What is e?

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Familiarity with basic calculus, including Taylor Series, is assumed. If you've never done calculus before, please read Differentiation – In a Nutshell.

We start by defining what e is.

e. The constant e is defined such that the derivative of $f(x) = e^x$ at $x = 0$ is 1.

This is the entire definition of e – **any other results *follow* from this definition**.[1] Obviously, e is unique and exists since the derivative increases continuously as x increases.[2]

Now we want to show that the derivative of e^x is e^x in general. Fortunately, this is very easy with the limit definition of the derivative.

Derivative of e^x. The derivative of $f(x) = e^x$ is e^x.

Proof. By the limit definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x \cdot 1,$$

where the last equation follows by the definition of e.

That's all fine and well, but now we have to show that the only function that satisfies $f'(x) = f(x)$ is e^x.[3]

$f'(x) = f(x)$. The only functions that satisfy $f'(x) = f(x)$ are of the form $k \cdot e^x$.

[1] You can define it the other way around and work backwards, but I am of the opinion that this is the most natural and convenient definition.

[2] Here is a short proof that this is actually true: Note that the derivative of $f(x) = k^x$ is $\lim_{\Delta x \to 0} \frac{k^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x \ln k} - 1}{\Delta x \ln k} \ln k = \ell k$, where ℓ is the derivative of e^x at 0. The base of e is actually arbitrary; it doesn't matter. You can replace it with 10 or whatever you want.

[3] This actually isn't exactly true because of addition and scalar multiplication, but bear with me for a moment.

Proof. Note that the Maclaurin Series of $f(x)$ can be expressed as

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots.$$

Since $f'(x) = f(x)$, then $f^{(n)}(0) = f(0)$, we can express the sum as

$$f(x) = f(0)(\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots) = f(0)e^x.$$

Now we have to show that this is actually the Maclaurin Series. Fortunately, this is the easy part; we just proved that there is exactly one function with $f(0) = 1$ that satisfies $f'(x) = f(x)$, and we proved before that e^x also satsfies this.[a] So the Maclaurin Series with $f(0) = 1$ could only represent $f(x) = e^x$, and $f(0)$ just accounts for the constant factor k.

[a] You technically have to still show $e^0 = 1$, but that's just a trivial detail.

A corollary of this is the following definition of e.

e as a sum. The constant e is defined as

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

One final thing you should know about e.

e as a limit. The constant e is equal to the value $\lim\limits_{n\to\infty}(1 + \frac{1}{n})^n$.

Proof. Note that expanding gives us

$$\lim_{n\to\infty} \sum_{i=0}^{n} (\frac{1}{n})^i \binom{n}{i}.$$

Since n approaches infinity, we only need to care about two things: the constant factor and the order. Therefore, this sum is equal to

$$\lim_{n\to\infty} \sum_{i=0}^{n} (\frac{1}{n})^i \frac{n!}{(n-i)!i!} \sim \lim_{n\to\infty} \sum_{i=0}^{n} (\frac{1}{n})^i \frac{n^i}{i!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} = e.$$

Exercise. Find the value of $\lim\limits_{n\to\infty}(1 + \frac{1}{2n})^{3n}$.