GEN 242: Linear Algebra

Chapter 1: Vectors

Solutions Guide

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Answers

Norm, Length, or Magnitude of a Vector

1.a $\|\vec{u}\| = \|(1,0,1)\| = \sqrt{2}$

1.b $\|\vec{v}\| = \|(2,1,-2)\| = 3$

1.c $\|\vec{w}\| = \|(3,0,-4)\| = 5$

1.d $\|\vec{s}\| = \|(1,-1,1)\| = \sqrt{3}$

1.e $\|\vec{m}\| = \left\|\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\| = 1$

Normalized Vectors

2.a $\hat{u} = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

2.b $\hat{v} = \frac{(2,1,-2)}{3} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

2.c $\widehat{w} = \frac{(3,0,-4)}{5} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right)$

2.d $\hat{s} = \frac{(1,-1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

2.e $\widehat{m} = \frac{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)}{1} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

Vectors Direction

3.a
$$\hat{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$
$$-\hat{u} = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

3.b
$$\hat{v} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$
$$-\hat{v} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

3.c
$$\widehat{w} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right)$$
$$-\widehat{w} = \left(-\frac{3}{5}, 0, \frac{4}{5}\right)$$

3.d
$$\hat{s} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$
$$-\hat{s} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

3.e
$$\widehat{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$$
$$-\widehat{m} = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$$

4.
$$\|\vec{v}\| = \left\|\left(\sqrt{3}, 0, 1\right)\right\| = 2$$
$$\hat{v} = \frac{\left(\sqrt{3}, 0, 1\right)}{2} = \left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$$

5.
$$\|\vec{v}\| = \|(-1,0,1)\| = \sqrt{2}$$
$$\hat{v} = \frac{(-1,0,1)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Co-linear and Parallel Vectors

6.a
$$(3,6,0) = 3(1,2,0)$$
$$\vec{q} = 3\vec{p} \rightarrow \vec{q} = k\vec{p}, k \in \mathbb{R} \rightarrow \vec{p} \parallel \vec{q}$$

6.b
$$(8,0,-20) = 4(2,0,-5)$$
$$\vec{q} = 4\vec{p} \rightarrow \vec{q} = k\vec{p}, k \in \mathbb{R} \rightarrow \vec{p} \parallel \vec{q}$$

6.c
$$(12,-30,18) = -6(-2,5,-3)$$
$$\vec{q} = -6\vec{p} \rightarrow \vec{q} = k\vec{p}, k \in \mathbb{R} \rightarrow \vec{p} \parallel \vec{q}$$

6.d
$$(6,9,15) = 3(2,3,5)$$
$$\vec{p} = 3\vec{q} \rightarrow \vec{p} = k\vec{q}, k \in \mathbb{R} \rightarrow \vec{p} \parallel \vec{q}$$

Building a Vector From 2 Vertices

7.a $$\overrightarrow{AB} = (1-2, 1-1, 1-0) = (-1,0,1)$$
$$\|\overrightarrow{AB}\| = \sqrt{2}$$

7.b $$\overrightarrow{AB} = (1-3, 0-0, 1-4) = (-2,0,-3)$$
$$\|\overrightarrow{AB}\| = \sqrt{13}$$

7.c $$\overrightarrow{AB} = (1-1, 1-0, 0-0) = (0,1,0)$$
$$\|\overrightarrow{AB}\| = 1$$

8. $$\|\overrightarrow{p_1p_2}\| = \sqrt{(1-2)^2 + (5-5)^2 + (1-4)^2} = \sqrt{10}$$

Vectors Addition

9.a $-2\vec{A} + 3\vec{B} = -2(2,-5,1) + 3(1,-2,-1) = (-1,4,-5)$

9.b $-\vec{A} + \vec{B} = -(2,-5,1) + (1,-2,-1) = (-1,3,-2)$

9.c $-\vec{A} + 3\vec{B} + \vec{C} = -(2,-5,1) + 3(1,-2,1) + (1,1,0) = (2,0,-4)$

9.d $-\vec{B} - \vec{C} + \vec{A} = -(1,-2,-1) - (1,1,0) + (2,-5,1) = (0,-4,2)$

9.e $-\vec{A} + \vec{B} + 2\vec{C} = -(2,-5,1) + (1,-2,1) + 2(1,1,0) = (1,5,-2)$

Dot Product of Two Vectors

10.a $(2,-1,3) \cdot (0,1,3) = 8$

10.b $(1,-2,0) \cdot (-2,4,0) = -10$

10.c $(0,-1,3) \cdot (0,3,1) = 0$

10.d $(3,-1,4) \cdot (1,1,2) = 10$

Angle Between Two Vectors

11.a $\theta_{(2,-1,3)}^{(0,1,3)} \approx 47.5°$

11.b $\theta_{(1,-2,0)}^{(-2,4,0)} = 180°$

11.c $\theta_{(0,-1,3)}^{(0,3,1)} = 90°$

11.d $\theta_{(3,-1,4)}^{(1,1,2)} \approx 36.8°$

Type of Angle Between Two Vectors

12.a $(2,-1,3)\cdot(0,1,3)=8\rightarrow\vec{A}\cdot\vec{B}>0\rightarrow$ acute

12.b $(1,-2,0)\cdot(-2,4,0)=-10\rightarrow\vec{A}\cdot\vec{B}<0\rightarrow$ obtuse

12.c $(0,-1,3)\cdot(0,3,1)=0\rightarrow$ right

12.d $(1,-1,3)\cdot(1,3,1)=10\rightarrow\vec{A}\cdot\vec{B}>0\rightarrow$ acute

Orthogonal or Perpendicular Vectors

13.a $(2,-1,3)\cdot(0,3,1)=0+(-3)+3=0\rightarrow\vec{a}\perp\vec{b}$

13.b $(-1,-2,0)\cdot(-2,1,0)=2+(-2)+0=0\rightarrow\vec{a}\perp\vec{b}$

13.c $(0,-1,3)\cdot(0,3,1)=0+(-3)+3=0\rightarrow\vec{a}\perp\vec{b}$

13.d $(3,-1,1)\cdot(1,1,-2)=3+(-1)+(-2)=0\rightarrow\vec{a}\perp\vec{b}$

Vector Components

14.a $Comp_{(1,1,1)}^{(1,2,1)}=\frac{4}{\sqrt{3}}$

14.b $Comp_{\hat{\imath}+2\hat{\jmath}-\hat{k}}^{3\hat{\imath}-2\hat{\jmath}+\hat{k}}=\frac{-2}{\sqrt{6}}$

14.c $Comp_{\hat{\imath}-\hat{k}}^{5\hat{\imath}+\hat{\jmath}}=\frac{5}{\sqrt{2}}$

14.d $Comp_{(-2,3,1)}^{(1,0,2)}=0$

Vector Projection

15.a $Proj_{(1,1,1)}^{(1,2,1)}=\left(\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$

15.b $Proj_{\hat{\imath}+2\hat{\jmath}-\hat{k}}^{3\hat{\imath}-2\hat{\jmath}+\hat{k}}=\left(-\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right)$

15.c $Proj_{\hat{\imath}-\hat{k}}^{5\hat{\imath}+\hat{\jmath}}=\left(\frac{5}{2},0,-\frac{5}{2}\right)$

15.d $Proj_{(-2,3,1)}^{(1,0,2)}=(0,0,0)$

Vector Rejection (Perpendicular Vector of $\vec{\boldsymbol{a}}$ to $\vec{\boldsymbol{b}}$)

16.a $\vec{a}_{\perp}{}_{(0,1,3)}^{(2,-1,3)}=\left(2,-\frac{9}{5},\frac{3}{5}\right)$

16.b $\vec{a}_{\perp}{}_{(-2,4,0)}^{(1,-2,0)}=(0,0,0)=\vec{0}$

16.c $\vec{a}_{\perp}{}_{(0,3,1)}^{(0,-1,3)}=(0,-1,3)$

Cross Product of two Vectors

17.a $(2,-1,3)\times(0,1,3)=(-6,-6,2)$

17.b $(1,-2,0)\times(-2,4,0)=(0,0,0)$

17.c $(0,-1,3)\times(0,3,1)=(-10,0,0)$

17.d $(3,-1,4)\times(1,1,2)=(-6,-2,4)$

18.a $(2\hat{\imath}) \times \hat{\jmath} = 2\hat{k}$

18.b $(\hat{\imath} \times \hat{k}) \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\imath}$

18.c $(\hat{\imath} \times \hat{\imath}) \cdot (\hat{\imath} \times \hat{\jmath}) = 0$

18.d $\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \hat{\imath} + 2\hat{\jmath}$

18.e $(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = 5\hat{\imath} - 5\hat{\jmath} - \hat{k}$

18.f $\hat{\imath} \times (\hat{\jmath} \times \hat{k}) = \vec{0}$

18.g $\hat{k} \cdot (\hat{\jmath} \times \hat{k}) = 0$

18.h $(\hat{\imath} \times \hat{k}) \times (\hat{\jmath} \times \hat{\imath}) = \hat{\imath}$

19.a $\vec{C} = (2,1,1) \times (-1,2,2) = (0,-5,5)$

19.b $\vec{C} = (1,0,1) \times (2,3,5) = (-3,-3,3)$

19.c $\vec{C} = (1,0,0) \times (0,1,0) = (0,0,1)$

19.d $\vec{C} = (3,-1,1) \times (1,1,-2) = (1,7,4)$

Vector Differentiation

20.a $\frac{d}{dt}\left[3t^2\hat{\imath} + t^3\hat{\jmath} - (t^2 - t^3)\hat{k}\right] = 6t\hat{\imath} + 3t^2\hat{\jmath} - (2t - 3t^2)\hat{k}$

20.b $\frac{d}{dt}\left(3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}\right) = 6t\hat{\imath} + 12t^2\hat{\jmath} - 6\hat{k}$

20.c $\frac{d}{dt}[(t^2, \cos(t), 7)] = (2t, -\sin(t), 0)$

20.d $\frac{d}{dt}[(t, 4, -6t)] = (1,0,-6)$

Partial Differentiation of Vectors

21.a $\vec{u} = (x + y^2, z + x, xz^2) \rightarrow \begin{matrix} \frac{\partial \vec{u}}{\partial x} = (1,1,z^2) \\ \frac{\partial \vec{u}}{\partial y} = (2y, 0,0) \\ \frac{\partial \vec{u}}{\partial z} = (0,1,2xz) \\ \frac{\partial^2 \vec{u}}{\partial x \partial y} = (0,0,0) \end{matrix}$

21.b $\vec{u} = (x^3 + y^3, zx, z^2 + y) \rightarrow \begin{matrix} \frac{\partial \vec{u}}{\partial x} = (3x^2, z, 0) \\ \frac{\partial \vec{u}}{\partial y} = (2y, 0,1) \\ \frac{\partial \vec{u}}{\partial z} = (0, x, 2z) \\ \frac{\partial^2 \vec{u}}{\partial x \partial y} = (0,0,0) \end{matrix}$

21.c $\vec{u} = (x^2 + y^2 + z^2, z, xz^2 + 2) \rightarrow \begin{matrix} \frac{\partial \vec{u}}{\partial x} = (2x, 0, z^2) \\ \frac{\partial \vec{u}}{\partial y} = (2y, 0,0) \\ \frac{\partial \vec{u}}{\partial z} = (2z, 1,2xz) \\ \frac{\partial^2 \vec{u}}{\partial x \partial y} = (0,0,0) \end{matrix}$

Vector Integration

22.a $\int_1^2 \left(3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}\right)dt = 7\hat{\imath} + 15\hat{\jmath} - 9\hat{k}$

22.b $\int_1^2 \left(t^2\hat{\imath} + 4t^3\hat{\jmath} - \hat{k}\right)dt = 2\hat{\imath} + 36\hat{\jmath} - \hat{k}$

22.c $\int_1^2 (1, \cos(t), \sin(t))dt \approx (1, .068, .956)$

22.d $\int_1^2 \left(2t\hat{\imath} + \hat{k}\right) = 2\hat{\imath} + \hat{k}$

Homogeneous Systems of Linear Equations

23.a $\begin{cases} x+y-z=0 \\ 2x+3y+z=0 \\ x-y+2z=0 \end{cases}$ is homogeneous.

23.b $\begin{cases} x+3y-z=5 \\ x+3y+8z=0 \\ x-y+2z=0 \end{cases}$ is not homogeneous.

23.c $\begin{cases} x+y-z=1 \\ 3y+z=0 \\ z=0 \end{cases}$ is not homogeneous.

System Consistency

24.a $\begin{cases} x+y-z=1 \\ 2x+3y+z=6 \\ x-y+2z=2 \end{cases}$ is consistent.

24.b $\begin{cases} x+3y-z=5 \\ x+3y+8z=0 \\ 0z=0 \end{cases}$ is consistent.

24.c $\begin{cases} x+y-z=1 \\ 3y+z=0 \\ 0z=4 \end{cases}$ is not consistent.

Free Variables and Leading Unknowns (Pivots)

25.a $\begin{cases} x+y-z=1 \\ 3y+z=0 \end{cases}$ z is a free variable. x and y are the leading unknowns.

25.b $\begin{cases} x+3y-z+s-2t=5 \\ 2y+8z+2s+5=4 \\ s+2t=1 \end{cases}$ z and t are free variables. x and y and s are the leading unknowns.

25.c $x+y-z=1$ y and z are free variables. x is the only leading unknown.

Gaussian Elimination

26.a $\begin{cases} x + 2y = 4 \\ 2x + y = 5 \end{cases} \rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$

26.b $\begin{cases} x - 3y = -2 \\ 5x + y = 6 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$

26.c $\begin{cases} x + 3y = 8 \\ 3x + y = 16 \end{cases} \rightarrow \begin{cases} x = 5 \\ y = 1 \end{cases}$

26.d $\begin{cases} x + y - z = 1 \\ 2x + 3y + z = 6 \\ x - y + 2z = 2 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$

26.e $\begin{cases} x + 3y - z = 7 \\ 2x + 3y + z = 8 \\ 3x - y + 2z = 1 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 0 \end{cases}$

26.f $\begin{cases} x + y - z = 0 \\ 5x - 3y + z = 2 \\ 3x - 2y + z = 2 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$

Subspaces

27.a $\{(3x, 5y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

27.b $\{(x, y + 1) : x \in \mathbb{R}, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

27.c $\{10x : x \in \mathbb{R}\}$ is ***not*** a subspace of $\mathbb{R}^2$.

Linear Combination

28.a $(0,2)$ is a linear combination of $\{(1,3), (2,4)\}$.

28.b $(3,0)$ is a linear combination of $\{(1,0), (0,2)\}$.

28.c $(5,2)$ is a linear combination of $\{(1,0), (0,1)\}$.

28.d $(1,2,0)$ is a linear combination of $\{(1,0,0), (0,1,0)\}$.

Linear Independence

29.a $\{(1,3), (2,3)\}$ is linearly independent.

29.b $\{(6,4), (12,8)\}$ is linearly dependent.

29.c $\{(1,5), (3,4)\}$ is linearly independent.

29.d $\{(1,1,0), (1,2,1), (1,1,1)\}$ is linearly independent.

29.e $\{(1,1,1), (1,2,0), (0,-1,1)\}$ is linearly dependent.

29.f $\{(1,2,3), (3,2,9), (5,2,-1)\}$ is linearly independent.

29.g $\{(1,2,3), (3,2,1), (0,4,8)\}$ is linearly dependent.

Basis of a Vector Space

30.a $\{(1,3),(2,3)\}$ is a basis for $\mathbb{R}^2$.

30.b $\{(6,4),(12,8)\}$ is not a basis for $\mathbb{R}^2$.

30.c $\{(1,5),(3,4)\}$ is a basis for $\mathbb{R}^2$.

30.d $\{(1,1,0),(1,2,1),(1,1,1)\}$ is a basis for $\mathbb{R}^3$.

30.e $\{(1,1,1),(1,2,0),(0,-1,1)\}$ is not a basis for $\mathbb{R}^3$.

30.f $\{(1,2,3),(3,2,9),(5,2,-1)\}$ is a basis for $\mathbb{R}^3$.

30.g $\{(1,2,3),(3,2,1),(0,4,8)\}$ is not a basis for $\mathbb{R}^3$.

Dimension of a Vector Space

31.a $dim(\{(1,3),(2,3)\}) = 2$

31.b $dim(\{(1,1,0),(1,2,1),(1,1,1)\}) = 3$

31.c $dim(\{1,x,x^2,x^3,x^4\}) = 5$

31.d $dim(\{(1,0,0,0),(0,2,0,0),(0,0,1,0),(0,0,0,3)\}) = 4$

Inner Product Space

32.a $\langle \vec{a},\vec{c}\rangle = \langle (2,1,2),(1,-1,1)\rangle = 3$

32.b $\langle \vec{b},\vec{c}\rangle = \langle (1,0,-1),(1,-1,1)\rangle = 0$

32.c $\langle 5\vec{a}-2\vec{b},\vec{c}\rangle = \langle 5(2,1,2)-2(1,0,-1),(1,-1,1)\rangle = 15$

32.d $\sqrt{\langle \vec{a},\vec{a}\rangle} = \sqrt{\langle\{(2,1,2),(2,1,2)\rangle} = 3$

33.a $\langle 5x^2,x^3\rangle = 0$

33.b $\|f\| = \sqrt{\int_{-1}^{1}(5x^2)(5x^2)\,dx} = \sqrt{10}$

33.c $\hat{f} = \frac{5x^2}{\|f\|} = \frac{\sqrt{10}}{2}x^2$

34.a $\langle x,x+2\rangle = \frac{4}{3}$

34.b $\|f\| = \sqrt{\int_{0}^{1}(x)(x)\,dx} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx .577$

34.c $\hat{f} = \frac{x}{\|f\|} = \sqrt{3}\cdot x$

35.a $\langle f, g\rangle = \int_0^{\pi/2} [\cos(x)][\sin(x)]\, dx = -\frac{1}{2}$

35.b $\|f\| = \sqrt{\langle \cos(x), \cos(x)\rangle} = \frac{\sqrt{\pi}}{2}$

35.c $\hat{f} = \frac{\cos(x)}{\|f\|} = \frac{2}{\sqrt{\pi}}\cos(x)$

36. $\langle 1 + 2x + x^2 + x^3, 1 + 5x^2 + x^3\rangle = 7$

37. $\langle 1 + 2x - x^2 + 3x^3, 1 + x - 2x^2 + 4x^3\rangle = 17$

Solutions

Norm, Length, or Magnitude of a Vector

Problem 1

Calculate the length (norm) of the following vectors:

1.a $\vec{u} = (1,0,1)$

$$\|\vec{u}\| = \sqrt{u_x{}^2 + u_y{}^2 + u_z{}^2}$$

$$\|\vec{u}\| = \sqrt{(1)^2 + (0)^2 + (1)^2}$$

$$\boxed{\|\vec{u}\| = \sqrt{2}}$$

1.b $\vec{v} = (2,1,-2)$

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$$

$$\|\vec{v}\| = \sqrt{(2)^2 + (1)^2 + (-2)^2}$$

$$\|\vec{v}\| = \sqrt{9}$$

$$\boxed{\|\vec{v}\| = 3}$$

1.c $\vec{w} = (3,0,-4)$

$$\|\vec{w}\| = \sqrt{w_x{}^2 + w_y{}^2 + w_z{}^2}$$

$$\|\vec{w}\| = \sqrt{(3)^2 + (0)^2 + (-4)^2}$$

$$\|\vec{w}\| = \sqrt{25}$$

$$\boxed{\|\vec{w}\| = 5}$$

1.d $\vec{s} = (1,-1,1)$

$$\|\vec{s}\| = \sqrt{s_x{}^2 + s_y{}^2 + s_z{}^2}$$

$$\|\vec{s}\| = \sqrt{(1)^2 + (1)^2 + (1)^2}$$

$$\boxed{\|\vec{s}\| = \sqrt{3}}$$

1.e $\vec{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$$\|\vec{m}\| = \sqrt{m_x{}^2 + m_y{}^2 + m_z{}^2}$$

$$\|\vec{m}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + (0)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$\|\vec{m}\| = \sqrt{1}$$

$$\boxed{\|\vec{m}\| = 1}$$

Normalized Vectors

Problem 2

Normalize the following vectors:

2.a $\vec{u} = (1,0,1)$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\hat{u} = \frac{(1,0,1)}{\sqrt{2}}$$

$$\boxed{\hat{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$$

2.b $\vec{v} = (2,1,-2)$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{(2,1,-2)}{3}$$

$$\boxed{\hat{v} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)}$$

2.c $\vec{w} = (3,0,-4)$

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|}$$

$$\hat{w} = \frac{(3,0,-4)}{5}$$

$$\boxed{\hat{w} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right)}$$

2.d $\vec{s} = (1,-1,1)$

$$\hat{s} = \frac{\vec{s}}{\|\vec{s}\|}$$

$$\hat{s} = \frac{(1,-1,1)}{\sqrt{3}}$$

$$\boxed{\hat{s} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

2.e $\vec{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$$\hat{m} = \frac{\vec{m}}{\|\vec{m}\|}$$

$$\hat{m} = \frac{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)}{1}$$

$$\boxed{\hat{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)}$$

Vectors Direction

Problem 3

Find the direction and opposite direction of the following vectors:

Vector direction is the same as the vector's normalized form; answers are taken from Problem #2, above.

3.a $\vec{u} = (1,0,1)$

$$\boxed{\begin{gathered}\hat{u} = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) \\ -\hat{u} = \left(-\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right)\end{gathered}}$$

3.b $\vec{v} = (2,1,-2)$

$$\boxed{\begin{gathered}\hat{v} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \\ -\hat{v} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)\end{gathered}}$$

3.c $\vec{w} = (3,0,-4)$

$$\boxed{\begin{gathered}\hat{w} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right) \\ -\hat{w} = \left(-\frac{3}{5}, 0, \frac{4}{5}\right)\end{gathered}}$$

3.d $\vec{s} = (1,-1,1)$

$$\boxed{\begin{gathered}\hat{s} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ -\hat{s} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\end{gathered}}$$

3.e $\vec{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$$\boxed{\begin{gathered}\hat{m} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \\ -\hat{m} = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)\end{gathered}}$$

Problem 4

Find the direction and speed of a car moving with velocity $\vec{v} = (\sqrt{3}, 0, 1)$ m/s.

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

$$\|\vec{v}\| = \sqrt{\left(\sqrt{3}\right)^2 + (0)^2 + (1)^2}$$

$$\|\vec{v}\| = \sqrt{4}$$

$\boxed{\|\vec{v}\| = 2}$ m/s (speed)

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{(\sqrt{3}, 0, 1)}{2}$$

$\boxed{\hat{v} = \left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)}$ (direction)

Problem 5

Find the direction and speed of a ball moving with velocity $\vec{v} = (-1,0,1)$ m/s.

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (0)^2 + (1)^2}$$

$\boxed{\|\vec{v}\| = \sqrt{2}}$ m/s (speed)

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{(-1,0,1)}{\sqrt{2}}$$

$\boxed{\hat{v} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$ (direction)

Co-linear and Parallel Vectors

Problem 6

Given the following pairs of vectors, show that each pair is collinear.

6.a $\vec{p} = (1,2,0)$ and $\vec{q} = (3,6,0)$

$\vec{q} = (3,6,0)$

$\vec{q} = 3(1,2,0)$

$\vec{q} = 3\vec{p}$

$\vec{q} = k\vec{p}$

$\vec{p} \parallel \vec{q}$

6.b $\vec{p} = (2,0,-5)$ and $\vec{q} = (8,0,-20)$

$\vec{q} = (8,0,-20)$

$\vec{q} = 4(2,0,-5)$

$\vec{q} = 4\vec{p}$

$\vec{q} = k\vec{p}$

$\vec{p} \parallel \vec{q}$

6.c $\vec{p} = (-2,5,-3)$ and $\vec{q} = (12,-30,18)$

$\vec{q} = (12,-30,18)$

$\vec{q} = -6(-2,5,-3)$

$\vec{q} = -6\vec{p}$

$\vec{q} = k\vec{p}$

$\vec{p} \parallel \vec{q}$

6.d $\vec{p} = (6,9,15)$ and $\vec{q} = (2,3,5)$

$\vec{p} = (6,9,15)$

$\vec{p} = 3(2,3,5)$

$\vec{p} = 3\vec{q}$

$\vec{p} = k\vec{q}$

$\vec{p} \parallel \vec{q}$

Building a Vector From Two Vertices

Problem 7

For each of the following pairs of vertices, find the vector between them and its length:

7.a $A = (2,1,0)$ and $B = (1,1,1)$

$$\overrightarrow{AB} = B - A$$
$$\overrightarrow{AB} = (1,1,1) - (2,1,0)$$
$$\overrightarrow{AB} = (1-2, 1-1, 1-0)$$
$$\boxed{\overrightarrow{AB} = (-1,0,1)}$$

$$\|\overrightarrow{AB}\| = \sqrt{(ab)_x{}^2 + (ab)_y{}^2 + (ab)_z{}^2}$$
$$\|\overrightarrow{AB}\| = \sqrt{(-1)^2 + (0)^2 + (1)^2}$$
$$\boxed{\|\overrightarrow{AB}\| = \sqrt{2}}$$

7.b $A = (3,0,4)$ and $B = (1,0,1)$

$$\overrightarrow{AB} = B - A$$
$$\overrightarrow{AB} = (1,0,1) - (3,0,4)$$
$$\overrightarrow{AB} = (1-3, 0-0, 1-4)$$
$$\boxed{\overrightarrow{AB} = (-2,0,-3)}$$

$$\|\overrightarrow{AB}\| = \sqrt{(ab)_x{}^2 + (ab)_y{}^2 + (ab)_z{}^2}$$
$$\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + (0)^2 + (-3)^2}$$
$$\boxed{\|\overrightarrow{AB}\| = \sqrt{13}}$$

7.c $A = (1,0,0)$ and $B = (1,1,0)$

$$\overrightarrow{AB} = B - A$$
$$\overrightarrow{AB} = (1,1,0) - (1,0,0)$$
$$\overrightarrow{AB} = (1-1, 1-0, 0-0)$$
$$\boxed{\overrightarrow{AB} = (0,1,0)}$$

$$\|\overrightarrow{AB}\| = \sqrt{(ab)_x{}^2 + (ab)_y{}^2 + (ab)_z{}^2}$$
$$\|\overrightarrow{AB}\| = \sqrt{(0)^2 + (1)^2 + (0)^2}$$
$$\|\overrightarrow{AB}\| = \sqrt{1}$$
$$\boxed{\|\overrightarrow{AB}\| = 1}$$

Problem 8

What is the distance between Ann, at $\vec{p}_1 = (2,5,4)$, and Paul, at $\vec{p}_2 = (1,5,1)$?

$\overrightarrow{p_1p_2} = p_2 - p_1$

$\overrightarrow{p_1p_2} = (1,5,1) - (2,5,4)$

$\overrightarrow{p_1p_2} = (1-2, 5-5, 1-4)$

$\overrightarrow{p_1p_2} = (-1,0,-3)$

$\|\overrightarrow{p_1p_2}\| = \sqrt{{(p_1p_2)_x}^2 + {(p_1p_2)_z}^2 + {(p_1p_2)_z}^2}$

$\|\overrightarrow{p_1p_2}\| = \sqrt{{(-1)_x}^2 + {(0)_z}^2 + {(-3)_z}^2}$

$\boxed{\|\overrightarrow{p_1p_2}\| = \sqrt{10}}$

Vectors Addition

Problem 9

Given the vectors $\vec{a} = (2,-5,1)$, $\vec{b} = (1,-2,-1)$, and $\vec{c} = (1,1,0)$, calculate the following:

9.a $-2\vec{a} + 3\vec{b}$

$-2\vec{a} + 3\vec{b} = -2(2,-5,1) + 3(1,-2,-1)$

$-2\vec{a} + 3\vec{b} = (-4,10,-2) + (3,-6,-3)$

$-2\vec{a} + 3\vec{b} = \big(-4+3, 10+(-6), -2+(-3)\big)$

$\boxed{-2\vec{a} + 3\vec{b} = (-1,4,-5)}$

9.b $-\vec{a} + \vec{b}$

$-\vec{a} + \vec{b} = -(2,-5,1) + (1,-2,-1)$

$-\vec{a} + \vec{b} = (-2,5,-1) + (1,-2,-1)$

$-\vec{a} + \vec{b} = \big(-2+1, 5+(-2), -1+(-1)\big)$

$\boxed{-\vec{a} + \vec{b} = (-1,3,-2)}$

9.c $-\vec{a} + 3\vec{b} + \vec{c}$

$-\vec{a} + 3\vec{b} + \vec{c} = -(2,-5,1) + 3(1,-2,-1) + (1,1,0)$

$-\vec{a} + 3\vec{b} + \vec{c} = (-2,5,-1) + (3,-6,-3) + (1,1,0)$

$-\vec{a} + 3\vec{b} + \vec{c} = (-2 + 3 + 1,5 + (-6) + 1,-1 + (-3) + 0)$

$\boxed{-\vec{a} + 3\vec{b} + \vec{c} = (2,0,-4)}$

9.d $-\vec{b} - \vec{c} + \vec{a}$

$-\vec{b} - \vec{c} + \vec{a} = -(1,-2,-1) - (1,1,0) + (2,-5,1)$

$-\vec{b} - \vec{c} + \vec{a} = (-1,2,1) - (1,1,0) + (2,-5,1)$

$-\vec{b} - \vec{c} + \vec{a} = (-1 - 1 + 2,2 - 1 + (-5),1 - 0 + 1)$

$\boxed{-\vec{b} - \vec{c} + \vec{a} = (0,-4,2)}$

9.e $-\vec{a} + \vec{b} + 2\vec{c}$

$-\vec{a} + \vec{b} + 2\vec{c} = -(2,-5,1) + (1,-2,-1) + 2(1,1,0)$
$-\vec{a} + \vec{b} + 2c = (-2,5,-1) + (1,-2,-1) + (2,2,0)$
$-\vec{a} + \vec{b} + 2c = (-2 + 1 + 2,5 + (-2) + 2,-1 + (-1) + 0)$
$\boxed{-\vec{a} + \vec{b} + 2\vec{c} = (1,5,-2)}$

Dot Product of Two Vectors

Problem 10

Calculate the dot product of the following vectors:

10.a $\vec{a} = (2, -1, 3)$ and $\vec{b} = (0, 1, 3)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (2)(0) + (-1)(1) + (3)(3)$

$\vec{a} \cdot \vec{b} = 0 + (-1) + 9$

$\boxed{\vec{a} \cdot \vec{b} = 8}$

10.b $\vec{a} = (1, -2, 0)$ and $\vec{b} = (-2, 4, 0)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (1)(-2) + (-2)(4) + (0)(0)$

$\vec{a} \cdot \vec{b} = -2 + (-8) + 0$

$\boxed{\vec{a} \cdot \vec{b} = -10}$

10.c $\vec{a} = (0, -1, 3)$ and $\vec{b} = (0, 3, 1)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (0)(0) + (-1)(3) + (3)(1)$

$\vec{a} \cdot \vec{b} = 0 + (-3) + 3$

$\boxed{\vec{a} \cdot \vec{b} = 0}$

10.d $\vec{a} = (3, -1, 4)$ and $\vec{b} = (1, 1, 2)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (3)(1) + (-1)(1) + (4)(2)$

$\vec{a} \cdot \vec{b} = 3 + (-1) + 8$

$\boxed{\vec{a} \cdot \vec{b} = 10}$

Angle Between Two Vectors

Problem 11

For the following pairs of vectors, calculate the angle between them:

11.a $\vec{a} = (2, -1, 3)$ and $\vec{b} = (0,1,3)$

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}\right)$$

$$\theta = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \cdot \sqrt{{b_x}^2 + {b_z}^2 + {b_z}^2}}\right)$$

$$\theta = \cos^{-1}\left[\frac{(2)(0) + (-1)(1) + (3)(3)}{\sqrt{(2)^2 + (-1)^2 + (3)^2} \cdot \sqrt{(0)^2 + (1)^2 + (3)^2}}\right]$$

$$\theta = \cos^{-1}\left[\frac{0 + (-1) + 9}{\sqrt{4 + 1 + 9} \cdot \sqrt{0 + 1 + 9}}\right]$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{14} \cdot \sqrt{10}}\right)$$

$$\theta = \cos^{-1}\left(\frac{8}{2\sqrt{35}}\right)$$

$$\boxed{\theta \approx 47.5°}$$

11.b $\vec{a} = (1, -2, 0)$ and $\vec{b} = (-2,4,0)$

$$\theta = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \cdot \sqrt{{b_x}^2 + {b_z}^2 + {b_z}^2}}\right)$$

$$\theta = \cos^{-1}\left[\frac{(1)(-2) + (-2)(4) + (0)(0)}{\sqrt{(1)^2 + (-2)^2 + (0)^2} \cdot \sqrt{(-2)^2 + (4)^2 + (0)^2}}\right]$$

$$\theta = \cos^{-1}\left[\frac{-2 + (-8) + 0}{\sqrt{1 + 4 + 0} \cdot \sqrt{4 + 16 + 0}}\right]$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{5} \cdot \sqrt{20}}\right)$$

$$\theta = \cos^{-1}\left(-\frac{10}{10}\right)$$

$$\boxed{\theta = 180°}$$

11.c $\vec{a} = (0, -1{,}3)$ and $\vec{b} = (0{,}3{,}1)$

$$\theta = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \cdot \sqrt{{b_x}^2 + {b_z}^2 + {b_z}^2}}\right)$$

$$\theta = \cos^{-1}\left[\frac{(0)(0) + (-1)(3) + (3)(1)}{\sqrt{(0)^2 + (-1)^2 + (3)^2} \cdot \sqrt{(0)^2 + (3)^2 + (1)^2}}\right]$$

$$\theta = \cos^{-1}\left[\frac{0 + (-3) + 3}{\sqrt{0 + 1 + 9} \cdot \sqrt{0 + 9 + 1}}\right]$$

$$\theta = \cos^{-1}\left(\frac{0}{\sqrt{10} \cdot \sqrt{10}}\right)$$

$$\theta = \cos^{-1}(0)$$

$$\boxed{\theta = 90°}$$

11.d $\vec{a} = (3, -1{,}4)$ and $\vec{b} = (1{,}1{,}2)$

$$\theta = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \cdot \sqrt{{b_x}^2 + {b_z}^2 + {b_z}^2}}\right)$$

$$\theta = \cos^{-1}\left[\frac{(3)(1) + (-1)(1) + (4)(2)}{\sqrt{(3)^2 + (-1)^2 + (4)^2} \cdot \sqrt{(1)^2 + (1)^2 + (2)^2}}\right]$$

$$\theta = \cos^{-1}\left[\frac{3 + (-1) + 8}{\sqrt{9 + 1 + 16} \cdot \sqrt{1 + 1 + 4}}\right]$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{26} \cdot \sqrt{6}}\right)$$

$$\theta = \cos^{-1}\left(\frac{10}{2\sqrt{39}}\right)$$

$$\boxed{\theta \approx 36.8°}$$

Type of Angle Between Two Vectors

Problem 12

For the following pairs of vectors, determine the type of angle between them, without directly computing it:

12.a $\vec{a} = (2, -1,3)$ and $\vec{b} = (0,1,3)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (2)(0) + (-1)(1) + (3)(3)$

$\vec{a} \cdot \vec{b} = 0 + (-1) + 9$

$\vec{a} \cdot \vec{b} = 8$

Since $\vec{a} \cdot \vec{b} > 0$, the angle between them must be **acute**.

12.b $\vec{a} = (1, -2,0)$ and $\vec{b} = (-2,4,0)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (1)(-2) + (-2)(4) + (0)(0)$

$\vec{a} \cdot \vec{b} = -2 + (-8) + 0$

$\vec{a} \cdot \vec{b} = -10$

Since $\vec{a} \cdot \vec{b} < 0$, the angle between them must be **obtuse**.

12.c $\vec{a} = (0, -1,3)$ and $\vec{b} = (0,3,1)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (0)(0) + (-1)(3) + (3)(1)$

$\vec{a} \cdot \vec{b} = 0 + (-3) + 3$

$\vec{a} \cdot \vec{b} = 0$

Since $\vec{a} \cdot \vec{b} = 0$, the angle between them must be **right**.

12.d $\vec{a} = (3, -1,4)$ and $\vec{b} = (1,1,2)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (3)(1) + (-1)(1) + (4)(2)$

$\vec{a} \cdot \vec{b} = 3 + (-1) + 8$

$\vec{a} \cdot \vec{b} = 10$

Since $\vec{a} \cdot \vec{b} > 0$, the angle between them must be **acute**.

Orthogonal or Perpendicular Vectors

Problem 13

For the following pairs of vectors, show that each pair is orthogonal.

The dot product of orthogonal vectors is zero.

13.a $\vec{a} = (2, -1,3)$ and $\vec{b} = (0,3,1)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (2)(0) + (-1)(3) + (3)(1)$

$\vec{a} \cdot \vec{b} = 0 + (-3) + 3$

$\vec{a} \cdot \vec{b} = 0$

Since $\vec{a} \cdot \vec{b} = 0$, the angle between them must be right, meaning $\vec{a} \perp \vec{b}$.

13.b $\vec{a} = (-1, -2,0)$ and $\vec{b} = (-2,1,0)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (-1)(-2) + (-2)(1) + (0)(0)$

$\vec{a} \cdot \vec{b} = 2 + (-2) + 0$

$\vec{a} \cdot \vec{b} = 0$

Since $\vec{a} \cdot \vec{b} = 0$, the angle between them must be right, meaning $\vec{a} \perp \vec{b}$.

13.c $\vec{a} = (0, -1,3)$ and $\vec{b} = (0,3,1)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (0)(0) + (-1)(3) + (3)(1)$

$\vec{a} \cdot \vec{b} = 0 + (-3) + 3$

$\vec{a} \cdot \vec{b} = 0$

Since $\vec{a} \cdot \vec{b} = 0$, the angle between them must be right, meaning $\vec{a} \perp \vec{b}$.

13.d $\vec{a} = (3, -1,1)$ and $\vec{b} = (1,1, -2)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$\vec{a} \cdot \vec{b} = (3)(1) + (-1)(1) + (1)(-2)$

$\vec{a} \cdot \vec{b} = 3 + (-1) + (-2)$

$\vec{a} \cdot \vec{b} = 0$

Since $\vec{a} \cdot \vec{b} = 0$, the angle between them must be right, meaning $\vec{a} \perp \vec{b}$.

Vector Component

Problem 14

For the following pairs of vectors, calculate $Comp_{\vec{v}}^{\vec{u}}$:

14.a $\vec{U} = (1,2,1)$ and $\vec{V} = (1,1,1)$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{u_x v_x + u_y v_y + u_z v_z}{\sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{(1)(1) + (2)(1) + (1)(1)}{\sqrt{(1)^2 + (1)^2 + (1)^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{1 + 2 + 1}{\sqrt{1 + 1 + 1}}$$

$$\boxed{Comp_{\vec{v}}^{\vec{u}} = \frac{4}{\sqrt{3}}}$$

14.b $\vec{U} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{V} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{u_x v_x + u_y v_y + u_z v_z}{\sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{(3)(1) + (-2)(2) + (1)(-1)}{\sqrt{(1)^2 + (2)^2 + (-1)^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{3 + (-4) + (-1)}{\sqrt{1 + 4 + 1}}$$

$$\boxed{Comp_{\vec{v}}^{\vec{u}} = \frac{-2}{\sqrt{6}}}$$

14.c $\vec{U} = 5\hat{\imath} + \hat{\jmath}$ and $\vec{V} = \hat{\imath} - \hat{k}$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{u_x v_x + u_y v_y + u_z v_z}{\sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{(5)(1) + (1)(0) + (0)(-1)}{\sqrt{(1)^2 + (0)^2 + (-1)^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{5 + 0 + 0}{\sqrt{1 + 0 + 1}}$$

$$\boxed{Comp_{\vec{v}}^{\vec{u}} = \frac{5}{\sqrt{2}}}$$

14.d $\vec{U} = (1,0,2)$ and $\vec{V} = (-2,3,1)$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{u_x v_x + u_y v_y + u_z v_z}{\sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{(1)(-2) + (0)(3) + (2)(1)}{\sqrt{(-2)^2 + (3)^2 + (1)^2}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{-2 + 0 + 2}{\sqrt{4 + 9 + 1}}$$

$$Comp_{\vec{v}}^{\vec{u}} = \frac{0}{\sqrt{14}}$$

$$\boxed{Comp_{\vec{v}}^{\vec{u}} = 0}$$

Vector Projection

Problem 15

For the following pairs of vectors, calculate $Proj_{\vec{v}}^{\hat{u}}$:

15.a $\vec{U} = (1,2,1)$ and $\vec{V} = (1,1,1)$

$$Proj_{\vec{v}}^{\vec{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \hat{v}$$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$Proj_{\vec{v}}^{\hat{u}} = \frac{4}{\sqrt{3}} \cdot \frac{(1,1,1)}{\sqrt{3}}$$

$$Proj_{\vec{v}}^{\hat{u}} = \frac{4}{3} \cdot (1,1,1)$$

$$\boxed{Proj_{\vec{v}}^{\hat{u}} = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)}$$

15.b $\vec{U} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{V} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \hat{v}$$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$Proj_{\vec{v}}^{\hat{u}} = -\frac{2}{\sqrt{6}} \cdot \frac{(1,2,-1)}{\sqrt{6}}$$

$$Proj_{\vec{v}}^{\hat{u}} = -\frac{2}{6} \cdot (1,2,-1)$$

$$Proj_{\vec{v}}^{\hat{u}} = -\frac{1}{3} \cdot (1,2,-1)$$

$$\boxed{Proj_{\vec{v}}^{\hat{u}} = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right)}$$

15.c $\vec{U} = 5\hat{\imath} + \hat{\jmath}$ and $\vec{V} = \hat{\imath} - \hat{k}$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \hat{v}$$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$Proj_{\vec{v}}^{\hat{u}} = \frac{5}{\sqrt{2}} \cdot \frac{(1,0,-1)}{\sqrt{2}}$$

$$Proj_{\vec{v}}^{\hat{u}} = \frac{5}{2} \cdot (1,0,-1)$$

$$\boxed{Proj_{\vec{v}}^{\hat{u}} = \left(\frac{5}{2}, 0, -\frac{5}{2}\right)}$$

15.d $\vec{U} = (1,0,2)$ and $\vec{V} = (-2,3,1)$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \hat{v}$$

$$Proj_{\vec{v}}^{\hat{u}} = Comp_{\vec{v}}^{\vec{u}} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$Proj_{\vec{v}}^{\hat{u}} = 0 \cdot \frac{(-2,3,1)}{\sqrt{14}}$$

$$Proj_{\vec{v}}^{\hat{u}} = 0 \cdot (1,0,-1)$$

$$\boxed{Proj_{\vec{v}}^{\hat{u}} = (0,0,0)}$$

Vector Rejection (Perpendicular Vector of $\vec{a}$ to $\vec{b}$)

Problem 16

For the following pairs of vectors, calculate the perpendicular [rejection] vector of $\vec{a}$ to $\vec{b}$. That is, calculate $\vec{a}_\perp = \vec{a} - Proj_{\vec{b}}^{\vec{a}} = \vec{a} - (\vec{a} \cdot \hat{b})\hat{b}$:

16.a $\vec{a} = (2, -1, 3)$ and $\vec{b} = (0, 1, 3)$

$$\vec{a}_\perp = \vec{a} - (\vec{a} \cdot \hat{b})\hat{b}$$

$$\hat{b} = \frac{\vec{b}}{\|\vec{b}\|}$$

$$\hat{b} = \frac{\vec{b}}{\sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}}$$

$$\hat{b} = \frac{(0,1,3)}{\sqrt{(0)^2 + (-1)^2 + (3)^2}}$$

$$\hat{b} = \frac{(0,1,3)}{\sqrt{0 + 1 + 9}}$$

$$\hat{b} = \frac{(0,1,3)}{\sqrt{10}}$$

$$\hat{b} = \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = \vec{a} - \left(a_x b_{n,x} + a_y b_{n,y} + a_z b_{n,z}\right) \cdot \hat{b}$$

$$\vec{a}_\perp = (2, -1, 3) - \left[(2)(0) + (-1)\left(\frac{1}{\sqrt{10}}\right) + (3)\left(\frac{3}{\sqrt{10}}\right)\right] \cdot \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (2, -1, 3) - \left[0 + \left(-\frac{1}{\sqrt{10}}\right) + \frac{9}{\sqrt{10}}\right] \cdot \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (2, -1, 3) - \frac{8}{\sqrt{10}} \cdot \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (2, -1, 3) - \cdot \left(0, \frac{8}{10}, \frac{24}{10}\right)$$

$$\vec{a}_\perp = (2, -1, 3) - \cdot \left(0, \frac{4}{5}, \frac{12}{5}\right)$$

$$\vec{a}_\perp = \left(2 - 0, -1 - \frac{4}{5}, 3 - \frac{12}{5}\right)$$

$$\boxed{\vec{a}_\perp = \left(2, -\frac{9}{5}, \frac{3}{5}\right)}$$

Check:

$$\vec{a}_\perp \cdot \vec{b} = a_{\perp,x} b_x + a_{\perp,y} b_y + a_{\perp,z} b_z$$

$$\vec{a}_\perp \cdot \vec{b} = (2)(0) + \left(-\frac{9}{5}\right)(1) + \left(\frac{3}{5}\right)(3)$$

$$\vec{a}_\perp \cdot \vec{b} = 0 + \left(-\frac{9}{5}\right) + \frac{9}{5}$$

$$\vec{a}_\perp \cdot \vec{b} = 0$$

$$\vec{a}_\perp \perp \vec{b}$$

16.b $\vec{a} = (1,-2{,}0)$ and $\vec{b} = (-2{,}4{,}0)$

$$\vec{a}_\perp = \vec{a} - (\vec{a}\cdot\hat{b})\hat{b}$$

$$\hat{b} = \frac{\vec{b}}{\|\vec{b}\|}$$

$$\hat{b} = \frac{\vec{b}}{\sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}}$$

$$\hat{b} = \frac{(-2{,}4{,}0)}{\sqrt{(-2)^2 + (4)^2 + (0)^2}}$$

$$\hat{b} = \frac{(-2{,}4{,}0)}{\sqrt{4 + 16 + 0}}$$

$$\hat{b} = \frac{(-2{,}4{,}0)}{\sqrt{20}}$$

$$\hat{b} = \frac{(-2{,}4{,}0)}{2\sqrt{5}}$$

$$\hat{b} = \left(\frac{-2}{2\sqrt{5}}, \frac{4}{2\sqrt{5}}, \frac{0}{2\sqrt{5}}\right)$$

$$\hat{b} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\vec{a}_\perp = \vec{a} - \left(a_x b_{n,x} + a_y b_{n,y} + a_z b_{n,z}\right)\cdot\hat{b}$$

$$\vec{a}_\perp = (1,-2{,}0) - \left[(1)\left(-\frac{1}{\sqrt{5}}\right) + (-2)\left(\frac{2}{\sqrt{5}}\right) + (0)(0)\right]\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\vec{a}_\perp = (1,-2{,}0) \text{—} \frac{1}{\sqrt{5}} + \left(-\frac{4}{\sqrt{5}}\right) + 0\text{⍰}\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\vec{a}_\perp = (1,-2{,}0) \text{—} \frac{5}{\sqrt{5}}\text{⍰}\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\vec{a}_\perp = (1,-2{,}0) + \frac{5}{\sqrt{5}}\cdot\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$\vec{a}_\perp = (1,-2{,}0) + \left(-\frac{5}{5}, \frac{10}{5}, 0\right)$$

$$\vec{a}_\perp = (1,-2{,}0) + (-1{,}2{,}0)$$

$$\vec{a}_\perp = [1 + (-1), -2 + 2{,}0 + 0]$$

$$\boxed{\vec{a}_\perp = (0{,}0{,}0) = \vec{0}}$$

Check:

$\vec{b} = (-2{,}4{,}0)$

$\vec{b} = -2(1,-2{,}0)$

$\vec{b} = -2\vec{a}$

$\vec{b} = k\vec{a},\ k \neq 0$

16.c $\vec{a} = (0, -1, 3)$ and $\vec{b} = (0, 3, 1)$

$$\vec{a}_\perp = \vec{a} - (\vec{a} \cdot \hat{b})\hat{b}$$

$$\hat{b} = \frac{\vec{b}}{\|\vec{b}\|}$$

$$\hat{b} = \frac{\vec{b}}{\sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}}$$

$$\hat{b} = \frac{(0,3,1)}{\sqrt{(0)^2 + (3)^2 + (1)^2}}$$

$$\hat{b} = \frac{(0,3,1)}{\sqrt{0 + 9 + 1}}$$

$$\hat{b} = \frac{(0,3,1)}{\sqrt{10}}$$

$$\hat{b} = \left(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = \vec{a} - \left(a_x b_{n,x} + a_y b_{n,y} + a_z b_{n,z}\right) \cdot \hat{b}$$

$$\vec{a}_\perp = (0, -1, 3) - \left[(0)(0) + (-1)\left(\frac{3}{\sqrt{10}}\right) + (3)\left(\frac{1}{\sqrt{10}}\right)\right] \cdot \left(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (0, -1, 3) - \left[0 + \left(-\frac{3}{\sqrt{10}}\right) + \frac{3}{\sqrt{10}}\right] \cdot \left(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (0, -1, 3) - 0 \cdot \left(0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$\vec{a}_\perp = (0, -1, 3) - 0$$

$$\boxed{\vec{a}_\perp = (0, -1, 3)}$$

Check:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{a} \cdot \vec{b} = (0)(0) + (-1)(3) + (3)(1)$$

$$\vec{a} \cdot \vec{b} = 0 + (-3) + 3$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \perp \vec{b}$$

Cross Product of Two Vectors

Problem 17

For the following pairs of vectors, calculate their cross product:

17.a $\vec{a} = (2,-1,3)$ and $\vec{b} = (0,1,3)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 0 & 1 & 3 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} -1 & 3 \\ 1 & 3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\hat{k}$$

$$\vec{a}\times\vec{b} = [(-1)(3)-(1)(3)]\hat{\imath} - [(2)(3)-(0)(3)]\hat{\jmath} + [(2)(1)-(0)(-1)]\hat{k}$$

$$\vec{a}\times\vec{b} = (-3-3)\hat{\imath} - (6-0)\hat{\jmath} + (2-0)\hat{k}$$

$$\boxed{\vec{a}\times\vec{b} = -6\hat{\imath} - 6\hat{\jmath} + 2\hat{k}}$$

17.b $\vec{a} = (1,-2,0)$ and $\vec{b} = (-2,4,0)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 0 \\ -2 & 4 & 0 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} -2 & 0 \\ 4 & 0 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}\hat{k}$$

$$\vec{a}\times\vec{b} = [(-2)(0)-(4)(0)]\hat{\imath} - [(1)(0) — 2)(0)]\hat{\jmath} + [(1)(4) — 2)(-2)]\hat{k}$$

$$\vec{a}\times\vec{b} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (4-4)\hat{k}$$

$$\boxed{\vec{a}\times\vec{b} = \vec{0}}$$

17.c $\vec{a} = (0, -1, 3)$ and $\vec{b} = (0, 3, 1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -1 & 3 \\ 0 & 3 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & -1 \\ 0 & 3 \end{vmatrix} \hat{k}$$

$$\vec{a} \times \vec{b} = [(-1)(1) - (3)(3)]\hat{\imath} - [(0)(1) - (0)(3)]\hat{\jmath} + [(0)(3) - (0)(-1)]\hat{k}$$

$$\vec{a} \times \vec{b} = (-1 - 9)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\boxed{\vec{a} \times \vec{b} = -10\hat{\imath}}$$

17.d $\vec{a} = (3, -1, 4)$ and $\vec{b} = (1, 1, 2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} \hat{k}$$

$$\vec{a} \times \vec{b} = [(-1)(2) - (1)(4)]\hat{\imath} - [(3)(2) - (1)(4)]\hat{\jmath} + [(3)(1) - (1)(-1)]\hat{k}$$

$$\vec{a} \times \vec{b} = (-2 - 4)\hat{\imath} - (6 - 4)\hat{\jmath} + [3 — 1) ⍰ \hat{k}$$

$$\boxed{\vec{a} \times \vec{b} = -6\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}$$

Problem 18

Simplify the following operations:

18.a $(2\hat{\imath}) \times \hat{\jmath}$

$$(2\hat{\imath}) \times \hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2i_x & 2i_y & 2i_z \\ j_x & j_y & j_z \end{vmatrix}$$

$$(2\hat{\imath}) \times \hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$(2\hat{\imath}) \times \hat{\jmath} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \hat{k}$$

$$(2\hat{\imath}) \times \hat{\jmath} = [(0)(0) - (1)(0)]\hat{\imath} - [(2)(0) - (0)(0)]\hat{\jmath} + [(2)(1) - (0)(0)]\hat{k}$$

$$(2\hat{\imath}) \times \hat{\jmath} = (0 - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (2 - 0)\hat{k}$$

$$\boxed{(2\hat{\imath}) \times \hat{\jmath} = 2\hat{k}}$$

18.b $\left(\hat{\imath} \times \hat{k}\right) \times (\hat{\imath} \times \hat{\jmath})$

$$\left(\hat{\imath} \times \hat{k}\right) \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\jmath} \times \hat{k}$$

$$\boxed{\left(\hat{\imath} \times \hat{k}\right) \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\imath}}$$

18.c $(\vec{\imath} \times \vec{\imath}) \cdot (\vec{\imath} \times \vec{\jmath})$

$$(\vec{\imath} \times \vec{\imath}) \cdot (\vec{\imath} \times \vec{\jmath}) = \hat{0} \cdot \vec{k}$$

$$\boxed{(\vec{\imath} \times \vec{\imath}) \cdot (\vec{\imath} \times \vec{\jmath}) = 0}$$

18.d $\hat{k} \times (2\hat{\imath} - \hat{\jmath})$

$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \hat{k} \times [2(1,0,0) - (0,1,0)]$

$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \hat{k} \times [(2,0,0) - (0,1,0)]$

$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \hat{k} \times (2,-1,0)$

$$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{vmatrix}$$

$$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \hat{k}$$

$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = [(0)(0) — 1)(1)\square\hat{\imath} - [(0)(0) - (2)(1)]\hat{\jmath} + [(0)(-1) - (2)(0)]\hat{k}$

$\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = [0 — 1)\square\hat{\imath} - (0 - 2)\hat{\jmath} + (0 - 0)\hat{k}$

$\boxed{\hat{k} \times (2\hat{\imath} - \hat{\jmath}) = \hat{\imath} + 2\hat{\jmath}}$

18.e $(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k})$

$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = [(1,0,0) + (0,1,0)] \times [(1,0,0) + 5(0,0,1)]$

$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = [(1,0,0) + (0,1,0)] \times [(1,0,0) + (0,0,5)]$

$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = (1,1,0) \times (1,0,5)$

$$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 5 \end{vmatrix}$$

$$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 0 \\ 1 & 5 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \hat{k}$$

$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = [(1)(5) - (0)(0)]\hat{\imath} - [(1)(5) - (1)(0)]\hat{\jmath} + [(1)(0) - (1)(1)]\hat{k}$

$(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = (5 - 0)\hat{\imath} - (5 - 0)\hat{\jmath} + (0 - 1)\hat{k}$

$\boxed{(\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 5\hat{k}) = 5\hat{\imath} - 5\hat{\jmath} - \hat{k}}$

18.f $\hat{\imath} \times (\hat{\jmath} \times \hat{k})$

$\hat{\imath} \times (\hat{\jmath} \times \hat{k}) = \hat{\imath} \times \hat{\imath}$

$\boxed{\hat{\imath} \times (\hat{\jmath} \times \hat{k}) = \vec{0}}$

18.g $\hat{k} \cdot (\hat{\jmath} \times \hat{k})$

$$\hat{k} \cdot (\hat{\jmath} \times \hat{k}) = \hat{k} \cdot \hat{\imath}$$

$$\hat{k} \cdot (\hat{\jmath} \times \hat{k}) = k_x i_x + k_y i_y + k_z i_z$$

$$\hat{k} \cdot (\hat{\jmath} \times \hat{k}) = (0)(1) + (0)(0) + (1)(0)$$

$$\boxed{\hat{k} \cdot (\hat{\jmath} \times \hat{k}) = 0}$$

The $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ unit vectors are perpendicular to each other. The dot product of perpendicular vectors is always 0.

18.h $(\hat{\imath} \times \hat{k}) \times (\hat{\jmath} \times \hat{\imath})$

$$(\hat{\imath} \times \hat{k}) \times (\hat{\jmath} \times \hat{\imath}) = -\hat{\jmath} \times -\hat{k}$$

$$\boxed{(\hat{\imath} \times \hat{k}) \times (\hat{\jmath} \times \hat{\imath}) = \hat{\imath}}$$

Problem 19

For each of the following pairs of vectors, find a vector $\vec{C}$ that is orthogonal to both:

An orthogonal vector to a vector pair may be found by calculating the cross product of the vector pair.

19.a $\vec{A} = (2,1,1)$ and $\vec{B} = (-1,2,2)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \hat{k}$$

$$\vec{A} \times \vec{B} = [(1)(2) - (2)(1)]\hat{\imath} - [(2)(2) — 1)(1) \square \hat{\jmath} + [(2)(2) — 1)(1) \square \hat{k}$$

$$\vec{A} \times \vec{B} = (2 - 2)\hat{\imath} - [4 — 1) \square \hat{\jmath} + [4 — 1) \square \hat{k}$$

$$\vec{A} \times \vec{B} = 0\hat{\imath} - 5\hat{\jmath} + 5\hat{k}$$

$$\boxed{\vec{A} \times \vec{B} = (0, -5, 5)}$$

19.b $\vec{A} = (1,0,1)$ and $\vec{B} = (2,3,5)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 3 & 5 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 0 & 1 \\ 3 & 5 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \hat{k}$$

$$\vec{A} \times \vec{B} = [(0)(5) - (3)(1)]\hat{\imath} - [(1)(5) - (2)(1)]\hat{\jmath} + [(1)(3) - (2)(0)]\hat{k}$$

$$\vec{A} \times \vec{B} = (0 - 3)\hat{\imath} - (5 - 2)\hat{\jmath} + (3 - 0)\hat{k}$$

$$\vec{A} \times \vec{B} = -3\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

$$\boxed{\vec{A} \times \vec{B} = (-3, -3, 3)}$$

19.c $\vec{A} = (1,0,0)$ and $\vec{B} = (0,1,0)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k}$$

$$\vec{A} \times \vec{B} = [(0)(0) - (1)(0)]\hat{\imath} - [(1)(0) - (0)(0)]\hat{\jmath} + [(1)(1) - (0)(0)]\hat{k}$$

$$\vec{A} \times \vec{B} = (0 - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (1 - 0)\hat{k}$$

$$\vec{A} \times \vec{B} = 0\hat{\imath} - 0\hat{\jmath} + 1\hat{k}$$

$$\boxed{\vec{A} \times \vec{B} = (0,0,1)}$$

19.d $\vec{A} = (3,-1,1)$ and $\vec{B} = (1,1,-2)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} \hat{k}$$

$$\vec{A} \times \vec{B} = [(-1)(-2) - (1)(1)]\hat{\imath} - [(3)(-2) - (1)(1)]\hat{\jmath} + [(3)(1) - (1)(-1)]\hat{k}$$

$$\vec{A} \times \vec{B} = (2 - 1)\hat{\imath} — 6 - 1 ⍰ \hat{\jmath} + [3 — 1) ⍰ \hat{k}$$

$$\vec{A} \times \vec{B} = 1\hat{\imath} — 7 ⍰ \hat{\jmath} + 4\hat{k}$$

$$\boxed{\vec{A} \times \vec{B} = (1,7,4)}$$

Vector Differentiation

Problem 20

For each of the following vectors, calculate $\frac{d\vec{a}}{dt}$:

20.a $\vec{a} = 3t^2\hat{\imath} + t^3\hat{\jmath} - (t^2 - t^3)\hat{k}$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left[3t^2\hat{\imath} + t^3\hat{\jmath} - (t^2 - t^3)\hat{k}\right]$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(3t^2)\hat{\imath} + \frac{d}{dt}(t^3)\hat{\jmath} - \frac{d}{dt}(t^2 - t^3)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 3 \cdot \frac{d}{dt}(t^2)\hat{\imath} + \frac{d}{dt}(t^3)\hat{\jmath} - \left[\frac{d}{dt}(t^2) - \frac{d}{dt}(t^3)\right]\hat{k}$$

$$\frac{d\vec{a}}{dt} = 3 \cdot (2t^{2-1})\hat{\imath} + (3t^{3-1})\hat{\jmath} - (2t^{2-1} - 3t^{3-1})\hat{k}$$

$$\frac{d\vec{a}}{dt} = 6t^1\hat{\imath} + 3t^2\hat{\jmath} - (2t^1 - 3t^2)\hat{k}$$

$$\boxed{\frac{d\vec{a}}{dt} = 6t\hat{\imath} + 3t^2\hat{\jmath} - (2t - 3t^2)\hat{k}}$$

20.b $\vec{a} = 3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left(3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}\right)$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(3t^2)\hat{\imath} + \frac{d}{dt}(4t^3)\hat{\jmath} - \frac{d}{dt}(6t)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 3 \cdot \frac{d}{dt}(t^2)\hat{\imath} + 4 \cdot \frac{d}{dt}(t^3)\hat{\jmath} - 6 \cdot \frac{d}{dt}(t^1)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 3 \cdot (2t^{2-1})\hat{\imath} + 4 \cdot (3t^{3-1})\hat{\jmath} - 6 \cdot t^{1-1}\hat{k}$$

$$\frac{d\vec{a}}{dt} = 6t^1\hat{\imath} + 12t^2\hat{\jmath} - 6t^0\hat{k}$$

$$\frac{d\vec{a}}{dt} = 6t\hat{\imath} + 12t^2\hat{\jmath} - 6(1)\hat{k}$$

$$\boxed{\frac{d\vec{a}}{dt} = 6t\hat{\imath} + 12t^2\hat{\jmath} - 6\hat{k}}$$

20.c $\vec{a} = (t^2, \cos(t), 7)$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left[t^2\hat{\imath} + \cos(t)\,\hat{\jmath} - 7\hat{k}\right]$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(t^2)\hat{\imath} + \frac{d}{dt}[\cos(t)]\hat{\jmath} - \frac{d}{dt}(7t^0)\hat{k}$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(t^2)\hat{\imath} + \frac{d}{dt}[\cos(t)]\hat{\jmath} - 7 \cdot \frac{d}{dt}(t^0)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 2t^{2-1}\hat{\imath} + [-\sin(t)]\hat{\jmath} - 7 \cdot (0t^{0-1})\hat{k}$$

$$\frac{d\vec{a}}{dt} = 2t^1\hat{\imath} - \sin(t)\,\hat{\jmath} - 7 \cdot (0)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 2t\hat{\imath} - \sin(t)\,\hat{\jmath} - 0\hat{k}$$

$$\boxed{\frac{d\vec{a}}{dt} = (2t, -\sin(t), 0)}$$

20.d $\vec{a} = (t, 4, -6t)$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}\left(t\hat{\imath} + 4\hat{\jmath} - 6t\hat{k}\right)$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(t^1)\hat{\imath} + \frac{d}{dt}(4t^0)\hat{\jmath} - \frac{d}{dt}(6t^1)\hat{k}$$

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(t^1)\hat{\imath} + 4 \cdot \frac{d}{dt}(t^0)\hat{\jmath} - 6 \cdot \frac{d}{dt}(t^1)\hat{k}$$

$$\frac{d\vec{a}}{dt} = 1t^{1-1}\hat{\imath} + 4 \cdot (0t^{0-1})\hat{\jmath} - 6 \cdot (1t^{1-1})\hat{k}$$

$$\frac{d\vec{a}}{dt} = t^0\hat{\imath} + 3(0)t^2\hat{\jmath} - 6t^0\hat{k}$$

$$\frac{d\vec{a}}{dt} = 1\hat{\imath} + 0\hat{\jmath} - 6\hat{k}$$

$$\boxed{\frac{d\vec{a}}{dt} = (1,0,-6)}$$

Partial Differentiation of Vectors

Problem 21

For each of the following vectors, calculate $\frac{\partial\vec{u}}{\partial x}, \frac{\partial\vec{u}}{\partial y}, \frac{\partial\vec{u}}{\partial z}$, and $\frac{\partial^2\vec{u}}{\partial x\partial y}$:

21.a $\vec{u}\left(u_x, u_y, u_z\right) = (x + y^2, z + x, xz^2)$

$$\frac{\partial\vec{u}}{\partial x} = \frac{\partial}{\partial x}\left[(x + y^2)\hat{\imath} + (z + x)\hat{\jmath} + xz^2\hat{k}\right]$$

$$\frac{\partial\vec{u}}{\partial x} = \frac{\partial}{\partial x}[(x + y^2)\hat{\imath}] + \frac{\partial}{\partial x}[(z + x)\hat{\jmath}] + \frac{\partial}{\partial x}\left(xz^2\hat{k}\right)$$

$$\frac{\partial\vec{u}}{\partial x} = (1 + 0)\hat{\imath} + (0 + 1)\hat{\jmath} + z^2\hat{k}$$

$$\frac{\partial\vec{u}}{\partial x} = 1\hat{\imath} + 1\hat{\jmath} + z^2\hat{k}$$

$$\boxed{\frac{\partial\vec{u}}{\partial x} = (1,1,z^2)}$$

$$\frac{\partial \vec{u}}{\partial y} = \frac{\partial}{\partial y}\left[(x+y^2)\hat{\imath} + (z+x)\hat{\jmath} + xz^2\hat{k}\right]$$

$$\frac{\partial \vec{u}}{\partial y} = \frac{\partial}{\partial y}[(x+y^2)\hat{\imath}] + \frac{\partial}{\partial y}[(z+x)\hat{\jmath}] + \frac{\partial}{\partial y}\left(xz^2\hat{k}\right)$$

$$\frac{\partial \vec{u}}{\partial y} = (0+2y)\hat{\imath} + (0+0)\hat{\jmath} + 0\hat{k}$$

$$\frac{\partial \vec{u}}{\partial y} = 2y\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{\partial \vec{u}}{\partial y} = (2y,0,0)}$$

$$\frac{\partial \vec{u}}{\partial z} = \frac{\partial}{\partial z}\left[(x+y^2)\hat{\imath} + (z+x)\hat{\jmath} + xz^2\hat{k}\right]$$

$$\frac{\partial \vec{u}}{\partial z} = \frac{\partial}{\partial z}[(x+y^2)\hat{\imath}] + \frac{\partial}{\partial z}[(z+x)\hat{\jmath}] + \frac{\partial}{\partial z}\left(xz^2\hat{k}\right)$$

$$\frac{\partial \vec{u}}{\partial z} = (0+0)\hat{\imath} + (1+0)\hat{\jmath} + 2xz\hat{k}$$

$$\frac{\partial \vec{u}}{\partial z} = 0\hat{\imath} + 1\hat{\jmath} + 2xz\hat{k}$$

$$\boxed{\frac{\partial \vec{u}}{\partial z} = (0,1,2xz)}$$

$$\frac{\partial^2 \vec{u}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left[(x+y^2)\hat{\imath} + (z+x)\hat{\jmath} + xz^2\hat{k}\right]$$

$$\frac{\partial^2 \vec{u}}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left[(x+y^2)\hat{\imath} + (z+x)\hat{\jmath} + xz^2\hat{k}\right]\right\}$$

$$\frac{\partial^2 \vec{u}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2y\hat{\imath} + 0\hat{\jmath} + 0\hat{k}\right)$$

$$\frac{\partial^2 \vec{u}}{\partial x \partial y} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{\partial^2 \vec{u}}{\partial x \partial y} = \vec{0}}$$

21.b $\vec{u}(u_x, u_y, u_z) = (x^3 + y^2, zx, z^2 + y)$

$$\frac{\partial \vec{u}}{\partial x} = \frac{\partial}{\partial x}\left[(x^3 + y^2)\hat{\imath} + zx\hat{\jmath} + (z^2 + y)\hat{k}\right]$$

$$\frac{\partial \vec{u}}{\partial x} = (3x^2 + 0)\hat{\imath} + z\hat{\jmath} + (0 + 0)\hat{k}$$

$$\frac{\partial \vec{u}}{\partial x} = 3x^2\hat{\imath} + z\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{\partial \vec{u}}{\partial x} = (3x^2, z, 0)}$$

$$\frac{\partial \vec{u}}{\partial y} = \frac{\partial}{\partial y}\left[(x^3 + y^2)\hat{\imath} + zx\hat{\jmath} + (z^2 + y)\hat{k}\right]$$

$$\frac{\partial \vec{u}}{\partial y} = (0 + 2y)\hat{\imath} + 0\hat{\jmath} + (0 + 1)\hat{k}$$

$$\frac{\partial \vec{u}}{\partial y} = 2y\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

$$\boxed{\frac{\partial \vec{u}}{\partial y} = (2y, 0,1)}$$

$$\frac{\partial \vec{u}}{\partial z} = \frac{\partial}{\partial z}\left[(x^3 + y^2)\hat{\imath} + zx\hat{\jmath} + (z^2 + y)\hat{k}\right]$$

$$\frac{\partial \vec{u}}{\partial z} = (0 + 0)\hat{\imath} + x\hat{\jmath} + (2z + 0)\hat{k}$$

$$\frac{\partial \vec{u}}{\partial z} = 0\hat{\imath} + x\hat{\jmath} + 2z\hat{k}$$

$$\boxed{\frac{\partial \vec{u}}{\partial z} = (0, x, 2z)}$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y} = \frac{\partial^2}{\partial x\partial y}\left[(x^3+y^3)\hat{\imath} + zx\hat{\jmath} + (z^2+y)\hat{k}\right]$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left[(x^3+y^3)\hat{\imath} + zx\hat{\jmath} + (z^2+y)\hat{k}\right]\right\}$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y} = \frac{\partial}{\partial x}\left(3y^2\hat{\imath} + 0\hat{\jmath} + 1\hat{k}\right)$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{\partial^2\vec{u}}{\partial x\partial y} = \hat{0}}$$

21.c $\vec{u}(u_x, u_y, u_z) = (x^2+y^2+z^2, z, xz^2+2)$

$$\frac{\partial\vec{u}}{\partial x} = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)\hat{\imath} + z\hat{\jmath} + (xz^2+2)\hat{k}\right]$$

$$\frac{\partial\vec{u}}{\partial x} = (2x+0+0)\hat{\imath} + 0\hat{\jmath} + (z^2+0)\hat{k}$$

$$\frac{\partial\vec{u}}{\partial x} = 2x\hat{\imath} + 0\hat{\jmath} + z^2\hat{k}$$

$$\boxed{\frac{\partial\vec{u}}{\partial x} = (2x, 0, z^2)}$$

$$\frac{\partial\vec{u}}{\partial y} = \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)\hat{\imath} + z\hat{\jmath} + (xz^2+2)\hat{k}\right]$$

$$\frac{\partial\vec{u}}{\partial y} = (0+2y+0)\hat{\imath} + 0\hat{\jmath} + (0+0)\hat{k}$$

$$\frac{\partial\vec{u}}{\partial y} = 2y\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{\partial\vec{u}}{\partial y} = (2y, 0,0)}$$

$$\frac{\partial\vec{u}}{\partial z}=\frac{\partial}{\partial z}\left[(x^2+y^2+z^2)\hat{\imath}+z\hat{\jmath}+(xz^2+2)\hat{k}\right]$$

$$\frac{\partial\vec{u}}{\partial z}=(0+0+2z)\hat{\imath}+1\hat{\jmath}+(2xz+0)\hat{k}$$

$$\frac{\partial\vec{u}}{\partial z}=2z\hat{\imath}+1\hat{\jmath}+2xz\hat{k}$$

$$\boxed{\frac{\partial\vec{u}}{\partial z}=(2z,1,2xz)}$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y}=\frac{\partial^2}{\partial x\partial y}\left[(x^2+y^2+z^2)\hat{\imath}+z\hat{\jmath}+(xz^2+2)\hat{k}\right]$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y}=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left[(x^2+y^2+z^2)\hat{\imath}+z\hat{\jmath}+(xz^2+2)\hat{k}\right]\right\}$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y}=\frac{\partial}{\partial x}\left(2y\hat{\imath}+0\hat{\jmath}+0\hat{k}\right)$$

$$\frac{\partial^2\vec{u}}{\partial x\partial y}=0\hat{\imath}+0\hat{\jmath}+0\hat{k}$$

$$\boxed{\frac{\partial^2\vec{u}}{\partial x\partial y}=\hat{0}}$$

Vector Integration

Problem 22

For each of the following vectors, calculate $\int_1^2 \vec{a}(t)dt$:

22.a $\vec{a} = 3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}$

$$\int_1^2 \vec{a}(t)dt = \int_1^2 \left(3t^2\hat{\imath} + 4t^3\hat{\jmath} - 6t\hat{k}\right)dt$$

$$\int_1^2 \vec{a}(t)dt = t^3\hat{\imath} + t^4\hat{\jmath} - 3t^2\hat{k}\Big|_1^2$$

$$\int_1^2 \vec{a}(t)dt = [(2)^3 - (1)^3]\hat{\imath} + [(2)^4 - (1)^4]\hat{\jmath} - 3[(2)^2 - (1)^2]\hat{k}$$

$$\int_1^2 \vec{a}(t)dt = (8 - 1)\hat{\imath} + (16 - 1)\hat{\jmath} - 3(4 - 1)\hat{k}$$

$$\boxed{\int_1^2 \vec{a}(t)dt = 7\hat{\imath} + 15\hat{\jmath} - 9\hat{k}}$$

22.b $\vec{a} = t^2\hat{\imath} + 4t^3\hat{\jmath} - \hat{k}$

$$\int_1^2 \vec{a}(t)dt = \int_1^2 \left(t^2\hat{\imath} + 4t^3\hat{\jmath} - \hat{k}\right)dt$$

$$\int_1^2 \vec{a}(t)dt = 2t\hat{\imath} + 12t^2\hat{\jmath} - t\hat{k}\Big|_1^2$$

$$\int_1^2 \vec{a}(t)dt = [2(2) - 2(1)]\hat{\imath} + [12(2)^2 - 12(1)^2]\hat{\jmath} - [(2) - (1)]\hat{k}$$

$$\int_1^2 \vec{a}(t)dt = [2(2) - 2(1)]\hat{\imath} + [12(4) - 12(1)]\hat{\jmath} - [(2) - (1)]\hat{k}$$

$$\int_1^2 \vec{a}(t)dt = (4 - 2)\hat{\imath} + (48 - 12)\hat{\jmath} - (2 - 1)\hat{k}$$

$$\boxed{\int_1^2 \vec{a}(t)dt = 2\hat{\imath} + 36\hat{\jmath} - \hat{k}}$$

22.c $\vec{a} = (1, \cos(t), \sin(t))$

$$\int_1^2 \vec{a}(t)dt = \int_1^2 \left[\hat{\imath} + \cos(t)\,\hat{\jmath} + \sin(t)\,\hat{k}\right]dt$$

$$\int_1^2 \vec{a}(t)dt = t\hat{\imath} + \sin(t)\,\hat{\jmath} + [-\cos(t)]\hat{k}\Big|_1^2$$

$$\int_1^2 \vec{a}(t)dt = [(2) - (1)]\hat{\imath} + [\sin(2) - \sin(1)]\hat{\jmath} - [\cos(2) - \cos(1)]\hat{k}$$

$$\int_1^2 \vec{a}(t)dt \approx (2 - 1)\hat{\imath} + (.909 - .841)\hat{\jmath} - (-.416 - .540)\hat{k}$$

$$\int_1^2 \vec{a}(t)dt \approx \hat{\imath} + .068\hat{\jmath} + .956\hat{k}$$

$$\boxed{\int_1^2 \vec{a}(t)dt \approx (1, .068, .956)}$$

22.d $\vec{a} = 2t\hat{\imath} + \hat{k}$

$$\int_1^2 \vec{a}(t)dt = \int_1^2 \left(2t\hat{\imath} + \hat{k}\right)dt$$

$$\int_1^2 \vec{a}(t)dt = t^2\hat{\imath} + t\hat{k}\Big|_1^2$$

$$\int_1^2 \vec{a}(t)dt = [(2)^2 - (1)^2]\hat{\imath} + [(2) - (1)]\hat{k}$$

$$\int_1^2 \vec{a}(t)dt = (4 - 2)\hat{\imath} + (2 - 1)\hat{k}$$

$$\boxed{\int_1^2 \vec{a}(t)dt = 2\hat{\imath} + \hat{k}}$$

Homogeneous Systems of Linear Equations

Problem 23

For each of the following systems of linear equations, determine if the system is homogeneous:

For a system of linear equations to be homogeneous, the constant term of each equation must be zero.

23.a $\begin{cases} x+y-z=0 \\ 2x+3y+z=0 \\ x-y+2z=0 \end{cases}$ Homogeneous.

23.b $\begin{cases} x+3y-z=5 \\ x+3y+8z=0 \\ x-y+2z=0 \end{cases}$ **Not** homogeneous due to first equation.

23.c $\begin{cases} x+y-z=1 \\ 3y+z=0 \\ z=0 \end{cases}$ **Not** homogeneous due to first equation.

System Consistency

Problem 24

For each of the following systems of linear equations, determine if it is inconsistent or consistent:

An inconsistent system of linear equations has no solution and may be identified if it shows a contradiction when placed in row-echelon form.

24.a $\begin{cases} x+y-z=1 \\ 2x+3y+z=6 \\ x-y+2z=2 \end{cases}$

$$\begin{cases} x+y-z=1 \\ 2x+3y+z=6 \\ x-y+2z=2 \end{cases} \xrightarrow{-2E_1+E_2} \begin{array}{r} -2x-2y+2z=-2 \\ +\ \underline{2x+3y+z=6} \\ y+3z=4 \end{array}$$

$$\begin{cases} x+y-z=1 \\ y+3z=4 \\ x-y+2z=2 \end{cases} \xrightarrow{-E_1+E_3} \begin{array}{r} -x-y+z=-1 \\ +\ \underline{x-y+2z=2} \\ -2y+3z=1 \end{array}$$

$$\begin{cases} x+y-z=1 \\ y+3z=4 \\ -2y+3z=1 \end{cases} \xrightarrow{2E_2+E_3} \begin{array}{r} 2y+6z=8 \\ \underline{+-2y+3z=1} \\ 9z=9 \end{array}$$

$$\begin{cases} x+y-z=1 \\ y+3z=4 \\ 9z=9 \end{cases} \xrightarrow{E_3/9} z=1$$

$\begin{cases} x+y-z=1 \\ y+3z=4 \\ z=1 \end{cases}$ Appears **consistent**; no contradictions.

24.b $\begin{cases} x+3y-z=5 \\ x+3y+8z=0 \\ 0z=0 \end{cases}$

$$\begin{cases} x+3y-z=5 \\ x+3y+8z=0 \\ 0z=0 \end{cases} \xrightarrow{E_2+(-E_1)} \begin{array}{r} x+3y+8z=0 \\ +\ \ -x-3y+z=-5 \\ \hline 9z=-5 \end{array}$$

$$\begin{cases} x+3y-z=5 \\ 9z=-5 \\ 0z=0 \end{cases}$$

$$\begin{cases} x+3y-z=5 \\ 9z=-5 \xrightarrow{E_2/9} z=-\dfrac{5}{9} \\ 0=0 \end{cases}$$

$$\begin{cases} x+3y-z=5 \\ z=-\frac{5}{9} \\ 0=0 \end{cases}$$

Appears **consistent**; no contradictions.

24.c $\begin{cases} x+y-z=1 \\ 3y+z=0 \\ 0z=4 \end{cases}$

Inconsistent due to third equation; $0 \neq 4$.

Free Variables and Leading Unknowns (Pivots)

Problem 25

For each of the following systems of linear equations, identify the free variables and the leading unknowns:

25.a $\begin{cases} \boldsymbol{x}+y-z=1 \\ 3\boldsymbol{y}+z=0 \end{cases}$

By inspection, Leading unknowns: x, y
Free variables: z

25.b $\begin{cases} \boldsymbol{x}+3y-z+s-2t=5 \\ 2\boldsymbol{y}+8z+2s+t=4 \\ \boldsymbol{s}+2t=1 \end{cases}$

By inspection, Leading unknowns: x, y, s
Free variables: z, t

25.c $\boldsymbol{x}+y-z=1$

By inspection, Leading unknowns: x
Free variables: y, z

Gaussian Elimination

Problem 26

Solve the following systems of linear equations using Gaussian elimination:

26.a $\begin{cases} x + 2y = 4 \\ 2x + y = 5 \end{cases}$

$$\begin{cases} x + 2y = 4 \\ 2x + y = 5 \end{cases} \xrightarrow{E_2 + (-2E_1)} \begin{array}{r} 2x + y = 5 \\ +(-2x - 4y) = -8 \\ \hline -3y = -3 \end{array}$$

$$\begin{cases} x + 2y = 4 \\ \quad -3y = -3 \end{cases} \xrightarrow{-E_2/3} y = 1$$

$$\begin{cases} x + 2y = 4 \\ \qquad y = 1 \end{cases}$$

If $y = -1$, we can substitute this into the first equation:

$x + 2(1) = 4$

$x + 2 = 4$

$x + 2 - 2 = 4 - 2$

$x = 2$

$\boxed{(2,1)}$

26.b $\begin{cases} x - 3y = -2 \\ 5x + y = 6 \end{cases}$

$$\begin{cases} x - 3y = -2 \\ 5x + y = 6 \end{cases} \xrightarrow{E_2 - 5E_1} \begin{array}{r} 5x + y = 6 \\ +(-5x + 15y) = 10 \\ \hline 16y = 16 \end{array}$$

$$\begin{cases} x - 3y = -2 \\ \quad 16y = 16 \end{cases} \xrightarrow{E_2/16} y = 1$$

$$\begin{cases} x - 3y = -2 \\ \qquad y = 1 \end{cases}$$

If $y = 1$, we can substitute into the first equation:

$x - 3(1) = -2$

$x - 3 = -2$

$x - 3 + 3 = -2 + 3$

$x = 1$

$\boxed{(1,1)}$

26.c $\begin{cases} x + 3y = 8 \\ 3x + y = 16 \end{cases}$

$$\begin{cases} x + 3y = 8 \\ 3x + y = 16 \end{cases} \xrightarrow{E_2 - 3E_1} \begin{array}{r} 3x + y = 16 \\ \underline{+(-3x - 9y) = -24} \\ -8y = -8 \end{array}$$

$$\begin{cases} x + 3y = 8 \\ -8y = -8 \xrightarrow{-E_2/8} y = 1 \end{cases}$$

$$\begin{cases} x + 3y = 8 \\ y = 1 \end{cases}$$

If $y = 1$, we can substitute into the first equation:

$x + 3(1) = 8$

$x + 3 = 8$

$x + 3 - 3 = 8 - 3$

$x = 5$

$\boxed{(5,1)}$

26.d $\begin{cases} x + y - z = 1 \\ 2x + 3y + z = 6 \\ x - y + 2z = 2 \end{cases}$

$$\begin{cases} x + y - z = 1 \\ 2x + 3y + z = 6 \\ x - y + 2z = 2 \end{cases} \xrightarrow{E_2 + (-2E_1)} \begin{array}{r} 2x + 3y + z = 6 \\ \underline{+(-2x - 2y + 2z) = -2} \\ y + 3z = 4 \end{array}$$

$$\begin{cases} x + y - z = 1 \\ y + 3z = 4 \\ x - y + 2z = 2 \end{cases} \xrightarrow{E_3 + (-E_1)} \begin{array}{r} x - y + 2z = 2 \\ \underline{+(-x - y + z) = -1} \\ -2y + 3z = 1 \end{array}$$

$$\begin{cases} x + y - z = 1 \\ y + 3z = 4 \\ -2y + 3z = 1 \end{cases} \xrightarrow{E_3 + 2E_2} \begin{array}{r} -2y + 3z = 1 \\ \underline{+2y + 6z = 8} \\ 9z = 9 \end{array}$$

$$\begin{cases} x + y - z = 1 \\ y + 3z = 4 \\ 9z = 9 \xrightarrow{E_3/9} z = 1 \end{cases}$$

$$\begin{cases} x + y - z = 1 \\ y + 3z = 4 \\ z = 1 \end{cases}$$

If $z = 1$, we can substitute into Equation 2:

$y + 3(1) = 4$

$y+3=4$

$y+3-3=4-3$

$y=1$

If $y=1$ and $z=1$, we can substitute into Equation 1:

$x+(1)-(1)=1$

$x=1$

$\boxed{(1,1,1)}$

26.e $\begin{cases} x+3y-z=7 \\ 2x+3y+z=8 \\ 3x-y+2z=1 \end{cases}$

$\begin{cases} x+3y-z=7 \\ 2x+3y+z=8 \\ 3x-y+2z=1 \end{cases} \xrightarrow{E_2-2E_1} \begin{array}{r} 2x+3y+z=8 \\ \underline{+(-2x-6y+2z)=-14} \\ -3y+3z=-6 \end{array}$

$\begin{cases} x+3y-z=7 \\ -3y+3z=-6 \xrightarrow{-E_2/3} y-z=2 \\ 3x-y+2z=1 \end{cases}$

$\begin{cases} x+3y-z=7 \\ y-z=2 \\ 3x-y+2z=1 \end{cases} \xrightarrow{E_3+(-3E_1)} \begin{array}{r} 3x-y+2z=1 \\ \underline{+(-3x-9y+3z)=-21} \\ -10y+5z=-20 \end{array}$

$\begin{cases} x+3y-z=7 \\ y-z=2 \\ -10y+5z=-20 \end{cases} \xrightarrow{E_3+10E_2} \begin{array}{r} -10y+5z=-20 \\ \underline{+10y-10z=20} \\ -5z=0 \end{array}$

$\begin{cases} x+3y-z=7 \\ y-z=2 \\ -5z=0 \xrightarrow{-E_3/5} z=0 \end{cases}$

$\begin{cases} x+3y-z=7 \\ y-z=2 \\ z=0 \end{cases}$

If $z=0$, we can substitute into the second equation:

$y-(0)=2$

$y=2$

If $y=2$ and $z=0$, we can substitute into the first equation:

$x+3(2)-(0)=7$

$x+6=7$

$x + 6 - 6 = 7 - 6$

$x = 1$

$\boxed{(1,2,0)}$

26.f $\begin{cases} x + y - z = 0 \\ 5x - 3y + z = 2 \\ 3x - 2y + z = 2 \end{cases}$

$$\begin{cases} x + y - z = 0 \\ 5x - 3y + z = 2 \\ 3x - 2y + z = 2 \end{cases} \xrightarrow{E_2 + (-5E_1)} \begin{array}{r} 5x - 3y + z = 2 \\ +(-5x - 5y + 5z) = 0 \\ \hline -8y + 6z = 2 \end{array}$$

$$\begin{cases} x + y - z = 0 \\ -8y + 6z = 2 \\ 3x - 2y + z = 2 \end{cases} \xrightarrow{-E_2/8} y - \frac{3}{4}z = -\frac{1}{4}$$

$$\begin{cases} x + y - z = 0 \\ y - \frac{3}{4}z = -\frac{1}{4} \\ 3x - 2y + z = 2 \end{cases} \xrightarrow{E_3 + (-3E_1)} \begin{array}{r} 3x - 2y + z = 2 \\ +(-3x - 3y + 3z) = 0 \\ \hline -5y + 4z = 2 \end{array}$$

$$\begin{cases} x + y - z = 0 \\ y - \frac{3}{4}z = -\frac{1}{4} \\ -5y + 4z = 2 \end{cases} \xrightarrow{E_3 + 5E_2} \begin{array}{r} -5y + 4z = 2 \\ +5y - \frac{15}{4}z = -\frac{5}{4} \\ \hline \frac{1}{4}z = \frac{3}{4} \end{array}$$

$$\begin{cases} x + y - z = 0 \\ y - \frac{3}{4}z = -\frac{1}{4} \\ \frac{1}{4}z = \frac{3}{4} \end{cases} \xrightarrow{4E_3} z = 3$$

$$\begin{cases} x + y - z = 0 \\ y - \frac{3}{4}z = -\frac{1}{4} \\ z = 3 \end{cases}$$

If $z = 3$, we can substitute into the second equation:

$y - \frac{3}{4}(3) = -\frac{1}{4}$

$y - \frac{9}{4} = -\frac{1}{4}$

$y - \frac{9}{4} + \frac{9}{4} = -\frac{1}{4} + \frac{9}{4}$

$y = \frac{8}{4} = 2$

If $y = 2$ and $z = 3$, we can substitute into the first equation:

$x + (2) - (3) = 0$

$x - 1 = 0$

$x - 1 + 1 = 0 + 1$

$x = 1$

$\boxed{(1,2,3)}$

Subspaces

Problem 27

Determine if any of the following sets are subspaces of $\mathbb{R}^2$.

Four-step process to determine subspace status:

- Is the set a subset of the larger space?
- Does $\vec{0}$ exist in the set?
- For arbitrary $\vec{p}$ and $\vec{q}$ in the set, is $\vec{p} + \vec{q}$ also in the set?
- For arbitrary $\vec{p}$ in the set, is $k\vec{p}$ also in the set?

27.a Is $W = \{(3x, 5y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

Subset:

The elements of W are all 2-tuples, so W is a subset of $\mathbb{R}^2$.

Identity property:

$(0,0) = (3x_1, 5y_1)$

$3x_1 = 0$ $\quad\quad$ $5y_1 = 0$

$x_1 = 0 \in \mathbb{R}$ $\quad\quad$ $y_1 = 0 \in \mathbb{R}$

Set W contains $\vec{0}$.

Addition closure:

Let $\vec{p} = (3x_1, 5y_1)$ and $\vec{p} = (3x_2, 5y_2)$:

$\vec{p} + \vec{q} = (3x_1 + 3x_2, 5y_1 + 5y_2)$

$\vec{p} + \vec{q} = \left(3(x_1 + x_2), 5(y_1 + y_2)\right)$

$\vec{p} + \vec{q} = (3x_3, 5y_3): x \in \mathbb{R}, y \in \mathbb{R}$

$\vec{p} + \vec{q} \in W$ Closed under addition.

Multiplication closure:

Let $\vec{p} = (3x_1, 5y_1)$:

$k\vec{p} = k(3x_1, 5y_1)$

$k\vec{p} = (3kx_1, 5ky_1)$

$k\vec{p} = \left(3(kx_1), 5(ky_1)\right)$

$k\vec{p} = (3x_2, 5y_2): x_2 \in \mathbb{R}, y_2 \in \mathbb{R}$

$k\vec{p} \in W$ Closed under scalar multiplication.

W is a subspace of $\mathbb{R}^2$.

27.b Is $W = \{(x, y+1) : x \in \mathbb{R}, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

Subset:

The elements of W are all 2-tuples, so W is a subset of $\mathbb{R}^2$.

Identity Property:

$(0,0) = (x_1, y_1 + 1)$

$x_1 = 0 \in \mathbb{R}$ $\quad$ $y_1 + 1 = 0$

$y_1 = -1 \in \mathbb{R}$

Set W contains $\vec{0}$.

Addition closure.

Let $\vec{p} = (x_1, y_1 + 1)$ and $\vec{q} = (x_2, y_2 + 1)$:

$\vec{p} + \vec{q} = \left(x_1 + x_2, (y_1 + 1) + (y_2 + 1)\right)$

$\vec{p} + \vec{q} = (x_1 + x_2, (y_1 + y_2) + 2)$

$\vec{p} + \vec{q} = \left((x_1 + x_2), (y_1 + y_2 + 1) + 1\right)$

$\vec{p} + \vec{q} = (x_3, y_3 + 1): x_3 \in \mathbb{R}, y_3 \in \mathbb{R}$

$\vec{P} + \vec{Q} \in W$ Closed under addition.

Multiplication closure:

Let $\vec{p} = (x_1, y_1 + 1)$:

$k\vec{p} = k(x_1, y_1 + 1\ )$

$k\vec{p} = \big(kx_1, k(y_1 + 1)\big)$

$k\vec{p} = (kx_1, ky_1 + k)$

$k\vec{p} = (kx_1, ky_1 + k - 1 + 1)$

$k\vec{p} = (kx_1, (ky_1 + k - 1) + 1)$

$k\vec{p} = (x_2, y_2 + 1): x_2 \in \mathbb{R}, y_2 \in \mathbb{R}$

$k\vec{p} \in W$ Closed under scalar multiplication.

W is a subspace of $\mathbb{R}^2$.

27.c Is $W = \{10x : x \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

Subset:

The elements of W are all 1-tuples, so W is ***not*** a subset of $\mathbb{R}^2$.

W is ***not*** a subspace of $\mathbb{R}^2$.

Linear Combination

Problem 28

For the following vector sets, determine whether $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

28.a $\vec{w} = (0,2)$, $\vec{u} = (1,3)$, $\vec{v} = (2,4)$

$x\vec{u} + y\vec{v} = \vec{w}$

$a(1,3) + b(2,4) = (0,2)$

$$\begin{cases} a + 2b = 0 \\ 3a + 4b = 2 \end{cases}$$

$$\begin{cases} a + 2b = 0 \\ 3a + 4b = 2 \end{cases} \xrightarrow{E_2 + (-3E_1)} \begin{array}{r} 3a + 4b = 2 \\ \underline{+(-3a - 6b) = 0} \\ -2b = 2 \end{array}$$

$$\begin{cases} a + 2b = 0 \\ \quad -2b = 2 \end{cases} \xrightarrow{-E_2/2} b = -1$$

$$\begin{cases} a + 2b = 0 \\ \quad\quad b = -1 \end{cases}$$

If $b = -1$, we can substitute into the first equation:

$a + 2(-1) = 0$

$a-2=0$

$a=2$

$(2,-1) \to 2\vec{u}-\vec{v}=\vec{w}$ linear combination

28.b $\vec{w}=(3,0), \vec{u}=(1,0), \vec{v}=(0,2)$

$a\vec{u}+b\vec{v}=\vec{w}$

$a(1,0)+b(0,2)=(3,0)$

$\begin{cases} a+0b=3 \\ 0a+2b=0 \end{cases}$

$\begin{cases} a=3 \\ 2b=0 \end{cases}$

$\begin{cases} a=3 \\ 2b=0 \xrightarrow{E_2=E_2/2} b=0 \end{cases}$

$\begin{cases} a=3 \\ b=0 \end{cases}$

$(3,0) \to 3\vec{u}=\vec{w}$ linear combination

28.c $\vec{w}=(5,2), \vec{u}=(1,0), \vec{v}=(0,1)$

$a\vec{u}+b\vec{v}=\vec{w}$

$a(1,0)+b(0,1)=(5,2)$

$\begin{cases} a+0b=5 \\ 0a+b=2 \end{cases}$

$\begin{cases} a=5 \\ b=2 \end{cases}$

$(5,2) \to 5\vec{u}+2\vec{v}=\vec{w}$ linear combination

28.d $\vec{w}=(1,2,0), \vec{u}=(1,0,0), \vec{v}=(0,1,0)$

$a\vec{u}+b\vec{v}=\vec{w}$

$a(1,0,0)+b(0,1,0)=(1,2,0)$

$\begin{cases} a+0b=1 \\ 0a+b=2 \\ 0a+0b=0 \end{cases}$

$\begin{cases} a=1 \\ b=2 \\ 0=0 \end{cases}$

$(1,2,0) \to \vec{u}+2\vec{v}=\vec{w}$ linear combination

Linear Independence

Problem 29

For the following vector sets, determine if the vectors are linearly dependent or independent:

If a set of vectors are linearly independent, the only set of coefficients (c_n) for which the sum of products of the coefficients and the individual vectors will equal the zero vector is zero. That is $c_n = 0$.

Declare coefficient variables, create summation, create equations for each vector component, solve resultant system.

29.a $\vec{a} = (1,3)$ and $\vec{b} = (2,3)$

$c_1\vec{a} + c_2\vec{b} = \hat{0}$

$c_1(1,3) + c_2(2,3) = (0,0)$

$$\begin{cases} c_1 + 2c_2 = 0 \\ 3c_1 + 3c_2 = 0 \end{cases} \xrightarrow{E_2 + (-3E_1)} \begin{array}{r} 3c_1 + 3c_2 = 0 \\ +(-3c_1 - 6c_2) = 0 \\ \hline -3c_2 = 0 \end{array}$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ \quad -3c_2 = 0 \end{cases} \xrightarrow{-E_2/3} c_2 = 0$$

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_2 &= 0 \end{aligned}$$

If $c_2 = 0$, we can substitute into the first equation:

$c_1 + 2(0) = 0$

$c_1 + 0 = 0$

$c_1 = 0$

$c_1 = c_2 = 0$

Therefore, these two vectors are linearly **independent**.

29.b $\vec{a} = (6,4)$ and $\vec{b} = (12,8)$

$c_1\vec{a} + c_2\vec{b} = \hat{0}$

$c_1(6,4) + c_2(12,8) = (0,0)$

$$\begin{cases} 6c_1 + 12c_2 = 0 \xrightarrow{E_1/6} c_1 + 2c_2 = 0 \\ 4c_1 + 8c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ 4c_1 + 8c_2 = 0 \end{cases} \xrightarrow{E_2 + (-4E_1)} \begin{array}{r} 4c_1 + 8c_2 = 0 \\ +(-4c_1 - 8c_2) = 0 \\ \hline 0 = 0 \end{array}$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ \quad 0 = 0 \end{cases}$$

There is an infinite number of solutions, so these two vectors are linearly **dependent**.

29.c $\vec{a} = (1,5)$ and $\vec{b} = (3,4)$

$c_1\vec{a} + c_2\vec{b} = \hat{0}$

$c_1(1,5) + c_2(3,4) = (0,0)$

$$\begin{cases} c_1 + 3c_2 = 0 \\ 5c_1 + 4c_2 = 0 \end{cases} \xrightarrow{E_2 + (53E_1)} \begin{array}{r} 5c_1 + 4c_2 = 0 \\ \underline{+(-5c_1 - 15c_2) = 0} \\ -11c_2 = 0 \end{array}$$

$$\begin{cases} c_1 + 3c_2 = 0 \\ -11c_2 = 0 \end{cases} \xrightarrow{-E_2/11} c_2 = 0$$

$$\begin{cases} c_1 + 3c_2 = 0 \\ c_2 = 0 \end{cases}$$

If $c_2 = 0$, we can substitute into the first equation:

$c_1 + 3(0) = 0$

$c_1 + 0 = 0$

$c_1 = 0$

$c_1 = c_2 = 0$

Therefore, these two vectors are linearly **independent**.

29.d $\vec{a} = (1,1,0)$, $\vec{b} = (1,2,1)$, and $\vec{c} = (1,1,1)$

$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$

$e_1(1,1,0) + e_2(1,2,1) + e_3(1,1,1) = (0,0,0)$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1 + 2e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \end{cases} \xrightarrow{E_2 + (-E_1)} \begin{array}{r} e_1 + 2e_2 + e_3 = 0 \\ \underline{+(-e_1 - e_2 - e_3) = 0} \\ e_2 = 0 \end{array}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 = 0 \\ e_2 + e_3 = 0 \end{cases} \xrightarrow{E_3 + (-E_2)} \begin{array}{r} e_2 + e_3 = 0 \\ \underline{+(-e_2) = 0} \\ e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 = 0 \\ e_3 = 0 \end{cases}$$

If $e_2 = 0$ and $e_3 = 0$, we can substitute into the first equation:

$e_1 + (0) + (0) = 0$

$e_1 = 0$

$e_1 = e_2 = e_2 = 0$

Therefore, these three vectors are linearly **independent**.

29.e $\vec{a} = (1,1,1)$, $\vec{b} = (1,2,0)$, and $\vec{c} = (0,-1,1)$

$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$

$e_1(1,1,1) + e_2(1,2,0) + e_3(0,-1,1) = (0,0,0)$

$$\begin{cases} e_1 + e_2 = 0 \\ e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} E_1 \leftrightarrow E_2$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_2 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{E_2+(-E_1)} \begin{array}{r} e_1 + e_3 = 0 \\ +(-e_1 - 2e_2 + e_3) = 0 \\ \hline -2e_2 + 2e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ -2e_2 + 2e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{-E_2/2} e_2 - e_3 = 0$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{E_3+(-E_1)} \begin{array}{r} e_1 + e_3 = 0 \\ +(-e_1 - 2e_2 + e_3) = 0 \\ \hline -2e_2 + 2e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ -2e_2 + 2e_3 = 0 \end{cases} \xrightarrow{E_3+(2E_2)} \begin{array}{r} -2e_2 + 2e_3 = 0 \\ +\ 2e_2 - 2e_3 = 0 \\ \hline 0 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ 0 = 0 \end{cases}$$

There is an infinite number of solutions, so these two vectors are linearly **dependent**.

29.f $\vec{a} = (1,2,3)$, $\vec{b} = (3,2,9)$, and $\vec{c} = (5,2,-1)$

$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$

$e_1(1,2,3) + e_2(3,2,9) + e_3(5,2,-1) = (0,0,0)$

$$\begin{cases} e_1 + 3e_2 + 5e_3 = 0 \\ 2e_1 + 2e_2 + 2e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{E_2+(-2E_1)} \begin{array}{r} 2e_1 + 2e_2 + 2e_3 = 0 \\ +(-2e_1 - 6e_2 - 10e_3) = 0 \\ \hline -4e_2 - 8e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 + 5e_3 = 0 \\ -4e_2 - 8e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{-E_2/4} e_2 + 2e_3 = 0$$

$$\begin{cases} e_1 + 3e_2 + 5e_3 = 0 \\ e_2 + 2e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{E_3+(-3E_1)} \begin{array}{r} 3e_1 + 9e_2 - e_3 = 0 \\ +(-3e_1 - 9e_2 - 15e_3) = 0 \\ \hline -16e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 + 5e_3 = 0 \\ e_2 + 2e_3 = 0 \\ -16e_3 = 0 \xrightarrow{-E_3/16} e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 + 5e_3 = 0 \\ e_2 + 2e_3 = 0 \\ e_3 = 0 \end{cases}$$

If $e_3 = 0$, we can substitute into the second equation:

$e_2 + 2(0) = 0$

$e_2 + 0 = 0$

$e_2 = 0$

If $e_2 = 0$ and $e_3 = 0$, we can substitute into the first equation:

$e_1 + 3(0) + 5(0) = 0$

$e_1 + 0 + 0 = 0$

$e_1 = 0$

$e_1 = e_2 = e_3 = 0$ linearly **independent**

29.g $\vec{a} = (1,2,3)$, $\vec{b} = (3,2,1)$, and $\vec{c} = (0,4,8)$

$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$

$e_1(1,2,3) + e_2(3,2,1) + e_3(0,4,8) = (0,0,0)$

$$\begin{cases} e_1 + 3e_2 = 0 \\ 2e_1 + 2e_2 + 4e_3 = 0 \xrightarrow{E_2+(-2E_1)} \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases} \qquad \begin{array}{r} 2e_1 + 2e_2 + 4e_3 = 0 \\ +\, (-2e_1 - 6e_2) = 0 \\ \hline -4e_2 + 4e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ -4e_2 + 4e_3 = 0 \xrightarrow{-E_2/4} e_2 - e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \xrightarrow{E_3+(-3E_1)} \end{cases} \qquad \begin{array}{r} 3e_1 + e_2 + 8e_3 = 0 \\ +(-3e_1 - 9e_2) = 0 \\ \hline -8e_2 + 8e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ -8e_2 + 8e_3 = 0 \xrightarrow{E_3+8E_2} \end{cases} \qquad \begin{array}{r} -8e_2 + 8e_3 = 0 \\ +\, 8e_2 - 8e_3 = 0 \\ \hline 0 = 0 \end{array}$$

There is an infinite number of solutions, so these two vectors are linearly **dependent**.

Basis of a Vector Space

Problem 30

For the following vector sets, determine if the set is a basis for the subsequent set:

Check that the vector set is linearly independent.

Show that any arbitrary vector can be expressed as multiples of the vectors in the subject set.

30.a $\vec{a} = (1,3)$ and $\vec{b} = (2,3)$ for $\mathbb{R}^2$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} = \hat{0}$$

$$e_1(1,3) + e_2(2,3) = (0,0)$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ 3e_1 + 3e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ 3e_1 + 3e_2 = 0 \end{cases} \xrightarrow{E_2+(-3E_1)} \begin{array}{r} 3e_1 + 3e_2 = 0 \\ +(-3e_1 - 6e_2) = 0 \\ \hline -3e_2 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ -3e_2 = 0 \end{cases} \xrightarrow{-E_2/3} e_2 = 0$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ e_2 = 0 \end{cases}$$

If $e_2 = 0$, we can substitute into the first equation:

$$e_1 + 2(0) = 0$$

$$e_1 + 0 = 0$$

$$e_1 = 0$$

$e_1 = e_2 = 0 \Rightarrow$ linearly independent

Span:

$$\vec{v} = (r, s)$$

$$x(1,3) + y(2,3) = (r, s)$$

$$\begin{cases} x + 2y = r \\ 3x + 3y = s \end{cases}$$

$$\begin{cases} x + 2y = r \\ 3x + 3y = s \end{cases} \xrightarrow{E_2+(-3E_1)} \begin{array}{r} 3x + 3y = s \\ +(-3x - 6y) = -3r \\ \hline -3y = s - 3r \end{array}$$

$$\begin{cases} x + 2y = r \\ -3y = s - 3r \end{cases} \xrightarrow{-E_2/3} y = r - \frac{1}{3}s$$

$$\begin{cases} x + 2y = r \\ y = r - \frac{1}{3}s \end{cases}$$

If $y = r - \frac{1}{3}s$, we can substitute into the first equation:

$$x + 2\left(r - \frac{1}{3}s\right) = r$$

$$x + 2r - \frac{2}{3}s = r$$

$$x + 2r - \frac{2}{3}s + \frac{2}{3}s = r + \frac{2}{3}s$$

$$x + 2r = r + \frac{2}{3}s$$

$$x + 2r - 2r = r - 2r + \frac{2}{3}s$$

$$x = \frac{2}{3}s - r$$

$\left(\frac{2}{3}s - r, r - \frac{1}{3}s\right) \Rightarrow \vec{a}$ and $\vec{b}$ span $\mathbb{R}^2$

Since $\vec{a}$ and $\vec{b}$ are linearly independent and span $\mathbb{R}^2$, they do form a basis for $\mathbb{R}^2$.

30.b $\vec{a} = (6,4)$ and $\vec{b} = (12,8)$ for $\mathbb{R}^2$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} = \hat{0}$$

$$e_1(6,4) + e_2(12,8) = (0,0)$$

$$\begin{cases} 6e_1 + 12e_2 = 0 \\ 4e_1 + 8e_2 = 0 \end{cases}$$

$$\begin{cases} 6e_1 + 12e_2 = 0 \xrightarrow{E_1/6} e_1 + 2e_2 = 0 \\ 4e_1 + 8e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ 4e_1 + 8e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ 4e_1 + 8e_2 = 0 \end{cases} \xrightarrow{E_2+(-4E_1)} \begin{array}{r} 4e_1 + 8e_2 = 0 \\ +(-4e_1 - 8e_2) = 0 \\ \hline 0 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 = 0 \\ 0 = 0 \end{cases}$$

There is an infinite number of solutions, so $\vec{a}$ and $\vec{b}$ are not linearly independent, so they **cannot be a basis** for $\mathbb{R}^2$.

30.c $\vec{a} = (1{,}5)$ and $\vec{b} = (3{,}4)$for $\mathbb{R}^2$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} = \hat{0}$$

$$e_1(1{,}5) + e_2(3{,}4) = (0{,}0)$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ 5e_1 + 4e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ 5e_1 + 4e_2 = 0 \xrightarrow{E_2 + (-5E_1)} \end{cases} \begin{array}{r} 5e_1 + 4e_2 = 0 \\ \underline{+(-5e_1 - 15e_2) = 0} \\ -11e_2 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ -11e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ -11e_2 = 0 \xrightarrow{-E_2/11} e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 = 0 \end{cases}$$

If $e_2 = 0$, we can substitute into the first equation:

$$e_1 + 3(0) = 0$$

$$e_1 + 0 = 0$$

$$e_1 = 0$$

Since $e_1 = e_2 = 0$, $\vec{a}$ and $\vec{b}$ are linearly independent.

Span:

$$\vec{v} = (r, s)$$

$$x(1{,}5) + y(3{,}4) = (r, s)$$

$$\begin{cases} x + 3y = r \\ 5x + 4y = s \end{cases}$$

$$\begin{cases} x + 3y = r \\ 5x + 4y = s \xrightarrow{E_2 + (-5E_1)} \end{cases} \begin{array}{r} 5x + 4y = s \\ \underline{+(-5x - 15y) = -5r} \\ -11y = s - 5r \end{array}$$

$$\begin{cases} x + 3y = r \\ -11y = s - 5r \end{cases}$$

$$\begin{cases} x + 3y = r \\ -11y = s - 5r \xrightarrow{-E_2/11} y = \frac{5}{11}r - \frac{1}{11}s \end{cases}$$

$$\begin{cases} x + 3y = r \\ y = \frac{5}{11}r - \frac{1}{11}s \end{cases}$$

Since $y = \frac{5}{11}r - \frac{1}{11}s$, we can substitute into the first equation:

$$x + 3\left(\frac{5}{11}r - \frac{1}{11}s\right) = r$$

$$x + \frac{15}{11}r - \frac{3}{11}s = r$$

$$x + \frac{15}{11}r - \frac{3}{11}s + \frac{3}{11}s = r + \frac{3}{11}s$$

$$x + \frac{15}{11}r - \frac{15}{11}r = r - \frac{15}{11}r + \frac{3}{11}s$$

$$x = \frac{3}{11}s - \frac{4}{11}r$$

$\left(\frac{3}{11}s - \frac{4}{11}r, \frac{5}{11}r - \frac{1}{11}s\right) \Rightarrow \vec{a}$ and $\vec{b}$ span $\mathbb{R}^2$.

Since $\vec{a}$ and $\vec{b}$ are linearly independent and span $\mathbb{R}^2$, $\vec{a}$ and $\vec{b}$ **are a basis** for $\mathbb{R}^2$.

30.d $\vec{a} = (1,1,0)$, $\vec{b} = (1,2,1)$, and $\vec{c} = (1,1,1)$ for $\mathbb{R}^3$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$$

$$e_1(1,1,0) + e_2(1,2,1) + e_3(1,1,1) = (0,0,0)$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1 + 2e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1 + 2e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \end{cases} E_2 \leftrightarrow E_3$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \\ e_1 + 2e_2 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \\ e_1 + 2e_2 + e_3 = 0 \end{cases} \xrightarrow{E_3 + (-E_1)} \begin{array}{r} e_1 + 2e_2 + e_3 = 0 \\ +(-e_1 - e_2 - e_3) = 0 \\ \hline e_2 = 0 \end{array}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \\ e_2 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \\ e_2 = 0 \end{cases} \xrightarrow{E_3 + (-E_2)} \begin{array}{r} e_2 = 0 \\ +(-e_2 - e_3) = 0 \\ \hline -e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ \quad e_2 + e_3 = 0 \\ \quad\quad -e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ \quad e_2 + e_3 = 0 \\ \quad\quad -e_3 = 0 \xrightarrow{-E_3} e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ \quad e_2 + e_3 = 0 \\ \quad\quad e_3 = 0 \end{cases}$$

Since $e_3 = 0$, we can substitute into the second equation:

$e_2 + (0) = 0$

$e_2 = 0$

Since $e_2 = 0$ and $e_3 = 0$, we can substitute into the first equation:

$e_1 + (0) + (0) = 0$

$e_1 = 0$

Since $e_1 = e_2 = e_3 = 0$, $\vec{a}$ and $\vec{b}$ and $\vec{c}$ are linearly independent.

Span:

$\vec{v} = (r, s, t)$

$x(1,1,0) + y(1,2,1) + z(1,1,1) = (r, s, t)$

$$\begin{cases} x + y + z = r \\ x + 2y + z = s \\ y + z = t \end{cases}$$

$$\begin{cases} x + y + z = r \\ x + 2y + z = s \xrightarrow{E_2 + (-E_1)} \\ y + z = t \end{cases} \qquad \begin{array}{r} x + 2y + z = s \\ -x - y - z = -r \\ \hline y = s - r \end{array}$$

$$\begin{cases} x + y + z = r \\ y = s - r \\ y + z = t \end{cases}$$

$$\begin{cases} x + y + z = r \\ y = s - r \\ y + z = t \xrightarrow{E_3 + (-E_2)} \end{cases} \qquad \begin{array}{r} y + z = t \\ -y = r - s \\ \hline z = r + t - s \end{array}$$

$$\begin{cases} x + y + z = r \\ y = s - r \\ z = r + t - s \end{cases}$$

If $y = s - r$ and $z = r + t - s$, we can substitute into the first equation:

$x + (s - r) + (r + t - s) = r$

$x + s - r + r + t - s = r$

$x - r + r + s - s + t = r$

$x + t = r$

$x = r - t$

$(r - t, s - r, r + t - s) \Rightarrow$ this vector set spans $\mathbb{R}^3$.

Since this vector set is linearly independent and spans $\mathbb{R}^3$, it **is a basis** for $\mathbb{R}^3$.

30.e $\vec{a} = (1,1,1)$, $\vec{b} = (1,2,0)$, and $\vec{c} = (0,-1,1)$ for $\mathbb{R}^3$

Linear independence:

$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$

$e_1(1,1,1) + e_2(1,2,0) + e_3(0,-1,1) = (0,0,0)$

$$\begin{cases} e_1 + e_2 = 0 \\ e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + e_2 = 0 \\ e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} E_1 \leftrightarrow E_2$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_2 = 0 \\ e_1 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_1 + e_2 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{E_2 + (-E_1)} \begin{array}{r} e_1 + e_2 = 0 \\ \underline{+(-e_1 - 2e_2 + e_3) = 0} \\ -e_2 + e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ -e_2 + e_3 = 0 \\ e_1 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ -e_2 + e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{-E_2} e_2 - e_3 = 0$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ e_1 + e_3 = 0 \end{cases} \xrightarrow{E_3 + (-E_1)} \begin{array}{r} e_1 + e_3 = 0 \\ \underline{-e_1 - 2e_2 + e_3 = 0} \\ -2e_2 + 2e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ -2e_2 + 2e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 2e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ -2e_2 + 2e_3 = 0 \end{cases} \xrightarrow{E_3+2E_2} \begin{array}{r} -2e_2 + 2e_3 = 0 \\ 2e_2 - 2e_3 = 0 \\ \hline 0 = 0 \end{array}$$

There is an infinite number of solutions for this system of equations, so this vector set is not linearly independent.

Since this vector set is not linearly independent, it is **not a basis** for $\mathbb{R}^3$.

30.f $\vec{a} = (1,2,3)$, $\vec{b} = (3,2,9)$, and $\vec{c} = (5,2,-1)$ for $\mathbb{R}^3$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$$

$$e_1(1,2,3) + e_2(3,2,9) + e_3(5,2,-1) = (0,0,0)$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ 2e_1 + 2e_2 + 2e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ 2e_1 + 2e_2 + 2e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{E_2+(-2E_1)} \begin{array}{r} 2e_1 + 2e_2 + 2e_3 = 0 \\ -2e_1 - 6e_2 + 2e_3 = 0 \\ \hline -4e_2 + 4e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ -4e_2 + 4e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ -4e_2 + 4e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{-E_2/4} e_2 - e_3 = 0$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ 3e_1 + 9e_2 - e_3 = 0 \end{cases} \xrightarrow{E_3+(-3E_1)} \begin{array}{r} 3e_1 + 9e_2 - e_3 = 0 \\ -3e_1 - 9e_2 + 3e_3 = 0 \\ \hline 2e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ 2e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ 2e_3 = 0 \end{cases} \xrightarrow{E_3/2} e_3 = 0$$

$$\begin{cases} e_1 + 3e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \\ e_3 = 0 \end{cases}$$

Since $e_3 = 0$, we can substitute into the second equation:

$e_2 - (0) = 0$

$e_2 = 0$

Since $e_2 = 0$ and $e_3 = 0$, we can substitute into the first equation:

$e_1 + 3(0) - (0) = 0$

$e_1 = 0$

Since $e_1 = e_2 = e_3 = 0$, this vector set is linearly independent.

Span:

$\vec{v} = (r, s, t)$

$x(1,2,3) + y(3,2,9) + z(5,2,-1) = (r, s, t)$

$$\begin{cases} x + 3y + 5z = r \\ 2x + 2y + 2z = s \\ 3x + 9y - z = t \end{cases}$$

$$\begin{cases} x + 3y + 5z = r \\ 2x + 2y + 2z = s \\ 3x + 9y - z = t \end{cases} \xrightarrow{E_2 + (-2E_1)} \begin{array}{r} 2x + 2y + 2z = s \\ -2x - 6y - 10z = -2r \\ \hline -4y - 8z = s - 2r \end{array}$$

$$\begin{cases} x + 3y + 5z = r \\ -4y - 8z = s - 2r \\ 3x + 9y - z = t \end{cases}$$

$$\begin{cases} x + 3y + 5z = r \\ -4y - 8z = s - 2r \\ 3x + 9y - z = t \end{cases} \xrightarrow{-E_2/4} y + 2z = \frac{1}{2}r - \frac{1}{4}s$$

$$\begin{cases} x + 3y + 5z = r \\ y + 2z = \frac{1}{2}r - \frac{1}{4}s \\ 3x + 9y - z = t \end{cases}$$

$$\begin{cases} x + 3y + 5z = r \\ y + 2z = \frac{1}{2}r - \frac{1}{4}s \\ 3x + 9y - z = t \end{cases} \xrightarrow{E_3 + (-3E_1)} \begin{array}{r} 3x + 9y - z = t \\ -3x - 9y - 15z = -3r \\ \hline -16z = t - 3r \end{array}$$

$$\begin{cases} x + 3y + 5z = r \\ y + 2z = \frac{1}{2}r - \frac{1}{4}s \\ -16z = t - 3r \end{cases}$$

$$\begin{cases} x + 3y + 5z = r \\ y + 2z = \frac{1}{2}r - \frac{1}{4}s \\ -16z = t - 3r \end{cases} \xrightarrow{-E_3/16} z = \frac{3}{16}r - \frac{1}{16}t$$

$$\begin{cases} x + 3y + 5z = r \\ y + 2z = \frac{1}{2}r - \frac{1}{4}s \\ z = \frac{3}{16}r - \frac{1}{16}t \end{cases}$$

If $z = \frac{3}{16}r - \frac{1}{16}t$, we can substitute into the second equation:

$$y + 2\left(\frac{3}{16}r - \frac{1}{16}t\right) = \frac{1}{2}r - \frac{1}{4}s$$

$$y + \frac{3}{8}r - \frac{1}{8}t = \frac{1}{2}r - \frac{1}{4}s$$

$$y = \frac{1}{2}r - \frac{3}{8}r - \frac{1}{4}s + \frac{1}{8}t$$

$$y = \frac{1}{8}r - \frac{1}{4}s + \frac{1}{8}t$$

If $y = \frac{1}{8}r - \frac{1}{4}s + \frac{1}{8}t$ and $z = \frac{3}{16}r - \frac{1}{16}t$, we can substitute into the first equation:

$$x + 3\left(\frac{1}{8}r - \frac{1}{4}s + \frac{1}{8}t\right) + 5\left(\frac{3}{16}r - \frac{1}{16}t\right) = r$$

$$x + \frac{3}{8}r - \frac{3}{4}s + \frac{3}{8}t + \frac{15}{16}r - \frac{5}{16}t = r$$

$$x + \frac{3}{8}r + \frac{15}{16}r - \frac{3}{4}s + \frac{3}{8}t - \frac{5}{16}t = r$$

$$x + \frac{21}{16}r - \frac{3}{4}s + \frac{1}{16}t = r$$

$$x = r - \frac{21}{16}r + \frac{3}{4}s - \frac{1}{16}t$$

$$x = \frac{3}{4}s - \frac{5}{16}r - \frac{1}{16}t$$

$\left(\frac{3}{4}s - \frac{5}{16}r - \frac{1}{16}, \frac{1}{8}r - \frac{1}{4}s + \frac{1}{8}t, \frac{3}{16}r - \frac{1}{16}t\right) \Rightarrow$this vector set spans $\mathbb{R}^3$.

Since this vector set is linearly independent and spans $\mathbb{R}^3$, it **is a basis** for $\mathbb{R}^3$.

30.g $\vec{a} = (1,2,3)$, $\vec{b} = (3,2,1)$, and $\vec{c} = (0,4,8)$ for $\mathbb{R}^3$

Linear independence:

$$e_1\vec{a} + e_2\vec{b} + e_3\vec{c} = \hat{0}$$

$$e_1(1,2,3) + e_2(3,2,1) + e_3(0,4,8) = (0,0,0)$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ 2e_1 + 2e_2 + 4e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ 2e_1 + 2e_2 + 4e_3 = 0 \xrightarrow{E_2+(-2E)} \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases} \quad \begin{array}{r} 2e_1 + 2e_2 + 4e_3 = 0 \\ -2e_1 - 6e_2 = 0 \\ \hline -4e_2 + 4e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ -4e_2 + 4e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ -4e_2 + 4e_3 = 0 \xrightarrow{-E_2/4} e_2 - e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ 3e_1 + e_2 + 8e_3 = 0 \xrightarrow{E_3+(-3E_1)} \end{cases} \quad \begin{array}{r} 3e_1 + e_2 + 8e_3 = 0 \\ -3e_1 - 9e_2 = 0 \\ \hline -8e_2 + 8e_3 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ -8e_2 + 8e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ -8e_2 + 8e_3 = 0 \xrightarrow{-E_3/8} e_2 - e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \end{cases}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ e_2 - e_3 = 0 \xrightarrow{E_3+(-E_2)} \end{cases} \quad \begin{array}{r} e_2 - e_3 = 0 \\ -e_2 + e_3 = 0 \\ \hline 0 = 0 \end{array}$$

$$\begin{cases} e_1 + 3e_2 = 0 \\ e_2 - e_3 = 0 \\ 0 = 0 \end{cases}$$

There is an infinite number of solutions for this system of equations, so this vector set is not linearly independent. So this set is **not a basis** for $\mathbb{R}^3$.

Dimension of a Vector Space

Problem 31

For the following subspace bases, determine the dimension:

The dimension of a subspace basis is the number of vectors in the basis.

31.a $B = \{\vec{a}, \vec{b}\} = \{(1,3), (2,3)\}$

By inspection $\boxed{dim(B) = 2}$.

31.b $B = \{\vec{a}, \vec{b}, \vec{c}\} = \{(1,1,0), (1,2,1), (1,1,1)\}$

By inspection, $\boxed{dim(B) = 3}$.

31.c $B = \{1, x, x^2, x^3, x^4\}$

By inspection, $\boxed{dim(B) = 5}$.

31.d $B = \{\vec{a}, \vec{b}, \vec{c}, \vec{d}\} = \{(1,0,0,0), (0,2,0,0), (0,0,1,0), (0,0,0,3)\}$

By inspection, $\boxed{dim(B) = 4}$.

Inner Product Space

Problem 32

Given $\vec{a} = (2,1,2)$, $\vec{b} = (1,0,-1)$, and $\vec{c} = (1,-1,1)$, compute the following inner products:

32.a $\langle \vec{a}, \vec{c} \rangle$

$$\langle \vec{a}, \vec{c} \rangle = a_x c_x + a_y c_y + a_z c_z$$

$$\langle \vec{a}, \vec{c} \rangle = (2)(1) + (1)(-1) + (2)(1)$$

$$\langle \vec{a}, \vec{c} \rangle = 2 + (-1) + 2$$

$$\boxed{\langle \vec{a}, \vec{c} \rangle = 3}$$

32.b $\langle \vec{b}, \vec{c} \rangle$

$$\langle \vec{b}, \vec{c} \rangle = b_x c_x + b_y c_y + b_z c_z$$

$$\langle \vec{b}, \vec{c} \rangle = (1)(1) + (0)(-1) + (-1)(1)$$

$$\langle \vec{b}, \vec{c} \rangle = 1 + 0 + (-1)$$

$$\boxed{\langle \vec{b}, \vec{c} \rangle = 0}$$

32.c $\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle$

$$\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = (5a_x - 2b_x)c_x + \left(5a_y - 2b_y\right)c_y + (5a_z - 2b_z)c_z$$

$$\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = [5(2) - 2(1)](1) + [5(1) - 2(0)](-1) + [5(2) - 2(-1)](1)$$

$$\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = (10 - 2)(1) + (5 - 0)(-1) + (10 + 2)(1)$$

$$\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = (8)(1) + (5)(-1) + (12)(1)$$

$$\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = 8 - 5 + 12$$

$$\boxed{\langle 5\vec{a} - 2\vec{b}, \vec{c} \rangle = 15}$$

32.d $\sqrt{\langle \vec{a}, \vec{a} \rangle}$

$$\sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

$$\sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{(2)^2 + (1)^2 + (2)^2}$$

$$\sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{4 + 1 + 4}$$

$$\sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{9}$$

$$\boxed{\sqrt{\langle \vec{a}, \vec{a} \rangle} = 3}$$

Problem 33

Given $f(x) = 5x^2$ and $g(x) = x^3$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, find:

33.a $\langle f, g \rangle$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

$$\langle f, g \rangle = \int_{-1}^{1} (5x^2)(x^3)dx$$

$$\langle f, g \rangle = \int_{-1}^{1} 5x^5 dx$$

$$\langle f, g \rangle = \frac{5}{6}x^6\Big|_{-1}^{1}$$

$$\langle f, g \rangle = \left[\frac{5}{6}(1)^6\right] - \left[\frac{5}{6}(-1)^6\right]$$

$$\langle f, g \rangle = \frac{5}{6} - \frac{5}{6}$$

$$\boxed{\langle f, g \rangle = 0}$$

33.b $\|f\|$

$$\|f\| = \sqrt{f, f}$$

$$\|f\| = \sqrt{\langle 5x^2, 5x^2 \rangle}$$

$$\|f\| = \sqrt{\int_{-1}^{1} (5x^2)(5x^2)dx}$$

$$\|f\| = \sqrt{\int_{-1}^{1} 25x^4 dx}$$

$$\|f\| = \sqrt{\frac{25}{5}x^5\Big|_{-1}^{1}}$$

$$\|f\| = \sqrt{5x^5|_{-1}^{1}}$$

$$\|f\| = \sqrt{[5(1)^5] - [5(-1)^5]}$$

$$\|f\| = \sqrt{5 - (-5)}$$

$$\boxed{\|f\| = \sqrt{10}}$$

33.c $\hat{f}$

$$\hat{f} = \frac{f(x)}{\|f\|}$$

$$\hat{f} = \frac{5x^2}{\sqrt{10}}$$

$$\hat{f} = \frac{5\sqrt{10}x^2}{10}$$

$$\boxed{\hat{f} = \frac{\sqrt{10}}{2}x^2}$$

Problem 34

Given $f(x) = x$ and $g(x) = x + 2$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$, find:

34.a $\langle f, g \rangle$

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

$$\langle f, g \rangle = \int_0^1 (x)(x+2)dx$$

$$\langle f, g \rangle = \int_0^1 (x^2 + 2x)dx$$

$$\langle f, g \rangle = \frac{1}{3}x^3 + x^2 \Big|_0^1$$

$$\langle f, g \rangle = \left[\frac{1}{3}(1)^3 + (1)^2\right] - \left[\frac{1}{3}(0)^3 + (0)^2\right]$$

$$\boxed{\langle f, g \rangle = \frac{4}{3}}$$

34.b $\|f\|$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$$\|f\| = \sqrt{\langle x, x \rangle}$$

$$\|f\| = \sqrt{\int_0^1 x^2 dx}$$

$$\|f\| = \sqrt{\frac{x^3}{3}\Big|_0^1}$$

$$\|f\| = \sqrt{\left[\frac{(1)^3}{3}\right] - \left[\frac{(0)^3}{3}\right]}$$

$$\|f\| = \sqrt{\frac{1}{3} - 0}$$

$$\|f\| = \sqrt{\frac{1}{3}}$$

$$\boxed{\|f\| = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx .577}$$

34.c $\hat{f}$

$$\hat{f} = \frac{f(x)}{\|f\|}$$

$$\hat{f} = \frac{x}{1/\sqrt{3}}$$

$$\boxed{\hat{f} = \sqrt{3} \cdot x}$$

Problem 35

Given $f(x) = \cos(x)$ and $g(x) = \sin(x)$ with inner product $\langle f, g \rangle = \int_0^{\pi/2} f(x)g(x)\,dx$:

35.a Find $\langle f, g \rangle$.

$$\langle f, g \rangle = \int_0^{\pi/2} [\cos(x)][\sin(x)]\,dx$$

$$u = \cos(x)$$

$$\frac{du}{dx} = \sin(x)$$

$$dx = \frac{du}{\sin(x)}$$

$$\langle f, g \rangle = \int_{x=0}^{x=\pi/2} u\sin(x) \cdot \frac{du}{\sin(x)}$$

$$\langle f, g \rangle = \int_{x=0}^{x=\pi/2} u\,du$$

$$\langle f, g \rangle = \frac{1}{2}u^2\Big|_{x=0}^{x=\pi/2}$$

$$\langle f, g \rangle = \frac{1}{2}\cos^2(x)\Big|_0^{\pi/2}$$

$$\langle f, g \rangle = \left[\frac{1}{2}\cos^2\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\cos^2(0)\right]$$

$$\langle f, g \rangle = \left[\frac{1}{2}(0)^2\right] - \left[\frac{1}{2}(1)^2\right]$$

$$\boxed{\langle f, g \rangle = -\frac{1}{2}}$$

35.b $\|f\|$.

$$\|f\| = \sqrt{\langle f, f\rangle}$$

$$\|f\| = \sqrt{\langle \cos(x), \cos(x)\rangle}$$

$$\|f\| = \sqrt{\int_0^{\pi/2} \cos^2(x)\, dx}$$

$$\|f\| = \sqrt{\int_0^{\pi/2} \left[\frac{1+\cos(2x)}{2}\right] dx}$$

$$\|f\| = \sqrt{\int_0^{\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right] dx}$$

$$u = 2x$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

$$\|f\| = \sqrt{\int_{x=0}^{x=\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos(u)\right] \cdot \frac{du}{2}}$$

$$\|f\| = \sqrt{\int_{x=0}^{x=\pi/2} \left[\frac{1}{4} + \frac{1}{4}\cos(u)\right] du}$$

$$\|f\| = \sqrt{\frac{1}{4}u + \frac{1}{4}\sin(u)\Big|_{x=0}^{x=\pi/2}}$$

$$\|f\| = \sqrt{\frac{1}{4}(2x) + \frac{1}{4}\sin(2x)\Big|_0^{\pi/2}}$$

$$\|f\| = \sqrt{\frac{1}{2}x + \frac{1}{4}\sin(2x)\Big|_0^{\pi/2}}$$

$$\|f\| = \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right)\right] - \left[\frac{1}{2}(0) + \frac{1}{4}\sin\big(2(0)\big)\right]}$$

$$\|f\| = \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin(\pi)\right] - \left[\frac{1}{2}(0) + \frac{1}{4}\sin(0)\right]}$$

$$\|f\| = \sqrt{\left[\frac{\pi}{4} + \frac{1}{4}(0)\right] - \left[0 + \frac{1}{4}(0)\right]}$$

$$\|f\| = \sqrt{\left[\frac{\pi}{4} + 0\right] - [0 + 0]}$$

$$\|f\| = \sqrt{\frac{\pi}{4}}$$

$$\boxed{\|f\| = \frac{\sqrt{\pi}}{2}}$$

35.c $\hat{f}$

$$\hat{f} = \frac{f(x)}{\|f\|}$$

$$\hat{f} = \frac{\cos(x)}{\frac{\sqrt{\pi}}{2}}$$

$$\hat{f} = \cos(x) \cdot \frac{2}{\sqrt{\pi}}$$

$$\boxed{\hat{f} = \frac{2}{\sqrt{\pi}}\cos(x) = \frac{2\sqrt{\pi}}{\pi}\cos(x)}$$

Problem 36

Given $p = 1 + 2x + x^2 + x^3$ and $q = 1 + 5x^2 + x^3$, compute $\langle p, q \rangle$.

Inner Product of Polynomial Space

$p = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$	$q = b_0x^0 + b_1x^1 + b_2x^2 + b_3x^3$
$a_0 = 1$	$b_0 = 1$
$a_1 = 2$	$b_1 = 0$
$a_2 = 1$	$b_2 = 5$
$a_3 = 1$	$b_3 = 1$

$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$

$\langle p, q \rangle = (1)(1) + (2)(0) + (1)(5) + (1)(1)$

$\langle p, q \rangle = 1 + 0 + 5 + 1$

$\boxed{\langle p, q \rangle = 7}$

Problem 37

- Given $p = 1 + 2x - x^2 + 3x^3$ and $q = 1 + x - 2x^2 + 4x^3$, compute $\langle p, q \rangle$.

$p = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$	$q = b_0x^0 + b_1x^1 + b_2x^2 + b_3x^3$
$a_0 = 1$	$b_0 = 1$
$a_1 = 2$	$b_1 = 1$
$a_2 = -1$	$b_2 = -2$
$a_3 = 3$	$b_3 = 4$

$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$

$\langle p, q \rangle = (1)(1) + (2)(1) + (-1)(-2) + (3)(4)$

$\langle p, q \rangle = 1 + 2 + 2 + 12$

$\boxed{\langle p, q \rangle = 17}$