Barycentric Coordinates

ϵ Books*

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Abstract

We've all heard of the term "barycentric coordinates" a couple of times, but this is one of the huge leaps that take incredible amounts of determination to make. This is my shot at bridging this gap.

Don't think for a second that barycentric coordinates are elegant. This is a powerful tool, but it is still fundamentally a bash technique. This technique also does have flaws; cyclic quadrilaterals in particular make life miserable.[1]

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[1]See Section 5.7 of http://web.evanchen.cc/handouts/bary/bary-full.pdf.

Overview

1 Preliminaries

1.1 Notation

Our reference triangle is $\triangle ABC$. We denote $\angle A = \angle CAB, \angle B = \angle ABC, \angle C = \angle BCA, a = BC, b = CA, c = AB$.

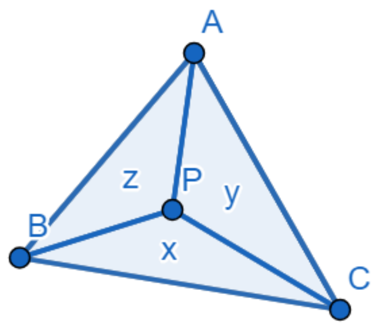
Also, let $\diamond P$ denote the mass of P.

1.2 Area Definition

Definition 1. *(Signed Areas)* We use *signed areas* for this entire section. What this means that if A, B, C goes counterclockwise, then $[ABC]$ is positive, and if A, B, C goes clockwise, then $[ABC]$ is negative.

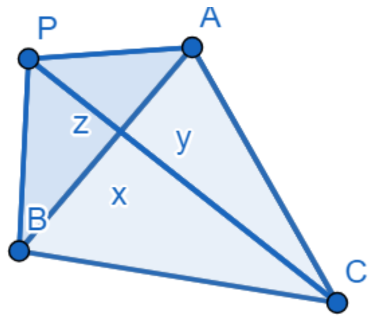
We first consider the most intuitive definition of barycentric coordinates, which is the area definition. With barycentric coordinates, our reference is a triangle, not a pair of perpendicular axes.

Definition 2. *(Area Ratios)* The barycentric coordinates of P with respect to $\triangle ABC$ are (x, y, z) such that $[BCP] : [CAP] : [ABP] = x : y : z$, and $x+y+z = 1$.



It is very important to note that the order of the vertices of $\triangle ABC$ matter. The coordinates of the same point P are different with respect to $\triangle ABC$ and $\triangle ACB$, for example. (In fact, any permutation of the vertices corresponds to a permutation of the coordinates.) For convenience, we can express this as an ordered ratio; $(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z})$ can easier be expressed as $(x : y : z)$. (This is because **normalizing,** or the process of making $x + y + z = 1$ is fairly easy, and even at some times not needed.)

Another important thing to note is that barycentric coordinates always must add to 1. Let's see what happens if we put P outside the triangle.



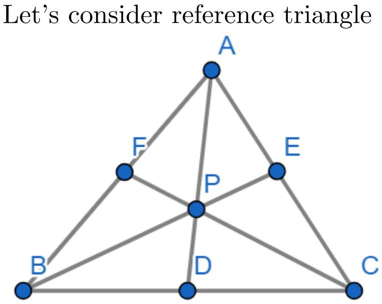
All of a sudden, the sum of the areas of the triangles is not $[ABC]$. This seems like a problem, until we notice that in the diagram above, $[ABC] = x + y - |z|$. This is the reason that signed areas are necessary.

1.3 Mass Points Definition

Remark 1. We use directed lengths as well for this entire section.

Barycentrics are known as an extension of mass points, which may not be apparent at first glance. But the analogy is certainly valid.

Let's consider reference triangle $\triangle ABC$ and a point P.



Draw an altitude from A to BC, and draw one from P to BC. Clearly, since the two triangles have the same base, the ratio of $[ABC] : [PBC]$ is the same as the ratio of their altitudes. And $PD : AD = a : a + b + c$, where $P = a + b + c, A = a$. We can do this with the other two vertices, which should illustrate the point.

We can generalize to zero and negative values if we just use directed lengths.

Thus, if $\triangle ABC$ has masses $\diamond A, \diamond B, \diamond C$, then P has barycentrics $(\frac{1}{\diamond A} : \frac{1}{\diamond B} : \frac{1}{\diamond C})$. (This is due to the reciprocal nature of mass points!)

Definition 3. P has barycentric coordinates (x, y, z) such that $\frac{BD}{DC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x}$.[2]

Remark 2. This can be used to prove Ceva's and Menelaus's very easily.

[2]See http://www.aquatutoring.org/TYMCM%20Mass%20Points.pdf for an introduction to mass points.

1.4 Vector Definition

Let $\vec{A}, \vec{B}, \vec{C}, \vec{P}$ be the vectors[3] with heads A, B, C, P and tail O for some arbitrary O. (We omit O because our choice of O. is irrelevant. Sometimes, we will define O as P for convenience; other times, we will not specify O.)

Definition 4. *(Vectors)* Then,

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}, x + y + z = 1$$

where (x, y, z) denotes the barycentric coordinates of P.

1.5 Normalization

We say coordinates x, y, z are normalized if $x + y + z = 1$. If they are not normalized, this means that we have expressed it in the form $(kx : ky : kz)$, where (x, y, z) is normalized, but $(kx : ky : kz)$ are not necessarily.

2 Common Points

Theorem 1. (Centroid) *The areal[4] coordinates of the centroid of $\triangle ABC$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

Proof. Trivial by the area definition. □

Theorem 2. (Symmedian Point)[5] *The barycentric coordinates of the symmedian point are $(a^2 : b^2 : c^2)$.*

Proof. It is a well-known property of the symmedian point that its distance to the sides of the triangle is proportional to the lengths of the triangle, i.e. its trilinear coordinates[6] are $(a : b : c)$. The conversion is straightforward; we get the barycentric coordinates as $(a^2 : b^2 : c^2)$. □

Theorem 3. (Incenter) *The barycentric coordinates of the incenter of $\triangle ABC$ are $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$.*

Proof. We use the area definition. We note

$$[BCP] = \frac{ar}{2}, [CAP] = \frac{br}{2}, [ABP] = \frac{cr}{2},$$

[3]Numerous resources exist for vectors; you could go to Evan's bary paper and see Appendix A, or you could email me for the chapter on vectors in my book.

[4]Areal coordinates are another name for barycentric coordinates. The only difference is that areal coordinates must be normalized!

[5]AKA Lemoine Point. See http://forumgeom.fau.edu/FG2008volume8/FG200812.pdf for basic properties.

[6]Trilinear coordinates are just ratios of the distances from P to the sides. Like barycentric coordinates, trilinear coordinates are homogenous, i.e. $(a : b : c)$ is the same as $(ka : kb : kc)$, though for some cases you will need to normalize them.

and $[ABC] = \frac{(a+b+c)r}{2}$. Dividing yields the barycentric coordinates of our incenter as

$$(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}),$$

as desired. $\square$

Theorem 4. (Excenter) *The A-excenter has barycentric coordinates $(-a : b : c)$, and symmetric expressions exist for the B and C excenters.*

Proof. Trivially, the perpendiculars from the $A - excenter$ to sides a, b, c all have the same absolute distance. Using $\frac{bh}{2}$ (4.2), we get

$$[BCE_A] : [CAE_A] : [ABE_A] = \frac{-ar}{2} : \frac{br}{2} : \frac{cr}{2} = -a : b : c,$$

as desired. $\square$

Theorem 5. (Orthocenter) *The barycentric coordinates of the orthocenter of $\triangle ABC$ are* $(\tan A : \tan B : \tan C)$.

Proof. Let H be the orthocenter. We use the area definition. WLOG, the circumcenter has diameter 2, then $a = \sin A, b = \sin B, c = \sin C$. Then we can let D, E, F be the foots of the altitudes of A, B, C, respectively. Then we use right $\triangle ABD$ and note that $BC = \cos B, BD = \sin C \cos B, HD = \cos B \cos C, [BCH] = \frac{\sin A \cos B \cos C}{2}$. We use symmetry and note $[ACH] = \frac{\sin B \cos A \cos C}{2}, [ABH] = \frac{\sin C \cos A \cos B}{2}$. This implies

$$[BCH] : [ACH] : [ABH] =$$

$$\sin A \cos B \cos C : sin B \cos A \cos C : \sin C \cos A \cos B =$$

$$\tan A : \tan B : \tan C,$$

as desired. (The last transition comes from dividing everything by $\cos A \cos B \cos C$.) $\square$

Theorem 6. (Circumcenter) *The barycentric coordinates of the circumcenter of $\triangle ABC$ are* $(\sin 2A : \sin 2B : \sin 2C)$.

Proof. Let the circumcenter be O. We use the area definition. Note that $OA = OB = OC$. Then we use the Inscribed Angle Theorem and angle chase; this means $\angle BOC = 2A, \angle COA = 2B, \angle AOB = 2C$. Then we use $[ABC] = \frac{1}{2}ab \sin C$ (4.4) to get

$$[BCO] : [CAO] : [ABO] =$$

$$r^2 \sin 2A : r^2 \sin 2B : r^2 \sin 2C =$$

$$\sin 2A : \sin 2B : \sin 2C,$$

as desired. $\square$

Remark 3. The orthocenter and circumcenter are not that nice. However, there is some useful stuff you can do with them!

3 Area, Lines, and Circles

3.1 Area

Theorem 7. (Area, Collinearity, and Concurrency) *Given three points P,Q,R with normalized coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ respectively,*

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Proof. We use Cartesian Coordinates; all points written in Cartesian Coordinates will be expressed as $[x, y, z]$ for clarity.

Choose O not in the plane determined by A,B,C, such that $O = [0, 0, 0], A = [1, 0, 0], B = [0, 1, 0], C = [0, 0, 1]$. (We are using a three-dimensional coordinate system!) Then we note the form of the plane containing ABC has the equation $x + y + z = 1$. (This corresponds to the normalized coordinates of any point in the plane!) Then let the parallelepiped that A,B,C spans (remember their tails are $O = [0, 0, 0]$, which this time cannot be ignored due to no preservation of generality) be denoted as P_{ABC}, and similarly, let P_{PQR} denote the parallelepiped spanned by $\vec{P}, \vec{Q}, \vec{R}$. Then we use the determinant definition of volume and note that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \frac{P_{PQR}}{P_{ABC}}.$$

Then we note that by the definition of a parallelepiped, $\frac{[P_{PQR}]}{[P_{ABC}]} = \frac{2[PQR]h}{2[ABC]h} = \frac{[PQR]}{[ABC]}$, as desired. □

Corollary 1. (Collinearity) *Three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Proof. Clearly, three points P, Q, R (which is what we shall assign our points as) are collinear if and only if $[PQR] = 0$. The only way for $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$ is if $[PQR] = 0$, or if P, Q, R are collinear, as desired. □

Remark 4. This proof is fairly trivial from the Area theorem, but this tool has lots of power.

Corollary 2. (Collinearity Again) *Three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Proof. Determinants satisfy the property that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & x_1+y_1+z_1 \\ x_2 & y_2 & x_2+y_2+z_2 \\ x_3 & y_3 & x_3+y_3+z_3 \end{vmatrix}.$$

Noting that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = x_3 + y_3 + z_3 = 1$ completes the proof. □

Remark 5. In most cases, calculation will be less painful.[7]

Theorem 8. (Concurrency)

Lines $u_i x + v_i y + w_i z = 0$ for $i = 1, 2, 3$ concur if and only if

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0.$$

Proof. This is basically solving a system of equations; trivial by Gaussian Elimination. □

Corollary 3. (Concurrency Again) *Lines $u_i x + v_i y + w_i z = 0$ for $i = 1, 2, 3$ concur if and only if*

$$\begin{vmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{vmatrix} = 0.$$

Proof. Determinants satisfy the property that

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & u_1+v_1+w_1 \\ u_2 & v_2 & u_2+v_2+w_2 \\ u_3 & v_3 & u_3+v_3+w_3 \end{vmatrix}.$$

Noting that $u_1 + v_1 + w_1 = u_2 + v_2 + w_2 = u_3 + v_3 + w_3 = 1$ completes the proof. □

3.2 Lines

Theorem 9. (Line) *The general equation of a line is $dx + ey + fz = 0$.*

Proof. It is well-known two points determine a line. Let the two points be $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$. By the Concurrency Theorem, we desire for any point (x, y, z) on the line to satisfy

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The determinant is $xy_1z_2 + yz_1x_2 + zx_1y_2 - zy_1x_2 - z_1y_2x - x_1yz_2$. Since $x_1, y_1, z_1, x_2, y_2, z_2$ are constant, this factors out to $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(y_1x_2 - x_1y_2)$, which is enough to finish our proof. □

[7]See corollary 13 of http://web.evanchen.cc/handouts/bary/bary-full.pdf.

Corollary 4. (Line Passing Through Given Points) *The equation of a line passing through* $(x_1, y_1, z_1), (x_2, y_2, z_2)$ *is*

$$x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(y_1x_2 - x_1y_2).$$

Proof. The proof for this is already in the proof of the general form of a line. □

Corollary 5. (Line Through a Vertex) *A line that passes through* A *has general equation* $\frac{y}{z} = k$ *for constant* k*. Symmetric expressions exist for* B, C*.*

Proof. Note that $(0, d, 1-d)$ represents the point our line intersects BC. We use $(1, 0, 0), (0, d, 1-d)$, substitute, and get $y(1-d) + z(-d) = 0$, or $y(1-d) = zd$, or $\frac{y}{z} = \frac{d}{1-d}$. Since d is constant for any given line passing through A, we are done. □

Definition 5. *(Displacement Vector)* The displacement vector $\vec{PQ}$ of $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$.

Remark 6. Our displacement vector has coordinates that sum to 0.

Lemma 1. *When* $\vec{O} = \vec{0}, \vec{A} \cdot \vec{A} = R^2$. *(R is the circumradius.)*

Proof. Let O be the circumcenter. Then this is trivially true, as $\vec{A}^2 = ||A||^2 = R^2$. □

Lemma 2. *When* $\vec{O} = \vec{0}, \vec{A} \cdot \vec{B} = R^2 - \frac{c^2}{2}$. *(R is the circumradius.)*

Proof. Again, let O be the circumcenter. Then

$$\vec{A} \cdot \vec{B} = R^2 \cos \angle AOB = R^2 \cos 2\angle ACB = R^2(1 - 2\sin^2 C) = R^2 - \frac{1}{2}(2R \sin C)^2 = R^2 - \frac{c^2}{2}.$$

(This comes from Inscribed Angle Theorem, the obvious $\sin^2 x + \cos^2 x = 1$ formula, and the Extended Law of Sines.) □

Theorem 10. (Evan's Favorite Forgotten Trick)[8] *Consider* $\overrightarrow{MN}$ *and* $\overrightarrow{PQ}$ *with coordinates* $(x_1, y_1, z_1), (x_2, y_2, z_2)$ *respectively. Then* $MN \perp PQ$ *if and only if*

$$a^2(y_1z_2 + z_1y_2) + b^2(z_1x_2 + x_1z_2) + c^2(x_1y_2 + y_1x_2) = 0.$$

Proof. Let $\vec{O} = \vec{0}$. Then we use the vector perpendicularity condition and note that it is necessary and sufficient for

$$(x_1\vec{A} + y_1\vec{B} + z_1\vec{C}) \cdot (x_2\vec{A} + y_2\vec{B} + z_2\vec{C}) = 0.$$

Expansion yields

$$\sum_{cyc}(x_1x_2\vec{A} \cdot \vec{A}) + \sum_{cyc}((x_1y_2 + x_2y_1)\vec{A} \cdot \vec{B}) = 0.$$

[8] Hi Evan!

Applying Lemma 1 and Lemma 2 gives us

$$\sum_{cyc}(x_1x_2R^2) + \sum_{cyc}((x_1y_2 + x_2y_1)(R^2 - \frac{c^2}{2})),$$

which implies

$$R^2(\sum_{cyc}(x_1x_2) + \sum_{cyc}(x_1y_2 + y_1x_2)) = \frac{1}{2}\sum_{cyc}((x_1y_2 + x_2y_1)(c^2)),$$

leading to

$$R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = R^2 \cdot 0 \cdot 0 = \frac{1}{2}\sum_{cyc}((x_1y_2 + x_2y_1)(c^2))$$

$$0 = \sum_{cyc}((x_1y_2 + x_2y_1)(c^2)),$$

as desired. □

Corollary 6. ($BC \perp PQ$) *Given displacement vector* $\overrightarrow{PQ} = (x_1, y_1, z_1)$,

$$BC \perp PQ$$

if and only if

$$a^2(z_1 - y_1) + x_1(c^2 - b^2) = 0.$$

Proof. We note that displacement vector $\overrightarrow{BC}$ has coordinates $(0, 1, -1)$. By Evan's,

$$a^2(z_1 - y_1) + z_1(c^2 - b^2) = 0,$$

which comes directly by substitution. □

Corollary 7. *The perpendicular bisector of BC can be expressed as*

$$a^2(z - y) + x(c^2 - b^2) = 0.$$

Proof. Note that $\overrightarrow{BC}$ has coordinates $(0, 1, -1)$ and that any point on PQ must at have the form $(0-x, \frac{1}{2}-y, \frac{1}{2}-z)$. (This comes by plugging the midpoint D in and any arbitrary point on the line P. We can let P have coordinates (x, y, z).)

Plugging this into Evan's yields $a^2(\frac{1}{2} - y - \frac{1}{2} + z) + b^2(-x) + c^2(x) = 0$. Simplifying yields

$$a^2(z - y) + x(c^2 - b^2) = 0,$$

as desired. □

Theorem 11. (Strong Evan) *Given M, N, P, Q, let*

$$\overrightarrow{MN} = x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO}$$

$$\overrightarrow{PQ} = x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}.$$

If $x_i + y_i + z_i = 0$ for either $i = 1, i = 2$, then $MN \perp PQ$ if and only $0 = a^2(y_1z_2 + z_1y_2) + b^2(z_1x_2 + x_1z_2) + c^2(x_1y_2 + y_1x_2) = 0.$

Proof. Our EFFT Lemmas still hold, since $\overrightarrow{PQ}$ can be shifted.

We note it is necessary and sufficient for

$$(x_1\vec{A} + y_1\vec{B} + z_1\vec{C}) \cdot (x_2\vec{A} + y_2\vec{B} + z_1\vec{C}) = 0.$$

Using our lemmas, expanding, and doing the same stuff as we did for normal EFFT, we get

$$R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = \frac{1}{2}\sum_{cyc}((x_1y_2 + x_2y_1)(c^2)).$$

Since at least one of $x_i + y_i + z_i = 0$, we're done. $\square$

Theorem 12. (Distance Formula) *Given displacement vector* $\overrightarrow{PQ} = (x, y, z)$,

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy.$$

Proof. We use the fact that $PQ^2 = (x\vec{A} + y\vec{B} + z\vec{C}) \cdot (x\vec{A} + y\vec{B} + z\vec{C})$. This yields

$$|PQ|^2 = (x\vec{A} + y\vec{B} + z\vec{C}) \cdot (x\vec{A} + y\vec{B} + z\vec{C}) =$$
$$(x + y + z)(|A|^2x + |B|^2y + |C|^2z) - yz|B - C|^2 - xz|A - C|^2 - xy|A - B|^2 =$$
$$-a^2yz - b^2zx - c^2xy = |PQ|^2,$$

as desired. (The reason we can get rid of $x + y + z$ is because $x + y + z = 0$ by definition.) $\square$

3.3 Circles

Theorem 13. (Circle) *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for constants u, v, w.

Proof. Let the circle have center (i, j, k) and radius r. Then we use the Distance formula and note that this is

$$-a^2(y - j)(z - k) - b^2(z - k)(x - i) - c^2(x - i)(y - j) = r^2.$$

Expanding yields

$$-a^2yz - b^2zx - c^2xy + Lx + My + Nz = C$$

for constants L, M, N, C. Since $x + y + z = 1$, we rewrite the righthand side as $C(x + y + z)$, and subtracting yields

$$-a^yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0,$$

where $u = L - C, v - M - C, w = N - C$. $\square$

Remark 7. The hideous general form of a circle means that we have to intelligently use barycentric coordinates on circles. Blind usage of them will get us nowhere. However, the circles we want to use will usually be much nicer.

Corollary 8. (Circumcircle) *The circumcircle can be represented as*

$$a^2yz + b^2zx + c^2xy = 0.$$

Proof. Three points define a unique circle; we know from the Circle theorem that this is in the form of a circle, so plugging in points A, B, C and noticing they satisfy the proof is enough. $\square$

4 Examples

1. Find the barycentric coordinates of A, B, C.

   Solution: Obviously if it is A, then $[BPA] = [ABC]$ and the other two triangles have area 0. Generalizing for B, C, this means that $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$. This is a very important tool for barycentric coordinates.

2. Why did we define the barycentric coordinates of P as $(\frac{\triangle[CPB]}{\triangle[ABC]}, \frac{\triangle[APC]}{\triangle[ABC]}, \frac{\triangle[BPA]}{\triangle[ABC]})$? Why can't we use $\triangle[BPC]$ in place of $\triangle[CPB]$?

   Solution: Because the areas are signed, so order does matter. Of course, we could've done $[BCP]$ or $[PBC]$ for the first triangle and so on.

3. Does our definition work if $\triangle[ABC]$ is clockwise?

   Solution: Yes! If P is in the "interior side" of AB then A, B, P is clockwise. This means the negatives cancel out, which is nice.

4. Find the midpoint of BC in barycentric coordinates.

   Solution: The areas of $[BPA]$ and $[APC]$ are equivalent, and $[CPB] = 0$, so the midpoint is $(0, \frac{1}{2}, \frac{1}{2})$.

5. Prove Ceva's Theorem using barycentric coordinates.

   Solution: The mass points definition of barycentric coordinates states that $\frac{BD}{DC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x}$, where D, E, F are the intersections of AP with BC, CA, AB respectively. Since AD, BE, CF are concurrent cevians by definition, then $\frac{z}{y} \cdot \frac{x}{z} \cdot \frac{y}{x} = 1$, as desired.

   Alternatively, note that D, E, F must have barycentric coordinates of the form $(0, d, 1-d), (1-e, 0, e), (f, 1-f, 0)$. Lines AD, BE, CF have equations $\frac{z}{y} = \frac{1-d}{d}, \frac{x}{z} = \frac{1-e}{e}, \frac{y}{x} = \frac{1-f}{f}$, implying that $1 = \frac{(1-d)(1-e)(1-f)}{def}$, as desired.

6. Prove Menelaus' Theorem using barycentric coordinates.

   Solution: Let D, E, F be on BC, CA, AB and let D, E, F have coordinates $(0, d, 1-d), (1-e, 0, e), (f, 1-f, 0)$. This means that our statement is equivalent to $|\frac{def}{(1-d)(1-e)(1-f)}| = 1$, when using directed lengths. We then note the equation of FD is

$$\begin{vmatrix} x & 0 & f \\ y & d & 1-f \\ z & 1-d & 0 \end{vmatrix} = 0,$$

for some arbitrary x, y, z. This implies

$$y(1-d)f - x(1-f)(1-d) - zdf = 0,$$

or

$$zfd = yf(1-d) - x(1-f)(1-d).$$

Since x, y, z are arbitrary, we just plug in the coordinates of E to get $def = -(1-d)(1-e)(1-f)$ (the negative comes through the way we direct our lengths). Dividing both sides by $(1-d)(1-e)(1-f)$ and taking absolute values completes the proof.

7. Find the equation of line BC.

   Solution: Substituting the points $(0,1,0), (0,0,1)$ into our corollary gives us $x = 0$. (Can you generalize for AB, BC by providing equations for them?)

8. Find the equation for the $A - median$ of $\triangle ABC$.

   Solution: Since a median is a cevian, the $A - median$ passes through A. We use our last corollary and note that the $A - median$ intersects BC at $(0, \frac{1}{2}, \frac{1}{2}) = (0, d, 1-d)$ where $d = \frac{1}{2}$. Thus, $\frac{y}{z} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$, implying the equation is just $y = z$.

9. Consider $\triangle ABC$ with $AB = 13, BC = 15, CA = 14$. If M is the midpoint of BC and P is a point on AC such that $MP \perp AC$, find MP.

   Solution: This is a warning against mindlessly using barycentric coordinates whenever feasible.

   This is the well-known $13 - 14 - 15$ triangle, so the B altitude has length 12. Using similar triangles, we see there's a ratio of $1/2$, so $MP = 6$.

   You'll see that the barycentric solution involves ugly systems, EFFT, and distance formula, which is very not nice.

5 Exercises

1. Prove that the angle bisectors/medians/altitudes/perpendicular bisectors are concurrent.

2. Prove Stewart's Theorem.

3. Prove that given the vector definition, the area definition and mass points definition is consistent.

4. Prove the centroid divides the median by a $2:1$ ratio.

5. Consider rectangle $ABCD$ with $AB = 6, BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM, CN intersect AC at X, Y. Find XY.

6. Consider $\triangle ABC$ with $AB = 7, BC = 8, AC = 6$. Let AD be the angle bisector of $\angle BAC$ and let E be the midpoint of AC. If BE and AD intersect at G, find AG.

7. (TST 2003 #2) Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that
$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC].$$

8. Consider $\triangle ABC$ with $\angle A = 45°, \angle B = 60°$, and with circumcenter O. If BO intersects CA at E and CO intersects AB at F, find $\frac{[AFE]}{[ABC]}$.

9. (ISL 1998 G5) Let ABC be a triangle. Let D, E, F be the reflections of A, B, C in BC, AC, AB respectively. Show that D, E, F are collinear if and only if $OH = 2R$.

10. (2012 ELMO Proposal) Let ABC be a triangle with orthocenter H and incenter I. Let D and E be the feet of the perpendiculars from A to BC and from C to AB, and let the incircle of the triangle touch the sides AC and AB at F and G, respectively. If I lies on DE and H lies on FG, show that (a) angle ACB is 60 degrees, and (b) $BG = 2AG$.

6 Hints and Comments

Hints and comments have been "encrypted" with ROT13.[9]

1. Guvf vf n fgnaqneq rkrepvfr jvgu onfvpnyyl nalguvat gung erzbgryl vaibyirf pbapheerapl.

2. Hfr gur qvfgnapr sbezhyn.

3. Lbh bayl arrq gb cebir bar bs gurfr. Gur bgure sbyybjf rnfvyl sebz cebivat gur svefg.

4. Hfr znff cbvagf.

5. V'q gryy lbh gb hfr znff cbvagf, ohg fvzvynevgl vf orggre.

[9] This means that any letter has been replaced with the letter of the alphabet 13 letters after. The encryption and decryption is the same. See https://cryptii.com/rot13 for a decryptor.

6. Guvf fubhyq or ebhgvar. Gurer'f n orggre jnl guna Fgrjneg'f gb pnyphyngr gur yratgu gubhtu.
7. Hfr znff cbvagf. Onelpragevp gb gevyvarne vaibyirf htyl senpgvbaf, ohg vg vf fgvyy fgenvtugsbejneq.
8. Gurer vf n znff cbvagf qrsvavgvba bs onelpragevp pbbeqvangrf. Hfr vg.
9. Qba'g or nsenvq gb znxr zber cbvagf.
10. Sbe neovgenel privna NQ, fcyvg natyr N. Chg fghss va grezf bs fvarf nf jryy.

7 Extra

Here are some extra things that might be of use.

7.1 Barycentric to Cartesian/Complex

Barycentrics to Cartesian is very straightforward using the vector interpretation. If $A = (x_a, x_y), B = (x_b, y_b), C = (x_c, y_c)$ and $P = (x, y, z)$, where $x+y+z = 1$, then P has Cartesian Coordinates $(xx_a + yx_b + xz_c, xy_a + yy_b + yz_c)$. Similarly, if $A = a_x + a_y i, B = b_x + b_y i, C = c_x + c_y i$, then $P = xA + yB + cZ$. This means we can theoretically find the orthocenter/incenter/centroid/circumcenter easily, but Cartesian or complex coordinates are usually better for this. Nonetheless, if you ever need to, this is how you can convert.

7.2 Conway Notation

Definition 6. *(Conway Notation)* Given $\triangle ABC$, let $S = 2[ABC]$. Then let $S_\theta = S\cot\theta$, and $S_\theta S_\alpha = S_{\theta\alpha}$.

Here are a few essential facts that lead to interesting results.

Theorem 14. (S_A) *For reference* $\triangle ABC$, $S_A = \frac{-a^2+b^2+c^2}{2}$. *Cyclic variations hold.*

Corollary 9. (S_B+S_C) *As an exercise, prove* $S_B+S_C = a^2$ *and cyclic variants.*

Corollary 10. $(\sum\limits_{cyc} S_{AB})$ *As another exercise (a little harder this time), prove*

$$S_{AB} + S_{BC} + S_{CA} = S^2.$$

Theorem 15. (Conway Formula) *Given* P *with directed* $\measuredangle PBC = \alpha$ *and* $\measuredangle BCP = \beta$, P *has barycentric coordinates*

$$(-a^2 : S_C + S_\beta : S_B + S_\alpha).$$

Proof. An outline will be provided.

Use the Law of Sines on $\triangle PBC$, and express BP, CP in terms of α, β. Doing $\frac{1}{2}ab\sin C$ on the three triangles (use area definition) and clearing denominators finishes the proof.

The details are left as an exercise. $\square$

7.3 Other Points

These points won't show up as much, but you might get lucky.

Theorem 16. (Appolonius Point) *The Appolonius Point*[10] *has barycentric coordinates* $a^3\cos^2(\frac{B-C}{2}) : b^3\cos^2(\frac{C-A}{2}) : c^3\cos^2(\frac{A-B}{2})$.

Theorem 17. (Feuerbach Point) *The barycentric coordinates of the Feuerbach Point are* $((2s-a)(b-c)^2 : (2s-b)(c-a)^2 : (2s-c)(a-b)^2)$.

Theorem 18. (Fermat Point 1) *The First Fermat Point has barycentric coordinates* $(a\csc(A+\frac{\pi}{3}) : b\csc(B+\frac{\pi}{3}) : c\csc(C+\frac{\pi}{3}))$.

Theorem 19. (Gregonne Point) *The Gregonne point has barycentric coordinates* $(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c})$.

Theorem 20. (H) *In Conway Notation, the orthocenter has barycentric coordinates* $(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C})$.

Theorem 21. (Isogonal Conjugate) *The isogonal conjugate of* $P = (x : y : z)$ *has barycentric coordinates* $(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z})$.

Proof. Have P^* be the isogonal conjugate of P. Let AP, AP^* intersect BC at D, E. It is a property of isogonal conjugates that $\frac{BD \cdot BE}{CE \cdot CD} = \frac{c^2}{b^2}$. This implies $\frac{BD}{CD} = \frac{c^2 \cdot CE}{b^2 \cdot BE}$, and by the mass points definition, this means $BD \cdot CD = \frac{y}{b^2} \cdot \frac{c^2}{z}$. Thus, the barycentrics of $P*$ satisfy $\frac{z*}{y*} = \frac{c^2/z}{b^2/y}$ by the mass points definition, and symmetrically, $\frac{y*}{x*} = \frac{b^2/y}{a^2/x}$, implying $(x* : y* : z*) = (a^2x, b^2y, c^2z)$, as desired. $\square$

Theorem 22. (Isotomic Conjugate) *The isotomic conjugate of* $P = (x : y : z)$ *has barycentric coordinates* $(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$.

Proof. It is a property of isotomic conjugates that if P,Q are isotomic conjugates, then $[ABP][ABQ] = [CAP][CAQ] = [BCP][BCQ]$. Thus $x[ABQ] = y[CAQ] = z[BCQ]$, and the area definition completes the proof. $\square$

Theorem 23. (Nagel Point) *The Nagel Point has barycentric coordinates* $(s-a : s-b : s-c)$.

Theorem 24. (Nine Point Center) *The Nine Point Center has barycentric coordinates* $(a\cos B - C : b\cos C - A : c\cos A - B)$.

Theorem 25. (O) *In terms of side lengths* a, b, c *and Conway Notation, the circumcenter has barycentric coordinates* $(a^2S_A : b^2S_B : c^2S_C)$.

[10] The isogonal conjugate of the isotomic conjugate of $X(12)$. See http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

7.4 Lines

Theorem 26. (Euler Line) *The Euler Line has equation*

$$S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z = 0.$$

7.5 Circles

Theorem 27. (Incircle) *The incircle has equation*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(s^2x + (s - c)^2y + (s - b)^2z) = 0.$$

7.6 Parting Shots

1. Prove the Gregonne and Nagel points are isotomic conjugates.

2. Prove that the only point that is its own isogonal conjugate are the incenters and excenters.

3. Prove that the only point that is its own isotomic conjugate is the centroid.

4. (ISL 2001 G1) Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

5. Let triangle ABC have circumcenter O and incenter I. D, E are on CB, CA such that $AD \perp CB, BE \perp AC$. AI intersects CB at P and BI intersects AC at Q.Prove that P, O, Q are colinear if and only if D, I, E are collinear.

6. (IMO 2004 #5) In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

   Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.