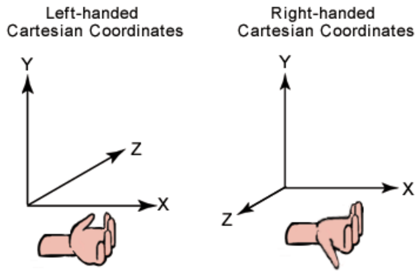
<u>Chapter8: Vector and Matrix Differential Calculus</u>.

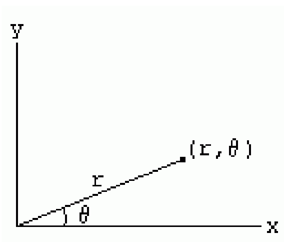
1. Coordinate systems

   a) Rectangular Cartesian Coordinate Systems



In this coordinates, the vectors that span the frame are the $\vec{i}$, $\vec{j}$, $\vec{k}$ vectors.The left-handed system is used in DirectX and the right-handed system in OpenGL.The right-handed system is the standard frame used in Math and in Physics. The frame $\{\vec{i},\vec{j},\vec{k}\}$ is said to be an orthonormal frame since $\vec{i} \perp \vec{j} \perp \vec{k}$ and $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$.

   b) Polar Coordinate



In polar coordinate, a point is represented by the pair (θ, r) where r is the line between the point and the origin O, and θ is the angle between r and the x-axis in radian such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ or $\vec{p}(x,y) = (r\cos(\theta),\, r\sin(\theta))$

This is a conversion from polar to Cartesian coordinates. To convert back to polar coordinate we calculate

$$\tan(\theta) = \left(\frac{y}{x}\right) \text{ or } \theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ and } r = \sqrt{x^2 + y^2}.$$

<u>Converting from rectangular (cartesian) coordinate to polar coordinate</u>

Example 1.b.1: Write the Cartesian coordinates of point $\vec{p}(1,1)$ in polar coordinate .

We need the pair (θ, r), $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right) \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

So $\vec{p}(1,1)$ is $\left(\frac{\pi}{4}, \sqrt{2}\right)$

TODO: Go to Activity and solve question 1

<u>Converting from Polar coordinate to rectangular (cartesian) coordinate</u>

Example 1.b.2 : Write the polar point $\vec{p}(\pi, 2)$ in Cartesian coordinates.

Answer : using $\vec{p}(x,y) = (r\cos(\theta),\, r\sin(\theta))$ in Cartesian coordinates with $\theta = \pi \ and \ r = 2$ we get :

$$\vec{p}(x,y) = (2\cos(\pi),\, 2\sin(\pi)) = (-2, 0)$$

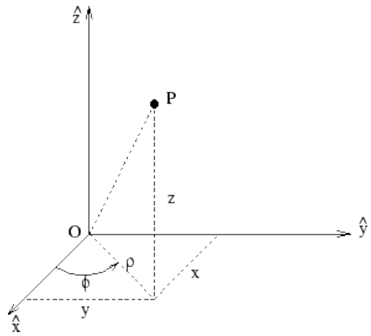
Example 1.b.3: Write the polar point $\vec{p}\left(\frac{\pi}{4}, 4\right)$ in Cartesian coordinates.

using $\vec{p}(x,y) = (r\cos(\theta),\, r\sin(\theta))$ in Cartesian coordinates with $\theta = \pi \ and \ r = 2$ we get :

$$\vec{p}(x,y) = \left(4\cos\left(\frac{\pi}{4}\right),\, 4\sin\left(\frac{\pi}{4}\right)\right) = \left(4\frac{\sqrt{2}}{2}, 4\frac{\sqrt{2}}{2}\right) = \left(2\sqrt{2}, 2\sqrt{2}\right).$$

TODO: Go to Activity and solve question 2

c) Cylindrical Coordinate Systems



In cylindrical coordinate , a point is represented by the triple (ρ,ϕ,z) such that

$$\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases}$$ where $0\le\phi\le 2\pi$, $\rho>0,\ -\infty<z<+\infty$ .This is a conversion from cylindrical to Cartesian coordinates. To convert from Cartesian to cylindrical coordinate we calculate

$$\rho=\sqrt{x^2+y^2} \qquad \phi=\tan^{-1}\left(\frac{y}{x}\right) \quad \text{and} \quad z=z$$ ϕ reads phi

Example 1.c.1: change $\left(3,\frac{\pi}{2},1\right)$ from cylindrical to Cartesian coordinates.

Answer: using $\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases}$ with $(\rho,\phi,z)=\left(3,\frac{\pi}{2},1\right)$ ➔ $\rho=3$, $\phi=\frac{\pi}{2}$ and $z=1$ we have

$$\begin{cases} x=3\cos\left(\frac{\pi}{2}\right)=3(0)=0 \\ y=3\sin\left(\frac{\pi}{2}\right)=3(1)=3 \\ z=1 \end{cases}$$ ➔ $\left(3,\frac{\pi}{2},1\right)$ in Cartesian coordinates is $(0,3,1)$

Example 1.c.2: change $\left(2,\frac{\pi}{6},5\right)$ from cylindrical to Cartesian coordinates.

Answer: using $\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases}$ with $(\rho,\phi,z)=\left(2,\frac{\pi}{6},5\right)$ ➔ $\rho=2$, $\phi=\frac{\pi}{6}$ and $z=5$ we have

$$\begin{cases} x=2\cos\left(\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}\right)=\sqrt{3} \\ y=2\sin\left(\frac{\pi}{6}\right)=2\left(\frac{1}{2}\right)=1 \\ z=5 \end{cases}$$ ➔ $\left(2,\frac{\pi}{6},5\right)$ in Cartesian coordinates is $\left(\sqrt{3},1,5\right)$

TODO: Go to Activity and solve question 3

Example 1.c.3: Change $(1,1,2)$ from Cartesian to cylindrical coordinates.

Answer : use $\rho=\sqrt{x^2+y^2}$ $\quad\phi=\tan^{-1}\left(\frac{y}{x}\right)$ and $z=z$

From $(1,1,2)$, $x=1$, $y=1$, *and* $z=2$ ➔ $\rho=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$,

$$\phi=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4}\quad(45^o)$$

So $(1,1,2)$ is $(\rho,\phi,z)=\left(\sqrt{2},\frac{\pi}{4},2\right)$ in cylindrical coordinates

Example 1.c.4: Change $(0,4,12)$ from Cartesian to cylindrical coordinates.

Answer : use $\rho=\sqrt{x^2+y^2}$ $\quad \phi=\tan^{-1}\left(\frac{y}{x}\right)$ and $z=z$

From $(0,4,12)$, $x=0$, $y=4$, $and\ z=12$ ➔ $\rho=\sqrt{0^2+4^2}=\sqrt{16}=4$,

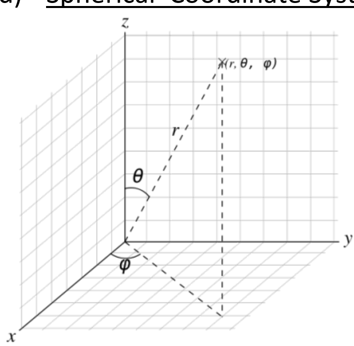
We cannot use $\phi=\tan^{-1}\left(\frac{y}{x}\right)$ because $x=0$. From $\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases}$ we use $x=\rho\cos\phi$ ➔

$\cos\phi=\frac{x}{\rho}=\frac{0}{4}=0$ and $\phi=\cos^{-1}(0)=\frac{\pi}{2}$ (90^o)

So $(0,4,12)$ is $(\rho,\phi,z)=\left(4,\frac{\pi}{2},12\right)$ in cylindrical coordinates

TODO: Go to Activity and solve question 4

d) Spherical Coordinate Systems



In spherical coordinate, a point is represented by the triple (r,θ,ϕ) such that

$\begin{cases} x=r\sin\theta\cos\phi \\ y=r\sin\theta\sin\phi \\ z=r\cos\theta \end{cases}$ where $0\le\theta\le\pi$, $0\le\phi\le 2\pi$, $r>0$.

This is a conversion from spherical to Cartesian coordinates. To convert back to spherical coordinate we calculate

$$r=\sqrt{x^2+y^2+z^2} \qquad \phi=\tan^{-1}\left(\frac{y}{x}\right) \qquad \theta=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Example 1.d.1 : change $\left(3,\frac{\pi}{2},\frac{\pi}{4}\right)$ from spherical to Cartesian coordinates.

Answer : we have $(r,\theta,\phi)=\left(3,\frac{\pi}{2},\frac{\pi}{4}\right)$ ➔ $r=3,\ \theta=\frac{\pi}{2},\ and\ \ \phi=\frac{\pi}{4}$ plugged in

$$\begin{cases} x=r\sin\theta\cos\phi=3\sin\frac{\pi}{2}\cos\frac{\pi}{4}=3(1)\left(\frac{\sqrt{2}}{2}\right)=\frac{3\sqrt{2}}{2} \\ y=r\sin\theta\sin\phi=3\sin\frac{\pi}{2}\ \sin\frac{\pi}{4}=3(1)\left(\frac{\sqrt{2}}{2}\right)=\frac{3\sqrt{2}}{2} \\ z=r\cos\theta=3\cos\frac{\pi}{2}=3(0)=0 \end{cases} ➔ \begin{cases} x=\frac{3\sqrt{2}}{2} \\ y=\frac{3\sqrt{2}}{2} \\ z=0 \end{cases}$$

So $\left(3,\frac{\pi}{2},\frac{\pi}{4}\right)$ is $\left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2},0\right)$ in Cartesian coordinates.

Example 1.d.2 : change $\left(2,\pi,\frac{\pi}{3}\right)$ from spherical to Cartesian coordinates.

Answer : we have $(r,\theta,\phi) = \left(2,\pi,\frac{\pi}{3}\right)$ ➔ $r=2,\ \theta=\pi\ ,\ and\ \ \phi=\frac{\pi}{3}$ plugged in

$$\begin{cases} x = r\sin\theta\cos\phi = 2\sin\pi\cos\frac{\pi}{3} = 2(0)\left(\frac{\sqrt{3}}{2}\right) = 0 \\ y = r\sin\theta\sin\phi = 2\sin\pi\ \sin\frac{\pi}{3} = 2(0)\left(\frac{1}{2}\right) = 0 \\ z = r\cos\theta = 2\cos\pi = 2(-1) = -2 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ z=-2 \end{cases}$$

So $\left(2,\pi,\frac{\pi}{3}\right)$ is $(0,0,-2)$ in Cartesian coordinates.

TODO: Go to Activity and solve question 5

Example 1.d.3 : change (1,1,0) from Cartesian to spherical coordinates.

Answer: (1,1,0) ➔ x=1 , y=1, z=0 we need to find $r,\ \theta$ and ϕ

Using $r=\sqrt{x^2+y^2+z^2} = \sqrt{1^2+1^2+0} = \sqrt{2}$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

and $\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{0}{\sqrt{1^2+1^2+0^2}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

so (1,1,0) is $(r,\theta,\phi) = \left(\sqrt{2},\frac{\pi}{2},\frac{\pi}{4}\right)$ in spherical coordinates

Example 1.d.4 : change $\left(2\sqrt{3},6,4\right)$ from Cartesian to spherical coordinates.

Answer: $\left(2\sqrt{3},6,4\right)$ ➔ $x=2\sqrt{3},\ y=6,\ z=4$ we need to find $r,\ \theta$ and ϕ

Using $r=\sqrt{\left(2\sqrt{3}\right)^2+6^2+4^2} = \sqrt{12+36+16} = \sqrt{64} = 8$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{2\sqrt{3}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

and $\theta = \cos^{-1}\left(\frac{4}{\sqrt{\left(2\sqrt{3}\right)^2+6^2+4^2}}\right) = \cos^{-1}\left(\frac{4}{8}\right) = \cos^{-1}(0.5) = \frac{\pi}{3}$

so $\left(2\sqrt{3},6,4\right)$ is $(r,\theta,\phi) = \left(8,\frac{\pi}{3},\frac{\pi}{3}\right)$ in spherical coordinates

TODO: Go to Activity and solve question 6

~~Changing form Cylindrical Coordinate to Spherical Coordinate Example~~

Example 1.d.5:

Example 1.d.6

~~Changing form Spherical Coordinate to Cylindrical Coordinate Examples~~

Example 1.d.7

Example 1.d.8

2. Vector Differentiation

Given a vector $\vec{u}(x,y,z)=(x(t),y(t),z(t))$ = $x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$, the derivation of $\vec{u}$ with respect to t is a vector $\frac{d\vec{u}}{dt}$ = $\dot{x}(t)\vec{i}+\dot{y}(t)\vec{j}+\dot{z}(t)\vec{k}$.

If $\vec{a}$ and $\vec{b}$ are differentiable vectors, then

a) $\frac{d}{dt}(\vec{a}\pm\vec{b})=\frac{d\vec{a}}{dt}\pm\frac{d\vec{b}}{dt}$

b) $\frac{d}{dt}(\vec{a}\bullet\vec{b})=\frac{d\vec{a}}{dt}\bullet\vec{b}+\vec{a}\bullet\frac{d\vec{b}}{dt}$

c) $\frac{d}{dt}(\vec{a}\times\vec{b})=\frac{d\vec{a}}{dt}\times\vec{b}+\vec{a}\times\frac{d\vec{b}}{dt}$

Example: Given $\vec{u}=3t^2\vec{i}+t^3\vec{j}-2t^5\vec{k}$

$$\frac{d\vec{u}}{dt}=\frac{d(3t^2)}{dt}\vec{i}+\frac{d(t^3)}{dt}\vec{j}-\frac{d(2t^5)}{dt}\vec{k}=(6t)\vec{i}+(\ 3t^2)\vec{j}-(10t^4)\vec{k}=(6t\ ,\ 3t^2,-10t^4\ ).$$

3. Time-Derivative of a vector on a Rotating Frame (Optional)

Let $\vec{u}$ (t) and $\vec{u}_0(t)$ be the vectors respectively in world and in a rotating body frame coordinates, such that $\vec{u}(t)=R(t)\cdot\vec{u}_0(t)$. If the body frame coordinate rotates with angular velocity $\vec{\omega}$,then the time derivative of $\vec{u}$ (t) in a fixed coordinate system (world) is related to its time derivative in a rotating frame(body frame) by the following equation : $\boxed{\left(\frac{d\vec{u}}{dt}\right)_{world}=\left(\frac{d\vec{u}}{dt}\right)_{rot}+\vec{\omega}\times\vec{u}=\frac{D\vec{u}}{Dt}+\vec{\omega}\times\vec{u}}$, where $\left(\frac{d\vec{u}}{dt}\right)_{rot}$ = time derivative of $\vec{u}(t)$ in the rotating frame (rigidbody space), $\left(\frac{d\vec{u}}{dt}\right)_{rot}=\left(\frac{d\vec{u}}{dt}\right)_{body}=\frac{D\vec{u}}{Dt}=R\frac{d\vec{u}_0(t)}{dt}$, $R=R(t)$ is body space world orientation.

If $\vec{u}_0(t)$ is constant vector with respect to the time in body space, then $\left(\frac{d\vec{u}}{dt}\right)_{rot}=\vec{0}$ and the above equation becomes $\boxed{\left(\frac{d\vec{u}}{dt}\right)_{world}=\vec{\omega}\times\vec{u}}$. note that $\left(\frac{d\vec{u}}{dt}\right)_{rot}=\left(\frac{d\vec{u}}{dt}\right)_{body}$

Example 3.1 : A vector $\vec{u}$ with world coordinate $\vec{u}$ =(1,0,1) is on a rotating disk with $\vec{\omega}$ (0,1,0)rad/s

a) Find $\frac{d\vec{u}}{dt}$ if its coordinates are constant in body space(disk frame)

$$\left(\frac{d\vec{u}}{dt}\right)_{world}=\vec{\omega}\times\vec{u}=\vec{j}\times\left(\vec{i}+\vec{k}\right)=\vec{i}-\vec{k}=(1,0,-1)\ \text{ since }\ \left(\frac{d\vec{u}}{dt}\right)_{rot}=\vec{0}\ \text{ from }\ \vec{u}\ \text{ in body frame.}$$

b) Find $\vec{u}$ (world coordinate) and $\frac{d\vec{u}}{dt}$ if its coordinates are $\vec{u}_0=(t^2,t,3)$ in body space (disk frame) whose orientation is $R_z\left(\pi/2\right)$.

$$\vec{u}=R_z(\pi/2)\vec{u}_0=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}t^2\\t\\3\end{pmatrix}=\begin{pmatrix}-t\\t^2\\3\end{pmatrix}$$

$$\frac{D\vec{u}}{Dt}=R\frac{d\vec{u}_0(t)}{dt}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\cdot\frac{d}{dt}\begin{pmatrix}t^2\\t\\3\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}2t\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\2t\\0\end{pmatrix},\qquad \vec{\omega}\times\vec{u}=(3,0,t)$$

$$\left(\frac{d\vec{u}}{dt}\right)_{world}=\frac{D\vec{u}}{Dt}+\vec{\omega}\times\vec{u}=\vec{\omega}\times\vec{u}=(-1,2t,0)+(3,0,t)=(2,2t,t).$$

4. Partial Differentiation

let f(x,y) be a function of 2 variables, then the partial derivative with respect to x is

$\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=f_x$, similarly a partial derivative with respect to y is

$$\frac{\partial f}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=f_y$$

For higher order and mixed derivatives we have .

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=f_{xx}\quad,\quad \frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=f_{yy}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=(f_y)_x\quad,\quad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=(f_x)_y\ ,$$

$$f_{xyz}=(f_{xy})_z=((f_x)_y)_z=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)=\frac{\partial^3 f}{\partial z\partial y\partial x}$$

Note that $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ if f(x,y) has continuous second partial derivatives.

Example: if $f(x,y)=4x^3y^2-3x^2+y+5$, $\frac{\partial f}{\partial x}=12x^2y^2-6x$ and $\frac{\partial f}{\partial y}=8x^3y+1$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(8x^3y+1)=24x^2y$$

Theorem: let f=f(x,y,z,t) be a scalar function that depends on the variables x=x(t),y=y(t) z=z(t)

and the parameter t ; then the derivative of f with respect to t is

$$\boxed{\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}+\frac{\partial f}{\partial t}}$$

And the differential of f is $\boxed{\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \frac{\partial f}{\partial t}\Delta t}$

Example: find $\frac{df}{dt}$ if f(x,y,z,t)= $x^2 + y^3 + z^2 + t^3$ where x(t)=t^2+1 y(t)= t-2t^2 z(t)=t

$\frac{\partial f}{\partial x}$=2x , $\frac{\partial f}{\partial y}$=3$y^2$, $\frac{\partial f}{\partial z}$=2z , $\frac{\partial f}{\partial t}$=3$t^2$ $\frac{dx}{dt}$=2t , $\frac{dy}{dt} = 1-4t$, $\frac{dz}{dt}$=1

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$ =2x(2t)+ 3y^2(1-4t)+2z(1)+3t^2

=4xt + 3y^2(1-4t)+2z+3t^2

5. <u>The Gradient of a Scalar Field</u>

Given a scalar field $f(x,y,z)$ with existing and continuous partial derivatives, we define the gradient of $f(x,y,z)$ as $\boxed{\overrightarrow{grad}(f) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}$. We can also write $\overrightarrow{grad}(f) = \vec{\nabla} f$ = $\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) f$ where $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ is called the “del” or “gradient” or vector “nabla” operator.

Physical interpretation: The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase. The gradient can also be used to measure how a scalar field changes in a given direction $\hat{v}$ by taking a dot product

$\left(\vec{\nabla} \cdot f\right) \bullet \hat{v}$ = $\overrightarrow{grad}(f) \bullet \hat{v}$

Example 5.1 : Calculate $\overrightarrow{grad}(f) = \vec{\nabla} f$ if $f\left(x, y, z,\right) = x^2 y + y^3 + z^2$

Answer : $\frac{\partial f}{\partial x} = 2xy$; $\frac{\partial f}{\partial y} = x^2 + 3y^2$; $\frac{\partial f}{\partial z} = 2z$ therefore

$$\overrightarrow{grad}(f) = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = 2xy\vec{i} + (x^2 + 3y^2)\vec{j} + 2z\vec{k}$$

or $\overrightarrow{grad}(f) = \left(2xy\ ,\ x^2 + 3y^2,\ 2z\right)$.

Example 5.2 : Calculate $\overrightarrow{grad}(f) = \vec{\nabla} f$ if $f\left(x, y, z,\right) = xy + z$

Answer : $\frac{\partial f}{\partial x} = y$; $\frac{\partial f}{\partial y} = x$; $\frac{\partial f}{\partial z} = 1$ therefore

$$\overrightarrow{grad}(f) = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = y\vec{i} + x\vec{j} + \vec{k}$$

or $\overrightarrow{grad}(f) = \left(y\ ,\ x,\ 1\right)$.

TODO: Go to Activity and solve question 7

In Engineering and Physics , gradient vector is used in many physical laws such as :

- Force field $\vec{f}\left(x,y,z\right)$ and potential energy $u\left(x,y,z\right)$: $\vec{f}(x,y,z)=-\vec{\nabla}u(x,y,z)$
- Electric field $\vec{E}\left(x,y,z\right)$ and electric potential $V\left(x,y,z\right)$: $\vec{E}(x,y,z)=-\vec{\nabla}V(x,y,z)$
- Heat flow $\vec{H}\left(x,y,z\right)$ and temperature $T\left(x,y,z\right)$: $\vec{H}=-k\nabla T$,
  $k = thermal\, conductivity$

Example 5.3: Calculate the force field $\vec{f}=-\vec{\nabla}u$ if the potential energy field is $u\left(x,y,z\right)=x^2+y^2+z^2$

Answer:

$$\frac{\partial u}{\partial x}=2x \;;\; \frac{\partial u}{\partial y}=2y \;;\; \frac{\partial u}{\partial z}=2z \quad \text{therefore}$$

$$\vec{\nabla}u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}=2x\vec{i}+2y\vec{j}+2z\vec{k}$$

So $\vec{f}=-\vec{\nabla}u=-2x\vec{i}-2y\vec{j}-2z\vec{k}$

Example 5.4: Calculate the force field $\vec{E}=-\vec{\nabla}V$ if the potential energy field is $V\left(x,y,z\right)=2x-y$

Answer:

$$\frac{\partial V}{\partial x}=2 \;;\; \frac{\partial V}{\partial y}=-1 \;;\; \frac{\partial V}{\partial z}=0 \quad \text{therefore}$$

$$\vec{\nabla}V=\frac{\partial V}{\partial x}\vec{i}+\frac{\partial V}{\partial y}\vec{j}+\frac{\partial V}{\partial z}\vec{k}=2\vec{i}-\vec{j}$$

So $\vec{E}=-\vec{\nabla}V=-2\vec{i}+\vec{j}$

Example 5.5 Calculate the heat flow $\vec{H}=-k\nabla T$ if the temperature field is $T\left(x,y,z\right)=5+xyz$, $k=10$

Answer:

$$\frac{\partial T}{\partial x}=yz \;;\; \frac{\partial T}{\partial y}=xz \;;\; \frac{\partial T}{\partial z}=xy \quad \text{therefore}$$

$$\vec{\nabla}T=\frac{\partial T}{\partial x}\vec{i}+\frac{\partial T}{\partial y}\vec{j}+\frac{\partial T}{\partial z}\vec{k}=yz\vec{i}+xz\vec{j}+xy\vec{k}$$

So $\vec{H}=-k\vec{\nabla}T=-10\left(yz\vec{i}+xz\vec{j}+xy\vec{k}\right)$

TODO: Go to Activity and solve question 8

Convective operator $\vec{u}\bullet\vec{\nabla}$ and Convective Derivative

Another important operator from the gradient operator is the convective operator $\left(\vec{u}\bullet\vec{\nabla}\right)$

$\vec{u}\bullet\vec{\nabla}=u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}$, or $\vec{u}\bullet\vec{\nabla}=u_x\frac{\partial(\;)}{\partial x}+u_y\frac{\partial(\;)}{\partial y}+u_z\frac{\partial(\;)}{\partial z}$ where $\vec{u}=(u_x,u_y,u_z)$ is the velocity field .

The convective derivative, denoted $\frac{D}{Dt}$, also called material derivative is the derivative with respect to a moving coordinate system of a physical entity (temperature, pressure, density) of a material (fluid) subject to a space-time dependent velocity field $\vec{u}=\left(\vec{x},t\right)$.We write $\frac{D}{Dt}(\;)=\frac{\partial}{\partial t}(\;)+\left(\vec{u}\cdot\vec{\nabla}\right)(\;)$

Some interesting physical entities are : Temperature $T=T\left(\vec{x},t\right)$ with $\frac{DT}{Dt}=\frac{\partial T}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)T$,

pressure $p=p\left(\vec{x},t\right)$ with $\frac{Dp}{Dt}=\frac{\partial p}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)p$, mass density with $\rho=\rho\left(\vec{x},t\right)$ $\frac{D\rho}{Dt}=\frac{\partial \rho}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\rho$

- **Convective Derivative of a Time-independent Scalar Field** $f=f\left(\vec{x}\right)$ **:**

Given a scalar field $f=f(\vec{x})=f(x,y,z)$, its convective derivative is

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+(\vec{u}\cdot\vec{\nabla})f=(\vec{u}\bullet\vec{\nabla})f=\left(u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}\right)f=u_x\cdot\frac{\partial f}{\partial x}+u_y\cdot\frac{\partial f}{\partial y}+u_z\cdot\frac{\partial f}{\partial z}$$

Since $f=f(\vec{x})=f(x,y,z)$ is not time-dependent , we have $\frac{\partial f}{\partial t}=0$

Example 5.6: Calculate the convective derivative of the scalar field $f(x,y,z)=x^2+y^2+xz$ if the velocity field is $\vec{u}(1,2y,2)$.

Answer:

$$\frac{Df}{Dt}=(\vec{u}\bullet\vec{\nabla})f=\left(u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}\right)f=u_x\cdot\frac{\partial f}{\partial x}+u_y\cdot\frac{\partial f}{\partial y}+u_z\cdot\frac{\partial f}{\partial z}$$

$$=u_x\cdot\frac{\partial}{\partial x}\left(x^2+y^2+xz\right)+u_y\cdot\frac{\partial}{\partial y}\left(x^2+y^2+xz\right)+u_z\cdot\frac{\partial}{\partial z}\left(x^2+y^2+xz\right)$$

$$=u_x\left(2x+z\right)+u_y\cdot\left(2y\right)+u_z\cdot\left(x\right)=(1)\left(2x+z\right)+(2y)\cdot\left(2y\right)+(2)\cdot\left(x\right)$$

$$=2x+z+4y^2+2x=4x+4y^2+z$$

Example 5.7: Calculate the convective derivative of the temperature field $T(x,y,z)=x^2+y^2+z^2$ if the velocity field is $\vec{u}=(1,2,1)$.

Answer:

$$\frac{DT}{Dt}=(\vec{u}\bullet\vec{\nabla})T=\left(u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}\right)T=u_x\cdot\frac{\partial T}{\partial x}+u_y\cdot\frac{\partial T}{\partial y}+u_z\cdot\frac{\partial T}{\partial z}$$

$$=u_x\cdot\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)+u_y\cdot\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)+u_z\cdot\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)$$

$$=u_x\cdot\left(2x\right)+u_y\cdot\left(2y\right)+u_z\cdot\left(2z\right)=(1)\cdot\left(2x\right)+(2)\cdot\left(2y\right)+(1)\cdot\left(2z\right)=2x+4y+2z$$

TODO: Go to Activity and solve question 9

- **Convective Derivative of a Time-Dependent Scalar Field** $f = f(\vec{x},t)$

Example 5.6: Calculate the convective derivative of $f(\vec{x},t) = f(x,y,z,t) = x^2 + y^2 + xz + 2t$ if the velocity field is $\vec{u}(1,2y,2)$.

Answer:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{u}\bullet\vec{\nabla})f = 2 + \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)f = 2 + u_x\cdot\frac{\partial f}{\partial x} + u_y\cdot\frac{\partial f}{\partial y} + u_z\cdot\frac{\partial f}{\partial z}$$

$$= 2 + u_x\cdot\frac{\partial}{\partial x}\left(x^2+y^2+xz+2t\right) + u_y\cdot\frac{\partial}{\partial y}\left(x^2+y^2+xz+2t\right) + u_z\cdot\frac{\partial}{\partial z}\left(x^2+y^2+xz+2t\right)$$

$$= 2 + 2x + z + 4y^2 + 2x = 2 + 4x + 4y^2 + z$$

- **Convective Derivative of a Time-Independent Vector field** $\vec{F}$

Given a vector field $\vec{F}$, its convective derivative is

$$\frac{D\vec{F}}{Dt} = \frac{\partial\vec{F}}{\partial t} + (\vec{u}\bullet\vec{\nabla})\vec{F} = (\vec{u}\bullet\vec{\nabla})\vec{F} = u_x\cdot\frac{\partial\vec{F}}{\partial x} + u_y\cdot\frac{\partial\vec{F}}{\partial y} + u_z\cdot\frac{\partial\vec{F}}{\partial z}$$

where $\vec{u}\bullet\vec{\nabla} = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$ is our convective derivative operator, and $\frac{\partial\vec{F}}{\partial t} = \vec{0}$ since $\vec{F}$ is not time t dependent.

Example 5.8: Let $\vec{F} = (x+y^2, y, xz)$, calculate $\frac{D\vec{F}}{Dt}$ if the velocity field is $\vec{u}(x^2, y, 5z)$

Answer: with $\frac{\partial\vec{F}}{\partial t} = \vec{0}$ $\qquad \frac{D\vec{F}}{Dt} = \frac{\partial\vec{F}}{\partial t} + (\vec{u}\bullet\vec{\nabla})\vec{F} = (\vec{u}\bullet\vec{\nabla})\vec{F}$

$$\frac{D\vec{F}}{Dt} = (\vec{u}\bullet\vec{\nabla})\vec{F} = \left(u_x\cdot\frac{\partial}{\partial x} + u_y\cdot\frac{\partial}{\partial y} + u_z\cdot\frac{\partial}{\partial z}\right)\vec{F} = u_x\cdot\frac{\partial\vec{F}}{\partial x} + u_y\cdot\frac{\partial\vec{F}}{\partial y} + u_z\cdot\frac{\partial\vec{F}}{\partial z}$$

$$= x^2\bullet\frac{\partial}{\partial x}(x+y^2, y, xz) + y\bullet\frac{\partial}{\partial y}(x+y^2, y, xz) + 5z\bullet\frac{\partial}{\partial z}(x+y^2, y, xz)$$

$$= x^2\cdot(1,0,z) + y\cdot(2y,1,0) + 5z\cdot(0,0,x) = (x^2,0,x^2z) + (2y^2,y,0) + (0,0,5xz)$$

$$= (x^2+2y^2, y, x^2z+5xz)$$

Example 5.9: Let $\vec{a} = (x^2, y, 5z)$, find $\frac{D\vec{a}}{Dt} = (\vec{u}\bullet\nabla)\vec{a}$ if the velocity field is $\vec{u} = (x, 3, y)$

Answer: $$\frac{D\vec{a}}{Dt} = (\vec{u}\bullet\vec{\nabla})\vec{a} = \left(u_x\cdot\frac{\partial}{\partial x} + u_y\cdot\frac{\partial}{\partial y} + u_z\cdot\frac{\partial}{\partial z}\right)\vec{a} = u_x\cdot\frac{\partial\vec{a}}{\partial x} + u_y\cdot\frac{\partial\vec{a}}{\partial y} + u_z\cdot\frac{\partial\vec{a}}{\partial z}$$

$$= x\cdot\frac{\partial}{\partial x}(x^2,y,5z) + 3\cdot\frac{\partial}{\partial y}(x^2,y,5z) + y\cdot\frac{\partial}{\partial z}(x^2,y,5z)$$

$$= x\cdot(2x,0,0) + 3\cdot(0,1,0) + y\cdot(0,0,5) = (2x^2,0,0) + (0,3,0) + (0,0,5y) = (2x^2,3,5y)$$

- **Convective Derivative of a Time-Dependent Vector field $\vec{F}$.**

  If a vector field $\vec{F} = \vec{F}(\vec{x}, t)$ is time-dependent then $\dfrac{D\vec{F}}{Dt} = \dfrac{\partial \vec{F}}{\partial t} + (\vec{u} \bullet \vec{\nabla})\vec{F}$

  Example 5.10: Let $\vec{a} = (x^2, y,\ 5z + t^2)$, find $\dfrac{D\vec{a}}{Dt}$ if the velocity field is $\vec{u} = (x, 3, y)$

  Answer :

$$\frac{D\vec{a}}{Dt} = \frac{\partial \vec{a}}{\partial t} + (\vec{u} \bullet \vec{\nabla})\vec{a} = \frac{\partial \vec{a}}{\partial t} + \left(u_x \cdot \frac{\partial}{\partial x} + u_y \cdot \frac{\partial}{\partial y} + u_z \cdot \frac{\partial}{\partial z} \right)\vec{a} = \frac{\partial \vec{a}}{\partial t} + u_x \cdot \frac{\partial \vec{a}}{\partial x} + u_y \cdot \frac{\partial \vec{a}}{\partial y} + u_z \cdot \frac{\partial \vec{a}}{\partial z}$$

$$= (0, 0, 2t) + x \cdot \frac{\partial}{\partial x}(x^2, y,\ 5z + t^2) + 3 \cdot \frac{\partial}{\partial y}(x^2, y,\ 5z + t^2) + y \cdot \frac{\partial}{\partial z}(x^2, y,\ 5z + t^2)$$

$$= (0, 0, 2t) + x \cdot (2x, 0, 0) + 3 \cdot (0, 1, 0) + y \cdot (0, 0, 5)$$

$$= (0, 0, 2t) + (2x^2, 0, 0) + (0, 3, 0) + (0, 0, 5y) = (2x^2, 3,\ 5y + 2t)$$

6. The Curl of a Vector Field

If $\vec{u} = (u_x, u_y, u_z) = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$ then the Curl of $\vec{u}$ is

Curl $\vec{u}$ = $\vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$ with $\vec{\nabla} = \dfrac{\partial}{\partial x}\vec{i} + \dfrac{\partial}{\partial y}\vec{j} + \dfrac{\partial}{\partial z}\vec{k}$

Physical interpretation: if $\vec{u}$ represents the velocity field of a flowing fluid at a point $\vec{p}(x, y, z)$, then Curl $\vec{u}$ represents the measure of the fluid tendency to rotate about an axis that has the same direction as Curl $\vec{u}$.

Example 6.1: Given $\vec{u}(u_x, u_y, u_z) = 3x^2 y \cdot \vec{i} + xy \cdot \vec{j} + z^3\vec{k}$, compute $curl\,\vec{u} = \vec{\nabla} \times \vec{u}$.

➔ Curl $\vec{u}$ = $\vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_y & u_z \end{vmatrix}\vec{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ u_x & u_z \end{vmatrix}\vec{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ u_x & u_y \end{vmatrix}\vec{k}$

$= \left(\dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} \right)\vec{i} - \left(\dfrac{\partial u_z}{\partial x} - \dfrac{\partial u_x}{\partial z} \right)\vec{j} + \left(\dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \right)\vec{k}$, with $u_x = 3x^2 y$, $u_y = xy$, $and\ \ u_z = z^3$

$= \left(\dfrac{\partial(z^3)}{\partial y} - \dfrac{\partial(xy)}{\partial z} \right)\vec{i} - \left(\dfrac{\partial(z^3)}{\partial x} - \dfrac{\partial(3x^2 y)}{\partial z} \right)\vec{j} + \left(\dfrac{\partial(xy)}{\partial x} - \dfrac{\partial(3x^2 y)}{\partial y} \right)\vec{k}$

$= (0 - 0)\vec{i} - (0 - 0)\vec{j} + \left(y - 3x^2\right)\vec{k} = \left(y - 3x^2\right)\vec{k} = (0, 0, y - 3x^2)$

Example 6.2: Given $\vec{u} = xz\vec{i} + 5y\vec{j} + xz^2\vec{k}$, compute $curl\,\vec{u} = \vec{\nabla} \times \vec{u}$.

Answer : $\vec{u} = xz\vec{i} + 5y\vec{j} + xz^2\vec{k}$ ➔ $u_x = xz,\ u_y = 5y \ and \ u_z = xz^2$

$$\text{Curl } \vec{u} = \vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & 5y & xz^2 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5y & xz^2 \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz & xz^2 \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & 5y \end{vmatrix} \vec{k}$$

$$= \left(\frac{\partial (xz^2)}{\partial y} - \frac{\partial (5y)}{\partial z} \right) \vec{i} - \left(\frac{\partial (xz^2)}{\partial x} - \frac{\partial (xz)}{\partial z} \right) \vec{j} + \left(\frac{\partial (5y)}{\partial x} - \frac{\partial (xz)}{\partial y} \right) \vec{k}$$

$$= (0-0)\vec{i} - \left(z^2 - x\right)\vec{j} + (0-0)\vec{k} = \left(x - z^2\right)\vec{j} = \left(0, x - z^2, 0\right)$$

Example 6.3: Given $\vec{u} = x^2z\vec{i} + 3xy\vec{j} + yz\vec{k}$, compute $curl\,\vec{u} = \vec{\nabla} \times \vec{u}$.

Answer: $\vec{u} = x^2z\vec{i} + 3xy\vec{j} + yz\vec{k}$ ➔ $u_x = x^2z,\ u_y = 3xy \ and \ u_z = yz$

$$\text{Curl } \vec{u} = \vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & 3xy & yz \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & yz \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2z & yz \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2z & 3xy \end{vmatrix} \vec{k}$$

$$= \left(\frac{\partial (yz)}{\partial y} - \frac{\partial (3xy)}{\partial z} \right) \vec{i} - \left(\frac{\partial (yz)}{\partial x} - \frac{\partial (x^2z)}{\partial z} \right) \vec{j} + \left(\frac{\partial (3xy)}{\partial x} - \frac{\partial (x^2z)}{\partial y} \right) \vec{k}$$

$$= (z-0)\vec{i} - \left(0 - x^2\right)\vec{j} + (3y-0)\vec{k} = z\vec{i} + x^2\vec{j} + 3y\vec{k} = \left(z, x^2, 3y\right)$$

Example 6.4: Given $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$, compute $curl\,\vec{u} = \vec{\nabla} \times \vec{u}$.

Answer: : $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$ ➔ $u_x = y,\ u_y = z \ and \ u_z = x$

$$\text{Curl } \vec{u} = \vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & x \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & z \end{vmatrix} \vec{k}$$

$$= \left(\frac{\partial (x)}{\partial y} - \frac{\partial (z)}{\partial z} \right) \vec{i} - \left(\frac{\partial (x)}{\partial x} - \frac{\partial (y)}{\partial z} \right) \vec{j} + \left(\frac{\partial (z)}{\partial x} - \frac{\partial (y)}{\partial y} \right) \vec{k}$$

$$= (0-1)\vec{i} - (1-0)\vec{j} + (0-1)\vec{k} = -\vec{i} - \vec{j} - \vec{k} = (-1, -1, -1)$$

TODO: Go to Activity and solve question 10

TODO: Go to Activity and solve question 11

7. <u>The Divergence of a Vector Field</u>

The divergence of a vector field is defined as the dot product of the del $\vec{\nabla}$ operator and the vector field $\vec{u} = (u_x, u_y, u_z) = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$,that is

$$\boxed{\vec{\nabla}\bullet\vec{u} = div\,\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}}$$

Physical interpretation: if $\vec{u}$ represents the velocity field of a flowing fluid, then div $\vec{u}$ represents the net rate of change of the fluid mass flowing out (div $\vec{u}$ (p)>0) or sinking in (div $\vec{u}$ (p)<0) at a point p(x,y,z).It's the measure of the fluid compressibility(measure of relative fluid volume change as a response to pressure change).

If div $\vec{u}$ =0 ➔ fluid is said to incompressible.

Example 7.1: Calculate $div\,\vec{u}$ if the velocity field is $\vec{u} = 2z\vec{i} + y\vec{j} + x^2\vec{k} = (2z, y, x^2)$

Answer: $\vec{\nabla}\bullet\vec{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\bullet\left(u_x, u_y, u_z\right) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial(2z)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(x^2)}{\partial z} = 1$

$div\,\vec{u} > 0$ ➔ fluid mass flowing out at constant rate

Example 7.2: Calculate $div\,\vec{u}$ if the velocity field is $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$

Answer: $\vec{\nabla}\bullet\vec{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\bullet\left(y, z, x\right) == \frac{\partial(y)}{\partial x} + \frac{\partial(z)}{\partial y} + \frac{\partial(x)}{\partial z} = 0$

$div\,\vec{u} = 0$ ➔ fluid is incompressible

Example 7.3 : Calculate $div\,\vec{u}$ if the velocity field is $\vec{u} = xz\vec{i} + 5y\vec{j} + xz^2\vec{k}$.

Is the flow a source or a sink at the point at the point $\vec{p} = (1, -2, -5)$

Answer: $\vec{\nabla}\bullet\vec{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\bullet\left(xz, 5y, xz^2\right) = \frac{\partial(xz)}{\partial x} + \frac{\partial(5y)}{\partial y} + \frac{\partial(xz^2)}{\partial z} = z + 5 + 2xz$

$\vec{\nabla}\vec{u}(\vec{p}) = \vec{\nabla}\vec{u}(1, -2, -5) = (-5) + 5 + 2(1)(-5) = -10 < 0$ ➔ we have a sink at $\vec{p}$ (1,-2,-5)

TODO: Go to Activity and solve question 12

TODO: Go to Activity and solve question 13

8. <u>The Laplacian of a Scalar or Vector Field</u>

Given a scalar field $f(x, y, z)$ with existing and continuous partial derivatives, we define

the Laplacian of f by $\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. From the previous formula, one can see that

the Laplacian operator is $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

For a vector field $\vec{u} = (u_x, u_y, u_z) = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$ we define the Laplacian of $\vec{u}$ by

$\boxed{\vec{\nabla}^2\vec{u} = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{u}\right) = \left(\vec{\nabla}^2 u_x\right)\cdot\vec{i} + \left(\vec{\nabla}^2 u_y\right)\cdot\vec{j} + \left(\vec{\nabla}^2 u_z\right)\cdot\vec{k}}$ that is a vector. $\vec{\nabla}^2\vec{u} = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{u}\right) = \overrightarrow{grad}\left(div\,\vec{u}\right)$

.

Example 8.1: Given the scalar field $f\left(x,y,z\right)= x^2y+ y^3+ z^2$ find Laplacian of $f\left(x,y,z\right)$

Answer: $\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(2xy\right)=2y$, $\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(x^2+ 3y^2\right)=6y$,

$$\frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial z}\left(2z\right)=2$$

So the Laplacian is $\vec{\nabla}^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=2y+6y+2= 8y + 2$

Example 8.2: Given the scalar field $f\left(x,y,z\right)=x^2+3y^2-z^2$ find Laplacian of $f\left(x,y,z\right)$

Answer: $\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(2x\right)=2$, $\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(6y\right)=6$,

$$\frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial z}\left(-2z\right)=-2$$

So the Laplacian is $\vec{\nabla}^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=2+6-2= 6$

TODO: Go to Activity and solve question 14

TODO: Go to Activity and solve question 15

Example 8.3: Given the vector field $\vec{u}(u_x,u_y,u_z)=(3x^2y,\ xy,z^3)$ find Laplacian of $\vec{u}$, $\vec{\nabla}^2\vec{u}$.

Answer : here we have $u_x=3x^2y$, $u_y=xy \ and \ u_z=z^3$, $\vec{\nabla}^2\vec{u}=\left(\vec{\nabla}^2u_x\right)\cdot\vec{i}+\left(\vec{\nabla}^2u_y\right)\cdot\vec{j}+\left(\vec{\nabla}^2u_z\right)\cdot\vec{k}$

With $u_x=3x^2y$

$$\vec{\nabla}^2u_x=\frac{\partial^2u_x}{\partial x^2}+\frac{\partial^2u_x}{\partial y^2}+\frac{\partial^2u_x}{\partial z^2}=\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}\right)=\frac{\partial}{\partial x}\left(6xy\right)+\frac{\partial}{\partial y}\left(3x^2\right)+\frac{\partial}{\partial z}\left(0\right)$$
$$=6y+0+0=6y$$

With $u_y=xy$

$$\vec{\nabla}^2u_y=\frac{\partial^2u_y}{\partial x^2}+\frac{\partial^2u_y}{\partial y^2}+\frac{\partial^2u_y}{\partial z^2}=\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial z}\right)=\frac{\partial}{\partial x}\left(y\right)+\frac{\partial}{\partial y}\left(x\right)+\frac{\partial}{\partial z}\left(0\right)$$
$$=0+0+0=0$$

$u_z=z^3$

$$\vec{\nabla}^2u_z=\frac{\partial^2u_z}{\partial x^2}+\frac{\partial^2u_z}{\partial y^2}+\frac{\partial^2u_z}{\partial z^2}=\frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial z}\right)=\frac{\partial}{\partial x}\left(0\right)+\frac{\partial}{\partial y}\left(0\right)+\frac{\partial}{\partial z}\left(3z^2\right)$$
$$=0+0+6z=6z$$

So the Laplacian of vector field $\vec{u}$ is $\vec{\nabla}^2\vec{u}=\left(\vec{\nabla}^2u_x\right)\cdot\vec{i}+\left(\vec{\nabla}^2u_y\right)\cdot\vec{j}+\left(\vec{\nabla}^2u_z\right)\cdot\vec{k}=6y\vec{i}+6z\vec{k}$

$\vec{\nabla}^2\vec{u}=(6y,0,6z)$.

Example 8.4: Given the vector field $\vec{u} = x^2\vec{i} + xy\vec{j} - 3xz\vec{k} = (x^2 , xy , -3xz^2)$ find Laplacian of $\vec{u}$, $\vec{\nabla}^2\vec{u}$.

Answer :

here we have $u_x = x^2$, $u_y = xy$ and $u_z = -3xz^2$, $\vec{\nabla}^2\vec{u} = \left(\vec{\nabla}^2 u_x\right)\cdot\vec{i} + \left(\vec{\nabla}^2 u_y\right)\cdot\vec{j} + \left(\vec{\nabla}^2 u_z\right)\cdot\vec{k}$

With $u_x = x^2$

$$\vec{\nabla}^2 u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}\right) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)$$

$$= 2 + 0 + 0 = 2$$

With $u_y = xy$

$$\vec{\nabla}^2 u_y = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial z}\right) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0)$$

$$= 0 + 0 + 0 = 0$$

With $u_z = -3xz^2$

$$\vec{\nabla}^2 u_z = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial z}\right) = \frac{\partial}{\partial x}(-3z) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-6xz)$$

$$= 0 + 0 - 6x = -6x$$

So the Laplacian of vector field $\vec{u}$ is $\vec{\nabla}^2\vec{u} = \left(\vec{\nabla}^2 u_x\right)\cdot\vec{i} + \left(\vec{\nabla}^2 u_y\right)\cdot\vec{j} + \left(\vec{\nabla}^2 u_z\right)\cdot\vec{k} = -6x\vec{k} = (0, 0, -6x)$

$\vec{\nabla}^2\vec{u} = (0, 0, -6x)$.

TODO: Go to Activity and solve question 16

TODO: Go to Activity and solve question 17

Useful Gradient ,Divergent and Curl Properties

- $\vec{\nabla}\bullet(\vec{\nabla}\times\vec{u}) = 0$, div(curl $\vec{u}$)=0
- $\vec{\nabla}\times\vec{\nabla}f = \vec{0}$, curl(grad f)= $\vec{0}$
- $\vec{\nabla}\bullet\vec{\nabla}f = \vec{\nabla}^2 f$, div(grad f)= Laplacian f
- $\vec{\nabla}\times(f\cdot\vec{u}) = f\cdot(\vec{\nabla}\times\vec{u}) + (\vec{\nabla}f)\times\vec{u}$
- $\vec{\nabla}\bullet(\vec{u}\cdot\vec{v}^t) = \vec{v}(\vec{\nabla}\bullet\vec{u}) + (\vec{u}\bullet\vec{\nabla})\vec{v}$
- $\vec{\nabla}\bullet(f\cdot\vec{u}) = f\cdot\vec{\nabla}\bullet\vec{u} + \vec{u}\bullet\vec{\nabla}f$
- $\vec{\nabla}(f\cdot g) = g\cdot\vec{\nabla}f + f\cdot\vec{\nabla}g$
- $(\vec{u}\bullet\vec{\nabla})\vec{u} = (\vec{\nabla}\times\vec{u})\times\vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})$

9. Vector Functions

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{pmatrix}$ a $n \times 1$ column vector and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_m \end{pmatrix}$ $m \times 1$ column vector,

we define the vector function $\vec{y}$ as function $\vec{x}$ to be $\vec{y} = \vec{y}(\vec{x})$.

Example 9.1:

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad \text{where} \quad \begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = x_1^2 - x_2 \\ y_3 = x_2 \\ y_4 = x_1 x_2 \end{cases} \quad \text{with} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In this note we will use a numerator layout; that is a layout according to $\vec{y}$ and $\vec{x}^T$ (transpose of $\vec{x}$)also known as the Jacobian formulation . That is vectors are defined as column vectors like $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ and

$$\frac{\partial \vec{y}}{\partial \vec{x}} = \frac{\partial \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}{\partial \begin{bmatrix} x_1 & x_2 \end{bmatrix}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} \end{pmatrix}$$, the order of the resulting matrix in general is :

order of ($\vec{y}$) by order of ($\vec{x}^T$)=($m \times 1$) by ($1 \times n$)= $m \times n$.

So from the above example , the matrix obtained has order $(4 \times 1) \times (1 \times 2) = 4 \times 2$.

10. <u>Derivative of a vector with respect to a vector.</u>

The derivative of a $m \times 1$ vector $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ with respect to a $n \times 1$ vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is

the $m \times n$ matrix $\dfrac{\partial \vec{y}}{\partial \vec{x}} = \dfrac{\partial \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}{\partial \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdot\ \cdot\ \cdot & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdot\ \cdot\ \cdot & \frac{\partial y_2}{\partial x_n} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdot\ \cdot\ \cdot & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$ known as the matrix Jacobian.

<u>Example 10.1</u>: if $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ where $\begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = x_1^2 - x_2 \\ y_3 = x_2 \\ y_4 = x_1 x_2 \end{cases}$ with $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, calculate $\dfrac{\partial \vec{y}}{\partial \vec{x}}$.

Answer:

With $\begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = x_1^2 - x_2 \\ y_3 = x_2 \\ y_4 = x_1 x_2 \end{cases}$, $\dfrac{\partial \vec{y}}{\partial \vec{x}} = \dfrac{\partial \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}{\partial \begin{bmatrix} x_1 & x_2 \end{bmatrix}} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2x_1 & -1 \\ 0 & 1 \\ x_2 & x_1 \end{pmatrix}$

Example 10.2: Calculate $\frac{\partial \vec{w}}{\partial \vec{u}}$, if $\vec{w}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ where $\begin{cases} w_1 = xy+z \\ w_2 = x^2+y^2+z \\ w_3 = 2x+y+z^3 \end{cases}$ with $\vec{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Answer:

This will result into a 3x1 by 1x3=3x3 matrix

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \frac{\partial \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}}{\partial \begin{bmatrix} x & y & z \end{bmatrix}} = \begin{pmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} & \frac{\partial w_1}{\partial z} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} & \frac{\partial w_2}{\partial z} \\ \frac{\partial w_3}{\partial x} & \frac{\partial w_3}{\partial y} & \frac{\partial w_3}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x & 1 \\ 2x & 2y & 1 \\ 2 & 1 & 3z^2 \end{pmatrix}$$

Example 10.3: Calculate $\frac{\partial \vec{w}}{\partial \vec{u}}$, if $\vec{w}=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ where $\begin{cases} w_1 = 3x-y+z \\ w_2 = x+y+5z \end{cases}$ with $\vec{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Answer:

This will result into a 2x1 by 1x3=2x3 matrix

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \frac{\partial \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}{\partial \begin{bmatrix} x & y & z \end{bmatrix}} = \begin{pmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} & \frac{\partial w_1}{\partial z} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} & \frac{\partial w_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

TODO: Go to Activity and solve question 18
TODO: Go to Activity and solve question 19
TODO: Go to Activity and solve question 20

11. Derivative of a scalar s with respect to a vector.

The derivative of scalar value $s=s(\vec{x})$ with respect to a $n\times 1$ vector $\vec{x}$ is the $1\times n$ row vector

$$\frac{\partial s}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial s}{\partial x_1} & \frac{\partial s}{\partial x_2} & . & . & . & \frac{\partial s}{\partial x_n} \end{bmatrix}$$

Example 11.1: if $s=s(\vec{x})=(x_1-a)^2+(x_2-b)^2+(x_3-c)^2-4$ where $\vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, calculate $\frac{\partial s}{\partial \vec{x}}$.

Answer: This will result into a 1x1 by 1x3= 1x3 row vector

$$\frac{\partial s}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial s}{\partial x_1} & \frac{\partial s}{\partial x_2} & \frac{\partial s}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2(x_1-a) & 2(x_2-b) & 2(x_3-c) \end{bmatrix}$$

Example 11.2: if $s = s(\vec{u}) = x + xy + y^2 + z^2$ where $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, calculate $\frac{\partial s}{\partial \vec{x}}$.

Answer:

This will result into a 1x1 by 1x3= 1x3 row vector

$$\frac{\partial s}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \end{bmatrix} = \begin{bmatrix} 1+y & x+2y & 2z \end{bmatrix}$$

Example 11.3: if $s = s(\vec{u}) = x + 2y + z$ where $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, calculate $\frac{\partial s}{\partial \vec{x}}$.

Answer:

This will result into a 1x1 by 1x3= 1x3 row vector

$$\frac{\partial s}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

TODO: Go to Activity and solve question 21

TODO: Go to Activity and solve question 22

TODO: Go to Activity and solve question 23

12. Vector Gradient or Jacobian

Let f($\vec{x}$) be a differentiable scalar function of n variables with

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{pmatrix}$, then the vector gradient of f($\vec{x}$) with respect to $\vec{x}$ is $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{x}}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & . & . & \frac{\partial f}{\partial x_n} \end{bmatrix}$

It is a 1xn row vector (important!), also called the Jacobian.

Example 12.1: : Calculate $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{x}}} f$, if $f(\vec{x}) = f(x_1, x_2, x_3) = \frac{x_1^{\ 2}}{4} + \frac{x_2^{\ 2}}{9} + \frac{x_3^{\ 2}}{25} - 1$

Answer : $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{x}}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{2} & \frac{2x_2}{9} & \frac{2x_3}{25} \end{bmatrix}$ or $\frac{x_1}{2}\hat{i} + \frac{2x_2}{9}\hat{j} + \frac{2x_3}{25}\hat{k}$

Example 12.2: Calculate $\frac{\partial f}{\partial \vec{v}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{v}}} f$ if $f(\vec{v}) = xyz$ with $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Answer: $\frac{\partial f}{\partial \vec{v}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{v}}} f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} yz & xz & xy \end{bmatrix}$ or $yz\hat{i} + xz\hat{j} + xy\hat{k}$

Example 12.3: Calculate $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \overrightarrow{\nabla_{\vec{x}}} f$ if $f(\vec{x}) = x_1 + 5x_2 - 7x_3$ with $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Answer: $\frac{\partial f}{\partial \vec{x}} = \vec{\nabla} f = \overrightarrow{\nabla_x} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 5 & -7 \end{bmatrix}$ or $\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$

13. Derivative of a vector with respect to a scalar s

Given $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_n \end{pmatrix}$, $\frac{\partial \vec{y}}{\partial s} = \begin{bmatrix} \frac{\partial y_1}{\partial s} \\ \frac{\partial y_2}{\partial s} \\ . \\ . \\ . \\ \frac{\partial y_m}{\partial s} \end{bmatrix}$

Example: if $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ where $\begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = x_1^2 - x_2 \\ y_3 = x_2 \\ y_4 = x_1 x_2 \end{cases}$

Then $\frac{\partial \vec{y}}{\partial x_1} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_3}{\partial x_1} \\ \frac{\partial y_4}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2x_2 \\ 0 \\ x_2 \end{bmatrix}$

14. Matrix formulation:

Let $\vec{a}=(a_1,a_2,a_3)$ *and* $\vec{b}=(b_1,b_2,b_3)$, in matrix form they will be column vectors $\vec{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

We want here to define the matrix expression of both the vector product and cross product in matrix form.

Dot product : $\vec{a}\cdot\vec{b}=\vec{a}^t\vec{b}=\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in matrix form

Example 14.1: if $f(x,y,z)=2x+y+3z$ with $\vec{c}=\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then in matrix form we

have $$f=f\ (\vec{v})=\vec{c}^t\vec{v}=\begin{pmatrix} 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=2x+y+3z$$

Cross product: $\vec{a}\times\vec{b}=\psi(\vec{a})\vec{b}$ in matrix form where $\psi(\vec{a})=skew(\vec{a})=\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$

Quadric form: Let $f(x,y)=ax^2+2bxy+cy^2$ then the matrix expression of the quadric form is

$f(x,y)=ax^2+2bxy+cy^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. If $\vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $=A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ then

$f(x,y)=f(\vec{x})=\vec{x}^t\cdot A\cdot\vec{x}$

Example 14.2 : Express the quadric forms in matrix form .

a) $f(x,y)=3x^2+4xy+5y^2$
b) $f(x,y)=4x^2+3xy+y^2$
c) $f(x,y,z)=x^2+6xy+4xz+3y^2+2yz+2z^2$
d) $f(x,y,z)=5x^2-2xy+8xz+3y^2+6yz+7z^2$

Answer:

a) $f(x,y)=3x^2+4xy+5y^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ or $f(x,y)=f(\vec{x})=\vec{x}^t\cdot A\cdot\vec{x}$ with

$\vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $A=\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$

b) $f(x,y)=4x^2+3xy+y^2=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ or $f(x,y)=f(\vec{x})=\vec{x}^t\cdot A\cdot\vec{x}$ with

$\vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $A=\begin{pmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$

c) $f(x,y,z) = x^2 + 6xy + 4xz + 3y^2 + 2yz + 2z^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

or $f(x,y) = f(\vec{x}) = \vec{x}^t \cdot A \cdot \vec{x}$ with $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

d) $f(x,y,z) = 5x^2 - 2xy + 8xz + 3y^2 + 6yz + 7z^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 5 & -1 & 4 \\ -1 & 3 & 3 \\ 4 & 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

or $f(x,y) = f(\vec{x}) = \vec{x}^t \cdot A \cdot \vec{x}$ with $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 5 & -1 & 4 \\ -1 & 3 & 3 \\ 4 & 3 & 7 \end{pmatrix}$

15. Identities

- If $\vec{u}$ is not a function of $\vec{x}$ then $\dfrac{\partial \vec{u}}{\partial \vec{x}} = 0_{m\times n}$ (zero matrix)

  Example 15.1: if $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\dfrac{\partial \vec{u}}{\partial \vec{x}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- $\dfrac{\partial \vec{x}}{\partial \vec{x}} = I_{n\times n}$ (identity matrix)

  Example 15.2: if $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\dfrac{\partial \vec{x}}{\partial \vec{x}} = \dfrac{\partial \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\partial \begin{bmatrix} x & y & z \end{bmatrix}} = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  Given a matrix A that is not a function of $\vec{x}$,

- $\dfrac{\partial}{\partial \vec{x}} \left(A\vec{x} \right) = A$

  Example 15.3: Given $\begin{cases} w_1 = 2x + 8y \\ w_2 = 5x + 3y \end{cases}$ ➔ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\vec{x}$

  $\dfrac{\partial}{\partial \vec{x}} \left(A\vec{x} \right) = A \dfrac{\partial \vec{x}}{\partial \vec{x}} = A \cdot I = A = \begin{pmatrix} 2 & 8 \\ 5 & 3 \end{pmatrix}$

- $\dfrac{\partial}{\partial \vec{x}} \left(\vec{x}^T A \right) = A^T$
- $\dfrac{\partial}{\partial \vec{x}} \left(c\vec{y} \right) = c \dfrac{\partial \vec{y}}{\partial \vec{x}}$ c is a scalar not function of $\vec{x}$ but $\vec{y} = \vec{y}(\vec{x})$
- $\dfrac{\partial}{\partial \vec{x}} \left(\vec{a} \bullet \vec{y} \right) = \dfrac{\partial}{\partial \vec{x}} \left(\vec{a}^t \vec{y} \right) = \vec{a}^t \dfrac{\partial \vec{y}}{\partial \vec{x}}$ $\vec{a}$ is not function of $\vec{x}$ but $\vec{y} = \vec{y}(\vec{x})$ and in matrix $\vec{a} \bullet \vec{y} = \vec{a}^t \vec{y}$

Example 15.4: let $g(x,y,z)=2x+3y+5z$, if $\vec{a}=\begin{pmatrix}2\\3\\5\end{pmatrix}$ and $\vec{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ then

$$g(x,y,z)=2x+3y+5z=\begin{pmatrix}2 & 3 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\vec{a}^t\vec{x}$$ or $g(\vec{x})=\vec{a}^t\vec{x}$ and

$$\frac{\partial}{\partial\vec{x}}(g)=\frac{\partial}{\partial\vec{x}}\left(\vec{a}^t\vec{x}\right)=\vec{a}^t\frac{\partial\vec{x}}{\partial\vec{x}}=\vec{a}^t=\begin{pmatrix}2 & 3 & 5\end{pmatrix}$$

- $\frac{\partial}{\partial\vec{x}}\left(\vec{x}^T A\vec{x}\right)=\vec{x}^T(A+A^T)$ or $\frac{\partial}{\partial\vec{x}}\left(\vec{x}^T A\vec{x}\right)=2\vec{x}^T A$ if A is symmetric ($A^T=A$)

Example 15.5 : Let $f(x,y)=3x^2+4xy+5y^2$, we want to calculate $\frac{\partial f}{\partial\vec{x}}$

$f(x,y)=3x^2+4xy+5y^2=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}3 & 2\\2 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ now we express $f(x,y)$ in term of vector $\vec{x}$, that is

$f(\vec{x})=\vec{x}^t\cdot A\cdot\vec{x}$ with $\vec{x}=\begin{pmatrix}x\\y\end{pmatrix}$ and $A=\begin{pmatrix}3 & 2\\2 & 5\end{pmatrix}$.

$\frac{\partial f(\vec{x})}{\partial\vec{x}}=\frac{\partial f}{\partial\vec{x}}\left(\vec{x}^T\cdot A\cdot\vec{x}\right)=\vec{x}^T\left(A+A^T\right)=2\vec{x}^T A$ A is symmetric ➔ $A^T=A$

$$\frac{\partial f(\vec{x})}{\partial\vec{x}}=2\vec{x}^T A=2\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}3 & 2\\2 & 5\end{pmatrix}=\begin{bmatrix}6x+4y & 4x+10y\end{bmatrix}$$

TODO: Go to Activity and solve question 24

TODO: Go to Activity and solve question 25

- $\frac{\partial(\vec{a}\bullet\vec{x})}{\partial\vec{x}}=\frac{\partial(\vec{a}^t\vec{x})}{\partial\vec{x}}=\vec{a}^t$ $\vec{a}$ is not function of $\vec{x}$ and in matrix $\vec{a}\bullet\vec{x}=\vec{a}^t\vec{x}$
- $\frac{\partial\vec{x}\bullet\vec{x}}{\partial\vec{x}}=\frac{\partial\vec{x}^T\vec{x}}{\partial\vec{x}}=\frac{\partial\vec{x}^T\cdot I\cdot\vec{x}}{\partial\vec{x}}=\vec{x}^T(\mathrm{I}+\mathrm{I}^T)=\vec{x}^T(2\,\mathrm{I})=2\vec{x}^T$
- $\frac{\partial(\vec{u}\bullet\vec{v})}{\partial\vec{x}}=\frac{\partial(\vec{u}^T\vec{v})}{\partial\vec{x}}=\vec{v}^T\frac{\partial\vec{u}}{\partial\vec{x}}+\vec{u}^T\frac{\partial\vec{v}}{\partial\vec{x}}$ where $\vec{u}=\vec{u}(\vec{x})$ and $\vec{v}=\vec{v}(\vec{x})$
- $\frac{\partial(\vec{u}+\vec{v})}{\partial\vec{x}}=\frac{\partial\vec{u}}{\partial\vec{x}}+\frac{\partial\vec{v}}{\partial\vec{x}}$
- $\frac{\partial\|\vec{x}-\vec{a}\|}{\partial\vec{x}}=\frac{(\vec{x}-\vec{a})^t}{\|\vec{x}-\vec{a}\|}$ (order =1xn) $\vec{a}$ is not function of $\vec{x}$
- $\frac{\partial(\vec{u}\times\vec{v})}{\partial\vec{v}}=\frac{\partial[\psi(\vec{u})\vec{v}]}{\partial\vec{v}}=\psi(\vec{u})$ where $\psi(\vec{u})=skew(\vec{u})=\begin{pmatrix}0 & -u_z & u_y\\u_z & 0 & -u_x\\-u_y & u_x & 0\end{pmatrix}$ skew symmetric matrix of u.

- $$\frac{\partial \|\vec{\mathrm{u}} \times \vec{v}\|}{\partial \vec{v}} = \frac{(\vec{\mathrm{u}} \times \vec{v})^T}{\|\vec{\mathrm{u}} \times \vec{v}\|} \psi(\vec{\mathrm{u}}) \quad \text{where} \quad \psi(\vec{\mathrm{u}}) = skew(\vec{\mathrm{u}}) = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

The rule is to convert the expressions in matrix form

16. Application to Non-linear Optimization with Lagrange Multipliers (optional)

17. ☺