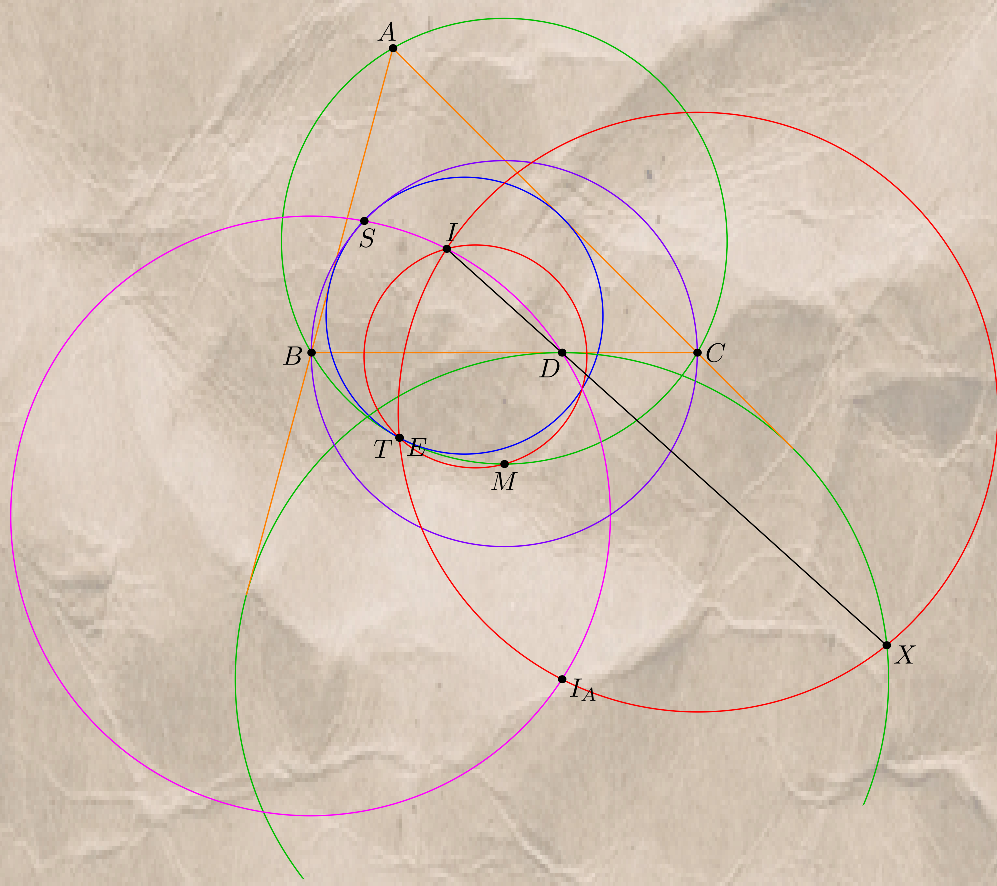
PREVIEW

Dennis Chen's

Exploring Euclidean Geometry



Dennis Chen

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Exploring Euclidean Geometry V2

Dennis Chen

June 1, 2020

Introduction

This is a preview of Exploring Euclidean Geometry V2. It contains the first four chapters, which constitute the entirety of the first part.

This should be a good introduction for those training for computational geometry questions. This book may be somewhat rough on beginners, so I do recommend using some slower-paced books as a supplement, but I believe the explanations should be concise and clear enough to understand. In particular, a lot of other texts have unnecessarily long proofs for basic theorems, while this book will try to prove it as clearly and succintly as possible.

There aren't a ton of worked examples in this section, but the check-ins should suffice since they're just direct applications of the material.

Contents

Part A

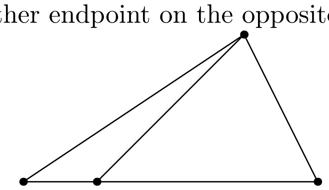
The Basics

Chapter 1

Triangle Centers

We define the primary four triangle centers, their corresponding lines, and define a cevian.

Definition 1. In a triangle, a cevian is a line segment with a vertex of the triangle as an endpoint and its other endpoint on the opposite side.



1.1 Incenter

The corresponding cevian is the *interior* angle bisector.

Definition 2. The interior angle bisector of $\angle CAB$ is the line that bisects $\angle CAB$.

The interior angle bisector of $\angle CAB$ is also the locus of points equidistant from lines AB and AC.

Theorem 1.1.1: Angle Bisector Equidistant from Both Sides

In $\angle CAB$, $\angle PAB = \angle PAC$ if and only if $\delta(P, AB) = \delta(P, AC)$.

Proof

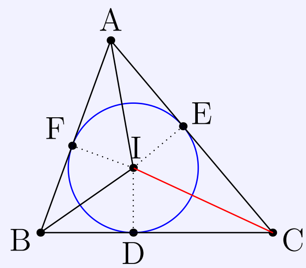
Let the feet of the altitudes from P to AB, AC be X, Y. Then note that either of these conditions imply $\triangle APX \cong \triangle APY$, which in turn implies the other condition.

Theorem 1.1.2: Incenter

There is a point I that the angle bisectors of $\triangle ABC$ concur at. Furthermore, I is equidistant from sides AB, BC, CA.

Proof

Recall that a point is on the angle bisector of $\angle CAB$ if and only if $\delta(P, AB) = \delta(P, AC)$. Let the angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at I. Then $\delta(P, CA) = \delta(P, AB)$ and $\delta(P, AB) = \delta(P, BC)$, so $\delta(P, BC) = \delta(P, CA)$, implying that I lies on the angle bisector of $\angle BCA$.
Since $\delta(P, AB) = \delta(P, BC) = \delta(P, CA)$, the circle with radius $\delta(P, AB)$ centered at I is inscribed in $\triangle ABC$.



1.2 Centroid

The corresponding cevian is the median.

Definition 3. The midpoint of segment AB is the unique point M that satisfies the following:

(a) M is on AB.

(b) $AM = MB$.

Definition 4. The A-median of $\triangle ABC$ is the line segment that joins A with the midpoint of BC.

Theorem 1.2.1: Centroid

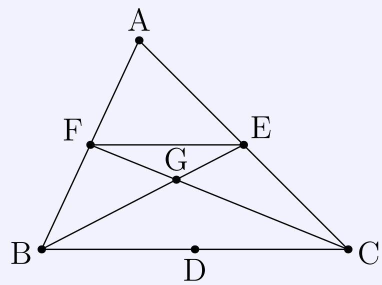
The medians AD, BE, CF of $\triangle ABC$ concur at a point G. Furthermore, the following two properties hold:

(a) $\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = 2$.

(b) $[BGD] = [CGD] = [CGE] = [AGE] = [AGF] = [BGF]$.

Proof

Let BE intersect CF at G. Since $\triangle AFE \sim \triangle ABC$, $FE \parallel BC$. Thus $\triangle BCG \sim \triangle EFG$ with a ratio of $\frac{BC}{EF} = 2$, so $\frac{BG}{GE} = 2$.
Similarly let BE intersect AD at G'. Repeating the above yields $\frac{BG'}{G'E} = 2$. Thus G and G' are the same point, and the medians are concurrent.



1.3 Circumcenter

A perpendicular bisector is not a cevian, but it is still a special line in triangles.

Definition 5. The perpendicular bisector of a line segment AB is the locus of points X such that $AX = BX$.

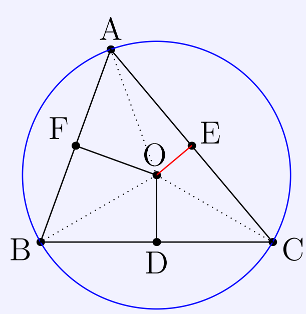
The circumcenter is the unique circle that contains points A, B, C.

Theorem 1.3.1: Circumcenter

There is a point O that the perpendicular bisectors of BC, CA, AB concur at. Furthermore, O is the center of (ABC).

Proof

Let the perpendicular bisectors of AB, BC intersect at O. By the definition of a perpendicular bisector, $AO = BO$ and $BO = CO$. But this implies $CO = AO$, so O lies on the perpendicular bisector of CA. Since $AO = BO = CO$, the circle centered at O with radius AO circumscribes $\triangle ABC$.



1.4 Orthocenter

The corresponding cevian is the altitude.

Definition 6. The A-altitude of $\triangle ABC$ is the line through A perpendicular to BC.

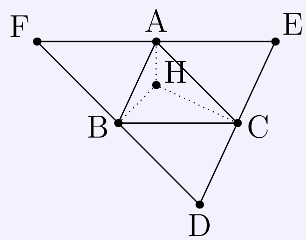
Definition 7. We call the *foot* from A to BC the point H where the A-altitude intersects BC.

Theorem 1.4.1: Orthocenter

The altitudes of $\triangle ABC$ concur.

Proof: Piggyback

We will be piggybacking on the proof for the circumcenter.
Let the line through B parallel to AC and the line through C parallel to AB intersect at D. Define E, F similarly. Note that $FA = BC = AE$, so the A altitude of $\triangle ABC$ is the perpendicular bisector of DE. Since the circumcenter exists, the orthocenter must too.



1.5 Exercises

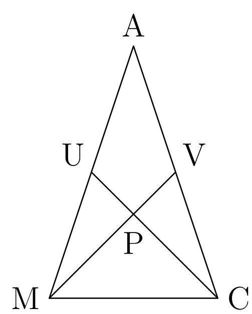
1.5.1 Check-ins

1. Prove that a triangle is equilateral if and only if its incenter is the same point as its circumcenter.

2. Consider $\triangle ABC$ with incenter I. Prove that $\angle BIC = 90^\circ + \frac{1}{2}\angle BAC$. **Hints**: 47

3. Consider $\triangle ABC$ with circumcenter O. If $AO = 20$ and $BC = 32$, find $[BOC]$.

4. (AMC 10A 2020/12) Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



1.5.2 Problems

1. Consider $\triangle ABC$ with medians BE, CF. If BE and CF are perpendicular, find $\frac{b^2+c^2}{a^2}$.**Hints**: 14 17
Solution: 4

2. (Brazil 2007) Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is 40°. Determine the measure of the angle $\angle OBC$.

1.5.3 Challenges

1. Three congruent circles $\omega_1, \omega_2, \omega_3$ concur at P. Let ω_1 intersect ω_2 at $A \neq P$, let ω_2 intersect ω_3 at $B \neq P$, and let ω_3 intersect ω_1 at $C \neq P$. What triangle center is P with respect to $\triangle ABC$?

2. Let ABC be an isosceles triangle with $AB = AC$. If ω is inscribed in ABC and the orthocenter of ABC lies on ω, find $\frac{AB}{BC}$.

3. Let G be the centroid of $\triangle ABC$. If $\angle BGC = 90^\circ$, find the maximum value $\sin A$ can take. **Hints**: 48

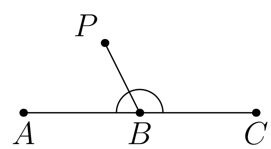
Chapter 2

Angle Chasing

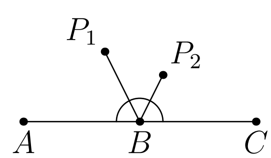
You can angle chase to show points are collinear or lines are concurrent, lines are parallel, a line is tangent to a circle, or four points are cyclic. In computational contests, you may be asked to find an angle for easier problems and angle chasing can reveal more about the configuration for harder problems.

2.1 Collinearity and Concurrency

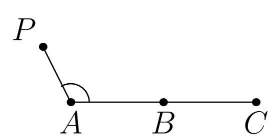
A line has measure 180°. This means A, B, C, are collinear if and only if for any point P, $\angle ABP + \angle PBC = 180^\circ$. This is one of the main ways to prove points are collinear.



This holds for more than one point too. For the right configuration, A, B, C are collinear if and only if for points $P_1, P_2, \ldots, P_n$, $\angle ABP_1 + \angle P_1BP_2 + \cdots + \angle P_nBC = 180^\circ$. (Directed angles can be used to avoid configuration issues.)



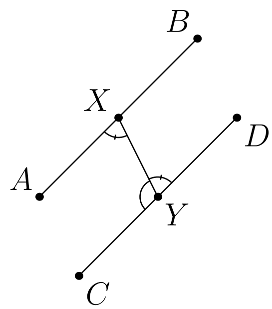
A similar condition is that A, B, C are collinear if and only if for any point P, $\angle PAB = \angle PAC$.



2.2 Parallel Lines

Consider parallel lines AB and CD. Then for X on segment AB and Y on segment CD,

$$\angle AXY = 180^\circ - \angle CXY = \angle DXY.$$



2.3 Angle Chasing in Circles

We begin with some definitions.

Definition 8. A chord is a line segment formed by two distinct points on a circle.

Definition 9. A secant is a line that intersects a circle twice.

Definition 10. A tangent is a line that intersects a circle once.

Definition 11. The measure of $\overset{\frown}{\mathrm{AB}}$ of circle with center O is the measure of $\angle AOB$. Unless specified, this means the minor arc, or the smaller arc.

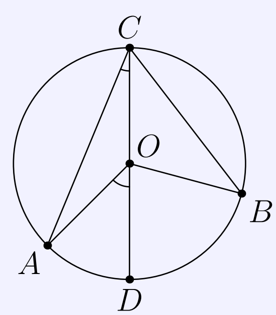
Now we present three important theorems.

Theorem 2.3.1: Inscribed Angle

Let A, B be points on a circle with center O.
If C is a point on minor arc AB, then $\angle ACB = \frac{\angle AOB}{2}$.
If C is a point on major arc AB, then $\angle ACB = 180^\circ - \frac{\angle AOB}{2}$.

Proof

Let D be the antipode of C. Then $\angle ACD = \frac{180^\circ - \angle AOC}{2} = \frac{\angle AOD}{2}$. Thus addition or subtraction, depending on whether O is inside acute angle $\angle ACB$, of $\angle ACD$ and $\angle BCD$ will yield the result.

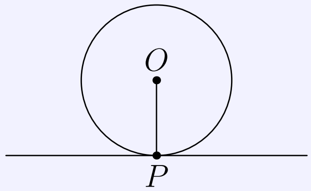


Theorem 2.3.2: Tangent Perpendicular to Radius

Consider circle ω with center O and point P on ω. If ℓ is the tangent to ω through P, then ℓ is perpendicular to OP.

Proof

This is identical to the claim that P is the point on ℓ with the smallest distance to O. We prove this is true by contradiction. Assume this is not true. Then there is some point X on ℓ such that $OX < OP$, implying that ℓ intersects ω twice, contradiction.

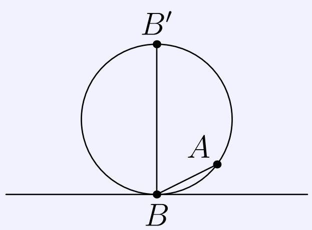


Theorem 2.3.3: Tangent Angle

Consider circle ω with center O and points A, B on ω. Let ℓ be the tangent to ω through B and let θ be the acute angle between AB and ℓ. Then $\theta = \frac{\angle AOB}{2}$.

Proof

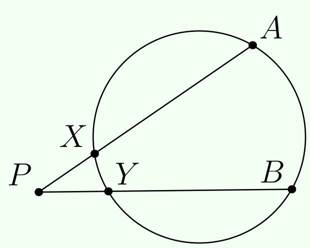
Let B' be the antipode of B. Then note that $\theta = 90^\circ - \angle ABB' = \frac{180^\circ - \angle AOB'}{2} = \frac{\angle AOB}{2}$.



A corollary of this theorem is that if C is some point on $\overset{\frown}{AB}$, then $\theta = \angle ACB$.
With the Inscribed Angle Theorem in mind, try to prove these two theorems.

Theorem 2.3.4: Angle of Secants/Tangents

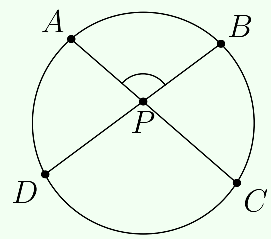
Let lines AX and BY intersect at P such that A, X, P and B, Y, P are collinear in that order. Then $\angle APB = \frac{\angle AOB - \angle XOY}{2}$.



Hints: 23 27 38

Theorem 2.3.5: Angle of Chords

Let chords AC, BD intersect at P. Then $\angle APB = \frac{\angle AOB + \angle COD}{2}$.



Hints: 18

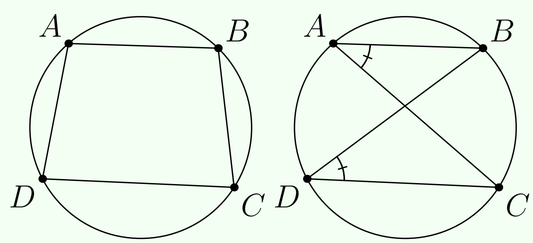
2.4 Cyclic Quadrilaterals

Here's a very important application of the Inscribed Angle Theorem.

Theorem 2.4.1: Cyclic Quadrilaterals

Any one of the three implies the other two:

1. Quadrilateral $ABCD$ is cyclic.
2. $\angle ABC + \angle ADC = 180°$.
3. $\angle BAC = \angle BDC$.



2.5 Summary

2.5.1 Theory

1. Supplementary Angles
   - $A, B, C,$ are collinear if and only if for any point P, $\angle ABP + \angle PBC = 180°$.
   - This is generalizable to more points.
   - A, B, C are collinear if and only if for any point P, $\angle PAB = \angle PAC$.
2. Parallel Lines
   - For parallel lines AB, CD and points X and Y on AB and CD respectively, $\angle AXY = 180° - \angle CXY = \angle DXY$.

3. Inscribed Angle Theorem
   - ✦ The measure of an angle is half the measure of the subtended arc.
   - ✦ Proved by considering the case where one leg of the angle is a diameter and angle chasing, and generalizing.
   - ✦ Thale's Theorem: In the special case where the feet of the angle form a diameter of the circle, the angle is $90°$. The converse also holds.
4. Tangent Perpendicular to Radius
   - ✦ This is important. Remember it.
5. Tangent Angle
   - ✦ When you see circles and an angle condition with a tangent, keep this in mind.
   - ✦ This proves points are concyclic.
6. Cyclic Quadrilaterals
   - ✦ Angles on opposite sides are supplementary.
   - ✦ Angles on the same side are congruent.

2.5.2 Tips and Strategies

1. Proving collinearity and concurrency for lines can basically be switched around at will.
2. One way to prove concurrency of three figures is to let two of them intersect at a point P, and prove the third passes through P.
3. If two lines are parallel, then it's probably an important part of the problem.

2.6 Exercises

2.6.1 Check-ins

1. Prove $\triangle ABC$ satisfies $\angle A + \angle B + \angle C = 180°$. **Hints**: 9

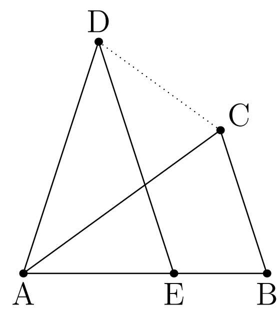
2. Prove that the sum of the interior angles of an $n-gon$ is $180(n-2)$. **Hints**: 16 1

3. (Brazil 2004) In the figure, ABC and DAE are isosceles triangles ($AB = AC = AD = DE$) and the angles BAC and ADE have measures $36°$.

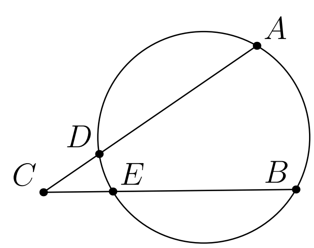
   (a) Using geometric properties, calculate the measure of angle $\angle EDC$.

   (b) Knowing that $BC = 2$, calculate the length of segment DC.

   (c) Calculate the length of segment AC.



4. If $\angle ABC = 60°$ and $\angle CAB = 70°$, find $\overset{\frown}{\mathrm{AB}} - \overset{\frown}{\mathrm{DE}}$.



5. (a) Given that A, B, C, and D are all on the same circle, that BE is the angle bisector of $\angle ABC$, that $\angle AEB = \angle CEB$, and that $\angle ADC = 50°$, find $\angle BAC$.

   (b) Given points A, B, C, D, E such that BE is the angle bisector of $\angle ABC$, $\angle AEB = \angle CEB$, $\angle BAC + \angle BDC = \angle ABD + \angle ACD$, and $\angle ADC = 48°$, find $\angle BCA$.

6. Consider any cyclic pentagon $ABCDE$. If P is the center of $(ABCDE)$, then prove that $ABCP$ is never cyclic.

2.6.2 Problems

1. Consider rectangle $ABCD$ with $AB = 6$, $BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM and BN intersect AC at X and Y respectively. Find XY.

2. (AMC 10A 2019/13) Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$? **Hints**: 40

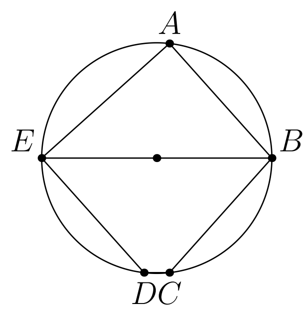
3. (Miquel's Theorem) Consider $\triangle ABC$ with D on BC, E on CA, and F on AB. Prove that (AEF), (BFD), and (CDE) concur. **Hints**: 35

4. Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle DCE$ intersect at $P \neq D$. If $\angle A = 50°, \angle B = 35°$, find $\angle DPE$.

5. Let circles ω_1 and ω_2 intersect at X, Y. Let line ℓ_1 passing through X intersect ω_1 at A and ω_2 at C, and let line ℓ_2 passing through Y intersect ω_1 at B and ω_2 at D. If ℓ_1 intersects ℓ_2 at P, prove that $\triangle PAB \sim \triangle PCD$. **Hints**: 37

6. (Simson's Theorem) Consider $\triangle ABC$ and point P, and let X, Y, Z be the feet of the altitudes from P to BC, CA, AB. Prove that X, Y, Z are collinear if and only if P is on (ABC). **Hints**: 33

7. (AMC 10B 2011/17) In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



8. (Formula of Unity 2018) A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$. **Solution**: 5

9. Consider square $ABCD$ and some point P outside $ABCD$ such that $\angle APB = 90°$. Prove that the angle bisector of $\angle APB$ also bisects the area of $ABCD$. **Hints**: 6 **Solution**: 1

10. (IMO 2006/1) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

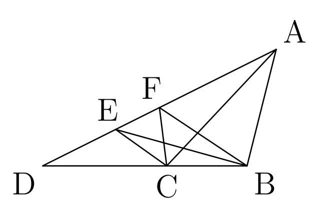
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

2.6.3 Challenges

1. (MAST Diagnostic 2020) Consider parallelogram $ABCD$ with $AB = 7$, $BC = 6$. Let the angle bisector of $\angle DAB$ intersect BC at X and CD at Y. Let the line through X parallel to BD intersect AD at Q. If $QY = 6$, find $\cos \angle DAB$. **Hints**: 32 8 **Solution**: 9

2. (Memorial Day Mock AMC 10 2018/21) In the following diagram, $m\angle BAC = m\angle BFC = 40°$, $m\angle ABF = 80°$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?



Hints: 46 15

3. (FARML 2012/6) In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD. **Hints**: 2 4 **Solution**: 7

4. (ISL 1994/G1) C and D are points on a semicircle. The tangent at C meets the extended diameter of the semicircle at B, and the tangent at D meets it at A, so that A and B are on opposite sides of the center. The lines AC and BD meet at E. F is the foot of the perpendicular from E to AB. Show that EF bisects angle CFD. **Hints**: 44 29 11 **Solution**: 8

5. Consider $\triangle ABC$ with D on line BC. Let the circumcenters of $\triangle ABD$ and $\triangle ACD$ be M, N, respectively. Let the circumcircle of $\triangle MND$ intersect the circumcircle of $\triangle ACD$ again at $H \neq D$. Prove that A, M, H are collinear. **Hints**: 41 43

6. (APMO 1999/3) Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR. **Hints**: 22 19 7 5 **Solution**: 10

7. Let K_1 and K_2 be circles that intersect at two points A and B. The tangents to K_1 at A and B intersect at a point P inside K_2, and the line BP intersects K_2 again at C. The tangents to K_2 at A and C intersect at a point Q, and the line QA intersects K_1 again at D.

   Prove that QP is perpendicular to PD if and only if the centre of K_2 lies on K_1. **Hints**: 28 12 **Solution**: 3

8. (IMO 2000/1) Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **Hints**: 26 13 **Solution**: 11

Chapter 3

Power of a Point

There's only one theorem, so this will be a short chapter. The only prerequisites are angle chasing theorems in circles.

3.1 Power of a Point

The Power of a Point theorem helps us length chase in circles. The proof is a result of similar triangles, and its uses are numerous in lower-level competitions. If you already know this theorem, feel free to skip this section.

Theorem 3.1.1: Power of a Point

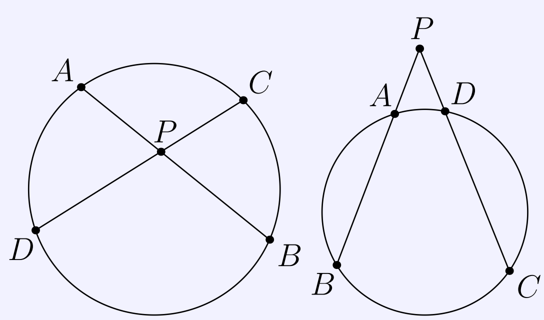
Let line ℓ_1 intersect circle ω at A, B and line ℓ_2 intersect ω at C, D. Then $PA \cdot PB = PC \cdot PD$.

Proof: Similar Triangles

There are two cases here: Either P is inside of ω or outside of ω.
If P is inside of ω, then note that by Inscribed Angle, $\angle PAC = \angle PDB$ and $\angle = \angle PAB$, so $\triangle PAC \sim \triangle PDB$.
If P is outside of ω, then without loss of generality, let $PA \leq PB$ and $PC \leq PD$. Then note $\angle PAC = 180^\circ - \angle CAB = \angle PDB$ and $\angle PCA = 180^\circ - \angle ACD = \angle PBD$, so $\triangle PAC \sim \triangle PDB$.
To finish, note that the similarity implies

$$\frac{PA}{PC} = \frac{PD}{PB}$$

$$PA \cdot PB = PC \cdot PB.$$



P inside ω and P outside ω.

In the case of tangency, $A = B$ is the point of tangency. (You can think of a tangent line intersecting a circle twice at the same point.)

3.2 Bisector Lemma

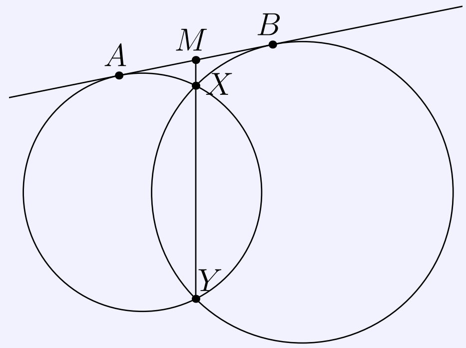
This is a very powerful fact that kills a lot of earlier computational geometry problems involving circles.

Theorem 3.2.1: Bisector Lemma

Let ω_1 and ω_2 intersect at X and Y, and let ℓ be a line tangent to ω_1 and ω_2. If ℓ intersects ω_1 at A and ω_2 at B, then XY bisects AB.

Proof: Power of a Point

Let XY intersect AB at P. Then by Power of a Point, $PX^2 = PA \cdot PB = PY^2$.



3.3 Summary

3.3.1 Theory

1. Power of a Point
   - If lines ℓ_1, ℓ_2 through P intersect circle ω at A, B and C, D, respectively, then $PA \cdot PB = PC \cdot PD$.
   - This is a consequence of similar triangles.
2. Bisector Lemma
   - The common chord of two circles bisects the common external tangent.

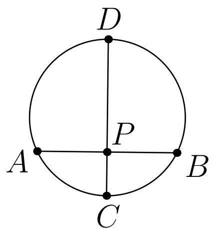
3.3.2 Tips and Strategies

1. If there are two circles and you're in doubt, use Bisector Lemma. (This even applies for some easier olympiad problems.)

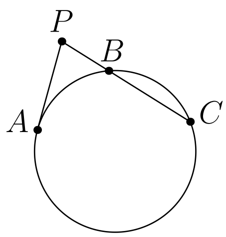
3.4 Exercises

3.4.1 Check-ins

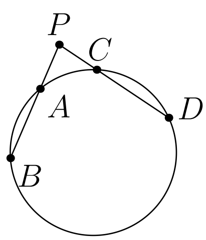
1. Let chords AB and CD in circle ω intersect at P. If $AP = BP = 4$ and $CP = 2$, find DP.



2. Let the tangent through A to circle ω intersect line ℓ that passes through B, C on ω at P. If $BP < CP$, $AP = 4$, and $BC = 6$, find BP.



3. Let line ℓ_1 that passes through A, B on circle ω intersect line ℓ_2 that passes through C, D on ω at P. If $PA = 6$, $AB = 12$, and $PC = 3$, find CD.

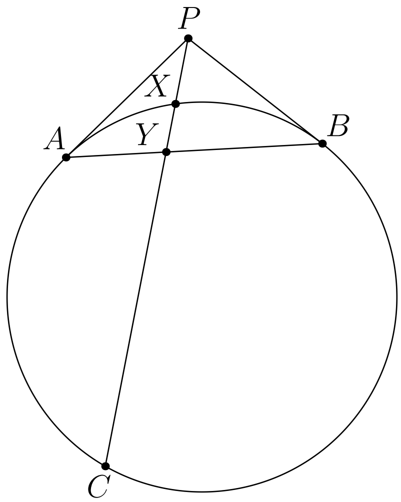


4. Prove that the lengths of the two tangents from a point to a circle are equal. **Hints**: 21 24

5. Let $\triangle ABC$ have a right angle at C and let P be the foot of the altitude from C to AB. If the foot of the altitude from P to AC is X and the foot from P to BC is Y, then prove that $AXYB$ is cyclic.

3.4.2 Problems

1. Let PA and PB be tangents to circle ω, and let line ℓ through P intersect ω at X and C and AB at Y. If $PA = 4$, $PC = 8$, $AY = 1$, and $XY = 1$, find the area of $\triangle PAB$.



2. Consider two externally tangent circles ω_1, ω_2. Let them have common external tangents AC, BD such that A, B are on ω_1 and C, D are on ω_2. Let AC intersect BD at P, and let the common internal tangent intersect AC and BD at X and Y. If $\frac{[PCD]}{[PAB]} = \frac{1}{25}$, find $\frac{[PCD]}{[PXY]}$.

3. (Mandelbrot 2012) Let A and B be points on the lines $y = 3$ and $y = 12$, respectively. There are two circles passing through A and B that are also tangent to the x axis, say at P and Q. Suppose that $PQ = 2012$. Find AB.

4. (Parody) Consider a coordinate plane with two circles tangent to the x axis at X, Y, respectively. If the circles intersect at P, Q, and $XY = 8$, is it possible for P to lie on $y = 3$ and Q to lie on $y = 12$?

5. (e-dchen Mock MATHCOUNTS) Consider chord AB of circle ω centered at O. Let P be a point on segment AB such that $AP = 2$ and $BP = 8$. If $\angle APO = 150°$, what is the area of ω?

☢ 3.5 Challenges

1. (Geometry Bee 2019) Circles O_1 and O_2 are constructed with O_1 having radius of 2, O_2 having radius of 4, and O_2 passing through the point O_1. Lines ℓ_1 and ℓ_2 are drawn so they are tangent to both O_1 and O_2. Let O_1 and O_2 intersect at points P and Q. Segment $\overline{EF}$ is drawn through P and Q such that E lies on ℓ_1 and F lies on ℓ_2. What is the length of $\overline{EF}$?

Chapter 4

Lengths and Areas in Triangles

4.1 Lengths

There are a couple of important lengths in a triangle. These are the lengths of cevians, the inradius/exradius, and the circumradius.

4.1.1 Law of Cosines and Stewart's

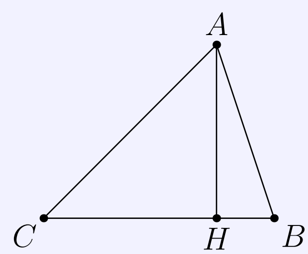
We discuss how to find the third side of a triangle given two sides and an included angle, and use this to find a general formula for the length of a cevian.

Theorem 4.1.1: Law of Cosines

Given $\triangle ABC$, $a^2 + b^2 - 2ab\cos C = c^2$.

Proof

Let the foot of the altitude from A to BC be H. Then note that $A = b\sin C$, $CH = b\cos C$, and $BH = |a - b\cos C|$. (The absolute value is because $\angle B$ can either be acute or obtuse.) Then note by the Pythagorean Theorem, $(b\sin C)^+2 + (a - b\cos C)^2 = a^2 + b^2 - 2ab\cos C = c^2$.



Theorem 4.1.2: Stewart's Theorem

Consider $\triangle ABC$ with cevian AD, and denote $BD = m$, $CD = n$, and $AD = d$. Then $man + dad = bmb + cnc$.

Proof

We use the Law of Cosines. Note that

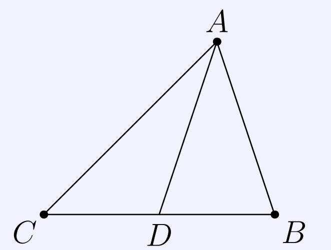
$$\cos\angle ADB = \frac{d^2+m^2-c^2}{2dm} = -\frac{d^2+n^2-b^2}{2dn} = -\cos\angle ADC.$$

Multiplying both sides by $2dmn$ yields

$$c^2n - d^2n - m^2n = -bm^2 + d^2m + mn^2$$

$$b^2m + c^2n = mn(m+n) + d^2(m+n)$$

$$bmb + cnc = man + dad.$$



Here are two corollaries that will save you a lot of time in computational contests.

Theorem 4.1.3: Length of Angle Bisector

In $\triangle ABC$ with angle bisector AD, denote $BD = x$ and $CD = y$. Then

$$AD = \sqrt{ab - xy}.$$

Theorem 4.1.4: Length of Median

In $\triangle ABC$ with median AD,

$$AD = \frac{\sqrt{2b^2+2c^2-a^2}}{2}.$$

4.1.2 Law of Sines and the Circumradius

The Law of Sines is a good way to length chase with a lot of angles.

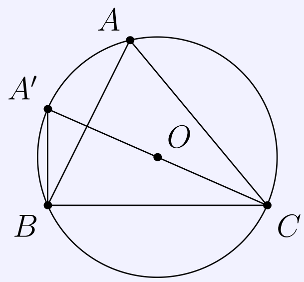
Theorem 4.1.5: Law of Sines

In $\triangle ABC$ with circumradius R,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof

We only need to prove that $\frac{a}{\sin A} = 2R$, and the rest will follow.
Let the line through B perpendicular to BC intersect (ABC) again at A'. Then note that $A'C = 2R$ by Thale's. By the Inscribed Angle Theorem, $\sin\angle CA'B = \sin A$, so $\frac{a}{\sin A} = \frac{a}{\sin\angle CA'B} = \frac{a}{\frac{a}{2R}} = 2R$.



Other texts will call this the Extended Law of Sines. But the Extended Law of Sines has a better proof than the "normal" Law of Sines, and redundancy is bad.

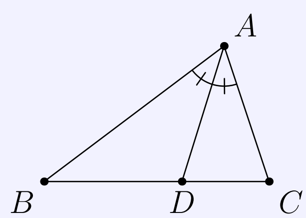
The Law of Sines gives us the Angle Bisector Theorem.

Theorem 4.1.6: Angle Bisector Theorem

Let D be the point on BC such that $\angle BAD = \angle DAC$. Then $\frac{AB}{BD} = \frac{AC}{CD}$.

Proof

By the Law of Sines, $\frac{\sin\angle ADB}{\sin\angle BAD} = \frac{AB}{BD}$ and $\frac{\sin\angle ADC}{\sin\angle CAD} = \frac{AC}{CD}$. But note that $\angle BAD = \angle ADC$ and $\angle BAD + \angle CAD = 180°$, so $\frac{AD}{BD} = \frac{AC}{CD}$.



In fact, the Angle Bisector Theorem can be generalized in what is known as the ratio lemma.

Theorem 4.1.7: Ratio Lemma

Consider $\triangle ABC$ with point P on BC. Then $\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}$.

The proof is pretty much identical to the proof for Angle Bisector Theorem.

Proof

By the Law of Sines, $BP = \frac{c\sin\angle BAP}{\sin\angle APB}$ and $CP = \frac{b\sin\angle CAP}{\sin\angle APC}$. Since $\sin\angle APB = \sin\angle APC$,

$$\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}.$$

Note that this remains true even if P is on the *extension* of BC.

4.1.3 The Incircle, Excircle, and Tangent Chasing

We provide formulas for the inradius, exradii, and take a look at some uses of the Two Tangent Theorem. Recall that the Two Tangent Theorem states that if the tangents from P to ω intersect ω at A, B, then $PA = PB$.

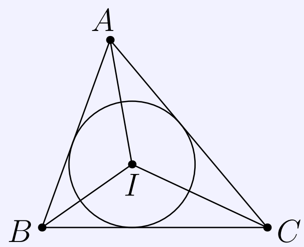
Theorem 4.1.8: rs

In $\triangle ABC$ with inradius r,

$$[ABC] = rs.$$

Proof

Note that $[ABC] = r \cdot \frac{a+b+c}{2} = rs$.



A useful result of the incircle is that the length of the tangents from A is $s - a$. Similar results hold for the B, C tangents to the incircle.

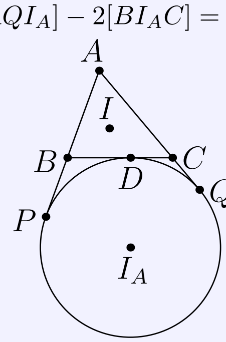
Theorem 4.1.9: $r_a(s-a)$

In $\triangle ABC$ with A exradius r_a,

$$[ABC] = r_a(s-a).$$

Proof

Let AB, AC be tangent to the A excircle at P, Q, respectively. Then note that by Two Tangent Theorem, $PB = BD$ and $DC = CQ$. Thus $[ABC] = [API_A] + [AQI_A] - 2[BI_AC] = r_a \cdot \frac{s+s-2a}{2} = r_a(s-a)$.



Also note that $AP = c + BD = b + CD = AQ$, so $BD = s - c$ and $CD = s - b$.

Keep these area and length conditions in mind when you see incircles and excircles.

4.1.4 Concurrency with Cevians

We discuss Ceva's Theorem, Menelaus Theorem, and mass points, three ways to look at concurrent cevians. Very rarely do problems involving concurrency with cevians appear on higher level contests, but they're fairly common in the AMC 8 and MATHCOUNTS. This is also a good tool to have for when you need it.

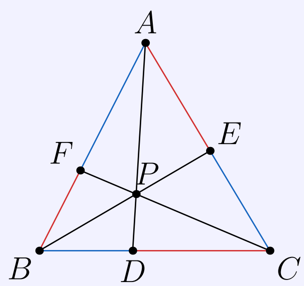
Theorem 4.1.10: Ceva's Theorem

In $\triangle ABC$ with cevians AD, BE, CF, they concur if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

Proof

Let the point of concurrency be P. Note that $\frac{[ABD]}{[ADC]} = \frac{[PBD]}{[PDC]} = \frac{BD}{DC}$, so $\frac{[BPA]}{[APC]} = \frac{BD}{DC}$. Thus,

$$\frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = \frac{[CPB]}{[BPA]} \cdot \frac{[APC]}{[CPB]} \cdot \frac{[BPA]}{[APC]} = 1.$$



A good way to remember what goes in the numerator and denominator is by looking at the colors and thinking about them alternating.

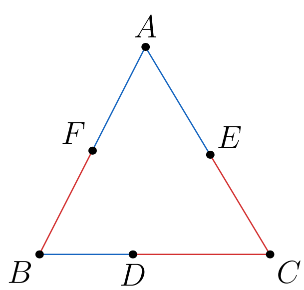
We present an example of what not to do.

Example 4.1.1: Order Mixed Up

Consider $\triangle ABC$ with D, E, F on BC, CA, AB respectively, such that $BD = 4$, $DC = 6$, $AE = 6$, $EC = 4$, and $AF = BF = 5$. Are AD, BE, and CF concurrent?

Solution: Bogus

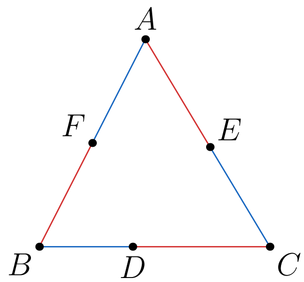
Yes. Note that $\frac{4}{6} \cdot \frac{6}{4} \cdot \frac{5}{5} = 1$.



This is not right, as the order of the lengths is messed up (intentionally) in the problem statement. (Also note the colors are messed up.) We now present the correct solution.

Solution: Correct

No. Note that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{4}{6} \cdot \frac{4}{6} \cdot \frac{5}{5} = \frac{4}{9}$, which is not 1.



Theorem 4.1.11: Menelaus

Consider $\triangle ABC$ with D, E, F on lines BC, CA, AB, respectively. Then D, E, F are collinear if $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = 1$.

This looks very similar to Ceva - in fact, the letters just switched. Instead of the line segments cycling through D, E, F, they now cycle through A, B, C.

Proof

Draw a line through A parallel to DE and let it intersect BC at P. Then note that $\triangle ABP \sim \triangle FBD$ and $\triangle ECD \sim \triangle ACP$, so

$$\frac{AF}{FB} = \frac{PD}{DB}$$

$$\frac{EC}{EA} = \frac{DC}{DP}$$

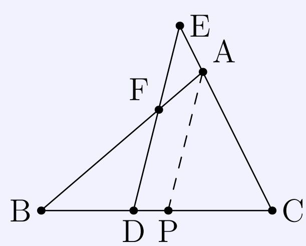
Multiplying the two together yields

$$\frac{AF}{FB} \cdot \frac{BD}{DC} = \frac{EA}{CE},$$

which implies that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

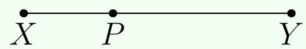
as desired.



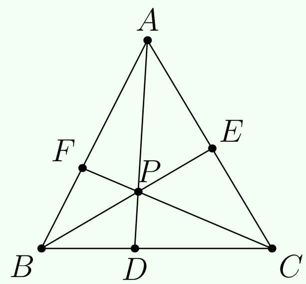
The converse states that $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = -1$, *where all lengths are directed.* (The directed lengths are necessary. In the original theorem, fixing D, E leaves two possible locations for F, only one of which actually lies on DE.)

Theorem 4.1.12: Mass Points

Consider segment XY with P on XY. Then assign *masses* $\diamond X, \diamond Y$ to points X, Y such that $\frac{XP}{YP} = \frac{\diamond Y}{\diamond X}$.



Now consider cevians AD, BE, CF of $\triangle ABC$ that concur at some point P. Then $\frac{AP}{PD} = \frac{\diamond B + \diamond C}{\diamond A}$. This means that for P on XY, we can define $\diamond P = \diamond X + \diamond Y$.



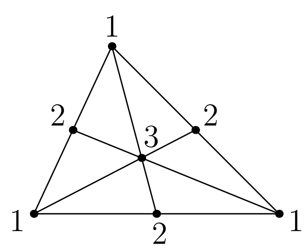
This is a direct application of Ceva's and Menelaus. This is somewhat abstract without an example, so we present the centroid as an example.

Example 4.1.2: Centroid

Assign masses to $\triangle ABC$, its midpoints, and its centroid.

Solution

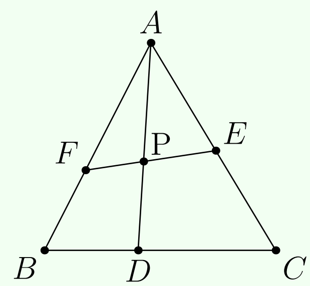
Note $\diamond A = \diamond B = \diamond C$. Without loss of generality, let $\diamond A = 1$.
Then note that since $\diamond X + \diamond Y = \diamond P$ for P on segment XY, $\diamond D = \diamond B + \diamond C = 2$. Similarly, $\diamond E = \diamond F = 2$, and $\diamond G = \diamond A + \diamond D = 1 + 2 = 3$.



Theorem 4.1.13: Mass Points with Transversals

Consider $\triangle ABC$ with points D, E, F on sides BC, CA, AB, and let AD intersect FE at P. Then $\diamond A = \diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA}$.
This is equivalent to

$$\frac{AP}{PD} = \frac{\diamond B + \diamond C}{\diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA}} = \frac{BC}{CD \cdot \frac{BF}{FA} + BD \cdot \frac{CE}{EA}}.$$



The classic analogy is having A_1 on AB and A_2 on AC, and adding $\diamond A_1 + \diamond A_2$ where the masses are taken with respect to AB and AC individually.

You can prove this with Law of Cosines. We present the outline of the proof (the actual algebraic manipulations are very long; this is just a demonstration that it can be proven true).

Proof: Outline

There is exactly one value of AP such that

$$FP + PE = FE,$$

where

$$FP = \sqrt{AF^2 + AP^2 - 2 \cdot AF \cdot AP \cos \angle BAD}$$
$$PE = \sqrt{AE^2 + AP^2 - 2 \cdot AE \cdot AP \cos \angle CAD}$$
$$FE = \sqrt{AE^2 + AF^2 - 2 \cdot AE \cdot AF \cos \angle BAC},$$

and all you have to do is verify

$$AG = \frac{BC \cdot GD}{CD \cdot \frac{BF}{FA} + BD \cdot \frac{CE}{EA}}$$

indeed works.

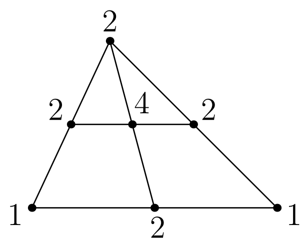
As an example, we use a midsegment and a median.

Example 4.1.3: Midsegment

Assign masses to $\triangle ABC$, A-midsegment EF, median AD, and the point P that lies on AD and EF.

Solution

Note $\diamond A = \diamond B \cdot \frac{BF}{FA} + \diamond C \cdot \frac{CE}{EA} = \diamond B + \diamond C$. Without loss of generality, let $\diamond B = \diamond C = 1$. Then $\diamond A = 2$. Also note that $\diamond D = \diamond B + \diamond C = 2$ and $\diamond P = \diamond A + \diamond D = 4$.



4.2 Areas

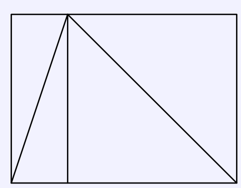
There are a variety of methods to find area. For harder problems, computing the area in two different ways can give useful information about the configuration.

Theorem 4.2.1: $\frac{bh}{2}$

The area of a triangle is $\frac{bh}{2}$.

Proof

The area of each right triangle is half of the area of the rectangle it is in.



Theorem 4.2.2: rs

The area of a triangle is rs, where r is the inradius and s is the semiperimeter.

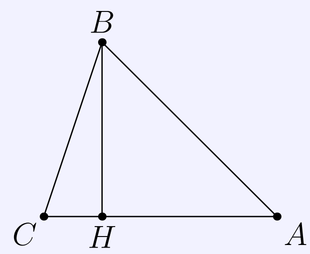
We have already proved this in Length Chasing - but we mention this theorem again because it is useful for area too.

Theorem 4.2.3: $\frac{1}{2}ab\sin C$

The area of a triangle is $\frac{1}{2}ab\sin C$, where a, b are side lengths and C is the included angle.

Proof

Drop an altitude from B to AC and let it have length h. Then note $\frac{1}{2} \cdot a \sin C \cdot b = \frac{1}{2} \cdot hb = \frac{bh}{2}$.



We present a useful corollary of this theorem.

Theorem 4.2.4: $\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$

Let P, A, X be on ℓ_1 and P, B, Y be on ℓ_2. Then $\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$.

Proof

Note $\frac{[PAB]}{[PXY]} = \frac{\frac{1}{2} \cdot PA \cdot PB \cdot \sin\theta}{\frac{1}{2} \cdot PX \cdot PY \cdot \sin\theta} = \frac{PA \cdot PB}{PX \cdot PY}$, where $\theta = \angle APB$.

This works for all configurations since $\sin\theta = \sin(180 - \theta)$.

Theorem 4.2.5: Heron's Formula

In $\triangle ABC$ with sidelengths a, b, c such that $s = \frac{a+b+c}{2}$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof

Since $\cos C = \frac{a^2+b^2-c^2}{2ab}$, the Pythagorean Identity gives us

$$\sin C = \sqrt{\frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4a^2b^2}} = \sqrt{\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2b^2}}.$$

So

$$\frac{1}{2}ab\sin C = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Another form of Heron's is

$$\frac{1}{[ABC]} = \sqrt{(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(-\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(\frac{1}{x}-\frac{1}{y}+\frac{1}{z})(\frac{1}{x}+\frac{1}{y}-\frac{1}{z})},$$

the proof of which is one of the exercises.

Theorem 4.2.6: $\frac{abc}{4R}$

In $\triangle ABC$ with side lengths a, b, c and circumradius R,

$$[ABC] = \frac{abc}{4R}.$$

Proof

Note that $[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}ab \cdot \frac{c}{2R} = \frac{abc}{4R}$.

4.3 Summary

4.3.1 Theory

1. Law of Cosines
   - ✦ $a^2 + b^2 - 2ab\cos C = c^2$.
2. Stewart's Theorem
   - ✦ $man + dad = bmb + cnc$.
   - ✦ $\sqrt{bc - xy}$ gives the length of angle bisector AD.
   - ✦ $\frac{\sqrt{2b^2+2c^2-a^2}}{2}$ gives the length of median AD.
3. Law of Sines
   - ✦ $\frac{a}{\sin A} = 2R$.
4. Angle Bisector Theorem and Ratio Lemma
   - ✦ If AD bisects $\angle BAC$, then $\frac{AB}{BD} = \frac{AC}{CD}$.
   - ✦ Generally, $\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}$.
5. Tangents
   - ✦ Two Tangent Theorem
   - ✦ The tangent is perpendicular to the radius.
   - ✦ $[ABC] = rs$.
   - ✦ $[ABC] = r_a(s-a)$.
   - ✦ Lengths of tangents to the incircle are $s-a, s-b, s-c$.
   - ✦ Lengths of tangents to the excircle are also $s-a, s-b, s-c$ (but in a different order).
6. Concurrency and Collinearity
   - ✦ Ceva's states $\frac{AF}{FB} \cdot \frac{BE}{EC} \cdot \frac{CD}{DA} = 1$.
   - ✦ Menelaus states $\frac{DB}{BF} \cdot \frac{FA}{AE} \cdot \frac{EC}{CD} = 1$.
7. Mass Points

- ✦ $\frac{XP}{YP} = \frac{\diamond Y}{\diamond X}$.
- ✦ $\diamond X + \diamond Y = \diamond P$.

8. Area

   - ✦ $\frac{bh}{2}$
   - ✦ rs
   - ✦ $\frac{1}{2}ab\sin C$
   - ✦ Heron's ($\sqrt{s(s-a)(s-b)(s-c)}$)

4.3.2 Tips and Strategies

1. Use the Law of Sines and Law of Cosines when convenient angles exist.

   - ✦ These can be supplementary, congruent, special, etc.
   - ✦ Use Stewart's when angles are not explicitly present but you need to find a cevian's length anyway.

2. If you have tangents, do length chasing. You will need it.
3. $\frac{1}{2}ab\sin C$ gives ratios of areas. (In general, whenever angles are the same or supplementary, use $\frac{1}{2}ab\sin C$ to get information.)
4. Use two methods to calculate area.

   - ✦ This can give you information about a problem; after all, area doesn't change. So now you can set two seemingly unrelated things equal.

4.4 Exercises

4.4.1 Check-ins

1. Find the inradius of the triangles with the following lengths:

   (a) $3, 4, 5$

   (b) $5, 12, 13$

   (c) $13, 14, 15$

   (d) $5, 7, 8$

   (These are arranged by difficulty. All of these are good to know.)

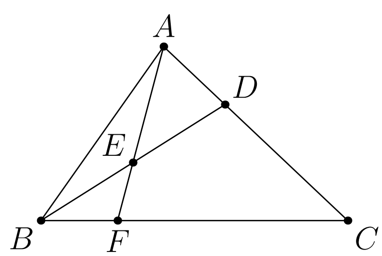
2. Prove that in a right triangle with legs of length a, b and hypotenuse with length c, $r = \frac{a+b-c}{2}$.

3. In $\triangle ABC$, $AB = 5$, $BC = 12$, and $CA = 13$. Points D, E are on BC such that $BD = DC$ and $\angle BAE = \angle CAE$. Find $[ADE]$. **Hints**: 34 **Solution**: 2

4. (Gergonne Point) Let the incircle of $\triangle ABC$ be tangent to BC, CA, AB at D, E, F, respectively. Prove that AD, BE, CF concur. **Hints**: 25

5. (Nagel Point) Let the A excircle of $\triangle ABC$ be tangent to BC at D, and define E, F similarly. Prove that AD, BE, CF concur. **Hints**: 31

6. (AMC 8 2019/24) In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



7. Consider $\triangle ABC$ where X, Y are on BC, CA such that $\frac{BX}{CX} = \frac{1}{4}$, $\frac{CY}{YA} = \frac{2}{3}$. If AX, BY intersect at Z, find $\frac{AZ}{ZX}$.

8. Given $\triangle ABC$ with E, F on line segments AC, AB such that $AE : EC = BF : FA = 1 : 3$ and median AD that intersects EF at G, $AG : GD$.

9. A triangle has side lengths $4, 8, x$ and area $3\sqrt{15}$. Find x.

10. Find the sum of the altitudes of a triangle with side lengths $5, 7, 8$.

11. Let $\angle BAC = 30°$ and let P be the midpoint of AC. If $\angle BPC = 45°$, what is $\angle ABC$? **Hints**: 36

12. Given $\triangle ABC$, find $\sin A \sin B \sin C$ in terms of $[ABC]$ and abc.

4.4.2 Problems

1. Consider $\triangle ABC$ with $AB = 7, BC = 8, AC = 6$. Let AD be the angle bisector of $\angle BAC$ and let E be the midpoint of AC. If BE and AD intersect at G, find AG.

2. Find the maximum area of a triangle with two of its sides having lengths $10, 11$.

3. Consider trapezoid $ABCD$ with bases AB and CD. If AC and BD intersect at P, prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ is at least half the area of trapezoid $ABCD$.

4. Consider rectangle $ABCD$ such that $AB = 2$ and $BC = 1$. Let X, Y trisect AB. Then let DX and DY intersect AC at P and Q, respectively. What is the area of quadrilateral $XYQP$?

5. Consider $\triangle ABC$ with altitudes of lengths x, y, z. Prove that
$$\frac{1}{[ABC]} = \sqrt{(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(-\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(\frac{1}{x}-\frac{1}{y}+\frac{1}{z})(\frac{1}{x}+\frac{1}{y}-\frac{1}{z})}.$$
Hints: 45

6. (Autumn Mock AMC 10) Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

7. Consider $\triangle ABC$ such that $AB = 8$, $BC = 5$, and $CA = 7$. Let AB and CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$. **Hints**: 49

8. Consider $\triangle ABC$ with an area of 60, inradius of 3, and circumradius of $\frac{17}{2}$. Find the side lengths of the triangle.

9. (AIME I 2019/2) In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

10. (PUMaC 2016) Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle\ DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

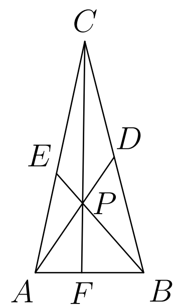
11. (AIME I 2001/4) In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

4.4.3 Challenges

1. (CIME 2020) An excircle of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90°$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

2. Consider ABC with $\angle A = 45°, \angle B = 60°$, and with circumcenter O. If BO intersects CA at E and CO intersects AB at F, find $\frac{[AFE]}{[ABC]}$.

3. (AIME 1989/15) Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure at right). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.



4. (AIME II 2019/11) Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hints**: 39

5. (AIME II 2016/10) Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

6. (AIME II 2005/14) In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.

7. (AIME I 2019/11) In $\triangle ABC$, the sides have integers lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

8. (ART 2019/6) Consider unit circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. If $XP = MQ$, find AT. **Hints**: 42 20 3 30 **Solution**: 6

9. (AIME I 2020/13) Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

10. (USAMO 1999/6) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles. **Hints**: 10

11. (CIME 2019) Let $\triangle ABC$ be a triangle with circumcenter O and incenter I such that the lengths of the three segments AB, BC and CA form an increasing arithmetic progression in this order. If $AO = 60$ and $AI = 58$, then the distance from A to BC can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 5

Hints

1. Pick a point. Draw all the diagonals connected to that point.
2. Look for similar triangles.
3. Reflect Y about XB to get Y'.
4. Prove $\triangle ABC \sim \triangle EDC \sim \triangle EBA$.
5. Reflect P about the midpoint of AB.
6. Add stuff so that the angle bisector of $\angle APB$ the diagonal of a square as well.
7. Prove that $ABRQ$ is cyclic.
8. Remember that QX and DB are parallel. How does this help you find QD?
9. Draw a line through A parallel to BC.
10. F is a *specific* point.
11. Look for similar triangles.
12. There is a cyclic quadrilateral with O_2 on it.
13. Let MN intersect AB at O.
14. Look at $\triangle GBC$.
15. Use Tangent/Secant to set up a system of equations.
16. Reduce the problem to a bunch of triangles.
17. You can get BE and BF (via Stewart's), so you can get BG and CG.
18. Note $\angle APB = 180^\circ - \angle BAP - \angle ABP$.
19. R seems somewhat pesky. Can you find other stuff R is involved with?
20. How can you find the proportions of the lengths with the knowledge that $OX = OM$?
21. Let the point be P and let P be tangent to the circle at A and B.
22. Show that $BP = BR$.
23. How can you express $\frac{\angle DOE}{2}$ and $\frac{\angle AOB}{2}$?
24. Use power of a point to relate PA and PB.
25. Two Tangent Theorem.
26. What is the foot of the perpendicular from E to PQ?
27. Draw AE.

28. Look for collinear points.
29. Draw in the center of the semicircle.
30. Find the area of $\triangle XYY'$ in two ways.
31. Two Tangent Theorem.
32. How would you find BX and DY?
33. There are three more cyclic quadrilaterals.
34. We know the height. What else do we need?
35. Let (BFD) intersect (CDE) at P.
36. Drop an altitude from B to CA.
37. What information do cyclic quadrilaterals give you?
38. Look at $\angle AEC$.
39. Use the tangent angle condition to angle chase.
40. What is $\angle BCD$?
41. Look for similar and congruent triangles.
42. What does $XP = MQ$ really mean?
43. What does $\triangle AMN \sim \triangle DMN \sim \triangle ABC$ tell you?
44. Where do BC and DA meet?
45. Note $x = \frac{[ABC]}{2a}$.
46. Look for cyclic quadrilaterals.
47. Look at $\triangle BIC$.
48. Have you found $\frac{b^2+c^2}{a^2}$ yet?
49. Find $\frac{[AT_BT_C]}{[ABC]}$.

Chapter 6

Solutions

1. Let Q, R, S be the rotations of P about O by $90^\circ, 180^\circ, 270^\circ$ counterclockwise. Note that PR is the angle bisector of $\angle APB$ and PR bisects the area of $[PQRS]$. Since the area we added to both halves of $ABCD$ is the same, PR also bisects $ABCD$.

2. Note that $BD = 6$ and $BE = \frac{5}{5+13} \cdot 12 = \frac{10}{3}$, so $DE = 6 - \frac{10}{3} = \frac{8}{3}$. Thus $[ADE] = \frac{1}{2} \cdot 5 \cdot \frac{8}{3} = \frac{20}{3}$.

3. Say the center of K_1 is O_1 and the center of K_2 is O_2. Obviously AO_2CQ is cyclic since $\angle QAO_2 = \angle QCO_2 = 90^\circ$. Now note $\angle QCP = 180 - \angle BAC$, and $\angle PAB = \angle ABP = \angle ABC = \angle QAC$, so P also lies on this circle. Thus $\angle O_2PQ = 90^\circ$. Note
$$\angle ABO_2 = \frac{180^\circ - \angle AO_2B}{2} = 90^\circ - \angle ACB = 90^\circ - \angle ACP = 90^\circ - \angle AQP = 90^\circ - \angle DQP,$$
and $\angle O_2DA = \angle O_2BA$ if and only if O_2 lies on (ABD), or K_1. Then $\angle O_2DA = \angle O_2BA$ implies that $\angle DPQ = 90^\circ$, as desired.

4. By Stewart's, $BE = \frac{\sqrt{2c^2+2a^2-b^2}}{2}$ and $CF = \frac{\sqrt{2a^2+2b^2-c^2}}{2}$, so $a^2 = BG^2 + CF^2 = \frac{4}{9}(\frac{2c^2+2a^2-b^2}{4} + \frac{2a^2+2b^2-c^2}{4}) = \frac{4a^2+b^2+c^2}{9}$.

   Thus, $5a^2 = b^2 + c^2$ and $\frac{b^2+c^2}{a^2} = 5$.

5. Note that $\angle MBO = 30^\circ = \angle NAO$, $\angle ANO = 180^\circ - \angle AMO = \angle BMO$, and $AO = BO$, so $\triangle BMO \cong \triangle ANO$. Thus $AN = BM$.

6. Let O be the center of the circle. Notice that this implies that $OM = OX$. We claim that if $BM = x$, then $XT = x$ as well.

   By the Pythagorean Theorem, $OM = \sqrt{x^2+1}$. Since $OM = OX$, $AX = \sqrt{x^2+1} - 1$. Then by Power of a Point, $XT = \sqrt{XA \cdot XB} = (\sqrt{x^2+1} - 1)(\sqrt{x^2+1} + 1) = x$, as desired.

   Also, by the Pythagorean Theorem, $BX = \sqrt{5}x$.

   We have is a semicircle with a known radius inscribed within a right triangle. Knowing the proportions of the triangle motivates reflecting about BX to use $[ABC] = rs$.

   Let the reflection of Y about BX be Y' Then notice $[YXY'] = 2\sqrt{5}x^2$, by $\frac{bh}{2}$. But also notice by $[ABC] = rs$, $[YXY'] = 5x$. Since the area of a triangle is the same no matter how it is computed, $2\sqrt{5}x^2 = 5x$, implying $x = \frac{\sqrt{5}}{2}$.

   Drop an altitude from T to BX, and let the foot be T'. Notice that $\triangle YBX \sim \triangle TT'X$ with a ratio of $3:1$. Thus $TT' = \frac{\sqrt{5}}{3}$ and $TX = \frac{5}{6}$. Then notice $T'A = T'X - AX$. Since $BX = \frac{5}{2}$ and $BA = 2$, $AX = \frac{1}{2}$. Thus $T'A = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$. By the Pythagorean Theorem, $TA = \sqrt{(\frac{1}{3})^2 + (\frac{\sqrt{5}}{3})^2} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$, which is our answer.

7. Angle chase to find $\triangle ABC \sim \triangle EDC \sim \triangle EBA$. So $BE = 7 \cdot \frac{7}{10} = \frac{49}{10}$, implying $CE = 10 - \frac{49}{10} = \frac{51}{10}$, and $CD = \frac{10}{8} \cdot \frac{51}{10} = \frac{51}{8}$, implying $AD = 8 - \frac{51}{8} = \frac{13}{8}$.

8. The key observation is that AD, BC, EF concur.

   Let AD and BC intersect at P and let Q be the foot of the altitude from P to AB. Also let the semicircle have center O. Now note
$$\triangle PAQ \sim \triangle OAD$$
$$\triangle PBQ \sim \triangle OBC$$
   so $\frac{AQ}{QB} \cdot \frac{BC}{CP} \cdot \frac{PD}{DA} = 1$. Since AC, BD, PQ concur, Q is actually F, and AC, BD, PF concur.

   Now note
$$\angle OCP = \angle ODP = \angle OFP = 90^\circ,$$
   so $OFCPD$ is cyclic. Thus
$$\angle COP = \angle DOP$$
$$\angle CFP = \angle DFP.$$

9. Note that $\triangle ABX$ and $\triangle ADY$ are isosceles, so $BX = 7$ and $DY = 6$. Now also note that $XQDB$ is a parallelogram, so $QD = BX = 7$. Now note $\angle QDY = \angle DAB$, so it suffices to find $\cos \angle QDY$.

   Now note that $QY = DY = 6$ and $QD = 7$. Thus dropping the altitude from Y to QD gives us $\cos \angle QDY = \frac{\frac{7}{2}}{6} = \frac{7}{12}$.

10. Note that
$$\angle BPR = \angle BAP + \angle ABP = \angle AQP + \angle PBQ = \angle AQB$$
    and that
$$\angle BRP = \angle RPC + \angle RCP = 180^\circ - \angle APC + \angle BCP = \angle AQP + \angle BQP = \angle AQB,$$
    so $\angle BPR = \angle BRP$.

    Now note
$$\angle AQB = \angle BAP + \angle ABP = 180^\circ - \angle APB = \angle BPR = \angle BRP,$$
    so $ABRQ$ is cyclic.

    Now reflect P about the midpoint of AB to get P'. Then note
$$\angle PQR = \angle P'QR = \angle P'AR = \angle P'AB + \angle BAR = \angle ABR + \angle BAP = \angle BPR,$$
    so BP is tangent to (PQR), as desired.

11. Notice $\angle EAB = \angle ACM = \angle ANM = \angle BAM$ and $\angle EBA = \angle ABM$, so $\triangle EAB \cong \triangle MAB$, implying that AB is the perpendicular bisector of EM. So $\angle EMP = \angle EMQ = 90^\circ$, and it suffices to show that $PM = MQ$.

    Let MN intersect AB at O. Note that $AO = BO$, so $PM = MQ$ by similar triangles.