1-Dimensional Steady Conduction

One of the simplest transport processes encountered in many engineering applications is 1-dimensional conduction. One-dimensional conduction will be used to discuss some of fundamentals associated with the numerical method introduced in this course. All the development discussed in this chapter is equally pertinent to more complex transport processes presented in later chapters. One-dimensional conduction equation may be obtained from the general form of transport equation as discussed.

With $\phi = e$, $\Gamma = k/c_v$, and $\mathbf{V} = 0$, we get an energy equation

$$\frac{\partial(\rho e)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k}{c_v}\frac{\partial e}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k}{c_v}\frac{\partial e}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{k}{c_v}\frac{\partial e}{\partial z}\right) + S \tag{1}$$

For incompressible substance, $\rho =$ constant, $C_v = C_p = C$, and $de = CdT$. Thus, Eq. (1) can be written as

$$\rho C\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + S \tag{2}$$

Note that we have not made any assumption on the specific heat, C. That is C can be a function of space and temperature.

Assuming the temperature variation is in x-coordinate alone, Eq. (2) reduces to 1-dimensional transient conduction equation,

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S \tag{3}$$

For 1-dimensional steady conduction, this further reduces to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + S = 0 \tag{4}$$

Eq. (4) is a simple transport equation which describes steady state energy balance when the energy is transported by diffusion (conduction) alone in 1-dimensional space.

Finite Volume Equation

Finite difference approximation to Eq. (4) can be obtained by a number of different approaches. We will consider a control volume method [1]. We will use notations and symbols commonly adopted for finite volume method (see Fig.1). Rewriting Eq. (4), we have

$$-\frac{d}{dx}(J_x) + S = 0 \tag{5}$$

where $J_x = -kdT/dx$ is the conduction flux in the x-direction. We are not making any assumption on the conductivity and the source, so that they may have different values at different control volumes and may depend on temperature as well.

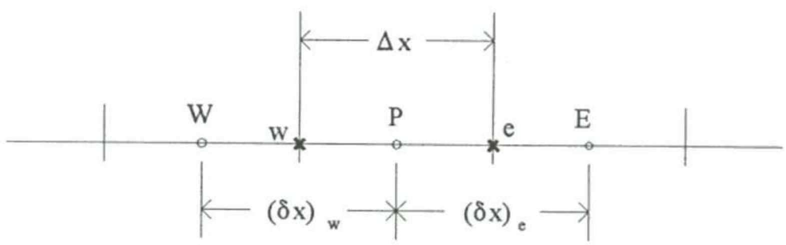


Figure 1 1-dimensional uniform control volumes

Integrating Eq. (5) over a control volume containing P, we have

$$\int_0^1\int_0^1\int_w^e \left(-\frac{dJ_x}{dx} + S\right) dxdydz = (1 \times 1)\left[\int_w^e -dJ_x + \int_w^e Sdx\right] \tag{6}$$

Carrying out the integration for the first term in the RHS of Eq. (6), we get

$$\int_w^e -dJ_x = -[(J_x)_e - (J_x)_w]$$

$$= -\left(-k\frac{dT}{dx}\right)_e + \left(-k\frac{dT}{dx}\right)_w \tag{7}$$

$$= k_e\left(\frac{dT}{dx}\right)_e - k_w\left(\frac{dT}{dx}\right)_w$$

Diffusion flux at the interfaces "e" and "w" can be approximated by

$$k_e \left(\frac{dT}{dx}\right)_e = k_e \frac{T_E - T_P}{(\delta x)_e} \tag{8a}$$

and

$$k_w \left(\frac{dT}{dx}\right)_w = k_w \frac{T_P - T_W}{(\delta x)_w} \tag{8b}$$

Eqns. (8a) and (8b) are second-order accurate when uniform control volumes are used. If control volumes are non-uniform, these expressions must be modified to reflect the different grid sizes. However, we will still use Eqns.(8a) and (8b) to evaluate diffusion flux even for non-uniform control volumes for the sake of simplicity in resulting finite volume equations. Accuracy will not be affected significantly as long as the grid sizes adjacent to each other are not extremely different. Physical meaning of this approximation is to assume a piecewise linear temperature variation between the temperatures in two adjacent control volumes as shown in Fig. 2.

Integration of the second term in the RHS of Eq. (6) yields

$$\int_w^e S dx = \bar{S} \Delta x \tag{9}$$

where $\bar{S}$ represents the average value of S in the given control volume. Physically, this assumption implies a stepwise variation of source as shown in Fig. 2. Putting these approximations into Eq. (6), we obtain

$$\frac{k_e (1 \text{ x } 1)}{(\delta x)_e}(T_E - T_P) - \frac{k_w (1 \text{ x } 1)}{(\delta x)_w}(T_P - T_E) + \bar{S}(1 \text{ x } 1)\Delta x = 0$$

Rearranging this, we have

$$a_P T_P = a_E T_E + a_W T_W + b \tag{10}$$

where

$$a_E = \frac{k_e(1x1)}{(\delta x)_e} \tag{11a}$$

$$a_W = \frac{k_w(1x1)}{(\delta x)_w} \tag{11b}$$

$$a_P = a_E + a_W \tag{11c}$$

and

$$b = \bar{S}(1x1)\Delta x \tag{11d}$$

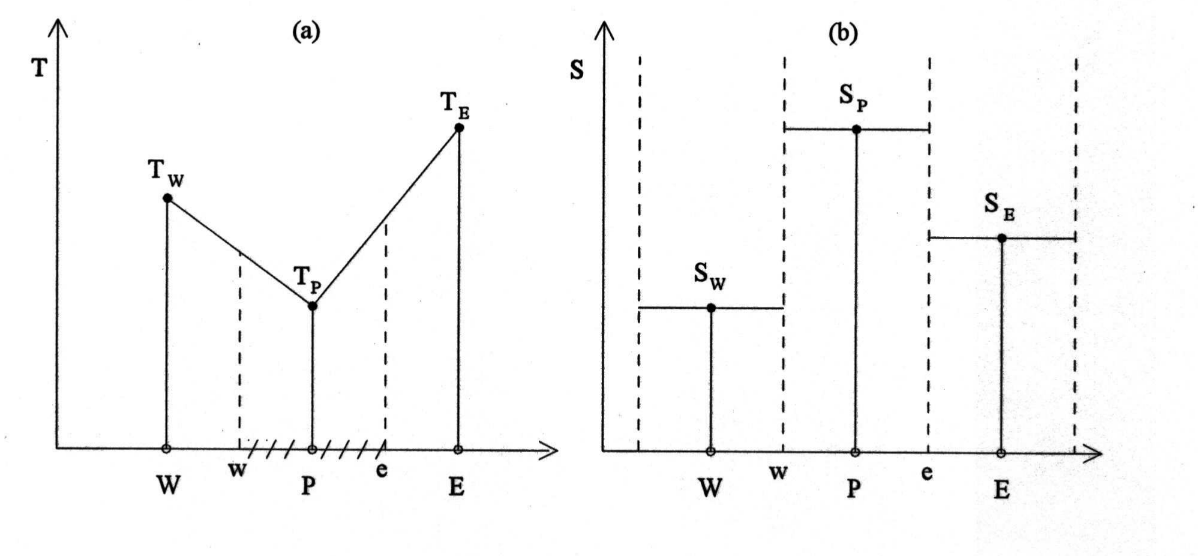


Figure 2 Piecewise linear temperature between adjacent control volumes (a) and step-wise uniform source in a control volume (b)

Eq. (10) is the finite volume equation that describes energy conservation for 1-dimensional steady state conduction. Temperature T_P of the control volume under consideration is influenced by temperature in the neighboring control volumes, T_W at the west and T_E at the east. Coefficients a_E, and a_W represent conductance between the control volume under consideration and the adjacent control volumes. Conductance is proportional to the interfacial conductivities and the cross-sectional area perpendicular to the x-coordinate (1x1) and is inversely proportional to the distances between two temperature nodes. All coefficients appearing in Eq. (10) are positive. Thus if we increase the temperatures in the neighboring volumes, temperature T_P will also increase, reflecting realistic physical situations. If coefficients are not positive, however, increasing neighboring temperature will decrease T_P. Having positive coefficients in finite volume equations is essential for a successful numerical simulation. In convection, these coefficients are not always positive and require some special treatments to make them positive (discussed in a later chapter). In Eq. (10), b represents generation or destruction of energy in the control volume under consideration.

Finite Volume Equation in Index Notations

Computer programming requires that the finite volume equations be written in index notations. Following practices in defining the control volumes and associated index notations are adopted.

Each control volumes are defined by identifying each control volume surfaces by assigning their locations in x, i.e., x_1, x_2, ...,x_{i-1}, x_i, x_{i+1},.......x_N, x_{N+1}, x_{N+2}, and x_{N+3} as shown in Fig. 3. Temperature and thermal conductivity are defined explicitly at the center of each control volume. There are N+2 control volumes. Two control volumes at the end of calculation domain are fictitious control volumes with zero thickness. They are introduced to handle the boundary conditions.

In terms of these notations, finite volume equation can be written as, $2 \le i \le N+1$,

$$a_i T_i = b_i T_{i+1} + c_i T_{i-1} + d_i \tag{12}$$

where

$$b_i = \frac{k_e(1x1)}{(\delta x)_e} = \frac{k_{i+1/2}(1x1)}{0.5(\Delta x_i + \Delta x_{i+1})} \tag{12a}$$

$$c_i = \frac{k_w(1x1)}{(\delta x)_w} = \frac{k_{i-1/2}(1x1)}{0.5(\Delta x_{i-1} + \Delta x_i)} \tag{12b}$$

$$a_i = b_i + c_i \tag{12c}$$

and

$$d_i = \overline{S_i}\Delta x_i(1x1) \tag{12d}$$

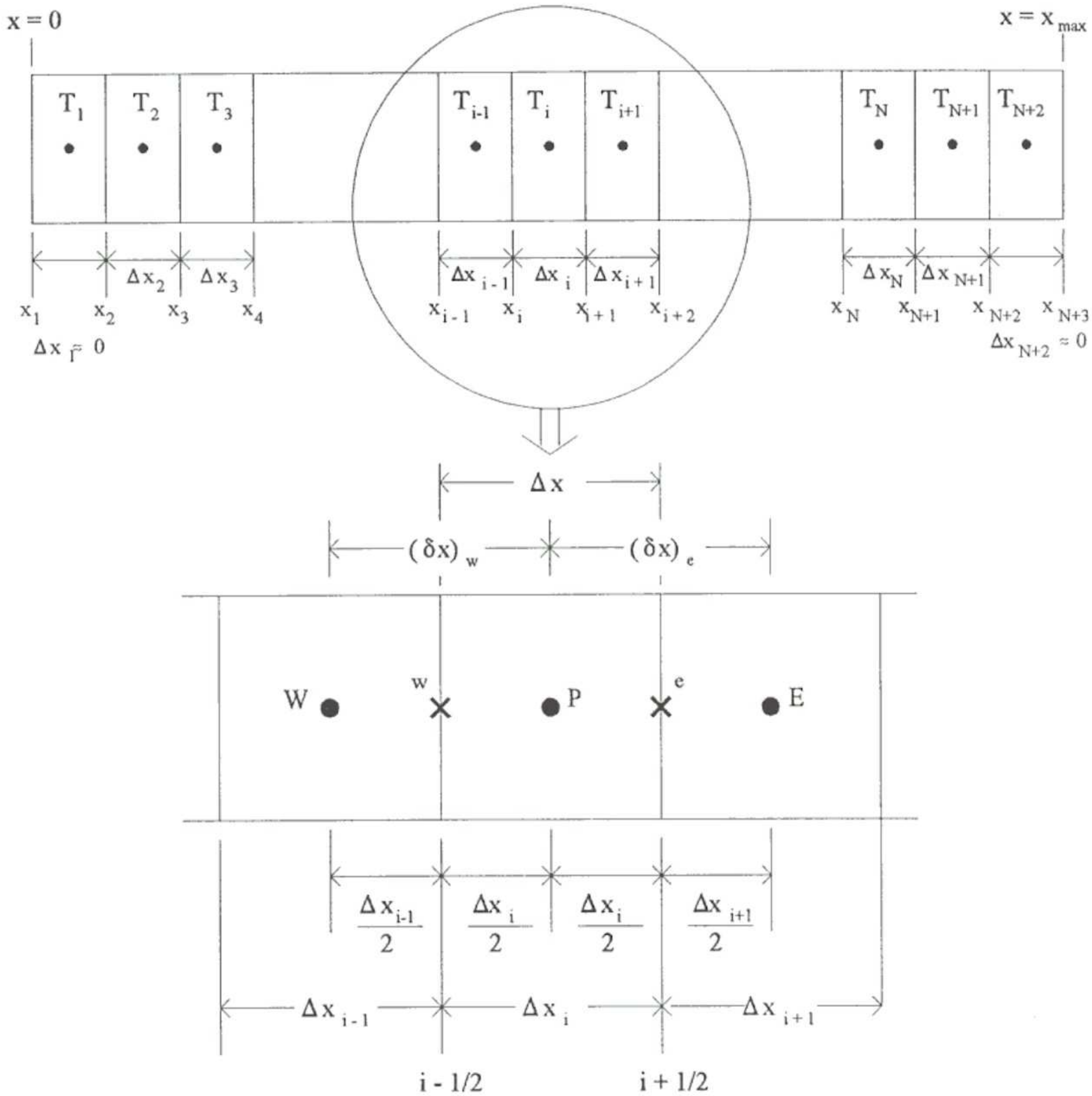


Figure 3 Index notations

Writing Eq. (12) for every control volumes, $2 \leq i \leq N+1$, we have N simultaneous equations:

$$
\begin{aligned}
a_2 T_2 &= b_2 T_3 + c_2 T_1 + d_2 \\
a_3 T_3 &= b_3 T_4 + c_3 T_2 + d_3 \\
&\cdots\cdots\cdots\cdots \\
a_{i-1} T_{i-1} &= b_{i-1} T_i + c_{i-1} T_{i-2} + d_{i-1} \\
a_i T_i &= b_i T_{i+1} + c_i T_{i-1} + d_i \\
a_{i+1} T_{i+1} &= b_{i+1} T_{i+2} + c_{i+1} T_i + d_{i+1} \\
&\cdots\cdots\cdots\cdots \\
a_N T_N &= b_N T_{N+1} + c_N T_{N-1} + d_N \\
a_{N+1} T_{N+1} &= b_{N+1} T_{N+2} + c_{N+1} T_N + d_{N+1}
\end{aligned}
\tag{13}
$$

where temperatures T_1 and T_{N+2} are given by the boundary conditions. Unknown temperatures, T_2, T_3, ………,T_N, and T_{N+1} are found by solving Eq.(13) simultaneously. Since Eq. (13) is in a tri-diagonal matrix form, Tri-Diagonal-Matrix-Algorithm (TDMA) can be used. This will be described later.

Boundary Conditions

Specification of boundary temperatures, T_1 and T_{N+2} depends upon the physical boundary conditions at the ends of calculation domain. Let us consider various boundary conditions at the left boundary.

Case 1:

Boundary temperature, T_B, is given. This is the simplest boundary condition. Then $T_1 = T_B$.

Case 2:

Heat flux at the boundary, q_{in}'', is given. Boundary temperature (T_1) is not known, however. We can find a relation between the given heat flux and the temperature inside the calculation domain by considering an energy balance. Consider the fictitious control volume (Fig. 4) and taking energy balance we get

$$\frac{\partial}{\partial t}(\rho c A \Delta x_1 T_1) = q_{in}'' A - q_{out}'' A \tag{14}$$

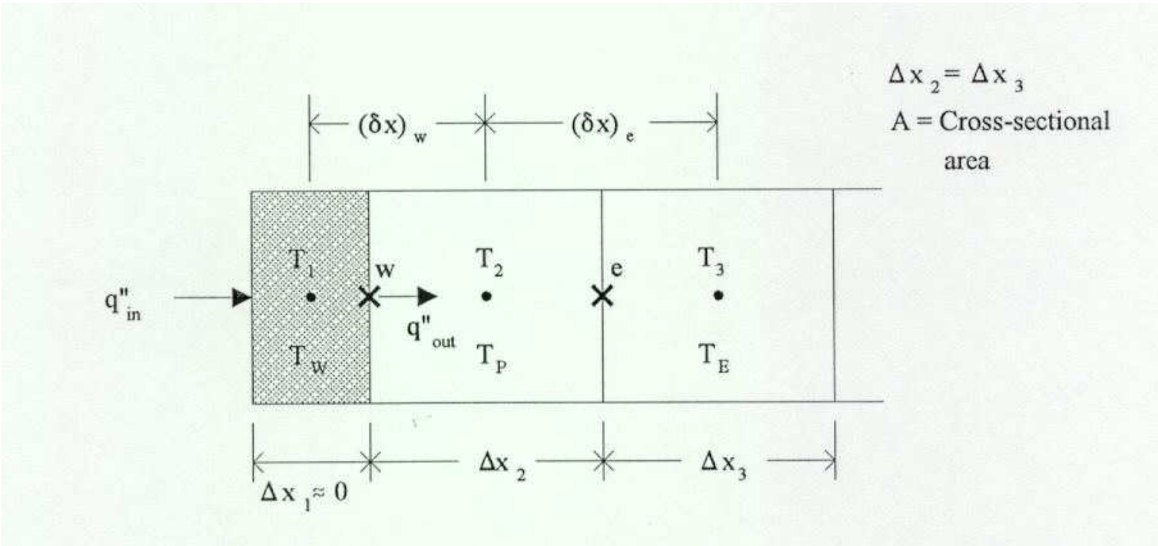


Figure 4 A known heat flux boundary at x=0

The first term in Eq. (14) is zero since there is no mass in this volume. Now evaluating the conduction flux through the right surface $(J_x)_w$ and equating with the given heat flux, q''_{in}, we get

$$q''_{in} = -k_2\left[\frac{T_2 - T_1}{0.5(\Delta x_1 + \Delta x_2)}\right] \tag{15}$$

Solving for the unknown boundary temperature, T_1, we have

$$T_1 = T_2 + \frac{1}{2}\left[\frac{\Delta x_2}{k_2} q''_{in}\right] \tag{16}$$

If $q''_{in} = 0$, i.e., an insulated boundary, Eq. (16) reduces to

$$T_1 = T_2 \tag{17}$$

Case 3:

Convective heat transfer coefficient, h_f and fluid temperature, T_f are given. This is similar to Case 2 except the convection flux is given by

$$q''_{conv} = h_f(T_f - T_1) \tag{18}$$

Note that the convection flux is positive in the x-direction if T_f is greater than T_1. By taking an energy balance at the fictitious control volume at x=0, we obtain

$$h_f (T_f - T_1) = -k_2 \left[\frac{T_2 - T_1}{0.5(\Delta x_1 + \Delta x_2)} \right] \tag{19}$$

Solving for T_1, we have

$$T_1 = \frac{h_f T_f + \dfrac{2}{\Delta x_2} k_2 T_2}{h_f + \dfrac{2}{\Delta x_2} k_2} \tag{20}$$

If $h_f >> k_2$, Eq. (20) reduces to $T_1 = T_f$, and if $h_f << k_2$, Eq.(20) reduces to Eq.(17), as expected. Similar approach can be used at the right boundary.

For the three boundary conditions considered so far, it is possible to solve explicitly for the boundary temperature T_1 . Suppose we have a radiative boundary at x=0. Then the energy balance gives

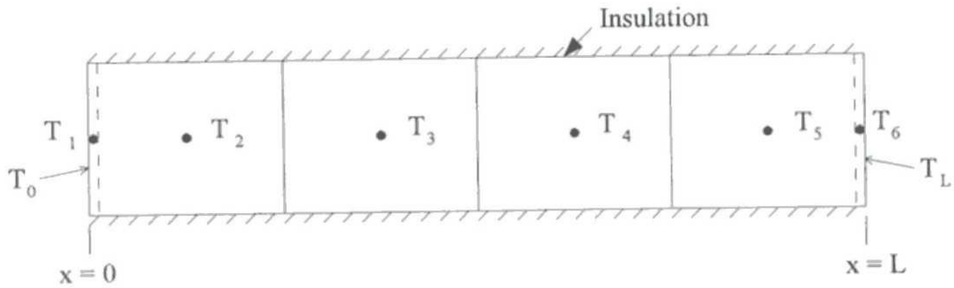
$$\sigma\varepsilon(T_f^4 - T_1^4) = -k_2 \left[\frac{T_2 - T_1}{0.5(\Delta x_1 + \Delta x_2)} \right] \tag{21}$$

We cannot solve for T_1 explicitly from this equation. A more general treatment of such nonlinear boundary conditions will be discussed in detail later in this chapter.

We have formulated finite volume equation for one dimensional conduction and considered how to incorporate some boundary conditions. Let us examine few example problems to reinforce our understanding of the proposed numerical method.

Example 1:

Consider a steady conduction in 1-dimensional bar with known temperature at x=0 (T_1 = 0 K) and x=L (T_6 =16 K). The length of the bar is 8.0 m, conductivity is constant at 1.5 W/K.m, and there is a uniform heat source that generates 3.0 W/m^3. We would like to find the temperature at the center of 4 equal control volumes as shown.



Solution:

Evaluating the coefficients at i=2, 3, 4 and 5, we have

$$b_2 = \frac{(1.5)(1x1)}{0.5(2+2)} = \frac{1.5}{2}; c_2 = \frac{(1.5)(1x1)}{0.5(0+2)} = 1.5; a_2 = \frac{4.5}{2}$$

$$b_3 = \frac{1.5}{2}; c_3 = \frac{1.5}{2}; a_3 = \frac{3.0}{2}$$

$$b_4 = \frac{1.5}{2}; c_4 = \frac{1.5}{2}; a_4 = \frac{3.0}{2}$$

$$b_5 = 1.5; c_5 = \frac{1.5}{2}; a_5 = \frac{4.5}{2}$$

Source terms are, for all i's; $d_i = \overline{S_i}\Delta x_i (1x1) = (3.0\ W/\ m^3)(2\ mx1\ m^2) = 6\ W$

With these coefficients and source terms, Eq. (13) becomes, after multiplied by 2,

$$4.5T_2 = 1.5T_3 + 3.0T_1 + 12 \quad (a.1)$$
$$3.0T_3 = 1.5T_4 + 1.5T_2 + 12 \quad (a.2)$$
$$3.0T_4 = 1.5T_5 + 1.5T_3 + 12 \quad (a.3)$$
$$4.5T_5 = 3.0T_6 + 1.5T_4 + 12 \quad (a.4)$$

Substituting the given boundary temperatures, $T_1=0$ and $T_6=16$, Eqns.(a.1) ~ (a.4) become

$$4.5T_2 = 1.5T_3 + 12 \tag{b.1}$$
$$3T_3 = 1.5T_4 + 1.5T_2 + 12 \tag{b.2}$$
$$3T_4 = 1.5T_5 + 1.5T_3 + 12 \tag{b.3}$$
$$4.5T_5 = 1.5T_4 + 60 \tag{b.4}$$

These equations can be solved by forward and backward substitution method. To do that lets rewrite these equations as

$$T_2 = \frac{1}{3}T_3 + \frac{12}{4.5} \tag{c.1}$$
$$T_3 = \frac{1}{2}T_4 + \frac{1}{2}T_2 + 4 \tag{c.2}$$
$$T_4 = \frac{1}{2}T_5 + \frac{1}{2}T_3 + 4 \tag{c.3}$$
$$T_5 = \frac{1}{3}T_4 + \frac{60}{4.5} \tag{c.4}$$

Forward substitution begins by introducing T_2 into (c.2),

$$T_3 = \frac{1}{2}T_4 + \frac{1}{2}(\frac{1}{3}T_3 + \frac{12}{4.5}) + 4$$

Rearranging this

$$T_3 = \frac{3}{5}T_4 + 6.4 \tag{d}$$

Substituting this into (c.3), we get

$$T_4 = \frac{5}{7}T_5 + \frac{72}{7} \tag{e}$$

Finally substituting T_4 into (c.4) and solving for T_5, we have $T_5 = 22$. Backward substitution determines $T_4=26$, $T_3=22$ and $T_2=10$.

Exact solution of this problem is

$$T(x) = T_0 + (T_L - T_0)\frac{x}{L} + \frac{S}{2k}(L-x)\,x \tag{f}$$

With the given conditions, Eq. (f) becomes

$$T(x) = 10\,x - x^2 \qquad (g)$$

Numerical solution is one degree larger than the exact values at all control volumes. This is due to small number of control volume used in the solution.

To check the solution further, let's calculate the energy balance. Energy generated in the bar is

$$q_{gen} = (3\,W/\,m^3)(8\,m x 1\,m^2) = 24\,W$$

$$q_0 = q_{x=0} = -k\frac{T_2 - T_1}{\Delta x_2 / 2}(1\,x\,1) = (-1.5)\frac{10 - 0}{2/2}(1) = -15\,W$$

$$q_L = q_{x=L} = -k\frac{T_6 - T_5}{\Delta x_5 / 2}(1\,x\,1) = -1.5\frac{16 - 22}{2/2}(1) = 9\,W$$

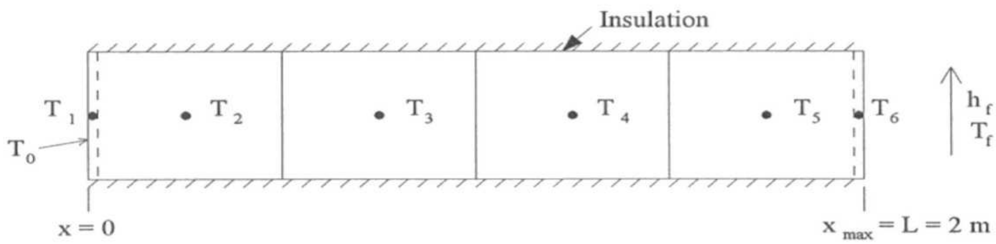
Thus

$$q_{out} = -q_0 + q_L = 15\,W + 9\,W = 24\,W$$

We see the energy conservation is satisfied.

Example 2:

An aluminum bar is exposed to a convective boundary at x=L and a constant temperature at x=0. Calculate steady-state temperature at four locations as shown. Assume k=14 W/m.K, T_0=373 K, T_f=298 K and h_f=10 W/m^2.K.



Solution:

Evaluating the coefficients for each i, we have

$$b_2 = \frac{(14)(1)}{0.5(0.5 + 0.5)} = 28;\ c_2 = \frac{(14)(1)}{0.5(0 + 0.5)} = 56;\ a_2 = 84$$

$$b_3 = 28;\ c_3 = 28;\ a_3 = 56$$

$$b_4 = 28;\ c_4 = 28;\ a_4 = 56$$

$$b_5 = \frac{(14)(1)}{0.5(0.5 + 0)} = 56;\ c_5 = 28;\ a_5 = 84$$

There is no source, thus $d_i=0$, for all i's.

With these coefficients, simultaneous equations are

$$84\ T_2 = 28\ T_3 + 56\ T_1$$

$$56\ T_3 = 28\ T_4 + 28\ T_2 \qquad \text{(a.1-a.4)}$$

$$56\ T_4 = 28\ T_5 + 28\ T_3$$

$$84\ T_5 = 56\ T_6 + 28\ T_4$$

Boundary temperature T_1 is $T_0=373$ K. T_6 is determined by energy balance at the right boundary. Thus, with N=4,

$$T_6 = \frac{h_f T_f + \dfrac{2}{\Delta x_5} k_5 T_5}{h_f + 2\dfrac{k_5}{\Delta x_5}} \tag{b}$$

$$= 45.15 + 0.84848 T_5$$

Substituting T_0 into (a.1),

$$84 T_2 = 28 T_3 + 20888 \tag{a.1'}$$

Substituting (b) into (a.4) and rearranging, we have

$$36.49 T_5 = 28 T_4 + 2528.4 \tag{a.4'}$$

Solving (a.1'), (a.2), (a.3) and (a.4') simultaneously, we obtain the numerical solutions. Exact solution of the given problem is T(x) = -22.06 x + 373. Table below shows a comparison of these results.

node	numerical	exact
1	373	373
2	367.47	367.49
3	356.41	356.46
4	345.35	345.43
5	334.29	334.40
6	328.79	328.88

Energy balance can be used to check the accuracy of numerical solution. Energy entering through the left boundary by conduction is

$$q_0'' = -k_2 \left[\frac{T_2 - T_1}{\dfrac{\Delta x_2}{2}} \right] = (-14)\left[\frac{367.47 - 373}{0.5(0.5)} \right] = 309.7\, W/m^2$$

Energy leaving through the right boundary due to convection is

$$q_L'' = h_f\,(T_6 - T_f) = (10)(328.79 - 298) = 307.9\,W/m^2$$

About 0.58 % error is observed in energy conservation.

Example 3

Consider the same physical and boundary conditions as given in Example 2. But allow uniform energy generation in the aluminum bar at a rate of 100 W/m^3. Only change we have to make is to add a source term b_i=(100 W/m^3)(0.5 m)(1m^2)=50 W.

Solution:

The simultaneous equations are:

$$\begin{aligned} 84\,T_2 &= 28\,T_3 + 20888 + 50 \\ 56\,T_3 &= 28\,T_4 + 28\,T_2 + 50 \\ 56\,T_4 &= 28\,T_5 + 28\,T_3 + 50 \\ 36.49\,T_5 &= 28\,T_4 + 2528.4 + 50 \end{aligned}$$

Solving these we get,

T_1=373;	T_2=369.99;	T_3=362.18
T_4=352.59	T_5=341.21;	T_6=334.66

Energy balance is

$$q_0 = -(14)\left[\frac{369.99 - 373}{0.5(0.5)}\right] = 168.56\,W$$

$$q_L = q_{conv}''(A) = (10)(334.66 - 298)(1.0) = 369.6\,W$$

and

$$q_{gen} = \dot{q}(\text{volume}) = (100\,W/m^3)(2\,m \times 1\,m^2) = 200\,W$$

Thus, the error in energy balance is

$$\text{Error} = \frac{q_{gen} + q_{in} - q_{out}}{q_{out}} = \frac{200 + 168.56 - 369.6}{369.6} = -0.028 \ (2.8 \ \%)$$

To improve the accuracy we need to have more control volumes.

Three example problems shown above illustrate the basic methodology of the finite volume numerical method. We find that there are basically no differences in formulating the numerical solutions for all three problems. Only differences appear in the evaluation of source terms and the boundary temperatures. Thus a general solution approach for any combinations of boundary conditions and source terms can be formulated. This flexibility of numerical method is an advantage over the exact analytical method.

Source Term Linearization

We have considered source terms that remain constant throughout the domain. There are many types of source terms that are not uniform but depend on dependent variables. Let us revisit Example 2 considered in a previous section. If we remove the insulation around the aluminum bar and subject its surface to the same convective boundary condition imposed at x=L, energy balance equation becomes [2],

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{h_f P}{A}(T_f - T) \tag{22}$$

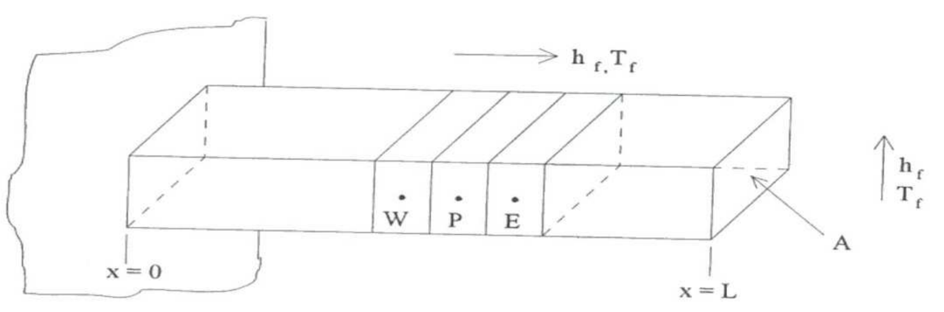


Figure 5 A rectangular fin

where P is the wetted perimeter of the cross section and A is the cross section area.

Thus we see that source term for this case is

$$S = \frac{h_f P}{A} T_f - \frac{h_f P}{A} T = S_C + S_P T \tag{23}$$

where S_c is the constant part and S_p is the slope of temperature dependency. This is a linear equation in T. Thus source term is already in a linear form. In many cases however, we have to linearize the source term for successful numerical calculations. With a linear form of source term, steady 1-D conduction equation can be written as

$$-\frac{dJ_X}{dx} + S_C + S_P T = 0 \tag{24}$$

Integrating the source term in Eq. (24) over a control volume surrounding "p" as we did before, we obtain

$$\int_0^1 \int_0^1 \int_w^e (S_C + S_P T)\ dxdydz = S_C \Delta x + S_P T_P \Delta x$$

The resulting finite volume equation is

$$a_P T_P = a_E T_E + a_W T_W + b \tag{25}$$

where

$$a_E = \frac{k_e (1x1)}{(\delta x)_e}$$

$$a_W = \frac{k_w (1x1)}{(\delta x)_w} \tag{26.a-d}$$

$$a_P = a_E + a_W - S_P \Delta x(1x1)$$

and

$$b = S_C \Delta x(1x1)$$

The difference between Eq. (10) and Eq. (25) is due to the linearized source term. If $S_p=0$, they are the same equation. In linearizing the source term, it is absolutely necessary to have $S_p \leq 0$. Otherwise a_P can become negative, and the resulting numerical solution can become nonphysical [1].

Example 4

Reexamine Example 2 with a change such that energy loss through the surface due to convection is included. Use the same physical and boundary conditions. Assume 1 by 1 cross sectional area of the bar.

Solution:

Linearized source term is

$$S = \frac{h_f P}{A} T_f - \frac{h_f P}{A} T = S_C + S_P T_P$$

Thus, for all i's,

$$S_{C_i} = (\frac{h_f P}{A} T_f)_i = \frac{(10 \ \mathrm{W/m^2K})(4\,\mathrm{m})}{1\,\mathrm{m}^2}(298 \ \mathrm{K}) = 11920 \ \mathrm{W/m^3}$$

and

$$S_{P_i} = -(\frac{h_F P}{A})_i = -40 \ \mathrm{W/m^3K}$$

$$d_i = S_{C_i} \Delta x_i (A_i) = (11920\,\mathrm{W/m^3})(0.5\,\mathrm{m})(1\,\mathrm{m^2}) = 5960 \ \mathrm{W}$$

$$-S_{P_i} \Delta x_i A_i = -(-40\,\mathrm{W/m^3K})(0.5\,\mathrm{m})(1\,\mathrm{m^2}) = 20\,\mathrm{W/K}$$

With these changes, simultaneous equations become (see example 2)

$$(84 + 20)\, T_2 = 28\, T_3 + 20888 + 5960$$
$$(56 + 20)\, T_3 = 28\, T_4 + 28\, T_2 + 5960$$
$$(56 + 20)\, T_4 = 28\, T_5 + 28\, T_3 + 5960$$
$$(36.49 + 20)\, T_5 = 28\, T_4 + 2528.4 + 5960$$

Solving these, we have

$$T_1 = 373;\ T_2 = 343.83;\ T_3 = 318.24$$
$$T_4 = 307.11;\ T_5 = 302.49;\ T_6 = 301.81$$

Taking an energy balance, we have

$$q_{x=0} = (1\mathrm{x}1)(-14)(\frac{343.83 - 373}{0.5(0.5)}) = 1633.52\ \mathrm{W}$$

$$q_{x=L} = (1\mathrm{x}1)(10\ \mathrm{W/m^2K})(301.81 - 298)\ \mathrm{K} = 38.1\ \mathrm{W}$$

$$q_{wall} = \sum_{2}^{5} A_{S_i} h_f (T_i - T_f) = (20\ m^2)(45.83 + 20.24 + 9.11 + 4.49) = 1593.4\ W$$

We see about 0.12 % errors in the energy balance.

Further Considerations on Source Term Linearization

The source terms are, in general, nonlinear function of dependent variables. Suppose we have a source term as

$$S = 4 - 5T^3 \tag{27}$$

This is a nonlinear equation since coefficient ($-5T^2$) depends on temperature itself. Several methods can be used to linearize this expression.

<u>Method 1:</u> Let $S_c = 4 - 5(T_p^*)^3$ and $S_p = 0$. T_p^* is the known temperature from the previous calculation.

<u>Method 2</u>: Let $S_c = 4$ and $S_p = -5(T_p^*)^2$.

<u>Method 3</u>: Use the Taylor series expansion of S(T) about T_p^*.

$$\begin{aligned} S(T_P) &= S(T_P^*) + \left(\frac{dS}{dT}\right)_{T_P^*}(T_P - T_P^*) + \ldots\ldots \\ &= 4 - 5T_P^{*3} + (0 - 15T^2)_{T_P^*}(T_P - T_P^*) + \ldots\ldots \\ &= 4 - 5T_P^{*3} + 15T_P^{*3} - 15T_P^{*2}T_P + \ldots \\ &= 4 + 10T_P^{*3} - 15T_P^{*2}T_p \end{aligned}$$

Thus $S_c = 4 + 10(T_p^*)^3$ and $S_p = --15(T_p^*)^2$. Graphical interpretation of three methods is shown in Fig. 6. Methods 1 and 2 are underestimating the dependency of S on T while method 3 gives an accurate dependency. Note that T_p and T_p^* are very close.

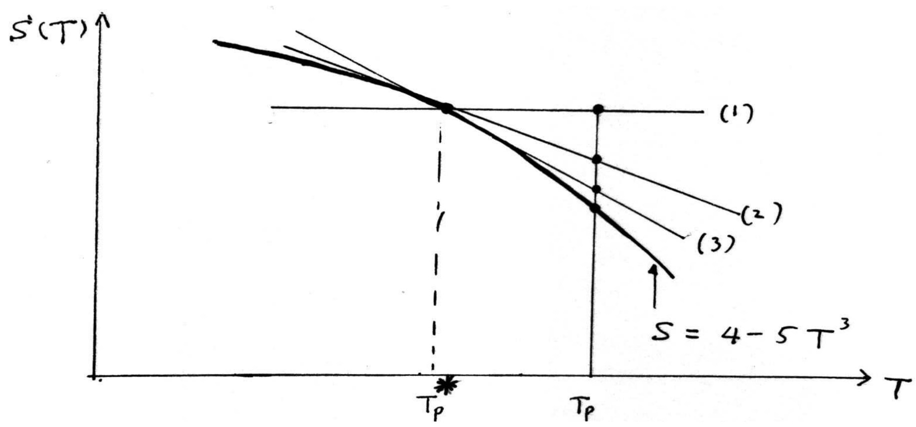


Figure 6 Graphical representation of linearized source term

Source term linearization can be used to specify the desired values of dependent variable at any control volumes. Suppose we like to specify a temperature at an interior control volume, say $T_p = T_{desired}$. To accomplish this let

$$S_c = 10^{20} \text{ x } T_{desired} \text{ and } S_p = -10^{20}.$$

With this choice of source term, we have

$$T_P = \frac{a_E T_E + a_W T_W + b}{a_P} = \frac{a_E T_E + a_W T_W + S_C \Delta x(1x1)}{a_E + a_W - S_P \Delta x(1x1)}$$

$$= \frac{S_C \Delta x(1x1)}{- S_P \Delta x(1x1)} = \frac{10^{20} T_{desired}}{-(-10^{20})} = T_{desired}$$

This is a very useful method to specify a value of dependent variable inside a calculation domain.

Source term linearization can also be used to stabilize numerical calculation when the physics dictates that the dependent variable should remain always positive, such as density, absolute temperature, and energy. Suppose we want to calculate the absolute temperature, T, for a problem in which source term is given by $S = S_{const}$ and S_{const} is a very large negative number. Under certain conditions, T_p can go negative because of negative large source term. To prevent such occurrence, let the source term be linearized by

$$S = 0 + S_{const} \frac{T_P}{T_P^*}$$

Then

$$S_C = 0 \quad ; \quad S_P = \frac{S_{const}}{T_P^*} \leq 0$$

Thus T_p can approach zero but never becomes negative.

Interfacial Conductivity

Up to this point of our discussions, it is implicitly assumed that the conductivity at the interfaces between two control volumes is known and is equal to the conductivity of material in an adjacent control volume. This is perfectly acceptable if conductivity is constant and uniform throughout the material as has been the case for all example problems. If conductivity is not uniform due to different materials or due to a dependency on temperature, interfacial conductivity should be carefully evaluated to ensure conservational principle.

Consider two control volumes with different conductivity as shown in the Fig. 7. It is very tempting to evaluate the conductivity at the interface "e" by linear interpolation. That is

$$k_e = k_P + (k_E - k_P)\frac{(\delta x)_{e^-}}{(\delta x)_e} \tag{28}$$

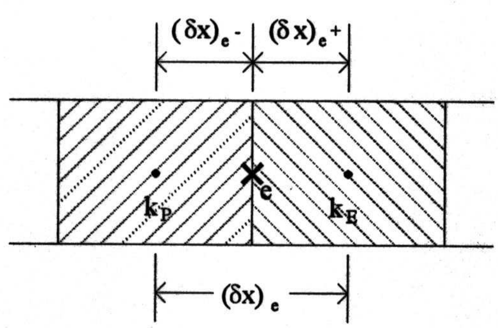


Figure 7 Interface conductivity at east interface

If two control volumes are the same size, Eq.(28) reduces to $k_e = 0.5\,(k_P + k_E)$, which is the arithmetic mean value of conductivity of two adjacent control volumes. This approach is simple yet it can lead to nonphysical solutions. As an example, let $k_E=0.$ (insulator), then conduction flux at the interface should be zero. But, $k_e = 0.5k_P$ and the heat flux at the interface is

$$q_e \;=\; -\,k_e \frac{T_E - T_P}{(\delta x)_e} = -0.5\,k_P \frac{T_E - T_P}{(\delta x)_e} \neq 0$$

which is not physically correct. This will introduce an energy sink or source through the interface "e".

To derive a physically correct interfacial conductivity, we consider an energy balance at the interface. Define a fictitious control volume of zero thickness at the interface (Fig. 8). Energy entering from the left surface is, using the first-order accurate difference scheme,

$$q_{e-} = -\,k_P \frac{T_e - T_P}{(\delta x)_{e-}} \tag{29a}$$

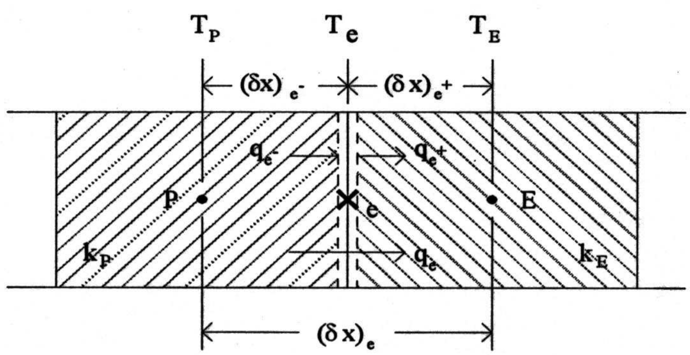


Figure 8 Energy balance at the interface control volume

And the energy leaving through the right surface is

$$q_{e+} = -\,k_E \frac{T_E - T_e}{(\delta x)_{e+}} \tag{29b}$$

where T_e is the temperature at the interface. Rearranging Eqs. (29a) and (29b), we have

$$(q_{e-})\frac{(\delta x)_{e-}}{k_P} = T_P - T_e \tag{30a}$$

and

$$(q_{e+})\frac{(\delta x)_{e+}}{k_E} = T_e - T_E \tag{30b}$$

Adding Eqs.(30a) and (30b) and noting that $q_{e-} = q_{e+} = q_e$, we have

$$q_e = \frac{T_P - T_E}{\frac{(\delta x)_{e-}}{k_P} + \frac{(\delta x)_{e+}}{k_E}} = -\frac{k_P k_E (\delta x)_e}{k_E(\delta x)_{e-} + k_P(\delta x)_{e+}} \cdot \frac{T_E - T_P}{(\delta x)_e} \tag{31a}$$

Conduction flux through the interface "e" is expressed as

$$q_e = -k_e \frac{T_E - T_P}{(\delta x)_e} \tag{31b}$$

Comparing Eqs.(31a) and (31b) we conclude that

$$k_e = \frac{k_P k_E (\delta x)_e}{k_E(\delta x)_{e-} + k_P(\delta x)_{e+}} \tag{32}$$

If two control volumes are of the equal size, Eq.(32) becomes

$$k_e = \frac{2 k_P k_E}{k_E + k_P} \tag{33}$$

which is the harmonic mean value of adjacent conductivities. If $k_E=0$, then $k_e=0$ as expected. Temperature at the interface "e" can be calculated by using Eq.(30a) and Eq.(30b),

$$T_e = \frac{\frac{k_P}{(\delta x)_{e-}} T_P + \frac{k_E}{(\delta x)_{e+}} T_E}{\frac{k_P}{(\delta x)_{e-}} + \frac{k_E}{(\delta x)_{e+}}} \tag{34}$$

All diffusion coefficients at the interface "e" will be evaluated by this method. Thus

$$\Gamma_e = \frac{(\delta x)_e}{(\delta x)_{e-} / \Gamma_P + (\delta x)_{e+} / \Gamma_E} = \frac{\Gamma_P \Gamma_E (\delta x)_e}{\Gamma_E (\delta x)_{e-} + \Gamma_P (\delta x)_{e+}} \tag{35}$$

This formulation of diffusion coefficient at the interface allows the present numerical method work for conduction in composite materials, fluid flow in complex geometries, conjugate heat transfer and heat transfer with phase changes and other complex transport processes.

Interfacial conductivity including a contact surface resistance, R_t, can be obtained by the same method (Fig. 9). Heat flux from the left, through the contact surface and to the right surface is, respectively;

$$q_{e-} = -k_P \frac{T_{e-} - T_P}{(\delta x)_{e-}}$$

$$q_R = -\frac{1}{R_t}(T_{e^+} - T_{e^-})$$

and

$$q_{e^+} = -k_E \frac{T_E - T_{e^+}}{(\delta x)_{e^+}}$$

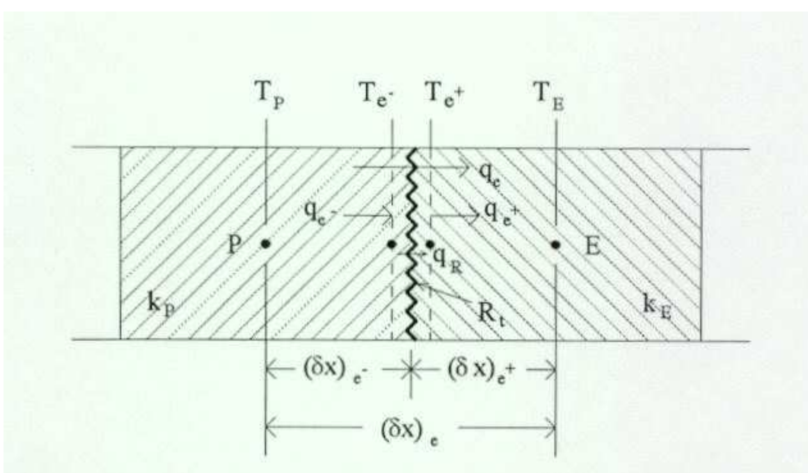


Figure 9 Contact Resistance

Eliminating T_{e-} and T_{e+} from these equations with $q_{e-} = q_{e+} = q_R = q_e$, we get

$$k_e = \frac{k_P k_E (\delta x)_e}{k_E (\delta x)_{e^-} + k_P (\delta x)_{e^+} + k_E k_P R_t} \tag{36}$$

Temperatures at the contact surface are (left as an exercise),

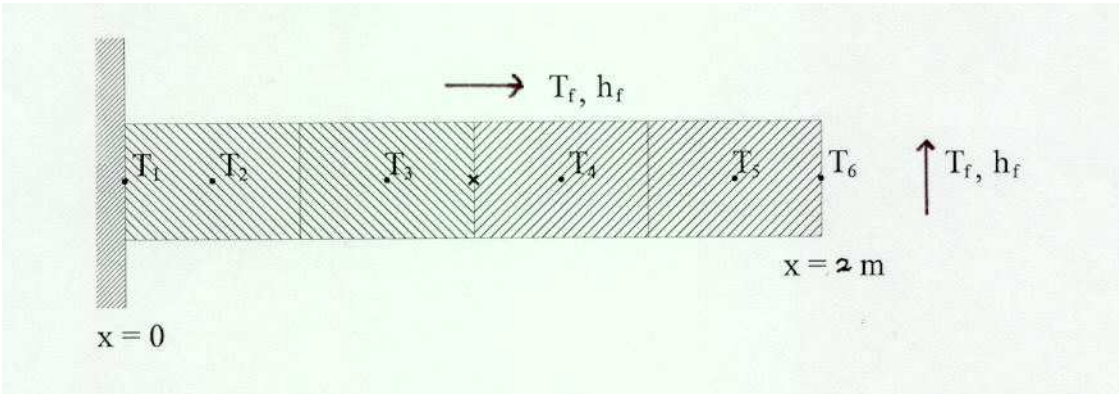
$$T_{e-} = \frac{\frac{k_P}{(\delta x)_{e^-}} T_P + \frac{k_E}{(\delta x)_{e^+}} T_E + R_t \frac{k_P}{(\delta x)_{e^-}} \frac{k_E}{(\delta x)_{e^+}} T_P}{\frac{k_P}{(\delta x)_{e-}} + \frac{k_E}{(\delta x)_{e^+}} + R_t \frac{k_P}{(\delta x)_{e^-}} \frac{k_E}{(\delta x)_{e^+}}} \tag{37a}$$

and

$$T_{e^+} = \frac{\frac{k_P}{(\delta x)_{e^-}} T_P + \frac{k_E}{(\delta x)_{e^+}} T_E + R_t \frac{k_P}{(\delta x)_{e^-}} \frac{k_E}{(\delta x)_{e^+}} T_E}{\frac{k_P}{(\delta x)_{e^-}} + \frac{k_E}{(\delta x)_{e^+}} + R_t \frac{k_P}{(\delta x)_{e^-}} \frac{k_E}{(\delta x)_{e^+}}} \tag{37b}$$

Example 5

Consider a fin problem as discussed in Example 4 with two changes. Conductivity of the bar is 14 W/m.K for the first half of the bar and 24 W/m.K for the remaining half. Energy is being generated uniformly at 100 W/m^3. All other conditions remain same as in Example 4; length of the bar is 2 m, cross sectional area is 1.0 m^2, base temperature is 373 K and all surfaces are exposed to convective heat transfer with h_f=10 W/m^2.K and T_f=298 K.



Solution:

Differential equation for this problem is given by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \frac{h_f P}{A}(T_f - T) + \dot{q} = 0$$

Therefore, the source term is

$$S = \dot{q} + \frac{h_f P}{A} T_f - \frac{h_f P}{A} T$$

Recall that

$$a_E = \frac{k_e (1 x 1)}{(\delta x)_e}$$

where the interface conductivity at 'e' is given by Eq.(36), i.e.,

$$k_e = \frac{k_E k_P (\delta x)_e}{(\delta x)_{e^-} k_E + (\delta x)_{e^+} k_P}$$

Substituting k_e, a_E becomes

$$a_E = \frac{k_E k_P (1 x 1)}{(\delta x)_{e^-} k_E + (\delta x)_{e^+} k_P}$$

Thus

$$b_i = a_E = \frac{k_{i+1} k_i (1 x 1)}{\frac{1}{2} \Delta x_i k_{i-1} + \frac{1}{2} \Delta x_{i+1} k_i} = \frac{2 k_i k_{i+1} (1x1)}{\Delta x_i k_{i+1} + \Delta x_{i+1} k_i}$$

and

$$c_i = a_W = \frac{2 k_{i-1} k_i (1x1)}{\Delta x_{i-1} k_i + \Delta x_i k_{i-1}}$$

Evaluating the coefficients for each i=2,..,5, we have

$$b_2 == \frac{2 k_2 k_3 (1 x 1)}{\Delta x_2 k_3 + \Delta x_3 k_2} = \frac{(2)(14)(14)(1 x 1)}{0.5(14) + 0.5(14)} = 28$$

$$c_2 = \frac{2 k_1 j_2 (1 x 1)}{\Delta x_1 k_2 + \Delta x_2 k_1} = \frac{(2)(14)(1 x 1)}{0 + 0.5(14)} = 56$$

$$b_3 = 35.368; \quad c_3 = 28$$

$$b_4 = 48; \quad c_4 = 35.368$$

$$b_5 = 96; \quad c_5 = 48$$

From the source term, we have

$$Sp_i \Delta x_i (1x1) = -\frac{h_f P}{A} \Delta x_i (1x1) = -20$$

Thus the coefficients $a_i = b_i + c_i - Sp_i \Delta x_i (1x1)$ are

$$a_2 = b_2 + c_2 - S_{P_2} \Delta x_2 (1x1) = 28 + 56 - (-20) = 84 + 20 = 104$$

$$a_3 = 83.368; \quad a_4 = 103.368; \quad a_5 = 144 + 20 = 164$$

Constant parts of the source terms are, for all i's,

$$d_i = S_{C_i} \Delta x_i (1x1) = (\dot{q} + \frac{h_f P}{A} T_f) \Delta x_i (1x1) = (100 + 11920)(0.5)(1x1) = 6010$$

Thus the simultaneous equations are

$$104\,T_2 = 28\,T_3 + 56\,T_1 + 6010$$

$$83.368\,T_3 = 35.368\,T_4 + 28\,T_2 + 6010$$

$$103.368\,T_4 = 48\,T_5 + 35.368\,T_3 + 6010$$

$$164\,T_5 = 96\,T_6 + 48\,T_4 + 6010$$

Boundary conditions are, $T_1 = 373$ K and

$$T_6 = \frac{h_f T_f + \frac{2}{\Delta x_5} k_5 T_5}{h_f + \frac{2}{\Delta x_5} k_5}$$

$$= 28.11 + 0.9056\,T_5$$

Substituting the boundary conditions and after rearranging, we have

$$104T_2 = 28T_3 + 26898$$

$$83.368\,T_3 = 35.368\,T_4 + 28\,T_2 + 6010$$

$$103.368\,T_4 = 48\,T_5 + 35.368\,T_3 + 6010$$

$$77.06\,T_5 = 48\,T_4 + 8708.56$$

Solving for the unknown temperatures, we get

$$T_2 = 344.51; T_3 = 318.96; T_4 = 309.18; T_5 = 305.60$$

Boundary temperature T_6 is

$$T_6 = 28.11 + 0.9056 * 305.6 = 304.86$$

Temperature at the interface between two different materials is given by Eq.(34),

$$T_e = \frac{\dfrac{k_3}{0.5\Delta x_3} T_3 + \dfrac{k_4}{0.5\Delta x_4} T_4}{\dfrac{k_3}{0.5\Delta x_3} + \dfrac{k_4}{0.5\Delta x_4}} = 312.78 \text{ K}$$

This temperature is lower than the linearly interpolated temperature (314.07 K). Fig. E5 shows the temperature distribution.

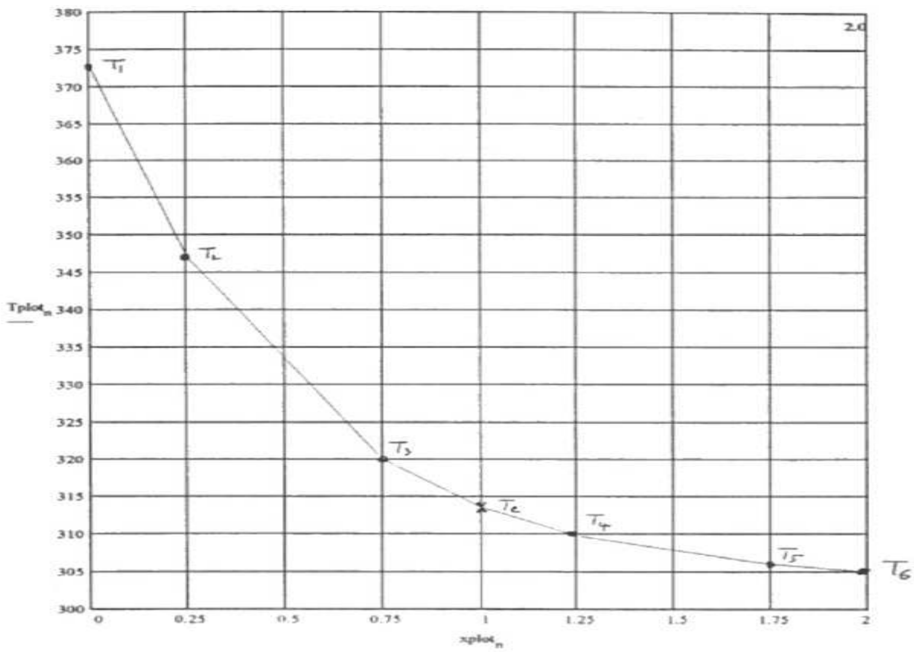


Figure E5 Temperature distribution

To check the energy balance, we calculate

$$q_{x=0} = -k_2\left[\frac{T_2 - T_1}{0.5\Delta x_2}\right](1x1) = 1595.44 \quad W$$

$$q_{gen} = \dot{q}\,(\text{volume}) = (100\ \text{W / m}^3)(2\ \text{m x}\ 1\,\text{m}^2) = 200\ \text{W}$$

$$q_{x=L} = h_f\,(T_6 - T_f)(1x1) = 68.6\ W$$

$$q_{surface} = \sum_{i=2}^{5} h_f\, As_i\,(T_i - T_f) = 1725\ W$$

Thus, $q_{in} + q_{gen} = 1795.44$ W, and $q_{out} = 1793.6$ W. The error is about 0.1 %.

Nonlinearity

Finite volume equation, Eq.(13) can be solved for the unknown temperatures if all coefficients and source terms are explicitly known values. For the linear problems, coefficients and source terms are known. Therefore, simultaneous equations are to be solved only once for the unknown temperatures. For the nonlinear problems, however, coefficients and source terms cannot be evaluated exactly because they depend on the temperatures that are unknown.

Suppose the thermal conductivity is a function of temperature, $k = k(T)$. Actually, thermal conductivity of most materials depends strongly on the temperature. To evaluate the coefficients, we need to know the value of thermal conductivity, which, in turn, requires the known temperatures. To initiate the calculation, we have to assume temperature, T^* and evaluate $k = k(T^*)$. With the guessed coefficients, simultaneous equations can be solved for the unknown temperature, T. If newly calculated temperature, T, is close enough to the guessed temperature T^*, we have solution. If not, let $T^* = T$ and repeat the calculation until

$$\text{Error} = \text{abs}\left(\frac{T - T^*}{T}\right) < \varepsilon \qquad (38)$$

where ε is an error tolerance.

Nonlinearity can also arise from the nonlinear source terms. Returning to Example 4 we note that source term contains a convective coefficient, h_f. Convective coefficient depends, in general, on the film temperature that is the average of the unknown surface temperature (T) and the fluid temperature (T_f).

Nonlinear boundary conditions also make the problem nonlinear. Returning to Example 2, and consider the convective boundary at x=L. Recall that the boundary temperature T_6 was given by (Eq.(b) in Example 2),

$$T_6 = 45.15 + 0.84848T_5$$

which contains an unknown temperature T_5. This is an example of nonlinear boundary conditions.

In Example 2, T_6 was substituted into the finite volume equation for the fifth (i=5) control volume and manually rearranged to eliminate the nonlinearity through an algebraic manipulation;

$$36.49T_5 = 28T_4 + 2528.4,$$

which does not contain any unknown terms.

Without this algebraic step, however, finite volume equation for the fifth control volume would contain nonlinear term, T_6^*, which is

$$T^*_6 = 45.15 + 0.84848T^*_5$$

where * denotes values at the previous iteration. Substituting this into Eq.(b.4) in Example 2, we have

$$\begin{aligned} 84T_5 &= 28T_4 + 56T_6^* \\ &= 28T_4 + 56(45.15 + 0.84848T_5^*) \end{aligned}$$

which contains nonlinear source terms due to T_5^*. Since source term contains the unknown temperatures, simultaneous equations must be solved iteratively until solution converges within a prescribed tolerance. That is

$$\text{abs}\left(\frac{T_5^* - T_5}{T_5}\right) \le \varepsilon$$

Nonlinear boundary conditions, such as radiative boundary condition, require such iterations since the boundary temperature often cannot be explicitly expressed in terms of the interior temperatures.

Nonlinearity is quite common in most transport processes involving fluid flow. Numerical solution of nonlinear problems do not entail additional new concepts other than that simultaneous equations must be solved many times until converged solutions are obtained. This is a task computer can handle rather rapidly.

Tri-Diagonal-Matrix-Algorithm (TDMA)

By introducing the boundary conditions at both ends of calculation domain, N-simultaneous equations can be put into a tri-diagonal matrix form. Tri-diagonal form of matrix equations can be easily solved by the forward and backward substitution method as has been utilized in our example problems. Thomas algorithm is a method based on the same principle and is useful to solve a large tri-diagonal matrix [1].

Finite volume equation in index notation can be rearranged into the following form;

$$-c_i T_{i-1} + a_i T_i - b_i T_{i+1} = d_i \tag{39}$$

for $i=2,3,\ldots i-1,i,i+1,\ldots,N,N+1$. Implementing the boundary conditions at $x=0$ and $x=L$, and expanding Eq.(39), we get a tri-diagonal matrix:

$$\begin{pmatrix}
a_2 & -b_2 & & & & & & & \\
-c_3 & a_3 & -b_3 & & & & & & \\
 & -c_4 & a_4 & -b_4 & & & & & \\
 & & \cdot & \cdot & \cdot & & & & \\
 & & & \cdot & \cdot & \cdot & & & \\
 & & & & -c_i & a_i & -b_i & & \\
 & & & & & \cdot & \cdot & \cdot & \\
 & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & -c_N & a_N & -b_N \\
 & & & & & & & -c_{N+1} & a_{N+1}
\end{pmatrix}
\begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ \cdot \\ \cdot \\ T_i \\ \cdot \\ \cdot \\ T_N \\ T_{N+1} \end{pmatrix}
=
\begin{pmatrix} d'_2 \\ d_3 \\ d_4 \\ \cdot \\ \cdot \\ d_i \\ \cdot \\ \cdot \\ d_N \\ d'_{N+1} \end{pmatrix} \tag{40}$$

where d_2' and d_{N+1}' are

$$d'_2 = d_2 + c_2 T_1 \tag{41a}$$

and

$$d'_{N+1} = d_{N+1} + b_{N+1} T_{N+2} \tag{41b}$$

due to the boundary conditions at $x=0$ and $x=L$. Note that $c_2=b_{N+1}=0$ after implementing the boundary conditions.

In the forward substitution, we seek a relation

$$T_i = P_i T_{i+1} + Q_i \tag{42}$$

right after we obtain

$$T_{i-1} = P_{i-1} T_i + Q_{i-1} \tag{43}$$

Substituting Eq.(43) into Eq.(39) for T_{i-1}, we have

$$-c_i(P_{i-1}T_i + Q_{i-1}) + a_iT_i - b_iT_{i+1} = d_i$$

Solving for T_i, we get

$$T_i = \frac{b_i}{a_i - c_i P_{i-1}} T_{i+1} + \frac{d_i + c_i Q_{i-1}}{a_i - c_i P_{i-1}} \tag{44}$$

Now comparing Eq.(42) and Eq.(44), we obtain two recursive relations

$$P_i = \frac{b_i}{a_i - c_i P_{i-1}} \tag{45a}$$

and

$$Q_i = \frac{d_i + c_i Q_{i-1}}{a_i - c_i P_{i-1}} \tag{45b}$$

Forward substitution is to evaluate P_i and Q_i for $i=2,3,\ldots,N+1$;

$$i = 2 \qquad P_2 = \frac{b_2}{a_2 - c_2 P_1} = \frac{b_2}{a_2} \quad ; \quad Q_2 = \frac{d'_2 + c_2 Q_1}{a_2 - c_2 P_1} = \frac{d'_2}{a_2}$$

$$i = 3 \qquad P_3 = \frac{b_3}{a_3 - c_3 P_2} \quad ; \quad Q_3 = \frac{d_3 + c_3 Q_2}{a_3 - c_3 P_2}$$

....

$$i = i \qquad P_i = \frac{b_i}{a_i - c_i P_{i-1}} \quad ; \quad Q_i = \frac{d_i + c_i Q_{i-1}}{a_i - c_i P_{i-1}}$$

....

$$i = N \qquad P_N = \frac{b_N}{a_N - c_N P_{N-1}} \quad ; \quad Q_N = \frac{d_N + c_N Q_{N-1}}{a_N - c_N P_{N-1}}$$

$$i = N+1 \qquad P_{N+1} = \frac{b_{N+1}}{a_{N+1} - C_{N+1} P_N} = 0 \quad ; \quad Q_{N+1} = \frac{d_{N'+1} + c_{N+1} Q_N}{a_{N+1} - c_{N+1} P_N}$$

With the known P_i and Q_i, Eq.(42) yields

$$T_2 = P_2 T_3 + Q_2$$

$$T_3 = P_3 T_4 + Q_3$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$T_i = P_i T_{i+1} + Q_i$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$T_N = P_N T_{N+1} + Q_N$$

$$T_{N+1} = P_{N+1} T_{N+2} + Q_{N+1} = 0 + Q_{N+1}$$

Since Q_{N+1} is known, T_{N+1} is found from the last equation. Then backward substitution yields T_N, T_{N-1},…., T_3, and T_2.

TDMA is a simple method to use and requires small computer memory space that is proportional to N. It can be used for ADI (Alternating-Direction-Implicit) method commonly adopted for the multi-dimensional problems.

Quasi 1-Dimensional Conduction

In 1-dimensional conduction along the x-direction, the area perpendicular to the heat flow direction remains constant, i.e., A(x)=constant. We have assumed (1x1) cross sectional area in all of our discussions so far. There are many situations in which area changes along the heat flow direction. If temperature variation perpendicular to the x-direction is very small compared to the variation in the x-direction, temperature variation perpendicular to the x-direction may be neglected. In this case, heat transfer becomes a quasi-one-dimensional in x-direction and the steady state heat conduction equation becomes

$$\frac{1}{A}\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + S = 0 \qquad (46)$$

By integrating Eq.(46), it can be easily shown that

$$a_P T_P = a_E T_E + a_W T_W + b \tag{47}$$

where

$$a_E = \frac{k_e A_e}{(\delta x)_e} \tag{48a}$$

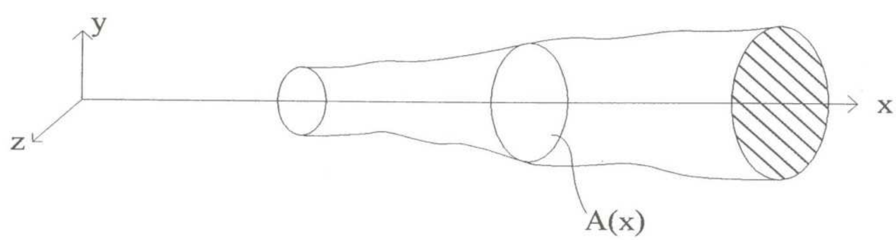
$$a_W = \frac{k_w A_w}{(\delta x)_w} \tag{48b}$$

$$a_P = a_E + a_W - S_P A_P \Delta x \tag{48c}$$

and

$$b = S_C A_P \Delta x \tag{48d}$$

This is a generalization of 1-dimensional finite volume representation of heat conduction equation. The area A(x) can be any arbitrary values. If $A(x)=x$, 1-D cylindrical coordinate and if $A(x)=x^2$, 1-D spherical coordinate system result.



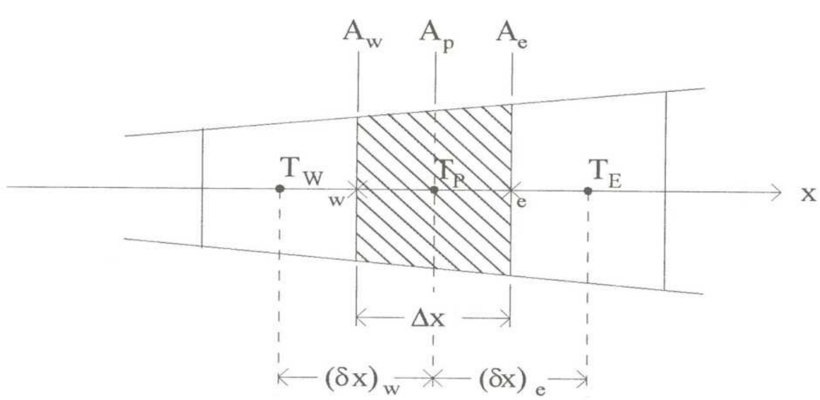


Figure 10 Quasi-one-dimensional conduction

Steady 1-Dimensional Conduction Program

We are now in a position to write a computer program that can solve the general 1-dimensional steady conduction problems. Conductivity of the material may vary throughout the material and may also depend on the temperature as well. The cross-sectional area can be arbitrary. Variety of source terms can be included.

Details of a computer program do vary significantly from one to another, reflecting the styles of individuals who write the programs even though the programs are based on the same numerical method and are designed to solve the same physical problem. There is absolutely no possibility that two programs can be identical in every detail.

A MATLAB program based on the methodology discussed in previous sections is created. This program is named as std1da.m, which solves 1-dimensional steady conduction problems. The flow chart shown in Fig. 11 shows the structure of the program. Detail of the program is illustrated by using an application to conduction through a circular fin.

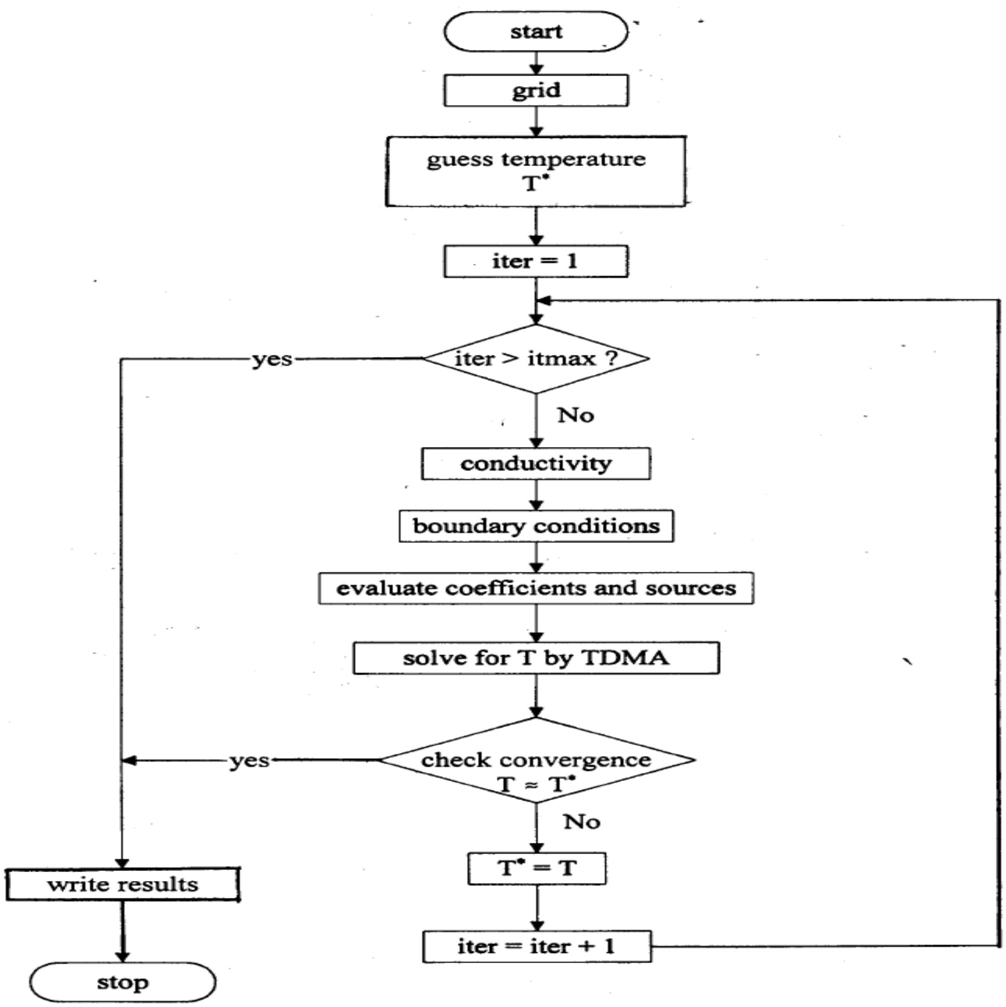


Figure 11 Flow chart for steady 1-D conduction

Description of the problem

Governing differential equation for 1-D heat conduction through a varying cross sectional area is

$$\frac{1}{A}\frac{d}{dx}(Ak\frac{dT}{dx}) + \frac{h_f P}{A}(T_f - T) + \dot{q} = 0 \tag{49}$$

where A(x) is the cross sectional area, P is the wetted perimeter of the cross-section and $\dot{q}$ is the source term. For the present example problem, cross sectional area is of circular shape with 2.5 cm-dia. The fin is made of copper. The length is 1.2 m and the conductivity is assumed to be constant at 401 W/m.K. The base temperature is 473 K and the fin surfaces including the right side boundary are exposed to air at 298 K with a constant convective coefficient of 10 W/m^2K, and $\dot{q} = 0$. The purpose of this example is to highlight the ingredients of the finite volume method discussed so far and compare the results with known exact solution. It is always a good idea to compare numerical solutions with available data either exact or experimental.

Exact solution of this problem is given by [2]

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (\frac{h_f}{mk})\sinh m(L-x)}{\cosh mL + (\frac{h_f}{mk})\sinh mL}$$

where

$$m^2 = \frac{h_f P(x)}{kA(x)}\ ;\ \theta_b = T_B - T_f\ ;\ and \qquad \theta = T - T_f$$

Structure of the program

Number of control volumes used in this example is 10 and they are uniform in sizes. Cross sectional area is also constant, ac(x)=$4.91 \times 10^{-4}\ m^2$. After defining the geometry of the problem, initial temperature is assumed at 373 K. This initial temperature can be arbitrary value. If we assumed initial temperature close to the steady solution, number of iterations will decrease. Temperature, tep= T^* is the previous iteration level temperature saved to check the convergence of the solution. ITER is the iteration counter. Maximum number of iteration is set to maxiter=200. Thermal conductivity of the fin is next given in the program. It is assumed to be constant in this example. Boundary conditions at both ends of the fin are next prescribed. At x=0, $T_1 = T_b = 473$ K. At x=L, a convective boundary condition is used and temperature T_{N+2} is found by energy balance which contains unknown temperature, T_{N+1}. A better treatment of nonlinear boundary condition is discussed in the next section. Conductance between the control volumes are next evaluated by using Eqns.(11a,b), where interfacial conductivities are evaluated by Eq.(33). For the present problem, of course, interfacial conductivity remains constant since the material is homogeneous and conductivity is assumed to be temperature independent. Constant part (S_C) and temperature dependent part (S_P) of source terms are

$$Sc = \left(\frac{h_f P}{A}\right) T_f + \dot{q} = 4.768x10^5 \text{ and } Sp = -\frac{h_f P}{A} = -1600.$$

and the source term b is then evaluated. The coefficients for the simultaneous equations are then calculated. Simultaneous equations are put into a tri-diagonal-matrix form by incorporating the boundary conditions. TDMA method is used to solve these set of simultaneous equations. After solution is obtained, convergence is tested by

$$\frac{abs(T_i - T_i^*)}{T_i} \le \varepsilon = 10^{-6}$$

for i=1,…,N+2. If convergence test is met, solution is obtained (iflag=0). Otherwise (iflag=1), the guessed temperatures are replaced by the just calculated temperatures and the solution steps are repeated until solution is converged or iteration count (iter) exceeds a preset maximum iteration number (maxiter). The latter option is needed to escape from an endless do-loop created by programming mistakes. At the end of calculation, number of iterations required to reach the solution and the temperatures at each control volumes are printed.

```
%std1da.m.
%steady, 1-dimensional conduction with varying cross section area
%,nonuniform condutivity and sources. Finite volume formulation
%using matlab program. (By Dr. S. Han, Sep 3, 2008)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%specify the number of control volumes
n=10;%number of control volumes
maxiter=100;%maximum iteration number
np1=n+1;
np2=n+2;
np3=n+3;
%define calculation domain
tl=1.2; %total length of the bar
delx=tl/n;
dx=ones(1,np2);
dx=delx*dx;
%replace fictitious boundary volume size
dx(1)=1.0e-10;
```

```
dx(np2)=1.0e-10;
%assign x-coordinate
x(1)=0;
for m=1:np2
    x(m+1)=x(m)+dx(m);
end
%define cross-sectional area
dia=0.025;
for i=1:np3
    ac(i)=pi/4*dia^2; %constant cross section
end
%prescribe intitial temperatures for all control volumes
for i=1:np2
    te(i)=373;
    tep(i)=te(i);
end
%iteration for convergence
iter=0;
iflag=1;
%iteration loop for the convergence
while iflag==1      % end is at the end of program *********
%prescribe thermal conductivity
    for i=1:np2
        tk(i)=401;
    end
%prescribe boundary temperature
    te(1)=473;%at the left boundary given temperature
    hf=10;% right boundary given flux
    tf=298;
    te(np2)=te(np1)-hf*(te(np2)-tf)/(tk(np1)/...
        (.5*dx(np1)));
%initialize the coefficients for tdma matrix
%evaluate the diffusion conductance and source terms
for i=2:np1
% diffusion conductance
    ke=tk(i)*tk(i+1)*(dx(i)+dx(i+1))/...
        (dx(i)*tk(i+1)+dx(i+1)*tk(i));
    de=2.0*ke*ac(i+1)/(dx(i)+dx(i+1));
    ae=de;
    kw=tk(i)*tk(i-1)*(dx(i)+dx(i-1))/...
        (dx(i-1)*tk(i)+dx(i)*tk(i-1));
    dw=2.0*kw*ac(i)/(dx(i-1)+dx(i));
    aw=dw;
%source term evaluation
%    sp=-1600;
%    sc=4.768e5;
    per=pi*dia;
     sp=-hf*per/ac(i);
     sc=hf*per/ac(i)*tf;
    ap=ae+aw-sp*dx(i)*0.5*(ac(i+1)+ac(i));
    b=sc*dx(i)*0.5*(ac(i)+ac(i+1));
%setting coefficients for tdma matrix
    ta(i)=ap;
    tb(i)=ae;
    tc(i)=aw;
```

```
    td(i)=b;
%modify using boundary conditions
    if i==2
        td(i)=td(i)+aw*te(1);
    elseif i==np1
        td(i)=td(i)+ae*te(np2);
    end
end
%solve the simultaneous equations by using tdma
nq=n;
nqp1=nq+1;
nqm1=nq-1;
%forward substitution
beta(2)=tb(2)/ta(2);
alpa(2)=td(2)/ta(2);
for i=3:nqp1
    beta(i)=tb(i)/(ta(i)-tc(i)*beta(i-1));
    alpa(i)=(td(i)+tc(i)*alpa(i-1))/...
        (ta(i)-tc(i)*beta(i-1));
end
%backward substitution
dum(nqp1)=alpa(nqp1);
for j=1:nqm1
    i=nqp1-j;
    dum(i)=beta(i)*dum(i+1)+alpa(i);
end
%end of tdma
%update the temperature
for i=2:np1
    te(i)=dum(i);
end
%check the convergence
for i=1:np2
    errote(i)=abs(te(i)-tep(i))/te(i);
end
error=1.0e-6;
    if (max(errote)>error)
    iter=iter+1;
    tep=te;
    iflag=1;
    else
    iflag=0;
    end
    if iter>maxiter
        break
    end
end % this end goes with the while iflag==1 at the top********
%compare with exact solution to make sure program works correctly
%locate the midpoint of control volumes
for i=1:np2
    xc(i)=0.5*(x(i)+x(i+1));
end
%
m=sqrt(hf*2/(tk(1)*1.25e-2));
thetab=te(1)-tf;
```

```
for i=1:np2
theta(i)=(cosh(m*(tl-xc(i)))+hf/(m*tk(1))*sinh(m*(tl-xc(i))))/...
    (cosh(m*tl)+hf/(m*tk(1))*sinh(m*tl));
end
theta=thetab*theta;
teexact=theta+tf;
teexact=teexact'
%print the results
fprintf('iteration number is %i \n',iter)
disp('steady state temperatures are')
fprintf('%9.3f \n',te')
%plot the result xc.vs.te
plot(xc,te,'-',xc,teexact,'--o')
grid on
title('steady state temp'),xlabel('x(m)'),ylabel('T(K)')
legend('numerical','exact')
```

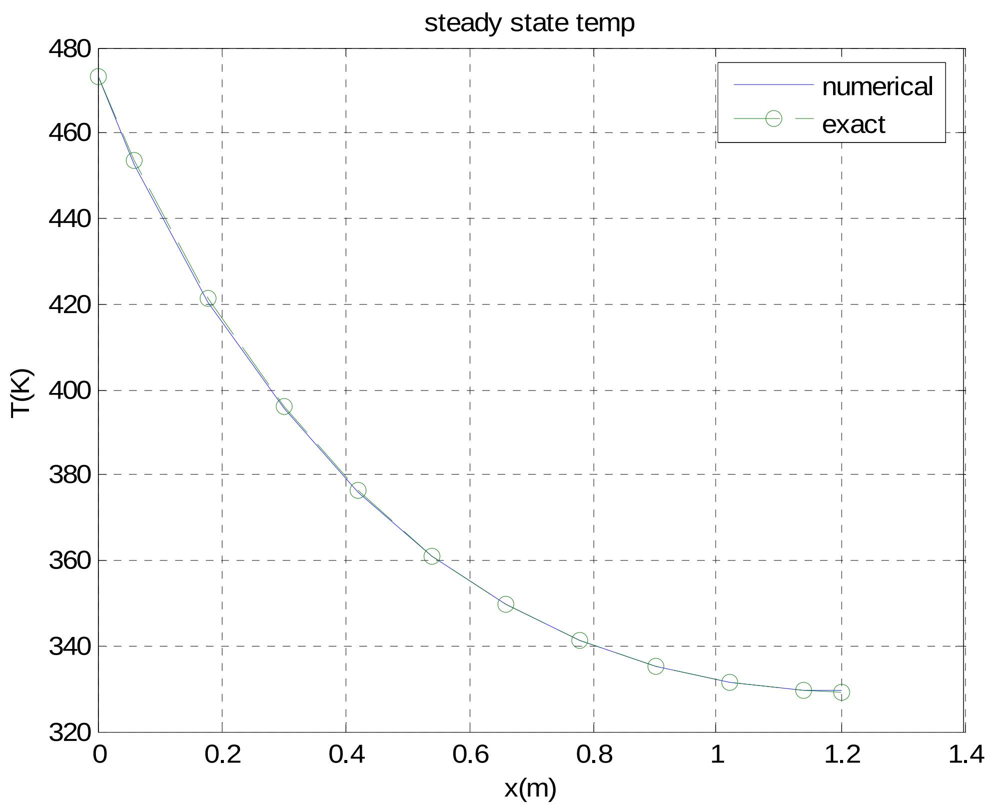


Figure 12. Comparison of numerical and exact solution

A steady state solution is obtained after 76 iterations. Numerical solution and exact solution are in good agreement as shown in fig. 12. A large number of iterations was required due to a nonlinear treatment of the convective boundary condition at x=L. A more general treatment of linearization of nonlinear boundary conditions is discussed in the next section.

Linearization of Boundary Conditions

Boundary temperatures, T_1 and T_{N+2} are not explicitly known when the boundary conditions are given in terms of a heat flux, q''. By taking an energy balance at x=0, it was previously shown that

$$T_1 = T_2 + \frac{1}{2}\left[\frac{\Delta x_2}{k_2} q''\right] \tag{50}$$

where q" is a boundary flux at x=0. Note that q" is assumed to be positive when it is directed along the positive x-direction. Linearized flux q" can be expressed by

$$q'' = q_c + q_P T_1 \tag{51}$$

where q_c is a constant and q_p is the slope of linearized flux equation. Note that we do not require q_P being less than zero in the boundary flux linearization. Substituting Eq.(51) into Eq. (50) and rearranging, we have

$$T_1 = \left[\frac{1}{1 - \frac{1}{2}\frac{\Delta x_2}{k_2} q_P}\right] T_2 + \frac{\frac{1}{2}\frac{\Delta x_2}{k_2} q_C}{1 - \frac{1}{2}\frac{\Delta x_2}{k_2} q_P} \tag{52}$$

Substituting Eq.(52) into the first equation in the simultaneous equations (13) and after rearranging , we get

$$a_2^{'} \; T_2 = b_2 T_3 + d_2^{'} \tag{53}$$

where a'_2 and d'_2 are the modified coefficients and source given by

$$a'_2 = a_2 - c_2 \left[\frac{1}{1 - \frac{1}{2}\frac{\Delta x_2}{k_2} q_p} \right] \tag{54a}$$

and

$$d'_2 = d_2 + c_2 \left[\frac{\frac{1}{2}\frac{\Delta x_2}{k_2} q_C}{1 - \frac{1}{2}\frac{\Delta x_2}{k_2} q_P} \right] \tag{54b}$$

Likewise when the boundary condition at x=L is given in terms of a heat flux, boundary temperature is given by

$$T_{N+2} = T_{N+1} - \frac{1}{2}[\frac{\Delta x_{N+1}}{k_{N+1}} q''] \tag{55}$$

The heat flux q" is expressed as

$$q'' = q_C + q_P T_{N+2}$$

Substituting this in Eq.(55) and solving for T_{N+2} , we have

$$T_{N+2} = \left[\frac{1}{1 + \frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}} q_P} \right] T_{N+1} - \frac{\frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}} q_C}{1 + \frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}} q_P} \tag{56}$$

Replacing T_{N+2} in the last equation in the simultaneous equations (13) and after rearrangement, we have

$$a'_{N+1} T_{N+1} = c_{N+1} T_N + d'_{N+1} \tag{57}$$

where

$$a'_{N+1} = a_{N+1} - b_{N+1}\left[\frac{1}{1+\frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}}q_P}\right] \tag{58a}$$

and

$$d'_{N+1} = d_{N+1} - b_{N+1}\left[\frac{\frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}}q_C}{1+\frac{1}{2}\frac{\Delta x_{N+1}}{k_{N+1}}q_P}\right] \tag{58b}$$

The fin problem discussed earlier is solved again with the linearized convective boundary at x=L. The convective heat flux boundary at x=L is

$$q_{conv}'' = h_f(T_{N+2} - T_f) = q_C + q_P T_{N+2} \tag{59}$$

Thus

$$q_C = -h_f T_f \quad \text{and} \quad q_P = h_f \tag{60}$$

Note that in writing the heat flux at x=L, heat flux directed along the positive x-direction is assumed to be positive. Hence Eq.(59) is used. Modifications are made on std1da.m program to reflect the linearized convective boundary at x=L. This modified program is called std1d.m. In this program, we introduce boundary types 1 (known temperature), 2 (known heat flux) and 3 (periodic). Numerical solution for the fin problem with boundary type 1 at x=0 (bx0=1) and boundary type 2 (bx1=2) was obtained with only 2 iterations which is a dramatic decrease from 76 required without boundary linearization. The copy of the program can be found from CAE lab website, ***www.cae.tntech.edu\~shan\me4730_2012***.

Periodic Boundary Condition

We have considered boundary conditions given by in terms of a known temperature (type 1 boundary) or a known heat flux (type 2 boundary), which may be nonlinear. There is one additional boundary condition, which is periodic (type 3 boundary), frequently occurs in transport processes. As an example, consider a closed loop of aluminum wire with uniform cross sectional area. Energy is generated non-uniformly in the wire and the surface is exposed to a convective environment as shown in the figure. To determine the steady state temperature in the wire, one has to define physical boundary and apply appropriate boundary conditions. There are no obvious physical boundaries since physical domain is an endless loop. If there is symmetry, symmetry boundary conditions can be applied. However, it is not clear whether there is symmetry in the present problem. If we measure temperature of the wire starting from any arbitrary point in the wire, temperature distribution along the wire may change as we move along the wire until we make a full circle. After one circle, temperature change will repeat the same distribution over and over. This type of boundary condition is called "periodic" boundary condition (Fig.13).

A lazy way of handling the periodic boundary condition is to let

$$T_1 = T_{N+1}^* \quad \text{and} \quad T_{N+2} = T_2^* \tag{61}$$

Where "*" means the temperatures at the previous iteration values and therefore act as nonlinear sources, resulting in many iterations for a convergence.

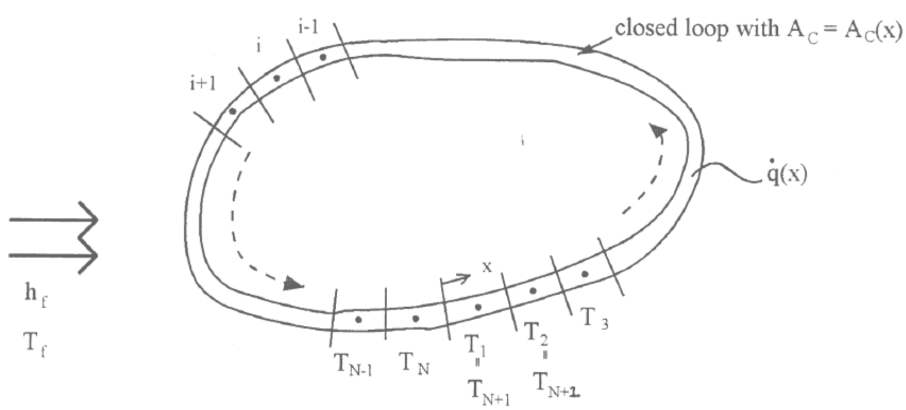


Figure 13 Closed loops with non-uniform energy generation

To accelerate the convergence, nonlinear source terms must be eliminated from the equations. This is accomplished by using a specialized TMDA which explicitly takes advantage of the periodic boundary condition. This method is called the Circular-Tri-diagonal-Matrix-Algorithm (CTDMA). A detail description of the CTDMA is presented in the Appendix [1]. An example is used to illustrate the procedure of using the CTDMA for a problem with a periodic boundary condition. The modified program std1d.m is designed also to handle periodic boundary condition as discussed below.

Problem description

To appreciate the CTDMA, let us consider a copper wire of 2.5 cm-dia forming a loop. The length of the loop is 1.2 m. Energy is generated uniformly at 10^5 W/m^3 in the half of the loop and uniformly at 2x10^5 W/m^3 in the remaining half of the loop. The conductivity of the copper is 401 W/m.K and the convective conditions are h_f=10 W/m^2.K and T_f=298 K. We want to calculate the steady-state temperature distribution in the copper loop.

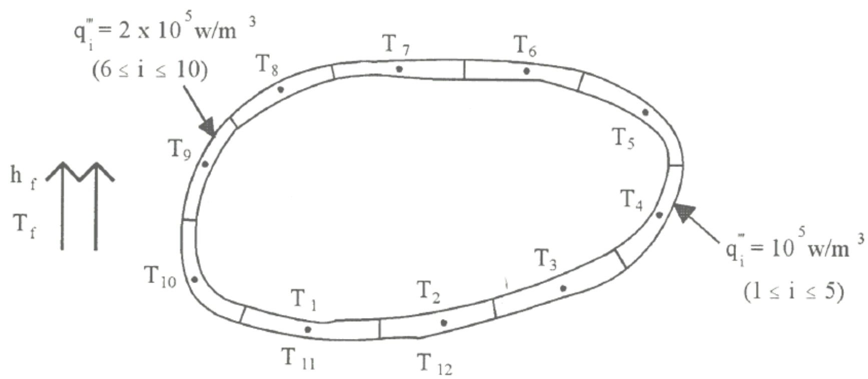


Figure 14 Copper wire loops with energy generation

Because of the periodic boundary condition, fictitious boundary control volumes are no longer needed. Instead, from the geometry, we have

$$\Delta x_1 = \Delta x_{N+1} \quad \text{and} \quad \Delta x_{N+2} = \Delta x_2 \tag{62}$$

The *CTDMA* solves for the temperatures T_2,........,and T_{N+1}. It is obvious from the geometry that

$$T_1 = T_{N+1} \quad \text{and} \quad T_{N+2} = T_2 \tag{63}$$

Note the changes made on implementing the boundary conditions for CTDMA routine and the CTDMA routine, which is added next to TDMA routine. These changes are highlighted in boxes. A new variable IPRIOD is introduced. IPRIOD=1 means a periodic boundary and IPRIOD=0 means a non-periodic boundary condition. The present problem calls for IPRIOD=1.

Numerical Results

Numerical results are obtained after 2 iterations. If we used a lazy man's approach calling TDMA instead of CTDMA, converged solution would take 47 iterations. (This is left as an exercise.) Numerical results show a periodic variation of temperature distribution along the loop as expected. To check the accuracy of the results, we can check the energy balance. This can be done conveniently by adding a few lines of statements at the end of the program. (This is left as an exercise).

```
ITER=         2
I=       1          2          3          4          5          6
I=       7          8          9          10         11         12
TE=   3.987E+02  3.949E+02  3.930E+02  3.930E+02  3.949E+02  3.987E+02
TE=   4.011E+02  4.023E+02  4.023E+02  4.011E+02  3.987E+02  3.949E+02
```

References

1. Patankar, S. V., Numerical Heat Transfer and Fluid Flow, Hemisphere , 1980.

2. Incropera, F. P., and DeWitt, D. P., Fundamentals of Heat Transfer, 3rd Ed., John Wiley &. Sons, 1990.

3. Patankar, S. V., Liu, C. H. and Sparrow, E. M.,"Fully Developed Flow and Heat Transfer in Ducts Having Streamwise-Periodic Variations of Cross-Sectional Area," ASME J. of Heat Transfer, Vol. 99, p180-186, 1977.

Problems

1. Starting from a 3-dimensional heat conduction equation written in the spherical coordinate system, show that 1-d steady state conduction in the radial direction is given by the following equation.

$$\frac{1}{r^2}\frac{d}{dr}(r^2 k\frac{dT}{dr}) + S = 0$$

Derive finite volume equation by integrating the above equation. Explain the physical meaning of each term appearing in the finite volume equation.

2. Starting from a 3-dimensional conduction equation written in the cylindrical coordinate system show that 1-d steady state conduction in the theta direction (θ) is given by

$$\frac{1}{r}\frac{d}{d\theta}(k\frac{1}{r}\frac{dT}{d\theta}) + S = 0$$

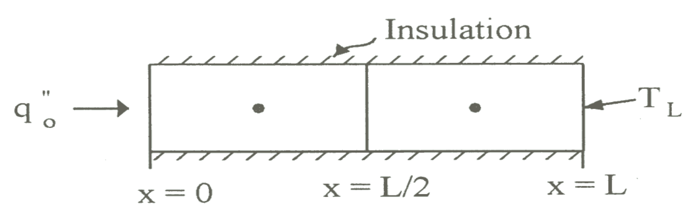
Derive finite volume equation by integrating the above equation. Explain the physical meanings of each term in the finite volume equation.

3. Convective and radiative boundary condition at $x=0$ is given by

$q''=h_f(T_f - T_1) + C(T_f^4 - T_1^4)$, where C is a constant.

Describe how you can implement this boundary condition into a finite volume equation.

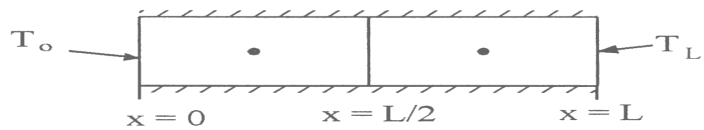
4. Consider 1-dimensional steady conduction as shown in the figure.



where $q_0=20$ W/m^2, L=8 m, $T_L=0$, S=4 W/m^3, k=2 W/m.K.

Calculate temperatures at two points as shown and check the energy balance.

5. Consider a 1-dimensional steady conduction through a bar as shown in the figure.



where L=8 m, T_0=0 , T_L=16 K, k=2 W/m.K, S=4 W/m^3.

Calculate the temperatures at two points and check the energy balance.

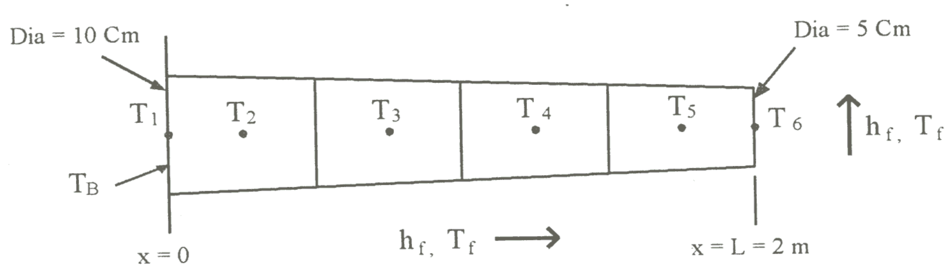
6. Angular direction momentum equation for a circular Couette flow has a source term (see problem 1 in chapter 2),

$$S = -\mu \frac{v_\theta}{r^2} - \frac{\rho v_r v_\theta}{r}$$

Discuss possible linearization schemes for this source term.

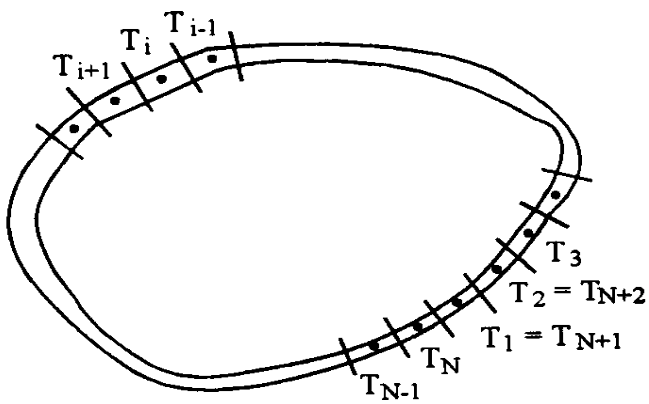
7. Derive interfacial conductivity including contact surface resistance and temperature as shown by Eqs.(36) and (37a,b).

8. Determine the temperature at 4 equally spaced points in a tapered cylindrical fin. The base is at 100 ^{0}C and the remaining surfaces are exposed to convective environment with h_f=10 W/m^2.K and T_f=25 ^{0}C. Assume constant conductivity of 400 W/m.K.



9. Consider 1-D conduction in a rod that is bent into a circular shape to form an endless loop. It thus has no exposed ends where boundary conditions can be prescribed. Indeed all grid points are internal grid points. If a cut is made in the loop, cyclic

boundary condition can be assumed at the cut. That is $T_1=T_{N+1}$ and $T_2=T_{N+2}$. How would you handle such cyclic boundary conditions? A special type of TDMA suitable for cyclic boundary conditions is discussed in Ref.[3] that is called circular TDMA. (See Appendix)



Project

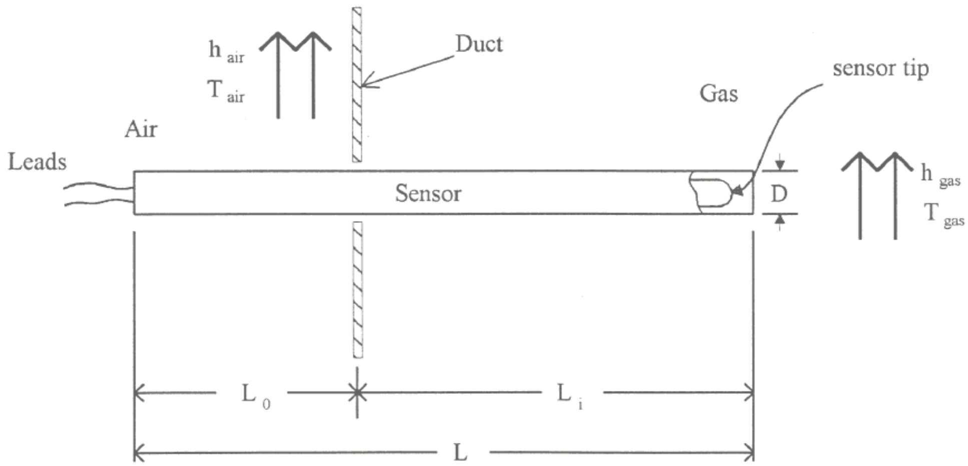
1. A probe of overall length L=20 cm and diameter D=1.25 cm is inserted through a duct wall such that a portion of its length, referred to as the immersion length L_i is in contact with a gas stream at 500 K that is to be measured. The remaining length of the probe is exposed to air at 298 K. The convective heat transfer coefficient in the duct is 1000 $W/m^2.K$ and in the air is 10 $W/m^2.K$. The conductivity of the probe material is 200 W/m.K and its emissivity is 0.1. [Ref. 2]

   Immersion error is defined by the temperature difference between the gas temperature in the duct and the temperature at the tip of the probe, $T_{err} = T_{tip} - T_{gas}$, which depends on the immersion length L_i.

   (a) Derive appropriate conduction equation for this problem.

   (b) By using the numerical method, calculate the temperature distribution in the probe when $L_i/L = 1.0$. Assume 12 uniform control volumes. Plot the results and check the energy balance.

(c) Estimate T_{err} as a function of L_i/L and plot the results. Use $L_i/L = 0.25, 0.5, 0.75$ and 1.0.



2. We want to evaluate triangular fin efficiencies by numerically solving steady heat conduction equation. Triangular fin geometry is shown in Fig. a. Fin efficiency is defined by $\eta_f = q_f/q_{max}$, where q_f is the actual heat transfer from the fin and q_{max} is the maximum heat transfer when the entire fin surface temperature is assumed to be at the base temperature. [Ref. 2]

   In order to verify our approach to this problem, we will first check the accuracy of numerical solution by solving a similar problem with a known solution. Consider a truncated rectangular fin as shown in Fig. B. Fin surfaces is insulated and temperatures at the base and tips are maintained at T_B and T_{L1}, respectively.

   (a) By integrating the quasi-one-dimensional steady conduction equation without source terms, show that heat transfer rate along the x-direction is given by

$$q_x = -\frac{Dtk(T_B - T_{L1})}{L\ln\left(1 - L\frac{1}{L}\right)}$$

   and the temperature distribution is given by

$$T(x) = T_B + \frac{q_x L}{Dtk}\ln(1 - \frac{x}{L}); 0 \le x \le L1$$

(b) Solve the above truncated fin problem numerically by using 10 uniform control volumes. Assume following values:

t=0.02 m; L=0.5 m; D=1.0 m; k=400 W/m.K; T_B=373 K; T_{L1}=298 K; L1=L/2

Compare the numerical results with the exact solution in terms of temperature distribution.

After checking the feasibility of numerical approach, we now return to the fin problem. Take the insulation off the fin surface and consider entire fin (not the truncated fin). The fin surface is now exposed to a convective cooling with known h_f and T_f.

(c) Show that the conduction equation is given by

$$\frac{1}{A}\frac{d}{dx}(Ak\frac{dT}{dx}) + \frac{h_f P}{A}(T_f - T) = 0$$

where A is the cross-sectional area and P is the wetted perimeter.

(d) Calculate the steady state temperatures by using 10 uniform control volumes with the following conditions. Check energy balance to see whether your solution is reasonable.

T=0.02 m; L=0.5 m; D=1.0 m; k=400 W/m.K; T_B=373 K; T_f=298 K; h_f=10 W/m^2.K

(e) Calculate the fin efficiency and compare with the results shown in Fig. C.

(f) By changing the conductivity while maintaining all other values, you can calculate the fin efficiency as a function of $L_c^{3/2}(h_f/kA_p)^{1/2}$ as shown in Fig. C. Obtain an efficiency curve and compare with the results in Fig.c.

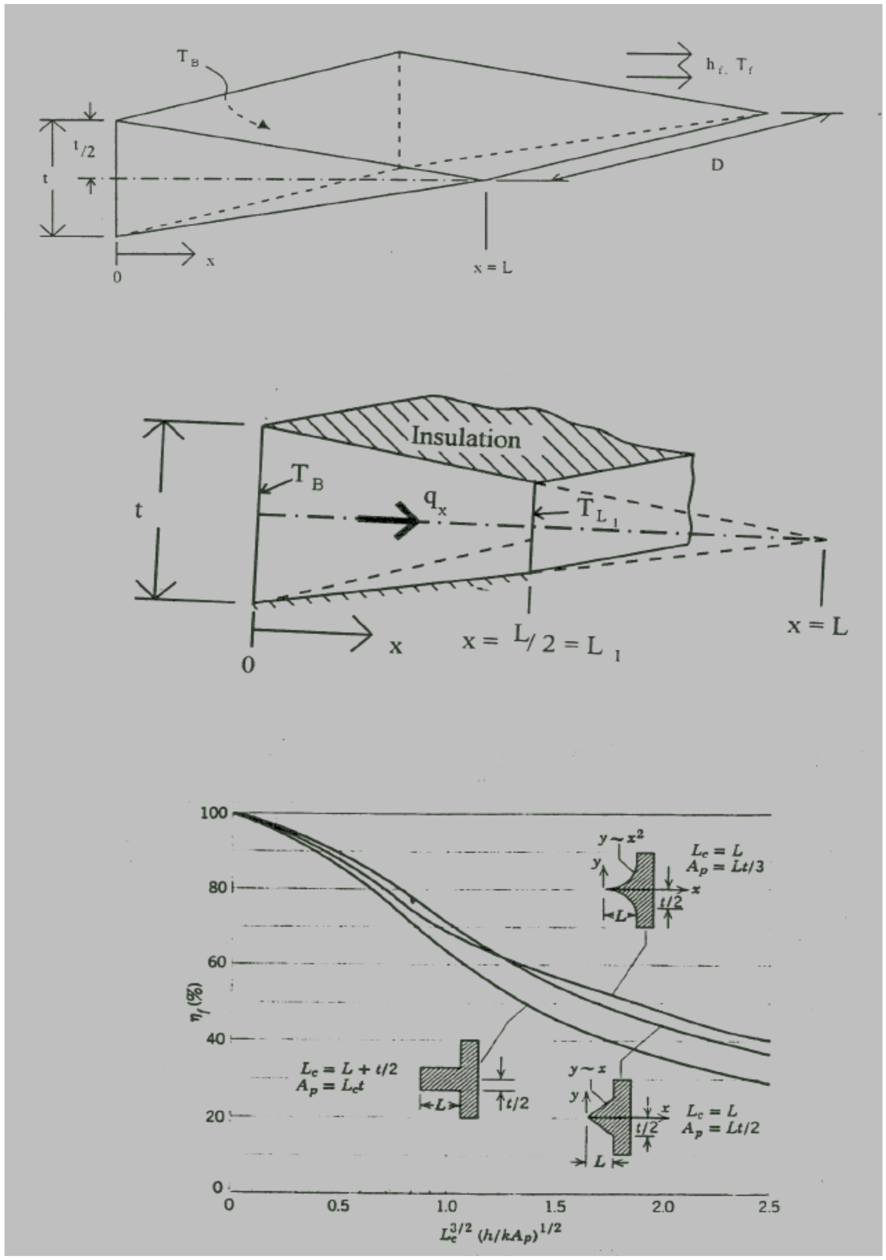
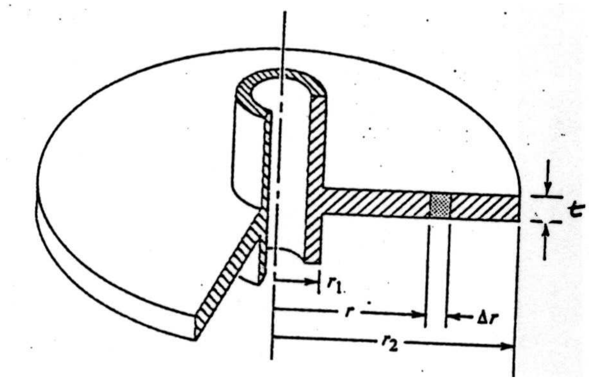


Figure (a) Triangular fin, (b) Truncated triangular fin, (c) Fin efficiency

4. Consider a circular fin with rectangular profile as shown in the diagram.



(a) Show that the steady conduction along the radial direction in the fin is described by the following equation

$$\frac{1}{A}\frac{d}{dr}\left(A\kappa\frac{dT}{dr}\right)+\frac{hP}{A}\left(T_f - T\right)=0$$

where A is the cross section area and P is the wetted perimeter of the cross section,

$$A = 2\pi rt \quad ; \quad P = 4\pi r$$

(b) Let r_1= 5 mm, r_2=20 mm, t=0.2 mm, κ=205 W/m.K, T_f=300 K, and h=8.2 W/m^2.K.

Calculate the temperature along the fin using the numerical method discussed in this chapter. Use 10 uniform control volumes. The base temperature is 380 K and the tip of the fin is insulated.

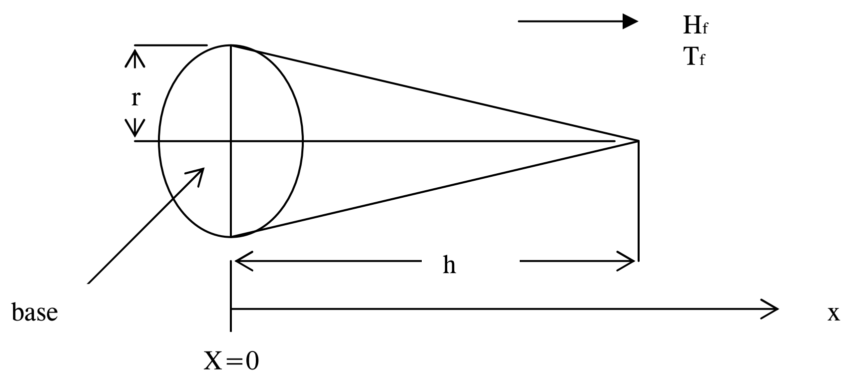
(c) Calculate the heat transfer rate fro, the base of the fin and evaluate the fin efficiency

$$\eta_f = \frac{q_{fin}}{q_{\max}}$$

(d) Repeat the calculations with several cases and obtain a fin efficiency curve. Compare with the known results.

(e) Add the radiation effects on the heat transfer and modify the governing equation in part (a). Let the emissivity varies 0.1 ~ 0.9. Compare with the results without radiation for part (b).

5. Consider a pin fin with circular cross section as shown in the figure below. The fin is made of pure iron. The total volume of the fin is fixed at 5.0e-7 m^3. The thermal conductivity of the material can be approximated by $\kappa(T) = 111 - 0.085T \quad W/mK$, where T is in Kelvin.



The base temperature is 800 K and the convective conditions are h_f=10 W/m^2 K and T_f=298 K.

We would like to solve this problem by modifying std1db.m. The radius of the fin base (r) changes as we change the height (h) since the total volume of the fin is fixed. The volume of the pin fin is given by $V = \frac{1}{3}\pi r^2 h$.

a. Optimize the pin height (h) that gives the maximum conduction heat transfer rate from the base by plotting $q_{x=0}$ as a function of h, varying h from 0.05 m to 0.15 m.

b. Fin efficiency can be defined by $\eta = q_{x=0} / q_{\max}$, where $q_{\max} = A_{side} h_f (T_1 - T_f)$. Plot η vs h. Is the maximum fin efficiency case also the maximum heat transfer rate case?