Dennis Chen's

Exploring Euclidean Geometry

Dennis Chen

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Exploring Euclidean Geometry V2

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Introduction

This is a preview of Exploring Euclidean Geometry V2. It contains the first five chapters, which constitute the entirety of the first part.

This should be a good introduction for those training for computational geometry questions. This book may be somewhat rough on beginners, so I do recommend using some slower-paced books as a supplement, but I believe the explanations should be concise and clear enough to understand. In particular, a lot of other texts have unnecessarily long proofs for basic theorems, while this book will try to prove it as clearly and succintly as possible.

There aren't a ton of worked examples in this section, but the check-ins should suffice since they're just direct applications of the material.

Contents

Part A

The Basics

Chapter 1

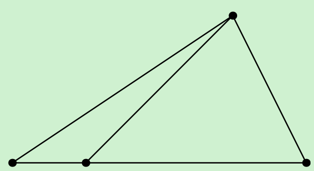
Triangle Centers

Empty your mind of any theories,
'Til all the facts are in.

Death Note Musical

We define the primary four triangle centers, their corresponding lines, and define a cevian.

Definition 1.1 (Cevian) In a triangle, a cevian is a line segment with a vertex of the triangle as an endpoint and its other endpoint on the opposite side.



1.1 Incenter

The corresponding cevian is the *interior* angle bisector.

Definition 1.2 (Interior Angle Bisector) The interior angle bisector of $\angle CAB$ is the line that bisects acute $\angle CAB$.

The interior angle bisector of $\angle CAB$ is also the locus of points inside $\angle CAB$ equidistant from lines AB and AC.

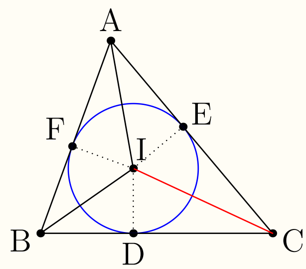
Fact 1.1 (Angle Bisector Equidistant from Both Sides) In $\angle CAB$, $\angle PAB = \angle PAC$ if and only if $\delta(P, AB) = \delta(P, AC)$.

Proof: Let the feet of the altitudes from P to AB, AC be X, Y. Then note that either of these conditions imply $\triangle APX \cong \triangle APY$, which in turn implies the other condition. ■

Theorem 1.1 (Incenter) There is a point I that the angle bisectors of $\triangle ABC$ concur at. Furthermore, I is equidistant from sides AB, BC, CA.

Proof: Recall that a point is on the angle bisector of $\angle CAB$ if and only if $\delta(P, AB) = \delta(P, AC)$. Let the angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at I. Then $\delta(P, CA) = \delta(P, AB)$ and $\delta(P, AB) = \delta(P, BC)$, so $\delta(P, BC) = \delta(P, CA)$, implying that I lies on the angle bisector of $\angle BCA$.

Since $\delta(P, AB) = \delta(P, BC) = \delta(P, CA)$, the circle with radius $\delta(P, AB)$ centered at I is inscribed in $\triangle ABC$.



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1.2 Centroid

The corresponding cevian is the median.

Definition 1.3 (Midpoint) The midpoint of segment AB is the unique point M that satisfies the following:

(a) M is on AB.

(b) $AM = MB$.

Definition 1.4 (Median) The A-median of $\triangle ABC$ is the line segment that joins A with the midpoint of BC.

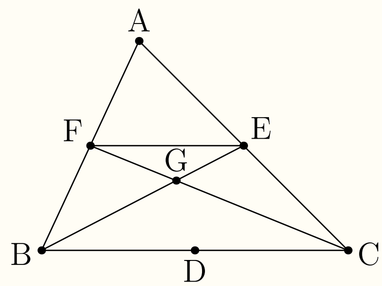
Theorem 1.2 (Centroid) The medians AD, BE, CF of $\triangle ABC$ concur at a point G. Furthermore, the following two properties hold:

(a) $\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = 2$.

(b) $[BGD] = [CGD] = [CGE] = [AGE] = [AGF] = [BGF]$.

Proof: Let BE intersect CF at G. Since $\triangle AFE \sim \triangle ABC$, $FE \parallel BC$. Thus $\triangle BCG \sim \triangle EFG$ with a ratio of $\frac{BC}{EF} = 2$, so $\frac{BG}{GE} = 2$.

Similarly let BE intersect AD at G'. Repeating the above yields $\frac{BG'}{G'E} = 2$. Thus G and G' are the same point, and the medians are concurrent.



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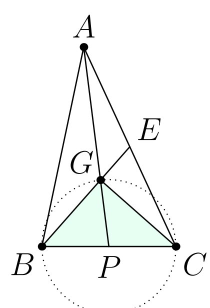
Example 1.1 (JMC 10 2020/20) Consider triangle ABC with medians $\overline{BE}$ and $\overline{CF}$ that intersect at G. If $AG = BC = 8$ and $CG = 6$, what is the length of $\overline{GE}$?

Solution: The key step is to realize that $\angle BGC$ is right. We can prove this in two ways.

The first way is let $\overline{AG}$ meet $\overline{BC}$ at point P. The length of $\overline{PG}$ is $\frac{8}{2} = 4$. So $BP = PG = PC = 4$ and $\triangle BGC$ can be circumscribed in a circle with diameter $\overline{BC}$. It follows that $\angle BGC = 90°$.

The second way is to extend point P to meet $\overline{GC}$ at midpoint M. Since $\triangle GPC$ is isosceles, $PG = PC$. We also have $BP = PC$ and $\angle PMC = 90°$. $\triangle PMC \sim \triangle BGC$ by side-angle-side and the rest follows.

By the Pythagorean theorem, $\overline{BG}$ has length $2\sqrt{7} \Rightarrow GE = \frac{2\sqrt{7}}{2} = \boxed{\sqrt{7}}$.



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1.3 Circumcenter

A perpendicular bisector is not a cevian, but it is still a special line in triangles.

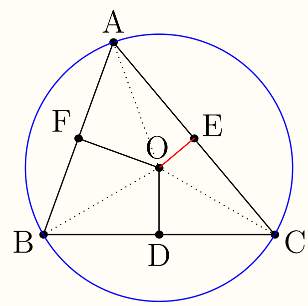
Definition 1.5 (Perpendicular Bisector) The perpendicular bisector of a line segment AB is the locus of points X such that $AX = BX$.

The circumcenter is the unique circle that contains points A, B, C.

Theorem 1.3 (Circumcenter) There is a point O that the perpendicular bisectors of BC, CA, AB concur at. Furthermore, O is the center of (ABC).

Proof: Let the perpendicular bisectors of AB, BC intersect at O. By the definition of a perpendicular bisector, $AO = BO$ and $BO = CO$. But this implies $CO = AO$, so O lies on the perpendicular bisector of CA.

Since $AO = BO = CO$, the circle centered at O with radius AO circumscribes $\triangle ABC$.



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1.4 Orthocenter

The corresponding cevian is the altitude.

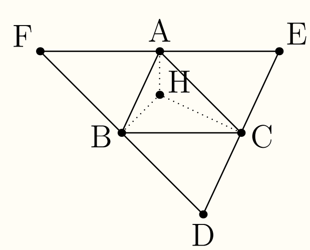
Definition 1.6 (Altitude) The A-altitude of $\triangle ABC$ is the line through A perpendicular to BC.

Definition 1.7 (Foot of Altitude) The foot of the altitude from A to BC is the point H where the A-altitude intersects BC.

Theorem 1.4 (Orthocenter) The altitudes of $\triangle ABC$ concur.

Proof: We will be piggybacking on the proof for the circumcenter.

Let the line through B parallel to AC and the line through C parallel to AB intersect at D. Define E, F similarly. Note that $FA = BC = AE$, so the A altitude of $\triangle ABC$ is the perpendicular bisector of DE. Since the circumcenter exists, the orthocenter must too.



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1.5 Summary

1.5.1 Theory

1. Angle Bisector

- ✦ The angle bisector bisects the angle, as the name suggests.
- ✦ It is also the locus of points equidistant from two lines.

2. Incenter
   - ✦ The incenter is the center of the incircle.
   - ✦ The incenter is equidistant from all sides.
   - ✦ The incenter is the concurrence point of the angle bisectors.
3. Median
   - ✦ The median joins the vertex to the midpoint of the opposite side.
4. Centroid
   - ✦ The centroid is the center of mass.
   - ✦ The centroid is the concurrence point of the medians.
   - ✦ The centroid G splits the medians in the ratio $\frac{AG}{GD} = 2$.
5. Perpendicular Bisector
   - ✦ The perpendicular bisector is the locus of points equidistant from the endpoints of a line segment.
   - ✦ The perpendicular bisector perpendicularly bisects the segment.
6. Circumcenter
   - ✦ The circumcenter is the center of the circumcircle.
   - ✦ The circumcenter is equidistant from all vertices.
   - ✦ The circumcenter is the concurrence point of the perpendicular bisectors.
7. Altitude
   - ✦ The altitude is the perpendicular line through one vertex to the opposite side.
   - ✦ The foot of the altitude is the intersection of the altitude with the side.
8. Orthocenter
   - ✦ The orthocenter is the concurrence point of the altitudes.

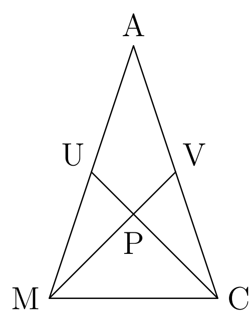
1.5.2 Tips and Strategies

1. Incenter
   - ✦ Just remember that the angle bisector bisects angles.
2. Centroid
   - ✦ Length chase with the length condition.
   - ✦ Take homotheties from the midpoint of a side with scale factor $\frac{1}{3}$ to send the opposite vertex to the centroid.
   - ✦ Remember that the medians split the triangle into six triangles of equal area.
3. Circumcenter
   - ✦ Use $AO = BO = CO$ to find isosceles triangles - this will help a lot with angle chasing.
4. Orthocenter
   - ✦ Watch out for right triangles and right angles.

1.6 Exercises

1.6.1 Check-ins

1. Prove that a triangle is equilateral if and only if its incenter is the same point as its circumcenter.
2. Consider $\triangle ABC$ with incenter I. Prove that $\angle BIC = 90^\circ + \frac{1}{2}\angle BAC$. **Hints**: 30
3. Prove that the perpendicular bisector of line segment AB is perpendicular to and bisects AB, if the perpendicular bisector is defined as the locus of points X such that $AX = BX$.
4. Consider $\triangle ABC$ with circumcenter O. If $AO = 20$ and $BC = 32$, find $[BOC]$.
5. (AMC 10A 2020/12) Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



6. Find the inradius and circumradius of an equilateral triangle with side length x.

1.6.2 Problems

1. (AMC 10B 2018/12) Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve? **Solution**: 1
2. Let $\triangle ABC$ have medians BM, CN, and let P and Q be the feet of the altitudes from B, C to CN, BM, respectively. Prove that the quadrilateral $MNPQ$ is cyclic. **Hints**: 61 3 17 **Solution**: 2
3. Consider $\triangle ABC$ with medians BE, CF. If BE and CF are perpendicular, find $\frac{b^2+c^2}{a^2}$.**Hints**: 4 12 **Solution**: 15
4. (Brazil 2007) Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is 40°. Determine the measure of the angle $\angle OBC$. **Hints**: 36 49 **Solution**: 11

1.6.3 Challenges

1. Three congruent circles $\omega_1, \omega_2, \omega_3$ concur at P. Let ω_1 intersect ω_2 at $A \neq P$, let ω_2 intersect ω_3 at $B \neq P$, and let ω_3 intersect ω_1 at $C \neq P$. What triangle center is P with respect to $\triangle ABC$?
2. Let ABC be an isosceles triangle with $AB = AC$. If ω is inscribed in ABC and the orthocenter of ABC lies on ω, find $\frac{AB}{BC}$.
3. Let G be the centroid of $\triangle ABC$. If $\angle BGC = 90^\circ$, find the maximum value $\sin A$ can take. **Hints**: 50

Chapter 2

Angle Chasing

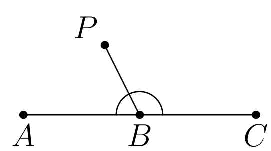
'Think simple' as my old master used to say — meaning reduce the whole of its parts into the simplest terms, getting back to first principles.

Frank Lloyd Wright

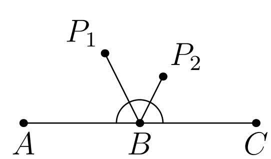
You can angle chase to show points are collinear or lines are concurrent, lines are parallel, a line is tangent to a circle, or four points are cyclic. In computational contests, you may be asked to find an angle for easier problems and angle chasing can reveal more about the configuration for harder problems.

2.1 Collinearity and Concurrency

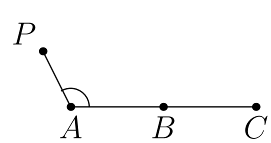
Fact 2.1 (Collinearity Condition) A line has measure $180°$. This means A, B, C, are collinear if and only if for any point P, $\angle ABP + \angle PBC = 180°$. This is one of the main ways to prove points are collinear.



This holds for more than one point too. For the right configuration, A, B, C are collinear if and only if for points $P_1, P_2, \ldots, P_n$, $\angle ABP_1 + \angle P_1BP_2 + \cdots + \angle P_nBC = 180°$. (Directed angles can be used to avoid configuration issues.)



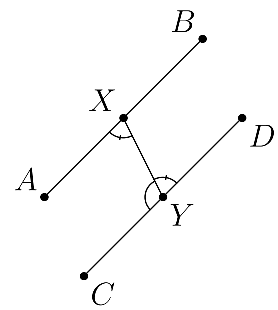
A similar condition is that A, B, C are collinear if and only if for any point P, $\angle PAB = \angle PAC$.



2.2 Parallel Lines

Fact 2.2 (Parallel Lines) Consider parallel lines AB and CD. Then for X on segment AB and Y on segment CD,

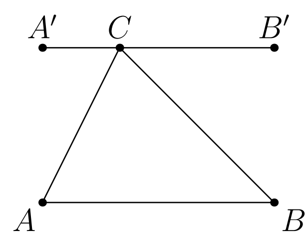
$$\angle AXY = 180^\circ - \angle CYX = \angle DYX.$$



Example 2.1 (180 Degrees in a Triangle) Prove that the angles of a triangle sum to 180 degrees.

Solution: Draw line ℓ through C parallel to AB, and let A', B' lie on ℓ such that $\angle ACA' = \angle CAB$ and $\angle BCB' = \angle CBA$. Then note

$$\angle A + \angle B + \angle C = \angle A'CA + \angle ACB + \angle BCB' = 180^\circ.$$



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2.3 Angle Chasing in Circles

We begin with some definitions.

Definition 2.1 (Chord) A chord is a line segment formed by two distinct points on a circle.

Definition 2.2 (Secant) A secant is a line that intersects a circle twice.

Definition 2.3 (Tangent) A tangent is a line that intersects a circle once.

Sometimes, it will be more convenient to think of a tangent as intersecting a circle twice at the same point, such as with Power of a Point.

Definition 2.4 (Measure of an Arc) The measure of $\overset{\frown}{\mathrm{AB}}$ of circle with center O is the measure of $\angle AOB$. Unless specified, this means the minor arc, or the smaller arc.

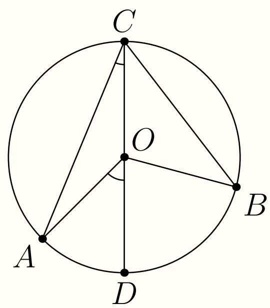
Now we present three important theorems.

Theorem 2.1 (Inscribed Angle) Let A, B be points on a circle with center O.

If C is a point on minor arc AB, then $\angle ACB = \frac{\angle AOB}{2}$.

If C is a point on major arc AB, then $\angle ACB = 180^\circ - \frac{\angle AOB}{2}$.

Proof: Let D be the antipode of C. Then $\angle ACD = \frac{180^\circ - \angle AOC}{2} = \frac{\angle AOD}{2}$. Thus addition or subtraction, depending on whether O is inside acute angle $\angle ACB$, of $\angle ACD$ and $\angle BCD$ will yield the result.



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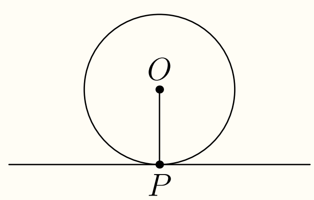
Theorem 2.2 (Tangent Perpendicular to Radius) Consider circle ω with center O and point P on ω. If ℓ is the tangent to ω through P, then ℓ is perpendicular to OP.

Proof: This is identical to the claim that P is the point on ℓ with the smallest distance to O. We prove this is true by contradiction. Assume this is not true. Then there is some point X on ℓ such that $OX < OP$, implying that ℓ intersects ω twice, contradiction.



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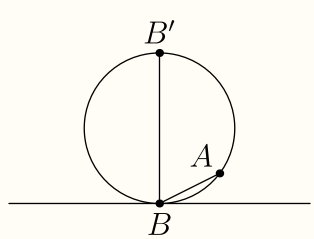
Theorem 2.3 (Tangent Angle) Consider circle ω with center O and points A, B on ω. Let ℓ be the tangent to ω through B and let θ be the acute angle between AB and ℓ. Then $\theta = \frac{\angle AOB}{2}$.

Proof: Let B' be the antipode of B. Then note that $\theta = 90^\circ - \angle ABB' = \frac{180^\circ - \angle AOB'}{2} = \frac{\angle AOB}{2}$.



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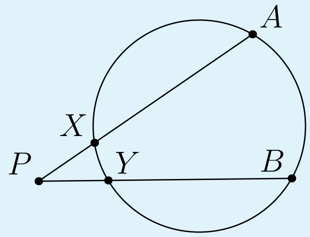
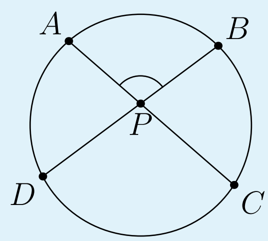
A corollary of this theorem is that if C is some point on $\widehat{AB}$, then $\theta = \angle ACB$.
With the Inscribed Angle Theorem in mind, try to prove these two theorems.

Theorem 2.4 (Angle of Secants/Tangents) Let lines AX and BY intersect at P such that A, X, P and B, Y, P are collinear in that order. Then $\angle APB = \frac{\angle AOB - \angle XOY}{2}$.



Hints: 64 42 11

Theorem 2.5 (Angle of Chords) Let chords AC, BD intersect at P. Then $\angle APB = \frac{\angle AOB + \angle COD}{2}$.



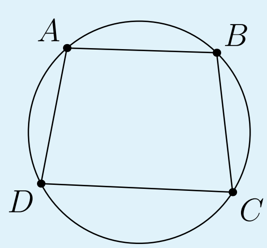
Hints: 2

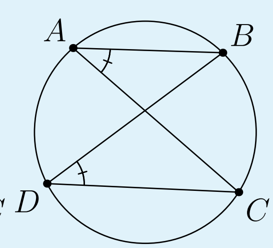
2.4 Cyclic Quadrilaterals

Here's a very important application of the Inscribed Angle Theorem.

Theorem 2.6 (Cyclic Quadrilaterals) Any one of the three implies the other two:

1. Quadrilateral $ABCD$ is cyclic.
2. $\angle ABC + \angle ADC = 180^\circ$.
3. $\angle BAC = \angle BDC$.





2.5 Examples

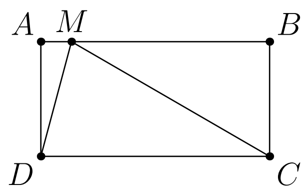
We present several examples of angle chasing problems, sorted by "flavor."

2.5.1 Computational Problems

This is a compilation of computational problems meant to serve as low-level examples for first-time readers. If this is your first time encountering the material, I strongly suggest you focus on this section.

Example 2.2 (AMC 10B 2011/18) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

Solution: Note that $\angle CMD = \angle AMD = \angle AMD = \angle MDC$, implying that $CM = CD = 6$. Thus $\angle BMC = 30°$, implying that $\angle AMD = 75°$.



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Example 2.3 Two circles ω_1, ω_2 intersect at P, Q. If a line intersects ω_1 at A, B and ω_2 at C, D such that A, B, C, D lie on the lie in that order, and P and Q lie on the same side of the line, compute the value of $\angle APC + \angle BQD$.

Solution: Without loss of generality, let P be closer to ℓ than Q. Note

$$\angle APC = 180 - \angle PAB - \angle BCP = \angle DCP - \angle PAB$$

$$\angle BQD = \angle BQP + \angle DQP.$$

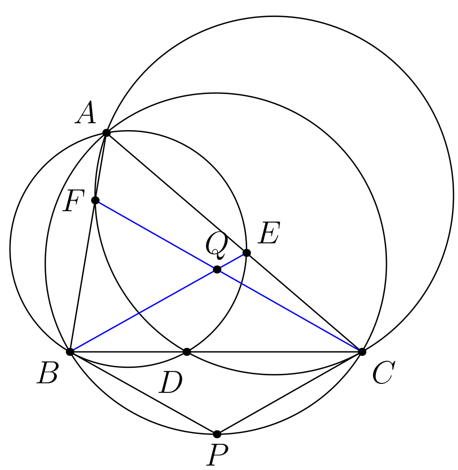
Since $\angle PAB = \angle BDP$, the sum is $\angle DCP + \angle DQP = 180$. ■

2.5.2 Construct the Diagram

These problems are very simple; just construct the diagram and the problem will solve itself for you.

Example 2.4 (USA EGMO TST 2020/4) Let ABC be a triangle. Distinct points D, E, F lie on sides BC, AC, and AB, respectively, such that quadrilaterals $ABDE$ and $ACDF$ are cyclic. Line AD meets the circumcircle of $\triangle ABC$ again at P. Let Q denote the reflection of P across BC. Show that Q lies on the circumcircle of $\triangle AEF$.

Solution: Note that Q is the intersection of BE and CF, since $\angle EBD = \angle CAP = \angle CBP$ and $\angle FCB = \angle BAO = \angle BCP$. Now note that $\angle BQC = \angle BPC = 180° - \angle A$.



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The motivation is just drawing the diagram – as soon as you figure out that Q lies on BE and CF, the problem solves itself from there.

Here's a slightly harder example.

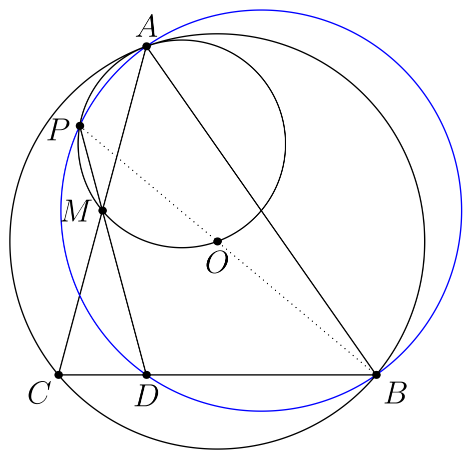
Example 2.5 (KJMO 2015/1) In an acute, scalene triangle $\triangle ABC$, let O be the circumcenter. Let M be the midpoint of AC. Let the perpendicular from A to BC be D. Let the circumcircle of $\triangle OAM$ hit DM at $P \neq M$. Prove that B, O, P are colinear.

Solution: Instead we show that the intersection of MD and BO, which we will call P', lies on (MAO). The central claim is that $PABD$ is cyclic.

Note $\angle PDA = \angle MDA = 90^\circ - \angle C$, and also note that $\angle PAD = \angle PAB - \angle DAB$. Note that $\angle PAB = \angle C$ since $\angle APB = 90^\circ$ and $\angle ABP = \angle ABO = 90^\circ - \angle C$ and $\angle BAD = 90^\circ - \angle B$. Thus $\angle PAD = \angle B + \angle C - 90^\circ$.

Now consider $\triangle PAD$. Note $\angle DPA = 180^\circ - (\angle PDA + \angle PAD) = 180^\circ - \angle B$. Thus $PABD$ is cyclic.

This implies that $\angle APO = \angle APB = \angle ADB = 90^\circ$. Since $\angle AMO = 90^\circ$ as well, we are done.



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This final example demonstrates the power of wishful thinking.

Example 2.6 (ISL 2010/G1) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Solution: We work in directed angles because there are plenty of configuration issues. (If you don't know what directed angles are, consult the chapter on them.)

Note that $AFPQ$ is cyclic, as

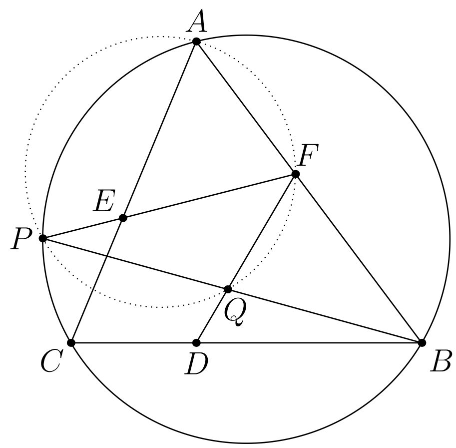
$$\measuredangle AFQ = \measuredangle BFD = \measuredangle ACB = \measuredangle APB = \measuredangle APQ.$$

Now note that

$$\measuredangle APQ = \measuredangle AFQ = \measuredangle BFD = \measuredangle ACB$$

$$\measuredangle PQA = \measuredangle PFA = \measuredangle EFA = \measuredangle ACB,$$

implying that $\measuredangle APQ = \measuredangle PQA$, or that $AP = AQ$.



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I personally thought this problem was harder than the other two, especially since the cyclic quadrilateral had an asymmetric structure with respect to the whole diagram.[1] We're inclined to look for cyclic quadrilaterals involving A, P, Q in some way because the problem is essentially equivalent to showing that $\angle AQP = \angle APQ$, and a little bit of experimentation shows that it's hard to show directly. The motivation for trying to prove F is the point on (APQ) is drawing in the circumcircles for both configurations, and noting that the second intersection point of them is F.

The rest of the motivation is quite straightforward – all you have to do afterwards is try to solve the problem with the assumption that $AFPQ$ is cyclic, and that part is fairly easy if you have any knowledge about the orthic triangle.

2.5.3 Tangent Angle Criterion

When tangent lines are given, you have to pay close attention the the tangent angle criterion.

Example 2.7 (British Math Olympiad Round 1 2000/1) Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

Solution: Note that $\angle RMQ = 180° - \angle RNQ = 180° - (\angle PNM + \angle QNM) = 180° - (\angle QPM + \angle PQM) = \angle PMQ$.

(This actually only takes care of the case where R is in between P and N. Can you show this is true for the other configuration as well?) ■

Let's expound on the motivation for this. We want to prove that PQ bisects $\angle PMN$, but it's quite hard to find the supplement of $\angle PMQ$ and $\angle RMQ$. This then motivates showing that $\angle PMQ = \angle RMQ$, because those angles seem more workable. We start by manipulating $\angle RMQ$ because it seems more unwieldy, and it feels like there are more ways to get to $\angle PMQ$ than $\angle RMQ$. (This part is personal preference, but a good rule of thumb is to try to manipulate the least independently defined points into the most independently defined points.[2])

The cyclic quadrilateral $RMNQ$ is the source of the only useful manipulation we can do with $\angle RMQ$, so we're pretty much forced into using it. Now looking at $\triangle PNQ$ as a whole motivates $\angle RNQ = 180° - (\angle PNM + \angle QNM)$, and at this point we want to start manipulating $\angle PMQ$. We're forced into doing $180° - (\angle QPM + \angle PQM) = \angle PMQ$, because tangent lines have lots of potential for angle chasing and it's the only place to go.

Now the rest of the problem will just come naturally by just trying things.

[1] This is explain by the entire diagram being asymmetric.

[2] A heuristic for the independence of a point is how much it would affect the diagram on GeoGebra if it was deleted.

2.5.4 Orthocenter

Sometimes a problem will ask you to prove that $AH \perp BC$ for some point H not on BC. This is generally difficult to do directly, and one of the more elementary methods used is to show that H is the orthocenter of $\triangle ABC$, or $BH \perp CA$ and $CH \perp AB$.

This is obviously not always going to be true, so make sure that this actually seems true before you try too hard to prove it.

Example 2.8 (Swiss Math Olympiad 2007/4) Let ABC be an acute-angled triangle with $AB > AC$ and orthocenter H. Let D be the projection of A on BC. Let E be the reflection of C about D. The lines AE and BH intersect at point S. Let N be the midpoint of AE and let M be the midpoint of BH. Prove that MN is perpendicular to DS.

Solution: We claim S is the orthocenter of $\triangle DEM$. To do this, it suffices to show that $SN \perp DM$ and $SM \perp DN$. Let H' be the second intersection of AH with (ABC).

Note that $DM \parallel BH'$ by a homothety about H, $\angle MAE = \angle DAC = 90° - \angle C$, and $\angle AMB = \angle C$, proving $SN \perp DM$.

Now note that $DN \parallel AC$ by a homothety about E, proving $SM \perp DN$. ■

2.6 Summary

2.6.1 Theory

1. Supplementary Angles
   - $A, B, C,$ are collinear if and only if for any point P, $\angle ABP + \angle PBC = 180°$.
   - This is generalizable to more points.
   - A, B, C are collinear if and only if for any point P, $\angle PAB = \angle PAC$.
2. Parallel Lines
   - For parallel lines AB, CD and points X and Y on AB and CD respectively, $\angle AXY = 180° - \angle CXY = \angle DXY$.
3. Inscribed Angle Theorem
   - The measure of an angle is half the measure of the subtended arc.
   - Proved by considering the case where one leg of the angle is a diameter and angle chasing, and generalizing.
   - Thale's Theorem: In the special case where the feet of the angle form a diameter of the circle, the angle is $90°$. The converse also holds.
4. Tangent Perpendicular to Radius
   - This is important. Remember it.
5. Tangent Angle
   - When you see circles and an angle condition with a tangent, keep this in mind.
   - This proves points are concyclic.
6. Cyclic Quadrilaterals
   - Angles on opposite sides are supplementary.
   - Angles on the same side are congruent.

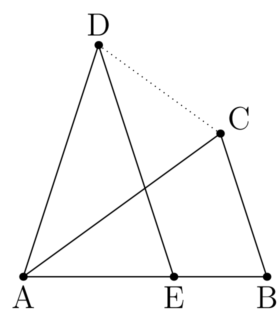
2.6.2 Tips and Strategies

1. Proving collinearity and concurrency for lines can basically be switched around at will.
2. One way to prove concurrency of three figures is to let two of them intersect at a point P, and prove the third passes through P.
3. If two lines are parallel, then it's probably an important part of the problem.
4. The same is true for tangent lines.
5. A nice way to show that $AH \perp BC$ is to show that H is the orthocenter of $\triangle ABC$, namely, $BH \perp CA$ and $CH \perp AB$.
   - ✦ This will not always be true - make sure that it seems true before you try too hard to prove it.

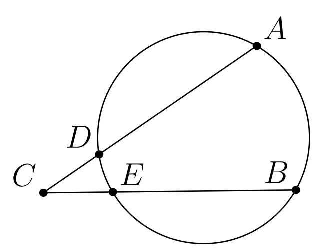
2.7 Exercises

2.7.1 Check-ins

1. Prove $\triangle ABC$ satisfies $\angle A + \angle B + \angle C = 180°$. **Hints**: 8
2. Prove that the sum of the interior angles of an $n - gon$ is $180(n - 2)$. **Hints**: 66 52
3. In $\triangle ABC$, let the feet of the altitude from B, C to CA, AB be E, F, respectively. Then prove that $(BCEF)$ is cyclic. **Solution**: 10
4. (Brazil 2004) In the figure, ABC and DAE are isosceles triangles ($AB = AC = AD = DE$) and the angles BAC and ADE have measures $36°$.
   (a) Using geometric properties, calculate the measure of angle $\angle EDC$.
   (b) Knowing that $BC = 2$, calculate the length of segment DC.
   (c) Calculate the length of segment AC.



5. If $\angle ABC = 60°$ and $\angle CAB = 70°$, find $\overset{\frown}{\mathrm{AB}} - \overset{\frown}{\mathrm{DE}}$.



6. (a) Given that A, B, C, and D are all on the same circle, that BE is the angle bisector of $\angle ABC$, that $\angle AEB = \angle CEB$, and that $\angle ADC = 50°$, find $\angle BAC$.
   (b) Given points A, B, C, D, E such that BE is the angle bisector of $\angle ABC$, $\angle AEB = \angle CEB$, $\angle BAC + \angle BDC = \angle ABD + \angle ACD$, and $\angle ADC = 48°$, find $\angle BCA$.
7. Consider any cyclic pentagon $ABCDE$. If P is the center of $(ABCDE)$, then prove that $ABCP$ is never cyclic.
8. Two circles ω_1, ω_2 intersect at P, Q. If a line intersects ω_1 at A, B and ω_2 at C, D such that A, B, C, D lie on the lie in that order, and P and Q lie on the same side of the line, compute $\angle APC + \angle BQD$. **Solution**: 6

2.7.2 Problems

1. Consider rectangle $ABCD$ with $AB = 6$, $BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM and BN intersect AC at X and Y respectively. Find XY.

2. (AMC 10A 2019/13) Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$? **Hints**: 51

3. (Miquel's Theorem) Consider $\triangle ABC$ with D on BC, E on CA, and F on AB. Prove that (AEF), (BFD), and (CDE) concur. **Hints**: 56

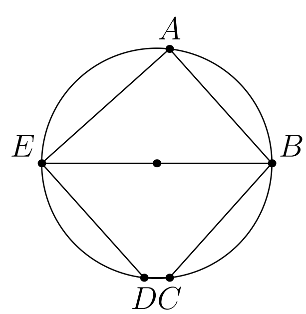
4. Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle EFA$ intersect at $P \neq D$. If $\angle A = 50°, \angle B = 35°$, find $\angle DPE$.

5. Let circles ω_1 and ω_2 intersect at X, Y. Let line ℓ_1 passing through X intersect ω_1 at A and ω_2 at C, and let line ℓ_2 passing through Y intersect ω_1 at B and ω_2 at D. If ℓ_1 intersects ℓ_2 at P, prove that $\triangle PAB \sim \triangle PCD$. **Hints**: 58

6. (Reim's Theorem) Let circles ω_1, ω_2 intersect at P, Q. Let line ℓ_1 passing through P intersect ω_1 again at A_1 and ω_2 again at A_2. Let B_1 be a point on ω_1 and B_2 be a point on ω_2. Then prove that $A_1B_1 \parallel A_2B_2$ if and only if Q lies on B_1B_2.

7. (Simson's Theorem) Consider $\triangle ABC$ and point P, and let X, Y, Z be the feet of the altitudes from P to BC, CA, AB. Prove that X, Y, Z are collinear if and only if P is on (ABC). **Hints**: 29

8. (AMC 10B 2011/17) In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4 : 5$. What is the degree measure of angle BCD?



9. (Formula of Unity 2018) A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$. **Solution**: 3

10. (Australia Math Olympiad 2019/3) Let A, B, C, D, E be five points in order on a circle K. Suppose that $AB = CD$ and $BC = DE$. Let the chords AD and BE intersect at the point P. Prove that the circumcentre of triangle AEP lies on K. **Hints**: 20 39 10 **Solution**: 17

11. Consider square $ABCD$ and some point P outside $ABCD$ such that $\angle APB = 90°$. Prove that the angle bisector of $\angle APB$ also bisects the area of $ABCD$. **Hints**: 19 **Solution**: 4

12. (USAJMO 2020/4) Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying $DA < AB = BC < CD$. Points E and F are chosen on sides CD and AB such that $BE \perp AC$ and $EF \parallel BC$. Prove that $FB = FD$.

13. (IMO 2006/1) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

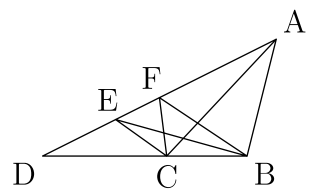
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

2.7.3 Challenges

1. (MAST Diagnostic 2020) Consider parallelogram $ABCD$ with $AB = 7$, $BC = 6$. Let the angle bisector of $\angle DAB$ intersect BC at X and CD at Y. Let the line through X parallel to BD intersect AD at Q. If $QY = 6$, find $\cos \angle DAB$. **Hints**: 32 26 **Solution**: 18

2. (Memorial Day Mock AMC 10 2018/21) In the following diagram, $m\angle BAC = m\angle BFC = 40°$, $m\angle ABF = 80°$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?



Hints: 40 48

3. (FARML 2012/6) In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD. **Hints**: 41 23 **Solution**: 20

4. (ISL 1994/G1) C and D are points on a semicircle. The tangent at C meets the extended diameter of the semicircle at B, and the tangent at D meets it at A, so that A and B are on opposite sides of the center. The lines AC and BD meet at E. F is the foot of the perpendicular from E to AB. Show that EF bisects angle CFD. **Hints**: 38 5 16 **Solution**: 9

5. Consider $\triangle ABC$ with D on line BC. Let the circumcenters of $\triangle ABD$ and $\triangle ACD$ be M, N, respectively. Let the circumcircle of $\triangle MND$ intersect the circumcircle of $\triangle ACD$ again at $H \neq D$. Prove that A, M, H are collinear. **Hints**: 60 57

6. (APMO 1999/3) Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR. **Hints**: 46 1 21 54 **Solution**: 5

7. Let K_1 and K_2 be circles that intersect at two points A and B. The tangents to K_1 at A and B intersect at a point P inside K_2, and the line BP intersects K_2 again at C. The tangents to K_2 at A and C intersect at a point Q, and the line QA intersects K_1 again at D.

   Prove that QP is perpendicular to PD if and only if the centre of K_2 lies on K_1. **Hints**: 43 59 **Solution**: 12

8. (IMO 2000/1) Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **Hints**: 15 63 **Solution**: 19

Chapter 3

Power of a Point

You're a wizard, Harry.

Harry Potter and the Philosopher's Stone

There's only one theorem, so this will be a short chapter. The only prerequisites are angle chasing theorems in circles.

3.1 Power of a Point

The Power of a Point theorem helps us length chase in circles. The proof is a result of similar triangles, and its uses are numerous in lower-level competitions. If you already know this theorem, feel free to skip this section.

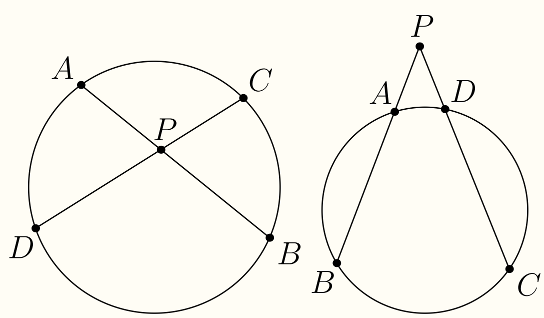
Theorem 3.1 (Power of a Point) Let line ℓ_1 intersect circle ω at A, B line ℓ_2 intersect ω at C, D, and ℓ_1 intersect ℓ_2 at P. Then $PA \cdot PB = PC \cdot PD$.

Proof: There are two cases here: Either P is inside of ω or outside of ω.

If P is inside of ω, then note that by Inscribed Angle, $\angle PAC = \angle PDB$ and $\angle = \angle PAB$, so $\triangle PAC \sim \triangle PDB$.

If P is outside of ω, then without loss of generality, let $PA \leq PB$ and $PC \leq PD$. Then note $\angle PAC = 180^\circ - \angle CAB = \angle PDB$ and $\angle PCA = 180^\circ - \angle ACD = \angle PBD$, so $\triangle PAC \sim \triangle PDB$.

To finish, note that the similarity implies $\frac{PA}{PC} = \frac{PD}{PB}$, or $PA \cdot PB = PC \cdot PB$.



P inside ω and P outside ω.

■

In the case of tangency, $A = B$ is the point of tangency. (You can think of a tangent line intersecting a circle twice at the same point.)

The Two Tangent Theorem is a corollary of Power of a Point. It states that the lengths of the two tangents from a point to a circle are equal.

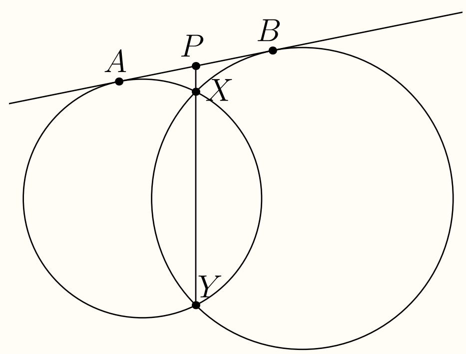
Fact 3.1 (Two Tangent Lemma) Let the tangents from point P to circle ω intersect ω at A, B. Then $PA = PB$. **Hints**: 13 **Solution**: 14

3.2 Bisector Lemma

This is a very powerful fact that kills a lot of earlier computational geometry problems involving circles.

Fact 3.2 (Bisector Lemma) Let ω_1 and ω_2 intersect at X and Y, and let ℓ be a line tangent to ω_1 and ω_2. If ℓ intersects ω_1 at A and ω_2 at B, then XY bisects AB.

Proof: Let XY intersect AB at P. Then by Power of a Point, $PX^2 = PA \cdot PB = PY^2$.



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3.3 Summary

3.3.1 Theory

1. Power of a Point

   ✦ If lines ℓ_1, ℓ_2 through P intersect circle ω at A, B and C, D, respectively, then $PA \cdot PB = PC \cdot PD$.

   ✦ This is a consequence of similar triangles.

2. Bisector Lemma

   ✦ The common chord of two circles bisects the common external tangent.

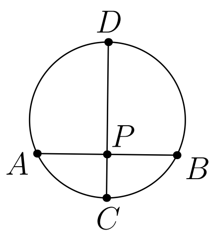
3.3.2 Tips and Strategies

1. If there are two circles and you're in doubt, use Bisector Lemma. (This even applies for some easier olympiad problems.)

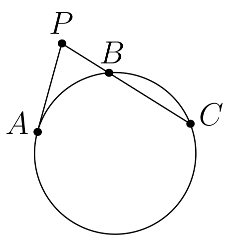
3.4 Exercises

3.4.1 Check-ins

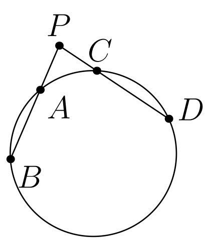
1. Let chords AB and CD in circle ω intersect at P. If $AP = BP = 4$ and $CP = 2$, find DP.



2. Let the tangent through A to circle ω intersect line ℓ that passes through B, C on ω at P. If $BP < CP$, $AP = 4$, and $BC = 6$, find BP.



3. Let line ℓ_1 that passes through A, B on circle ω intersect line ℓ_2 that passes through C, D on ω at P. If $PA = 6$, $AB = 12$, and $PC = 3$, find CD.



4. Prove that if $\triangle ABC$ has a right angle at C and the foot of the altitude from C to AB is P, then

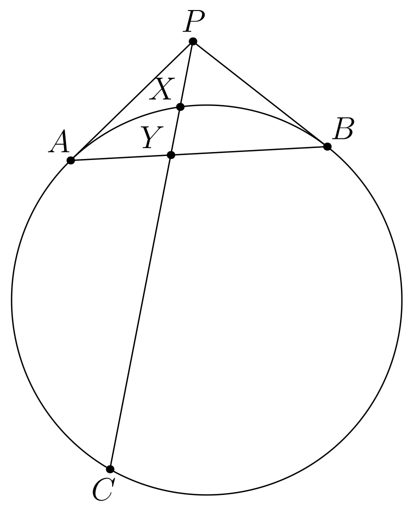
$$PC^2 = PA \cdot PB.$$

Hints: 33 47

5. Let $\triangle ABC$ have a right angle at C and let P be the foot of the altitude from C to AB. If the foot of the altitude from P to AC is X and the foot from P to BC is Y, then prove that $AXYB$ is cyclic.

3.4.2 Problems

1. Let PA and PB be tangents to circle ω, and let line ℓ through P intersect ω at X and C and AB at Y. If $PA = 4$, $PC = 8$, $AY = 1$, and $XY = 1$, find the area of $\triangle PAB$.



2. Consider two externally tangent circles ω_1, ω_2. Let them have common external tangents AC, BD such that A, B are on ω_1 and C, D are on ω_2. Let AC intersect BD at P, and let the common internal tangent intersect AC and BD at X and Y. If $\frac{[PCD]}{[PAB]} = \frac{1}{25}$, find $\frac{[PCD]}{[PXY]}$.

3. (Mandelbrot 2012) Let A and B be points on the lines $y = 3$ and $y = 12$, respectively. There are two circles passing through A and B that are also tangent to the x axis, say at P and Q. Suppose that $PQ = 2012$. Find AB.

4. (Parody) Consider a coordinate plane with two circles tangent to the x axis at X, Y, respectively. If the circles intersect at P, Q, and $XY = 8$, is it possible for P to lie on $y = 3$ and Q to lie on $y = 12$?

5. (e-dchen Mock MATHCOUNTS) Consider chord AB of circle ω centered at O. Let P be a point on segment AB such that $AP = 2$ and $BP = 8$. If $\angle APO = 150°$, what is the area of ω?

☢ 3.5 Challenges

1. (Geometry Bee 2019) Circles O_1 and O_2 are constructed with O_1 having radius of 2, O_2 having radius of 4, and O_2 passing through the point O_1. Lines ℓ_1 and ℓ_2 are drawn so they are tangent to both O_1 and O_2. Let O_1 and O_2 intersect at points P and Q. Segment $\overline{EF}$ is drawn through P and Q such that E lies on ℓ_1 and F lies on ℓ_2. What is the length of $\overline{EF}$?

2. (AMC 10B 2013/23) In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$? **Solution**: 7

3. (AIME I 2019/6) In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Chapter 4

Lengths and Areas in Triangles

> Again, you can't connect the dots looking forward; you can only connect them looking backward. So you have to trust that the dots will somehow connect in your future. You have to trust in something — your gut, destiny, life, karma, whatever. This approach has never let me down, and it has made all the difference in my life.
>
> *Steve Jobs*

4.1 Lengths

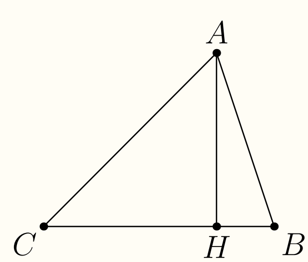
There are a couple of important lengths in a triangle. These are the lengths of cevians, the inradius/exradius, and the circumradius.

4.1.1 Law of Cosines and Stewart's

We discuss how to find the third side of a triangle given two sides and an included angle, and use this to find a general formula for the length of a cevian.

Theorem 4.1 (Law of Cosines) Given $\triangle ABC$, $a^2 + b^2 - 2ab\cos C = c^2$.

Proof: Let the foot of the altitude from A to BC be H. Then note that $A = b\sin C$, $CH = b\cos C$, and $BH = |a - b\cos C|$. (The absolute value is because $\angle B$ can either be acute or obtuse.) Then note by the Pythagorean Theorem, $(b\sin C)^2 + (a - b\cos C)^2 = a^2 + b^2 - 2ab\cos C = c^2$.



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Theorem 4.2 (Stewart's Theorem) Consider $\triangle ABC$ with cevian AD, and denote $BD = m$, $CD = n$, and $AD = d$. Then $man + dad = bmb + cnc$.

Proof: We use the Law of Cosines. Note that

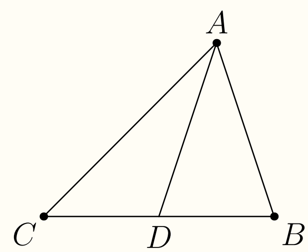
$$\cos\angle ADB = \frac{d^2+m^2-c^2}{2dm} = -\frac{d^2+n^2-b^2}{2dn} = -\cos\angle ADC.$$

Multiplying both sides by $2dmn$ yields

$$c^2n - d^2n - m^2n = -bm^2 + d^2m + mn^2$$

$$b^2m + c^2n = mn(m+n) + d^2(m+n)$$

$$bmb + cnc = man + dad.$$



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Here are two corollaries that will save you a lot of time in computational contests.

Fact 4.1 (Length of Angle Bisector) In $\triangle ABC$ with angle bisector AD, denote $BD = x$ and $CD = y$. Then

$$AD = \sqrt{bc - xy}.$$

Fact 4.2 (Length of Median) In $\triangle ABC$ with median AD,

$$AD = \frac{\sqrt{2b^2+2c^2-a^2}}{2}.$$

4.1.2 Law of Sines and the Circumradius

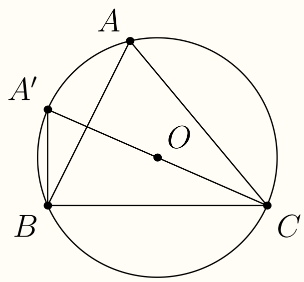
The Law of Sines is a good way to length chase with a lot of angles.

Theorem 4.3 (Law of Sines) In $\triangle ABC$ with circumradius R,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof: We only need to prove that $\frac{a}{\sin A} = 2R$, and the rest will follow.

Let the line through B perpendicular to BC intersect (ABC) again at A'. Then note that $A'C = 2R$ by Thale's. By the Inscribed Angle Theorem, $\sin \angle CA'B = \sin A$, so $\frac{a}{\sin A} = \frac{a}{\sin \angle CA'B} = \frac{a}{\frac{a}{2R}} = 2R$.



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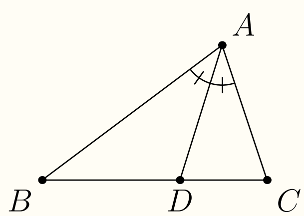
Other texts will call this the Extended Law of Sines. But the Extended Law of Sines has a better proof than the "normal" Law of Sines, and redundancy is bad.

The Law of Sines gives us the Angle Bisector Theorem.

Theorem 4.4 (Angle Bisector Theorem) Let D be the point on BC such that $\angle BAD = \angle DAC$. Then $\frac{AB}{BD} = \frac{AC}{CD}$.

Proof: By the Law of Sines, $\frac{\sin \angle ADB}{\sin \angle BAD} = \frac{AB}{BD}$ and $\frac{\sin \angle ADC}{\sin \angle CAD} = \frac{AC}{CD}$. But note that $\angle BAD = \angle ADC$ and $\angle BAD + \angle CAD = 180°$, so $\frac{AB}{BD} = \frac{AC}{CD}$.



■

In fact, the Angle Bisector Theorem can be generalized in what is known as the ratio lemma.

Theorem 4.5 (Ratio Lemma) Consider $\triangle ABC$ with point P on BC. Then $\frac{BP}{CP} = \frac{c \sin \angle BAP}{b \sin \angle CAP}$.

The proof is pretty much identical to the proof for Angle Bisector Theorem.

Proof: By the Law of Sines, $BP = \frac{c \sin \angle BAP}{\sin \angle APB}$ and $CP = \frac{b \sin \angle CAP}{\sin \angle APC}$. Since $\sin \angle APB = \sin \angle APC$,

$$\frac{BP}{CP} = \frac{c \sin \angle BAP}{b \sin \angle CAP}.$$

■

Note that this remains true even if P is on the *extension* of BC.

Here's a classic example that cleverly utilizes the Law of Sines.

Example 4.1 Show that $\triangle ABC$ is similar to the triangle with side lengths $\sin A, \sin B, \sin C$.

Solution: Note that $\sin A = \frac{a}{2R}$, so the similarity factor is $2R$. ■

We'll utilize this concept further in the next example.

Example 4.2 Consider $\triangle ABC$ with side lengths $AB = 13$, $BC = 5$, and $CA = 12$. Find the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$.

Solution: Note that $[ABC] = 60$ and the triangle with lengths $\sin A$, $\sin B$, and $\sin C$ is similar to $\triangle ABC$ with a scale factor of 13. Thus the desired area is $\frac{60}{13^2} = \frac{60}{169}$. ■

It's possible to just directly use the values of $\sin A$, $\sin B$, and $\sin C$, but this will not work for general triangles.

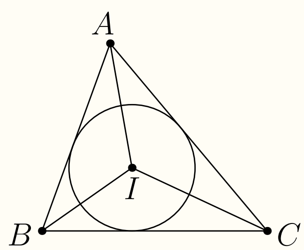
4.1.3 The Incircle, Excircle, and Tangent Chasing

We provide formulas for the inradius, exradii, and take a look at some uses of the Two Tangent Theorem. Recall that the Two Tangent Theorem states that if the tangents from P to ω intersect ω at A, B, then $PA = PB$.

Theorem 4.6 (rs) In $\triangle ABC$ with inradius r,

$$[ABC] = rs.$$

Proof: Note that $[ABC] = r \cdot \frac{a+b+c}{2} = rs$.



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A useful fact of the incircle is that the length of the tangents from A is $s - a$. Similar results hold for the B, C tangents to the incircle.

Fact 4.3 (Tangents to Incircle) Let the incircle of $\triangle ABC$ be tangent to BC, CA, AB at D, E, F. Then

$$AE = AF = s - a$$

$$BF = BD = s - b$$

$$CD = CE = s - c.$$

Proof: Note that by the Two Tangent Theorem, $AE = AF = x$, $BF = BD = y$, and $CD = CE = z$. Also note that

$$BD + CD = y + z = a$$

$$CE + EA = z + x = b$$

$$AF + FB = x + y = c.$$

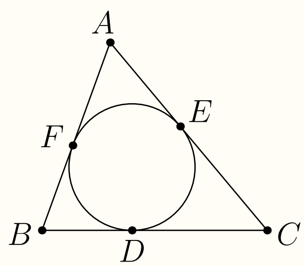
Adding these equations gives $2x + 2y + 2z = a + b + c = 2s$, implying $x + y + z = s$. Thus

$$x = AE = AF = s - a$$

$$y = BF = BD = s - b$$

$$z = CD = CE = s - c,$$

as desired.

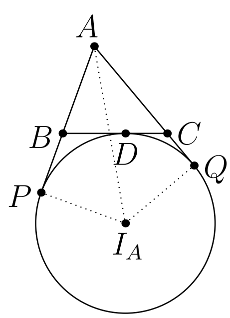


■

Theorem 4.7 ($r_a(s - a)$) In $\triangle ABC$ with A exradius r_a,

$$[ABC] = r_a(s - a).$$

Proof: Let AB, AC be tangent to the A excircle at P, Q, respectively, and let BC be tangent to the A excircle at D. Then note that by the Two Tangent Theorem, $PB = BD$ and $DC = CQ$. Thus $[ABC] = [API_A] + [AQI_A] - 2[BI_AC] = r_a \cdot \frac{s+s-2a}{2} = r_a(s - a)$. ■

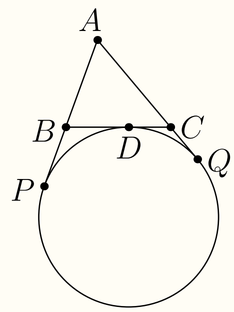


The proof also implies the following corollary.

Fact 4.4 (Tangents to Excircle) Let the A excircle of $\triangle ABC$ be tangent to BC at D. Then $BD = s - c$ and $CD = s - b$.

Analogous equations hold for the B and C excircles.

Proof: Let the A excircle be tangent to line AB at P and line AC at Q. Note that $AP = AB+BP = c+BD$ and $AQ = AC + CQ = b + CD$ by the Two Tangent Theorem. Applying the Two Tangent Theorem again gives $AP = AQ$, or $c + BD = b + CD$. Also note that $AP + AQ = b + c + BD + DC = 2s$, so $AP = AQ = s$ and $s = c + BD = b + CD$. Thus $BD = s - c$ and $CD = s - b$.



■

Keep these area and length conditions in mind when you see incircles and excircles.

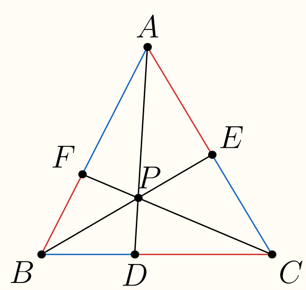
4.1.4 Concurrency with Cevians

We discuss Ceva's Theorem, Menelaus Theorem, and mass points, three ways to look at concurrent cevians. Very rarely do problems involving concurrency with cevians appear on higher level contests, but they're fairly common in the AMC 8 and MATHCOUNTS. This is also a good tool to have for when you need it.

Theorem 4.8 (Ceva's Theorem) In $\triangle ABC$ with cevians AD, BE, CF, they concur if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

Proof: Let the point of concurrency be P. Note that $\frac{[ABD]}{[ADC]} = \frac{[PBD]}{[PDC]} = \frac{BD}{DC}$, so $\frac{[BPA]}{[APC]} = \frac{BD}{DC}$. Thus,

$$\frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = \frac{[CPB]}{[BPA]} \cdot \frac{[APC]}{[CPB]} \cdot \frac{[BPA]}{[APC]} = 1.$$



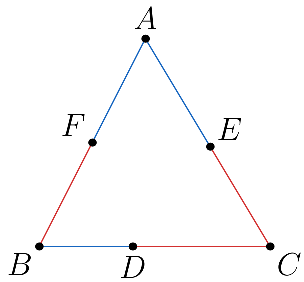
■

A good way to remember what goes in the numerator and denominator is by looking at the colors and thinking about them alternating.

We present an example of what not to do.

Example 4.3 (Order Mixed Up) Consider $\triangle ABC$ with D, E, F on BC, CA, AB respectively, such that $BD = 4$, $DC = 6$, $AE = 6$, $EC = 4$, and $AF = BF = 5$. Are AD, BE, and CF concurrent?

Solution (Bogus): Yes. Note that $\frac{4}{6} \cdot \frac{6}{4} \cdot \frac{5}{5} = 1$.



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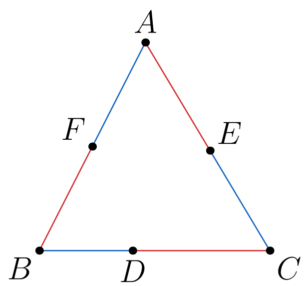
This is not right, as the order of the lengths is messed up (intentionally) in the problem statement. (Also note the colors are messed up.) We now present the correct solution.

Solution (Correct): No. Note that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{4}{6} \cdot \frac{4}{6} \cdot \frac{5}{5} = \frac{4}{9}$, which is not 1.



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There is also a trigonometric version of this theorem. Its proof is left as an exercise for the more experienced reader If you are encountering this for the first time, following the hints as a walkthrough is recommended.

Theorem 4.9 (Trigonometric Ceva) Let D, E, and F be points on sides BC, CA, and AB of $\triangle ABC$. Then AD, BE, and CF concur if and only if

$$\frac{\sin \angle CAD}{\sin \angle DAB} \cdot \frac{\sin \angle ABE}{\sin \angle EBC} \cdot \frac{\sin \angle BCF}{\sin \angle FCE} = 1.$$

Hints: 22 34 9 35 65 7 62

Here is a harder example that relies on the trigonometric form of Ceva.

Example 4.4 (Swiss Math Olympiad 2008/8) Let $ABCDEF$ be a convex hexagon inscribed in a circle. Prove that the diagonals AD, BE and CF intersect at one point if and only if

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Solution: Using Ceva's on $\triangle ACE$ gives us that AD, BE, and CF concur if and only if

$$\frac{\sin \angle AEB}{\sin \angle BEC} \cdot \frac{\sin \angle CAD}{\sin \angle DAE} \cdot \frac{\sin \angle ECF}{\sin \angle FCA} = 1.$$

But note that by the Law of Sines,

$$\frac{\sin \angle AEB}{\sin \angle BEC} = \frac{\frac{AB}{2R}}{\frac{BC}{2R}} = \frac{AB}{BC},$$

where R is the circumradius of $(ABCDEF)$, so this is equivalent to

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

■

Theorem 4.10 (Menelaus) Consider $\triangle ABC$ with D, E, F on lines BC, CA, AB, respectively. Then D, E, F are collinear if $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = 1$.

This looks very similar to Ceva - in fact, the letters just switched. Instead of the line segments cycling through D, E, F, they now cycle through A, B, C.

Proof: Draw a line through A parallel to DE and let it intersect BC at P. Then note that $\triangle ABP \sim \triangle FBD$ and $\triangle ECD \sim \triangle ACP$, so

$$\frac{AF}{FB} = \frac{PD}{DB}$$

$$\frac{EC}{EA} = \frac{DC}{DP}$$

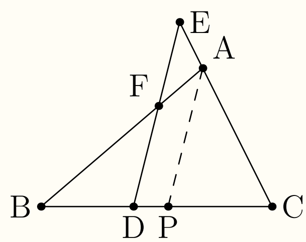
Multiplying the two together yields

$$\frac{AF}{FB} \cdot \frac{BD}{DC} = \frac{EA}{CE},$$

which implies that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

as desired.

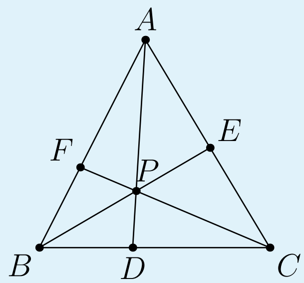


■

The converse states that $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = -1$, *where all lengths are directed.* (The directed lengths are necessary. In the original theorem, fixing D, E leaves two possible locations for F, only one of which actually lies on DE.)

Theorem 4.11 (Mass Points) Consider segment XY with P on XY. Then assign *masses* $\diamond X, \diamond Y$ to points X, Y such that $\frac{XP}{YP} = \frac{\diamond Y}{\diamond X}$.

Now consider cevians AD, BE, CF of $\triangle ABC$ that concur at some point P. Then $\frac{AP}{PD} = \frac{\diamond B + \diamond C}{\diamond A}$. This means that for P on XY, we can define $\diamond P = \diamond X + \diamond Y$.

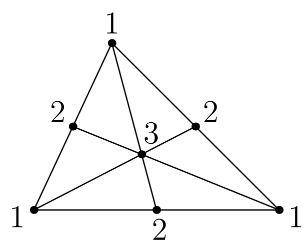


This is a direct application of Ceva's and Menelaus. This is somewhat abstract without an example, so we present the centroid as an example.

Example 4.5 (Centroid) Assign masses to $\triangle ABC$, its midpoints, and its centroid.

Solution: Note $\diamond A = \diamond B = \diamond C$. Without loss of generality, let $\diamond A = 1$.

Then note that since $\diamond X + \diamond Y = \diamond P$ for P on segment XY, $\diamond D = \diamond B + \diamond C = 2$. Similarly, $\diamond E = \diamond F = 2$, and $\diamond G = \diamond A + \diamond D = 1 + 2 = 3$.

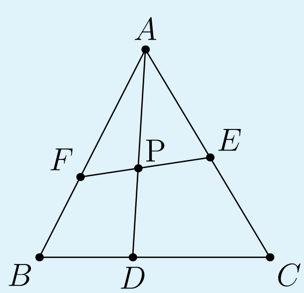


■

Theorem 4.12 (Mass Points with Transversals) Consider $\triangle ABC$ with points D, E, F on sides BC, CA, AB, and let AD intersect FE at P. Then $\diamond A = \diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA}$.

This is equivalent to

$$\frac{AP}{PD} = \frac{\diamond B + \diamond C}{\diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA}} = \frac{BC}{CD \cdot \frac{BF}{FA} + BD \cdot \frac{CE}{EA}}.$$



The classic analogy is having A_1 on AB and A_2 on AC, and adding $\diamond A_1 + \diamond A_2$ where the masses are taken with respect to AB and AC individually.

You can prove this with Law of Cosines. We present the outline of the proof (the actual algebraic manipulations are very long; this is just a demonstration that it can be proven true).

Proof: There is exactly one value of AP such that

$$FP + PE = FE,$$

where

$$FP = \sqrt{AF^2 + AP^2 - 2 \cdot AF \cdot AP \cos \angle BAD}$$
$$PE = \sqrt{AE^2 + AP^2 - 2 \cdot AE \cdot AP \cos \angle CAD}$$
$$FE = \sqrt{AE^2 + AF^2 - 2 \cdot AE \cdot AF \cos \angle BAC},$$

and all you have to do is verify

$$AG = \frac{BC \cdot GD}{CD \cdot \frac{BF}{FA} + BD \cdot \frac{CE}{EA}}$$

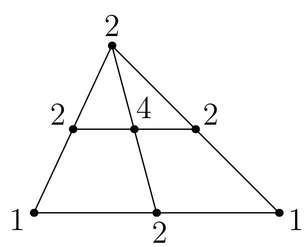
indeed works. ■

As an example, we use a midsegment and a median.

Example 4.6 (Midsegment) Assign masses to $\triangle ABC$, A-midsegment EF, median AD, and the point P that lies on AD and EF.

Solution: Note $\diamond A = \diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA} = \diamond B + \diamond C$. Without loss of generality, let $\diamond B = \diamond C = 1$. Then $\diamond A = 2$.

Also note that $\diamond D = \diamond B + \diamond C = 2$ and $\diamond P = \diamond A + \diamond D = 4$.



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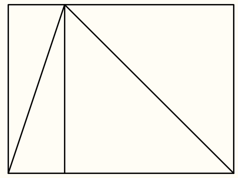
4.2 Areas

There are a variety of methods to find area. For harder problems, computing the area in two different ways can give useful information about the configuration.

Theorem 4.13 ($\frac{bh}{2}$) The area of a triangle is $\frac{bh}{2}$.

Proof: The area of each right triangle is half of the area of the rectangle it is in.



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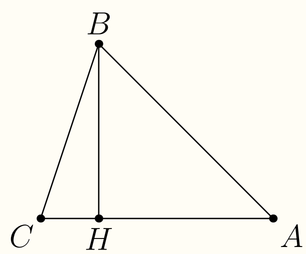
Theorem 4.14 (rs) The area of a triangle is rs, where r is the inradius and s is the semiperimeter.

We have already proved this in Length Chasing - but we mention this theorem again because it is useful for area too.

Theorem 4.15 ($\frac{1}{2}ab\sin C$) The area of a triangle is $\frac{1}{2}ab\sin C$, where a, b are side lengths and C is the included angle.

Proof: Drop an altitude from B to AC and let it have length h. Then note $\frac{1}{2} \cdot a\sin C \cdot b = \frac{1}{2} \cdot hb = \frac{bh}{2}$.



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We present a useful corollary of this theorem.

Fact 4.5 ($\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$) Let P, A, X be on ℓ_1 and P, B, Y be on ℓ_2. Then $\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$.

Proof: Note $\frac{[PAB]}{[PXY]} = \frac{\frac{1}{2} \cdot PA \cdot PB \cdot \sin\theta}{\frac{1}{2} \cdot PX \cdot PY \cdot \sin\theta} = \frac{PA \cdot PB}{PX \cdot PY}$, where $\theta = \angle APB$.
This works for all configurations since $\sin\theta = \sin(180 - \theta)$. ■

Theorem 4.16 ($\frac{abc}{4R}$) In $\triangle ABC$ with side lengths a, b, c and circumradius R,

$$[ABC] = \frac{abc}{4R}.$$

Proof: Note that $[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}ab \cdot \frac{c}{2R} = \frac{abc}{4R}$. ■

Heron's Formula can find the area of a triangle with *only* the side lengths.

Theorem 4.17 (Heron's Formula) In $\triangle ABC$ with sidelengths a, b, c such that $s = \frac{a+b+c}{2}$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof: Since $\cos C = \frac{a^2+b^2-c^2}{2ab}$, the Pythagorean Identity gives us

$$\sin C = \sqrt{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}} = \sqrt{\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2b^2}}.$$

So

$$\frac{1}{2}ab\sin C = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)} = \sqrt{s(s-a)(s-b)(s-c)}.$$

■

Heron's Formula has a reputation for being notoriously tricky to prove, but the proof isn't too bad if you consider what we're actually doing.

1. Use the Law of Cosines to find $\cos C$.
2. Use the Pythagorean Identity to find $\sin C$.
3. Use $\frac{1}{2}ab\sin C$ to find $[ABC]$.
4. Clean the expression up.

Fact 4.6 (Heron's with Altitudes) If x, y, z are the lengths of the altitudes of $\triangle ABC$,

$$\frac{1}{[ABC]} = \sqrt{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(-\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{x}-\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)}.$$

Hints: 37

4.3 Summary

4.3.1 Theory

1. Law of Cosines
   - $a^2+b^2-2ab\cos C = c^2$.
2. Stewart's Theorem
   - $man + dad = bmb + cnc$.
   - $\sqrt{bc-xy}$ gives the length of angle bisector AD.
   - $\frac{\sqrt{2b^2+2c^2-a^2}}{2}$ gives the length of median AD.
3. Law of Sines
   - $\frac{a}{\sin A} = 2R$.
4. Angle Bisector Theorem and Ratio Lemma
   - If AD bisects $\angle BAC$, then $\frac{AB}{BD} = \frac{AC}{CD}$.
   - Generally, $\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}$.

5. Tangents
   - Two Tangent Theorem
   - The tangent is perpendicular to the radius.
   - $[ABC] = rs$.
   - $[ABC] = r_a(s-a)$.
   - Lengths of tangents to the incircle from the vertices are $s-a, s-b, s-c$.
   - Lengths of tangents to the excircles from the vertices are also $s-a, s-b, s-c$ (but in a different order).
6. Concurrency and Collinearity
   - Ceva's states $\frac{AF}{FB} \cdot \frac{BE}{EC} \cdot \frac{CD}{DA} = 1$.
   - Menelaus states $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = 1$.
7. Mass Points
   - $\frac{XP}{YP} = \frac{\diamond Y}{\diamond X}$.
   - $\diamond X + \diamond Y = \diamond P$.
8. Area
   - $\frac{bh}{2}$
   - rs
   - $\frac{1}{2}ab\sin C$
   - $\frac{abc}{4R}$
   - Heron's ($\sqrt{s(s-a)(s-b)(s-c)}$)

4.3.2 Tips and Strategies

1. Use the Law of Sines and Law of Cosines when convenient angles exist.
   - These can be supplementary, congruent, special, etc.
   - Use Stewart's when angles are not explicitly present but you need to find a cevian's length anyway.
2. If you have tangents, do length chasing. You will need it.
3. $\frac{1}{2}ab\sin C$ gives ratios of areas. (In general, whenever angles are the same or supplementary, use $\frac{1}{2}ab\sin C$ to get information.)
4. Use two methods to calculate area.
   - This can give you information about a problem; after all, area doesn't change. So now you can set two seemingly unrelated things equal.

4.4 Exercises

4.4.1 Check-ins

1. Find the inradius of the triangles with the following lengths:

   (a) $3, 4, 5$

   (b) $5, 12, 13$

   (c) $13, 14, 15$

   (d) $5, 7, 8$

   (These are arranged by difficulty. All of these are good to know.)

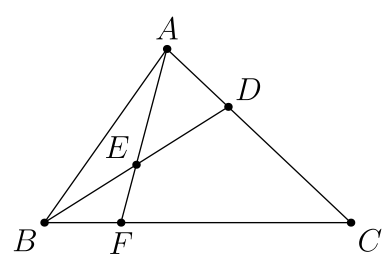
2. Prove that in a right triangle with legs of length a, b and hypotenuse with length c, $r = \frac{a+b-c}{2}$.

3. In $\triangle ABC$, $AB = 5$, $BC = 12$, and $CA = 13$. Points D, E are on BC such that $BD = DC$ and $\angle BAE = \angle CAE$. Find $[ADE]$. **Hints**: 44 **Solution**: 13

4. (Gergonne Point) Let the incircle of $\triangle ABC$ be tangent to BC, CA, AB at D, E, F, respectively. Prove that AD, BE, CF concur. **Hints**: 18

5. (Nagel Point) Let the A excircle of $\triangle ABC$ be tangent to BC at D, and define E, F similarly. Prove that AD, BE, CF concur. **Hints**: 24

6. (AMC 8 2019/24) In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



7. Consider $\triangle ABC$ where X, Y are on BC, CA such that $\frac{BX}{CX} = \frac{1}{4}, \frac{CY}{YA} = \frac{2}{3}$. If AX, BY intersect at Z, find $\frac{AZ}{ZX}$.

8. Given $\triangle ABC$ with E, F on line segments AC, AB such that $AE : EC = BF : FA = 1 : 3$ and median AD that intersects EF at G, $AG : GD$.

9. A triangle has side lengths $4, 8, x$ and area $3\sqrt{15}$. Find x.

10. Find the sum of the altitudes of a triangle with side lengths $5, 7, 8$.

11. Let $\angle BAC = 30°$ and let P be the midpoint of AC. If $\angle BPC = 45°$, what is $\angle ABC$? **Hints**: 45

12. Given $\triangle ABC$, find $\sin A \sin B \sin C$ in terms of $[ABC]$ and abc.

13. Let P be a point inside $\triangle ABC$, and let AP, BP, CP intersect BC, CA, AB at D, E, F. Let (DEF) intersect BC, CA, AB again at X, Y, Z. Prove that AX, BY, CZ concur. **Hints**: 53 **Solution**: 8

4.4.2 Problems

1. Consider $\triangle ABC$ with $AB = 7, BC = 8, AC = 6$. Let AD be the angle bisector of $\angle BAC$ and let E be the midpoint of AC. If BE and AD intersect at G, find AG.

2. Find the maximum area of a triangle with two of its sides having lengths $10, 11$.

3. Consider trapezoid $ABCD$ with bases AB and CD. If AC and BD intersect at P, prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ is at least half the area of trapezoid $ABCD$.

4. Consider rectangle $ABCD$ such that $AB = 2$ and $BC = 1$. Let X, Y trisect AB. Then let DX and DY intersect AC at P and Q, respectively. What is the area of quadrilateral $XYQP$?

5. (Autumn Mock AMC 10) Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

6. Consider $\triangle ABC$ such that $AB = 8$, $BC = 5$, and $CA = 7$. Let AB and CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$. **Hints**: 28

7. Consider $\triangle ABC$ with an area of 60, inradius of 3, and circumradius of $\frac{17}{2}$. Find the side lengths of the triangle.

8. (AIME I 2019/2) In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

9. (PUMaC 2016) Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m+n$.

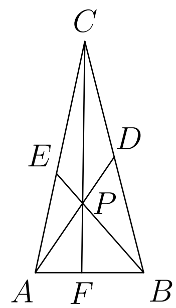
10. (AIME I 2001/4) In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

4.4.3 Challenges

1. (CIME 2020) An excircle of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90^\circ$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

2. Consider ABC with $\angle A = 45^\circ, \angle B = 60^\circ$, and with circumcenter O. If BO intersects CA at E and CO intersects AB at F, find $\frac{[AFE]}{[ABC]}$.

3. (AIME 1989/15) Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure at right). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.



4. (AIME II 2019/11) Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hints**: 55

5. (AIME II 2016/10) Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

6. (AIME II 2005/14) In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.

7. (AIME I 2019/11) In $\triangle ABC$, the sides have integers lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

8. (ART 2019/6) Consider unit circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. If $XP = MQ$, find AT. **Hints**: 27 14 6 31 **Solution**: 16

9. (AIME I 2020/13) Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

10. (USAMO 1999/6) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles. **Hints**: 25

11. (ISL 2003/G1) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

12. (CIME 2019) Let $\triangle ABC$ be a triangle with circumcenter O and incenter I such that the lengths of the three segments AB, BC and CA form an increasing arithmetic progression in this order. If $AO = 60$ and $AI = 58$, then the distance from A to BC can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 5

3D Geometry

Calculus is the most powerful weapon of thought yet devised by the wit of man.

Wallace B. Smith

The proofs in this section assume a basic understanding of calculus. It's fine if you don't; you can just ignore the proofs. But I believe 3D geometry is most naturally understood with calculus, and it is one of the most natural and elegant exercises of calculus.

5.1 Definitions

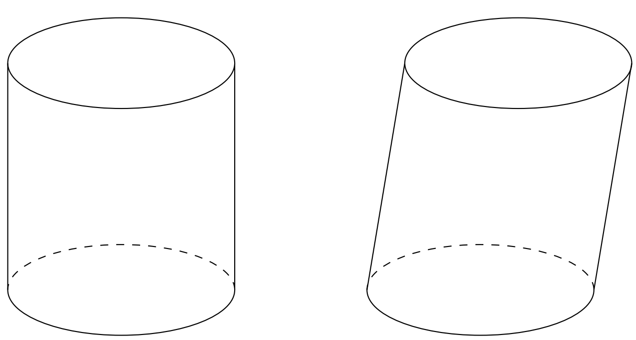
Before we *prove* that the formulas in all their glory, we need to define volume and surface area.

5.1.1 Volume

Definition 5.1 (Volume) The volume of a solid is the integral of the areas of all cross-sections made with planes parallel to a certain reference plane.

This basically means you sum up all of the cross-sections, similar to finding the area of a triangle. Usually our reference plane will be the base.

Fact 5.1 (Cavalieri's Principle) If in two solids of equal altitudes, the planes parallel to and at the same distance from their respective bases always create cross-sections with equal area, then the two solids have the same volume.



Fact 5.2 (Rearranging Principle) If you can rearrange a solid into another solid, the two solids have the same volume.

This means that if you have a rectangular prism attached to another, you can just find the sum of the volumes of the two rectangular prisms to find the volume of the joined solid. Similar methods apply for other combinations of solids.

5.1.2 Surface Area

Surface area is surprisingly hard to define formally. Intuitively, it is just the area of the surfaces.

Fact 5.3 (Additivity) The surface area of an object is the sum of the surface area of its parts.

Fact 5.4 (Surface Area of Flat Shapes) The surface area of a flat shape is the same as the area of the flat shape.

Fact 5.5 (Straight Lines) If part of the surface consists of lines, then the surface area of that part can be found by integrating the lengths of the lines.

As an example, consider the side of a cylinder or the lines joining the apex of a cone to the circumference of its base.

Fact 5.6 (Curves) If part of the surface consists of curves, then the surface area of that part can be found by integrating the lengths of the curves.

This will be useful for spheres.

☢ 5.2 Prisms and Cylinders

Prisms and cylinders are similar in that a cylinder is analogous to a prism with a continuous base. We also discuss some specific prisms, such as cubes, rectangular prisms, and parallelepipeds.

5.2.1 Prisms and Cylinders

We define a prism and a cylinder.

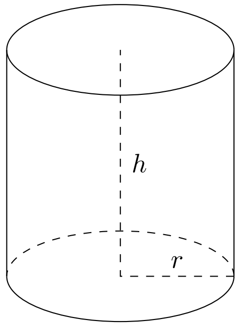
Definition 5.2 (Prism) A prism is a solid with two congruent parallel bases with parallelograms for the side faces.

An equivalent definition is that a prism is a solid with two bases that can be translated to each other.

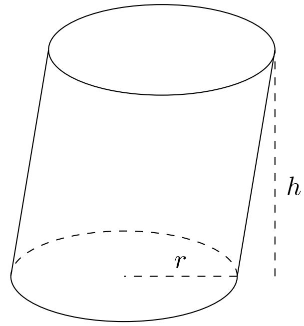
Definition 5.3 (Cylinder) A cylinder is a solid with two parallel circles as bases.

Definition 5.4 (Right Prism and Cylinder) A prism or cylinder is considered *right* if a line joining two corresponding points on the two bases is perpendicular to both bases.

Unless otherwise specified, prisms and cylinders are right. Regardless, the volume formulas *always* hold.



A right cylinder.



An oblique cylinder.

Theorem 5.1 (Volume of a Prism) The volume of a prism with a base of area B and a height of h is Bh.

Theorem 5.2 (Volume of a Cylinder) The volume of a cylinder with a base if area B and a height of h is Bh.

An alternative formulation of this is that the volume of a cylinder with radius r and height h is $\pi r^2 h$.

We prove both volume formulas in one fell swoop.

Proof: Let the reference plane be one of the bases. Then note that the cross-section always has area B over a height of h. Let k be the distance of the cross-section from the base. Then the volume is $\int\limits_0^h Bkdk = Bh$. ■

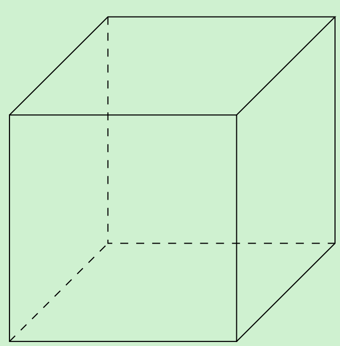
Theorem 5.3 (Surface Area of a Right Cylinder) The surface area of a right cylinder with radius r and height h is $2\pi r^2 + 2\pi rh$.

Proof: The two faces have area πr^2 each, and the lateral surface area (surface area of the sides) can be found by integrating the circumference of the base about the height. So the lateral surface area is $\int\limits_0^h 2\pi r dx = 2\pi rh$. Thus the total surface area is $2\pi r^2 + 2\pi rh$. ■

5.2.2 Cubes and Rectangular Prisms

We discuss cubes and rectangular prisms.

Definition 5.5 (Cube) A cube is a solid with 6 square faces.



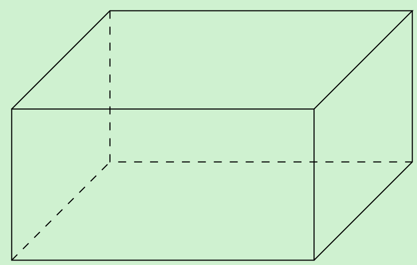
Theorem 5.4 (Volume of a Cube) The volume of a cube with side length x is x^3.

Proof: A cube is a prism with base area x^2 and height x, so its area is x^3. ■

Theorem 5.5 (Surface Area of a Cube) The surface area of a cube with side length x is $6x^2$.

Proof: There are six faces, and each face has surface area x^2. Thus the total surface area is $6x^2$. ■

Definition 5.6 (Rectangular Prism) A rectangular prism is a prism with rectangular bases.



In this chapter, a rectangular prism will always refer to a *right* rectangular prism. Unless otherwise specified, rectangular prisms are right. This is generally true in competitions as well.

Theorem 5.6 (Volume of a Right Rectangular Prism) The volume of a rectangular prism with side lengths l, w, h is lwh.

Proof: The base has area lw and the height is h, so the volume is lwh. ■

Theorem 5.7 (Surface Area of a Right Rectangular Prism) The surface area of a rectangular prism with side lengths l, w, h is $2(lw + wh + hl)$.

Proof: There are two faces with the dimensions $l \times w$, $w \times h$, and $h \times l$. Multiplication and addition finish. ■

Example 5.1 Three faces of a rectangular prism have areas $6, 10, 15$. Find the volume of the rectangular prism.

Solution: Let the side lengths be a, b, c. Note that

$$ab = 6$$

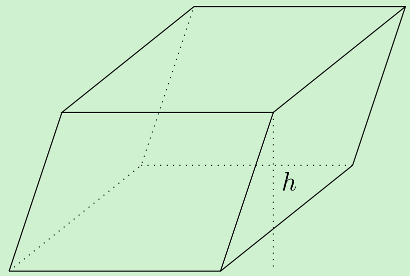
$$bc = 10$$

$$ca = 15,$$

and the volume of the prism is abc. Multiplying all of the expressions together gives us $(abc)^2 = 900$, or $abc = 30$. ■

5.2.3 Parallelepipeds

Definition 5.7 (Parallelepipeds) A parallelpiped is a solid with 6 parallelogram faces.



Note that all parallelepipeds are prisms.

Theorem 5.8 (Volume of a Parallelepiped) The volume of a parallelepiped with a base of area B and height h is Bh.

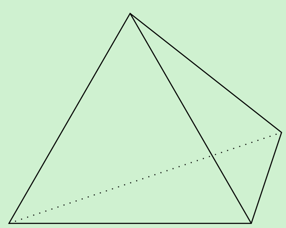
The surface area of a parallelepiped should also not be too hard to compute, though there is no nice general formula.

5.3 Pyramids and Cones

Pyramids and cones are similar in that a cylinder is analogous to a prism with a continuous base.

5.3.1 Pyramids

Definition 5.8 A pyramid is a solid with a polygonal base whose vertices are all joined to a point not in the plane of the base. This point is called the *apex*.



Theorem 5.9 (Volume of a Pyramid) The volume of a pyramid with a base of area B and a height of h is $\frac{Bh}{3}$.

Proof: Let the reference plane be the plane through the apex parallel to the base and let k be the distance of the cross-section from the reference plane. (The cross-section lies on the same side of the reference plane as the base.)

Then by similarity, the volume is $\int\limits_0^k B\frac{k^2}{h^2}dk = \frac{B}{h^2}\int\limits_0^k k^2 dk = \frac{B}{h^2} \cdot \frac{h^3}{3} = \frac{Bh}{3}$. ■

5.3.2 Cones

Definition 5.9 A cone is a solid with a circular base where every point on the base is joined to a point not in the plane of the base. This point is called the *apex*.

The volume formula is identical to the volume of a pyramid.

Theorem 5.10 (Volume of a Cone) The volume of a cone with a base of area B and a height of h is $\frac{Bh}{3}$. An alternative formulation of this is that the volume of a cone with radius r and height h is $\frac{\pi r^2 h}{3}$.

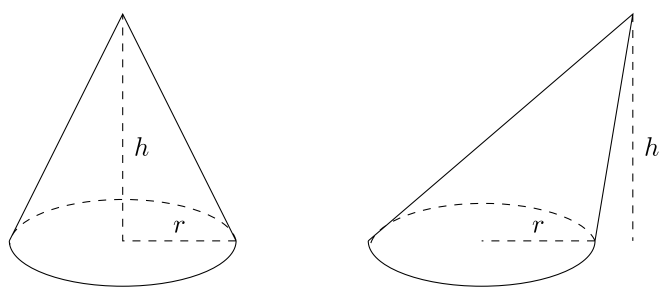
Proof: Let the reference plane be the plane through the apex parallel to the base and let k be the distance of the cross-section from the reference plane. (The cross-section lies on the same side of the reference plane as the base.)

Then by similarity, the volume is $\int\limits_0^k B\frac{k^2}{h^2}dk = \frac{B}{h^2}\int\limits_0^k k^2 dk = \frac{B}{h^2}\cdot\frac{h^3}{3} = \frac{Bh}{3}$.

To prove the alternate formulation, note that $B = \pi r^2$. ■

Definition 5.10 (Right and Oblique Cones) A cone is right if the line joining the center of it bases with the apex is perpendicular to the base, and oblique otherwise.

Unless otherwise specified, cones are right. This is generally true in competitions as well. Regardless, the volume formula *always* holds.



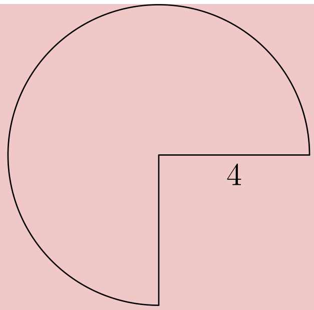
A right cone. *An oblique cone.*

Theorem 5.11 (Surface Area of a Right Cone) The surface area of a cone with radius r and height h is $\pi r^2 + 2\pi r\sqrt{r^2+h^2}$.

Proof: The area of the base is πr^2, and the lateral surface area can be found by integrating the slant height over the circumference. The lateral surface area is $\int\limits_0^{2\pi r}\sqrt{r^2+h^2}dx = 2\pi r\sqrt{r^2+h^2}$. (Note that the slant height is $\sqrt{r^2+h^2}$ by the Pythagorean Theorem.) Adding gives $\pi r^2 + 2\pi r\sqrt{r^2+h^2}$. ■

The surface area of an oblique cone is surprisingly hard to find.

Example 5.2 A three-quarter sector of a circle of radius 4 inches along with its interior is the 2-D net that forms the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



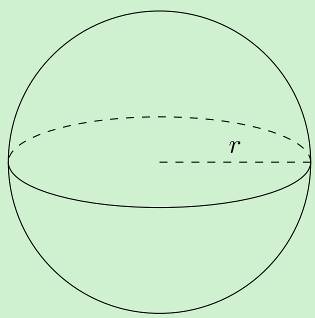
Solution: Note that the radius of the circle will become the slant height of the cone, and the circumference of the sector will become the circumference of the base of the cone.

This implies that the slant height is 4 and that the circumference of the base of the cone is 6π, or that the radius of the circle is 3. Thus the height of the cone is $\sqrt{7}$, and the volume of the cone is $\frac{3\sqrt{7}}{3} = \sqrt{7}$. ■

5.4 Spheres

Spheres are the 3 dimensional version of circles.

Definition 5.11 (Sphere) A sphere is the locus of points in space equidistant from a certain point. This point is called the center.



Theorem 5.12 (Volume of a Sphere) The volume of a sphere with radius r is $\frac{4\pi r^3}{3}$.

Proof: Instead we prove the volume of a hemisphere is $\frac{2\pi r^3}{3}$. Let the reference plane be the base of the hemisphere.

Then let the cross-section have a distance of k from the base. Then by the Pythagorean Theorem, the radius of the cross-section is $\sqrt{r^2 - k^2}$, so the area of the cross-section is $\pi(r^2 - k^2)$. Thus the volume is

$$\int_0^r \pi(r^2 - k^2)dk = \pi r^3 - \pi \int_0^r k^2 dk = \pi r^3 - \frac{\pi r^3}{3} = \frac{2\pi r^3}{3}.$$

Multiplying by 2 implies that the volume of the sphere is $\frac{4\pi r^3}{3}$. ■

Theorem 5.13 (Surface Area of a Sphere) The surface area of a sphere with radius r is $4\pi r^2$.

To prove this, we first make an observation of what the surface area of the sphere is.

Fact 5.7 (Surface Area of a Sphere) The surface area of a sphere is the integral of the circumferences of all cross-sections made with planes parallel to a certain reference plane.

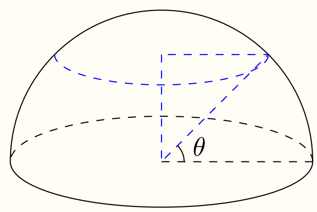
Now we can explicitly integrate.

Proof: We instead prove that the surface area of a hemisphere, not counting the base, is $2\pi r^2$.

We integrate about the arc of the circumference. Let θ be the angle a point on the cross-section forms with the radius containing the foot from the point onto the base. Then we integrate about $t = r\theta$. Note integrating the circumferences gives

$$\int_0^{\frac{\pi r}{2}} 2\pi r \cos \frac{t}{r} dt = 2\pi r^2.$$

Multiplying by 2 implies that the surface area of the sphere is $4\pi r^2$.



■

5.5 Cross-sections

Definition 5.12 (Cross-section) A cross-section of a solid is the intersection of the solid with a plane.

The word "solid" implies that the interior of the object is part of it.

Theorem 5.14 (Cross-section of a Sphere) The cross-section of a sphere is a circle.

You can also take cross-sections of a cube and a cone. The former is used sometimes in math competitions, and the latter produces a shape known as a *conic*.

Theorem 5.15 (Cross-section of a Cube) The cross-section of a cube can be a triangle, quadrilateral, pentagon, or hexagon.

The heuristical reason this is true is because a cube has 6 faces, and the plane can intersect the cube at any 6 of those faces. It's not too hard to construct any polygon with less than 7 sides.

Sometimes the entire problem is reduced significantly or just solved by taking the correct cross section.

Example 5.3 Inside a cone of radius 5 and height 12 there is a sphere inscribed. What is its radius?

Solution: Here is a walkthrough of the solution.

1. Take a cross section through the apex of the cone perpendicular to the base.
2. Now you have a triangle and its incircle. Finish with $[ABC] = rs$.

■

Example 5.4 (AMC 10A 2019/21) A sphere with center O has radius 6. A triangle with sides of length 15, 15, and 24 is situated in space so that each of its sides are tangent to the sphere. What is the distance between O and the plane determined by the triangle?

Solution: Here is a walkthrough of the solution.

1. This is not actually a 3D geometry problem.
2. Take a cross section of the sphere with the triangle.
3. Use $[ABC] = rs$ to figure out the radius of the cross-section.
4. Finish with the Pythagorean Theorem.

■

5.6 Miscellaneous Configurations

Here are some examples of miscellaneous techniques that can help solve 3D geometry problems.

5.6.1 Pythagorean Theorem

The Pythagorean Theorem still holds in 3 dimensions, and can be generalized to a 3 dimensional version by applying the two dimensional version twice.

Example 5.5 Consider unit cube $ABCDEFGH$, where $ABCD$ and $EFGH$ are opposite faces and AG, BH, CE, DF are space diagonals. Find the area of triangle AFH.

Solution: Note that $AF = FH = HA = \sqrt{2}$ by the Pythagorean Theorem, so the area is $\frac{(\sqrt{2})^2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. ■

5.6.2 Tangent Spheres

For problems with tangent spheres, remember the following fact. The essential idea is **just considering the centers of the spheres**.

Fact 5.8 (Tangency Point is Collinear with Centers) If two spheres with centers O_1, O_2 are tangent at T, then O_1, O_2, T are collinear.

This implies the following corollary.

Fact 5.9 (Distance Between Centers) Say two spheres Γ_1, Γ_2 with centers O_1, O_2 and radii r_1, r_2 are tangent at T. Then

$$\begin{cases} O_1O_2 = r_1 + r_2 \text{ if } \Gamma_2 \text{ is externally tangent to } \Gamma_1 \\ O_1O_2 = r_1 - r_2 \text{ if } \Gamma_2 \text{ is internally tangent to } \Gamma_1. \end{cases}$$

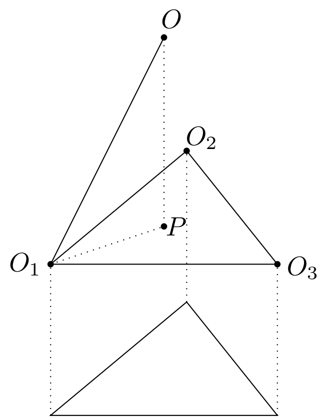
Using this in conjunction with the Pythagorean Theorem is enough to solve almost all problems with tangent spheres.

Example 5.6 (AMC 12A 2004/22) Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

Solution: Let's label some points. Let the centers of the unit spheres be O_1, O_2, O_3, let the center of the sphere of radius 2 be O, and let the foot of the perpendicular from O to $O_1O_2O_3$ be P. Note that $O_1O_2O_3$ is parallel to the horizontal plane with a distance of 1.

Note that $\triangle O_1O_2O_3$ is equilateral with side length 2, and by symmetry, P must be the center. Thus $PO_1 = \frac{2\sqrt{3}}{3}$. Since $OO_1 = 3$ by Fact 2, $OP = \sqrt{OO_1^2 - PO_1^2} = \sqrt{3^2 - (\frac{2\sqrt{3}}{3})^2} = \frac{\sqrt{69}}{3}$.

Since the distance from P to the horizontal plane is 1 and the tip of the sphere with radius 2 is 2 above O, the answer is $1 + 2 + \frac{\sqrt{69}}{3} = 3 + \frac{\sqrt{69}}{3}$.



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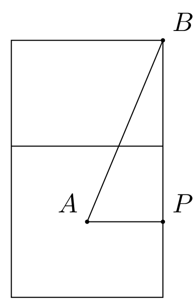
Notice that the entire solution was essentially just using the correct setup and the Pythagorean Theorem.

5.6.3 Unfolding

These are problems where you take the shortest path from one point to another on the surface of a solid.

Example 5.7 Consider a $10 \times 10 \times 7$ rectangular prism with A as the center of a 10×10 squares and B as a vertex of the opposite 10×10 square. If an ant crawls along the surface of the prism from A to B, what is the length of the shortest path he could take?

Solution: Unfold the square and lay it flat with the rectangle containing B. Since the shortest distance between two points is a line, $AB = \sqrt{AP^2 + BP^2} = \sqrt{(\frac{10}{2})^2 + (\frac{10}{2} + 7)^2} = 13$, where P is the foot of the altitude from A to the side of the square whose extension passes through V.



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5.7 Summary

5.7.1 Theory

1. Prisms and Cylinders

   - The volume is Bh.
   - The surface area of a cylinder is $2\pi r^2 + 2\pi rh$.
   - The volume of a cube is x^3.
   - The surface area of a cube is $6x^2$.

- ✦ The volume of a right rectangular prism is lwh.
- ✦ The surface area of a right rectangular prism is $2(lw + wh + hl)$.

2. Pyramids and Cones
   - ✦ The volume is $\frac{Bh}{3}$.
   - ✦ The surface area of a cylinder is

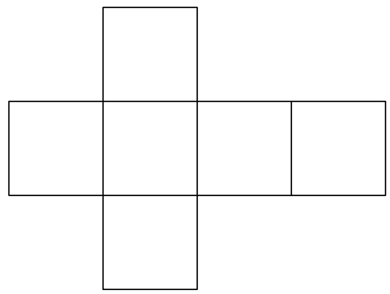
5.7.2 Tips and Tricks

1. Classifications
   - ✦ Think of prisms and cylinders as one group of solids, pyramids and cones as another, and spheres as their own thing.
   - ✦ The design of this chapter explicitly reflects this.
2. Techniques
   - ✦ Think in cross-sections whenever possible.
   - ✦ The Pythagorean Theorem still holds.
   - ✦ Tangent spheres can be reduced to points.
   - ✦ Problems where you find a path along a surface can be solved via unfolding.

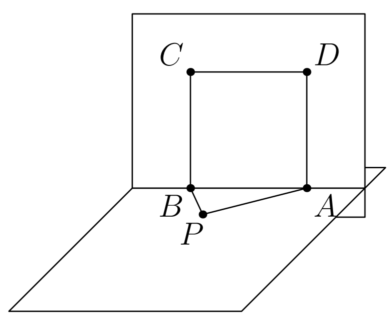
5.8 Exercises

5.8.1 Check-ins

1. Consider a rectangular prism whose base has an area of 40 and a height of 17. What is its volume?
2. Consider a cylinder with diameter 10 and height 7. What is its volume?
3. The net of a 3D figure is composed of 6 congruent squares and has a total area of 216 square inches. When the shape is folded to form a cube, how cubic inches are in its volume?



4. The numerical surface area and volume of a sphere are the same. What is the radius of this sphere?
5. Consider two right cylinders P and Q with the same volume. Cylinder P has a radius 30% longer than Cylinder Q. What percent larger is the height of Cylinder Q than that of Cylinder P?
6. A right rectangular prism with a volume of 32000 and a base width of 8 and a base length of 10. When the prism is cut by a plane parallel and equidistant to both bases, what is the combined surface area of the two remaining figures?
7. (AHSME 1996/9) Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3, PB = 4$ and $AB = 5$, what is PD?



8. (AMC 10A 2009/11) One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?
9. (AMC 12A 2008/8) What is the volume of a cube whose surface area is twice that of a cube with volume 1?

5.8.2 Problems

1. Consider rectangular prism $ABCDEFGH$ with dimensions $1 \times 1 \times \sqrt{3}$. Let AE, BF, CG, and DH be perpendicular to planes $ABCD$ and $EFGH$, and let $AE = BF = CG = DH = 1$. Furthermore, let $AB = 1$ and $BC = \sqrt{3}$. Find $\angle AGC$.

2. (AMC 12B 2008/18) On a sphere with a radius of 2 units, the points A and B are 2 units away from each other. Compute the distance from the center of the sphere to the line segment AB.

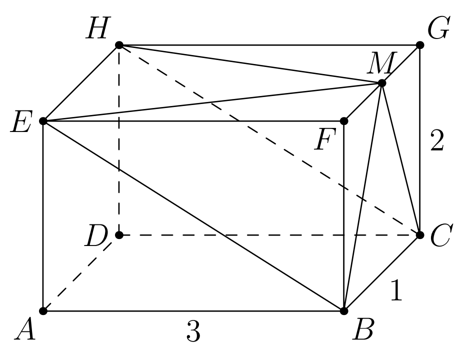
3. A parallelelepiped has coordinates $(0,0,0),(2,0,0),(1,\sqrt{3},0),(3,\sqrt{3},0)$ for one face and coordinates $(1,0,2),(3,0,2),(2,\sqrt{3},2),(4,\sqrt{3},2)$ for the opposite face. Find its surface area.

4. (AMC 12A 2005/22) A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?

5. (AMC 12B 2005/16) Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres.

6. Consider two externally spheres Γ_1 and Γ_2 with radii $12, r$, and consider cylinder Ω with radius 16 and height 25. If Γ_1 is tangent to a base and the circumference of Ω and Γ_2 is tangent to the opposite base and the circumference of Ω, find r.

7. (AMC 10B 2018/10) In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$, and $CG = 2$. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



8. (AIME I 2020/6) A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

9. (AIME 1984/9) In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

10. (AHSME 1996/28) On a $4 \times 4 \times 3$ rectangular parallelepiped, vertices A, B, and C are adjacent to vertex D. Find the distance from D to plane ABC.

5.8.3 Challenges

1. (AIME I 2011/12) A sphere is inscribed in the tetrahedron whose vertices are $A = (6,0,0), B = (0,4,0), C = (0,0,2)$, and $D = (0,0,0)$. The radius of the sphere is m/n, where m and n are relatively prime positive integers. Find $m+n$.

2. (AMC 10A 2013/22) Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

3. (AMC 12B 2004/19) A truncated cone has horizontal bases with radii 18 and 2. A sphere is tangent to the top, bottom, and lateral surface of the truncated cone. What is the radius of the sphere?

4. (AIME II 2020/7) Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies inside both cones. The maximum possible value for r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. (AIME I 2013/7) A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $m+n$.

6. (AIME II 2016/14) Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

Chapter 6

Hints

If anyone tells me it's a mistake to have hope, well then, I'll just tell them they're wrong. And I'll keep telling them 'til they believe! No matter how many times it takes.

Puella Magi Madoka Magica

1. R seems somewhat pesky. Can you find other stuff R is involved with?
2. Note $\angle APB = 180^\circ - \angle BAP - \angle ABP$.
3. Show that $GP \cdot GN = GQ \cdot GM$, where G is the centroid of $\triangle ABC$.
4. Look at $\triangle GBC$.
5. Draw in the center of the semicircle.
6. Reflect Y about XB to get Y'.
7. This implies $\frac{AB}{BD \sin \angle ADB} = \frac{AC}{DC \sin \angle CAD}$, or $\frac{AB}{CA} = \frac{BD \sin \angle ADB}{DC \sin \angle CAD}$.
8. Draw a line through A parallel to BC.
9. The Ratio Lemma will help you explicitly solve it.
10. $BCDP$ is a parallelogram.
11. Look at $\angle AEC$.
12. You can get BE and BF (via Stewart's), so you can get BG and CG.
13. Use power of a point to relate PA and PB.
14. How can you find the proportions of the lengths with the knowledge that $OX = OM$?
15. What is the foot of the perpendicular from E to PQ?
16. Look for similar triangles.
17. Remember that the centroid splits the median in a fixed ratio.
18. Two Tangent Theorem.
19. Add stuff so that the angle bisector of $\angle APB$ the diagonal of a square as well.
20. Equal sides mean equal arclengths.
21. Prove that $ABRQ$ is cyclic.
22. How can we relate lengths with angles?
23. Prove $\triangle ABC \sim \triangle EDC \sim \triangle EBA$.

24. Two Tangent Theorem.
25. F is a *specific* point.
26. Remember that QX and DB are parallel. How does this help you find QD?
27. What does $XP = MQ$ really mean?
28. Find $\frac{[AT_BT_C]}{[ABC]}$.
29. There are three more cyclic quadrilaterals.
30. Look at $\triangle BIC$.
31. Find the area of $\triangle XYY'$ in two ways.
32. How would you find BX and DY?
33. Draw the circumcircle of $\triangle ABC$.
34. We can use the Law of Sines to relate lengths with angles.
35. Find $\frac{\sin CAD}{\sin DAB}$.
36. Note that $BO = AO$.
37. Note $x = \frac{[ABC]}{2a}$.
38. Where do BC and DA meet?
39. Look for parallel lines.
40. Look for cyclic quadrilaterals.
41. Look for similar triangles.
42. Draw AE.
43. Look for collinear points.
44. We know the height. What else do we need?
45. Drop an altitude from B to CA.
46. Show that $BP = BR$.
47. Reflect C about AB.
48. Use Tangent/Secant to set up a system of equations.
49. $\triangle AOP$ is isosceles.
50. Have you found $\frac{b^2+c^2}{a^2}$ yet?
51. What is $\angle BCD$?
52. Pick a point. Draw all the diagonals connected to that point.
53. Use the Power of a Point and Ceva's to relate lengths.
54. Reflect P about the midpoint of AB.
55. Use the tangent angle condition to angle chase.
56. Let (BFD) intersect (CDE) at P.
57. What does $\triangle AMN \sim \triangle DMN \sim \triangle ABC$ tell you?
58. What information do cyclic quadrilaterals give you?
59. There is a cyclic quadrilateral with O_2 on it.
60. Look for similar and congruent triangles.

61. Use Power of a Point.
62. Multiply $\frac{AB}{CA} = \frac{BD \sin \angle ADB}{DC \sin \angle CAD}$ with symmetric expressions and finish with Ceva.
63. Let MN intersect AB at O.
64. How can you express $\frac{\angle DOE}{2}$ and $\frac{\angle AOB}{2}$?
65. By the Law of Sines, $\frac{AB}{\sin \angle ADB} = \frac{BD}{\sin DAB}$ and $\frac{AC}{\sin \angle ADC} = \frac{DC}{\sin \angle CAD}$.
66. Reduce the problem to a bunch of triangles.

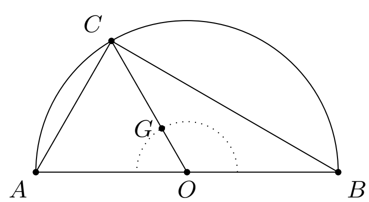
Chapter 7

Solutions

Regret your helplessness, and feel despair.

Psycho-Pass

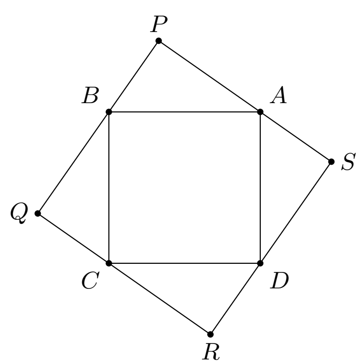
1. Let O be the center of the circle and G be the centroid of $\triangle ABC$. Since O is also the midpoint of AB and thus lies on CG, we're motivated to make use of the $2:1$ ratio. CO always has length 12, so it follows that GO always has length 4. This means that the locus of G is a circle with center O and radius 4 by definition, so the area is $16\pi \approx 50$.



2. Let the centroid be G. We want to prove that $XP \cdot XN = XQ \cdot XM$, which is equivalent to proving $XP \cdot XC = XQ \cdot XB$, as $\frac{XC}{XN} = \frac{XB}{XM} = 2$. Now note that $BPQC$ is cyclic as $\angle BPC = \angle BQC = 90^\circ$, which finishes the problem.

3. Note that $\angle MBO = 30^\circ = \angle NAO$, $\angle ANO = 180^\circ - \angle AMO = \angle BMO$, and $AO = BO$, so $\triangle BMO \cong \triangle ANO$. Thus $AN = BM$.

4. Let Q, R, S be the rotations of P about O by $90^\circ, 180^\circ, 270^\circ$ counterclockwise. Note that PR is the angle bisector of $\angle APB$ and PR bisects the area of $[PQRS]$. Since the area we added to both halves of $ABCD$ is the same, PR also bisects $ABCD$.



5. Note that
$$\angle BPR = \angle BAP + \angle ABP = \angle AQP + \angle PBQ = \angle AQB$$
and that
$$\angle BRP = \angle RPC + \angle RCP = 180^\circ - \angle APC + \angle BCP = \angle AQP + \angle BQP = \angle AQB,$$
so $\angle BPR = \angle BRP$.
Now note
$$\angle AQB = \angle BAP + \angle ABP = 180^\circ - \angle APB = \angle BPR = \angle BRP,$$
so $ABRQ$ is cyclic.
Now reflect P about the midpoint of AB to get P'. Then note
$$\angle PQR = \angle P'QR = \angle P'AR = \angle P'AB + \angle BAR = \angle ABR + \angle BAP = \angle BPR,$$
so BP is tangent to (PQR), as desired.

6. Without loss of generality, let P be closer to ℓ than Q. Note
$$\angle APC = 180 - \angle PAB - \angle BCP = \angle DCP - \angle PAB$$
$$\angle BQD = \angle BQP + \angle DQP.$$
Since $\angle PAB = \angle BDP$, the sum is $\angle DCP + \angle DQP = 180$.

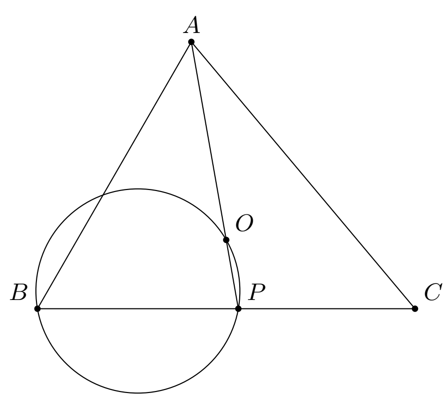
7. Let the circle with diameter AB be ω. Note ω intersects DE at D, F and AC at H where H is the foot of the altitude from B to AC.
Now note $EF \cdot ED = EH \cdot EA$, or $EF \cdot \frac{36}{5} = 3 \cdot \frac{48}{5}$. So $EF = 4$ and $DF = \frac{16}{5}$, so the answer is 21.

8. Note that $ZB \cdot FB = BX \cdot BD$, or $\frac{ZB}{BX} = \frac{FB}{BD}$. Cyclically multiplying finishes.

9. The key observation is that AD, BC, EF concur.
Let AD and BC intersect at P and let Q be the foot of the altitude from P to AB. Also let the semicircle have center O. Now note
$$\triangle PAQ \sim \triangle OAD$$
$$\triangle PBQ \sim \triangle OBC$$
so $\frac{AQ}{QB} \cdot \frac{BC}{CP} \cdot \frac{PD}{DA} = 1$. Since AC, BD, PQ concur, Q is actually F, and AC, BD, PF concur.
Now note
$$\angle OCP = \angle ODP = \angle OFP = 90^\circ,$$
so $OFCPD$ is cyclic. Thus
$$\angle COP = \angle DOP$$
$$\angle CFP = \angle DFP.$$

10. Note that $\angle BEC = \angle BFC = 90^\circ$.

11. Note that $BO = AO = AP$, so $\triangle AOP$ is isosceles. Thus $\angle OAP = 20^\circ$, implying $\angle AOC = 140^\circ$, and $\angle AOP = \angle APO = 80^\circ$, implying that $\angle AOB = 100^\circ$. So $\angle OBC = 360^\circ - 140^\circ - 100^\circ = 120^\circ$, or $\angle OBC = 30^\circ$.



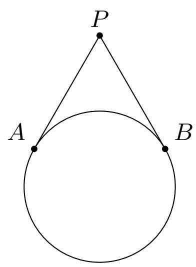
12. Say the center of K_1 is O_1 and the center of K_2 is O_2. Obviously AO_2CQ is cyclic since $\angle QAO_2 = \angle QCO_2 = 90°$. Now note $\angle QCP = 180 - \angle BAC$, and $\angle PAB = \angle ABP = \angle ABC = \angle QAC$, so P also lies on this circle. Thus $\angle O_2PQ = 90°$. Note

$$\angle ABO_2 = \frac{180° - \angle AO_2B}{2} = 90° - \angle ACB = 90° - \angle ACP = 90° - \angle AQP = 90° - \angle DQP,$$

and $\angle O_2DA = \angle O_2BA$ if and only if O_2 lies on (ABD), or K_1. Then $\angle O_2DA = \angle O_2BA$ implies that $\angle DPQ = 90°$, as desired.

13. Note that $BD = 6$ and $BE = \frac{5}{5+13} \cdot 12 = \frac{10}{3}$, so $DE = 6 - \frac{10}{3} = \frac{8}{3}$. Thus $[ADE] = \frac{1}{2} \cdot 5 \cdot \frac{8}{3} = \frac{20}{3}$.

14. Note that by Power of a Point, $PA^2 = PB^2$.



15. By Stewart's, $BE = \frac{\sqrt{2c^2+2a^2-b^2}}{2}$ and $CF = \frac{\sqrt{2a^2+2b^2-c^2}}{2}$, so $a^2 = BG^2 + CF^2 = \frac{4}{9}(\frac{2c^2+2a^2-b^2}{4} + \frac{2a^2+2b^2-c^2}{4}) = \frac{4a^2+b^2+c^2}{9}$.

Thus, $5a^2 = b^2 + c^2$ and $\frac{b^2+c^2}{a^2} = 5$.

16. Let O be the center of the circle. Notice that this implies that $OM = OX$. We claim that if $BM = x$, then $XT = x$ as well.

By the Pythagorean Theorem, $OM = \sqrt{x^2+1}$. Since $OM = OX$, $AX = \sqrt{x^2+1} - 1$. Then by Power of a Point, $XT = \sqrt{XA \cdot XB} = (\sqrt{x^2+1} - 1)(\sqrt{x^2+1} + 1) = x$, as desired.

Also, by the Pythagorean Theorem, $BX = \sqrt{5}x$.

We have is a semicircle with a known radius inscribed within a right triangle. Knowing the proportions of the triangle motivates reflecting about BX to use $[ABC] = rs$.

Let the reflection of Y about BX be Y' Then notice $[YXY'] = 2\sqrt{5}x^2$, by $\frac{bh}{2}$. But also notice by $[ABC] = rs$, $[YXY'] = 5x$. Since the area of a triangle is the same no matter how it is computed, $2\sqrt{5}x^2 = 5x$, implying $x = \frac{\sqrt{5}}{2}$.

Drop an altitude from T to BX, and let the foot be T'. Notice that $\triangle YBX \sim \triangle TT'X$ with a ratio of $3:1$. Thus $TT' = \frac{\sqrt{5}}{3}$ and $TX = \frac{5}{6}$. Then notice $T'A = T'X - AX$. Since $BX = \frac{5}{2}$ and $BA = 2$, $AX = \frac{1}{2}$. Thus $T'A = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$. By the Pythagorean Theorem, $TA = \sqrt{(\frac{1}{3})^2 + (\frac{\sqrt{5}}{3})^2} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$, which is our answer.

17. The central claim is that $BCDP$ is a parallelogram. Note that $ED = CB$ implies $EB \parallel DC$, and similarly, $AB = CD$ implies $AD \parallel BC$.

Note note $\angle AKE = 180° = 2\angle APE = 360° - 2\angle BPD = 360° - 2\angle BCD$. Also note that $2\angle BCD - \angle ACE = 180°$, so $\angle ACE = 2\angle BCD - 180°$.

Thus $\angle AKE + \angle ACE = 180°$, as desired.

18. Note that $\triangle ABX$ and $\triangle ADY$ are isosceles, so $BX = 7$ and $DY = 6$. Now also note that $XQDB$ is a parallelogram, so $QD = BX = 7$. Now note $\angle QDY = \angle DAB$, so it suffices to find $\cos \angle QDY$.

Now note that $QY = DY = 6$ and $QD = 7$. Thus dropping the altitude from Y to QD gives us $\cos \angle QDY = \frac{\frac{7}{2}}{6} = \frac{7}{12}$.

19. Notice $\angle EAB = \angle ACM = \angle ANM = \angle BAM$ and $\angle EBA = \angle ABM$, so $\triangle EAB \cong \triangle MAB$, implying that AB is the perpendicular bisector of EM. So $\angle EMP = \angle EMQ = 90°$, and it suffices to show that $PM = MQ$.

Let MN intersect AB at O. Note that $AO = BO$, so $PM = MQ$ by similar triangles.

20. Angle chase to find $\triangle ABC \sim \triangle EDC \sim \triangle EBA$. So $BE = 7 \cdot \frac{7}{10} = \frac{49}{10}$, implying $CE = 10 - \frac{49}{10} = \frac{51}{10}$, and $CD = \frac{10}{8} \cdot \frac{51}{10} = \frac{51}{8}$, implying $AD = 8 - \frac{51}{8} = \frac{13}{8}$.