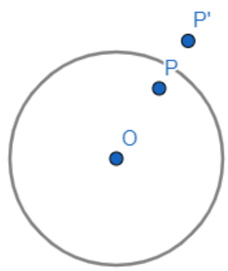
Exploring Euclidean Geometry

Inversion

Inversion about a circle is a useful *involution* that reveals a wealth of information about certain geometric configurations. Generally, an involution refers to a function f such that $f(f(x)) = x$ for all x. In this case, it refers to the fact that two inversions about the same circle yields the original diagram.

To invert a point P about circle ω with center O and radius r, we take the unique point P' such that that $\vec{OP'} = \vec{OP} \cdot (\frac{r}{|\vec{OP}|})^2$. (Note: This notation is vector notation, not rays.)



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1. Consider circle ω with center O and radius r. Prove that any inversion about ω such that P is sent to P', $\overline{OP} \cdot \overline{OP'} = r^2$.

2. Prove that for an inversion about circle ω centered at O that sends P to P', O, P, P' are collinear.

3. If you invert point P on circle ω about ω, what point do you get?

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1. Consider circle ω with center O and radius r. Prove that any inversion about ω such that P is sent to P', $\overline{OP} \cdot \overline{OP'} = r^2$.

Solution: Use the vector definition and take magnitudes. Notice that $\overline{OP'} = \overline{OP} \cdot \frac{r^2}{\overline{OP}^2} = \frac{r^2}{\overline{OP}}$, which implies that $\overline{OP} \cdot \overline{OP'} = r^2$, as desired.

2. Prove that for an inversion about circle ω centered at O that sends P to P', O, P, P' are collinear.

Solution: This is trivially true by the vector definition. If you multiply $\vec{OP}$ by a constant to get $\vec{OP'}$, then O, P, P' are collinear, which is just a property of vectors.

3. If you invert point P on circle ω about ω, what point do you get?

Solution: Let the center be O. Notice that $\overline{OP} = r$, so we want $\overline{OP'} = r$, implying $P = P'$ due to the collinearity condition. Thus you get P itself when inverting.

Consider circle ω with center O. It is relatively well known that inverting a circle not passing through O yields another circle, inverting a circle passing through O yields a line, and inverting a line yields a circle passing through O. However, we should prove this.

We glossed over what happens when we invert the center of a circle. First, we need to discuss the concept of the *point at infinity*. This discussion arises when we ask the question, "What happens if you invert the center of a circle about said circle?" The seemingly obvious answer is, "You can't." However, we instead let the point be "the point at infinity."

Here's an intuitive explanation why. For standard transformations, if two pre-images intersect, their images intersect. (This is because transformations are a function. The point of intersection cannot go to two different places, after all.) But two circles who pass through the center do intersect, but after an inversion, they become parallel lines. Instead of saying parallel lines do not intersect, we say they intersect at the *point at infinity*.

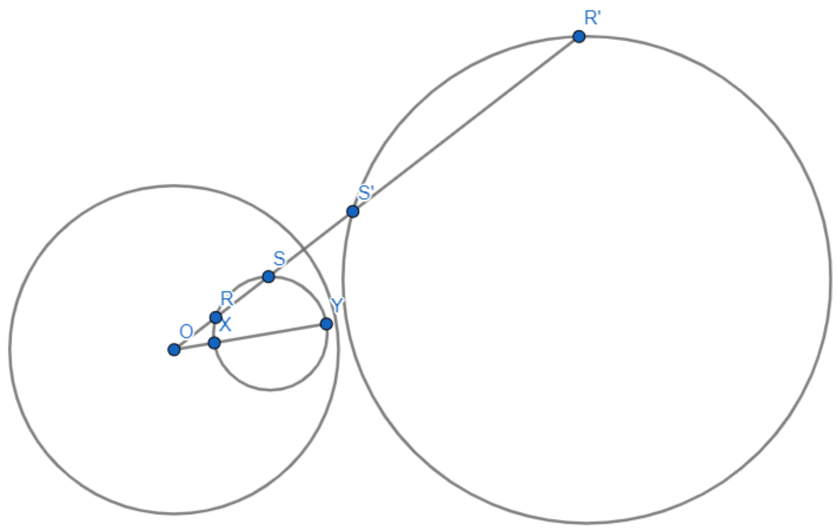
This answers two questions. Where does the center of the circle go? And where does the intersection point of two circles who intersect at the center go?

Circle to Circle (26.1)

Consider circle ω with center O. Then an inversion of a circle that does not pass through O about ω sends said circle to another circle.

Theorem 26.1's Proof

Let the initial circle be Γ. Then draw line OXY such that XY is a diameter of Γ. Then we draw arbitrary ray OR such that OR intercepts Γ at R, S. By the definition of inversion, $\overline{OR} \cdot \overline{OR'} = \overline{OS} \cdot \overline{OS'} = r^2$. Rearranging gets us $\overline{OR'} = \frac{r^2}{\overline{OR \cdot OS}} \cdot \overline{OS}$. By Power of a Point (3.2), $\overline{OR} \cdot \overline{OS} = \overline{OX} \cdot \overline{OY}$. So $\overline{OR'} = \frac{r^2}{\overline{OX} \cdot \overline{OY}} \cdot \overline{OS}$. At this point we notice that R' is the result of a dilation of S about O with scale factor $\frac{r^2}{\overline{OX} \cdot \overline{OY}}$, which has been established as a constant. As S traces out the circle, so does R', completing the proof.

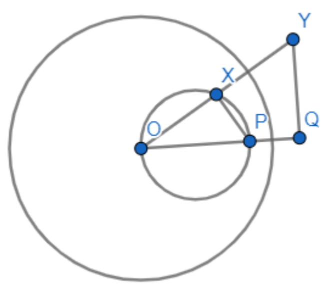


Circle to Line (26.2)

Consider circle ω with center O. Then consider circle Γ passing through O. An inversion of Γ about ω yields a line.

Theorem 26.2's Proof

Let OP be a diameter of Γ. Then let the inversion of P about ω be Q. Then pick any other point X on Γ and let its inversion about ω be Y. By the definition of inversion, $\overline{OP} \cdot \overline{OQ} = \overline{OX} \cdot \overline{OY} = r^2$. Rearranging yields $\frac{\overline{OP}}{\overline{OX}} = \frac{\overline{OY}}{\overline{OQ}}$. This implies $\triangle OPX \sim \triangle OYQ$. By the Inscribed Angle Theorem (1.1), $\angle OXP = 90^\circ$, implying $\angle OQY = 90^\circ$. So the locus of Y is the locus of points such that $OQ \perp QY$, which is a line.

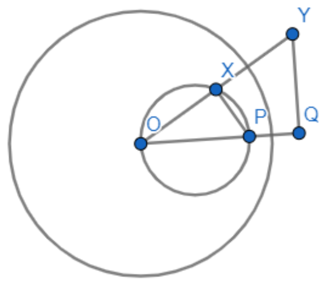


Line to Circle (26.3)

Consider circle ω with center O. Then the inversion of a line about ω yields a circle passing through O.

Theorem 26.3's Proof

Let the line be l. Drop a perpendicular from O to l, and let the foot be Q. Then pick some other point Y on l. Let the inverse of Q be P, and the inverse of Y be X. Then notice $\triangle OXP \sim \triangle OQY$. (This was proved in the proof of Theorem 26.2.) Thus, X is the locus of points such that $\angle OXP = 90^{\circ}$, which is also known as a circle.



Notice that these proofs are so basically identical, I reused the diagram.

Because of the nature of these transformations and the concept of the point at infinity, sometimes it is desirable to think of lines as circles with infinite radius. We can say three points determine a circle - a line is determined by two points and the point at infinity. Then we call the combination of lines and circles as *generalized circles*.

Now we have a tool to turn collinearity problems into concyclic problems and vice versa. Before we dive a little deeper with poles and polars, let's investigate some of the generic inversions.

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1. Consider circle ω with diameter AB. What do you get when you invert line AB about ω?

2. What about inverting segment AB?

3. Construct the inversion of a line.

4. Construct the inversion of a circle passing through the center of the circle of inversion.

5. Construct the inversion of a circle not passing through the center of the circle of inversion.

6. Consider $\triangle ABC$, and let its incircle touch BC, CA, AB at D, E, F, respectively. Prove that an inversion of the circumcircle of $\triangle ABC$ about the incircle of $\triangle ABC$ yields the nine-point circle of $\triangle DEF$.

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1. Consider circle ω with diameter AB. What do you get when you invert line AB about ω?

Solution: You get line AB. Let the center of ω be O. Then you can pair up all of the points on ray OA into X, X' such that $\overline{OX} \cdot \overline{OX'} = \overline{OA}^2$ with no leftover or overlap. You can do the same thing for ray OB.

2. What about inverting segment AB?

Solution: You get all of line AB except for segment AB. This is because the aforementioned pairs have a group of points inside and a group of points outside the circle. You take the points inside the circle and make them the points outside the circle. (Notice that A and B remain.)

3. Construct the inversion of a line.

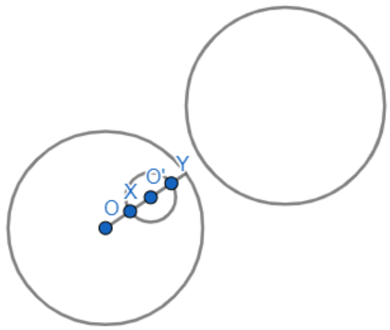
Solution: Let the circle of inversion ω have center O. Then choose the point Q on the line such that OQ is perpendicular to the line. Let P be the result of an inversion of Q about ω. Then draw the circle with diameter OP. (Notice how similar this looks to our diagram for Theorem 26.3?)

4. Construct the inversion of a circle passing through the center of the circle of inversion.

Solution: Let our circle of inversion be ω and let the circle we want to invert be Γ. Let P be on Γ such that OP is a diameter of Γ. Then invert P about ω to get Q. Draw the line passing Q perpendicular to OQ to get your inversion. (Again, notice this is the diagram for Theorem 26.2.)

5. Construct the inversion of a circle not passing through the center of the circle of inversion.

Solution: Let our circle of inversion be ω and let the circle we want to invert be Γ. Let the center of ω be O and let the center of Γ be O'. Then let OO' intersect Γ at X, Y. Then dilate Γ by a factor of $\frac{r^2}{\overline{OX} \cdot \overline{OY}}$, where r is the radius of ω.

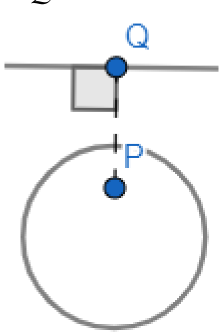


Notice that all these construction problems resulted from the **proofs** of the theorems!

6. Consider $\triangle ABC$, and let its incircle touch BC, CA, AB at D, E, F, respectively. Prove that an inversion of the circumcircle of $\triangle ABC$ about the incircle of $\triangle ABC$ yields the nine-point circle of $\triangle DEF$.

Solution: Since circles go to circles (26.1), we only need to prove three points of the circumcircle of $\triangle ABC$ belong on the nine-point circle of $\triangle DEF$. The easiest points to do this with are A, B, C. Notice that the inverse of A is the midpoint of EF as AI perpendicularly bisects EF. (This is because AE, AF are tangents to the incircle.) Analogously, the inverse of B and C are the midpoints of CA and AB, respectively. Since the midpoints of $\triangle DEF$ are on the nine-point circle of $\triangle DEF$, we are done.

Now we will discuss poles and polars. The *pole* of a point P with respect to ω is the point Q that results from an inversion about ω. (This is merely the inversion point.) The *polar* of point P with respect to ω is the line l through its pole Q such that $PQ \perp l$.



Here's a crucial theorem about polars that most books neglect to mention, let alone prove.

La Hire's (26.4)

If P lies on the polar of Q, then Q lies on the polar of P.

Theorem 26.4's Proof

By Power of a Point (3.2), P, P', Q, Q' are concyclic. Since P is on the polar of Q, $\angle PQ'Q = 90^\circ$. By the Inscribed Angle Theorem (1.1), $\angle PP'Q = \angle PQ'Q = 90^\circ$. Thus Q is on the polar of P.

This can not only be a useful tool in directly proving polars, but can help with proving collinearity concerning tangent lines, as you might see in one of the problems below.

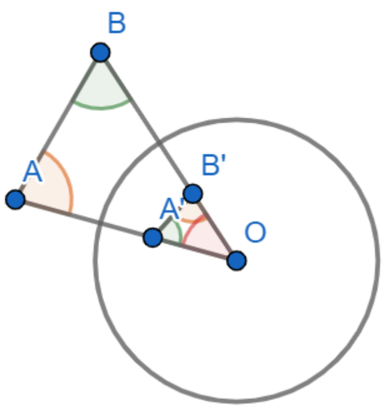
There's also a distance formula for inverted points based on similar triangles. Try to prove it yourself; it's very easy to do so.

Inversion Distance Formula (26.5)

Consider circle ω with center O and radius r and points A, B. Let A', B' be the results of inverting A, B around ω, respectively. Then $\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA}\cdot\overline{OB}}$.

Theorem 26.5's Proof

Note that $\triangle OAB \sim \triangle OB'A'$ as $\overline{OA'} = \frac{r^2}{\overline{OA}}$ and $\overline{OB'} = \frac{r^2}{\overline{OB}}$. Then notice that $\overline{OA} : \overline{OA'} = \overline{OA} : \frac{r^2}{\overline{OB}} = 1 : \frac{r^2}{\overline{OA}\cdot\overline{OB}}$. Since $\overline{AB} : \overline{A'B'} = 1 : \frac{r^2}{\overline{OA}\cdot\overline{OB}}$, $\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA}\cdot\overline{OB}}$, as desired.



As a final note, straight Cartesian coordinate bashing is possible using inversion. Without loss of generality, you should have ω be the circle $x^2 + y^2 = 1$, where you intend to invert around ω. The inversion will transform $P = (x, y)$ to $Q = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. I personally have not found any use for this, but perhaps someone out there can use coordinate inversion somehow. I find synthetic solutions better, and inversion in itself is already obscure enough already.

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1. Verify the coordinate transformation for inversion.

2. Let the poles of A, B with respect to ω be A', B'. Prove that $ABB'A'$ is cyclic.

3. Let P inside circle ω have pole Q. Then let there be points X, Y on ω such that QX, QY are tangent to ω. Prove that Q, X, Y are collinear.

4. Consider circle ω and points A, B. Let the tangents from A to ω intersect ω at A_1, A_2 and let the tangents from B to ω intersect ω at B_1, B_2. Let the midpoint of A_1A_2 be M_A, and let the midpoint of B_1B_2 be M_B. Prove that ABM_BM_A is cyclic.

5. Two circles ω and Γ with centers O and O' that intersect at X, Y are considered *orthogonal* if and only if $OX \perp O'X$ and $OY \perp O'Y$. Prove that if ω is orthogonal with Γ, then an inversion about ω preserves Γ.

6. Consider $\triangle PAB$ with circumcenter X. Then consider an inversion about some circle ω with center P. If A', B', X' are the poles of A, B, X with respect to ω, prove that X' is the result of reflecting P about $A'B'$.

7. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

8. Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

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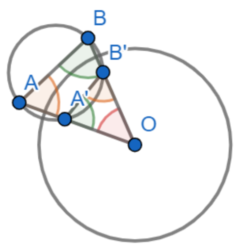
1. Verify the coordinate transformation for inversion.

Solution: Let $O = (0,0)$. Notice that Q is P dilated by $\frac{1}{x^2+y^2}$, so O, P, Q are collinear. Also, note that $\overline{OQ} = \frac{\overline{OP}}{x^2+y^2}$, and $\overline{OP} = \sqrt{x^2+y^2}$, so $\overline{OQ} = \frac{\sqrt{x^2+y^2}}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}$. Thus, $\overline{OP} \cdot \overline{OQ} = 1^2$, verifying that Q is indeed the result of inverting P about ω.

For those of you who want vocabulary practice, we can say Q is the pole of P with respect to ω.

2. Let the poles of A, B with respect to ω be A', B'. Prove that $ABB'A'$ is cyclic.

Solution: Since $\triangle OAB \sim \triangle OB'A'$, $\angle OAB = \angle OB'A'$. Thus, $ABB'A'$ is cyclic as desired.

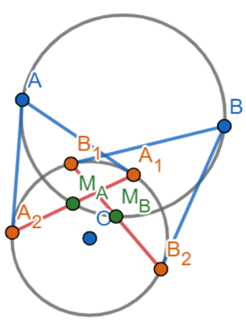


3. Let P inside circle ω have pole Q. Then let there be points X, Y on ω such that QX, QY are tangent to ω. Prove that Q, X, Y are collinear.

Solution: By similarity, $XP \perp OQ$ and $YP \perp OQ$. So XP and YP are either parallel or are the same line; since they intersect at P, they are the same line.

4. Consider circle ω and points A, B. Let the tangents from A to ω intersect ω at A_1, A_2 and let the tangents from B to ω intersect ω at B_1, B_2. Let the midpoint of A_1A_2 be M_A, and let the midpoint of B_1B_2 be M_B. Prove that ABM_BM_A is cyclic.

Solution: This is a direct result of Problem 2 and Problem 3. Notice that M_A and M_B are the poles of A, B with respect to ω, so the quadrilateral is cyclic.



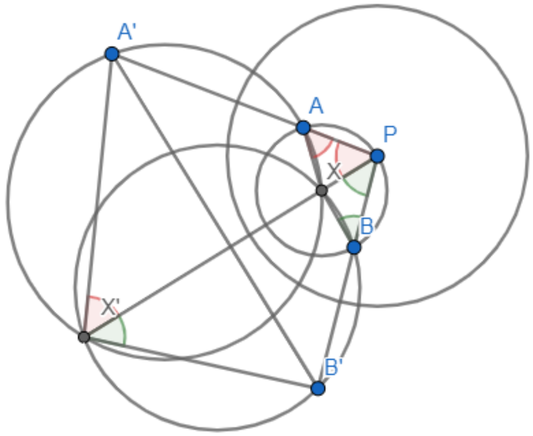
5. Two circles ω and Γ with centers O and O' that intersect at X, Y are considered *orthogonal* if and only if $OX \perp O'X$ and $OY \perp O'Y$. Prove that if ω is orthogonal with Γ, then an inversion about ω preserves Γ.

Solution: Let a line passing through O intersect Γ at P, Q. But by Power of a Point (3.2), $\overline{OX}^2 = \overline{OP} \cdot \overline{OQ}$, implying that Q is the polar of P with respect to ω. As P traces out Γ, so will Q.

6. Consider $\triangle PAB$ with circumcenter X. Then consider an inversion about some circle ω with center P. If A', B', X' are the poles of A, B, X with respect to ω, prove that X' is the result of reflecting P about $A'B'$.

Solution: By the definition of inversion, $\overline{PA} \cdot \overline{PA'} = \overline{PX} \cdot \overline{PX'}$. Applying Power of a Point (3.2) yields that $AA'X'X$ and $BB'X'X$ are cyclic quadrilaterals. Then notice that $\triangle PAX \sim \triangle PX'A'$.

Since X is the center of a circle, $\overline{AX} = \overline{PX}$. By similarity, $\overline{X'A'} = \overline{PA'}$, so $\angle PXA = \angle APX = \angle PX'A$. Similarly, $\angle BPX = \angle PX'B$. This implies that $\angle A'X'B' = \angle PA'B'$. Since P, X, X' are collinear, X' is the reflection of P about AB, as desired.

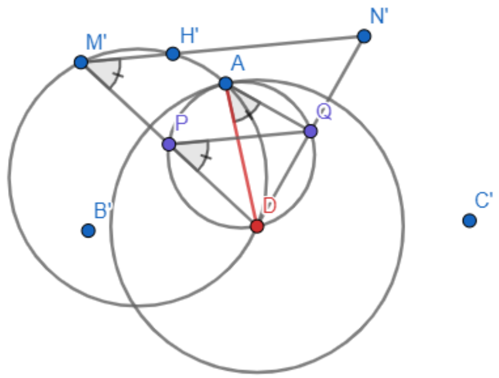


7. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

Solution: Invert about the circle with center D and radius DA. This sends B, C, M, N, H to B', C', M', N', H', respectively.

By Problem 4, M' is the reflection of D about AB' and N' is the reflection of D about AB'. Then notice that H' is the intersection of $M'N'$ and AC'.

Let P be the midpoint of DM' and let Q be the midpoint of DN'. Notice that by Problem 4, $\angle APD = \angle AQD = 90^\circ$. By Theorem 2.2, $APDQ$ is cyclic because $\angle APD + \angle AQD = 90^\circ + 90^\circ = 180^\circ$. So $\angle QAD = \angle QPD = \angle N'M'D = \angle H'M'D$, by Inscribed Angle (1.1) and because $\triangle DPQ \sim \triangle DM'N'$. Since $\angle H'AD = 180^\circ - \angle QAD = 180^\circ - \angle H'M'D$, we notice that $H'M'DA$ is cyclic, as desired.



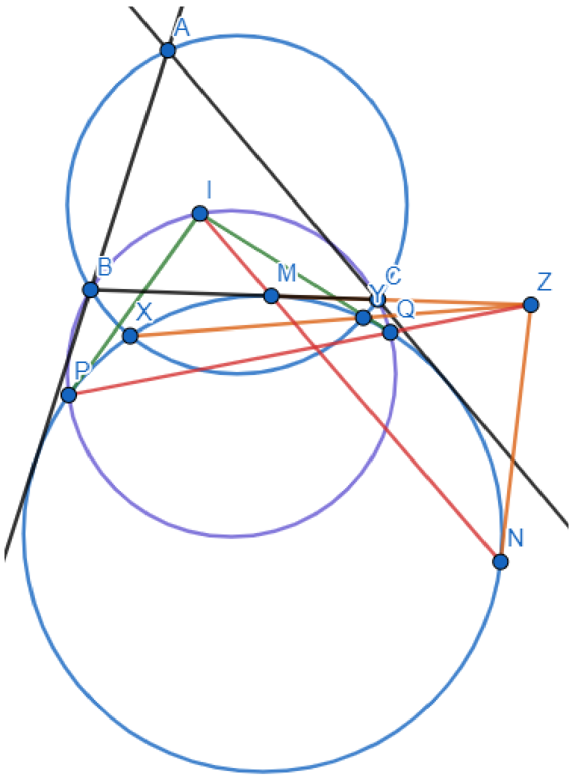
8. Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

Solution: Notice that we wish to prove that I lies on the polar of Z with respect to the A excircle, so we instead use La Hire's (26.4) by proving Z lies on the polar of I. (The motivation is problem 2. Notice that MN is the polar of Z with respect to the A excircle.)

Let the A excenter be I_A and let the tangents from I to the A excircle be P, Q. Then notice $\angle IPI_A = \angle IQI_A = 90^\circ = \angle IBI_A = \angle ICI_A$, implying that B, C, P, Q are concyclic.

Letting the circumcircle be ω_1, the excircle be ω_2 and the circumcircle of $BCPQ$ be ω_3, notice that $\pi(Z, \omega_1) = \pi(Z, \omega_2)$. But by the Radical Axis Theorem (4.2), Z is on the radical axis of ω_2, ω_3, also known as PQ.

Then notice that PQ is the polar of I with respect to the A excircle, so Z lies on the polar of $I,$ as desired. La Hire's (26.4) finishes the problem.



Fun fact: The diagram of the **problem** (not the solution, so points P, Q are left out) is the figure on the cover of this book.
