Transport Equations

Conservation of mass, momentum, energy and other thermodynamic properties are based on the physical observation of transport processes. Details on the derivation of various transport equations can be found in standard textbooks [1, 2]. Brief reviews on some of these equations are presented in conservational forms.

Conservation of Mass

Conservation of mass (also called continuity equation) is based on the fact that mass can neither be created nor be destroyed. Imagine a finite volume with a fixed shape placed in a flow stream as in Fig. 1. Mass conservation requires that the rate of mass accumulation in the volume (system) is equal to the rate of mass entering minus the rate of mass leaving the volume.

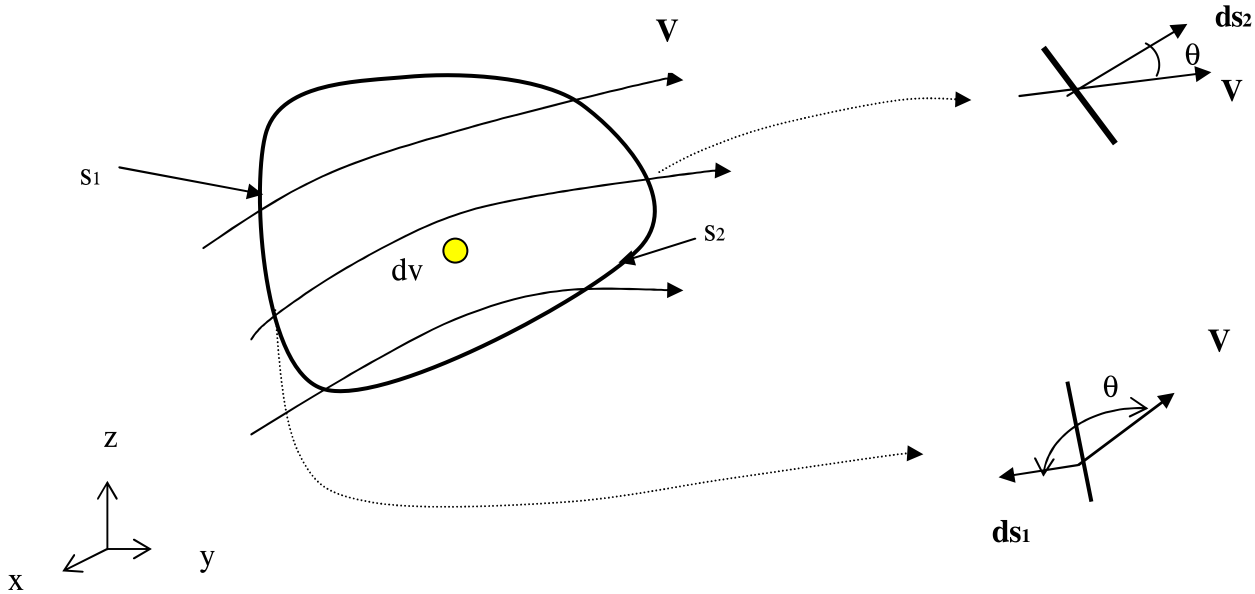

Figure 1 A control volume in a flow field: s_s (inflow surface) and s_s (outflow surface) and **ds** is unit outward normal vector on the surface

Integral form of mass conservation is

$$\frac{\partial}{\partial t}\int\int_{V}\int \rho\, dv = -\int\int_{s_1} \rho\, \vec{v}.\vec{d}s - \int\int_{s_2} \rho\, \vec{v}.\vec{d}s$$

Or,

$$\frac{\partial}{\partial t}\iiint_{v} \rho\, dv = -\iint_{s} \rho\, \vec{v}\cdot d\vec{s} \qquad (1)$$

The above equation expresses the integral form of mass conservation principle applied to a finite volume fixed in space. By employing a vector identity, surface integral can be replaced by volume integral, thus

$$\iint_{s} \rho\, \vec{v}\cdot d\vec{s} = \iiint_{v} \nabla\cdot(\rho\vec{v})\, dv$$

$$\iiint_{v} \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\vec{v})\right] dv = 0$$

To have this expression valid for all possible size control volumes, the integrand must be equal to zero, i.e.,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\vec{v}) = 0 \qquad (2)$$

This is the differential form of mass conservation valid at every point in the flow field. Note that this equation expresses conservation of mass with respect to a fixed volume in space. If the volume does move or changes its shape, additional terms should be included. We will not consider these complications.

Mass conservation principle can be written about a fixed mass rather than a fixed volume as considered above. Returning to the above equation and using a vector identity, we obtain

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0 \qquad (3)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \tag{4}$$

D/Dt is called substantial derivative, which accounts for temporal and convective contributions for the density change. Physically, this expression says that the mass of a fluid element made of a fixed set of molecules and atom is constant as the fluid element moves through space. Eq. (2) is in conservational form while Eq. (3) is in non-conservational form. It is preferred to use conservational form of transport equations in formulating numerical approximations.

Eulerian and Lagrangian Description

Description of transport processes focused on a fixed volume in space is called the Eulerian method. In Eulerian method, thus, we are observing flow property changes at a given volume. In Lagrangian method, on the other hand, we are following a fixed mass and observe property changes of the mass as it flows through space.

Physical interpretation of Eulerian and Lagrangian description of transport process may be illustrated by considering 1-dimensional continuity equations, i.e., Eqs.(2) and (3). Expanding Eq.(2) in Cartesian coordinate, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \cancel{\frac{\partial \rho v}{\partial y}} + \cancel{\frac{\partial \rho w}{\partial z}} = 0 \tag{5}$$

Now consider a unit control volume as defined in a pipe as shown in Fig. 2.

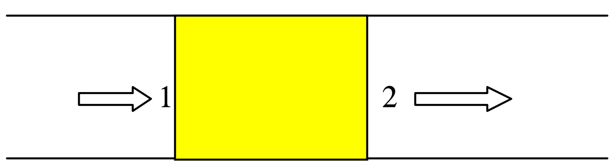

Figure 2 A unit fixed control volume in a pipe

The mass contained in the system is equal to its density (ρ). Eq. (5) simply states that to have a steady state in the system, i.e., $\frac{\partial \rho}{\partial t} = 0$ requires that

$$\frac{\partial \rho u}{\partial x} = 0 \quad \text{or} \quad \rho u = \text{constant} \tag{6}$$

In other words, the total mass in the system remains constant as long as $(\rho u)_1 = (\rho u)_2$. This is a rather obvious conclusion. We are focusing our attention on the control volume fixed in the space. This is the Eulerian description.

Returning to Eq. (5) and rearranging, we get

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0$$

$$\frac{D \rho}{D t} = -\rho \frac{\partial u}{\partial x} \tag{7}$$

The LHS of this equation is the rate of change of density of the system with fixed mass (not volume) as the system flows in space. Select a fixed mass system as shown in Fig. 2 and follow as it moves along the pipe. Eq. (7) says that the density of the system decreases as the fluid speed increases (top) and increases as the flow speed decreases (bottom). This is a direct consequence of conserving mass as the volume of the system changes. This is Lagrangian method in which we are following a system with a fixed mass.

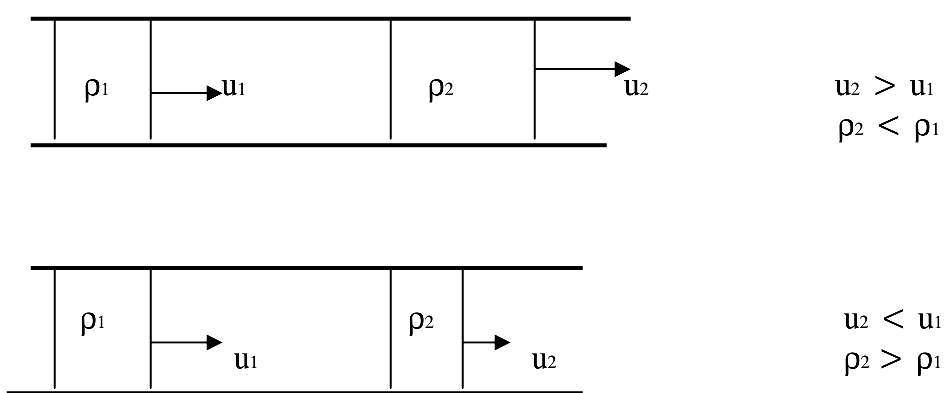

Figure 3 A fixed mass of fluid

It may not be so difficult to follow a fixed mass system in 1-D situations. However, Lagrangian description becomes impractical as the volume of the system changes in multi-dimensional problems. In numerical simulations, Eulerian description is preferred in most situations. Lagrangian description is sometimes used when there is a need to track discontinuities in flow fields, such as shocks and phase changes.

Conservation of Linear Momentum

Conservation of linear momentum is based on the Newton's second law of motion: the rate of change of linear momentum of a fixed mass system is equal to the summation of forces applied to the system. Equivalent form of conservation of linear momentum applied to a fixed volume in space can be derived by using the Reynolds' transport theorem. Let u be a vector component of linear velocity vector **V**, along say x-direction, then the conservational form of linear momentum (ρ u) along the x-direction can be expressed by

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho \vec{v} u) = \nabla \cdot (\mu \nabla u) + B_x - \frac{\partial p}{\partial x} + V_x \qquad (8)$$

The first term in the LHS of Eq. (8) is the rate of change of x-momentum per unit volume of fluid and the second term is the rate of x-momentum flowing in and out of control volume. The first term in the RHS is the viscous force, second term body force, the third term pressure force and the last term is the viscous force not included in the first term. Similar expressions can be written along the other directions.

Energy Conservation

Conservation of energy is based on the first law of thermodynamics which states that energy can neither be created nor be destroyed; it changes from one form to others. Energy conservation can be expressed in many forms. In terms of internal energy, it may be expressed for incompressible fluid or ideal gas as

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho \vec{v} e) = \nabla \cdot \left(\frac{k}{c_v} \nabla e\right) - p \nabla \cdot \vec{v} + \Phi + \dot{q} \qquad (9)$$

The first term is the rate of internal energy change per unit volume of fluid and the second term is the rate of internal energy flowing in and out of control volume. The RHS of Eq. (9) is the sum of rate of energy transfer to the volume by conduction, pressure work, viscous dissipation and energy generation.

General Transport Equations

Conservation of mass, momentum and energy equations have a common appearance that can be put into the following general transport equation;

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\vec{v}\phi) = \nabla\cdot(\Gamma\nabla\phi) + S \tag{10}$$

Physically this equation expresses the fact that the time rate of change of a property at a point in a flow field is contributed by convection due to flow motion, diffusion due to property gradient and production or destruction of properties due to the sources.

As much as possible, all transport equations will be cast into this general form. Numerical method that can solve this general transport equation can be applied to many different physical processes. In terms of general transport equation, conservations of mass, x-momentum and energy can be summarized as in Table 1.

Table 1 List of variables, diffusion coefficients and sources

Equation	ϕ	Γ	S
Mass	1	0	0
x-momentum	u	μ	$B_x + V_x - \frac{\partial p}{\partial x}$
Energy	e	$\frac{k}{c_v}$	$-p\nabla\cdot\vec{v} + \Phi + \dot{q}$

Expanding Eq. (10) in Cartesian, cylindrical and spherical coordinate system, we have [Hughes & Gaylord, 1964],

$$\begin{aligned}&\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) = \\ &\frac{\partial}{\partial x}(\Gamma\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(\Gamma\frac{\partial\phi}{\partial y}) + \frac{\partial}{\partial z}(\Gamma\frac{\partial\phi}{\partial z}) + S\end{aligned} \tag{11}$$

$$\frac{\partial\rho\phi}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\,\rho\,v_r\,\phi)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho\,v_\theta\,\phi)+\frac{\partial}{\partial z}(\rho\,v_z\,\phi)=$$
$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\Gamma\frac{\partial\phi}{\partial r})+\frac{1}{r}\frac{\partial}{\partial\theta}(\Gamma\frac{1}{r}\frac{\partial\phi}{\partial\theta})+\frac{\partial}{\partial z}(\Gamma\frac{\partial\phi}{\partial z})+S \qquad (12)$$

and

$$\frac{\partial\rho\phi}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\,v_r\,\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\rho\,v_\theta\,\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}(r\,\rho\,v_\phi\,\phi)=$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Gamma\frac{\partial\phi}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\Gamma\frac{\partial\phi}{\partial\theta})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\varphi}(\Gamma\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\varphi})+S \qquad (13)$$

References

1. White, Frank M., Fluid Mechanics, 2nd Edition, McGraw-Hill, 1986.

2. White, Frank M., Viscous Fluid Flow, McGraw-Hill,1974.

3. Hughes, W. F. and Gaylord, E. W., Basic Equations of Engineering Science, Schaum's Outline Series, McGraw-Hill, 1964.Problems

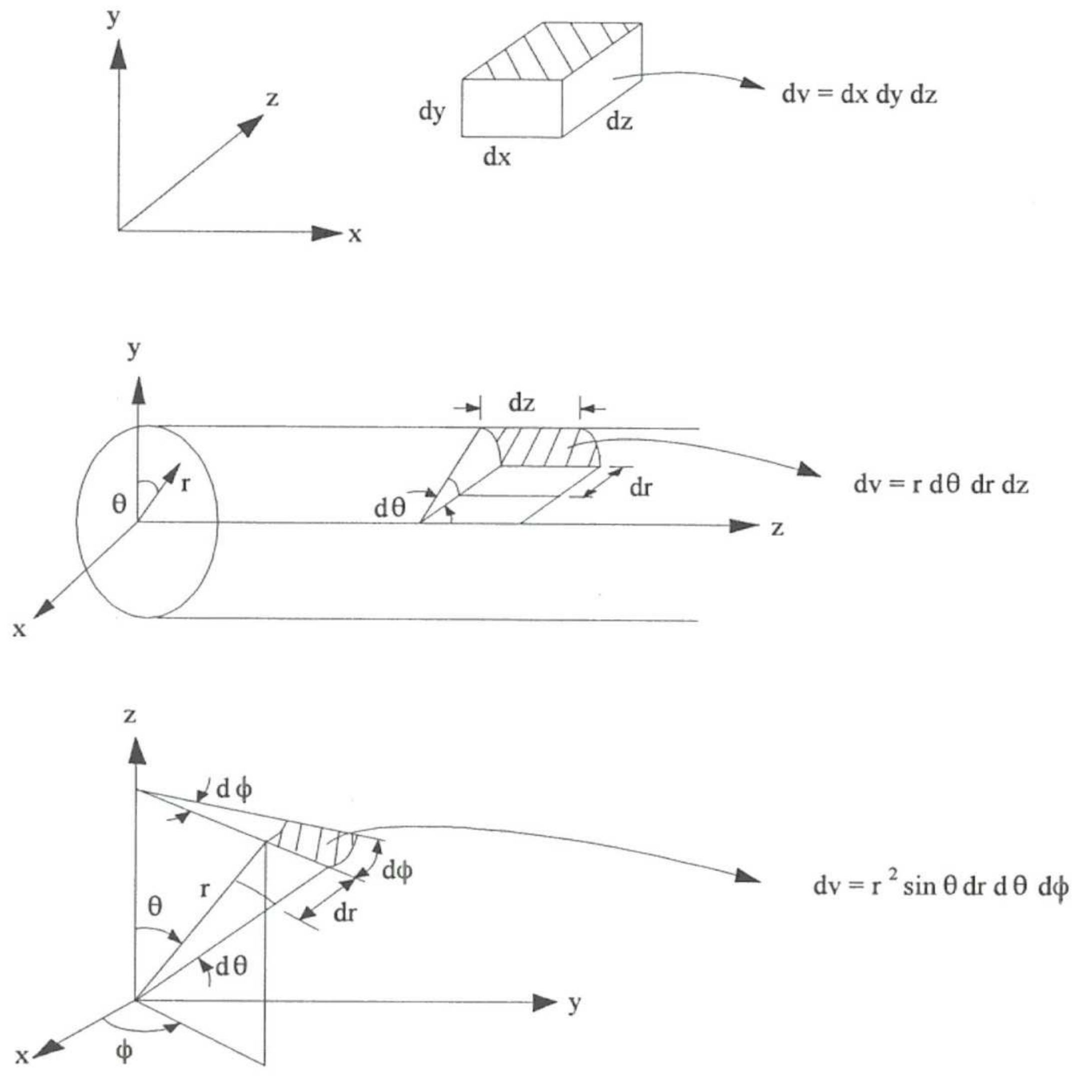

Figure 4 Cartesian, cylindrical and spherical coordinates

Problems

1. The azimuthal direction momentum equation in the cylindrical coordinate system for an incompressible flow may be expressed as

$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+F_\theta+\frac{2}{r}\frac{\partial}{\partial \theta}\left(\frac{\mu}{r}\frac{\partial v_\theta}{\partial \theta}\right)+$$

$$\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)$$

Show that this equation can be reduced to the following conservational form for a circular Couette flow between two concentric cylinders. Assume two dimensional in radial (r) and axial (z) directions. [hint: Use continuity equation.]

$$\frac{\partial \rho v_\theta}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho v_r v_\theta-\mu\frac{\partial v_\theta}{\partial r}\right)\right]+\frac{\partial}{\partial z}\left(\rho v_z v_\theta-\mu\frac{\partial v_\theta}{\partial z}\right)=$$

$$-\mu\frac{v_\theta}{r^2}-\frac{\rho v_r v_\theta}{r}$$

2. Starting from Eq. (9) with the definition of enthalpy, $h=e+p/\rho$, show that energy equation in terms of enthalpy may be expressed as, for incompressible or ideal gas flows,

$$\frac{\partial \rho h}{\partial t}+\nabla\cdot(\rho\vec{v}h)=\nabla\cdot\left(\frac{\kappa}{c_p}\nabla h\right)+\Phi+\dot{q}+\frac{Dp}{Dt}$$