Slanted Coordinate Axes

Dennis Chen

GQT

Slanted coordinate axes are very useful when dealing with parallelograms, since the coordinates of all the vertices will be very nice and obvious. They are also good at dealing with angle bisectors due to their simple and symmetrical equation. For easier problems, slanted coordinate axes reduce the amount of intelligence needed to actually solve the problem, and also reduce the difficulty of just coordinate bashing the answer/proof.

1 Theory

1.1 So What Are Cartesian Coordinates?

In Cartesian Coordinates we have an x axis and a y axis. Let them be ℓ_1 and ℓ_2 respectively. They form a $90°$ angle and they intersect at a point called the *origin*.

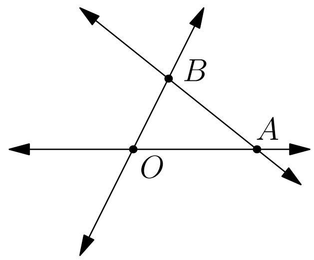
Let's talk about lines in Cartesian Coordinates first. A line can be described with two points. In a coordinate plane, these two points are the intersections with the axes ℓ_1 and ℓ_2 - we call them the x and y intercepts. Let the x intercept be A and the y intercept be B. Then let $OA = a$ and $OB = b$. We let the *coordinates* of the intercepts be $(a, 0)$ and $(0, b)$, depending on what axis it is on. Then the line through $(a, 0)$ and $(0, b)$ has equation $x/a + y/b = 1$, and any point (x, y) that satisfies this condition is on the line. Of course, we can describe a point as the intersection of two lines (though we really don't do that). We'll go over what coordinates really mean later.

Wait a minute - what does any of this have to do with the fact that the axes are perpendicular?

1.2 Setup

We see that coordinate planes can be generalized! This is really intuitive - we already know how to do this for perpendicular axes.

Use directed lengths. The axes are ℓ_1, ℓ_2, and the directed angle $\measuredangle(\ell_1, \ell_2) = \theta$. Let ℓ_1, ℓ_2 intersect at O. Then let our line ℓ intersect ℓ_1 at A and ℓ_2 at B. If $OA = a$ and $OB = b$ (remember directed lengths) then our line has equation $x/a + y/b = 1$, and the x and y intercepts are $(a, 0)$ and $(0, b)$, respectively.



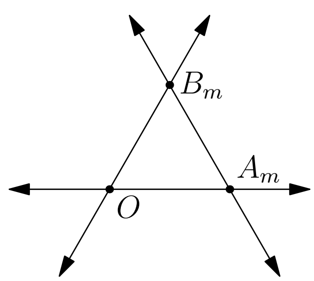
Then if we have two lines $x/a_1 + y/b_1 = 1$ and $x/a_2 + y/b_2 = 1$, we can solve for their intersection. Doing the algebra,

$$
\begin{aligned}
y &= -b_1/a_1 x + b_1 \\
y &= -b_2/a_2 x + b_2 \\
-b_1/a_1 x + b_1 &= -b_2/a_2 x + b_2 \\
x(b_2/a_2 - b_1/a_1) &= b_2 - b_1 \\
x &= \frac{b_2 - b_1}{b_2/a_2 - b_1/a_1} \\
y &= \frac{a_2 - a_1}{a_2/b_2 - a_1/b_1}
\end{aligned}
$$

(This looks suspiciously like slope-intercept, which it should. We used standard form to introduce slanted axes because they have the simplest definition.)

1.3 What is Slope?

Let ℓ intersect ℓ_1 and ℓ_2 at A and B, respectively. Then the slope of ℓ is $\frac{OB}{AO}$.



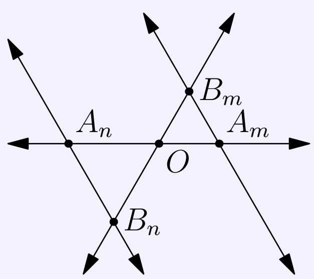
Theorem 1.1: Slope of Parallel Lines

Two lines are parallel if and only if they have the same slope.

Proof

We prove the if condition.
Let the axes be ℓ_1 and ℓ_2, and let the lines be m, n. Let m intersect ℓ_1 and ℓ_2 at A_m and B_m, respectively, and let n intersect ℓ_1 and ℓ_2 at A_m and B_m, respectively. By definition, $\frac{OB_m}{A_mO} = \frac{OB_n}{A_nO}$. Then by SAS similarity, $\triangle OA_mB_m \sim \triangle OA_nB_n$. Since A_m, A_n, O and B_m, B_n, O are collinear, $A_mB_m \parallel A_nB_n$.

We prove the only if condition.
Let the axes be ℓ_1 and ℓ_2, and let the lines be m, n. Let m intersect ℓ_1 and ℓ_2 at A_m and B_m, respectively, and let n intersect ℓ_1 and ℓ_2 at A_m and B_m, respectively. By definition, $A_mB_m \parallel A_nB_n$, so $\triangle OA_mOB_m \sim \triangle OA_nOB_n$ as A_m, A_n, O and B_m, B_n, O are collinear. Since the triangles are similar, $\frac{OB_m}{OA_m} = \frac{OB_n}{OA_n}$ as desired.



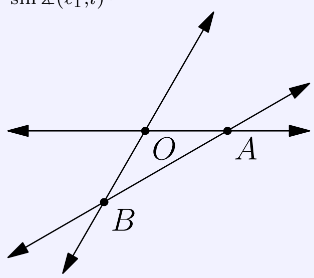
So what's the slope through an origin? We can't find it by our definition, but we can say that it has the same slope as any line parallel to it. Now that we've defined the slope of a point through the origin, we can ask ourselves what slope really means.

Theorem 1.2: Slope: Ratio of Sines

Consider line ℓ in coordinate plane with axes ℓ_1, ℓ_2. Then the slope of ℓ is $\frac{\sin \measuredangle(l,\ell_2)}{\sin \measuredangle(\ell_1,l)}$.

Proof

Let ℓ intersect the axes at A and B respectively. Then by the Law of Sines, $\frac{OB}{\sin\measuredangle(OAB)} = \frac{OA}{\sin\measuredangle(ABO)}$. Rearranging, $\frac{OB}{OA} = \frac{\sin\measuredangle(ABO)}{\sin\measuredangle(OAB)} = \frac{\sin\measuredangle(l,\ell_2)}{\sin\measuredangle(\ell_1,l)}$, as desired.



The final question is, what are the slopes of lines parallel to the axes? What does a line with slope 0 or slope "infinity" (for lack of a better term) look like? To answer this, we'll borrow an idea from projective geometry - the point at infinity.

If ℓ is a line parallel to ℓ_1, it intersects ℓ_1 at a point A_∞, a point at infinity, and ℓ_2 on some normal point B. Then the slope is $\frac{OB}{OA_\infty}$. Since A_∞ is the point at infinity, we want to find $\lim\limits_{OA_\infty \to \infty} \frac{OB}{OA_\infty}$, which is 0.

Similarly, if ℓ is parallel to ℓ_2, it intersects ℓ_1 at A and ℓ_2 at B_∞. Then the slope is $\lim\limits_{OB_\infty \to \infty} \frac{OB_\infty}{OA} = \infty$. But it's irresponsible to say the slope is "infinity." In Cartesian Coordinates, a line parallel to the y axis has an undefined slope, so here, a line parallel to ℓ_2 also has an undefined slope.

1.4 What do Coordinates Mean?

A point can be uniquely defined as the intersection of two lines. In Cartesian Coordinates, we may describe the point (a, b) as the intersection of $x = a$ and $y = b$. For a slanted coordinate system, we can do the same thing.

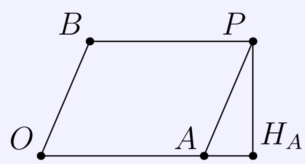
Let line m have equation $x = a$ and line n have equation $y = b$. Then let $O = (0, 0)$, $A = (a, 0)$, $B = (0, b)$, and $P = (a, b)$. Then notice that $OAPB$ is a parallelogram (as $AO \parallel BP$ and $BO \parallel AP$ by definition). Thus we can use some angle chasing to determine things such as the magnitude of a point and its distance from the axes.

Theorem 1.3: Point to Axes Distance

Let $P = (a, b)$ and let $\measuredangle(\ell_1, \ell_2) = \theta$. Then the distance from P to ℓ_1 is $a \sin\theta$ and the distance from P to ℓ_2 is $b \sin\theta$.

Proof

Let the foot of the altitude from P to ℓ_1 be H_A and let the foot of the altitude from P to ℓ_2 be H_B. Since $PA||BO$, $\measuredangle HAP = \measuredangle AOB = \theta$, so $H_AP = AP\sin\theta = a\sin\theta$, as desired.
Similarly, $H_BP = b\sin\theta$.



1.5 Distance and Circles

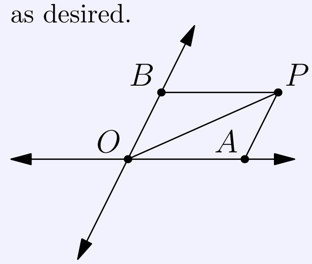
How do you find the distance between two points? We use the Law of Cosines.

Theorem 1.4: Distance Formula

Given two points $X = (a, b)$ and $Y = (a + \Delta a, b + \Delta b)$, $XY = \sqrt{(\Delta a)^2 + (\Delta b)^2 + 2\Delta a\Delta b\cos\theta}$.

Proof

Without loss of generality, let $A = (0, 0)$. (Anything else just is a translation.)
Let $A = (\Delta a, 0)$ and $B = (0, \Delta b)$. Angle chasing, we find that $\measuredangle OBP = 180-\theta$, as $OAPB$ is a parallelogram. Then by the Law of Cosines, $OP^2 = OB^2+BP^2-2\cdot OB\cdot BP\cos(180-\theta)$. Since $\cos(180-\theta) = -\cos\theta$, $OB = b$, and $BP = a$, $OP^2 = b^2 + a^2 + 2ab\cos\theta$, or $OP = \sqrt{a^2 + b^2 + 2ab\cos\theta}$, as desired.



A circle is the locus of points equidistant from a given point (called the center). So if we have center $P = (a, b)$ and a radius of r, our equation $r^2 = (x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\theta$.

1.6 Slopes of Lines Given Angles

In Cartesian Coordinates, two lines m, n with slopes p, q are perpendicular if and only if $pq = -1$ (with the exception of a vertical line). Is there a similar way to find if two lines are perpendicular in a slanted coordinate system?

Theorem 1.5: Slope of a Perpendicular Line

Given a line m such that $\measuredangle(\ell_1, m) = \alpha$, the slope of the line n perpendicular to m is $-\frac{\cos(\theta-\alpha)}{\cos\alpha}$.

Proof

Notice that $\measuredangle(\ell_1, n) = \alpha + 90°$. Thus the slope of n is $\frac{\sin(\theta-(\alpha+90°))}{\sin(\alpha+90°)} = \frac{-\cos(\theta-\alpha)}{\cos\alpha} = -\frac{\cos(\theta-\alpha)}{\cos\alpha}$, as desired.

This is interesting - but doesn't reduce to what we want it to. What if the reason the slopes of perpendicular lines multiply to -1 is because the angle between perpendicular lines is the same as the angle between the axes?

Theorem 1.6: Slope of Lines with Axes Angle

Given a line m such that $\measuredangle(\ell_1, m) = \alpha$, the slope of the line n such that $\measuredangle(m, n) = \theta$ is $-\frac{\sin(\alpha)}{\sin(\alpha+\theta)}$.

Proof

Notice that $\measuredangle(\ell_1, n) = \alpha + \theta$. Thus the slope of n is $\frac{\sin((\theta-\alpha)-\theta)}{\sin(\alpha+\theta)} = \frac{-\sin(\alpha)}{\sin(\alpha+\theta)} = -\frac{\sin(\alpha)}{\sin(\alpha+\theta)}$, as desired.

There's no exact match with the perpendicularity theorem for Cartesian Coordinates, but these results are useful nonetheless.

1.7 Special Lines and Angles

A few reasons we use slanted coordinates - angle bisectors have a really nice equation, and medians/centroids don't really change at all.

Theorem 1.7: Internal Angle Bisector

The equation of the angle bisector of $\measuredangle(\ell_1, \ell_2)$ is $x = y$.

Proof

The angle bisector of $\measuredangle(\ell_1, \ell_2)$ passes through the origin. Since $\alpha = \theta - \alpha = \frac{\theta}{2}$ by definition, the slope is $\frac{\sin(\frac{\theta}{2})}{\sin(\frac{\theta}{2})} = 1$. Thus, the equation is $x = y$, as desired.

(We use directed angles because we're talking about the internal angle bisector, not external.)

Similarly, a reflection of point (a, b) about the angle bisector yields (b, a), and two lines through the origin are bisected by the angle bisector if the product of their slopes is 1.

Theorem 1.8: Centroid

If $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$, then the centroid of $\triangle ABC$ has coordinates $(\frac{1}{3}(x_a + x_b + x_c), \frac{1}{3}(y_a + y_b + y_c))$.

Proof

Note that the midpoint of BC is $(\frac{1}{2}(x_b + x_c), \frac{1}{2}(y_b + y_c))$. We want the point $\frac{2}{3}$ of the way from (x_a, y_a) to $(\frac{1}{2}(x_b + x_c), \frac{1}{2}(y_b + y_c))$, which is $(\frac{1}{3}(x_a + x_b + x_c), \frac{1}{3}(y_a + y_b + y_c))$, as desired.

The proof is identical to the proof in Cartesian Coordinates.

Theorem 1.9: Circumcenter

If $A = (0, 0)$, $B = (b, 0)$, and $C = (0, c)$, the circumcenter of $\triangle ABC$ has coordinates $(\frac{b\sec\theta - c}{2(\sec\theta - \cos\theta)}, \frac{c\sec\theta - b}{2(\sec\theta - \cos\theta)})$.

Proof

We claim that the coordinates of the antipode of A are $(\frac{b\sec\theta - c}{\sec\theta - \cos\theta}, \frac{c\sec\theta - b}{\sec\theta - \cos\theta})$. Let the antipode be D. Note that by Thale's Theorem, $\angle ABD = \angle ACD = 90°$.
Let the line through B perpendicular to AB intersect AC at P, and the line through C perpendicular to AC intersect AB at Q. Then the slope of BP is $-\frac{AP}{AB} = -\sec\theta$. Similarly, the slope of CQ is $-\frac{AC}{AQ} = -\cos\theta$. So the equation of the line through B is $y = -\sec\theta(x - b)$ and the equation of the line through C is $y = -x\cos\theta + c$. Thus $-\sec\theta(x - b) = -x\cos\theta + c$, implying $x(\cos\theta - \sec\theta) = -b\sec\theta + c$, or $x = \frac{b\sec\theta - c}{\sec\theta - \cos\theta}$ and $y = \frac{c\sec\theta - b}{\sec\theta - \cos\theta}$, so $D = (\frac{b\sec\theta - c}{\sec\theta - \cos\theta}, \frac{c\sec\theta - b}{\sec\theta - \cos\theta})$.

This works well with circumcenter problems with one reference triangle and manageable slopes (like perpendicular or parallel lines).

2 Problems

1. (AIME I 2009/4) In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.

2. (HMMT 2019) Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that
$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

3. (January Mock AMC 10) Consider $\angle A$ with varying points B, C on opposite rays of the angle such that $1/AB + 1/AC$ is constant. Prove that all possible BC pass through a fixed point.

4. Given $\angle D$ and A, B, C on one ray and A', B', C' on the other, prove that if $A'B \parallel B'A$ and $A'C \parallel AC'$, then $BC' \parallel B'C$.

5. Consider parallelogram $ABCD$. Let M be the midpoint of BC and let N be the midpoint of CD. Let AM and BN intersect at P. If CP intersects AD at Q, prove that $\frac{QA}{AD} = 2$.

6. In quadrilateral $ABCD$, AB and CD intersect at P, and AD and BC intersect at Q. Let AC intersect PQ at X and let BD intersect PQ at Y. Prove that $\frac{PX}{XQ} = \frac{PY}{YQ}$.

7. A paper trapezoid has side lengths $AB = 3, BC = 8, CD = 9$, and $DA = 9$ with AB parallel to DC. Let E be a point on line segment $\overline{BC}$ such that, when C is folded over the crease $\overline{DE}$, C coincides exactly with A. What is the length of CE?

8. Let ℓ be a line through the centroid of $\triangle ABC$. If ℓ intersects AB at M and AC at N, prove that $AM \cdot NC + AN \cdot MB = AM \cdot AN$.

9. Sides AB, BC, CD, and DA of quadrilateral $ABCD$ are cut by a straight line at points K, L, M, N, respectively. Prove that $\frac{BL}{LC} \cdot \frac{AK}{KB} \cdot \frac{DN}{NA} \cdot \frac{CM}{MD} = 1$.

10. Consider lines ℓ_1, ℓ_2 such that $\measuredangle(\ell_1, \ell_2) = 75^\circ$, and let $d(X, \ell_1)$ and $d(X, \ell_2)$ denote the distance of point X to line ℓ_1 and point X to line ℓ_2, respectively. Let ℓ be the locus of points X such that $\frac{d(X,\ell_1)}{d(X,\ell_2)} = \frac{\sqrt{2}}{2}$.

    (a) Prove that ℓ is a line.

    (b) Find $\measuredangle(\ell_1, l)$.

11. Consider parallelogram $ABCD$ with $AB = 7$, $BC = 6$. Let the angle bisector of $\angle DAB$ intersect BC at X and CD at Y. Let the line through X parallel to BD intersect AD at Q. If $QY = 6$, find $\cos\angle DAB$.

12. (PUMAC 2018) Let $ABCD$ be a parallelogram such that $AB = 35$ and $BC = 28$. Suppose that $BD \perp BC$. Let ℓ_1 be the reflection of AC across the angle bisector of $\angle BAD$, and let ℓ_2 be the line through B perpendicular to CD. If ℓ_1 and ℓ_2 intersect at a point P, find PD.

13. Sides BA and CA of $\triangle ABC$ are extended through A to form rhombuses $BATR$ and $CAKN$. Let BN intersect RC at P. Let BN intersect AC at M and RC intersect AB at S. Let the line through M parallel to AB intersect BC at Q.

    (a) Prove that $AMQS$ is a rhombus.

    (b) Prove that $[BPC] = [ASPM]$. (This can't be done with slanted axes, but relies on the first result and is informative.)

14. (Sharygin Correspondence 2020) Let BB_1, CC_1 be altitudes of triangle ABC, and AD be the diameter of its circumcircle. The lines BB_1 and DC_1 meet at point E, and the lines CC_1 and DB_1 meet at point F. Prove that $\angle CAE = \angle BAF$.