Design Criteria for Marginal Stability

Trade Study Number – A0007
Conducted by Emilio Gordon
Began on 4/13/2019
Completed on 4/14/2019

Purpose of Trade Study (Forsee)

The purpose of TS A0007 is to derive criteria on the moment of inertia elements (J_1, J_2, J_3) that guarantee marginal passive gravity-gradient stabilization for a spacecraft in low earth orbit. The results of this trade study is to generate criteria for the possible geometries of the space craft that would enable the use of gravity gradient stabilization for periods when the payload is off, thus saving power and reducing the loads on the control system. This is only possible if gravity gradient is utilized to keep the satellite within an acceptable pointing range while the in eclipse.

Investigation

To begin, we define the following three constants:

$$a_1 = \frac{J_2 - J_3}{J_1} \qquad a_2 = \frac{J_1 - J_3}{J_2} \qquad a_3 = \frac{J_2 - J_1}{J_3}$$

From these three constants, six inequalities involving a_1 and a_3 can be derived that guarantee marginal passive gravity gradient stability for a space craft in low earth orbit. These six inequalities are as follows:

$$\begin{aligned}
&1. \quad & 1 + 3a_1 + a_1 a_3 > 0 \\
&2. \quad & (1 + 3a_1 + a_1 a_3)^2 - 16 a_1 a_3 > 0 \\
&3. \quad & a_1 a_3 > 0 \\
&4. \quad & a_1 > a_3 \\
&5. \quad & |a_1| < 1 \\
&6. \quad & |a_3| < 1
\end{aligned}$$

Inequalities 1,2 and 3 relate to the spacecrafts roll and yaw stability by using the equations of motion for roll and yaw. Inequality 4 relates to the spacecrafts pitch stability by using the equations of motion for pitch. The final two inequalities were derived using the definition of moments of inertia. For the complete derivation please see Appendix A of this trade study.

Figure 1 below shows the resulting constraints for a_1 and a_3. The shaded region represents the a_1 and a_3 values for stability. Each inequality region is labeled with respect to the related inequality equation listed above.

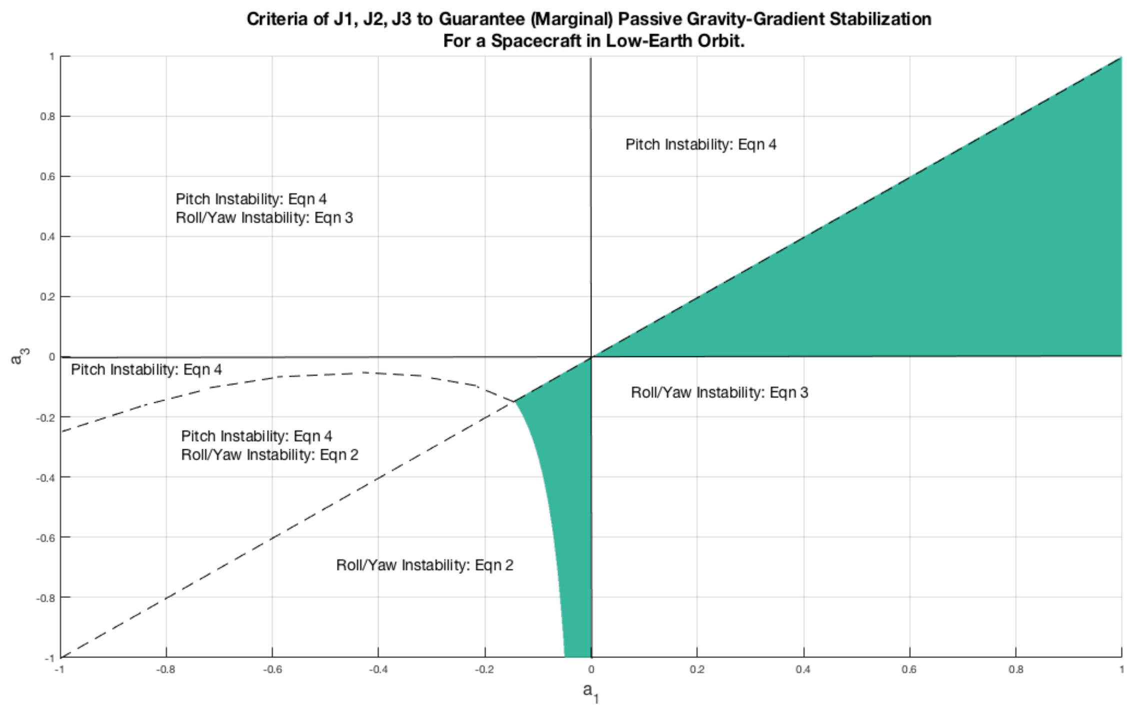


Figure 1: Criteria for J1, J2, J3 to guarantee (marginal) passive gravity -gradient stabilization for a spacecraft in low-earth orbit.

We see that if $a_1 = \frac{J_2 - J_3}{J_1}$ and $a_3 = \frac{J_2 - J_1}{J_3}$ then the craft must have an a_1 value following the shaded region of the plot which is directly dependent on the possible stable geometries for the craft.

These findings were verified by UIUC professor Dr. Koki Ho April 10th, 2018. These findings can be verified by the stability simulation from A0006.

Stability Configuration

A stability configuration option is tested by selecting the point $(a_1, a_3) = (0.8, 0.6)$ within the stability region. Solving for the moment of inertias, a J_3 is assumed and the others can be solved. With this configuration, we find that:

$$J_1 = 8$$
$$J_2 = 10.4$$
$$J_3 = 4$$

Those values plugged into the simulation with an initial state as follows:

$$[\ \theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\ ] = [\ 0.5, 0.5, 0.1, 0, 0, 0\ ]$$

Which are in units of radians and radians/second. As shown in Figure 2, the selected moment of inertia did result in marginal stability using only gravity gradient stabilization. It can be observed that the pitch stabilizes to 0 while the roll and yaw keep a periodic orientation.

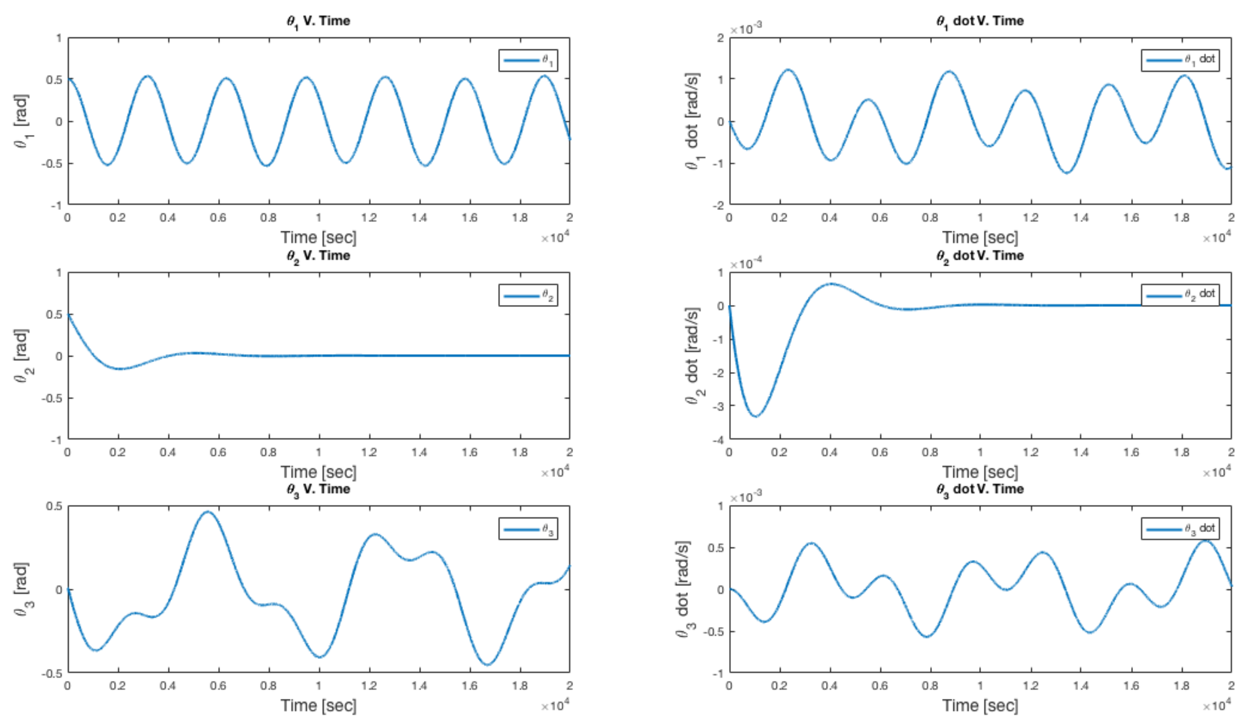


Figure 2: Stability configuration with a_3 = 0.6, a_1=0.8.

Unstable Configuration

An unstable configuration option was also tested by selecting the point $(a_1, a_3) = (0.8, -0.2)$ within the roll/yaw instability region. Solving for the moment of inertias, a J_3 is assumed and the others can be solved. With this configuration, we find that:

$$J_1 = 24$$
$$J_2 = 23.2$$
$$J_3 = 4$$

Those values plugged into the simulation with an initial state as follows:

$$[\ \theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\ ] = [\ 0.5, 0.5, 0.1, 0, 0, 0\ ]$$

Which are in units of radians and radians/second. As shown in Figure 3, the selected moment of inertia did result in an instability with the craft increasing its spin. Pitch is attempted to stabilize but is likely to have to high of an initial offset and therefore also increases its spin rate.

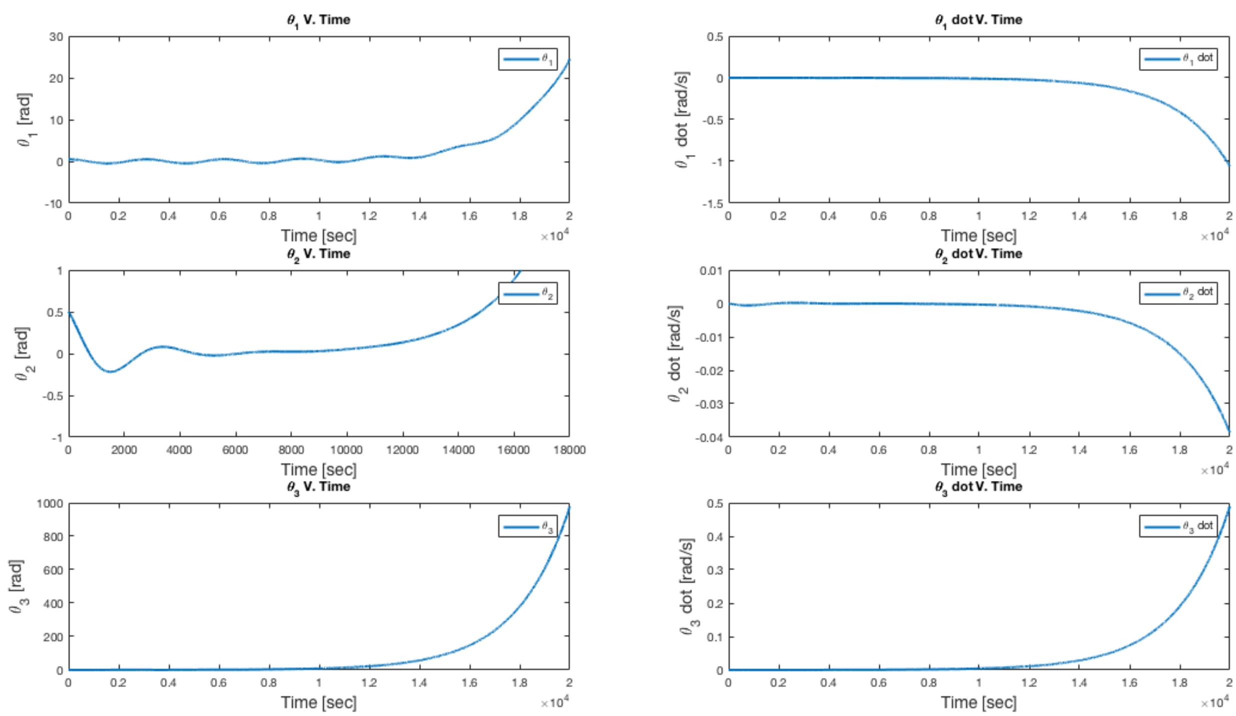


Figure 3: Unstable configuration with a_1 = 0.8, a_3 = -0.2

Record

From this trade study, criteria for designing a spacecraft with marginal gravity gradient stabilization was derived and tested. The results of this trade study are to be used for future structure trade studies in the design of a spacecraft that is marginally stable using only gravity gradient. After these structure trade studies are completed, a more accurate moment of inertia for the craft can be used for TS A0006.

Execute

Signatures:

Trade Study Supervisor: Emilio G

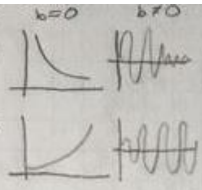
Subteam Liason(s): Lucas Alava Peña Rachit Singhvi

Team Lead: ______________________________

Appedix A:

$a_1 = \frac{\quad}{J_1}$ $a_2 = \frac{\quad}{J_2}$ $a_3 = \frac{\quad}{J_3}$

From notes
$\lambda = a \pm bi$ $\begin{cases} a < 0 \\ a > 0 \end{cases}$ b=0 b≠0



a < 0 — Stable
$\longrightarrow$ Marginally stable when $a = 0$, $b =$ anything
a > 0 — unstable

// From 2d ... $\dot{\omega}_2 = 3n^2 \frac{J_3 - J_1}{J_2} \theta_2 \rightarrow -3n^2 a_2 \theta_2 \rightarrow A = \begin{bmatrix} 0 & 1 \\ -3n^2 a_2 & 0 \end{bmatrix}$

// To analyse stability $\det(\lambda I - A) = 0 = \det\begin{pmatrix} \lambda & -1 \\ 3n^2 a_2 & \lambda \end{pmatrix} = \lambda^2 + 3n^2 a_2 = 0$

// Solve for λ ... $\lambda = \sqrt{-3n^2 a_2}$ $\longrightarrow$ This has to be imaginary ∴ we know its marginally stable since there is no a. ($a = 0$)

$-3n^2 a_2 < 0 \quad \therefore \underline{\underline{a_2 > 0}}$ But the solution cannot have a_2

// From 2e

// via matlab b/c I am lazy & tired

$$A = \begin{bmatrix} 0 & n & 1 & 0 \\ -n & 0 & 0 & 1 \\ -a_1 3n^2 & 0 & 0 & -a_1 n \\ 0 & 0 & a_3 n & 0 \end{bmatrix} \xrightarrow{\det(\lambda I - A)} = \det\begin{pmatrix} \lambda & -n & -1 & 0 \\ n & \lambda & 0 & -1 \\ a_1 3n^2 & 0 & \lambda & +a_1 n \\ 0 & 0 & -a_3 n & \lambda \end{pmatrix}$$

$$= \lambda \det\begin{pmatrix} \lambda & 0 & -1 \\ 0 & \lambda & a_1 n \\ 0 & -a_3 n & \lambda \end{pmatrix} + n \det\begin{pmatrix} n & 0 & -1 \\ a_1 3n^2 & \lambda & a_1 n \\ 0 & -a_3 n & \lambda \end{pmatrix} - 1 \det\begin{pmatrix} n & \lambda & -1 \\ a_1 3n^2 & 0 & a_1 n \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\lambda(\lambda(\lambda^2 + n^2 a_1 a_3)) + n(n(\lambda^2 + n^2 a_1 a_3) - 1(-3a_3 a_1 n^3)) + (\lambda(a_1 3n^2 \lambda)$$

$$\lambda^4 + \lambda^2 n^2 + 3a_1 \lambda^2 n^2 + 4a_1 a_3 n^4 + a_1 a_3 \lambda^2 n^2 = 0$$

$$\lambda^4 + \lambda^2 \underline{n^2(1 + 3a_1 + a_1 a_3)} + 4a_1 a_3 n^4 =$$

Lets say $\alpha = \lambda^2 \rightarrow \underbrace{\alpha^2}_{A=1} + \alpha \underbrace{n^2(1+3a_1+a_1a_3)}_{B} + \underbrace{4a_1a_3n^4}_{C} = 0$ // quadratic Eqn

Quadratic Formula $\alpha = \dfrac{-B \pm \sqrt{B^2-4AC}}{2A} = \dfrac{-B \pm \sqrt{B^2-4C}}{2}$

// since $\alpha = \lambda^2$

$\lambda = \sqrt{\alpha} \longrightarrow \therefore$

$\lambda = \sqrt{\dfrac{-B \pm \sqrt{B^2-4C}}{2}}$ // we want the inside term to be negative such that $\lambda = a + bi = bi$

// For this to be negative...

$B>0,\ B^2-4C>0,\ C>0 \rightarrow$ for marginal stability

// Lets look at each of these interms of the quadratic...

$\underline{B>0} \rightarrow n^2(1+3a_1+a_1a_3)>0 \longrightarrow \boxed{1+3a_1+a_1a_3>0}$ ①

$\underline{B^2-4C>0}\ n^4(1+3a_1+a_1a_3)^2 - 4(4a_1a_3n^4)$

$= n^4((1+3a_1+a_1a_3)^2 - 16a_1a_3) > 0 \rightarrow \boxed{(1+3a_1+a_1a_3)^2 - 16a_1a_3 > 0}$ ②

$\underline{C>0} \rightarrow 4a_1a_3n^4 > 0 \longrightarrow \boxed{a_1a_3>0}$ ③

// From earlier... $a_2>0$, we can solve this in terms of a_1 & a_3

$a_1 - a_3 = \dfrac{J_2-J_3}{J_1} - \dfrac{J_2-J_1}{J_3} = \dfrac{J_3(J_2-J_3) - J_1(J_2-J_1)}{J_1J_3} = \dfrac{J_3J_2 - J_3^2 - J_1J_2 + J_1^2}{J_1J_3}$

$= \dfrac{-J_3^2 + J_1^2 + J_2(J_3-J_1)}{J_1J_3}$ // $a_2 = \dfrac{J_1-J_3}{J_2} \rightarrow J_1 - J_3 = J_2a_2$ or $J_3 - J_1 = -J_2a_2$

$// = \cancel{\dfrac{-J_2^2a^2}{J_1J_3}} + \cancel{\dfrac{J_3^2+J_1^2}{J_1J_3}} = \dfrac{(J_1+J_3)(J_1-J_3) + J_2(J_3-J_1)}{J_1J_3} = \dfrac{(J_1-J_3)(J_1+J_3-J_2)}{J_1J_3}$

$= \dfrac{J_2a_2(J_1+J_3-J_2)}{J_1J_3} = a_1 - a_3 \quad \therefore \quad a_2 = \dfrac{J_1J_3(a_1-a_3)}{J_2(J_1+J_3-J_2)}$

so... since $a_2>0$, $\dfrac{J_1J_3(a_1-a_3)}{J_2(J_1+J_3-J_2)} > 0 \rightarrow a_1 - a_3 > 0$ so $\boxed{a_1>a_3}$ ④

// The final two inequalities are derived using MOI.

$$\text{MOI: } J = \int \hat{s}^T \hat{s}\, dm = \int \begin{bmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -S_3 & S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix} dm = \int \begin{bmatrix} S_2^2+S_3^2 & -S_1S_2 & -S_1S_3 \\ -S_1S_2 & S_1^2+S_3^2 & -S_2S_3 \\ -S_1S_3 & -S_2S_3 & S_2^2+S_1^2 \end{bmatrix} dm$$

$J_1 = \int (s_2^2 + s_3^2)\,dm$

$J_2 = \int (s_1^2 + s_3^2)\,dm$

$J_3 = \int (s_2^2 + s_1^2)\,dm$

$\to J_1 + J_2 = \int s_2^2 + 2s_3^2 + s_1^2 \, dm \to -J_3 \to$

so... $J_1 + J_2 - J_3 = \int (s_3^2 + s_1^2)\,dm = J_2$

$J_1 = J_3$ ∴ $J_1 + J_2 > J_3 \to J_1 > J_3 - J_2$ a

$J_3 + J_2 > J_1 \to J_3 > J_1 - J_2$ b

$J_1 + J_3 > J_2 \to J_1 > J_2 - J_3$ c

So if...

(a) $J_1 > J_3 - J_2$ then $1 > \frac{J_3 - J_2}{J_1}$ or $-1 > \frac{J_2 - J_3}{J_1} = a_1$

(c) $J_1 > J_2 - J_3$ then $1 > \frac{J_2 - J_3}{J_1} = a_1$ ∴ $-1 > a_1$, $1 > a_1$ $\boxed{|a_1| < 1}$ (5)

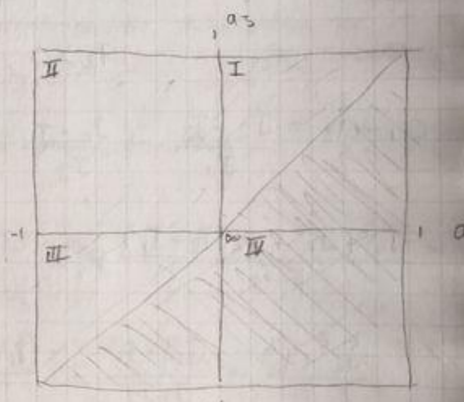
(b) $J_3 > J_1 - J_2$ then $1 > \frac{J_1 - J_2}{J_3}$ or $-1 > \frac{J_2 - J_1}{J_3} = a_3$

(c) $J_1 > J_2 - J_3$ or $J_3 > J_2 - J_1$ then $1 > \frac{J_2 - J_1}{J_3} = a_3$ $\boxed{|a_3| < 1}$ (6)

In Summary: 6 Inequalities

roll / yaw
1. $1 + 3a_1 + a_1 a_3 > 0$
2. $(1 + 3a_1 + a_1 a_3)^2 - 16 a_1 a_3 > 0$
3. $a_1 a_3 > 0$ Q1, Q3

Pitch
4. $a_1 > a_3$ Q1, Q3, Q4
5. $|a_1| < 1$
6. $|a_3| < 1$



See last Page for Plot. The shaded region represents the a_1 & a_3 values for stability. Each inequality region is labled with respect to the pink number in the solution above.

If $a_1 = \frac{J_2 - J_3}{J_1}$ & $a_3 = \frac{J_2 - J_1}{J_3}$ then the craft must have an a_1 following the shaded region of the plot which in turn affects the possible stable geometries for the craft.