Solutions to Differentiation

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Contents

1 Exercises

1.1 The Fundamentals

Exercise. Find the equation of the line tangent to $x^4 + 3x^2$ at $(2, 28)$.

Solution: Note that $f'(x) = 3x^3 + 6x$, so $f'(2) = 36$. Thus the point-slope equation of the line is

$$y - 28 = 44(x - 2).$$

Example. Find $\lim\limits_{x \to 0} \frac{\sin(2x)}{x + x^2}$.

Solution 1 (Maclaurin Series): Note that the Maclaurin Series of $\sin(2x)$ is $2x + O(x^3)$. Because the denominator has degree 2, and x approaches 0, we don't care about $O(x^3)$. So the limit is equivalent to

$$\lim_{x \to 0} \frac{2x}{x + x^2} = \lim_{x \to 0} \frac{2}{1 + x} = 2.$$

Solution 2 (Factoring): Note that this expression is equivalent to

$$\lim_{x \to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{1 + x} = 1 \cdot \frac{2}{1} = 2.$$

Solution 3 (L'Hopital's): Note that 1 is the smallest number such that $g^{(1)}(x) \neq 0$, where $g(x) = x + x^2$, so

$$\lim_{x \to 0} \frac{\sin(2x)}{x + x^2} = \frac{2\cos(2 \cdot 0)}{1 + 2 \cdot 0} = 2.$$

1.2 Laws of Differentiation

Exercise (AoPS Calculus, 3.6.3). Find $\frac{dy}{dx}$ if $x^2 + y = \ln(y^2 - 1)$.

Solution: We implicitly differentiate. Note that

$$2x + \frac{dy}{dx} = \frac{1}{y^2 - 1} 2y \frac{dy}{dx}$$

$$2x = \frac{dy}{dx} \frac{2y - (y^2 - 1)}{y^2 - 1} = \frac{dy}{dx} \frac{-y^2 + 2y + 1}{y^2 - 1}$$

$$\frac{2x(y^2 - 1)}{-y^2 + 2y + 1} = \frac{dy}{dx}.$$

Exercise (AoPS Calculus, 3.6.4). Find the slope of the tangent line to the curve $x \sin(x + y) = y \cos(x - y)$ at the point $(0, \frac{\pi}{2})$.

Solution: We implicitly differentiate. First use the product rule and note that

$$\sin(x + y) + x(\cos(x + y))' = y' \cos(x - y) + y(\cos(x - y))'.$$

By the Chain Rule, this implies

$$\sin(x+y) + x\cos(x+y)(x+y)' = y'\cos(x-y) - y\sin(x-y)(x-y)'.$$

Now the linearity of derivatives implies

$$\sin(x+y) + x\cos(x+y)(1+y') = y'\cos(x-y) - y\sin(x-y)(1-y').$$

Plug in $(x,y) = (0, \frac{\pi}{2})$ to get

$$\begin{aligned}
\sin(\frac{\pi}{2}) &= y'\cos(-\frac{\pi}{2}) - \frac{\pi}{2}\sin(-\frac{\pi}{2})(1-y') \\
1 &= \frac{\pi}{2}(1-y') \\
\frac{\pi}{2}y' &= \frac{\pi}{2} - 1 \\
y' &= 1 - \frac{2}{\pi}.
\end{aligned}$$

Thus the slope is $1 - \frac{2}{\pi}$.

1.3 Derivatives of Certain Functions

Exercise (Periodic Derivatives). If $f(x) = \sin x$, find $f'(x), f''(x), f'''(x)$, and $f''''(x)$. Do the same for $f(x) = \cos x$.

Solution: We already know from before that $f'(x) = \cos x$. Thus $f''(x) = -\sin x, f'''(x) = -\cos x$, and $f''''(x) = \sin x$.

A good way to think of this is that $f'(x) = \sin(x + \frac{\pi}{2})$. Then $f^{(n)}(x) = \sin(x + \frac{n\pi}{2})$, which intuitively explains why $f^{(n)}(x)$ has period 4.

Exercise (Derivatives of Reciprocal Functions). Given how the trigonometric derivatives for sin, cos, and tan were derived, determine and prove the derivatives of csc, sec, and cot.

Solution: The reciprocal rule murders the first wo:

$$\begin{aligned}
(\csc x)' &= \left(\frac{1}{\sin x}\right)' = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x \\
(\sec x)' &= \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \sec x \tan x.
\end{aligned}$$

We could also use the reciprocal rule on $\cot x$, but it's more convenient to just use the quotient rule:

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x.$$

Exercise. Find the Maclaurin Series of $x\cos x$.

Solution: Note that the Maclaurin Series of x is x, and the Maclaurin Series of $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$. Multiplying the two yields

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots.$$

Exercise (Derivative of Inverse of Reciprocal Trigonometric Functions). Find the derivative of $\operatorname{arccsc} x$, $\operatorname{arcsec} x$, and $\operatorname{arccot} x$.

Solution: We implicitly differentiate for all of these.

For the first one, let $f(x) = \operatorname{arccsc} x$, and note this implies $\csc y = x$. Then differentiating with respect to x gives

$$\begin{aligned}\left(\frac{1}{\sin y}\right)' y' &= 1\\ -\frac{\cos y}{\sin^2 y} y' &= 1\\ y' &= -\frac{\sin^2 y}{\cos y}.\end{aligned}$$

Since $\sin y = \frac{1}{x}$ and $\cos y = \sqrt{1 - \frac{1}{x^2}}$,

$$y' = -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = -\frac{1}{|x|\sqrt{x^2-1}}.$$ [1]

For the second one, let $f(x) = \operatorname{arcsec} x$, and note that this implies $\sec y = x$. Then differentiating with respect to x gives

$$\begin{aligned}\left(\frac{1}{\cos x}\right) y' &= 1\\ \frac{\sin y}{\cos^2 y} y' &= 1\\ y' &= \frac{\cos^2 y}{\sin y}.\end{aligned}$$

Since $\cos y = \frac{1}{x}$ and $\sin y = \sqrt{1 - \frac{1}{x^2}}$,

$$y' = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{|x|\sqrt{x^2-1}}.$$ [2]

For the last one, let $f(x) = \operatorname{arccot} x$, and note that this implies $\cot y = x$. Then differentiating with respect to x gives

$$\begin{aligned}\left(\frac{\cos y}{\sin y}\right) y' &= 1\\ \frac{-\sin^2 y - cos^2 y}{\sin y} y' = \frac{1}{\sin y} y' &= 1\\ y' &= \sin y.\end{aligned}$$

Since $x = \cot y$,

$$y' = \frac{1}{\sqrt{x^2+1}}.$$

[1] The absolute value appears because $x^2 \geq 0$ for obvious reasons, and we need to preserve this even after factoring out an x.

[2] See above.

Exercise. Find the derivative of $f(x) = \log_a(g(x))$.

Solution: Note that $f(x) = \frac{\ln(g(x))}{\ln a}$, so

$$f'(x) = \frac{(\ln(g(x)))'}{\ln a} = \frac{\frac{1}{g(x)} g'(x)}{\ln a} = \frac{g'(x)}{g(x) \ln a}.$$

2 Problems

2.1 Unsourced

Prove that the derivative of $f(x) = e^{g(x)}$ is $e^{g(x)}g'(x)$.

Solution: This follows directly from the chain rule.

2.2 HMMT

Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose that the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

Solution: Note that $f'(x) = 3x^2 + a$, so $f'(a) = f'(b)$ implies $3a^2 + a = 3b^2 + a$, or that $a = -b$. Thus $f(1) = 1 + a + b = 1$.

2.3 HMMT Calculus 2010/1

Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

Solution: Note that $p(x)$ must have leading term x^2, because by the Power Rule $\deg(p(x) - p'(x)) = \deg(p(x))$, and furthermore the leading coefficients are the same. So we have

$$\begin{aligned} p(x) &= x^2 + ax + b \\ p'(x) &= 2x + a \end{aligned}$$

and we want $p(x) - p'(x) = x^2 + x(a-2) + (b-a) = x^2 + 2x + 1$, or

$$\begin{aligned} a - 2 &= 2 \\ b - a &= 1, \end{aligned}$$

implying that $a = 4$ and $b = 5$. Therefore, $p(5) = 5^2 + 4 \cdot 5 + 5 = 50$.

2.4 HMMT Calculus 2010/3

Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeroes of $p'(x)$ are also zeros of $p(x)$. Find p. Note: monic means that the leading coefficient is 1.

Solution: There are either three distinct roots, two distinct roots, or one root. We look at all three cases.

If there are three distinct roots, then $p(x) = (x - r_1)(x - r_2)(x - r_3)$ and $p'(x) = 3x^2 - 2x(r_1 + r_2 + r_3) + (r_1 r_2 + r_2 r_3 + r_3 r_1)$. We can verify that this function has zeroes and that none of them are r_1, r_2, r_3, since the roots are distinct.

If there are two distinct roots, there are two copies of one root and one copy of another. So $p(x) = (x - r_1)^2(x - r_2)$, and by the Product Rule,

$$p'(x) = 2(x - r_1)(x - r_2) + (x - r_1)^2 = (x - r_1)(3x - r_1 - 2r_2).$$

Since $r_1 \neq r_2$, $\frac{r_1 + 2r_2}{3}$ cannot be either r_1 or r_2, since it is a weighted mean.

If there is one distinct root, then $p(x) = (x - r)^3$. Note that $p'(x) = 3(x - r)^2$, and the only root is $x = r$, so this satisfies the condition. Since $p(0) = 1$, we must have $r = -1$, or $p(x) = (x + 1)^3$.

2.5 Lemma of Mock JMO

Determine the minimum value $f(x) = e^x + \frac{1}{e^x}$ can take.

Solution: Note that $f'(x) = e^x - \frac{1}{e^x}$. It is somewhat easy to guess that the minimum occurs at $x = 0$, and to prove it, note that $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$. Thus $f(0) = 2$ is the minimum value it can take, and this minimum **is only achieved at** $x = 0$.

2.6 Unsourced

Find the derivative of $\frac{4^x}{4^x+1}$.

Solution: Let this function be $f(x)$. Note that $f(x) = 1 - \frac{1}{4^x+1}$, so $f'(x) = -\frac{d}{dx}(\frac{1}{4^x+1})$. By the Reciprocal Rule,

$$-\frac{d}{dx}(\frac{1}{4^x+1}) = \frac{\frac{d}{dx}(4^x+1)}{(4^x+1)^2} = \frac{4^x \ln 4}{(4^x+1)^2}.$$

2.7 MIT OCW

Show that, $g(h) = \frac{f(a+h)-f(a)}{h}$ has a removable discontinuity at h = 0 given that $f'(a)$ exists.

Solution: In order for g(h) to have a removable discontinuity at h = 0 it must follow two different rules. Firstly the $\lim_{h\to 0}$ for g(h) must exist. As the g(h) when evaluated for the limit leads to the equation $\lim_{h\to 0} g(h) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ it is therefore shown, through the limit definition of a limit that $\lim_{h\to 0} g(h) = f'(a)$. As we know, therefore, that $f'(a)$ exists at a we therefore know that the $\lim_{h\to 0} g(h)$ exists. When paired with the structure of the equation, with $\frac{f(a+h)-f(a)}{h}$ demonstrating a rational equation and when evaluated for $\lim_{h\to 0}$ giving $\frac{0}{0}$ there is clearly a removable discontinuity. Therefore based on the structure of the equation and what g(h) represents it can be surmised that at h = 0, $g(h)$ has a removable discontinuity.

2.8 HMMT

Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

Solution: Note that it is necessary (but not sufficient) for $f'(a) = 0$. Note that $f'(x) = 4x^3 - 3x^2 - 2x + a$, so

$$f'(a) = 4a^3 - 3a^2 - 2a + a = 4a^3 - 3a^2 - a = (a-1)a(4a+1).$$

Thus the possible values of a are $-\frac{1}{4}, 0, 1$.

Note that we require $a = f(a)$, so the only case left to check is $a = 1$. For $a = 1$, we have

$$f'(x) = 4x^3 - 3x^2 - 2x - 1 = (x+1)(4(x+\frac{1}{8})^2 - \frac{17}{16}),$$

so the other roots of $f'(x)$ are $x = -\frac{1\pm\sqrt{17}}{8}$. Since both of these roots are less than 1, and the leading coefficient of $f'(x)$ is positive, $f(1-\epsilon) < 0$ and $f(1+\epsilon) > 0$ for small $\epsilon > 0$, which are necessary for a minimum to be achieved.

So the answer is just $a = 1$.

2.9 HMMT

Compute $\lim_{x\to 0} \frac{e^{x\cos x}-1-x}{\sin(x^2)}$.

Solution: We use L'Hopital's Rule to completely butcher this problem. Note that

$$\lim_{x\to 0} \frac{e^{x\cos x}-1-x}{\sin(x^2)} = \lim_{x\to 0} \frac{e^{x\cos x}(\cos x - x\sin x)-1}{2x\cos(x^2)}{}^{3} = \lim_{x\to 0} \frac{1-x\sin x-1}{2x} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}.$$

2.10 MAST Diagnostic 2020/C10

Find the maximum value of k such that $(x+1)^4 \geq kx^3$ for all x.

We solve this problem in two separate ways: one with calculus and another with AM-GM.

Solution 1 (Calculus): Note that obviously $k \geq 0$, so $x \leq 0$ is not even a case worth considering since the left-hand side will be non-negative and the right-hand side will be non-positive.

This is equivalent to finding the minimum value of $f(x) = \frac{(x+1)^4}{x^3}$ over positive x. Note that the derivative of this function is, by the quotient/chain rules,

$$f'(x) = \frac{4(x+1)^3x^3 - 3(x+1)^4x^2}{x^6} = \frac{(x+1)^3(4x-3(x+1))}{x^4} = \frac{(x+1)^3(x-3)}{x^4}.$$

Note that $f'(x) > 0$ when $x > 3$ and $f'(x) < 0$ when $0 < x < 3$, so on the domain $(0,\infty)$, $f(x)$ is minimized when $x = 3$.

It is easy to verify that $x = 3$ gives $f(x) = \frac{64}{27}$, so that is our answer.

Solution 2 (AM-GM): Note that by AM-GM, $(\frac{x}{3}+\frac{x}{3}+\frac{x}{3}+1)^4 \geq 64\cdot\frac{x^3}{27}$ with equality at $\frac{x}{3} = 1$, so our maximum is $k = \frac{64}{27}$.[4]

2.11 Extension of C10

Find the range of values k such that $(x+1)^4 \geq kx^3$ for all x.

Solution: As we can probably infer from the solution above, the problem behaves differently for $k \geq 0$ and $k \leq 0$, and each of these cases only care about $x \geq 0$ and $x \leq 0$, respectively.

We have already done $x \geq 0$ – in that case, $k \leq \frac{64}{27}$ will work. So we do $x \leq 0$.

The extension is really not hard; when $x = -1$ we have $0 \geq k \cdot -1^3$, so we must have $k \geq 0$. Thus the range is $k \in [0, \frac{64}{27}]$.

2.12 Leibniz Rule

Given two nth differentiable functions f, g, prove that

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Solution: This is just algebraic manipulation with Taylor Series.

[3] We took the derivative of both sides of the fraction.

[4] The case where x is non-positive is not addressed in this solution, but it is completely trivial.

Note that the Taylor Series of $f(x)$ is $f(x+\epsilon) = f(x) + f'(x)\epsilon + \frac{f''(x)\epsilon^2}{2!} + \cdots$. A similar equation holds for $g(x+\epsilon)$. Take the product of the Taylor Series and note that the coefficient of the ϵ^n term can be expressed as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x)\epsilon^k}{k!} \cdot \frac{g^{(n-k)}(x)\epsilon^{n-k}}{(n-k)!},$$

and since

$$\epsilon^n \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} \cdot \frac{g^{(n-k)}(x)}{(n-k)!} = \frac{(fg)^{(n)}(x)\epsilon^n}{n!},$$

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

2.13 Mock USAJMO

Find, **with proof**, all real triples (a, b, c) satisfying

$$(2^{2a}+1)(2^{2b}+2)(2^{2c}+8) = 2^{a+b+c+5}.$$

Solution: We would like to make this expression symmetric to make it easier to work with; factoring out powers of 2 from the second and third terms in the left-hand side would make the problem a lot easier to work with. So note this expression is equivalent to

$$(2^{2a}+1)(2^{2b-1}+1)(2^{2c-3}+1) = 2^{a+b+c+1}.$$

Now we want to actually make this problem symmetric, so we substitute $y = b - \frac{1}{2}$ and $z = c - \frac{3}{2}$ to get

$$(2^{2a}+1)(2^{2y}+1)(2^{2z}+1) = 2^{a+y+z+3}$$

$$(2^a + \frac{1}{2^a})(2^y + \frac{1}{2^y})(2^z + \frac{1}{2^z}) = 2^3.$$

By **Lemma of Mock JMO**, the minimum value of $2^a + \frac{1}{2^a}$ is 2, achieved only when $a = 0$.[5][6] So we must have $a = y = z = 0$, or $(a, b, c) = (0, \frac{1}{2}, \frac{3}{2})$.

[5] Note that all exponential functions are the same; there's just a difference in the constant.

[6] Alternatively, you may cite AM-GM. Actually, I will directly prove this: $(\sqrt{2}^a - \frac{1}{\sqrt{2}^a})^2 \geq 0$ with equality at $\sqrt{2}^a - \frac{1}{\sqrt{2}^a} = 0$, or $a = 0$ – expanding the inequality and rearranging gives the desired $2^a + \frac{1}{2^a} \geq 2$.