Chapter 3 : Determinants and Eigen Space (Integrative Learning)

1) Definition

Given a matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we define the determinant of A as a real value function.

It is a scalar quantity associated with a squared matrix.

We write det(A)= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ using the absolute value sign | |

TODO➔ Go to Activity and solve question 1

2) Determinant of a 2x2 matrix

Given A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, det(A)= $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$

Example 2.1 : $\begin{vmatrix} 2 & 5 \\ 3 & 5 \end{vmatrix}$ = (2)*(5) - (3)*(5) = 10 - 15= -5

$$\begin{vmatrix} 2 & 9 \\ 1 & 10 \end{vmatrix} = (2)(10) - (1)(9) = 20 - 9 = 11$$

TODO➔ Go to Activity and solve question 2.1

3) Determinant of a 3x3 matrix

Given a matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, using the same technique to compute the cross product, but we return here a scalar instead of a vector, and $[a_{11} \quad a_{12} \quad a_{13}]$ as our target row, we get:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ + & - & + \end{vmatrix} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \cdot a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \cdot a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \cdot a_{13}$$

$$= a_{11} \cdot (a_{22} \cdot a_{33} - a_{32} \cdot a_{23}) - a_{12} \cdot (a_{21} \cdot a_{33} - a_{31} \cdot a_{23}) + a_{13} \cdot (a_{21} \cdot a_{32} - a_{31} \cdot a_{22})$$

Example 3.1: **Calculate the determinant** $\begin{vmatrix} 2 & 3 & 4 \\ 2 & 0 & 1 \\ 1 & 4 & 3 \end{vmatrix}$

Using $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ **as the target row**

$$\begin{vmatrix} 2 & 3 & 4 \\ 2 & 0 & 1 \\ 1 & 4 & 3 \\ + & - & + \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 2*(0*3-4*1)-3*(2*3-1*1)+4*(2*4-1*0)$$

$$= -8 -15 +32 = 9$$

Example 3.2: **Calculate the determinant** $\begin{vmatrix} 2 & 5 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{vmatrix}$

Using $\begin{bmatrix} 2 & 5 & -2 \end{bmatrix}$ **as the target row**

$$\begin{vmatrix} 2 & 5 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \\ + & - & + \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 2(3\times3-0\times1)-5(1\times3-1\times1)+(-2)(1\times0-1\times3)$$

$$=2(9)-5(2)-2(-3) = 18-10+6=14$$

Video reference : https://www.youtube.com/watch?v=21LWuY8i6Hw

TODO➔ Go to Activity and solve question 2.2

PYTHONIC: use Matrix.det() to compute a matrix determinant

```
import sympy  as sy
M=sy.Matrix([ [2,5,-2],[1,3,1],[1,0,3]], dtype='float')
sy.pprint(M)
#compute matrix determinant using det()
d=M.det()
print("\n Matrix determinant is {0}".format(d) )


```

```
⎡2  5  -2⎤
⎢        ⎥
⎢1  3  1 ⎥
⎢        ⎥
⎣1  0  3 ⎦

 Matrix determinant is 14
```

4) Minor determinants and cofactors

In $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we define the minor determinant at entry (i , j) to be the determinant obtained when a_{ij} is crossed out. That is the minor determinant at (1,1) is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{32} \cdot a_{23}$

The cofactor at entry (i,j) is defined as $c_{ij} = (-1)^{i+j} \bullet M_{ij}$, where M_{ij} =minor determinant at(i,j).

Example 4.1: in $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

the minor determinant at (2,3) is $M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$, and the cofactor at (2,3) is $c_{23} = (-1)^{2+3} \bullet M_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (-1) \cdot (3-4) = 1$; that is i=2 , j=3 in $c_{ij} = (-1)^{i+j} \bullet M_{ij}$.

the minor at (1,3) is $M_{13} = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 12 - 10 = 2$

and the cofactor at (1,3) is $c_{13} = (-1)^{1+3} \bullet M_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = (+1) \cdot (12-10) = 2$

TODO➔ Go to Activity and solve questions 3.1 and 3.2

Video Reference: https://www.youtube.com/watch?v=EcI4E15ElK0

5) Properties of Determinants

We assume A and B are square matrices of order n.

Theorem 1: $\det(A^t) = \det(A)$.

Example 5.1: $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $A^T = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ ➔ det(A)= $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$= -2 and $\det(A^T) = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$=10-12= -2

Theorem 2: $\det(A \bullet B) = \det(A) \bullet \det(B)$

Example 5.2: $A = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$; $A * B = \begin{pmatrix} 25 & 20 \\ 14 & 13 \end{pmatrix}$ ➔ det(A*B)=25*13-14*20=325-280=45

det(A)=12-3 =9 and det(B)=8-3=5 ➔ det(A)*det(B)=45;

Theorem 3: if any 2 rows(columns) in A are identical , then det(A)=0.

Example 5.3 : $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 1 & 2 & 3 \end{vmatrix}=0$ since row1=row3.

Theorem 4: if all the entries value in a row(column) of A are all zeroes, then det(A)=0.

Ex: $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 1 & 2 & 0 \end{vmatrix}=0$ since column3 is a zero-column.

Theorem 5: If A is a triangular or diagonal matrix , then det(A) is the product of the entry values on the main diagonal of A. That is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \bullet a_{22} \bullet a_{33}$$

Example 5.4:

$\begin{vmatrix} 3 & 6 & 2 \\ 0 & 8 & 3 \\ 0 & 0 & 10 \end{vmatrix}=3*8*10=240$ for an upper triangle matrix determinant

$\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 \\ 7 & 5 & 6 & 4 & 0 \\ 2 & 7 & 8 & 1 & 5 \end{vmatrix} = (2)\cdot(1)\cdot(3)\cdot(4)\cdot(5) = 120$ for a lower triangle matrix determinant

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (1)\cdot(2)\cdot(3)\cdot(4) = 24$ for a diagonal matrix triangle determinant

TODO➔ Go to Activity and solve question 4

Theorem 6: if a multiple of one row of A is added to another row to produce a matrix B, then det(B)=det(A).

Example 5.6: $\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}=(2)(5)-(2)(4)=10-8=2$, a multiple of row1 [2 4]

Is [6 12] = 3[2 4], then add to row2 [2 5]
to give **row2 → row2 + [6 12]=[**2 5] + [6 12] =[8 17]

and we have $\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 8 & 17 \end{vmatrix} = 34 - 32 = 2$

Theorem 7 : if 2 rows(columns) of A are interchanged to produce B, then det(B) = - det(A) .

Example 5.7: $\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}$=(2)(5)-(2)(4)=10-8=2. row1=[2 4] and row2=[2 5] .if interchanged then

row1=[2 5] and row2=[2 4] that is $\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} = 8 - 10 = -2 = -\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}$

Theorem 8: if one row(column) of A is multiplied by k to produce B, then det(B)=k*det(A).

Example 5.8: $\begin{vmatrix} 5 & 8 \\ 20 & 16 \end{vmatrix} = 5\begin{vmatrix} 1 & 8 \\ 4 & 16 \end{vmatrix} = (5)\bullet(8)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = (5)\bullet(8)\bullet(2)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 80\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 80(1-2) = -80$.

6) Adjoint Matrix

Given a matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the adjoint matrix of A is Adj(A)=

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^t = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}$$

Where c_{ij} is the cofactor at (i,j) , that is $c_{ij} = (-1)^{i+j} \bullet M_{ij} \qquad 1 \le i, j \le 3$

Example 6.1: Calculate Adj(A) if A= $\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

We compute all the 9 cofactors using $c_{ij} = (-1)^{i+j} \bullet M_{ij}$ where (i, j) is the entry.

$c_{11} = (-1)^{1+1} \bullet M_{11} = \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2$, $c_{12} = (-1)^{1+2} \bullet M_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$, $c_{13} = (-1)^{1+3} \bullet M_{13} = \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$

$c_{21} = (-1)^{2+1} \bullet M_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14$, $c_{22} = (-1)^{2+2} \bullet M_{22} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$, $c_{23} = (-1)^{2+3} \bullet M_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$

$c_{31} = (-1)^{3+1} \bullet M_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$, $c_{32} = (-1)^{3+2} \bullet M_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$, $c_{33} = (-1)^{3+3} \bullet M_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$

Once the 9 cofactors found, we build the matrix of cofactors (cofactor matrix):

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{pmatrix}$$ then taking the transpose of the cofactor

matrix gives us the adjoint matrix: Adj(A)= $\begin{pmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{pmatrix}^t = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}$

7) <u>Invertible Matrices</u>

Theorem 7.1: A square matrix A is invertible if and only if det(A) $\neq 0$.

Example 7.1 Check whether the matrices $A = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 4 \\ 2 & 1 \end{pmatrix}$ are invertible

$\det(A) = \begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} = 10 - 8 = 2$; since $\det(A) \neq 0$ ➔ A is invertible

$\det(B) = \begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 8 - 8 = 0$ ➔ B is not invertible.

TODO➔ Go to Activity and solve question 5

Theorem 7.2:
Let A be an invertible NxN matrix. Then the inverse of A is
$$A^{-1} = \frac{Adj(A)}{\det(A)} = \frac{1}{\det(A)} \cdot Adj(A)$$

Example 7.2: Given a matrix A= $\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

a) Show that A is invertible.
b) Compute the adjoint matrix of A.
c) Compute the inverse matrix of A.

Answer:

a) det(A)= $\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix} = 14$. Since det(A) $\neq 0$ ➔ A is invertible.

b) Adj(A)= $\begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}$ as computed in the previous example.

c) $A^{-1} = \frac{1}{\det(A)} \cdot Adj(A) = \frac{1}{14}\begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} = \begin{pmatrix} \frac{-1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & \frac{-1}{2} & \frac{1}{14} \\ \frac{5}{14} & \frac{-1}{2} & \frac{-3}{14} \end{pmatrix}$

Video Reference: https://www.youtube.com/watch?v=g7TFJUJXErU

TODO➔ Go to Activity and solve questions 6.1, 6.2, and 6.3

PYTHONIC: Use **Matrix.inv( )** to compute the inverse of a matrix

```
import sympy as sy
M=sy.Matrix( [ [2,1,3],[1,-1,1],[1,4,-2]])
print("Original matrix M: \n")
sy.pprint(M)
#compute the inverse using inv()
inverse_M=M.inv()
print("\n Inverse of matrix M \n:")
sy.pprint(inverse_M)
```

```
Original matrix M:

⎡2   1   3 ⎤
⎢          ⎥
⎢1  -1   1 ⎥
⎢          ⎥
⎣1   4  -2 ⎦

 Inverse of matrix M
:
⎡-1/7   1     2/7 ⎤
⎢                 ⎥
⎢3/14  -1/2  1/14 ⎥
⎢                 ⎥
⎣5/14  -1/2  -3/14⎦
```

<u>Theorem</u> : $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

$(A^t)^{-1} = (A^{-1})^t$ assuming A and B are invertible matrices.

8) <u>Inverse of 2-by-2 Matrix</u>

The inverse of a 2x2 A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $\det = a \cdot d - b \cdot c$

Example 8.1: A= $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$ det=2*6-3*3=12-9=3 ➔ $A^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix}$

Example 8.2 B=$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$ det=3*2 – 2*4=6-8=-2 ➔

$$B^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & \frac{-3}{2} \end{pmatrix}$$

TODO➔ Go to Activity and solve questions 7.1 , 7.2 and 7.3

Application: Solving a two linear equations with two unknowns using matrix form

Given the system of linear equations (1) $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ where a,b,c,d,e,f are constants and x and y the unknown terms. Equation can be expressed in matrix form as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$ ➔ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}$ since

$$\det\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = ad - bc$$

Example 8.3: solve $\begin{cases} 2x + 3y = -1 \\ 3x + 6y = 0 \end{cases}$

$\begin{cases} 2x + 3y = -1 \\ 3x + 6y = 0 \end{cases}$ in matrix form is $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ➔

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Remember the inverse of $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$ was previously computed above,

$$\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix}$$

Example 8.4: solve $\begin{cases} 4x - 3y = 9 \\ x + y = 4 \end{cases}$

$\begin{cases} 4x - 3y = 9 \\ x + y = 4 \end{cases}$ in matrix form is $\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$ ➔

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 21 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

TODO➔ Go to Activity and solve question 8
Reference: https://www.youtube.com/watch?v=T_aiofOSWfI

9) Matrix Inverse by Row Reduced Echelon Form(RREF)

The inverse of a matrix can be computed by row reduced echelon form if its reduced form does not contain a zero-row. That is given a matrix $M=\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$, we construct the following

augmented matrix $[M\,|\,I]=\left(\begin{array}{ccc:ccc} m_{11} & m_{12} & m_{13} & 1 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 1 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 & 1 \end{array}\right)$ where I is the identity

matrix $I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for 3x3 matrix or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for 2x2 matrix. The goal is to

reduce (rref) $[M\,|\,I]\longrightarrow[I\,|\,M^{-1}]$ where M^{-1} is the inverse of M.

<u>Example 9.1</u>: Compute the inverse of $M=\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ using RREF

We start with

$$\left(\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_3} \left(\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right) \begin{array}{c} \xrightarrow{-2R_3+R1} \\ \xrightarrow{R_3+R_2} \end{array} \left(\begin{array}{ccc:ccc} 1 & -1 & 0 & 5 & -2 & 4 \\ 0 & 2 & 0 & -4 & 2 & -2 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right) \xrightarrow{1/2R_2} \left(\begin{array}{ccc:ccc} 1 & -1 & 0 & 5 & -2 & 4 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & -1 & 0 & 5 & -2 & 4 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right) \Rightarrow M^{-1}=\begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -1 \\ -2 & 1 & -2 \end{pmatrix}$$

Example 9.2: Compute the inverse of $M=\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ using RREF

We start with

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{3}R_3]{-\frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right) \xrightarrow[R_3+R_2]{R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right) \Rightarrow$$

$$M^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

TODO➔ Go to Activity and solve questions 9.1 and 9.2

10) <u>Least Square Approximation</u> <u>**(optional)**</u>

Suppose we have the following data:

x	x_1	x_2	x_3	$\cdots$	x_n
y	y_1	y_2	y_3	$\cdots$	y_n

that is $(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots,(x_n,y_n)$ obtained experimentally .We want a model function $y=f(x)$ between x *and* y that best fits all the points above with minimal error a.
We can decide on the model function to be used as a:

Line : $y=a+bx$

Quadratic polynomial : $y=a+bx+cx^2$

Cubic polynomial : $y=a+bx+cx^2+dx^3$

Let's use the case of a line $y = f(x) = a + bx$. This is then to find a and b such that:

$$\begin{aligned} y_1 &= a + bx_1 \\ y_2 &= a + bx_2 \\ y_3 &= a + bx_3 \\ &\vdots \\ y_n &= a + bx_n \end{aligned} \quad \text{translated in matrix form will be} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Setting $\vec{u} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$, $M = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ we reduce the equation as $\vec{u} = M \cdot \vec{v}$

or $M \cdot \vec{v} = \vec{u}$.

Since M is not squared-matrix , we cannot compute its inverse to find the unknown vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$.

We will use a least square method to find an approximate solution as follows below:

- We multiply both sides of $M \cdot \vec{v} = \vec{u}$ by the transpose of M (M^t) to get $\mathrm{M}^t M \cdot \vec{v} = \mathrm{M}^t \vec{u}$
- $\mathrm{M}^t M$ is a square matrix, if $\det(\mathrm{M}^t M) \neq 0$ then $\left(\mathrm{M}^t M\right)^{-1} \left(\mathrm{M}^t M\right) \cdot \vec{v} = \left(\mathrm{M}^t M\right)^{-1} \mathrm{M}^t \vec{u}$
- And finally $\vec{v} = \left(\mathrm{M}^t M\right)^{-1} \mathrm{M}^t \vec{u}$

Example 1: Find the equation of the best fit straight line by the method of least squares to the

following data

x	2	3	4	5
y	3	5	3	6

we want to fit the data in $y = a + bx$ in order to find a and b :

$$\begin{aligned} 3 &= a + 2b \\ 5 &= a + 3b \\ 3 &= a + 4b \\ 6 &= a + 5b \end{aligned} \quad \text{in matrix form} \quad \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix} \quad \text{where}$$

$M = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ *and* $\vec{u} = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}$. That is $M \cdot \vec{v} = \vec{u}$

From $M^t M \cdot \vec{v} = M^t \vec{u}$ we have $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}$ ➔

$$\begin{pmatrix} 4 & 14 \\ 14 & 54 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 14 \\ 14 & 54 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 63 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 & 14 \\ 14 & 54 \end{pmatrix}^{-1} \begin{pmatrix} 17 \\ 63 \end{pmatrix}$$

But $\begin{pmatrix} 4 & 14 \\ 14 & 54 \end{pmatrix}^{-1} = \frac{1}{20} \begin{pmatrix} 54 & -14 \\ -14 & 4 \end{pmatrix}$ ➔ $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 54 & -14 \\ -14 & 4 \end{pmatrix} \begin{pmatrix} 17 \\ 63 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 36 \\ 14 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 0.7 \end{pmatrix}$

$a = 1.8$ *and* $b = 0.7$. So the best approximate equation of the line is
$\boxed{y = 1.8 + 0.7x}$

In the case of a quadratic polynomial : $y = a + bx + cx^2$.
This is then to find a, b and c such that:

$$\begin{aligned} y_1 &= a + bx_1 + cx_1^2 \\ y_2 &= a + bx_2 + cx_2^2 \\ y_3 &= a + bx_3 + cx_3^2 \\ &\vdots \\ y_n &= a + bx_n + cx_n^2 \end{aligned}$$

translated in matrix form will be $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Setting $\vec{u} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$, $M = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ we reduce the equation as $\vec{u} = M \cdot \vec{v}$ or $M \cdot \vec{v} = \vec{u}$.

Since M is not squared-matrix , we cannot compute its inverse to find the unknown vector $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

We will use a least square method to find an approximate solution as follows below:

- We multiply both sides of $M \cdot \vec{v} = \vec{u}$ by the transpose of $\left(\mathrm{M}^t\right)$ to get $\mathrm{M}^t M \cdot \vec{v} = \mathrm{M}^t \vec{u}$
- $\mathrm{M}^t M$ is a square matrix, if $\det(\mathrm{M}^t M) \neq 0$ then $\left(\mathrm{M}^t M\right)^{-1}\left(\mathrm{M}^t M\right) \cdot \vec{v} = \left(\mathrm{M}^t M\right)^{-1} \mathrm{M}^t \vec{u}$
- And finally $\vec{v} = \left(\mathrm{M}^t M\right)^{-1} \mathrm{M}^t \vec{u}$

Example 2: Find the equation of the best fit quadratic equation by the method of least squares to the following data

x	2	3	4	5
y	3	5	3	6

Answer:

$$\begin{aligned} 3 &= a + 2b + 4c \\ 5 &= a + 3b + 9c \\ 3 &= a + 4b + 16c \\ 6 &= a + 5b + 25c \end{aligned}$$

in matrix form we have $\begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ or

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}$$

From $\mathrm{M}^t M \cdot \vec{v} = \mathrm{M}^t \vec{u}$ we have $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 4 & 14 & 54 \\ 14 & 54 & 224 \\ 54 & 224 & 978 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 17 \\ 63 \\ 225 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 & 14 & 54 \\ 14 & 54 & 224 \\ 54 & 224 & 978 \end{pmatrix}^{-1}\begin{pmatrix} 17 \\ 63 \\ 225 \end{pmatrix}$$

Using python scipy to get the answer as $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{91}{20} \\ \frac{-21}{20} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 4.55 \\ -1.05 \\ 0.25 \end{pmatrix}$.

We finally have our equation: $y = 4.55 - 1.05x + 0.25x^2$

11) Cramer's Rule for a System of Linear Equations

Let's consider a system (Δ) of 3 equations with 3 unknowns x ,y and z as defined below.

$$(\Delta)\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Δ has a unique solution if and only if $\det(\Delta) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

then

$$x = \frac{\det_x(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\det(\Delta)} \quad y = \frac{\det_y(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\det(\Delta)} \quad \text{and} \quad z = \frac{\det_z(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\det(\Delta)}$$

Example: 11.1 Solve the following system of equations

$$(\Delta)\begin{cases} x - y = 5 \\ x + y + z = 0 \\ 2x + y + z = 2 \end{cases} \Rightarrow \det(\Delta) = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -1$$

So $x = \frac{\det_x(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} 5 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}}{-1} = \frac{-2}{-1} = 2$, $y = \frac{\det_y(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} 1 & 5 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix}}{-1} = \frac{3}{-1} = -3$

$$z = \frac{\det_z(\Delta)}{\det(\Delta)} = \frac{\begin{vmatrix} 1 & -1 & 5 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{vmatrix}}{-1} = \frac{-1}{-1} = 1$$

TODO➔ Go to Activity and solve question 10.1 and 10.2

12) <u>Vector Approach of the Cramer's Rule</u>

Given the system $(\Delta)\ \begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$ ➔ $x\cdot\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}+y\cdot\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}+z\cdot\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

or

$x\cdot\vec{a}+y\cdot\vec{b}+z\cdot\vec{c}=\vec{d}$ where $\vec{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\vec{c}=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and $\vec{d}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

where x ,y, and z are the Unknowns.

The determinant of (Δ) is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\det(\vec{a},\vec{b},\vec{c})=\vec{a}\bullet(\vec{b}\times\vec{c})$

using theorem 1.

Now , if $\det(\vec{a},\vec{b},\vec{c})\neq 0$ then the systems of equation $\boxed{x\cdot\vec{a}+y\cdot\vec{b}+z\cdot\vec{c}=\vec{d}}$ has solutions

$$x=\frac{\det_x(\Delta)}{\det(\Delta)}=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\det(\Delta)}=\frac{\begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}{\det(\Delta)}=\frac{\det(\vec{d},\vec{b},\vec{c})}{\det(\vec{a},\vec{b},\vec{c})}=\frac{\vec{d}\bullet(\vec{b}\times\vec{c})}{\vec{a}\bullet(\vec{b}\times\vec{c})}$$

$$y=\frac{\det_y(\Delta)}{\det(\Delta)}=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\det(\Delta)}=\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}{\det(\Delta)}=\frac{\det(\vec{a},\vec{d},\vec{c})}{\det(\vec{a},\vec{b},\vec{c})}=\frac{\vec{a}\bullet(\vec{d}\times\vec{c})}{\vec{a}\bullet(\vec{b}\times\vec{c})}$$

$$z=\frac{\det_z(\Delta)}{\det(\Delta)}=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\det(\Delta)}=\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}}{\det(\Delta)}=\frac{\det(\vec{a},\vec{b},\vec{d})}{\det(\vec{a},\vec{b},\vec{c})}=\frac{\vec{a}\bullet(\vec{b}\times\vec{d})}{\vec{a}\bullet(\vec{b}\times\vec{c})}$$

13) Linear Independence and Basis using Determinant

In this paragraph, we show how to use the determinant to prove that a set of vector are linearly independent , linearly dependent, or forming a basis .

Theorem 13.1: Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ be 3 vectors in $\mathbb{R}^3$, then

- $\vec{a}, \vec{b}$ *and* $\vec{c}$ are linearly independent if and only if $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$
- $\vec{a}, \vec{b}$ *and* $\vec{c}$ are linearly dependent if and only if $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
- $\vec{a}, \vec{b}$ *and* $\vec{c}$ form a basis for $\mathbb{R}^3$ if and only if $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Example 13.1: Are the vectors $\vec{a} = (1,2,1), \vec{b} = (0,1,2)$ *and* $\vec{c} = (3,1,1)$ linearly independent?

We compute $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + (0)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (1-2) + 3(4-1) = 8$

$\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \neq 0$ ➔ $\vec{a} = (1,2,1), \vec{b} = (0,1,2)$ *and* $\vec{c} = (3,1,1)$ are linearly independent and form a basis for $\mathbb{R}^3$.

Example 13.2: Are the vectors $\vec{a}=(1,1,4), \vec{b}=(5,2,5)$ *and* $\vec{c}=(4,1,1)$ are linearly dependent?

We compute $\det(\vec{a},\vec{b},\vec{c})=\begin{vmatrix}1 & 5 & 4\\ 1 & 2 & 1\\ 4 & 5 & 1\end{vmatrix}=1\begin{vmatrix}2 & 1\\ 5 & 1\end{vmatrix}-\begin{vmatrix}1 & 1\\ 4 & 1\end{vmatrix}(5)+\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}(4)$

$$=(2-5)-(1-4)(5)+(5-8)(4)=-3+15-12=0$$

$\det(\vec{a},\vec{b},\vec{c})=\begin{vmatrix}1 & 5 & 4\\ 1 & 2 & 1\\ 4 & 5 & 1\end{vmatrix}=0$ ➔ $\vec{a}=(1,1,4), \vec{b}=(5,2,5)$ *and* $\vec{c}=(4,1,1)$ are linearly

dependent , and do not form a basis for $\mathbb{R}^3$

TODO➔ Go to Activity and solve questions 11.1 , 11.2 , 12.1 and 12.2

14) Eigen values and Eigen Vectors

Given a matrix M, we want to find some scalar λ and vector $\vec{v}$ such that $M\cdot\vec{v}=\lambda\vec{v}$. λ (read **lambda**) is called the eigen value and its corresponding vector $\vec{v}$ is called the eigen vector.

We solve the characteristic equation $\det(M-\lambda\cdot I)=0$ to find λ.

Then the value of λ will be used to compute $\vec{v}$ using $M\cdot\vec{v}=\lambda\vec{v}$ or better $(M-\lambda I)\vec{v}=\vec{o}$.

a) Characteristics Equation (polynomial)

The characteristics equation is obtained by computing $\det(M-\lambda\cdot I)=0$ where I is the identity matrix like $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for a 2x2 matrix or

$I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ for a 3x3 matrix.

Example 14.1: Find the characteristics equation of $M=\begin{pmatrix}-5 & 2\\ 2 & -2\end{pmatrix}$

We start with $M-\lambda I=\begin{pmatrix}-5 & 2\\ 2 & -2\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-5 & 2\\ 2 & -2\end{pmatrix}+\begin{pmatrix}-\lambda & 0\\ 0 & -\lambda\end{pmatrix}=\begin{pmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{pmatrix}$

Then $\det(M-\lambda\cdot I)=0$ or $\begin{vmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{vmatrix}=0$ ➔ $(-5-\lambda)(-2-\lambda)-4=\lambda^2+7\lambda+6=0$

So the characteristics equation is $\lambda^2+7\lambda+6=0$.

Example 14.2: Find the characteristics equation of $M=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$

We start with $M-\lambda I=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}+\begin{pmatrix}-\lambda & 0\\ 0 & -\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda & 3\\ 0 & 1-\lambda\end{pmatrix}$

Then $\det(M-\lambda\cdot I)=0$ or $\begin{vmatrix}1-\lambda & 3\\ 0 & 1-\lambda\end{vmatrix}=0$ ➔ $(\lambda-1)^2=0$ or $\lambda-1=0$

So the characteristics equation is $\lambda-1=0$

Example 14.3: Find the characteristics equation of $M=\begin{pmatrix}2 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}$

$$M-\lambda I=\begin{pmatrix}2 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}2 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}+\begin{pmatrix}-\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & -\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & 1 & 1\\ 0 & 2-\lambda & 1\\ 0 & 0 & 1-\lambda\end{pmatrix}$$

Then $\det(M-\lambda\cdot I)=0$ or $\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 2-\lambda & 1\\ 0 & 0 & 1-\lambda\end{vmatrix}=0$ ➔ $(\lambda-2)^2(\lambda-1)=0$

So the characteristics equation is $(\lambda-2)^2(\lambda-1)=0$ or $\lambda^3-5\lambda^2+8\lambda-4=0$

PYTHONIC: use matrix.charpoly( ) to compute the characteristic (or polynomial) equation.

```python
import sympy as sy

M=sy.Matrix([ [2,1,1],[0,2 ,1] ,[0,0,1] ])
lamda=sy.symbols('lamda')
sy.pprint(M)
eigen_val=M.eigenvals()
print("characteristic or polynomial equation \n")
p=M.charpoly(lamda)
# to factor the polynomial using sympy.factor()
p=sy.factor(p)
sy.pprint(p)
# to factor the polynomial using sympy.expand()
p=sy.expand(p)
sy.pprint(p)
```

```
⎡2  1  1⎤
⎢       ⎥
⎢0  2  1⎥
⎢       ⎥
⎣0  0  1⎦
characteristic or polynomial equation

       2
(λ - 2) ⋅(λ - 1)
 3      2
λ  - 5⋅λ  + 8⋅λ - 4
```

TODO➔ Go to Activity and solve question 13.1

b) Eigen Values

Eigen Values are solution of the characteristics equation (or polynomial).

We solve $\det(M-\lambda\cdot I)=0$ for λ to get the eigen values.

Also we should remember how to solve a quadratic equation of the form

$ax^2+bx+c=0$

to find λ: $x_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $x_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ (formula 1)

Example 14.4 Find the eigen value of $M=\begin{pmatrix}-5 & 2\\ 2 & -2\end{pmatrix}$

From example 14.1 , the characteristics found was $\lambda^2+7\lambda+6=0$

Using the above formula , we have a=1 , b=7 , c=6 then

$$\lambda_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-7+\sqrt{7^2-4(6)}}{2}=\frac{-7+\sqrt{49-24}}{2}=\frac{-7+5}{2}=-1$$

$$\lambda_1=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-7-\sqrt{7^2-4(6)}}{2}=\frac{-7-\sqrt{49-24}}{2}=\frac{-7-5}{2}=-6$$

So the eigen values are $\lambda_1=-1 \text{ and } \lambda_2=-6$.

<u>Example 14.5</u> Find the eigen value of $M=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$

From example 14.2 , the characteristics found was $(\lambda-1)^2=0$ ➔ $\lambda-1=0$ ➔ $\lambda=1$

eigen value is $\lambda=1$ **with multiplicity 2** because of the power 2 on $(\lambda-1)^2$

Example 14.6: Find the eigen values of $M=\begin{pmatrix}2 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}$

From example 14.3 the characteristics equation of M was $(2-\lambda)^2(1-\lambda)=0$ ➔

the eigen values are $\lambda_1=1(\mathit{multiplicity}\ 1)\ \mathit{and}\ \lambda_2=2\ (\mathit{multiplicity}\ 2)$

PYTHONIC: To find the eigenvalues of a matrix,
use eigenvals. eigenvals returns a dictionary of eigenvalue :algebraic multiplicity pairs like :
{ eigenvalue : algebraic multiplicity }

```
import sympy as sy
M=sy.Matrix([ [2,1,1],[0,2 ,1] ,[0,0,1] ])
sy.pprint(M)
eigen_val=M.eigenvals()
print("\n")
print(eigen_val)

```

```
⎡2  1  1⎤
⎢       ⎥
⎢0  2  1⎥
⎢       ⎥
⎣0  0  1⎦


{1: 1, 2: 2}
```

Notice the { eigenvalue : algebraic multiplicity } pair .

Importing Properties:

- The eigen values from an upper, lower or diagonal matrix are the values from the matrix main diagonal.
  That is, given the following matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \quad \text{or} \quad A=\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{or} \quad A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

  Eigen values of A are: $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\lambda_3 = a_{33}$

  Characteristic polynomial: $f(\lambda)=(\lambda-a_{11})(\lambda-a_{22})(\lambda-a_{33})$

  Characteristic equation : $(\lambda-a_{11})(\lambda-a_{22})(\lambda-a_{33})=0$

  Example: if $A=\begin{pmatrix} 1 & 5 & 4 \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{pmatrix}$, then

  $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$

  Characteristic polynomial: $f(\lambda)=(\lambda-1)(\lambda-3)(\lambda-4)$

  Characteristic equation : $(\lambda-1)(\lambda-3)(\lambda-4)=0$

- The characteristic equation or polynomial can be obtained using the matrix determinant, cofactors and trace.

  for a 2x2 matrix $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

  characteristic polynomial: $f(\lambda)=\lambda^2 - tr(A)\lambda + \det(A)$

  characteristic equation: $\lambda^2 - tr(A)\lambda + \det(A) = 0$

  where $tr(A) = trace(A) = a_{11} + a_{22}$, $\det(A)=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

  Example: if $A=\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$,

  $tr(A) = -5 + (-2) = -7$, $\det(A)=\begin{vmatrix} -5 & 2 \\ 2 & -2 \end{vmatrix} = 10 - 4 = 6$

  characteristic polynomial: $f(\lambda)=\lambda^2 - (-7)\lambda + 6 = \lambda^2 + 7\lambda + 6$

  characteristic equation: $\lambda^2 + 7\lambda + 6 = 0$

For a 3x3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we have

characteristic polynomial:

$f(\lambda) = \lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A)$

characteristic equation: $\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A) = 0$

where $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

are respectively the cofactor at entry (1,1) (2,2) and (3,3).

$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $tr(A) = trace(A) = a_{11} + a_{22} + a_{33}$

Example: if $A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{pmatrix}$ then

$\det(A) = \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{vmatrix} = (1)(3)(4) = 12$, $tr(A) = 1 + 3 + 4 = 8$

$A_{11} = \begin{vmatrix} 3 & 9 \\ 0 & 4 \end{vmatrix} = 12$, $A_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 4 \end{vmatrix} = 4$, $A_{33} = \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} = 3$

characteristic polynomial

$f(\lambda) = \lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A)$ will be

$f(\lambda) = \lambda^3 - 8\lambda^2 + (12 + 4 + 3)\lambda - 12 = \lambda^3 - 8\lambda^2 + 19\lambda - 12$, and the

characteristic equation: $\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A) = 0$

will be $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

TODO➔ Go to Activity and solve question 13.2

c) Eigen Vectors

Each eigenvalue λ_i obtained from a characteristics equation has a corresponding eigen vector $\vec{v}_i$ satisfying the equation

$M \cdot \vec{v}_i = \lambda_i \vec{v}_i$ or $(M - \lambda_i I)\vec{v}_i = \vec{o}$

Example 14.7: Find the eigen value and corresponding eigen vectors of $M = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$

The characteristic equation is $\det(M - \lambda \cdot I) = 0$ or $\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$ ➔

$(-5-\lambda)(-2-\lambda) - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$ ➔ eigen value

$\lambda_1 = -1$ *and* $\lambda_2 = -6$ both with multiplicity 1.

For $\lambda_1 = -1$, we want to find $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $(M - \lambda_1 I)\vec{v}_1 = \vec{o}$ with $\lambda_1 = -1$

we have $(M - \lambda_1 I)\vec{v}_1 = \vec{o}$ ➔ $(M + I)\vec{v}_1 = \left[\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{v}_1 = \vec{o}$ ➔

$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o}$

that is $\begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases}$ ➔ $2x - y = 0$ ➔ $x = \frac{1}{2}y$ ➔ $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$,

So we pick $\vec{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ for eigen vector when $\lambda_1 = -1$

For $\lambda_2 = -6$, we want to find $\vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $(M - \lambda_2 I)\vec{v}_2 = \vec{o}$ with $\lambda_2 = -6$

$(M - \lambda_2 I)\vec{v}_2 = \vec{o}$ ➔ $(M + 6I)\vec{v}_2 = \left[\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right]\vec{v}_2 = \vec{o}$ ➔ $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o}$,

That is $\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$ ➔ $x + 2y = 0$ ➔ $x = -2y$ ➔ $\vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

So we pick $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for eigen vector when $\lambda_2 = -6$

Example 14.8 : Find the eigen vectors of $M = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

From example 14.3 the characteristics equation of M was
$(2-\lambda)^2(1-\lambda)=0$ ➔
the eigen values are $\lambda_1 = 1 \ and \ \lambda_2 = 2 \ \ (\lambda_2 \ is \ of \ order \ 2)$

For $\lambda_1 = 1$, we want to find $\vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $(M-\lambda_1 I)\vec{v}_1 = \vec{o}$ with $\lambda_1 = 1$

$$(M-\lambda_1 I)\vec{v}_1 = \vec{o} \ ➔ \ (M-I)\vec{v}_1 = \left[\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right]\vec{v}_1 = \vec{o} ➔$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Leading to $\begin{cases} x+y+z=0 \\ y+z=0 \end{cases}$ ➔ setting $z=t$ gives $y=-t$ and $x=0$.

So $\vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

So we pick $\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ for eigen vector when $\lambda_1 = 1$

For $\lambda_2 = 2$, we want to find $\vec{v}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $(M-\lambda_2 I)\vec{v}_2 = \vec{o}$ with $\lambda_2 = 2$

$$(M-\lambda_2 I)\vec{v}_2 = \vec{o} \ ➔ \ (M-2I)\vec{v}_2 = \left[\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right]\vec{v}_2 = \vec{o} ➔$$

$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ Leading to $\begin{cases} y+z=0 \\ z=0 \end{cases}$ ➔ setting $z=0$ gives $y=0$

$$\vec{v}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So we pick $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for eigen vector when $\lambda_2 = 2$

We finally list the eigen value – eigen vector pair :

$$\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ for } \lambda_1 = 1 \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda_2 = 2$$

PYTHONIC: To find the eigenvectors of a matrix, use eigenvects(). eigenvects returns a list of tuples of the form **(eigenvalue :algebraic multiplicity, [eigenvectors])**.

```python
import sympy as sy
from sympy.matrices import *
M=Matrix([ [2,1,1],[0,2 ,1] ,[0,0,1] ])
sy.pprint(M)
eigen_vector=M.eigenvects()
print("\n")
sy.pprint(eigen_vector)
```

```
⎡2  1  1⎤
⎢       ⎥
⎢0  2  1⎥
⎢       ⎥
⎣0  0  1⎦
```

```
⎡⎛      ⎡⎡0 ⎤⎤⎞  ⎛      ⎡⎡1⎤⎤⎞⎤
⎢⎜      ⎢⎢  ⎥⎥⎟  ⎜      ⎢⎢ ⎥⎥⎟⎥
⎢⎜1, 1, ⎢⎢-1⎥⎥⎟, ⎜2, 2, ⎢⎢0⎥⎥⎟⎥
⎢⎜      ⎢⎢  ⎥⎥⎟  ⎜      ⎢⎢ ⎥⎥⎟⎥
⎣⎝      ⎣⎣1 ⎦⎦⎠  ⎝      ⎣⎣0⎦⎦⎠⎦
```

TODO→ Go to Activity and solve questions 13.3

d) Caley-Hamilton Theorem

Theorem 14.1 :

Let $a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0$ be the characteristics equation of a matrix

A, then A will satisfy the characteristics equation :

$a_n A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + Ia_0 = 0$ where I is the identity matrix.

Example 1.9: it was shown $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ had characteristics equation

$(\lambda - 1)^2 = 0$ or

$\lambda^2 - 2\lambda + 1 = 0$. Do we have $A^2 - 2A + I = 0$?

$$A^2 - 2A + I = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^2 - 2\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -6 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Note that $A^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

15) Eigen Space (optional)

Given a $n \times n$ matrix A , the set $S_\lambda = \{\vec{v}_\lambda : \; A\vec{v}_\lambda = \lambda\vec{v}_\lambda \}$ of all eigenvectors corresponding to the eigenvalue λ is called the eigenspace of λ or λ-eigenspace . We will denote its basis E_λ ;

Also $S_\lambda = \{\vec{v}_\lambda : \; A\vec{v}_\lambda = \lambda\vec{v}_\lambda \}$ is a subspace of $\mathbb{R}^n$.

Example 15.1: What is the eigen space E_λ of $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ and its basis.

Answer :

Using $\lambda^2 - tr(A)\lambda + \det(A) = 0$ with $\det(A) = -4$, trace(A)=0

The characteristic equation becomes $\lambda^2 - 4 = 0$ with eigen values $\lambda_1 = 2 \;\; and \;\; \lambda_2 = -2$.

For $\lambda_1 = 2$, we want to find $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $(M - \lambda_1 I)\vec{v}_1 = \vec{o}$ with $\lambda_1 = 2$

we have $(M - \lambda_1 I)\vec{v}_1 = \vec{o}$ ➔ $(M - 2I)\vec{v}_1 = \left[\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{v}_1 = \vec{o}$ ➔ $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{o}$

that is $\begin{cases} -x+3y=0 \\ x-3y=0 \end{cases}$ ➔ $x-3y=0$ ➔ $x=3y$; setting $y=t$, we have $x=3t$

and $\vec{v}_1=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3t \\ t \end{pmatrix}=t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so we pick $\vec{v}_1=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for eigen vector when $\lambda_1=2$

so $\lambda_1-eigenspace$ is $E_{\lambda_1}=span\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$ and its basis is $\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$

For $\lambda_2=-2$, we want to find $\vec{v}_2=\begin{pmatrix} x \\ y \end{pmatrix}$ such that $(M-\lambda_2 I)\vec{v}_2=\vec{o}$ with $\lambda_2=-2$

we have $(M-\lambda_2 I)\vec{v}_2=\vec{o}$ ➔ $(M+I)\vec{v}_2=\left[\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}+2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{v}_2=\vec{o}$ ➔ $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\vec{o}$

that is $\begin{cases} 3x+3y=0 \\ x+y=0 \end{cases}$ ➔ $x+y=0$ ➔ $x=-y$; setting $y=t$, we have $x=-t$

and $\vec{v}_1=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -t \\ t \end{pmatrix}=t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so we pick $\vec{v}_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for eigen vector when $\lambda_2=-2$

so $\lambda_2-eigenspace$ is $E_{\lambda_2}=span\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ and its basis is $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

Example 15.2 What is the eigen space E_λ of $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and its basis

Answer:

The characteristic equation is $\lambda(\lambda-1)(\lambda-2)=0$ with root solutions $\lambda_1=0$, $\lambda_2=1$ and $\lambda_3=2$.

For $\lambda_1=0$, we want to find $\vec{v}_1=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $(M-\lambda_1 I)\vec{v}_1=\vec{o}$ with $\lambda_1=0$

we have $(M-\lambda_1 I)\vec{v}_1=\vec{o}$ ➔ $(M+0I)\vec{v}_1=M\vec{v}_1=\begin{pmatrix}1&1&1\\1&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$,

we row reduce $A=\begin{pmatrix}1&1&1&0\\1&1&-1&0\\0&0&1&0\end{pmatrix}\sim\begin{pmatrix}1&1&1&0\\0&0&-2&0\\0&0&1&0\end{pmatrix}\sim\begin{pmatrix}1&1&1&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$

$\begin{pmatrix}1&1&1\\1&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ becomes $\begin{pmatrix}1&1&1\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ leading to $\begin{cases}x+y+z=0\\ z=0\end{cases}$ ➔

$\begin{cases}x+y=0\\ z=0\end{cases}$ setting $y=t$ ➔ $x=-t$ and $\vec{v}_1=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-t\\t\\0\end{pmatrix}=t\begin{pmatrix}-1\\1\\0\end{pmatrix}$ so we pick

$\vec{v}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, and $\lambda_1-eigenspace$ is $E_{\lambda_1}=span\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$ and its basis is $\begin{pmatrix}-1\\1\\0\end{pmatrix}$

For $\lambda_2=1$, we want to find $\vec{v}_2=\begin{pmatrix}x\\y\\z\end{pmatrix}$ such that $(M-\lambda_2 I)\vec{v}_2=\vec{o}$ with $\lambda_2=1$

we have $(M-\lambda_2 I)\vec{v}_2=\vec{o}$ ➔ $(M-I)\vec{v}_2=\left[\begin{pmatrix}1&1&1\\1&1&-1\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$,

$\begin{pmatrix}0&1&1\\1&0&-1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ ➔ $\begin{cases}y+z=0\\ x-z=0\end{cases}$ setting $z=t$ ➔ $x=t$ and $y=-t$

$\vec{v}_2=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}t\\-t\\t\end{pmatrix}=t\begin{pmatrix}1\\-1\\1\end{pmatrix}$, we pick $\vec{v}_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\lambda_2-eigenspace$ is

$E_{\lambda_2}=span\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$ and its basis is $\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$

For $\lambda_3=2$, we want to find $\vec{v}_3=\begin{pmatrix}x\\y\\z\end{pmatrix}$ such that $(M-\lambda_3 I)\vec{v}_3=\vec{o}$ with $\lambda_3=2$

we have $(M-\lambda_3 I)\vec{v}_3=\vec{o}$ ➔ $(M-2I)\vec{v}_3=\left[\begin{pmatrix}1&1&1\\1&1&-1\\0&0&1\end{pmatrix}-\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}\right]\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$,

$$\begin{pmatrix}-1&1&1\\1&-1&-1\\0&0&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}-x+y+z=0\\x-y-z=0\\z=0\end{cases} \Rightarrow \begin{cases}x-y-z=0\\x-y-z=0\\z=0\end{cases} \Rightarrow \begin{cases}x-y-z=0\\z=0\end{cases}$$

$$\begin{cases}x-y=0\\z=0\end{cases}.$$

setting $y=t$, $x=t$ with $z=0$

so $\vec{v}_3=\begin{pmatrix}z\\y\\z\end{pmatrix}=\begin{pmatrix}t\\t\\0\end{pmatrix}=t\begin{pmatrix}1\\1\\0\end{pmatrix}$, we pick $\vec{v}_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\lambda_3-eigenspace$ is

$$E_{\lambda_3}=span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$ and its basis is $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ $\dim(E_{\lambda_3})=1$

16) Similar Matrix and Diagonalizable matrix (optional)

A 3×3 matrix A is said to be diagonalizable if there is a 3×3 invertible P such $D=P\cdot A\cdot P^{-1}$ where $D=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$ is a diagonal matrix,

$P=[\vec{v}_1\quad \vec{v}_2\quad \vec{v}_3]=\begin{pmatrix}v_{1x}&v_{2x}&v_{3x}\\v_{1y}&v_{2y}&v_{3y}\\v_{1z}&v_{2z}&v_{3z}\end{pmatrix}$ with $\lambda_1,\lambda_2,$ *and* λ_3 the eigen values

with their corresponding eigen vectors $\vec{v}_1=\begin{pmatrix}v_{1x}\\v_{1y}\\v_{1z}\end{pmatrix}$, $\vec{v}_2=\begin{pmatrix}v_{2x}\\v_{2y}\\v_{2z}\end{pmatrix}$, *and* $\vec{v}_3=\begin{pmatrix}v_{3x}\\v_{3y}\\v_{3z}\end{pmatrix}$

of A. This statement is also true for all $n\times n$ matrix.

Example 16.1: Diagonalize

17) Spectral Decomposition(optional)

18)