Mock AIME

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2020

1 Problems

180 minutes.

1. A car at Safeway leaves for the airport, and thirty minutes later, another car at the airport leaves for Safeway. Both cars arrive at their respective locations 5 hours after the first car leaves. It takes m minutes for the cars to meet after the first car has left. Find $\lfloor m \rfloor$.

2. Consider right triangle $\triangle ABC$ with a right angle at A, $AB = 24$, and $AC = 10$. The midpoint of AC is D, and a line perpendicular to BC is drawn through D, meeting BC at E. The area of quadrilateral $ABED$ can be expressed as $m + n$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

3. Jim is playing a game with the following rules:

   (a) There are 5 boxes. To win, you must guess the box that contains 100 dollars. Only one box contains 100 dollars.

   (b) You have three guesses.

   (c) Immediately after each guess, a box that does not contain the 100 dollars is randomly selected and revealed to not have the 100 dollars. (The box revealed may be the box you have selected, but a box already revealed will not be re-revealed.)

   With optimal play, the probability of picking the box with 100 in three guesses and winning is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Let the number of ordered pairs of positive integers (a, b) such that $\gcd(a, b) = 20$ and $\text{lcm}(a, b) = 19!$ be n. Find the remainder of n when divided by 1000.

5. Consider $\triangle ABC$ such that $AB = 8, BC = 5, CA = 7$. Let AB, AC be tangent to the incircle at X, Y, respectively. The area of ΔAXY can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and

n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

6. Let n be the number of 10 digit integers divisible by 5 whose sum of digits is also divisible by 5. Find the number of divisors of n.

7. Let S be a set defined recursively as follows:

   - Initially, the only elements in S are $12, 54$.
   - If a and b are in S, then ab is in S.

   The ninety-fifth smallest element of S is $2^a \cdot 3^b$. Find $2a+3b$.

8. Let $\lfloor x \rfloor$ be the largest integer such that $\lfloor x \rfloor \leq x$, and let $\{x\} = x - \lfloor x \rfloor$. How many values of x satisfy $x + \lfloor x \rfloor \cdot \{x\} = 23$?

9. Consider a monic cubic polynomial $P(x)$ with roots a, b, c such that $a, b, c \leq 1$. If $a+b+c = -8$ and there is one root at least 4 greater than another, find the maximum possible value of the sum of the coefficients of $P(x)$. (The constant term is counted.)

10. Two distinct roots of the equation $f(x) = 2x^2 - 8nx + 10x - n^2 + 35n - 76$, where the constant n is an integer, are positive primes. Find the sum of the roots of $f(x)$.

11. Let the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period 4 be S. Find the remainder when S is divided by 1000.

12. A secret spy organization needs to spread some secret knowledge to all of its members. In the beginning, only 1 member is *informed.* Every informed spy will call an uninformed spy such that every informed spy is calling a different uninformed spy. After being called, an uninformed spy becomes informed. The call takes 1 minute, but since the spies are running low on time, they call the next spy directly afterward. However, to avoid being caught, after the third call an informed spy makes, the spy stops calling. How many minutes will it take for every spy to be informed, provided that the organization has 600 spies?

13. Consider unit circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. Given that $XP = MQ$, the length of AT can be written as $\frac{\sqrt{m}}{n}$ where m and n are positive integers, and m is square-free. Find $m+n$.

14. Santa Claus is putting 1000 identical toy trains into a red stocking, a green stocking, and a white stocking such that the amount of trains in the green stocking is divisible by 3 and the amount of trains in the white stocking is even. Mrs. Claus is putting 1000 identical elves into a red stocking,

a green stocking, and a white stocking such that the amount of elves in the green stocking is divisible by 3 and the amount of elves in the white stocking is odd. Find the positive difference between the amount of ways Santa Claus can put his trains in the stockings and the amount of ways Mrs. Claus can put her elves in the stockings.

15. Alice and Bob are playing a game with a cursed coin. They take turns flipping the coin, and the winner of the game is the first person to get heads. At first, its probability of coming up heads is $\frac{1}{2}$. However, after every flip, its probability of coming up heads is halved. For example, if Alice flips the coin, her probability of getting heads is $\frac{1}{2}$, and if Bob then flips the coin afterwards, his probability of getting heads is $\frac{1}{4}$. The probability of Alice winning if she flips first is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2 Answer Key

If you want solutions for any problems, email me at proofprogram@gmail.com or message dchenmathcounts on Art of Problem Solving.

1. 157
2. 699
3. 082
4. 256
5. 156
6. 240
7. 108
8. 012
9. 036
10. 007
11. 632
12. 010
13. 009
14. 167
15. 049