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Zero-sum game

In game theory and economic theory, a **zero-sum game** is a mathematical representation of a situation in which each participant's gain or loss of utility is exactly balanced by the losses or gains of the utility of the other participants. If the total gains of the participants are added up and the total losses are subtracted, they will sum to zero. Thus, cutting a cake, where taking a larger piece reduces the amount of cake available for others, is a zero-sum game if all participants value each unit of cake equally (see marginal utility).

In contrast, **non-zero-sum** describes a situation in which the interacting parties' aggregate gains and losses can be less than or more than zero. A zero-sum game is also called a *strictly competitive* game while non-zero-sum games can be either competitive or non-competitive. Zero-sum games are most often solved with the minimax theorem which is closely related to linear programming duality,[1] or with Nash equilibrium.

Contents

Definition

The zero-sum property (if one gains, another loses) means that any result of a zero-sum situation is Pareto optimal. Generally, any game where all strategies are Pareto optimal is called a conflict game.[2]

Zero-sum games are a specific example of constant sum games where the sum of each outcome is always zero. Such games are distributive, not integrative; the pie cannot be enlarged by good negotiation.

	Choice 1	Choice 2
Choice 1	-A, A	B, -B
Choice 2	C, -C	-D, D

Generic zero-sum game

Situations where participants can all gain or suffer together are referred to as non-zero-sum. Thus, a country with an excess of bananas trading with another country for their excess of apples, where both benefit from the transaction, is in a non-zero-sum situation. Other non-zero-sum games are games in which the sum of gains and losses by the players are sometimes more or less than what they began with.

The idea of Pareto optimal payoff in a zero-sum game gives rise to a generalized relative selfish rationality standard, the punishing-the-opponent standard, where both players always seek to minimize the opponent's payoff at a favorable cost to himself rather to prefer more than less. The punishing-the-opponent standard can be used in both zero-sum games (e.g. warfare game, chess) and non-zero-sum games (e.g. pooling selection games).[3]

Solution

For two-player finite zero-sum games, the different game theoretic solution concepts of Nash equilibrium, minimax, and maximin all give the same solution. If the players are allowed to play a mixed strategy, the game always has an equilibrium.

Example

Blue / Red	*A*	*B*	*C*
1	-30 / 30	10 / -10	-20 / 20
2	10 / -10	-20 / 20	20 / -20

A zero-sum game

A game's payoff matrix is a convenient representation. Consider for example the two-player zero-sum game pictured at right or above.

The order of play proceeds as follows: The first player (red) chooses in secret one of the two actions 1 or 2; the second player (blue), unaware of the first player's choice, chooses in secret one of the three actions A, B or C. Then, the choices are revealed and each player's points total is affected according to the payoff for those choices.

Example: Red chooses action 2 and Blue chooses action B. When the payoff is allocated, Red gains 20 points and Blue loses 20 points.

In this example game, both players know the payoff matrix and attempt to maximize the number of their points. Red could reason as follows: "With action 2, I could lose up to 20 points and can win only 20, and with action 1 I can lose only 10 but can win up to 30, so action 1 looks a lot better." With similar reasoning, Blue would choose action C. If both players take these actions, Red will win 20 points. If Blue anticipates Red's reasoning and choice of action 1, Blue may choose action B, so as to win 10 points. If Red, in turn, anticipates this trick and goes for action 2, this wins Red 20 points.

Émile Borel and John von Neumann had the fundamental insight that probability provides a way out of this conundrum. Instead of deciding on a definite action to take, the two players assign probabilities to their respective actions, and then use a random device which, according to these probabilities, chooses an action for them. Each player computes the probabilities so as to minimize the maximum expected point-loss independent of the opponent's strategy. This leads to a linear programming problem with the optimal strategies for each player. This minimax method can compute probably optimal strategies for all two-player zero-sum games.

For the example given above, it turns out that Red should choose action 1 with probability 4/7 and action 2 with probability 3/7, and Blue should assign the probabilities 0, 4/7, and 3/7 to the three actions A, B, and C. Red will then win 20/7 points on average per game.

Solving

The Nash equilibrium for a two-player, zero-sum game can be found by solving a linear programming problem. Suppose a zero-sum game has a payoff matrix M where element $M_{i,j}$ is the payoff obtained when the minimizing player chooses pure strategy i and the maximizing player chooses pure strategy j (i.e. the player trying to minimize the payoff chooses the row and the player trying to maximize the payoff chooses the column). Assume every element of M is positive. The game will have at least one Nash equilibrium. The Nash equilibrium can be found (see ref. [2], page 740) by solving the following linear program to find a vector u:

Minimize:

$$\sum_i u_i$$

Subject to the constraints:

$$u \geq 0$$
$$M u \geq 1.$$

The first constraint says each element of the u vector must be nonnegative, and the second constraint says each element of the $M u$ vector must be at least 1. For the resulting u vector, the inverse of the sum of its elements is the value of the game. Multiplying u by that value gives a probability vector, giving the probability that the maximizing player will choose each of the possible pure strategies.

If the game matrix does not have all positive elements, simply add a constant to every element that is large enough to make them all positive. That will increase the value of the game by that constant, and will have no effect on the equilibrium mixed strategies for the equilibrium.

The equilibrium mixed strategy for the minimizing player can be found by solving the dual of the given linear program. Or, it can be found by using the above procedure to solve a modified payoff matrix which is the transpose and negation of M (adding a constant so it's positive), then solving the resulting game.

If all the solutions to the linear program are found, they will constitute all the Nash equilibria for the game. Conversely, any linear program can be converted into a two-player, zero-sum game by using a change of variables that puts it in the form of the above equations. So such games are equivalent to linear programs, in general.

Universal solution

If avoiding a zero-sum game is an action choice with some probability for players, avoiding is always an equilibrium strategy for at least one player at a zero-sum game. For any two players zero-sum game where a zero-zero draw is impossible or non-credible after the play is started, such as Poker, there is no Nash equilibrium strategy other than avoiding the play. Even if there is a credible zero-zero draw after a zero-sum game is started, it is not better than the avoiding strategy. In this sense, it’s interesting to find reward-as-you-go in optimal choice computation shall prevail over all two players zero-sum games with regard to starting the game or not.[4]

Non-zero-sum

Psychology

The most common or simple example from the subfield of social psychology is the concept of "social traps". In some cases pursuing our personal interests can enhance our collective well-being, but in others personal interest results in mutually destructive behavior.

Complexity

It has been theorized by Robert Wright in his book *Nonzero: The Logic of Human Destiny*, that society becomes increasingly non-zero-sum as it becomes more complex, specialized, and interdependent.

Extensions

In 1944, John von Neumann and Oskar Morgenstern proved that any non-zero-sum game for n players is equivalent to a zero-sum game with $n + 1$ players; the $(n + 1)$th player representing the global profit or loss.[5]

Misunderstandings

Zero-sum games and particularly their solutions are commonly misunderstood by critics of game theory, usually with respect to the independence and rationality of the players, as well as to the interpretation of utility functions. Furthermore, the word "game" does not imply the model is valid only for recreational games.[1]

Politics is sometimes called zero sum.[6][7][8]

Zero-sum thinking

In psychology, zero-sum thinking refers to the perception that a situation is like a zero-sum game, where one person's gain is another's loss.

See also

- Bimatrix game
- Comparative advantage
- Dutch disease
- Gains from trade
- Lump of labour fallacy
- Positive-sum game
- Zero-sum thinking

References

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2. Bowles, Samuel (2004). *Microeconomics: Behavior, Institutions, and Evolution*. Princeton University Press. pp. 33–36. ISBN 0-691-09163-3.
3. Wenliang Wang (2015). Pooling Game Theory and Public Pension Plan. ISBN 978-1507658246. Chapter 1 and Chapter 4.
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6. Rubin, Jennifer (2013-10-04). "The flaw in zero sum politics" (https://www.washingtonpost.com/blogs/right-turn/wp/2013/10/04/the-flaw-in-zero-sum-politics/). *The Washington Post*. Retrieved 2017-03-08.
7. "Lexington: Zero-sum politics" (https://www.economist.com/news/united-states/21595973-voters-think-both-parties-are-telling-truth-about-how-awful-other-lot-are-zero-sum). *The Economist*. 2014-02-08. Retrieved 2017-03-08.
8. "Zero-sum game I Define Zero-sum game at" (http://www.dictionary.com/browse/zero-sum-game). *Dictionary.com*. Retrieved 2017-03-08.

Further reading

- *Misstating the Concept of Zero-Sum Games within the Context of Professional Sports Trading Strategies*, series *Pardon the Interruption* (2010-09-23) ESPN, created by Tony Kornheiser and Michael Wilbon, performance by Bill Simmons
- *Handbook of Game Theory - volume 2*, chapter *Zero-sum two-person games*, (1994) Elsevier Amsterdam, by Raghavan, T. E. S., Edited by Aumann and Hart, pp. 735–759, ISBN 0-444-89427-6
- *Power:Its Forms, Bases and Uses* (1997) Transaction Publishers, by Dennis Wrong

External links

- Play zero-sum games online (http://www.egwald.ca/operationsresearch/twoperson.php) by Elmer G. Wiens.
- Game Theory & its Applications (https://web.archive.org/web/20070518035304/http://www.le.ac.uk/psychology/amc/gtaia.html) - comprehensive text on psychology and game theory. (Contents and Preface to Second Edition.)
- A playable zero-sum game (http://economics-games.com/mixed-nash) and its mixed strategy Nash equilibrium.