§ 1 Sample Problems

1. A rectangle has sides A and B such that $B > A$. Two circles, both with centers O are in the $B \times A$ rectangle, with one circumscribing the rectangle and one tangent to two parallel sides inside the rectangle. Find B if the region between both circles has an area of 4π.

2. A real number a satisfies the equation $(2)^{\frac{1}{a}} \cdot (4)^{\frac{1}{a^2}} \cdot (8)^{\frac{1}{a^3}} \cdot (16)^{\frac{1}{a^4}} \cdots = \sqrt[9]{16}$. Find a.

3. Let T be the set of positive integer divisors of 27000. Find the number of positive integers that can be expressed as a product of 3 pairwise distinct elements of T.

4. A rectangle $ABCD$ has $AB = 6$ and $BC = 8$. Let M be the midpoint of $\overline{AD}$ and let N be the midpoint of $\overline{CD}$. Let $\overline{BM}, \overline{BN}$ intersect $\overline{AC}$ at X, Y. Find XY.

5. Consider polynomial $f(x) = (x-1)(x-2)\ldots(x-8)$. Let a, b be integers such that $a \neq b$, a, b are not roots of $f(x)$, and the remainder of $f(x)$ when divided by $x - a$ and $x - b$ are equal. What is $a + b$?

6. We define the function $f_n(k)$ for positive integers k and $n \neq 1$ as follows: - n is a positive integer that is divisible by distinct primes $p_1, p_2, p_3, \ldots p_a$. - $f_n(k) = r_1 r_2 r_3 \ldots r_a$ where r_i is the remainder when k is divided by p_i for $1 \leq i \leq a$. Let S be the set of positive integers n such that

$$\sum_{k=1}^{p_1 p_2 \cdots p_a} f_n(k) = 630.$$

   Find $\sum_{x \in S} \frac{1}{x}$.