Telescoping Remastered

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Abstract

Telescoping is the art of expressing a predictable sum or product in such a way that most of the expression cancels out. Formal definitions include series with a finite amount of terms for every partial sum, and so on, but the definition and method really is heuristic. Often the best way to see if something is a telescoping problem is to see if it cancels stuff out predictably.

Even though the definition is heuristic, we can precisely categorize problems as telescoping or not; usually, telescoping will be the only non-bash or feasible solution to a telescoping problem.

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Overview

1 Prior Knowledge

You should know summation notation and product notation. If you don't, that should be fine; the notation is really obvious and intuitive once you learn it.

For the definitions below, c is constant, i is meant to represent the initial term, and n can either be varying or constant, depending on the problem.

Definition 1. *(Summation Notation)* The expression

$$\sum_{i=c}^{n} f(i)$$

is equivalent to

$$f(c) + f(c+1) + f(c+2) + \cdots + f(n-1) + f(n).$$

Definition 2. *(Product Notation)* The expression

$$\prod_{i=c}^{n} f(i)$$

is equivalent to

$$f(c)f(c+1)f(c+2)\cdots f(n-1)f(n).$$

The $f(i)$ might be intimidating, but it really just generally covers all functions (expressions in terms) of i.

1.1 Notational Examples

Problem 1. Expand $\sum_{i=1}^{10} i$.

Since $f(i)$ is just i, $c = 1$, and $n = 10$, we have

$$\sum_{i=1}^{10} i = f(1) + f(2) + \cdots + f(10) = 1 + 2 + \cdots + 10 = 55.$$

Examples in $\prod$ notation should basically be equivalent to summation notation in terms of understandability, so we'll only present one example.

Problem 2. Expand $\prod_{i=1}^{10} i$.

It's basically the same; instead, we have

$$\prod_{i=1}^{10} i = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 10 = 10!$$

Remark 1. Note that we just swapped the addition signs for multiplication signs.

1.2 Notational Exercises

Problem 3. *(Triangular Numbers)* Write out the summation

$$\sum_{i=1}^{n},$$

and provide a general formula for this value in terms of n.

Problem 4. *(Get Used to Infinity)* Expand $\sum_{i=0}^{\infty} \frac{1}{2^n}$.

Remark 2. It's acceptable to use a trailing $\cdots$ after giving two or three terms for something that goes on infinitely!

Problem 5. Write $n!$ in product notation.

2 Theory

Telescoping really is just expressing $f(x)$ as an equivalent value that cancels nicely. Since this is a heuristic method, we'll mostly be teaching through problems, as the only "theory" is arithmetic.

2.1 A Trivial Example

Problem 6. Find $\frac{1}{2} \cdot \frac{2}{3} \cdot \cdots \cdot \frac{99}{100}$.

The solution consists of canceling all the 2's, 3's, $\cdots$ 98's, and 99's. This leaves us with just $\frac{1}{100}$.

Problem 7. *(A Natural Extension)* Find $\prod\limits_{i=1}^{n} \frac{i}{i+1}$ in terms of n.

WARNING. *If you can't understand either of these problems, this handout is going to be too difficult for you.*

2.2 Partial Fraction Decomposition

Definition 3. The partial fraction decomposition of $\frac{f(x)}{g(x)}$ can be expressed as $\sum \frac{f_i(x)}{g_i(x)}$, where $g_i(x)$ is a root of $g(x)$ with a smaller degree.

Remark 3. It is not always necessary to fully decompose a fraction; there are cases where going partway is fine. This is in the context of a problem; for the exercises, I recommend fully decomposing them.

We present an example that will be relevant to telescoping.

Problem 8. Decompose $\frac{1}{n(n+1)}$.

Solution: Note that we want to express this in the form of $\frac{K}{n} + \frac{J}{n+1}$ such that $\frac{K}{n} + \frac{J}{n+1} = \frac{K(n+1)+J(n)}{n(n+1)} = \frac{1}{n(n+1)}$. Co-efficient matching yields the system of equations

$$K + J = 0$$

$$K = 1,$$

implying $J = 1$. This means that our partial fraction decomposition is

$$\boxed{\frac{1}{n} - \frac{1}{n+1}}.$$

Remark 4. Generally, we will use co-efficient matching to do partial fraction decomposition. A large portion of telescoping problems will involve partial fraction decomposition.

We present the following extension as an exercise.

Problem 9. *A Natural Extension* What is the partial fraction decomposition of $\frac{1}{n(n+k)}$, in terms of k?

Finally, we present an example involving three variables.

Problem 10. Find the partial fraction decomposition of $\frac{1}{n(n+1)(n+2)}$.

Solution: Let the decomposition be $\frac{I}{n}+\frac{J}{n+1}+\frac{K}{n+2}$. This implies that

$$\frac{I(n+1)(n+2)+J(n)(n+2)+K(n)(n+1)}{n(n+1)(n+2)}=\frac{1}{n(n+1)(n+2)}.$$

We see that the following system results from co-efficient matching:

$$n^2(I+J+K)=0\rightarrow I+J+K=0$$

$$n(3I+2J+K)=0\rightarrow 3I+2J+K=0$$

$$2I=1.$$

Solving gives us $I=1/2, J=-1, K=1/2$, which means our partial fraction decomposition is

$$\boxed{\frac{1/2}{n}-\frac{1}{n+1}+\frac{1/2}{n+2}}.$$

WARNING. *Partial decomposition cannot be done on expressions such as* $\frac{1}{n^2}$ *because nothing differentiates* $\frac{I}{n}$ *from* $\frac{J}{n}$*!*

For fractions such as $\frac{1}{n^2(n+1)}$, we would decompose it as $\frac{I}{n+1}+\frac{J}{n^2}$.

2.3 Convergence and Divergence

A quick test for if a quantity $\sum_{i=c}^{n} f(i)$ can be telescoped is if the quantity $\sum_{i=c}^{\infty} f(i)$ diverges or converges.

Problem 11. *(Not a Telescope)* Find $\sum_{i=1}^{6}\frac{1}{i}$.

Bogus: We telescope cleverly. Note that $\frac{1}{i}=\frac{i+1}{i(i+1)}$. We abuse this fact to find that $\sum_{i=1}^{6}\frac{1}{i}=2\sum_{i=1}^{6}\frac{1}{i(i+1)}+\sum_{i=2}^{6}\frac{1}{i(i+1)}+\sum_{i=3}^{6}\frac{1}{i(i+1)}+\sum_{i=4}^{6}\frac{1}{i(i+1)}+\sum_{i=5}^{6}\frac{1}{i(i+1)}+\sum_{i=6}^{6}\frac{1}{i(i+1)}=2(1-1/7)+(1/2-1/7)+(1/3-1/7)+(1/4-1/7)+(1/5-1/7)+(1/6-1/7)$, which simplifies to $\sum_{i=1}^{6}\frac{1}{i}$.

This clearly does not help us. (If the above example went over your head, it's fine; it's an example of telescoping done incorrectly, not correctly.)

WARNING. *Don't telescope if it won't save you time!*

WARNING. *Don't try to telescope a divergent series!*

Let's take a look at this problem. We see that in this case, $f(i)=\frac{1}{i}$, and $c=1$.

Theorem 1. (Harmonic Series) *The infinite series* $\sum_{i=1}^{\infty} \frac{1}{i}$ *diverges.*

Remark 5. This theorem is very well-known; if you already are familiar with it, you can probably gloss over the rest of this section.

Proof. We note that

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots .$$

The inequality can be trivially achieved by comparing individual terms, and the second series is divergent because there are infinitely many groups summing up to $\frac{1}{2}$. $\square$

There are two important rules when dealing with divergent series.

1. If $\sum f(x) \geq \sum g(x)$ and $\sum g(x)$ diverges, then $\sum f(x)$ diverges.
2. If $\sum f(x) \leq \sum g(x)$ and $\sum (x)$ converges, then $\sum f(x)$ converges.

It goes without saying that these are infinite series.

Remark 6. Note that $\sum_{i=1}^{\infty} \frac{1}{n^2}$ converges (it converges to $\frac{\pi^2}{6}$), so any reasonable expression with a constant in the numerator and a degree ≥ 2 in the denominator should be telescopable with partial fraction decomposition.

Remark 7. For any $n > 1$, $\sum_{i=1}^{\infty} \frac{1}{i^n}$ will converge.

3 Examples

WARNING. *If you don't know summation and product notation by now, you may have a difficult time.*

Problem 12. Find $\prod_{i=3}^{10} \frac{i}{i+2}$.

Solution: We see this is $\frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \dots \cdot \frac{10}{12}$. Canceling the numbers from 5 to 10 yields our answer as

$$\frac{3 \cdot 4}{11 \cdot 12} = \boxed{\frac{1}{11}}.$$

Remark 8. It is frequently the case when telescoping that we will have more than two terms; this is perfectly fine and normal. (Though if you have too many terms when telescoping, it is possible you weren't supposed to telescope. Or that it's a particularly mean problem.)

Problem 13. *(Difference of Fractions)* Find $\sum_{i=1}^{19} \frac{1}{i(i+1)}$.

Solution: It would be really stupid to find this using arithmetic; we are all already good enough at it. Instead we look for something clever we can exploit; our technique comes from the fact that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Then we get

$$\sum_{i=1}^{19} \frac{1}{i(i+1)} = \sum_{i=1}^{19} \frac{1}{i} - \sum_{i=2}^{20} \frac{1}{i} = 1 - \frac{1}{20} = \boxed{\frac{1}{19}}.$$

Problem 14. Find $\sum_{n=1}^{21} \frac{1}{n(n+2)}$ to the nearest integer.

Solution: We will abuse the fact that

$$\frac{1}{n(n+2)} = \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2}).$$

Remark 9. The rest of the solution is just details.

This gives us

$$\sum_{n=1}^{21} \frac{1}{n(n+2)} = \frac{1}{2}(1 + 1/2 - 1/24 - 1/25).$$

Since

$$1/24, 1/25$$

are sufficiently small, we can state with confidence that $1+1/2-1/24-1/25 \geq 1 \to \frac{1}{2}(1 + 1/2 - 1/24 - 1/25) \geq 1/2$ implying that this sum rounds to $\boxed{1}$.

Problem 15. Find $\sum_{n=2}^{1000} \frac{1}{n^2-1}$.

Solution: This can be rewritten as

$$\sum_{n=1}^{999} \frac{1}{n(n+2)},$$

which simplifies to

$$\frac{1}{2}(\sum_{n=1}^{999} \frac{1}{n} - \sum_{n=3}^{1001} \frac{1}{n}) = \frac{1}{2}(1 + 1/2 - 1/1000 - 1/1001) = \boxed{\frac{1499499}{2002000}}.$$

Speaking of having lots of terms, we present the next problem (which is still not nice even after telescoping).

Problem 16. Find $\sum\limits_{n=1}^{1} 3\frac{1}{n(n+3)}$.

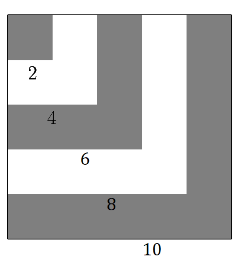
Solution: Abuse the fact that $\frac{1}{n(n+3)} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+3})$. Substituting yields

$$\sum_{n=1}^{1} 3\frac{1}{n(n+3)} = \frac{1}{3}(\sum_{n=1}^{1} 3\frac{1}{n} - \sum_{n=3}^{1} 5\frac{1}{n}) = \frac{1}{3}(1+1/2+1/3-1/14-1/15-1/16) = \boxed{\frac{2743}{5040}}.$$

Remark 10. I used this as an example because this was too ugly to be an exercise.

Now for something a little bit creative. This may not fit under the bill "telescoping," but in essence it is the same; clever manipulation leads to canceling terms out in a desirable way.

Problem 17. Find the shaded area.



Solution: The shaded area is

$$10^2 - 8^2 + 6^2 - 4^2 + 2^2 - 0^2 =$$

$$(10 - 8)(10 + 8) + (6 - 4)(6 + 4) + (2 - 0)(2 + 0) =$$

$$2(10 + 8) + 2(6 + 4) + 2(2 + 0) =$$

$$2(2 + 4 + 6 + 8 + 10) =$$

$$4(1 + 2 + 3 + 4 + 5) = 4 \cdot \frac{5(5 + 1)}{2} = 60.$$

Remark 11. The main idea was using difference of squares and canceling the mess. While this might not fit under the bill of telescoping, this is still similar enough and clever enough for me to include on the handout. (Keep in mind that next time, this won't be so easy to bash!)

Problem 18. Simplify $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$.

Solution: Note that in general, $1+a = \frac{1-a}{1-a^2}$. This means that this expression is condensed into

$$\frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1-x^{16}}{1-x^8} \cdot \frac{1-x^{32}}{1-x^{16}} = \frac{1-x^{32}}{1-x}.$$

Some basic knowledge of division yields our answer as

$$\sum_{n=0}^{31} x^n.$$

Remark 12. Telescoping isn't always straightforward! Be on the watch for clever manipulations.

We end the examples with a classic problem.

Problem 19. *(Going Halfway)* Find $\sum\limits_{n=0}^{\infty} \frac{1}{2^n}$.

Solution: We see that this is equivalent to $1 + 1/2 + 1/4 + \cdots$. Note that $1 = 2-1, 1/2 = 1-1/2, 1/4 = 1/2-1/4 \cdots$. Substituting yields

$$\sum_{n=0}^{\infty} = \sum_{n=-1}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} = \boxed{2}.$$

Remark 13. With a little bit of creativity, you should be able to do this for other infinite geometric series.

4 Exercises

Problem 20. Find a general formula for $\prod_{n=1}^{c} \frac{n}{n+1}$ for integer $c \geq 1$ in terms of c.

Problem 21. Find $\frac{1}{1\cdot 2} + \frac{2}{2\cdot 4} + \frac{3}{4\cdot 7} + \frac{4}{7\cdot 11} + \frac{5}{11\cdot 16}$.

Problem 22. Find $\prod_{n=1}^{1000} \frac{n}{n+2}$.

Problem 23. Simplify $\sum_{n=a}^{b} \frac{1}{n(n+k)}$ for constant a, b, k.

Problem 24. If $f(x) = \frac{x^2}{x^2-1}$, find $\prod_{n=1}^{50} f(n)$.

Problem 25. Find $\frac{1}{4} + \frac{1}{10} + \frac{1}{18} + \frac{1}{28} + \frac{1}{40} + \frac{1}{54} + \frac{1}{70} + \frac{1}{88} + \frac{1}{108}$.

Problem 26. Find $\sum_{n=1}^{13} \frac{1}{t(n)}$, where $t(n) = \sum_{i=1}^{n} i$ ($t(n)$ is the nth triangular number).

Problem 27. Find $\prod_{n=3}^{100} \frac{r^2-1}{r^2-4}$.

Problem 28. Find the exact value of $\sum_{i=1}^{132} \frac{1}{\sqrt{3i+1}+\sqrt{3(i+1)+1}}$.

Problem 29. What does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converge to?

Problem 30. What does $\prod_{n=2}^{\infty} (1 - \frac{1}{n^2})$ converge to?

Problem 31. Have $f(n) = \sum_{i=2}^{n} \frac{i-1}{i} \cdot (-1)^{i-1}$ and have $g(n) = \sum_{i=2}^{n} \frac{1}{i} \cdot (-1)^{i-1}$. Find

$$\sum_{n=1}^{49} \frac{1}{f(2n+1)} + \sum_{n=1}^{49} \frac{1}{g(2n+1)}.$$

Problem 32. Have $f(n) = \sum_{i=2}^{n} \frac{i-1}{i} \cdot (-1)^{i-1}$ and have $g(n) = \sum_{i=2}^{n} \frac{1}{i} \cdot (-1)^{i-1}$. Find

$$\sum_{n=2}^{100} f(n) + \sum_{n=2}^{100} g(n).$$

Problem 33. Find $\sum_{n=1}^{200} 5n(-1)^{200-n}$.

Problem 34. What does $\prod_{n=3}^{\infty} \frac{n^2-1}{n^2}$ converge to?