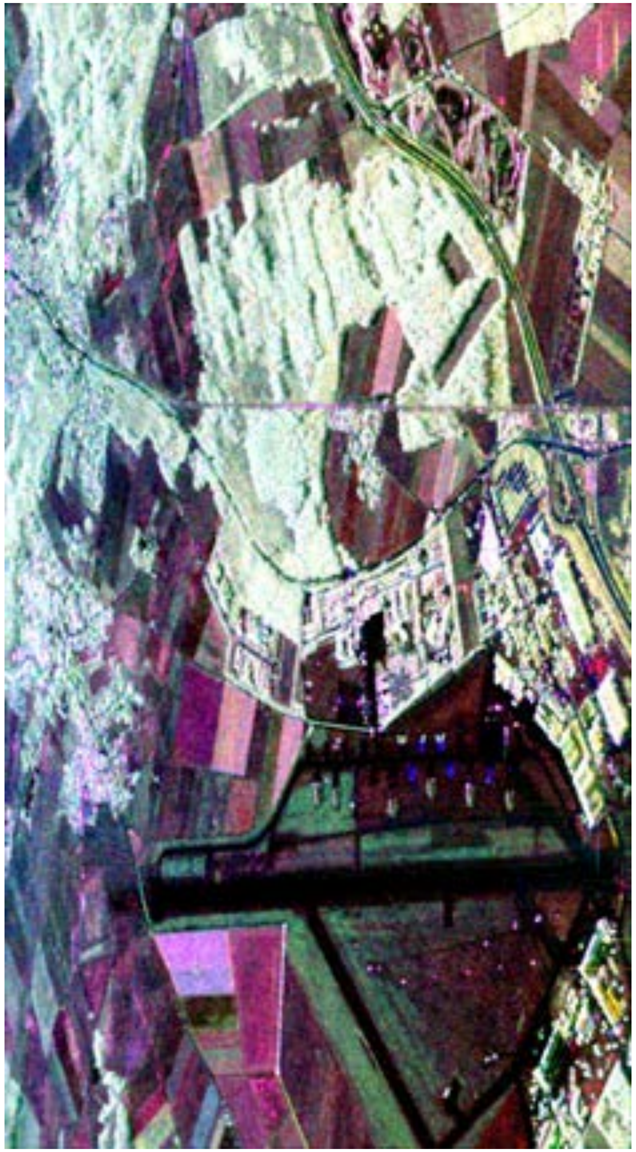
The extension of the scalar linear Lee filter presents limitations

Not based on the multiplicative-additive speckle noise model. This limits the capacity to reduce noise in those images areas characterized by low correlation ➡ The elements of the covariance matrix can be processed differently, but according to the right speckle noise model

The a priori information in the span image σ_n^2 is no longer a constant as the noise content in span depends on the data's correlation structure

Local Statistics Linear Filter

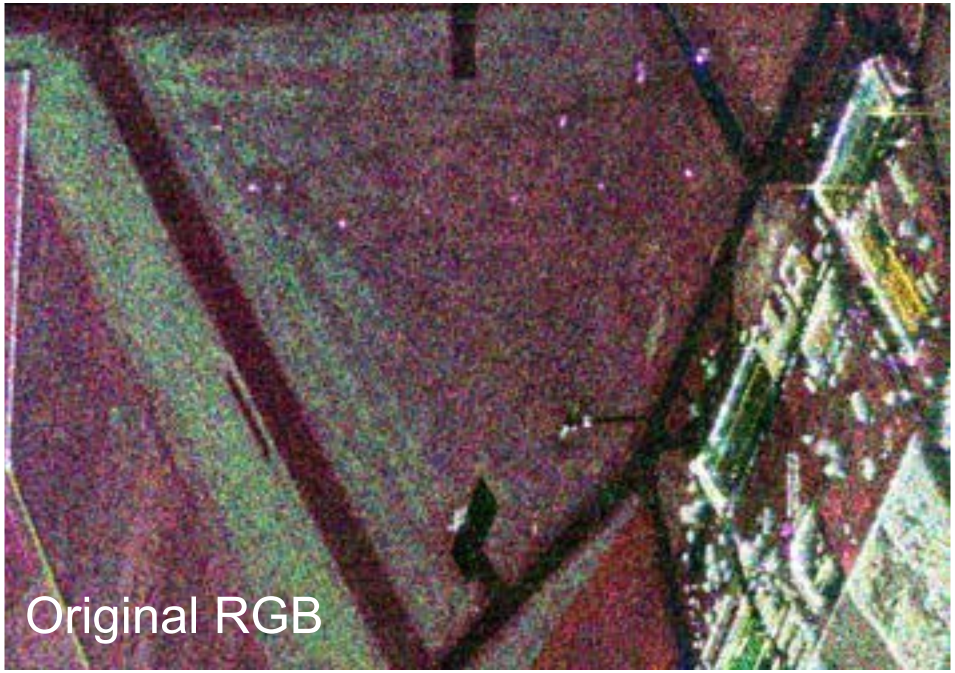


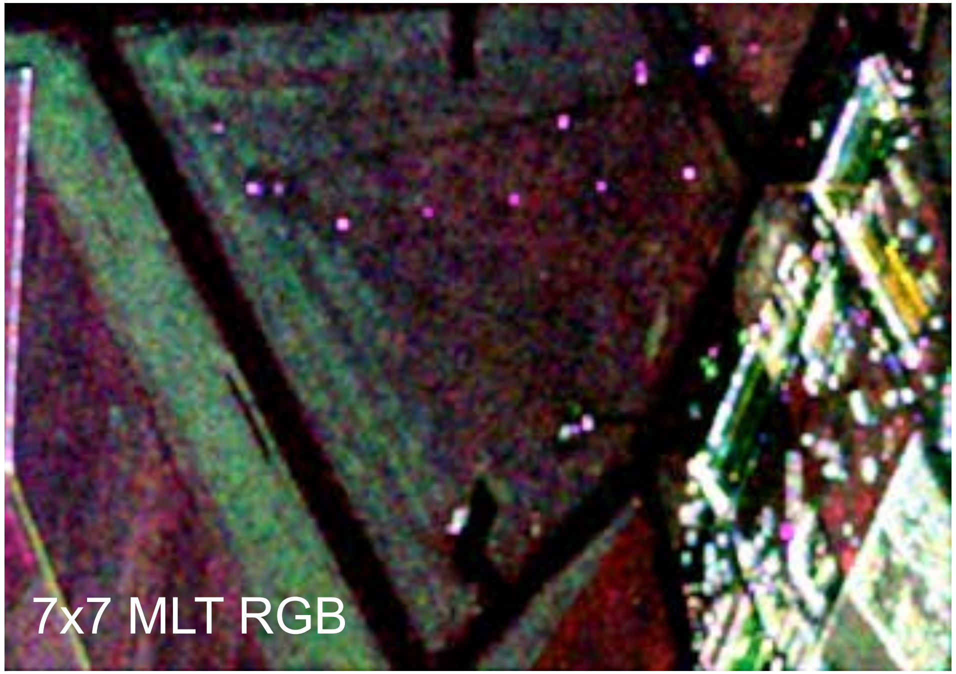


|Shh| |Shv| |Svv|

L-band (1.3 GHz) fully PolSAR data
E-SAR system. Oberpfaffenhofen test area (D)

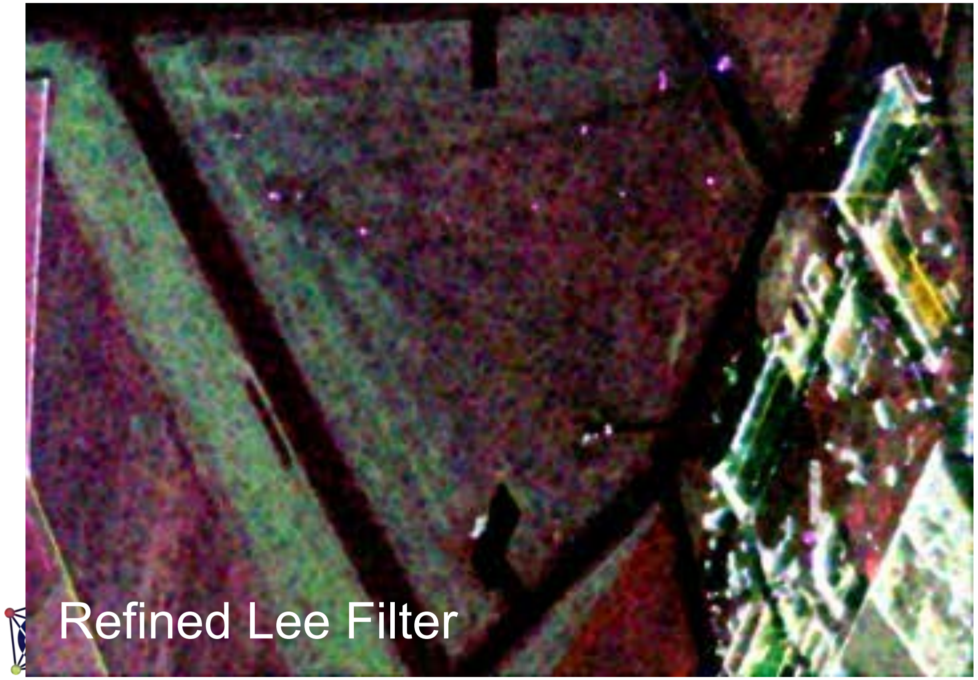
Local Statistics Linear Filter





|Shh| |Shv| |Svv|

L-band (1.3 GHz) fully PolSAR data
E-SAR system. Oberpfaffenhofen test area (D)



Co-funded by the European Union

Single-look Multidimensional Speckle Noise Model

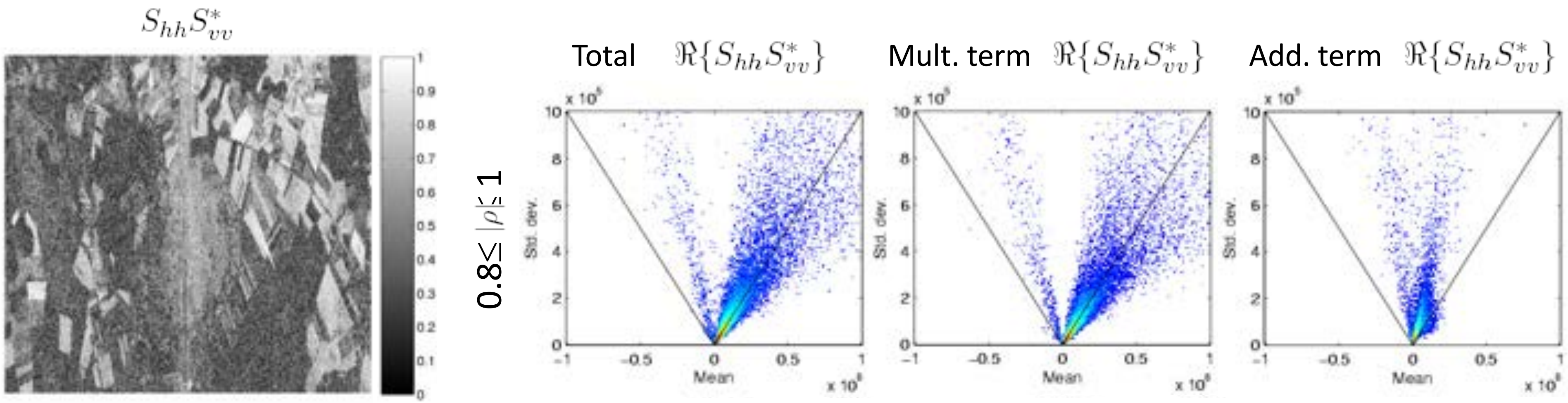
Hermitian product speckle noise model: $$S_i S_j^* = \underbrace{\psi \bar{z}_n \boxed{n_m} N_c e^{j\phi_x}}_{Multiplicative\ term} + \underbrace{\psi\left(|\rho| - N_c \bar{z}_n\right)e^{j\phi_x} + \psi \boxed{\left(n_{ar} + jn_{ai}\right)}}_{Additive\ term}$$

C. López-Martínez and X. Fàbregas, "Polarimetric SAR Speckle Noise Model"
IEEE TGRS, vol. 41, no. 10, pp. 2232 – 2242, Oct. 2003

Multiplicative speckle noise component: n_m ➡ Important for high coherence areas

Additive speckle noise component: $n_{ar}+jn_{ai}$ ➡ Important for low coherence areas

Combination controlled by complex coherence



Multilook Multidimensional Speckle Noise Model

Hermitian product speckle noise model: $\left\langle S_i S_j^* \right\rangle_n = \underbrace{\psi \boxed{n_m} \exp(j\phi_x)}_{Multiplicative\ term} + \underbrace{\psi\left(|\rho| - N_c \bar{z}_n\right)\exp(j\phi_x) + \psi \boxed{(n_{ar} + j n_{ai})}}_{Additive\ term}$

C. López-Martínez and E. Pottier, "Extended multidimensional speckle noise model and its implications on the estimation of physical information," *IGARSS 06*, Denver (CO) USA, July 2006

Multiplicative speckle noise component

- Dominant for high coherences
- Modulated by phase information

$$E\{n_m\} = N_c \bar{z}_n \qquad \sigma_{n_m}^2 = N_c^2 \frac{\left(1+|\rho|^2\right)}{2n}$$

Additive speckle noise component

- Dominant for low coherences
- Not affected by phase information

$$E\{n_{ar}\} = E\{n_{ai}\} = 0 \qquad \sigma_{n_{ar}}^2 = \sigma_{n_{ai}}^2 \simeq \frac{1}{2n}\left(1-|\rho|^2\right)^{1.32\sqrt{n}}$$

Effect of the approximations

- Mean value IS NOT approximated ➡ No loss of information

$$\lim_{n\to\infty}\left\{\psi n_m \exp(j\phi_x) + \psi\left(|\rho| - N_c \bar{z}_n\right)\exp(j\phi_x) + \psi\left(n_{ar} + j n_{ai}\right)\right\} = \psi|\rho|\exp(j\phi_x)$$

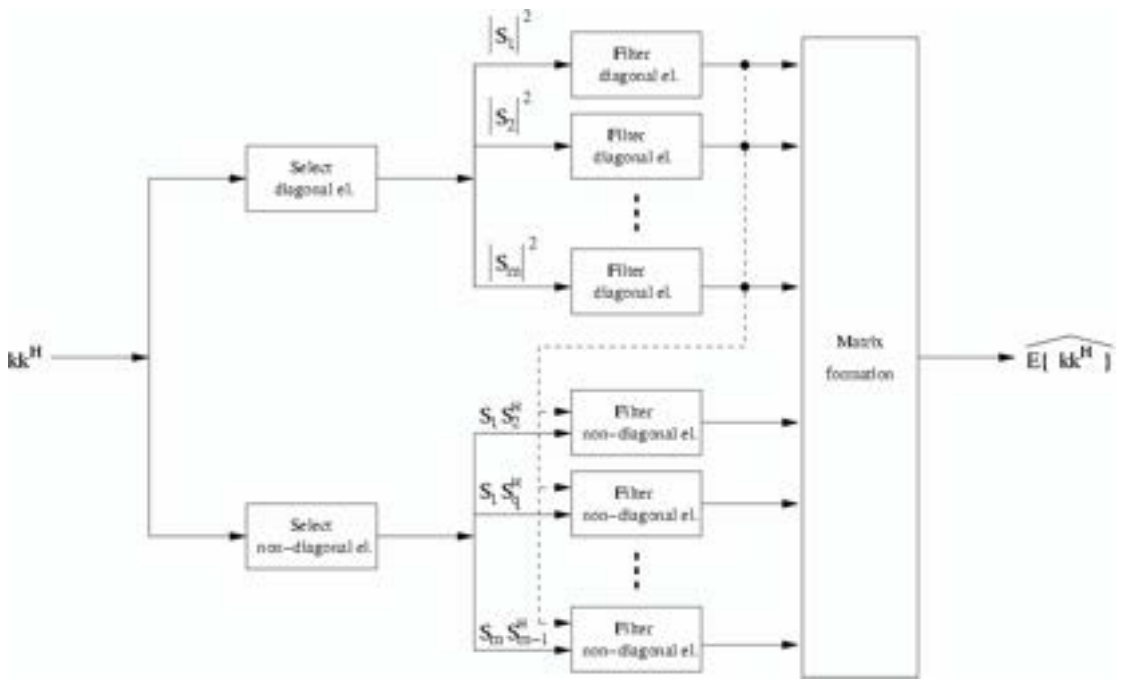
- Std. Dev. ARE approximated

Multidimensional Speckle Noise Filtering

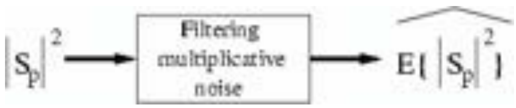
Define a multidimensional SAR data filtering strategy based on the multidimensional speckle noise model

Element to consider: Covariance matrix

- Diagonal element: Multiplicative noise source
- Non-diagonal element: Multiplicative and additive noise sources combined according to the complex correlation coefficient



Diagonal element processing

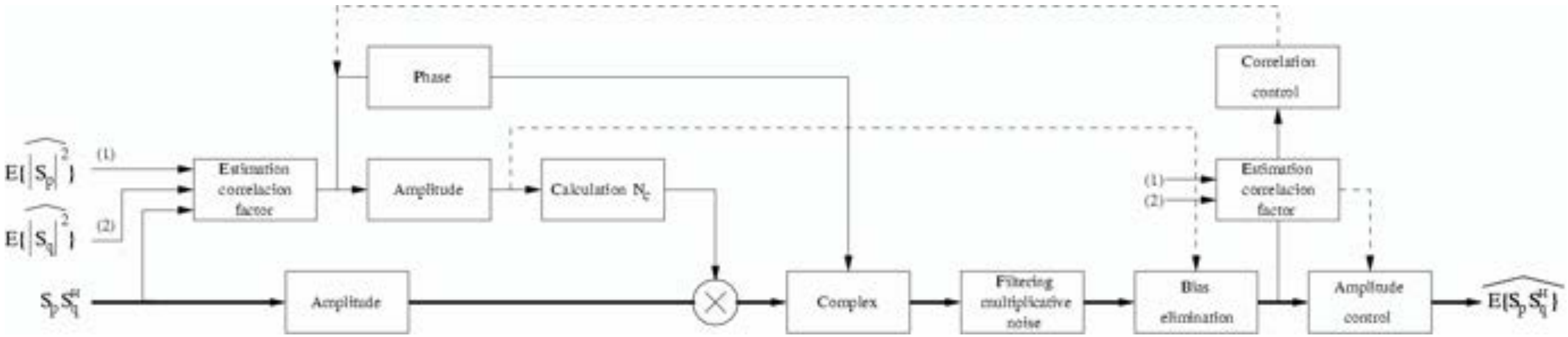


Any alternative to filter multiplicative noise can be considered

Non-iterative scheme

Off-diagonal element processing

The filter uses the Hermitian product speckle model: $S_i S_j^* = \underbrace{\psi \bar{z}_n n_m N_c e^{j\phi_x}}_{Multiplicative\ term} + \underbrace{\psi\left(|\rho| - N_c \bar{z}_n\right) e^{j\phi_x} + \psi\left(n_{ar} + j n_{ai}\right)}_{Additive\ term}$



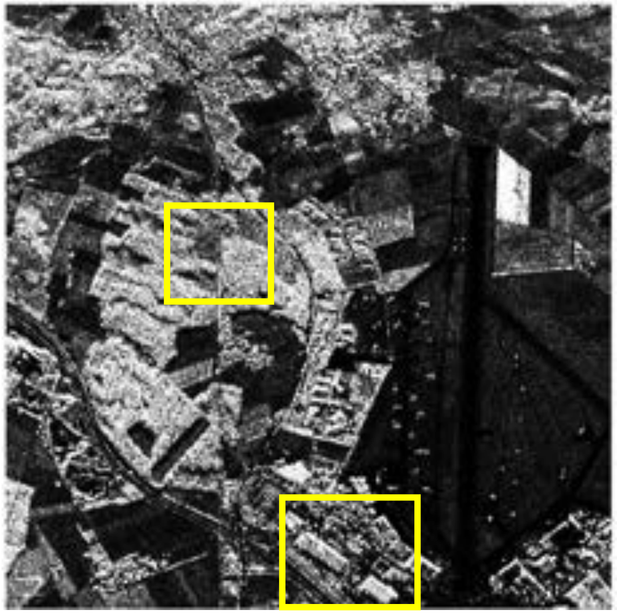
Iterative scheme to take benefit of the improved coherence estimation

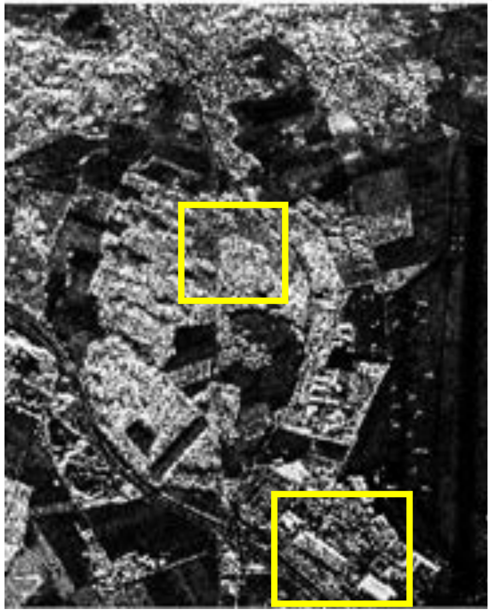
This strategy filters differently the covariance matrix elements

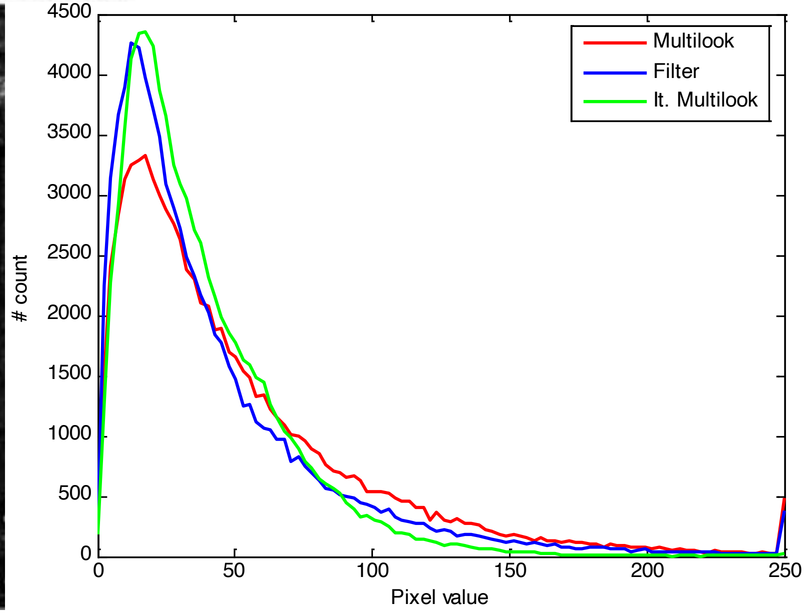
Results: Experimental PolSAR Data

Full-polar ESAR L-Band SAR data in Oberpfaffenhofen (DE)

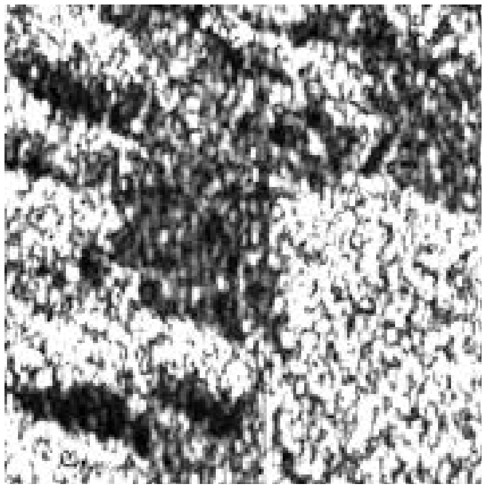
5x5 Multilook | **Filter** | **5x5 It. multilook**

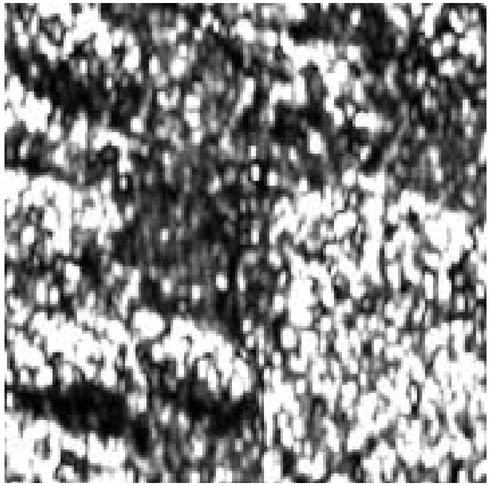


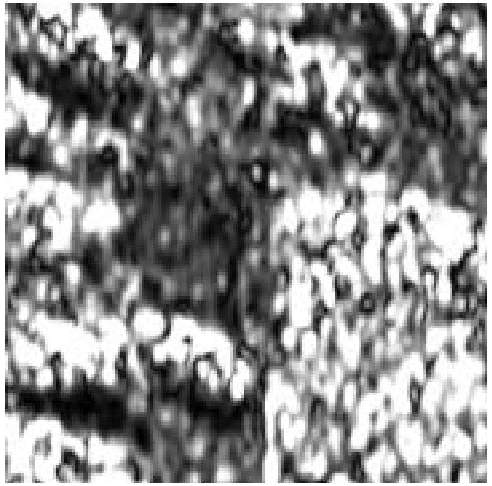




$$\left|S_{hh}S_{vv}^{*}\right|$$







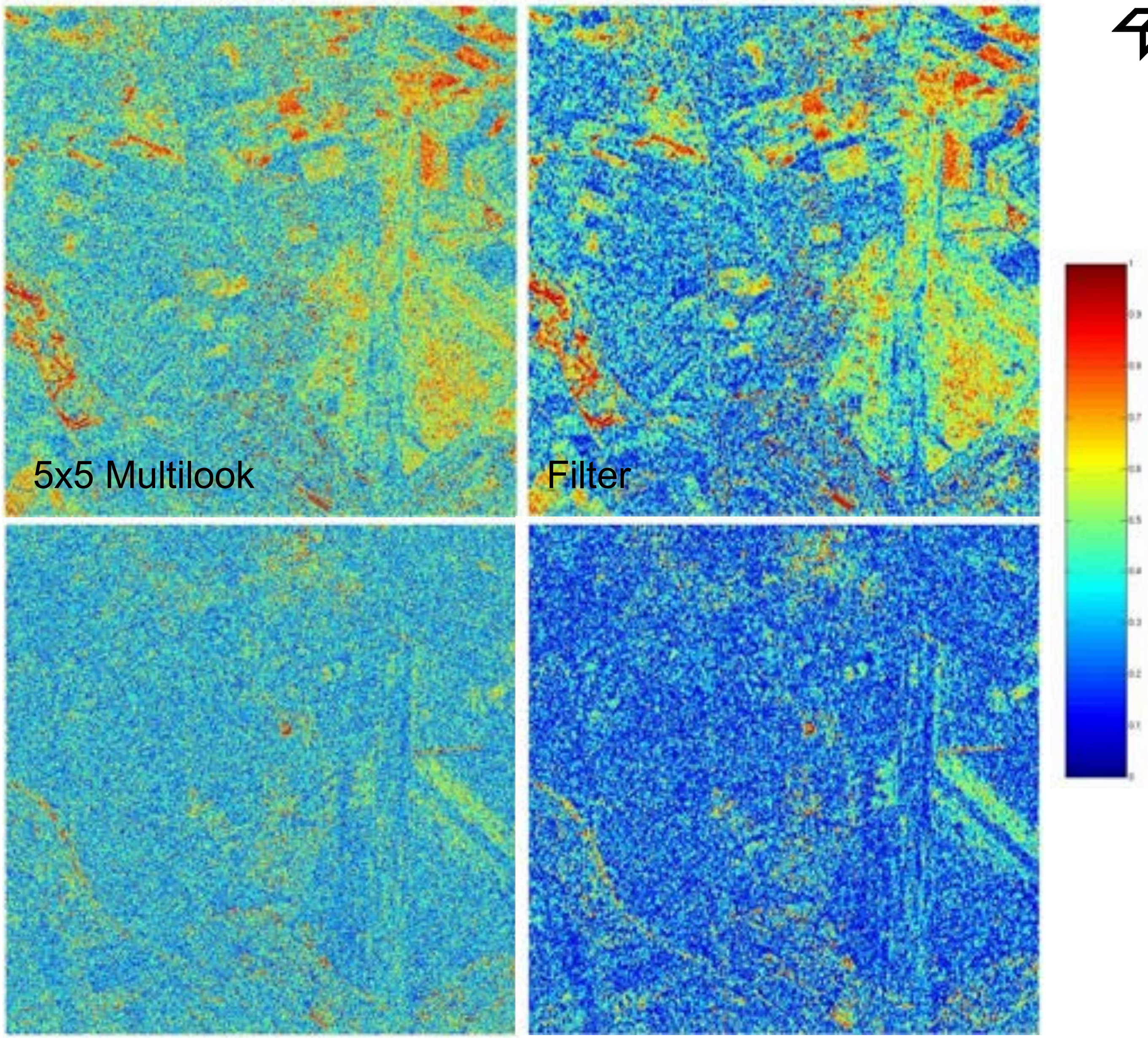
Results: Experimental PolSAR Data

Co-polar correlation

$\left|\rho_{hhvv}\right|$

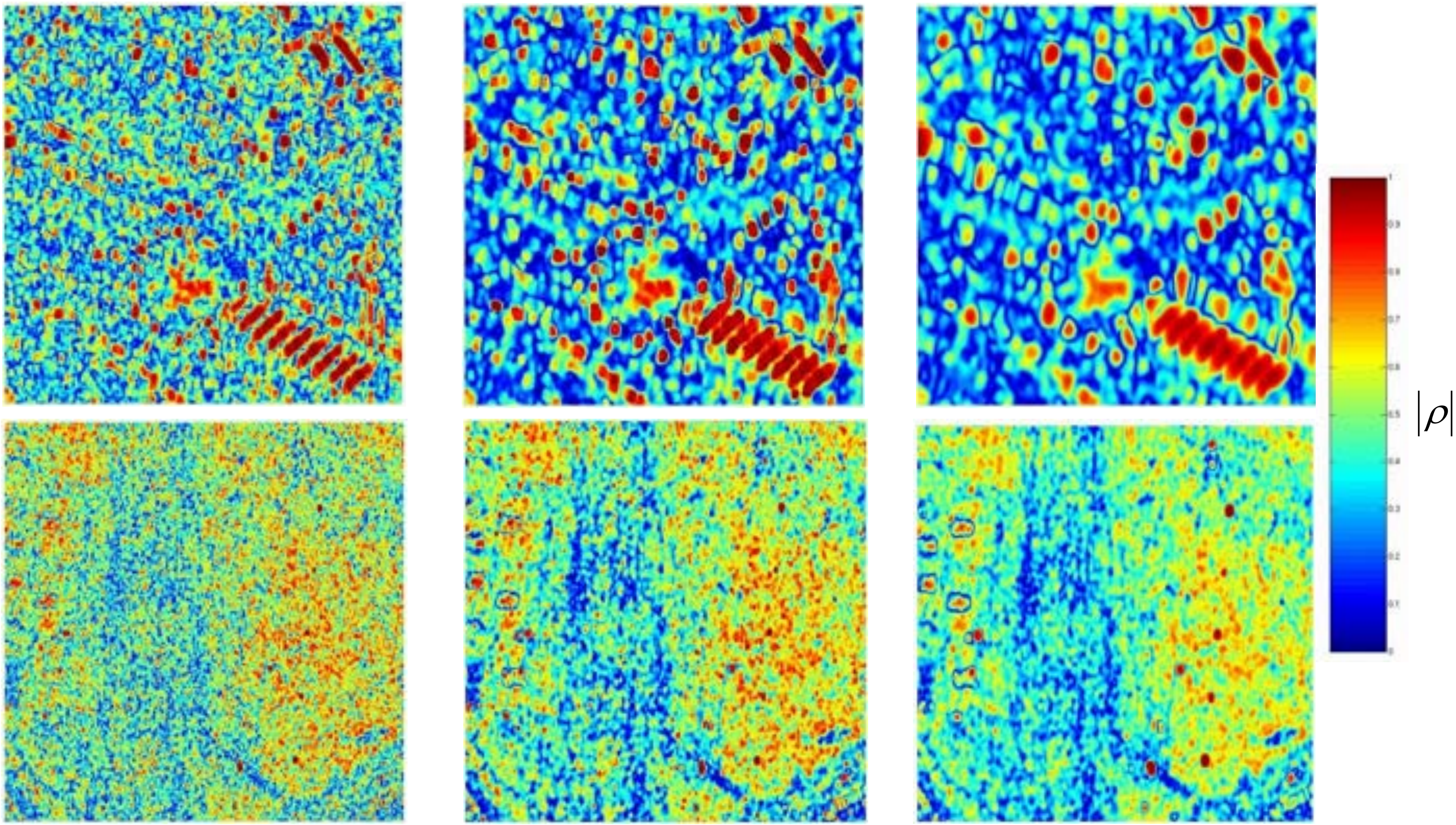
Cross-polar correlation

$\left|\rho_{hhhv}\right|$



Co-polar correlation $|\rho_{hhvv}|$: Details analysis

5x5 Multilook | Filter | 5x5 It. multilook

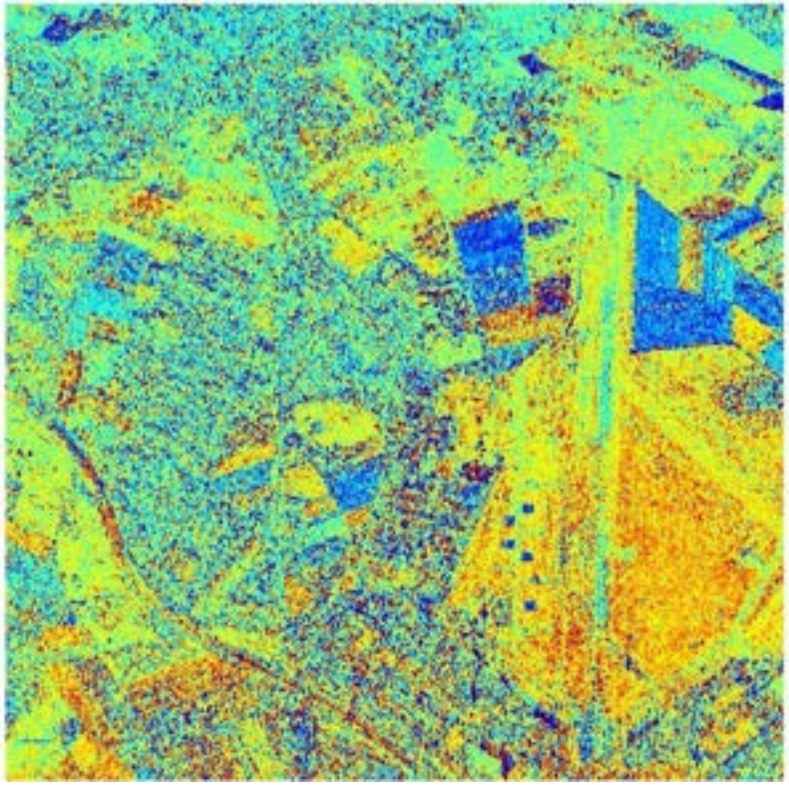


Results: Experimental PolSAR Data

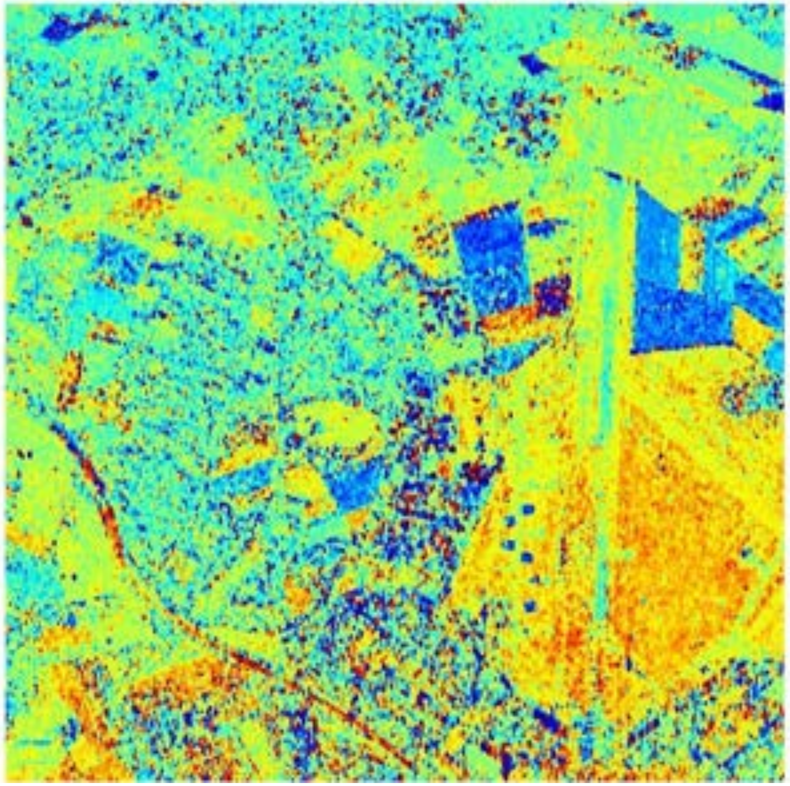
Co-polar correlation phase

$$\left|\rho_{hhvv}\right|$$

5x5 Multilook



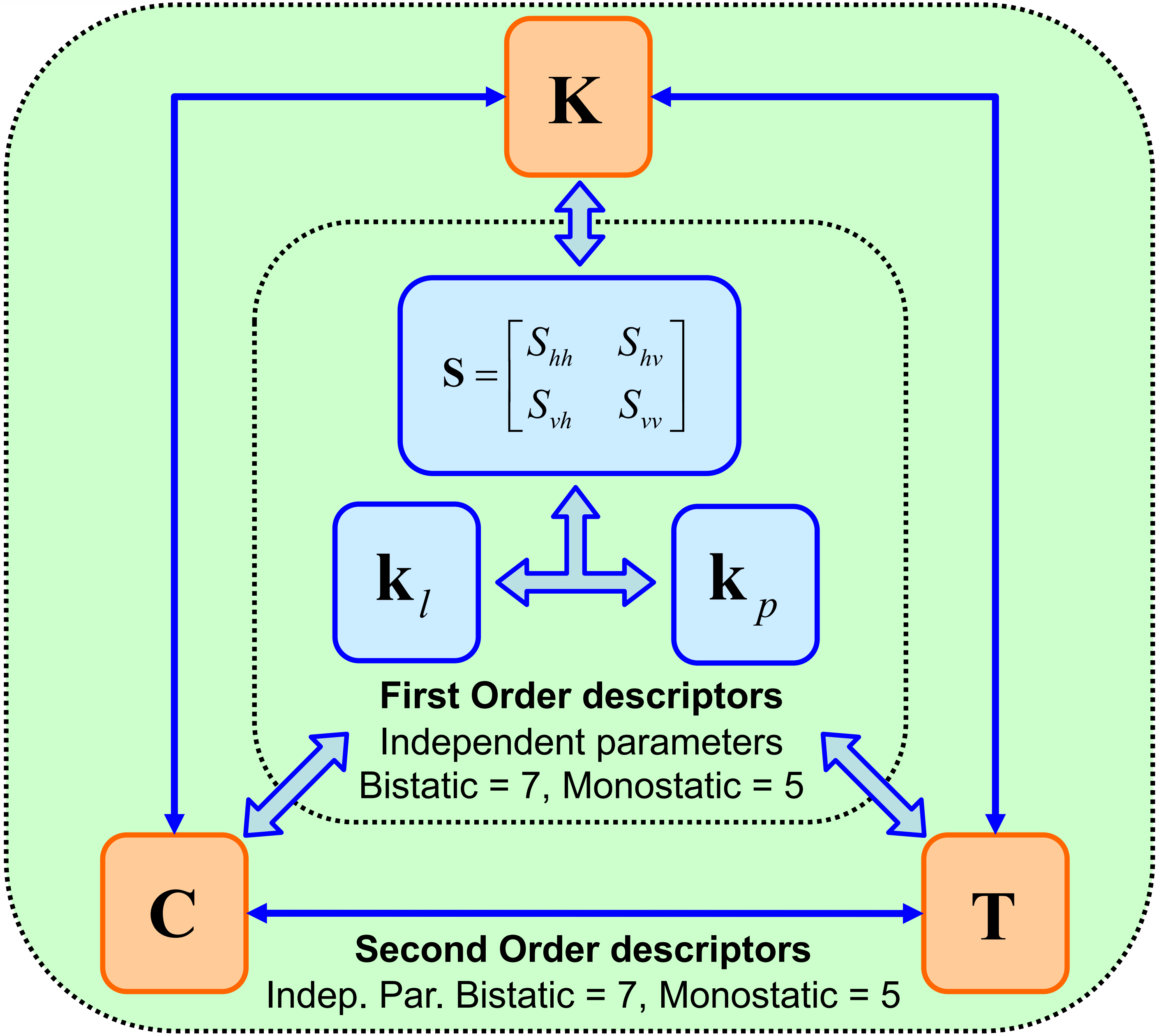
Filter



Description of pure or deterministic scatters

First and Second order descriptors contain the same information

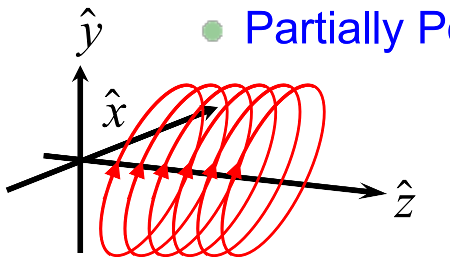
- All descriptors are equivalent
- Second order descriptors are Rank 1 matrices



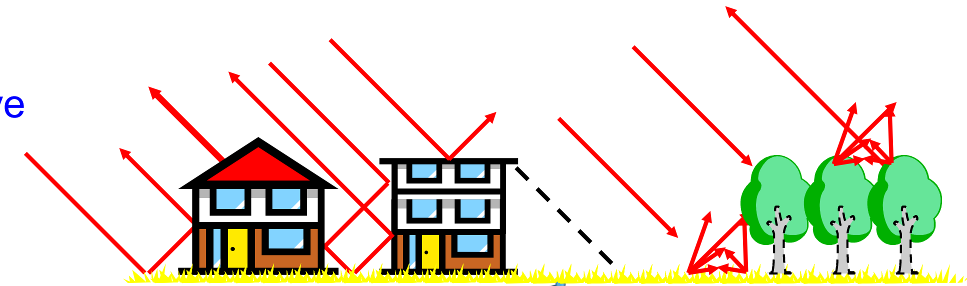
Partially Polarized Scattering

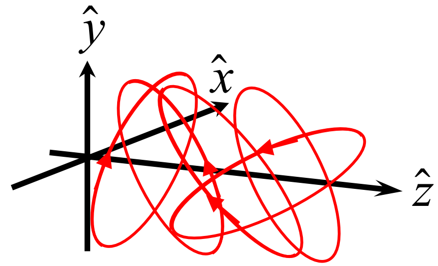
Electromagnetic fields depend on the microscopic arrangement of elementary scatters in the resolution cell, therefore, Scattering Polarimetry descriptors dependsalso on this effect

- Deterministic scatters: The scatering polarimetry descriptors do no vary, in space or time
  - scatering polarimetry descriptors are constant
  - Completely Polarized Scattering
- Distributed scatters: The electromagnetic field varies, in space or time, due to the complex arrangement of individual scatters
  - Scatering polarimetry descriptors vary randomly in time or space
  - Scatering polarimetry descriptors need to accommodate this random nature, i.e., stochastic descriptors are necessary
  - Partially Polarized Scattering



Completely Polarized Wave





Partially Polarized Wave

Equivalence of Scat. Pol. Descriptors

Description of distributted scatters

$$E\{S_{kl}\}=0 \quad \forall k,l$$
$$E\{\mathbf{S}\}=\mathbf{0}$$
$$E\{\mathbf{k}_l\}=\mathbf{0}$$
$$E\{\mathbf{k}_p\}=\mathbf{0}$$

First and Second order descriptors do not contain the same information

- Polarimetric scattering descriptors are not equivament
- Second order descriptors are Rank≥1 matrices

$$E\{\mathbf{K}\}\neq\mathbf{0}$$
$$E\{\mathbf{T}\}\neq\mathbf{0}$$
$$E\{\mathbf{C}\}\neq\mathbf{0}$$