Diagnostic Quiz

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1 Instructions

Take as long as you need on these problems and do as many as you can. For computational problems, provide numerical answers and include a short sketch of your solution. For proof problems, include a full solution. Once done, submit it through the MAST Diagnostic Form.

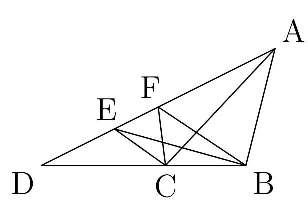
You may use books as reference/ask me or other people for help. (In particular, if you need reading to learn the material, please contact me.) All of these are my problems so I do think it'll be difficult to google them, but please don't try this all the same.

2 Problems

2.1 Computational

1. How many integer values of $1 \leq x \leq 100$ makes $x^2 + 8x + 5$ divisible by 10?

2. In the following diagram, $m\angle BAC = m\angle BFC = 40°$, $m\angle ABF = 80°$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?



3. Consider parallelogram $ABCD$ with $AB = 7$, $BC = 6$. Let the angle bisector of $\angle DAB$ intersect BC at X and CD at Y. Let the line through X parallel to BD intersect AD at Q. If $QY = 6$, find $\cos \angle DAB$.

4. Consider unit circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. If $XP = MQ$, find AT.

5. A secret spy organization needs to spread some secret knowledge to all of its members. In the beginning, only 1 member is *informed*. Every informed spy will call an uninformed spy such that every informed spy is calling a different uninformed spy. After being called, an uninformed spy becomes informed. The call takes 1 minute, but since the spies are running low on time, they call the next spy directly afterward. However, to avoid being caught, after the third call an informed spy makes, the spy stops calling. How many minutes will it take for every spy to be informed, provided that the organization has 600 spies?

6. Andy the unicorn is on a number line from 1 to 2019. He starts on 1. Each step, he randomly and uniformly picks an integer greater than the integer he is currently on, and goes to it. He stops when he reaches 2019. What is the probability he is ever on 1984?

7. Find $\sum\limits_{a=1}^{\infty} \frac{32a}{16a^4+24a^2+25}$.

8. Find the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period 4.

9. Santa Claus is putting n identical toy trains into a red stocking, a green stocking, and a white stocking such that there are an even amount of trains in the white stocking. Mrs. Claus is putting n identical toy elves into a red stocking, a green stocking, and a white stocking such that there are an odd amount of elves in the white stocking. Find, in terms of n, the positive difference between the amount of ways Santa Claus can put his trains in the stockings and the amount of ways Mrs. Claus can put her elves in the stockings.

10. Find the maximum value of k such that $(x+1)^4 \geq kx^3$ for all x.

2.2 Proof

1. Consider $\triangle ABC$, and let the feet of the B and C altitudes of the triangle be X, Y. Let XY intersect BC at P. Then prove that the circumcircles of $\triangle PBY$ and $\triangle PCX$ concur with AP.

2. Consider $\triangle ABC$ with D on line BC. Let the circumcenters of $\triangle ABD$ and $\triangle ACD$ be M, N, respectively. Let the circumcircle of $\triangle MND$ intersect the circumcircle of $\triangle ACD$ again at $H \neq D$. Prove that A, M, H are collinear.

3. Let $f(x) = x^2 - 12x + 36$. In terms of k, for $k \geq 2$, find the sum of all real n such that $f^{k-1}(n) = f^k(n)$.

4. Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z.

Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.