GEN 242: Linear Algebra

Chapter 2: Matrices

Solutions Guide

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Answers

Order of a Matrix

1.a $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 7\\ 6 & 7 & 5\end{bmatrix}$ is order $\boxed{3x3}$ (alternately, a square matrix of order 3).

1.b $\begin{bmatrix}1 & 1 & 3 & 5\\ 5 & -1 & 6 & 2\\ 7 & 8 & 9 & -2\\ 3 & 5 & 9 & 10\end{bmatrix}$ is order $\boxed{4x4}$ (alternately, a square matrix of order 4).

1.c $\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$ is order $\boxed{2x2}$ (alternately, a square matrix of order 2).

1.d $\begin{bmatrix}1 & 4 & 0 & 4\\ 1 & 8 & 3 & 1\end{bmatrix}$ is order $\boxed{2x4}$.

1.e $\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ is order $\boxed{3x1}$.

Trace of a Square Matrix

2.a $\text{Trace}\left(\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 7\\ 6 & 7 & 5\end{bmatrix}\right) = \boxed{11}$

2.b $\text{Trace}\left(\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}\right) = \boxed{5}$

2.c $\text{Trace}\left(\begin{bmatrix}1 & 1 & 3 & 5\\ 5 & -1 & 6 & 2\\ 7 & 8 & 9 & -2\\ 3 & 5 & 9 & 10\end{bmatrix}\right) = \boxed{19}$

2.d $\text{Trace}\left(\begin{bmatrix}1 & 7 & 5\\ 1 & 4 & 7\\ 1 & 7 & -5\end{bmatrix}\right) = \boxed{0}$

Transpose of a Matrix

3.a $\begin{bmatrix}1 & 5 & 7\\ 9 & 1 & 7\\ 0 & 7 & 1\end{bmatrix}^T = \boxed{\begin{bmatrix}1 & 9 & 0\\ 5 & 1 & 7\\ 7 & 7 & 1\end{bmatrix}}$

3.b $\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}^T = \boxed{\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}}$

3.c $\begin{bmatrix}1 & 1 & 3 & 5\\ 5 & -1 & 6 & 2\\ 7 & 8 & 9 & -2\\ 3 & 5 & 9 & 10\end{bmatrix}^T = \boxed{\begin{bmatrix}1 & 5 & 7 & 3\\ 1 & -1 & 8 & 5\\ 3 & 6 & 9 & 9\\ 5 & 2 & -2 & 10\end{bmatrix}}$

3.d $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 7\\ 6 & 7 & 5\end{bmatrix}^T = \boxed{\begin{bmatrix}1 & 2 & 6\\ 2 & 5 & 7\\ 3 & 7 & 5\end{bmatrix}}$

3.e $\begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}^T = \boxed{\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}}$

Matrix Entry Value

4. $M = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 5 & -1 & 6 & 2 \\ 7 & 8 & 9 & -2 \\ 3 & 5 & 9 & 10 \end{bmatrix} \rightarrow \boxed{\begin{matrix} m_{12} = 1 & m_{34} = -2 & m_{14} = 5 \\ m_{22} = -1 & m_{44} = 10 & m_{33} = 9 \end{matrix}}$

Column and Row Vectors

5.a $\vec{v} = (2,1,3) \rightarrow \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ column and $\vec{v} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$ row.

5.b $\vec{v} = (2,0,3,4) \rightarrow \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ column and $\vec{v} = \begin{bmatrix} 2 & 0 & 3 & 4 \end{bmatrix}$ row.

5.c $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is already in column format; $\vec{v} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ row.

5.d $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is already in column format; $\vec{v} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ row.

Symmetric Matrices

6.a $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ is **not** symmetric.

6.b $\begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$ is symmetric.

6.c $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$ is **not** symmetric.

6.d $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$ is symmetric.

6.e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is symmetric.

6.f Given symmetric matrix $M = \begin{bmatrix} 2 & x & y & 7 \\ 0 & 4 & z & t \\ 1 & 0 & 1 & u \\ v & 6 & 8 & 5 \end{bmatrix}$:

This question is mislabeled as 6.b on FSO.

$x = 0$ $\qquad z = 0$ $\qquad u = 8$

$y = 1$ $\qquad t = 6$ $\qquad v = 7$

Diagonal, Triangular, and Skew-Symmetric Matrices

7.a $\begin{bmatrix}1 & 2 & 3\\0 & 5 & 7\\0 & 0 & 5\end{bmatrix}$ is upper triangular.

7.b $\begin{bmatrix}1 & 0 & 0 & 0\\5 & 1 & 0 & 0\\7 & 8 & 9 & 0\\3 & 5 & 9 & 10\end{bmatrix}$ is lower triangular.

7.c $\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix}$ is diagonal.

7.d $\begin{bmatrix}0 & -2 & 3\\2 & 0 & -7\\-3 & 7 & 0\end{bmatrix}$ is skew-symmetric.

7.e

$$\operatorname{skew}\big((1,2,3)\big) = \begin{bmatrix}0 & -3 & 2\\3 & 0 & -1\\-2 & 1 & 0\end{bmatrix}$$

$$\operatorname{skew}\big((0,2,-1)\big) = \begin{bmatrix}0 & 1 & 2\\-1 & 0 & 0\\-2 & 0 & 0\end{bmatrix}$$

$$\operatorname{skew}\big((4,-2,3)\big) = \begin{bmatrix}0 & -3 & -2\\3 & 0 & -4\\2 & 4 & 0\end{bmatrix}$$

This question is mislabeled as 7.b on FSO.

Matrices Addition

Given $A = \begin{bmatrix}1 & 2 & 0\\0 & 5 & 1\\3 & 1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0 & 2\\3 & 2 & 1\\1 & 2 & 3\end{bmatrix}$:

8.a $A + B = \begin{bmatrix}2 & 2 & 2\\3 & 7 & 2\\4 & 3 & 4\end{bmatrix}$

8.b $2B - A = \begin{bmatrix}1 & -2 & 4\\6 & -1 & 1\\-1 & 3 & 5\end{bmatrix}$

8.c $B - B^T = \begin{bmatrix}0 & -3 & 1\\3 & 0 & -1\\-1 & 1 & 0\end{bmatrix}$

8.d $2A - 3A = \begin{bmatrix}-1 & -2 & 0\\0 & -5 & -1\\-3 & -1 & -1\end{bmatrix} = -A$

8.e $A^T + B = \begin{bmatrix}2 & 0 & 5\\5 & 7 & 2\\1 & 3 & 4\end{bmatrix}$

Matrix Form of Vector Dot Product

9.a $(0,2,-1) \cdot (1,2,3) = 1$

9.b $(3,2,-4) \cdot (2,2,-1) = 14$

Matrix Form of Vector Cross Product

10.a $(0,2,-1) \times (1,2,3) = \begin{bmatrix} 8 \\ -1 \\ -2 \end{bmatrix} = (8,-1,-2)$

10.b $(2,1,-1) \times (1,0,-1) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = (-1,1,-1)$

Matrix Multiplication

11.a $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 8 \\ 16 & 12 & 8 \\ 7 & 4 & 10 \end{bmatrix}$

11.b $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ 20 & 15 \end{bmatrix}$

11.c $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & 11 \end{bmatrix}$

11.d $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 2 \end{bmatrix}$

11.e $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \\ 5 & 7 & 11 \\ 10 & 8 & 14 \end{bmatrix}$

11.f $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 5$

11.g $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \\ 3 & 6 & 3 \end{bmatrix}$

Right and Left Vector-Matrix Multiplication

Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$:

12.a $A \cdot \vec{v} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$

12.b $\vec{v} \cdot A = \begin{bmatrix} 4 & 7 & 5 \end{bmatrix}$

12.c $A \cdot \vec{u} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$

12.d $\vec{u} \cdot A = \begin{bmatrix} 1 & 6 & 1 \end{bmatrix}$

12.e $B \cdot \vec{u} = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix}$

12.f $\vec{u} \cdot B = \begin{bmatrix} 10 & 3 & 1 \end{bmatrix}$

12.g $B \cdot \vec{w} = \begin{bmatrix} 13 \\ 5 \\ 7 \end{bmatrix}$

12.h $\vec{v} \cdot \vec{u}^T = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}$

Systems of Linear Equations and Augmented Matrices

13.a $\begin{cases} x + 2y = 5 \\ 2x - 3y = -4 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -3 & -4 \end{array}\right]$

13.b $\begin{cases} x + 2y = 7 \\ 5x - 3y = 9 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 5 & -3 & 9 \end{array}\right]$

13.c $\begin{cases} 2x + 3y = 16 \\ 2x - y = 8 \end{cases} \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 16 \\ 2 & -1 & 8 \end{array}\right]$

13.d $\begin{cases} 3x + y = 2 \\ 2x + y = 1 \end{cases} \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right]$

13.e $\begin{cases} x + y - 5z = -3 \\ x + y + z = 3 \\ 7x - y + 2z = 8 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{array}\right]$

13.f $\begin{cases} x + y + z = 2 \\ x - 3y + 2z = -4 \\ 5x - y + 3z = 8 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -3 & 2 & -4 \\ 5 & -1 & 3 & 8 \end{array}\right]$

13.g $\begin{cases} x + 3y + z = 4 \\ 2x - y + 2z = 1 \\ 3x - y + 2z = 3 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 2 & 3 \end{array}\right]$

13.h $\begin{cases} x + y - z = 6 \\ 2x + 3y + z = 7 \\ x - y + 2z = -2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -2 \end{array}\right]$

Identifying Row-Echelon Form

14.a $\begin{bmatrix}1 & 3 & 1\\0 & 0 & 0\\0 & 0 & 5\end{bmatrix}$ is **not** row-echelon.

14.b $\begin{bmatrix}8 & 4 & 1\\0 & 3 & 1\\0 & 0 & 0\end{bmatrix}$ is row-echelon.

14.c $\begin{bmatrix}1 & 4 & 1\\0 & 1 & 2\\0 & 2 & 0\end{bmatrix}$ is **not** row-echelon.

14.d $\begin{bmatrix}0 & 8 & 2\\2 & 3 & 4\\0 & 0 & 3\end{bmatrix}$ is **not** row-echelon.

14.e $\begin{bmatrix}1 & 4 & 2\\0 & 3 & 4\\0 & 0 & 2\end{bmatrix}$ is row-echelon.

14.f $\begin{bmatrix}0 & 5 & 3 & 0 & 7\\0 & 0 & 5 & 5 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ is row-echelon.

14.g $\begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$ is row-echelon.

14.h $\begin{bmatrix}0 & 0\\0 & 4\end{bmatrix}$ is **not** row-echelon.

14.i $\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$ is **not** row-echelon.

14.j $\begin{bmatrix}1 & 2 & 3\\0 & 3 & 5\end{bmatrix}$ is row-echelon.

Identifying Reduced Row-Echelon Form (RREF)

15.a $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ is RREF.

15.b $\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$ is **not** RREF.

15.c $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$ is **not** RREF.

15.d $\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix}$ is **not** RREF.

15.e $\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ is **not** RREF.

15.f $\begin{bmatrix}1 & 5 & 0 & 0 & 7\\0 & 0 & 1 & 5 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ is RREF.

15.g $\begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$ is **not** RREF.

15.h $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is RREF.

15.i $\begin{bmatrix}1 & 0 & 3\\0 & 1 & 5\end{bmatrix}$ is RREF.

15.j $\begin{bmatrix}1 & 0 & 0 & 0 & 7\\0 & 1 & 0 & 5 & 3\\0 & 0 & 1 & 0 & 0\end{bmatrix}$ is RREF.

Computing Row-Echelon Form

16.a $\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -4\\0 & 7\end{bmatrix}$

16.b $\begin{bmatrix}2 & 6\\6 & 3\end{bmatrix} \sim \begin{bmatrix}2 & 6\\0 & -15\end{bmatrix}$

16.c $\begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\6 & 7 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\0 & 0 & -8\end{bmatrix}$

16.d $\begin{bmatrix}1 & 2 & 3\\2 & 5 & 0\\3 & 0 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3\\0 & 1 & -6\\0 & 0 & -40\end{bmatrix}$

16.e $\begin{bmatrix}1 & 2 & 1\\2 & 3 & 1\\4 & 7 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\0 & -1 & -1\\0 & 0 & 0\end{bmatrix}$

Computing Reduced Row-Echelon Form

17.a $\begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

17.b $\begin{bmatrix} 1 & 2 & 7 \\ 5 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

17.c $\begin{bmatrix} 2 & 3 & 16 \\ 2 & -1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$

17.d $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

17.e $\begin{bmatrix} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}$

17.f $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 2 & -4 \\ 5 & -1 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

17.g $\begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

17.h $\begin{bmatrix} 1 & 1 & -1 & 6 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Solving Systems Using Reduced Row-Echelon Form

18.a $\begin{cases} x + 2y = 5 \\ 2x - 3y = -4 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$

18.b $\begin{cases} x + 2y = 7 \\ 5x - 3y = 9 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 2 \end{cases}$

18.c $\begin{cases} 2x + 3y = 16 \\ 2x - y = 8 \end{cases} \rightarrow \begin{cases} x = 5 \\ y = 2 \end{cases}$

18.d $\begin{cases} 3x + y = 2 \\ 2x + y = 1 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$

18.e $\begin{cases} x + y - 5z = -3 \\ x + y + z = 3 \\ 7x - y + 2z = 8 \end{cases} \rightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$

18.f $\begin{cases} x + y + z = 2 \\ x - 3y + 2z = -4 \\ 5x - y + 3z = 8 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 1 \\ z = -2 \end{cases}$

18.g $\begin{cases} x + 3y + z = 4 \\ 2x - y + 2z = 1 \\ 3x - y + 2z = 3 \end{cases} \rightarrow \begin{cases} x = 2 \\ y = 1 \\ z = -1 \end{cases}$

18.h $\begin{cases} x + y - z = 6 \\ 2x + 3y + z = 7 \\ x - y + 2z = -2 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 1 \\ z = -2 \end{cases}$

Rank of a Matrix

19.a $\text{Rank}\left(\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 0\\ 3 & 0 & 5\end{bmatrix}\right) = 3$

19.b $\text{Rank}\left(\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & 1\\ 3 & 2 & 2\end{bmatrix}\right) = 2$

19.c $\text{Rank}\left(\begin{bmatrix}1 & 1 & 2 & 3\\ 2 & 6 & 1 & 1\\ 3 & 4 & 3 & 4\end{bmatrix}\right) = 3$

19.d $\text{Rank}\left(\begin{bmatrix}1 & 1 & -1\\ 2 & 3 & -1\\ 3 & 1 & -5\end{bmatrix}\right) = 3$

Linear Dependence Using Matrix Echelon Form

20.a $\{(1,2,5), (2,4,1), (1,1,2)\}$ is linearly independent.

20.b $\{(1,4,3), (3,0,1), (1,1,2)\}$ is linearly independent.

20.c $\{(1,1,1), (1,2,0), (0,-1,1)\}$ is **not** linearly independent.

20.d $\{(1,1,1), (1,2,0), (0,-1,2)\}$ is linearly independent.

21.a $\{(1,2), (2,4)\}$ is linearly dependent.

21.b $\{(2,8), (2,5)\}$ is linearly **in**dependent.

22.a $\{1-x, 5-3x+2x^2, 1+3x-x^2\}$ is inearly **in**dependent.

22.b $\{1+x+x^2, x+2x^2, x^2\}$ is linearly **in**dependent.

Basis Using Matrix Reduced Row-Echelon Form

23.a $\{(2,8), (2,5)\}$ forms a basis for $\mathbb{R}^2$.

23.b $\{(1,3), (2,6)\}$ does **not** form a basis for $\mathbb{R}^2$.

24.a $\{(1,0,0), (1,1,0), (1,1,1)\}$ forms a basis for $\mathbb{R}^3$.

24.b $\{(1,2,3), (2,0,1), (3,2,2)\}$ forms a basis for $\mathbb{R}^3$.

24.c $\{(1,2,1), (1,7,-1), (2,1,3)\}$ forms a basis for $\mathbb{R}^3$.

24.d $\{(1,2,1), (5,2,3), (3,2,2)\}$ does **not** form a basis for $\mathbb{R}^3$.

25.a $\{1-x, 5-3x+2x^2, 1+3x-x^2\}$ forms a basis for P_2.

25.b $\{1+2x+x^2, 2+x^2, 3+2x+2x^2\}$ forms a basis for P_2.

25.c $\{1+x+x^2, x+2x^2, x^2\}$ forms a basis for P_2.

25.d $\{1-2x+3x^2, 5+6x-x^2, 3+2x+x^2\}$ does not for a basis for P_2.

Basis of a Matrix Row Space

26.a $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -40 \end{bmatrix} \right\}$ $\dim(\mathrm{rowsp}(A)) = 3$ $\mathrm{rank}(A) = 3$

26.b $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \right\}$ $\dim(\mathrm{rowsp}(A)) = 2$ $\mathrm{rank}(A) = 2$

26.c $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 1 \\ 3 & -4 & 3 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -27 \end{bmatrix} \right\}$ $\dim(\mathrm{rowsp}(A)) = 3$ $\mathrm{rank}(A) = 3$

26.d $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\dim(\mathrm{rowsp}(A)) = 2$ $\mathrm{rank}(A) = 2$

Basis of a Matrix Column Space

27.a $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}$ $\dim(\mathrm{colsp}(A)) = 3$ $\mathrm{rank}(A) = 3$

27.b $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$ $\dim(\mathrm{colsp}(A)) = 2$ $\mathrm{rank}(A) = 2$

27.c $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 1 \\ 3 & -4 & 3 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$ $\dim(\mathrm{colsp}(A)) = 3$ $\mathrm{rank}(A) = 2$

27.d $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ $\dim(\mathrm{colsp}(A)) = 2$ $\mathrm{rank}(A) = 2$

Basis of a Matrix Null Space

28.a $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \rightarrow \begin{matrix} \{\vec{0}\} \\ \dim(\text{Null}(A)) = 0 \end{matrix}$

28.b $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \rightarrow \begin{matrix} \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\} \\ \dim(\text{Null}(A)) = 1 \end{matrix}$

28.c $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{matrix} \{\vec{0}\} \\ \dim(\text{Null}(A)) = 0 \end{matrix}$

28.d $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix} \rightarrow \begin{matrix} \{\vec{0}\} \\ \dim(\text{Null}(A)) = 0 \end{matrix}$

28.e $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 5 & 1 \end{bmatrix} \rightarrow \begin{matrix} \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\} \\ \dim(\text{Null}(A)) = 1 \end{matrix}$

Coordinate of a Vector and Matrix

Given basis $B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$:

29.a $[2\hat{\imath} + 3\hat{\jmath} - \hat{k}]_B = (2,3,-1)$

29.b $[\hat{\imath} + \hat{\jmath} - \hat{k}]_B = (1,1,-1)$

29.c $[5\hat{\imath} - \hat{k}]_B = (5,0,-1)$

Given basis $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$:

30.a $\begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}_B = (2,2,4,3)$

30.b $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}_B = (1,2,1,0)$

30.c $\begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}_B = (0,4,2,1)$

30.d $\begin{bmatrix} 3 & -7 \\ 2 & 4 \end{bmatrix}_B = (3,-7,2,4)$

31.a $[5 - 4x + 7x^2 + 10x^3]_{B=\{1,x,x^2,x^3\}} = (5,-4,7,10)$

31.b $[-x + 3x^2]_{B=\{1,x,x^2,x^3\}} = (0,-1,3,0)$

31.c $[-x + 3x^2]_{B=\{1,x,x^2\}} = (0,-1,3)$

31.d $[2 - x + 7x^2]_{B=\{1,x,x^2\}} = (2,-1,7)$

32.a $[(2,-3)]_{B=\{(1,1),(3,4)\}} = (17,-5)$

32.b $[(8,7)]_{B=\{(1,2),(2,1)\}} = (2,3)$

32.c $[(-3,1)]_{B=\{(1,3),(2,1)\}} = (1,-2)$

32.d $[(1,2)]_{B=\{(1,1),(3,4)\}} = (-2,1)$

Change of Basis and Transition Matrix

Given the bases $S = \{\hat{\imath}, \hat{\jmath}\} = \{(1,0), (0,1)\}$ and $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,2), (2,5)\}$:

33.a $M_{B\leftarrow S} = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$

33.b $[(1,2)_S]_B = (1,4)_B$

33.c $M_{S\leftarrow B} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Given the bases $S = \{\hat{\imath}, \hat{\jmath}\} = \{(1,0), (0,1)\}$ and $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3), (1,4)\}$:

34.a $M_{B\leftarrow S} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

34.c $M_{S\leftarrow B} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

34.b $[(1,2)_S]_B = (2,-1)_B$

Given the bases $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3), (1,4)\}$ and $B' = \{\vec{v}_1, \vec{v}_2\} = \{(1,2), (2,5)\}$:

35.a $M_{B'\leftarrow B} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$

35.b $[(2,5)_B]_{B'} = (-17,12)_{B'}$

Given the bases $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3), (1,4)\}$ and $B' = \{\vec{v}_1, \vec{v}_2\} = \{(1,2), (1,1)\}$:

36.a $M_{B'\leftarrow B} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

36.b $[(3,1)_B]_{B'} = (9,-5)_{B'}$

Solutions

Order of a Matrix

Problem 1

Identify the order of the following matrices:

Matrix order is “(#rows)x(#columns)”.

1.a $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$

Matrix A has three rows and three columns.

$\boxed{\text{Order}(A) = 3\text{x}3}$

1.b $A = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 5 & -1 & 6 & 2 \\ 7 & 8 & 9 & -2 \\ 3 & 5 & 9 & 10 \end{bmatrix}$

Matrix A has four rows and four columns.

$\boxed{\text{Order}(A) = 4\text{x}4}$

1.c $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Matrix A has two rows and two columns.

$\boxed{\text{Order}(A) = 2\text{x}2}$

1.d $A = \begin{bmatrix} 1 & 4 & 0 & 4 \\ 1 & 8 & 3 & 1 \end{bmatrix}$

Matrix A has two rows and four columns.

$\boxed{\text{Order}(A) = 2\text{x}4}$

1.e $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Matrix A has three rows and one column.

$\boxed{\text{Order}(A) = 3\text{x}1}$

Trace of a Square Matrix

Problem 2

Calculate the trace of the following matrices:

Trace is the sum of the elements on a square matrix's main diagonal.

2.a $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$

$\text{Trace}(A) = a_{11} + a_{22} + a_{33}$

$\text{Trace}(A) = (1) + (5) + (5)$

$\boxed{\text{Trace}(A) = 11}$

2.b $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\text{Trace}(A) = a_{11} + a_{22}$

$\text{Trace}(A) = (2) + (3)$

$\boxed{\text{Trace}(A) = 5}$

2.c $A = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 5 & -1 & 6 & 2 \\ 7 & 8 & 9 & -2 \\ 3 & 5 & 9 & 10 \end{bmatrix}$

$\text{Trace}(A) = a_{11} + a_{22} + a_{33} + a_{44}$

$\text{Trace}(A) = (1) + (-1) + (9) + (10)$

$\boxed{\text{Trace}(A) = 19}$

2.d $A = \begin{bmatrix} 1 & 7 & 5 \\ 1 & 4 & 7 \\ 1 & 7 & -5 \end{bmatrix}$

$\text{Trace}(A) = a_{11} + a_{22} + a_{33} + a_{44}$

$\text{Trace}(A) = (1) + (4) + (-5)$

$\boxed{\text{Trace}(A) = 0}$

Transpose of a Matrix

Problem 3

Find the transpose for each of the following matrices:

3.a $A = \begin{bmatrix} 1 & 5 & 7 \\ 9 & 1 & 7 \\ 0 & 7 & 1 \end{bmatrix}$

This problem is mislabeled as 3.b on FSO.

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\boxed{A^t = \begin{bmatrix} 1 & 9 & 0 \\ 5 & 1 & 7 \\ 7 & 7 & 1 \end{bmatrix}}$

3.b $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\boxed{A^t = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}$

3.c $A = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 5 & -1 & 6 & 2 \\ 7 & 8 & 9 & -2 \\ 3 & 5 & 9 & 10 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$

$\boxed{A^t = \begin{bmatrix} 1 & 5 & 7 & 3 \\ 1 & -1 & 8 & 5 \\ 3 & 6 & 9 & 9 \\ 5 & 2 & -2 & 10 \end{bmatrix}}$

3.d $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\boxed{A^t = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 3 & 7 & 5 \end{bmatrix}}$

3.e $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \end{bmatrix}$

$\boxed{A^t = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}$

Matrix Entry Value

Problem 4

Given matrix $M = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 5 & -1 & 6 & 2 \\ 7 & 8 & 9 & -2 \\ 3 & 5 & 9 & 10 \end{bmatrix}$, find the following entry values:

$m_{12} \to \boxed{m_{12} = 1}$

$m_{22} \to \boxed{m_{22} = -1}$

$m_{34} \to \boxed{m_{34} = -2}$

$m_{44} \to \boxed{m_{44} = 10}$

$m_{14} \to \boxed{m_{14} = 5}$

$m_{33} \to \boxed{m_{33} = 9}$

Column and Row Vectors

Problem 5

Rewrite the following vectors using the alternative format (row to column, column to row):

5.a $\vec{v} = (2,1,3)$

$$\boxed{\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$$

5.b $\vec{v} = (2,0,3,4)$

$$\boxed{\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}}$$

5.c $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\boxed{\vec{v} = (1,2,-1)}$$

5.d $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\boxed{\vec{v} = (1,2)}$$

Symmetric Matrices

Problem 6

For the following matrices, identify each as either symmetric or asymmetric:

A matrix is symmetric if it is equal to its transpose.

6.a $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$A^t \neq A$$

A is asymmetric.

6.b $A = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$$

$$A^t = A$$

A is symmetric.

6.c $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 3 & 7 & 5 \end{bmatrix}$$

$$A^t \neq A$$

A is asymmetric.

6.d $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$A^t = A$

A is symmetric.

6.e $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$A^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

$A^t = A$

A is symmetric.

6.f If $M = \begin{bmatrix} 2 & x & y & 7 \\ 0 & 4 & z & t \\ 1 & 0 & 1 & u \\ v & 6 & 8 & 5 \end{bmatrix}$ is a symmetric matrix, what must be the values of:

x: $x = m_{12}$

$m_{12} = m_{21}$ symmetric matrix

$m_{12} = 0$

$\boxed{x = 0}$

y: $y = m_{13}$

$m_{13} = m_{31}$ symmetric matrix

$m_{13} = 1$

$\boxed{y = 1}$

z: $z = m_{23}$

$m_{23} = m_{32}$ symmetric matrix

$m_{23} = 0$

$\boxed{z = 0}$

t: $t = m_{24}$

$m_{24} = m_{42}$ symmetric matrix

$m_{24} = 6$

$\boxed{t = 6}$

u: $u = m_{34}$

$m_{34} = m_{43}$ symmetric matrix

$m_{34} = 8$

$\boxed{u = 8}$

v: $v = m_{41}$

$m_{41} = m_{14}$ symmetric matrix

$m_{41} = 7$

$\boxed{v = 7}$.

Diagonal, Triangular, and Skew-Symmetric Matrices

Problem 7

For the following matrices, identify each as diagonal, upper triangular, lower triangular, or skew-symmetric:

7.a $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

All elements below the main diagonal are zero. A is **lower triangular**.

7.b $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 7 & 8 & 9 & 0 \\ 3 & 5 & 9 & 10 \end{bmatrix}$

All elements above the main diagonal are zero. B is **upper triangular**.

7.c $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

All elements not on the main diagonal are zero. C is **diagonal**.

7.d $D = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -7 \\ -3 & 7 & 0 \end{bmatrix}$

All elements across the main diagonal are negations of each other, and all elements on the main diagonal are zero. D is **skew symmetric**.

7.e Write the skew symmetric matrix for each of the following vectors:

A vector (x, y, z) becomes the skew-symmetric matrix $\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$.

This problem is labeled 7.b on FSO.

$\vec{v} = (1,2,3)$

$$\text{skew}(\vec{v}) = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

$$\boxed{\text{skew}(\vec{v}) = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}}$$

$\vec{u} = (0,2,-1)$

$$\text{skew}(\vec{u}) = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$\text{skew}(\vec{u}) = \begin{bmatrix} 0 & -(-1) & 2 \\ -1 & 0 & -0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{skew}(\vec{u}) = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}}$$

$\vec{w} = (4,-2,3)$

$$\text{skew}(\vec{w}) = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

$$\text{skew}(\vec{w}) = \begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -4 \\ -(-2) & 4 & 0 \end{bmatrix}$$

$$\boxed{\text{skew}(\vec{w}) = \begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -4 \\ 2 & 4 & 0 \end{bmatrix}}$$

Matrices Addition

Problem 8

Given matrices $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, find the following:

8.a $A + B$

$$A + B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

$$A + B = \begin{bmatrix} (1) + (1) & (2) + (0) & (0) + (2) \\ (0) + (3) & (5) + (2) & (1) + (1) \\ (3) + (1) & (1) + (2) & (1) + (3) \end{bmatrix}$$

$$\boxed{A + B = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 7 & 2 \\ 4 & 3 & 4 \end{bmatrix}}$$

8.b $2B - A$

$$2B - A = 2\begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$2B - A = \begin{bmatrix} 2(1) & 2(0) & 2(2) \\ 2(3) & 2(2) & 2(1) \\ 2(1) & 2(2) & 2(3) \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$2B - A = \begin{bmatrix} 2 & 0 & 4 \\ 6 & 4 & 2 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$2B - A = \begin{bmatrix} 2-1 & 0-2 & 4-0 \\ 6-0 & 4-5 & 2-1 \\ 2-3 & 4-1 & 6-1 \end{bmatrix}$$

$$\boxed{2B - A = \begin{bmatrix} 1 & -2 & 4 \\ 6 & -1 & 1 \\ -1 & 3 & 5 \end{bmatrix}}$$

8.c $B - B^t$

$$B - B^t = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}^t$$

$$B - B^t = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$B - B^t = \begin{bmatrix} 1-1 & 0-3 & 2-1 \\ 3-0 & 2-2 & 1-2 \\ 1-2 & 2-1 & 3-3 \end{bmatrix}$$

$$\boxed{B - B^t = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}}$$

8.d $2A - 3A$

$$2A - 3A = 2\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$2A - 3A = \begin{bmatrix} 2(1) & 2(2) & 2(0) \\ 2(0) & 2(5) & 2(1) \\ 2(3) & 2(1) & 2(1) \end{bmatrix} - \begin{bmatrix} 3(1) & 3(2) & 3(0) \\ 3(0) & 3(5) & 3(1) \\ 3(3) & 3(1) & 3(1) \end{bmatrix}$$

$$2A - 3A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 10 & 2 \\ 6 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 0 \\ 0 & 15 & 3 \\ 9 & 3 & 3 \end{bmatrix}$$

$$2A - 3A = \begin{bmatrix} 2-3 & 4-6 & 0-0 \\ 0-0 & 10-15 & 2-3 \\ 6-9 & 2-3 & 2-3 \end{bmatrix}$$

$$\boxed{2A - 3A = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -5 & -1 \\ -3 & -1 & -1 \end{bmatrix} = -A}$$

8.e $A^t + B$

$$A^t + B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}^t + \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^t + B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^t + B = \begin{bmatrix} 1+1 & 0+0 & 3+2 \\ 2+3 & 5+2 & 1+1 \\ 0+1 & 1+2 & 1+3 \end{bmatrix}$$

$$\boxed{A^t + B = \begin{bmatrix} 2 & 0 & 5 \\ 5 & 7 & 2 \\ 1 & 3 & 4 \end{bmatrix}}$$

Matrix Form of the Vector Dot Product

Problem 9

For the following vector pairs, write the matrix dot product:

9.a $\vec{u} = (0,2,-1)$ and $\vec{v} = (1,2,3)$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = (0)(1) + (2)(2) + (-1)(3)$$

$$\vec{u} \cdot \vec{v} = 0 + 4 + (-3)$$

$$\boxed{\vec{u} \cdot \vec{v} = 1}$$

9.b $\vec{s} = (3,2,-4)$ nd $\vec{t} = (2,2,-1)$

$$\vec{s} \cdot \vec{t} = \begin{bmatrix} s_x & s_y & s_z \end{bmatrix} \cdot \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\vec{s} \cdot \vec{t} = \begin{bmatrix} 3 & 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{s} \cdot \vec{t} = (3)(2) + (2)(2) + (-4)(-1)$$

$$\vec{s} \cdot \vec{t} = 6 + 4 + 4$$

$$\boxed{\vec{s} \cdot \vec{t} = 14}$$

Matrix Form of the Vector Cross Product

Problem 10

For the following vector pairs, write the matrix cross product:

10.a $\vec{u} = (0,2,-1)$ and $\vec{v} = (1,2,3)$

$$\vec{u} \times \vec{v} = \text{skew}(\vec{u}) \cdot \vec{v}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -(-1) & (2) \\ (-1) & 0 & -(0) \\ -(2) & (0) & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} (0)(1) + (1)(2) + (2)(3) \\ (-1)(1) + (0)(2) + (0)(3) \\ (-2)(1) + (0)(2) + (0)(3) \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 + 2 + 6 \\ -1 + 0 + 0 \\ -2 + 0 + 0 \end{bmatrix}$$

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} 8 \\ -1 \\ -2 \end{bmatrix} = (8,-1,-2)}$$

Alternate:

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix} \hat{\imath} - \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} \hat{\jmath} + \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \hat{k}$$

$$\vec{u} \times \vec{v} = [(2)(3) - (2)(-1)]\hat{\imath} - [(0)(3) - (1)(-1)]\hat{\jmath} + [(0)(2) - (1)(2)]\hat{k}$$

$$\vec{u} \times \vec{v} = [6 - (-2)]\hat{\imath} - [0 - (-1)]\hat{\jmath} + [0 - 2]\hat{k}$$

$$\vec{u} \times \vec{v} = 8\hat{\imath} - \hat{\jmath} - 2\hat{k} = (8,-1,-2)$$

10.b $\vec{u} = (2,1,-1)$ and $\vec{v} = (1,0,-1)$

$$\vec{u} \times \vec{v} = \text{skew}(\vec{u}) \cdot \vec{v}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \cdot \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -(-1) & (1) \\ (-1) & 0 & -(2) \\ -(1) & (2) & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} (0)(1) + (1)(0) + (1)(-1) \\ (-1)(1) + (0)(0) + (-2)(-1) \\ (-1)(1) + (2)(0) + (0)(-1) \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 0 + 0 + (-1) \\ -1 + 0 + 2 \\ -1 + 0 + 0 \end{bmatrix}$$

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = (-1,1,-1)}$$

Alternate:

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \hat{\imath} - \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \hat{\jmath} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \hat{k}$$

$$\vec{u} \times \vec{v} = [(1)(-1) - (0)(-1)]\hat{\imath} - [(2)(-1) - (1)(-1)]\hat{\jmath} + [(2)(0) - (1)(1)]\hat{k}$$

$$\vec{u} \times \vec{v} = [-1 - 0]\hat{\imath} - [-2 - (-1)]\hat{\jmath} + [0 - 1]\hat{k}$$

$$\vec{u} \times \vec{v} = -\hat{\imath} + \hat{\jmath} - \hat{k} = (-1,1,-1)$$

Matrix Multiplication

Problem 11

For the following matrix pairs, multiply the first matrix by the second:

11.a $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} (1)(1) + (2)(3) + (0)(1) & (1)(0) + (2)(2) + (0)(2) & (1)(2) + (0)(1) + (2)(3) \\ (0)(1) + (5)(3) + (1)(1) & (0)(0) + (5)(2) + (1)(2) & (0)(2) + (5)(1) + (1)(3) \\ (3)(1) + (1)(3) + (1)(1) & (3)(0) + (1)(2) + (1)(2) & (3)(2) + (1)(1) + (1)(3) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 + 6 + 0 & 0 + 4 + 0 & 2 + 0 + 6 \\ 0 + 15 + 1 & 0 + 10 + 2 & 0 + 5 + 3 \\ 3 + 3 + 1 & 0 + 2 + 2 & 6 + 1 + 3 \end{bmatrix}$$

$$\boxed{A \cdot B = \begin{bmatrix} 7 & 4 & 8 \\ 16 & 12 & 8 \\ 7 & 4 & 10 \end{bmatrix}}$$

11.b $C = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$

$$C \cdot D = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} (2)(2) + (-1)(6) & (2)(6) + (-1)(3) \\ (1)(2) + (3)(6) & (1)(6) + (3)(3) \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} 4 + (-6) & 12 + (-3) \\ 2 + 18 & 6 + 9 \end{bmatrix}$$

$$\boxed{C \cdot D = \begin{bmatrix} -2 & 9 \\ 20 & 15 \end{bmatrix}}$$

11.c $E = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$

$$E \cdot F = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$E \cdot F = \begin{bmatrix} (1)(1) + (0)(1) + (3)(0) & (1)(3) + (0)(4) + (3)(1) \\ (2)(1) + (1)(1) + (1)(0) & (2)(3) + (1)(4) + (1)(1) \end{bmatrix}$$

$$E \cdot F = \begin{bmatrix} 1 + 0 + 0 & 3 + 0 + 3 \\ 2 + 1 + 0 & 6 + 4 + 1 \end{bmatrix}$$

$$\boxed{E \cdot F = \begin{bmatrix} 1 & 6 \\ 3 & 11 \end{bmatrix}}$$

11.d $G = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$

$$G \cdot H = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$$

$$G \cdot H = \begin{bmatrix} (1)(1) + (1)(3) + (3)(0) & (1)(0) + (1)(1) + (3)(1) \\ (0)(1) + (1)(3) + (1)(0) & (0)(0) + (1)(1) + (1)(1) \end{bmatrix}$$

$$G \cdot H = \begin{bmatrix} 1 + 3 + 0 & 0 + 1 + 3 \\ 0 + 3 + 0 & 0 + 1 + 1 \end{bmatrix}$$

$$\boxed{G \cdot H = \begin{bmatrix} 4 & 4 \\ 3 & 2 \end{bmatrix}}$$

11.e $I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$$I \cdot J = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$I \cdot J = \begin{bmatrix} (1)(2) + (0)(1) + (1)(1) & (1)(1) + (0)(2) + (1)(2) & (1)(2) + (0)(3) + (1)(3) \\ (1)(2) + (2)(1) + (1)(1) & (1)(1) + (2)(2) + (1)(2) & (1)(2) + (2)(3) + (1)(3) \\ (4)(2) + (1)(1) + (1)(1) & (4)(1) + (1)(2) + (1)(2) & (4)(2) + (1)(3) + (1)(3) \end{bmatrix}$$

$$I \cdot J = \begin{bmatrix} 2+0+1 & 1+0+2 & 2+0+3 \\ 2+2+1 & 1+4+2 & 2+6+3 \\ 8+1+1 & 4+2+2 & 8+3+3 \end{bmatrix}$$

$$\boxed{I \cdot J = \begin{bmatrix} 3 & 3 & 5 \\ 5 & 7 & 11 \\ 10 & 8 & 14 \end{bmatrix}}$$

11.f $K = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

$$K \cdot L = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$K \cdot L = (1)(2) + (2)(0) + (1)(3)$$

$$K \cdot L = 2 + 0 + 3$$

$$\boxed{K \cdot L = 5}$$

11.g $M = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

$$M \cdot N = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$M \cdot N = \begin{bmatrix} (2)(1) & (2)(2) & (2)(1) \\ (0)(1) & (0)(2) & (0)(1) \\ (3)(1) & (3)(2) & (3)(1) \end{bmatrix}$$

$$\boxed{M \cdot N = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \\ 3 & 6 & 3 \end{bmatrix}}$$

Right and Left Vector-Matrix Multiplication

Problem 12

Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, calculate:

12.a $A \cdot \vec{v}$ and $\vec{v} \cdot A$

$$A \cdot \vec{v} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$A \cdot \vec{v} = \begin{bmatrix} (2)(1) + (1)(2) + (0)(1) \\ (0)(1) + (3)(2) + (1)(1) \\ (3)(1) + (1)(2) + (0)(1) \end{bmatrix}$$

$$A \cdot \vec{v} = \begin{bmatrix} 2 + 2 + 0 \\ 0 + 6 + 1 \\ 3 + 2 + 0 \end{bmatrix}$$

$$\boxed{A \cdot \vec{v} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}}$$

$$\vec{v} \cdot A = \vec{v}^t \cdot A^t$$

$$\vec{v} \cdot A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^t \cdot \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}^t$$

$$\vec{v} \cdot A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{v} \cdot A = \begin{bmatrix} (1)(2) + (2)(1) + (1)(0) & (1)(0) + (2)(3) + (1)(1) & (1)(3) + (2)(1) + (1)(0) \end{bmatrix}$$

$$\vec{v} \cdot A = \begin{bmatrix} 2 + 2 + 0 & 0 + 6 + 1 & 3 + 2 + 0 \end{bmatrix}$$

$$\boxed{\vec{v} \cdot A = \begin{bmatrix} 4 & 7 & 5 \end{bmatrix}}$$

12.b $A \cdot \vec{u}$ and $\vec{u} \cdot A$

$$A \cdot \vec{u} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$A \cdot \vec{u} = \begin{bmatrix} (2)(0) + (1)(1) + (0)(3) \\ (0)(0) + (3)(1) + (1)(3) \\ (3)(0) + (1)(1) + (0)(3) \end{bmatrix}$$

$$A \cdot \vec{u} = \begin{bmatrix} 0 + 1 + 0 \\ 0 + 3 + 3 \\ 0 + 1 + 0 \end{bmatrix}$$

$$\boxed{A \cdot \vec{u} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}}$$

$$\vec{u} \cdot A = \vec{u}^t \cdot A^t$$

$$\vec{u} \cdot A = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}^t \cdot \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}^t$$

$$\vec{u} \cdot A = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{u} \cdot A = \begin{bmatrix} (0)(2) + (1)(1) + (3)(0) & (0)(0) + (1)(3) + (3)(1) & (0)(3) + (1)(1) + (3)(0) \end{bmatrix}$$

$$\vec{u} \cdot A = \begin{bmatrix} 0 + 1 + 0 & 0 + 3 + 3 & 0 + 1 + 0 \end{bmatrix}$$

$$\boxed{\vec{u} \cdot A = \begin{bmatrix} 1 & 6 & 1 \end{bmatrix}}$$

12.c $B \cdot \vec{u}$ and $\vec{u} \cdot B$

$$B \cdot \vec{u} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$B \cdot \vec{u} = \begin{bmatrix} (2)(0) + (1)(1) + (3)(3) \\ (1)(0) + (0)(1) + (1)(3) \\ (2)(0) + (1)(1) + (0)(3) \end{bmatrix}$$

$$B \cdot \vec{u} = \begin{bmatrix} 0 + 1 + 9 \\ 0 + 0 + 3 \\ 0 + 1 + 0 \end{bmatrix}$$

$$\boxed{B \cdot \vec{u} = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix}}$$

$$\vec{u} \cdot B = \vec{u}^t \cdot B^t$$

$$\vec{u} \cdot B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}^t \cdot \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^t$$

$$\vec{u} \cdot B = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\vec{u} \cdot B = \begin{bmatrix} (0)(2) + (1)(1) + (3)(3) & (0)(1) + (1)(0) + (3)(1) & (0)(2) + (1)(1) + (3)(0) \end{bmatrix}$$

$$\vec{u} \cdot B = \begin{bmatrix} 0 + 1 + 9 & 0 + 0 + 3 & 0 + 1 + 0 \end{bmatrix}$$

$$\boxed{\vec{u} \cdot B = \begin{bmatrix} 10 & 3 & 1 \end{bmatrix}}$$

12.d $B \cdot \vec{w}$

$$B \cdot \vec{w} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$B \cdot \vec{w} = \begin{bmatrix} (2)(3) + (1)(1) + (3)(2) \\ (1)(3) + (0)(1) + (1)(2) \\ (2)(3) + (1)(1) + (0)(2) \end{bmatrix}$$

$$B \cdot \vec{w} = \begin{bmatrix} 6 + 1 + 6 \\ 3 + 0 + 2 \\ 6 + 1 + 0 \end{bmatrix}$$

$$\boxed{B \cdot \vec{w} = \begin{bmatrix} 13 \\ 5 \\ 7 \end{bmatrix}}$$

12.e $\vec{v} \cdot \vec{u}^t$

$$\vec{v} \cdot \vec{u}^t = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}^t$$

$$\vec{v} \cdot \vec{u}^t = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

$$\vec{v} \cdot \vec{u}^t = \begin{bmatrix} (1)(0) & (1)(1) & (1)(3) \\ (2)(0) & (2)(1) & (2)(3) \\ (1)(0) & (1)(1) & (1)(3) \end{bmatrix}$$

$$\boxed{\vec{v} \cdot \vec{u}^t = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}}$$

Systems of Linear Equations and Augmented Matrices

Problem 13

For each of the following systems of linear equations, write the augmented matrix:

13.a $\begin{cases} x+2y=5 \\ 2x-3y=-4 \end{cases}$

$$\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -3 & -4 \end{array}\right]}$$

13.b $\begin{cases} x+2y=7 \\ 5x-3y=9 \end{cases}$

$$\begin{bmatrix} 1 & 2 \\ 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 5 & -3 & 9 \end{array}\right]}$$

13.c $\begin{cases} 2x+3y=16 \\ 2x-y=8 \end{cases}$

$$\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 2 & -1 & 8 \end{array}\right]}$$

13.d $\begin{cases} 3x+y=2 \\ 2x+y=1 \end{cases}$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right]}$$

13.e $\begin{cases} x+y-5z=-3 \\ x+y+z=3 \\ 7x-y+2z=8 \end{cases}$

$$\begin{bmatrix} 1 & 1 & -5 \\ 1 & 1 & 1 \\ 7 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 8 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{array}\right]}$$

13.f $\begin{cases} x+y+z=2 \\ x-3y+2z=-4 \\ 5x-y+3z=8 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -3 & 2 & -4 \\ 5 & -1 & 3 & 8 \end{array}\right]}$$

13.g $\begin{cases} x+3y+z=4 \\ 2x-y+2z=1 \\ 3x-y+2z=3 \end{cases}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 2 \\ 3 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 2 & 3 \end{array}\right]}$$

13.h $\begin{cases} x+y-z=6 \\ 2x+3y+z=7 \\ x-y+2z=-2 \end{cases}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -2 \end{array}\right]}$$

Identifying a Row-Echelon Form of a Matrix

Problem 14

For each of the following matrices, identify if it is in row echelon form:

Row-echelon form requires:

All all-zero rows are at the matrix's bottom.

The first non-zero number of a row is to the right of the first non-zero number in the row above.

14.a $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A is **not** in row-echelon form. Its single non-zero row is not at the matrix's bottom.

14.b $B = \begin{bmatrix} 8 & 4 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

B is in row-echelon form. Its single non-zero row is at the matrix's bottom. The first non-zero number of the second row is to the right of the first non-zero number in the first row.

14.c $C = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

C is **not** in row-echelon form. The first non-zero number in the third row is not located to the right of the first non-zero number in the second row.

14.d $D = \begin{bmatrix} 0 & 8 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

D is **not** in row-echelon form. The first non-zero number in the second row is not to the right of the first non-zero number in the first row.

14.e $E = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

E is in row-echelon form. The first non-zero number in each row is located to the right of the first non-zero number in the row above. There are no all-zero rows to consider.

14.f $F = \begin{bmatrix} 0 & 5 & 3 & 0 & 7 \\ 0 & 0 & 5 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

F is in row-echelon form. The single all-zero row is at the matrix's bottom. The first non-zero number in each row is located to the right of the first non-zero number in the row above.

14.g $G = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

G is in row-echelon form. The first non-zero number in the second row is located to the right of the first non-zero number in the first row. There are no all-zero rows to consider.

14.h $H = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

H is **not** in row-echelon form. The single all-zero row is not located at the matrix's bottom.

14.i $I = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

I is **not** in row-echelon form. The first non-zero number in the second row is not located to the right of the first non-zero number in the first row.

14.j $J = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}$

J is in row-echelon form. The first non-zero number in the second row is located to the right of the first non-zero number in the first row.

Identifying the Reduced Row-Echelon Form of a Matrix

Problem 15

For each of the following matrices, identify if it is in **reduced** row echelon form:

Reduced row-echelon form requires:

The matrix is in row-echelon form.

All all-zero rows are at the matrix's bottom.

The first non-zero number of a row is to the right of the first non-zero number in the row above.

The first non-zero number in each row has the value 1.

The first non-zero in each row is the only non-zero number in that entire column.

15.a $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A is in reduced row-echelon form. There are no all-zero rows to consider. The first non-zero number of each row is to the right of the first non-zero number of the row above. The first non-zero number of each row has the value 1. The first non-zero number in each row is the only non-zero number in that entire column.

15.b $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

B is **not** in reduced row-echelon form. The first non-zero number in the second row shares that column with a second non-zero number (in the first row).

15.c $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

C is **not** in reduced row-echelon form. The first non-zero number in the third row does not have a value of 1, and it shares that column with a second non-zero number (in the first row).

15.d $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

D is **not** in reduced row-echelon form. The single all-zero row is not located at the matrix's bottom.

15.e $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E is **not** in reduced row-echelon form. The first non-zero number in the first row does not have a value of 1.

15.f $F = \begin{bmatrix} 1 & 5 & 0 & 0 & 7 \\ 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

F is in reduced row-echelon form. Its single all-zero row is located at the matrix's bottom. The first non-zero number in the second row is to the right of the first non-zero number in the first row. The first non-zero numbers in each row has a value of one and are the only non-zero numbers in their respective columns.

15.g $G = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

G is **not** in reduced row-echelon form. The first non-zero number in the second row does not have a value of 1 and shares the column with another non-zero number.

15.h $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

H is in reduced row-echelon form. There are no all-zero rows to consider. The first non-zero number in the second row is to the right of the first non-zero number in the first row. The first non-zero numbers in both rows have values of 1 and are the only non-zero numbers in their respective columns.

15.i $I = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$

I is in reduced row-echelon form. There are no all-zero rows to consider. The first non-zero number in the second row is to the right of the first non-zero number in the first row. The first non-zero numbers in both rows have values of 1 and are the only non-zero numbers in their respective columns.

15.j $J = \begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

J is in reduced row-echelon form. There are no all-zero rows to consider. The first non-zero number in each row is located to the right of the first non-zero numbers in the rows above, has a value of 1, and is the only non-zero number in its respective column.

Computing the Row-Echelon Form of a Matrix

Problem 16

Convert the following matrices to row echelon form:

16.a $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 2-1 & -1-3 \\ 1 & 3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \xrightarrow[r_2 - r_1]{} \begin{bmatrix} 1 & -4 \\ 1-1 & 3-(-4) \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 1 & -4 \\ 0 & 7 \end{bmatrix}}$$

16.b $A = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$

$$A \rightarrow \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix} \xrightarrow[r_2 - 3r_1]{} \begin{bmatrix} 2 & 6 \\ 6-3(2) & 3-3(3) \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 2 & 6 \\ 0 & -15 \end{bmatrix}}$$

16.c $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 6 & 7 & 5 \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 6r_1} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 2 - 2(1) & 5 - 2(2) & 7 - 2(3) \\ 6 - 6(1) & 7 - 6(2) & 5 - 6(3) \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -13 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -13 \end{bmatrix} \xrightarrow{r_3 + 5r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 + 5(0) & -5 + 5(1) & -13 + 5(1) \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix}}$$

16.d $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 3r_1} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 2 - 2(1) & 5 - 2(2) & 0 - 2(3) \\ 3 - 3(1) & 0 - 3(2) & 5 - 3(3) \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{r_3 + 6r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 + 6(0) & -6 + 6(1) & -4 + 6(-6) \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -40 \end{bmatrix}}$$

16.e $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 7 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 7 & 3 \end{bmatrix} \underset{\xrightarrow{r_3-4r_1}}{\xrightarrow{r_2-2r_1}} \begin{bmatrix} 1 & 2 & 1 \\ 2-2(1) & 3-2(2) & 1-2(1) \\ 4-4(1) & 7-4(2) & 3-4(1) \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{r_3-r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0+0 & -1-(-1) & -1-(-1) \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Computing the Reduced Row-Echelon Form of a Matrix

Problem 17

Convert the following matrices to **reduced** row echelon form:

17.a $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & -4 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & -4 \end{bmatrix} \xrightarrow{r_2-2r_1} \begin{bmatrix} 1 & 2 & 5 \\ 2-2(1) & -3-2(2) & -4-2(5) \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -14 \end{bmatrix} \xrightarrow{-r_2/2} \begin{bmatrix} 1 & 2 & 5 \\ -0/2 & -(-7)/2 & -(-14)/2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1-2r_2} \begin{bmatrix} 1-2(0) & 2-2(1) & 5-2(2) \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}}$$

17.b $B = \begin{bmatrix} 1 & 2 & 7 \\ 5 & -3 & 9 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 5 & -3 & 9 \end{bmatrix} \xrightarrow{r_2 - 5r_1} \begin{bmatrix} 1 & 2 & 7 \\ 5 - 5(1) & -3 - 5(2) & 9 - 5(7) \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & -13 & -26 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & -13 & -26 \end{bmatrix} \xrightarrow{-r_2/13} \begin{bmatrix} 1 & 2 & 7 \\ -0/_{13} & -(-13)/_{13} & -(-26)/_{13} \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 2 - 2(1) & 7 - 2(2) \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{B \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}}$$

17.c $C = \begin{bmatrix} 2 & 3 & 16 \\ 2 & -1 & 8 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & 3 & 16 \\ 2 & -1 & 8 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 2 & 3 & 16 \\ 2 - 2 & -1 - 3 & 8 - 16 \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 3 & 16 \\ 0 & -4 & -8 \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 3 & 16 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{r_2/(-4)} \begin{bmatrix} 2 & 3 & 16 \\ 0/_{-4} & -4/_{-4} & -8/_{-8} \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 3 & 16 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 3 & 16 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \begin{bmatrix} 2 - 3(0) & 3 - 3(1) & 16 - 3(2) \\ 0 & 1 & 2 \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 0 & 10 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C \sim \begin{bmatrix} 2 & 0 & 10 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1/2} \begin{bmatrix} 2/_2 & 0/_2 & 10/_2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{C \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}}$$

17.d $D = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

$$D = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 3-2 & 1-1 & 2-1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$D \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$D \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 0 & 1 \\ 2-2(1) & 1-2(0) & 1-2(1) \end{bmatrix}$$

$$\boxed{D \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}}$$

17.e $E = \begin{bmatrix} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{bmatrix}$

$$E = \begin{bmatrix} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{bmatrix} \underset{r_3 - 7r_1}{\xrightarrow{r_2 - r_1}} \begin{bmatrix} 1 & 1 & -5 & -3 \\ 1-1 & 1-1 & 1-(-5) & 3-(-3) \\ 7-7(1) & -1-7(1) & 2-7(-5) & 8-7(-3) \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 0 & 6 & 6 \\ 0 & -8 & 35 & 29 \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 0 & 6 & 6 \\ 0 & -8 & 35 & 29 \end{bmatrix} \underset{r_3/(-8)}{\xrightarrow{r_2/6}} \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0/6 & 0/6 & 6/6 & 6/6 \\ 0/(-8) & -8/(-8) & 35/(-8) & 29/(-8) \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -\frac{35}{8} & -\frac{29}{8} \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 0 & 6 & 6 \\ 0 & 1 & -\frac{35}{8} & -\frac{29}{8} \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 1 & -\frac{35}{8} & -\frac{29}{8} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 1 & -\frac{35}{8} & -\frac{29}{8} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & -5 & -3 \\ 0 & 1 & -\frac{35}{8} & -\frac{29}{8} \\ 0 & 0 & 1 & 1 \end{bmatrix} \underset{r_2 + \frac{35}{8}r_3}{\xrightarrow{r_1 + 5r_3}} \begin{bmatrix} 1+5(0) & 1+5(0) & -5+5(1) & -3+5(1) \\ 0+\frac{35}{8}(0) & 1+\frac{35}{8}(0) & -\frac{35}{8}+\frac{35}{8}(1) & -\frac{29}{8}+\frac{35}{8}(1) \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1-0 & 1-1 & 0-0 & 2-\frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{E \sim \begin{bmatrix} 1 & 0 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}}$$

17.f $F = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 2 & -4 \\ 5 & -1 & 3 & 8 \end{bmatrix}$

$$F = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 2 & -4 \\ 5 & -1 & 3 & 8 \end{bmatrix} \underset{\xrightarrow{r_3 - 5r_1}}{\xrightarrow{r_2 - r_1}} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1-1 & -3-1 & 2-1 & -4-2 \\ 5-5(1) & -1-5(1) & 3-5(1) & 8-5(2) \end{bmatrix}$$

$$F \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & 1 & -6 \\ 0 & -6 & -2 & -2 \end{bmatrix}$$

$$F \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & 1 & -6 \\ 0 & -6 & -2 & -2 \end{bmatrix} \xrightarrow{r_2/(-4)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ {}^{0}/_{(-4)} & {}^{-4}/_{(-4)} & {}^{1}/_{(-4)} & {}^{-6}/_{(-4)} \\ 0 & -6 & -2 & -2 \end{bmatrix}$$

$$F \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{2} \\ 0 & -6 & -2 & -2 \end{bmatrix}$$

$$F \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{2} \\ 0 & -6 & -2 & -2 \end{bmatrix} \xrightarrow{r_3 + 6r_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{2} \\ 0+6(0) & -6+6(1) & -2+6\left(-\frac{1}{4}\right) & -2+6\left(\frac{3}{2}\right) \end{bmatrix}$$

$$F \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{2} \\ 0 & 0 & -\frac{7}{2} & 7 \end{bmatrix}$$

$$F\sim\begin{bmatrix}1&1&1&2\\0&1&-\frac{1}{4}&\frac{3}{2}\\0&0&-\frac{7}{2}&7\end{bmatrix}\xrightarrow{r_3\cdot\left(-\frac{2}{7}\right)}\begin{bmatrix}1&1&1&2\\0&1&-\frac{1}{4}&\frac{3}{2}\\0\cdot\left(-\frac{2}{7}\right)&0\cdot\left(-\frac{2}{7}\right)&-\frac{7}{2}\cdot\left(-\frac{2}{7}\right)&7\cdot\left(-\frac{2}{7}\right)\end{bmatrix}$$

$$F\sim\begin{bmatrix}1&1&1&2\\0&1&-\frac{1}{4}&\frac{3}{2}\\0&0&1&-2\end{bmatrix}$$

$$F\sim\begin{bmatrix}1&1&1&2\\0&1&-\frac{1}{4}&\frac{3}{2}\\0&0&1&-2\end{bmatrix}\xrightarrow{r_2+\frac{1}{4}r_3}\begin{bmatrix}1&1&1&2\\0+\frac{1}{4}(0)&1+\frac{1}{4}(0)&-\frac{1}{4}+\frac{1}{4}(1)&\frac{3}{2}+\frac{1}{4}(-2)\\0&0&1&-2\end{bmatrix}$$

$$F\sim\begin{bmatrix}1&1&1&2\\0&1&0&1\\0&0&1&-2\end{bmatrix}$$

$$F\sim\begin{bmatrix}1&1&1&2\\0&1&0&1\\0&0&1&-2\end{bmatrix}\xrightarrow{r_1-r_2-r_3}\begin{bmatrix}1-0-0&1-1-0&1-0-1&2-1-(-2)\\0&1&0&1\\0&0&1&-2\end{bmatrix}$$

$$\boxed{F\sim\begin{bmatrix}1&0&0&3\\0&1&0&1\\0&0&1&-2\end{bmatrix}}$$

17.g $G=\begin{bmatrix}1&3&1&4\\2&-1&2&1\\3&-1&2&3\end{bmatrix}$

$$G=\begin{bmatrix}1&3&1&4\\2&-1&2&1\\3&-1&2&3\end{bmatrix}\begin{matrix}\xrightarrow{r_2-2r_1}\\\xrightarrow{r_3-3r_1}\end{matrix}\begin{bmatrix}1&3&1&4\\2-2(1)&-1-2(3)&2-2(1)&1-2(4)\\3-3(1)&-1-3(3)&2-3(1)&3-3(4)\end{bmatrix}$$

$$G\sim\begin{bmatrix}1&3&1&4\\0&-7&0&-7\\0&-10&-1&-9\end{bmatrix}$$

$$G\sim\begin{bmatrix}1&3&1&4\\0&-7&0&-7\\0&-10&-1&-9\end{bmatrix}\xrightarrow{r_2/(-7)}\begin{bmatrix}1&3&1&4\\0/_{-7}&-7/_{-7}&0/_{-7}&-7/_{-7}\\0&-10&-1&-9\end{bmatrix}$$

$$G\sim\begin{bmatrix}1&3&1&4\\0&1&0&1\\0&-10&-1&-9\end{bmatrix}$$

$$G\sim\begin{bmatrix}1&3&1&4\\0&1&0&1\\0&-10&-1&-9\end{bmatrix}\xrightarrow{r_3+10r_2}\begin{bmatrix}1&3&1&4\\0&1&0&1\\0+10(0)&-10+10(1)&-1+10(0)&-9+10(1)\end{bmatrix}$$

$$G\sim\begin{bmatrix}1 & 3 & 1 & 4\\0 & 1 & 0 & 1\\0 & 0 & -1 & 1\end{bmatrix}$$

$$G\sim\begin{bmatrix}1 & 3 & 1 & 4\\0 & 1 & 0 & 1\\0 & 0 & -1 & 1\end{bmatrix}\xrightarrow{-r_3}\begin{bmatrix}1 & 3 & 1 & 4\\0 & 1 & 0 & 1\\-0 & -0 & -(-1) & -1\end{bmatrix}$$

$$G\sim\begin{bmatrix}1 & 3 & 1 & 4\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

$$G\sim\begin{bmatrix}1 & 3 & 1 & 4\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}\xrightarrow{r_1-3r_2-r_3}\begin{bmatrix}1-3(0)-0 & 3-3(1)-0 & 1-3(0)-1 & 4-3(1)-(-1)\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

$$\boxed{G\sim\begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}}$$

17.h $$H=\begin{bmatrix}1 & 1 & -1 & 6\\2 & 3 & 1 & 7\\1 & -1 & 2 & -2\end{bmatrix}$$

$$H=\begin{bmatrix}1 & 1 & -1 & 6\\2 & 3 & 1 & 7\\1 & -1 & 2 & -2\end{bmatrix}\begin{matrix}\xrightarrow{r_2-2r_1}\\\xrightarrow{r_3-r_1}\end{matrix}\begin{bmatrix}1 & 1 & -1 & 6\\2-2(1) & 3-2(1) & 1-2(-1) & 7-2(6)\\1-1 & -1-1 & 2-(-1) & -2-6\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & -2 & 3 & -8\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & -2 & 3 & -8\end{bmatrix}\xrightarrow{r_3+2r_2}\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0+2(0) & -2+2(1) & 3+2(3) & -8+2(-5)\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & 0 & 9 & -18\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & 0 & 9 & -18\end{bmatrix}\xrightarrow{r_3/9}\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0/9 & 0/9 & 9/9 & -18/9\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & 0 & 1 & -2\end{bmatrix}$$

$$H\sim\begin{bmatrix}1 & 1 & -1 & 6\\0 & 1 & 3 & -5\\0 & 0 & 1 & -2\end{bmatrix}\xrightarrow{r_2-3r_3}\begin{bmatrix}1 & 1 & -1 & 6\\0-3(0) & 1-3(0) & 3-3(1) & -5-3(-2)\\0 & 0 & 1 & -2\end{bmatrix}$$

$$H \sim \begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$H \sim \begin{bmatrix} 1 & 1 & -1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{r_1 - r_2 + r_3} \begin{bmatrix} 1-0+0 & 1-1+0 & -1-0+1 & 6-1+(-2) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\boxed{H \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}}$$

Solution of Systems of Linear Equations Using Reduced Row-Echelon Form

Problem 18

Solve each of the following systems of linear equations using a matrix in RREF:

18.a $\begin{cases} x + 2y = 5 \\ 2x - 3y = -4 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -3 & -4 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2-2(1) & -3-2(2) & -4-2(5) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -14 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -14 \end{array}\right] \xrightarrow{r_2/-7} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ {}^{0}/_{-7} & {}^{-7}/_{-7} & {}^{-14}/_{-7} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{cc|c} 1-2(0) & 2-2(1) & 5-2(2) \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\boxed{\begin{cases} x = 1 \\ y = 2 \end{cases}}$$

Check:

$$x + 2y = (1) + 2(2) = 1 + 4 = 5$$

$$2x - 3y = 2(1) - 3(2) = 2 - 6 = -4$$

18.b $\begin{cases} x+2y=7 \\ 5x-3y=9 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 5 & -3 & 9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 5 & -3 & 9 \end{array}\right] \xrightarrow{r_2-5r_1} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 5-5(1) & -3-5(2) & 9-5(7) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -13 & -26 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -13 & -26 \end{array}\right] \xrightarrow{r_2/-13} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0/_{-13} & -13/_{-13} & -26/_{-13} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cc|c} 1-2(0) & 2-2(1) & 7-2(2) \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$$\boxed{\begin{cases} x=3 \\ y=2 \end{cases}}$$

Check:

$(3)+2(2)=3+4=7$

$5(3)-3(2)=15-6=9$

18.c $\begin{cases} 2x+3y=16 \\ 2x-y=8 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 2 & -1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 2 & -1 & 8 \end{array}\right] \xrightarrow{r_2-r_1} \left[\begin{array}{cc|c} 2 & 3 & 16 \\ 2-2 & -1-3 & 8-16 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 0 & -4 & -8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 0 & -4 & -8 \end{array}\right] \xrightarrow{r_2/-4} \left[\begin{array}{cc|c} 2 & 3 & 16 \\ 0/_{-4} & -4/_{-4} & -8/_{-4} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 16 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{r_1-3r_2} \left[\begin{array}{cc|c} 2-3(0) & 3-3(1) & 16-3(2) \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 0 & 10 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 0 & 10 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{r_1/2} \left[\begin{array}{cc|c} 2/_2 & 0/_2 & 10/_2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

$$\boxed{\begin{cases} x=5 \\ y=2 \end{cases}}$$

Check:

$2x+3y=2(5)+3(2)=10+6=16$

$2x-y=2(5)-(2)=10-2=8$

18.d $\begin{cases} 3x + y = 2 \\ 2x + y = 1 \end{cases}$

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|c} 3-2 & 1-1 & 2-1 \\ 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2-2(1) & 1-2(0) & 1-2(1) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$\boxed{\begin{cases} x = 1 \\ y = -1 \end{cases}}$$

Check:

$3(1) + (-1) = 3 - 1 = 2$

$2(1) + (-1) = 2 - 1 = 1$

18.e $\begin{cases} x + y - 5z = -3 \\ x + y + z = 3 \\ 7x - y + 2z = 8 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 1 & 1 & 1 & 3 \\ 7 & -1 & 2 & 8 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 1-1 & 1-1 & 1-(-5) & 3-(-3) \\ 7 & -1 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 0 & 0 & 6 & 6 \\ 7 & -1 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 0 & 0 & 6 & 6 \\ 7 & -1 & 2 & 8 \end{array}\right] \xrightarrow{r_2/6} \left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 0/6 & 0/6 & 6/6 & 6/6 \\ 7 & -1 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \\ 7 & -1 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & -3 \\ 0 & 0 & 1 & 1 \\ 7 & -1 & 2 & 8 \end{array}\right] \xrightarrow[r_3 - 2r_2]{r_1 + 5r_2} \left[\begin{array}{ccc|c} 1+5(0) & 1+5(0) & -5+5(1) & -3+5(1) \\ 0 & 0 & 1 & 1 \\ 7-2(0) & -1-2(0) & 2-2(1) & 8-2(1) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 7 & -1 & 0 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 7 & -1 & 0 & 6\end{array}\right] \xrightarrow{r_3-7r_1} \left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 7-7(1) & -1-7(1) & 0-7(0) & 6-7(2)\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & -8 & 0 & -8\end{array}\right] \xrightarrow{r_3/(-8)} \left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{array}\right] \xrightarrow{r_1-r_3} \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{ccc|c}1 & 1-1 & 0-0 & 2-1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\boxed{\begin{cases}x = 1\\ y = 1\\ z = 1\end{cases}}$$

Check:

$x + y - 5z = (1) + (1) - 5(1) = 1 + 1 - 5 = -3$

$x + y + z = (1) + (1) + (1) = 1 + 1 + 1 = 3$

$7x - y + 2z = 7(1) - (1) + 2(1) = 7 - 1 + 2 = 8$

18.f Solve $\begin{cases} x + y + z = 2\\ x - 3y + 2z = -4\\ 5x - y + 3z = 8\end{cases}$.

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 1 & -3 & 2 & -4\\ 5 & -1 & 3 & 8\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 1 & -3 & 2 & -4\\ 5 & -1 & 3 & 8\end{array}\right] \xrightarrow{r_2-r_1} \xrightarrow{r_3-5r_1} \left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 1-1 & -3-1 & 2-1 & -4-2\\ 5-5(1) & -1-5(1) & 3-5(1) & 8-5(2)\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & -4 & 1 & -6\\ 0 & -6 & -2 & -2\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & -4 & 1 & -6\\ 0 & -6 & -2 & -2\end{array}\right] \xrightarrow{r_3/(-2)} \left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & -4 & 1 & -6\\ {}^{0}/_{-2} & {}^{-6}/_{-2} & {}^{-2}/_{-2} & {}^{-2}/_{-2}\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & -4 & 1 & -6\\ 0 & 3 & 1 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & -4 & 1 & -6\\ 0 & 3 & 1 & 1\end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0+0 & -4+3 & 1+1 & -6+1\\ 0 & 3 & 1 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & -5 \\ 0 & 3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & -5 \\ 0 & 3 & 1 & 1 \end{array}\right] \xrightarrow{r_2/(-1)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ {}^{0}\!/_{-1} & {}^{-1}\!/_{-1} & {}^{2}\!/_{-1} & {}^{-5}\!/_{-1} \\ 0 & 3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 3 & 1 & 1 \end{array}\right] \xrightarrow{r_3-3r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0-3(0) & 3-3(1) & 1-3(-2) & 1-3(5) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 7 & -14 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 7 & -14 \end{array}\right] \xrightarrow{r_3/7} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ {}^{0}\!/_{7} & {}^{0}\!/_{7} & {}^{7}\!/_{7} & {}^{-14}\!/_{7} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_2+2r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0+2(0) & 1+2(0) & -2+2(1) & 5+2(-2) \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_1-r_2-r_3} \left[\begin{array}{ccc|c} 1-0-0 & 1-1-0 & 1-0-1 & 2-1-(-2) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\boxed{\begin{cases} x = 3 \\ y = 1 \\ z = -2 \end{cases}}$$

Check:

$$x + y + z = (3) + (1) + (-2) = 3 + 1 - 2 = 2$$

$$x - 3y + 2z = (3) - 3(1) + 2(-2) = 3 + 3 - 4 = -4$$

$$5x - y + 3z = 5(3) - (1) + 3(-2) = 15 - 1 - 7 = 8$$

18.g Solve $\begin{cases} x+3y+z=4 \\ 2x-y+2z=1 \\ 3x-y+2z=3 \end{cases}$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 1 \\ 3 & -1 & 2 & 3 \end{array}\right] \xrightarrow[r_3-3r_1]{r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2-2(1) & -1-2(3) & 2-2(1) & 1-2(4) \\ 3-3(1) & -1-3(3) & 2-3(1) & 3-4(4) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & -7 & 0 & -7 \\ 3 & -1 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & -7 & 0 & -7 \\ 0 & -10 & -1 & -9 \end{array}\right] \xrightarrow{r_2/(-7)} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0/_{-7} & -7/_{-7} & 0/_{-7} & -7/_{-7} \\ 0 & -10 & -1 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -10 & -1 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -10 & -1 & -9 \end{array}\right] \xrightarrow{r_3+10r_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -10+10(1) & -1+10(0) & -9+10(1) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{r_3/(-1)} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0/_{-1} & 0/_{-1} & -1/_{-1} & 1/_{-1} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_1-3r_2-r_3} \left[\begin{array}{ccc|c} 1-3(0)-0 & 3-3(1)-0 & 1-3(0)-1 & 4-3(1)-(-1) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\boxed{\begin{cases} x=2 \\ y=1 \\ z=-1 \end{cases}}$$

Check:

$x+3y+z=(2)+3(1)+(-1)=2+3-1=4$

$2x-y+2z=2(2)-(1)+2(-1)=4-1-2=1$

$3x-y+2z=3(2)-(1)+2(-1)=6-1-2=3$

18.h Solve $\begin{cases} x+y-z=6 \\ 2x+3y+z=7 \\ x-y+2z=-2 \end{cases}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 2 & -2 \end{array}\right] \xrightarrow{r_2-2r_1} \xrightarrow{r_3-r_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2-2(1) & 3-2(1) & 1-2(-1) & 7-2(6) \\ 1-1 & -1-1 & 2-(-1) & -2-6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & -2 & 3 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & -2 & 3 & -8 \end{array}\right] \xrightarrow{r_3+2r_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0+2(0) & -2+2(1) & 3+2(3) & -8+2(-5) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 9 & -18 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 9 & -18 \end{array}\right] \xrightarrow{r_3/9} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ {}^{0}/_{9} & {}^{0}/_{9} & {}^{9}/_{9} & {}^{-18}/_{9} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_2-3r_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0-3(0) & 1-3(0) & 3-3(1) & -5-3(-2) \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_1-r_2+r_3} \left[\begin{array}{ccc|c} 1-0+0 & 1-1+0 & -1-0+1 & 6-1+(-2) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 01 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\boxed{\begin{cases} x=3 \\ y=1 \\ z=-2 \end{cases}}$$

Check:

$x+y-z=(3)+(1)-(-2)=3+1+2=6$

$2x+3y+z=2(3)+3(1)+(-2)=6+3-2=7$

$x-y+2z=(3)-(1)+2(-2)=3-1-4=-2$

Rank of a Matrix

Problem 19

Matrix rank is the number of non-zero rows in its row-echelon equivalent matrix.

19.a What is the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 5-2(2) & 0-2(3) \\ 3-3(1) & 0-3(2) & 5-3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{r_3+6r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0+6(0) & -6+6(1) & -4+6(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 14 \end{bmatrix}$$

Row Echelon form
3 nonzero rows

$$\boxed{\text{Rank}(A) = 3}$$

19.b What is the rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & 2 & 1 \\ 2-2(1) & 0-2(2) & 1-2(1) \\ 3-3(1) & 2-3(2) & 2-3(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix} \xrightarrow{r_3-r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0-0 & -4-(-4) & -1-(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Echelon form
2 non-zero rows

$$\boxed{\text{Rank}(A) = 2}$$

19.c What is the rank of $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 6 & 1 & 1 \\ 3 & 4 & 3 & 4 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 6 & 1 & 1 \\ 3 & 4 & 3 & 4 \end{bmatrix} \begin{matrix} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 3r_1} \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 - 2(1) & 6 - 2(1) & 1 - 2(2) & 1 - 2(3) \\ 3 - 3(1) & 4 - 3(1) & 3 - 3(2) & 4 - 3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & -3 & -5 \\ 0 & 1 & -3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & -3 & -5 \\ 0 & 1 & -3 & -5 \end{bmatrix} \xrightarrow{r_2 - 4r_3} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 - 4(0) & 4 - 4(1) & -3 - 4(-3) & -5 - 4(-5) \\ 0 & 1 & -3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 9 & 15 \\ 0 & 1 & -3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 9 & 15 \\ 0 & 1 & -3 & -5 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 9 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 9 & 15 \end{bmatrix}$$

Row Echelon form
3 non-zero rows

$\boxed{\text{Rank}(A) = 3}$

19.d What is the rank of $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \begin{matrix} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 3r_1} \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 - 2(1) & 3 - 2(1) & -1 - 2(1) \\ 3 - 3(1) & 1 - 3(1) & -5 - 3(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & -8 \end{bmatrix} \xrightarrow{r_3 + 2r_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 + 2(0) & -2 + 2(1) & -8 + 2(-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -14 \end{bmatrix}$$

Row Echelon form
3 non-zero rows

$\boxed{\text{Rank}(A) = 3}$

Linear Dependence Using Matrix Echelon Form

Problem 20

To determine linear independence, convert vector sets into matrix of transposed vectors, convert matrix to row-echelon form. If the matrix has no zero rows, the set is linearly independent.

20.a Are $\vec{u}_1 = (1,2,5)$, $\vec{u}_2 = (2,4,1)$, and $\vec{u}_3 = (1,1,2)$ linearly independent in $\mathbb{R}^3$?

$$M_a = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$M_a = \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix}$$

$$M_a = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 5 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 5 & 1 & 2 \end{bmatrix} \xrightarrow[r_3-5r_1]{r_2-2r_1} \begin{bmatrix} 1 & 2 & 1 \\ 2-2(1) & 4-2(2) & 1-2(1) \\ 5-5(1) & 1-5(2) & 2-5(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & -9 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & -9 & -3 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Row-Echelon form
No zero rows

This vector set is linearly independent.

Alternate:

$$|M_a| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 5 & 1 & 2 \end{vmatrix}$$

$$|M_a| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}(1) - \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix}(2) + \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}(1)$$

$$|M_a| = [(4)(2) - (1)(1)](1) - [(2)(2) - (5)(1)](2) + [(2)(1) - (5)(4)](1)$$

$$|M_a| = (8-1)(1) - (4-5)(2) + (2-20)(1)$$

$$|M_a| = (7)(1) - (-1)(2) + (-18)(1)$$

$$|M_a| = 7 + 2 - 18$$

$$|M_a| = -9$$

$$|M_a| \neq 0$$

Linearly independent.

20.b Are $\vec{v}_1 = (1,4,3)$, $\vec{v}_2 = (3,0,1)$, and $\vec{v}_3 = (1,1,2)$ linearly independent in $\mathbb{R}^3$?

$$M_b = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$$

$$M_b = \begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix}$$

$$M_b = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 4r_1} \begin{bmatrix} 1 & 3 & 1 \\ 4 - 4(1) & 0 - 4(3) & 1 - 4(1) \\ 3 - 3(1) & 1 - 3(3) & 2 - 3(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -12 & -4 \\ 0 & -8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -12 & -4 \\ 0 & -8 & -1 \end{bmatrix} \xrightarrow{r_2/-4} \begin{bmatrix} 1 & 3 & 1 \\ 0/_{-4} & -12/_{-4} & -4/_{-4} \\ 0 & -8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -8 & -1 \end{bmatrix} \xrightarrow{r_2 + \frac{3}{8}r_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 + \frac{3}{8}(0) & 3 + \frac{3}{8}(-8) & 1 + \frac{3}{8}(-1) \\ 0 & -8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & \frac{5}{8} \\ 0 & -8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & \frac{5}{8} \\ 0 & -8 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -1 \\ 0 & 0 & \frac{5}{8} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -1 \\ 0 & 0 & \frac{5}{8} \end{bmatrix}$$

Row-Echelon form
No zero rows.

This vector set is linearly independent.

Alternate:

$$|M_b| = \begin{vmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$
$$|M_b| = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}(1) - \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}(3) + \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix}(1)$$
$$|M_b| = [(0)(2) - (1)(1)](1) - [(4)(2) - (3)(1)](3) + [(4)(1) - (3)(0)](1)$$
$$|M_b| = (0 - 1)(1) - (8 - 3)(3) + (4 - 0)(1)$$
$$|M_b| = (-1)(1) - (5)(2) + (4)(1)$$
$$|M_b| = -1 - 10 + 4$$
$$|M_b| = -7$$
$$|M_b| \neq 0$$
Linearly independent.

20.c Are $\vec{w}_1 = (1,1,1)$, $\vec{w}_2 = (1,2,0)$, and $\vec{u}_3 = (0,-1,1)$ linearly independent in $\mathbb{R}^3$?

$$M_c = [\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3]$$

$$M_c = \begin{bmatrix} w_{1x} & w_{2x} & w_{3x} \\ w_{1y} & w_{2y} & w_{3y} \\ w_{1z} & w_{2z} & w_{3z} \end{bmatrix}$$

$$M_c = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & 0 \\ 1-1 & 2-1 & -1-0 \\ 1-1 & 0-1 & 1-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0+0 & -1+1 & 1+(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form
1 zero row

This matrix set is *not* linearly independent.

Alternate:

$$|M_c| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$|M_c| = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}(1) - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}(1) + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}(0)$$

$$|M_c| = [(2)(1) - (0)(-1)](1) - [(1)(1) - (1)(-1)](1) + [(1)(0) - (1)(2)](0)$$

$$|M_c| = (2-0)(1) - (1+1)(1) + (0-2)(0)$$

$$|M_c| = (2)(1) - (2)(1) + (-2)(0)$$

$$|M_c| = 2 - 2 + 0$$

$$|M_c| = 0$$

Not linearly independent.

20.d Are $\vec{x}_1 = (1,1,1)$, $\vec{x}_2 = (1,2,0)$, and $\vec{x}_3 = (0,-1,2)$ linearly independent in $\mathbb{R}^3$?

$$M_d = [\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3]$$

$$M_d = \begin{bmatrix} x_{1x} & x_{2x} & x_{3x} \\ x_{1y} & x_{2y} & x_{3y} \\ x_{1z} & x_{2z} & x_{3z} \end{bmatrix}$$

$$M_d = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow[\xrightarrow{r_3 - r_1}]{r_2 - r_1} \begin{bmatrix} 1 & 1 & 0 \\ 1-1 & 2-1 & -1-0 \\ 1-1 & 0-1 & 2-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0+0 & -1+1 & 2+(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-echelon form.
No zero rows.

This vector set is linearly independent.

Alternate:

$$|M_d| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$|M_d| = \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}(1) - \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}(1) + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}(0)$$

$$|M_d| = [(2)(2) - (0)(-1)](1) - [(1)(2) - (1)(-1)](1) + [(1)(0) - (1)(2)](0)$$

$$|M_d| = (4-0)(1) - [2-(-1)](1) + (0-2)(0)$$

$$|M_d| = (4)(1) - (3)(1) + (-2)(0)$$

$$|M_d| = 4 - 3 + 0$$

$$|M_d| = 1$$

$$|M_d| \neq 0$$

Linearly independent.

Problem 21

21.a Are $\vec{u}_1 = (1,2)$ and $\vec{u}_2 = (2,4)$ linearly independent in $\mathbb{R}^2$?

$M_a = [\vec{u}_1 \quad \vec{u}_2]$

$M_a = \begin{bmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{bmatrix}$

$M_a = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 2 \\ 2-2(1) & 4-2(2) \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Row-echelon form
1 zero row

This vector pair is ***not*** linearly independent.

Alternate:

$|M_a| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$|M_a| = (1)(4) - (2)(2)$

$|M_a| = 4 - 4$

$|M_a| = 0$

Not linearly independent.

21.b Are $\vec{v}_1 = (2,8)$ and $\vec{v}_2 = (2,5)$ linearly independent in $\mathbb{R}^2$?

$M_b = [\vec{v}_1 \quad \vec{v}_2]$

$M_b = \begin{bmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{bmatrix}$

$M_b = \begin{bmatrix} 2 & 2 \\ 8 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 8 & 5 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 2 & 2 \\ 8-4(2) & 5-4(2) \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}$

Row-echelon form.
No zero rows.

This vector pair is linearly independent.

Alternate:

$|M_a| = \begin{vmatrix} 2 & 2 \\ 8 & 5 \end{vmatrix}$

$|M_a| = (2)(5) - (8)(2)$

$|M_a| = 10 - 16$

$|M_a| = -6$

$|M_a| \neq 0$

Linearly independent.

Problem 22

22.a Is $\begin{cases} u = 1 - x \\ v = 5 - 3x + 2x^2 \\ w = 1 + 3x - x^2 \end{cases}$ linearly independent in P_2?

$$M_a = [[u]^t \quad [v]^t \quad [w]^t]$$

$$M_a = \begin{bmatrix} u_0 & v_0 & w_0 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix}$$

$$M_a = \begin{bmatrix} 1 & 5 & 1 \\ -1 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{r_2 + r_1} \begin{bmatrix} 1 & 5 & 1 \\ -1+1 & -3+5 & 3+1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0-0 & 2-2 & -1-2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Row-echelon form
No zero rows.

This polynomial set is linearly independent.

Alternate:

$$|M_a| = \begin{vmatrix} 1 & 5 & 1 \\ -1 & -3 & 3 \\ 0 & 2 & -1 \end{vmatrix}$$

$$|M_a| = \begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix}(1) - \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}(5) + \begin{vmatrix} -1 & -3 \\ 0 & 2 \end{vmatrix}(1)$$

$$|M_a| = [(-3)(-1) - (2)(3)](1) - [(-1)(-1) - (0)(-1)](5) + [(-1)(2) - (0)(-3)](1)$$

$$|M_a| = (3 - 6)(1) - (1 - 0)(5) + (-2 - 0)(1)$$

$$|M_a| = (-3)(1) - (1)(5) + (-2)(1)$$

$$|M_a| = -3 - 5 - 2$$

$$|M_a| = -10$$

$$|M_a| \neq 0$$

Linearly independent.

22.b Is $\begin{cases} a = 1 + x + x^2 \\ b = x + 2x^2 \\ c = x^2 \end{cases}$ linearly independent in P_2?

$M_b = [[a]^t \quad [b]^t \quad [c]^t]$

$$M_b = \begin{bmatrix} a_{a0} & b_{a0} & c_{a0} \\ a_{a1} & b_{a1} & c_{a1} \\ a_{a2} & b_{a2} & c_{a2} \end{bmatrix}$$

$$M_a = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 0 & 0 \\ 1-1 & 1-0 & 0-0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-1-2(0) & 2-0-2(1) & 1-0-2(0) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-echelon form No zero rows.

This polynomial set is linearly independent.

Alternate:

Alternate:

$$|M_b| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$|M_b| = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}(1) - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}(0) + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}(0)$$

$$|M_b| = [(1)(1) - (2)(0)](1) - [(1)(1) - (1)(0)](0) + [(1)(2) - (1)(1)](0)$$

$$|M_b| = (1-0)(1) - (1-0)(0) + (2-1)(0)$$

$$|M_b| = (1)(1) - (1)(0) + (1)(0)$$

$$|M_b| = 1 - 0 + 0$$

$$|M_b| = 1$$

$$|M_b| \neq 0$$

Linearly independent.

Basis Using Matrix Reduced Row-Echelon Form

Problem 23

If a vector set is a basis for $\mathbb{R}^2$, its constituent vectors will be linearly independent, and the set will span $\mathbb{R}^2$.

23.a Do $\vec{u}_1 = (2,8)$ and $\vec{u}_2 = (2,5)$ form a basis for $\mathbb{R}^2$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 2 & 2 \\ 8 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 8 & 5 \end{bmatrix} \underset{\xrightarrow{r_2 - 4r_1}}{\xrightarrow{r_1/2}} \begin{bmatrix} 2/2 & 2/2 \\ 8-4(2) & 5-4(2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$$

Row-echelon form
No zero rows.

Since vector set B has no zero rows in its row-echelon form, B is linearly independent.

Span:

If B spans $\mathbb{R}^2$, then any arbitrary vector in $\mathbb{R}^2$ may be expressed as a linear combination of the vectors in B.

$$\vec{w} = (x_1, y_1)$$

$$\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2$$

$$(x_1, y_1) = c_1(2,8) + c_2(2,5)$$

$$\begin{bmatrix} 2 & 2 \\ 8 & 5 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 2 & x_1 \\ 8 & 5 & y_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 2 & x_1 \\ 8 & 5 & y_1 \end{array}\right] \underset{\xrightarrow{r_2 - 4r_1}}{\xrightarrow{r_1/2}} \left[\begin{array}{cc|c} 2/2 & 2/2 & x_1/2 \\ 8-4(2) & 5-4(2) & y_1 - 4x_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2}x_1 \\ 0 & -3 & y_1 - 4x_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2}x_1 \\ 0 & -3 & y_1 - 4x_1 \end{array}\right] \xrightarrow{r_2/(-3)} \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2}x_1 \\ 0/_{-3} & -3/_{-3} & (y_1 - 4x_1)/_{-3} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2}x_1 \\ 0 & 1 & \frac{4}{3}x_1 - \frac{1}{3}y_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2}x_1 \\ 0 & 1 & \frac{4}{3}x_1 - \frac{1}{3}y_1 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|c} 1-0 & 1-1 & \frac{1}{2}x_1 - \left(\frac{4}{3}x_1 - \frac{1}{3}y_1\right) \\ 0 & 1 & \frac{4}{3}x_1 - \frac{1}{3}y_1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3}y_1 - \frac{5}{6}x_1 \\ 0 & 1 & \frac{4}{3}x_1 - \frac{1}{3}y_1 \end{array}\right]$$

$$\vec{w} = (x_1, y_1) = \left(\frac{1}{3}y_1 - \frac{5}{6}x_1\right)\vec{u}_1 + \left(\frac{4}{3}x_1 - \frac{1}{3}y_1\right)\vec{u}_2$$

Since arbitrary vector $\vec{w}$ can be represented as a linear combination of the vectors in set B, vector set B spans $\mathbb{R}^2$.

Since vector set B is both linearly independent and spans $\mathbb{R}^2$, it is a basis for $\mathbb{R}^2$.

23.b Do $\vec{u}_1 = (1,3)$ and $\vec{u}_2 = (2,6)$ form a basis for $\mathbb{R}^2$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 3 \\ 2-2(1) & 6-2(3) \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Row-echelon form.
One zero row.

Since vector set B has at least one zero row in its row-echelon form, B is not linearly independent.

Because vector set B is not linearly independent, it is not a basis for $\mathbb{R}^2$.

Problem 24

If a vector set is a basis for $\mathbb{R}^2$, its constituent vectors will be linearly independent, and the set will span $\mathbb{R}^2$.

24.a Do $\vec{u}_1 = (1,0,0)$, $\vec{u}_2 = (1,1,0)$, and $\vec{u}_3 = (1,1,1)$ form a basis for $\mathbb{R}^3$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \xrightarrow{r_2 - r_3} \begin{bmatrix} 1-0 & 1-1 & 1-1 \\ 0-0 & 1-0 & 1-1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row echelon form
No zero rows.

Since vector set B has no zero rows in its row-echelon form, B is linearly independent.

Span:

If B spans $\mathbb{R}^3$, then any arbitrary vector in $\mathbb{R}^3$ may be expressed as a linear combination of the vectors in B.

$$\vec{w} = (x_1, y_1, z_1)$$

$$\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

$$(x_1, y_1, z_1) = c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & 1 & 1 & y_1 \\ 0 & 0 & 1 & z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & 1 & 1 & y_1 \\ 0 & 0 & 1 & z_1 \end{array}\right] \xrightarrow{r_1 - r_2} \xrightarrow{r_2 - r_3} \left[\begin{array}{ccc|c} 1-0 & 1-1 & 1-1 & x_1 - y_1 \\ 0-0 & 1-0 & 1-1 & y_1 - z_1 \\ 0 & 0 & 1 & z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 - y_1 \\ 0 & 1 & 0 & y_1 - z_1 \\ 0 & 0 & 1 & z_1 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ y_1 - z_1 \\ z_1 \end{bmatrix}$$

$$\vec{w} = (x_1, y_1, z_1) = (x_1 - y_1)(1,0,0) + (y_1 - z_1)(1,1,0) + z_1(1,1,1)$$

Since arbitrary vector $\vec{w}$ can be represented as a linear combination of the vectors in set B, vector set B spans $\mathbb{R}^3$.

Since vector set B is both linearly independent and spans $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

24.b Do $\vec{u}_1 = (1,2,3)$, $\vec{u}_2 = (2,0,1)$, and $\vec{u}_3 = (3,2,2)$ form a basis for $\mathbb{R}^3$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{r_2-2r_1} \xrightarrow{r_3-3r_1} \begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 0-2(2) & 2-2(3) \\ 3-3(1) & 1-3(2) & 2-3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 & -5 & -7 \end{bmatrix} \xrightarrow{r_2/-4} \begin{bmatrix} 1 & 2 & 3 \\ {}^{0}/_{-4} & {}^{-4}/_{-4} & {}^{-4}/_{-4} \\ 0 & -5 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \xrightarrow{r_3+5r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0+5(0) & -5+5(1) & -7+5(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Row-echelon form.
No zero rows.

Since vector set B has no zero rows in its row-echelon form, B is linearly independent.

Span:

$$\vec{w} = (x_1, y_1, z_1)$$

$$\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

$$(x_1, y_1, z_1) = c_1(1,2,3) + c_2(2,0,1) + c_3(3,2,2)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 2 & 0 & 2 & y_1 \\ 3 & 1 & 2 & z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 2 & 0 & 2 & y_1 \\ 3 & 1 & 2 & z_1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - 3r_1} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 2-2(1) & 0-2(2) & 2-2(3) & y_1 - 2(x_1) \\ 3-3(1) & 1-3(2) & 2-3(3) & z_1 - 3x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & -4 & -4 & y_1 - 2x_1 \\ 0 & -5 & -7 & z_1 - 3x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & -4 & -4 & y_1 - 2x_1 \\ 0 & -5 & -7 & z_1 - 3x_1 \end{array}\right] \xrightarrow{r_2/-4} \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0/-4 & -4/-4 & -4/-4 & y_1 - 2x_1/-4 \\ 0 & -5 & -7 & z_1 - 3x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & -5 & -7 & z_1 - 3x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & -5 & -7 & z_1 - 3x_1 \end{array}\right] \xrightarrow{r_3 + 5r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0+5(0) & -5+5(1) & -7+5(1) & z_1 - 3x_1 + 5\left(\frac{1}{2}x_1 - \frac{1}{4}y_1\right) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & -2 & z_1 - \frac{1}{2}x_1 - \frac{5}{4}y_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & -2 & z_1 - \frac{1}{2}x_1 - \frac{5}{4}y_1 \end{array}\right] \xrightarrow{r_3/-2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0/-2 & 0/-2 & -2/-2 & z_1 - \frac{1}{2}x_1 - \frac{5}{4}y_1 / -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{ccc|c} 1-2(0) & 2-2(1) & 3-2(1) & x_1 - 2\left(\frac{1}{2}x_1 - \frac{1}{4}y_1\right) \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{2}y_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{2}y_1 \\ 0 & 1 & 1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 - r_3} \\ \xrightarrow{r_2 - r_3} \end{array} \left[\begin{array}{ccc|c} 1-0 & 0-0 & 1-1 & \frac{1}{2}y_1 - \left(\frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1\right) \\ 0-0 & 1-0 & 1-1 & \frac{1}{2}x_1 - \frac{1}{4}y_1 - \left(\frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1\right) \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{4}x_1 - \frac{1}{8}y_1 + \frac{1}{2}z_1 \\ 0 & 1 & 0 & \frac{1}{4}x_1 - \frac{7}{8}y_1 + \frac{1}{2}z_1 \\ 0 & 0 & 1 & \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}x_1 - \frac{1}{8}y_1 + \frac{1}{2}z_1 \\ \frac{1}{4}x_1 - \frac{7}{8}y_1 + \frac{1}{2}z_1 \\ \frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1 \end{bmatrix}$$

$$\begin{aligned} \vec{w} = (x_1, y_1, z_1) = &\left(-\frac{1}{4}x_1 - \frac{1}{8}y_1 + \frac{1}{2}z_1\right)(1,2,3) + \left(\frac{1}{4}x_1 - \frac{7}{8}y_1 + \frac{1}{2}z_1\right)(2,0,1) \\ &+ \left(\frac{1}{4}x_1 + \frac{5}{8}y_1 - \frac{1}{2}z_1\right)(3,2,2) \end{aligned}$$

Since arbitrary vector $\vec{w}$ can be represented as a linear combination of the vectors in set B, vector set B spans $\mathbb{R}^3$.

Since vector set B is both linearly independent and spans $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

24.c Do $\vec{u}_1 = (1,2,1)$, $\vec{u}_2 = (1,7,-1)$, and $\vec{u}_3 = (2,1,3)$ form a basis for $\mathbb{R}^3$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 7 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 7 & 1 \\ 1 & -1 & 3 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 2-2(1) & 7-2(1) & 1-2(2) \\ 1-1 & -1-1 & 3-2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{r_3 + \frac{2}{5}r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 + \frac{2}{5}(0) & -2 + \frac{2}{5}(5) & 6 + \frac{2}{5}(-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & \frac{24}{5} \end{bmatrix}$$

Row-echelon form.
No zero rows.

Since vector set B has no zero rows in its row-echelon form, B is linearly independent.

Span:

$$\vec{w} = (x_1, y_1, z_1)$$

$$\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

$$(x_1, y_1, z_1) = c_1(1,2,1) + c_2(1,7,-1) + c_3(2,1,3)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 7 & 1 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 2 & 7 & 1 & y_1 \\ 1 & -1 & 3 & z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 2 & 7 & 1 & y_1 \\ 1 & -1 & 3 & z_1 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 2-2(1) & 7-2(1) & 1-2(2) & y_1 - 2(x_1) \\ 1-1 & -1-1 & 3-2 & z_1 - x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 5 & -3 & y_1 - 2x_1 \\ 0 & -2 & 1 & z_1 - x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 5 & -3 & y_1 - 2x_1 \\ 0 & -2 & 1 & z_1 - x_1 \end{array}\right] \xrightarrow{r_2/5} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0/5 & 5/5 & -3/5 & y_1 - 2x_1/5 \\ 0 & -2 & 1 & z_1 - x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5}y_1 - \frac{2}{5}x_1 \\ 0 & -2 & 1 & z_1 - x_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5}y_1 - \frac{2}{5}x_1 \\ 0 & -2 & 1 & z_1 - x_1 \end{array}\right] \xrightarrow{r_3+2r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5}y_1 - \frac{2}{5}x_1 \\ 0+2(0) & -2+2(1) & 1+2\left(-\frac{3}{5}\right) & z_1 - x_1 + 2\left(\frac{1}{5}y_1 - \frac{2}{5}x_1\right) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5}y_1 - \frac{2}{5}x_1 \\ 0 & 0 & -\frac{1}{5} & -\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5}y_1 - \frac{2}{5}x_1 \\ 0 & 0 & -\frac{1}{5} & -\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1 \end{array}\right] \xrightarrow{r_2-3r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0-3(0) & 1-3(0) & -\frac{3}{5}-3\left(-\frac{1}{5}\right) & \frac{1}{5}y_1 - \frac{2}{5}x_1 - 3\left(-\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1\right) \\ 0 & 0 & -\frac{1}{5} & -\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & -\frac{1}{5} & -\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x_1 \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & -\frac{1}{5} & -\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1-r_2} \\ \\ \xrightarrow{-5r_3} \end{array} \left[\begin{array}{ccc|c} 1-0 & 1-1 & 2-0 & x_1 - (5x_1 - y_1 - 3z_1) \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ -5(0) & -5(0) & -5\left(-\frac{1}{5}\right) & -5\left(-\frac{9}{5}x_1 + \frac{2}{5}y_1 + z_1\right) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4x_1 + y_1 + 3z_1 \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & 1 & 9x_1 - 2y_1 - 5z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4x_1 + y_1 + 3z_1 \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & 1 & 9x_1 - 2y_1 - 5z_1 \end{array}\right] \xrightarrow{r_1-2r_3} \left[\begin{array}{ccc|c} 1-2(0) & 0-2(0) & 2-2(1) & -4x_1 + y_1 + 3z_1 - 2(9x_1 - 2y_1 - 5z_1) \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & 1 & 9x_1 - 2y_1 - 5z_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 14x_1 + 5y_1 + 13z_1 \\ 0 & 1 & 0 & 5x_1 - y_1 - 3z_1 \\ 0 & 0 & 1 & 9x_1 - 2y_1 - 5z_1 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 14x_1 + 5y_1 + 13z_1 \\ 5x_1 - y_1 - 3z_1 \\ 9x_1 - 2y_1 - 5z_1 \end{bmatrix}$$

$$\begin{aligned} \vec{w} = (x_1, y_1, z_1) = (14x_1 + 5y_1 + 13z_1)(1,2,1) + (5x_1 - y_1 - 3z_1)(1,7,-1) \\ + (9x_1 - 2y_1 - 5z_1)(2,1,3) \end{aligned}$$

Since arbitrary vector $\vec{w}$ can be represented as a linear combination of the vectors in set B, vector set B spans $\mathbb{R}^3$.

Since vector set B is both linearly independent and spans $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

24.d Do $\vec{u}_1 = (1,2,1)$, $\vec{u}_2 = (5,2,3)$, and $\vec{u}_3 = (3,2,2)$ form a basis for $\mathbb{R}^3$?

Linear independence:

$$[B] = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3]$$

$$[B] = \begin{bmatrix} u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ u_{1z} & u_{2z} & u_{3z} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 2 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 5 & 3 \\ 2-2(1) & 2-2(5) & 2-2(3) \\ 1-1 & 3-5 & 2-3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & -8 & -4 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & -8 & -4 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{r_2 - 4r_3} \begin{bmatrix} 1 & 5 & 3 \\ 0-4(0) & -8-4(-2) & -4-4(-1) \\ 0 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 5 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form.
One zero row.

Since vector set B has a zero row in its row-echelon form, B is not linearly independent.

Since vector set B is not linearly independent, it is **not** a basis for $\mathbb{R}^3$.

Problem 25

If a polynomial set is a basis for P_2, its constituent polynomials will be linearly independent, and the set will span P_2.

25.a Do the polynomials $u = 1 - x$, $v = 5 - 3x + 2x^2$, and $w = 1 + 3x - x^2$ form a basis for P_2?

Linear independence:

$$[P] = [u \quad v \quad w]$$

$$[P] = \begin{bmatrix} a_{0,u} & a_{0,v} & a_{0,w} \\ a_{1,u} & a_{1,v} & a_{1,w} \\ a_{2,u} & a_{2,v} & a_{2,w} \end{bmatrix}$$

$$[P] = \begin{bmatrix} 1 & 5 & 0 \\ -1 & -3 & 2 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ -1 & -3 & 2 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{r_2 + r_1} \begin{bmatrix} 1 & 5 & 0 \\ -1+1 & -3+5 & 2+0 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{r_3 - \frac{3}{2} r_2} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 2 \\ 0 - \frac{3}{2}(0) & 3 - \frac{3}{2}(2) & -1 - \frac{3}{2}(2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

Row-echelon form.
No zero rows.

Since polynomial set P has no zero rows in its row-echelon form, P is linearly independent.

Span:

$$p = a_{0,p} + a_{1,p}x + a_{2,p}x^2$$

$$p = c_1 u + c_2 v + c_3 w$$

$$a_{0,p} + a_{1,p}x + a_{2,p}x^2 = c_1(1 - x + 0x^2) + c_2(5 - 3x + 2x^2) + c_3(1 + 3x - x^2)$$

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_{0,p} \\ a_{1,p} \\ a_{2,p} \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ -1 & -3 & 3 & a_{1,p} \\ 0 & 2 & -1 & a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ -1 & -3 & 3 & a_{1,p} \\ 0 & 2 & -1 & a_{2,p} \end{array}\right] \xrightarrow{r_2+r_1} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ -1+1 & -3+5 & 3+1 & a_{1,p}+a_{0,p} \\ 0 & 2 & -1 & a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 2 & 4 & a_{0,p}+a_{1,p} \\ 0 & 2 & -1 & a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 2 & 4 & a_{0,p}+a_{1,p} \\ 0 & 2 & -1 & a_{2,p} \end{array}\right] \xrightarrow{r_3-r_2} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 2 & 4 & a_{0,p}+a_{1,p} \\ 0-0 & 2-2 & -1-4 & a_{2,p}-\left(a_{0,p}+a_{1,p}\right) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 2 & 4 & a_{0,p}+a_{1,p} \\ 0 & 0 & -5 & -a_{0,p}-a_{1,p}+a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 2 & 4 & a_{0,p}+a_{1,p} \\ 0 & 0 & -5 & -a_{0,p}-a_{1,p}+a_{2,p} \end{array}\right] \begin{array}{c} \xrightarrow{r_2/2} \\ \xrightarrow{r_3/-5} \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0/2 & 2/2 & 4/2 & \left(a_{0,p}+a_{1,p}\right)/2 \\ 0/-5 & 0/-5 & -5/-5 & \left(-a_{0,p}-a_{1,p}+a_{2,p}\right)/-5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 1 & 2 & \frac{1}{2}a_{0,p}+\frac{1}{2}a_{1,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a_{0,p} \\ 0 & 1 & 2 & \frac{1}{2}a_{0,p}+\frac{1}{2}a_{1,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right] \begin{array}{c} \xrightarrow{r_1-r_3} \\ \xrightarrow{r_2-2r_3} \end{array} \left[\begin{array}{ccc|c} 1-0 & 5-0 & 1-1 & a_{0,p}-\left(\frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p}\right) \\ 0-2(0) & 1-2(0) & 2-2(1) & \frac{1}{2}a_{0,p}+\frac{1}{2}a_{1,p}-2\left(\frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p}\right) \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & -\frac{4}{5}a_{0,p}-\frac{1}{5}a_{1,p}+\frac{1}{5}a_{2,p} \\ 0 & 1 & 0 & \frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & -\frac{4}{5}a_{0,p}-\frac{1}{5}a_{1,p}+\frac{1}{5}a_{2,p} \\ 0 & 1 & 0 & \frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right] \xrightarrow{r_1-5r_2} \left[\begin{array}{ccc|c} 1-5(0) & 5-5(1) & 0-5(0) & -\frac{4}{5}a_{0,p}-\frac{1}{5}a_{1,p}+\frac{1}{5}a_{2,p}-5\left(\frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p}\right) \\ 0 & 1 & 0 & \frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{13}{10}a_{0,p}-\frac{7}{10}a_{1,p}+\frac{11}{5}a_{2,p} \\ 0 & 1 & 0 & \frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p} \\ 0 & 0 & 1 & \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{13}{10}a_{0,p}-\frac{7}{10}a_{1,p}+\frac{11}{5}a_{2,p} \\ \frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p} \\ \frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p} \end{bmatrix}$$

$$p = \left(-\frac{13}{10}a_{0,p}-\frac{7}{10}a_{1,p}+\frac{11}{5}a_{2,p}\right)u + \left(\frac{1}{10}a_{0,p}+\frac{1}{10}a_{1,p}+\frac{2}{5}a_{2,p}\right)v + \left(\frac{1}{5}a_{0,p}+\frac{1}{5}a_{1,p}-\frac{1}{5}a_{2,p}\right)w$$

Since arbitrary polynomial p can be represented as a linear combination of polynomial set P, set P spans P_2.

Since polynomial set P is linearly independent and spans P_2, P is a basis for P_2.

25.b Do polynomials $u = 1 + 2x + x^2$, $v = 2 + x^2$, and $w = 3 + 2x + 2x^2$ form a basis for P_2?

Linear independence:

$$[P] = [u \quad v \quad w]$$

$$[P] = \begin{bmatrix} a_{0,u} & a_{0,v} & a_{0,w} \\ a_{1,u} & a_{1,v} & a_{1,w} \\ a_{2,u} & a_{2,v} & a_{2,w} \end{bmatrix}$$

$$[P] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 2 - 2(1) & 0 - 2(2) & 2 - 2(3) \\ 1 - 1 & 1 - 2 & 2 - 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{r_3 - \frac{1}{4}r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 - \frac{1}{4}(0) & -1 - \frac{1}{4}(-4) & -1 - \frac{1}{4}(-4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form.
One zero row.

Because polynomial set P has a zero row in its row-echelon form, P is not linearly independent.

Since polynomial set P is not linearly independent, it is not a basis for P_2.

25.c Do polynomials $u = 1 + x + x^2$, $v = x + 2x^2$, and $w = x^2$ form a basis for P_2?

Linear independence:

$$[P] = [u \quad v \quad w]$$

$$[P] = \begin{bmatrix} a_{0,u} & a_{0,v} & a_{0,w} \\ a_{1,u} & a_{1,v} & a_{1,w} \\ a_{2,u} & a_{2,v} & a_{2,w} \end{bmatrix}$$

$$[P] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 0 & 0 \\ 1-1 & 1-0 & 0-0 \\ 1-1 & 2-0 & 1-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0-2(0) & 2-2(1) & 1-2(0) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-echelon form.
No zero vectors.

Because vector set P has no zero rows in its row-echelon form, P is linearly independent.

Span:

$$p = a_{0,p} + a_{1,p}x + a_{2,p}x^2$$

$$p = c_1 u + c_2 v + c_3 w$$

$$a_{0,p} + a_{1,p}x + a_{2,p}x^2 = c_1(1 - x + 0x^2) + c_2(5 - 3x + 2x^2) + c_3(1 + 3x - x^2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_{0,p} \\ a_{1,p} \\ a_{2,p} \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 1 & 1 & 0 & a_{1,p} \\ 1 & 2 & 1 & a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 1 & 1 & 0 & a_{1,p} \\ 1 & 2 & 1 & a_{2,p} \end{array}\right] \xrightarrow{r_2 - r_1} \xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 1-1 & 1-0 & 0-0 & a_{1,p} - a_{0,p} \\ 1-1 & 2-0 & 1-0 & a_{2,p} - a_{0,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 0 & 1 & 0 & -a_{0,p} + a_{1,p} \\ 0 & 2 & 1 & -a_{0,p} + a_{2,p} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 0 & 1 & 0 & -a_{0,p} + a_{1,p} \\ 0 & 2 & 1 & -a_{0,p} + a_{2,p} \end{array}\right] \xrightarrow{r_3 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 0 & 1 & 0 & -a_{0,p} + a_{1,p} \\ 0 - 2(0) & 2 - 2(1) & 1 - 2(0) & -a_{0,p} + a_{2,p} - 2\left(-a_{0,p} + a_{1,p}\right) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_{0,p} \\ 0 & 1 & 0 & -a_{0,p} + a_{1,p} \\ 0 & 0 & 1 & a_{0,p} - 2a_{1,p} + a_{2,p} \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_{0,p} \\ -a_{0,p} + a_{1,p} \\ a_{0,p} - 2a_{1,p} + a_{2,p} \end{bmatrix}$$

$$p = a_{0,p}u + \left(-a_{0,p} + a_{1,p}\right)v + \left(a_{0,p} - 2a_{1,p} + a_{2,p}\right)w$$

Since arbitrary polynomial p can be represented as a linear combination of polynomial set P, set P spans P_2.

Since polynomial set P is linearly independent and spans P_2, P is a basis for P_2.

25.d Do polynomials $u = 1 - 2x + 3x^2$, $v = 5 + 6x - x^2$, and $w = 3 + 2x + x^2$ form a basis for P_2?

Linear independence:

$$[P] = [u \quad v \quad w]$$

$$[P] = \begin{bmatrix} a_{0,u} & a_{0,v} & a_{0,w} \\ a_{1,u} & a_{1,v} & a_{1,w} \\ a_{2,u} & a_{2,v} & a_{2,w} \end{bmatrix}$$

$$[P] = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow{r_2+2r_1} \xrightarrow{r_3-3r_1} \begin{bmatrix} 1 & 5 & 3 \\ -2+2(1) & 6+2(5) & 2+2(3) \\ 3-3(1) & -1-3(5) & 1-3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0+0 & -16+16 & -8+8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form.
One zero row.

Since polynomial set P has one zero row in its row-echelon form, P is not linearly independent.

Since polynomial set P is not linearly independent, P is **not** a basis for P_2.

Basis of a Matrix Row Space

Problem 26

To find the basis of a matrix's row space, find the matrix's row-echelon form. Each nonzero row in the row-echelon form is a vector in the row space's basis. The number of vectors in the basis is the dimension of the matrix's row space. The number of non-zero rows in the matrix's row-echelon form is the matrix's rank.

26.a Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$, find the basis of its row space, the dimension of its row space, and its rank.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 5-2(2) & 0-2(3) \\ 3-3(1) & 0-3(0) & 5-3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{r_3+6r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0+6(0) & -6+6(1) & -4+6(-6) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -40 \end{bmatrix}$$

Row-echelon form.

The non-zero rows in the row-echelon form of matrix A are the vectors in the basis of A's row space.

A basis of A's row space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -40 \end{bmatrix} \right\}}$.

The dimension of matrix A's row space is the number of vectors in the row space's basis.

$$\boxed{\dim(\text{rowsp}(A)) = 3}$$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$$\boxed{\text{rank}(A) = 3}$$

26.b Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, find the basis of its row space, the dimension of its row space, and its rank.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & 2 & 1 \\ 2-2(1) & 0-2(2) & 1-2(1) \\ 3-3(1) & 2-3(2) & 2-3(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix} \xrightarrow{r_3-r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0-0 & -4-(-4) & -1-(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form.

The non-zero rows in the row-echelon form of matrix A are the vectors in the basis of A's row space.

A basis of A's row space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} \right\}}$.

The dimension of matrix A's row space is the number of vectors in the row space's basis.

$\boxed{\dim(\text{rowsp}(A)) = 2}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 2}$

26.c Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 1 \\ 3 & -4 & 3 \end{bmatrix}$, find the basis of its row space, the dimension of its row space, and its rank.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 1 \\ 3 & -4 & 3 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-2r_1} \begin{bmatrix} 1 & -1 & 2 \\ 2-2(1) & 6-2(-1) & 1-2(2) \\ 3-3(1) & -4-3(-1) & 3-3(2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{r_2+8r_3} \begin{bmatrix} 1 & -1 & 2 \\ 0+8(0) & 8+8(-1) & -3+8(-3) \\ 0 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -27 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -27 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -27 \end{bmatrix}$$

Row-echelon form.

The non-zero rows in the row-echelon form of matrix A are the vectors in the basis of A's row space.

A basis of A's row space is $\boxed{\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -27 \end{bmatrix}\right\}}$.

The dimension of matrix A's row space is the number of vectors in the row space's basis.

$\boxed{\dim(\text{rowsp}(A)) = 3}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 3}$

26.d Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}$, find the basis of its row space, the dimension of its row space, and its rank.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 3r_1} \begin{bmatrix} 1 & 1 & -1 \\ 2 - 2(1) & 3 - 2(1) & -1 - 2(-1) \\ 3 - 3(1) & 1 - 3(1) & -5 - 3(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{r_3 + 2r_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 + 2(0) & -2 + 2(1) & -2 + 2(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form.

The non-zero rows in the row-echelon form of matrix A are the vectors in the basis of A's row space.

A basis of A's row space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}}$.

The dimension of matrix A's row space is the number of vectors in the row space's basis.

$\boxed{\dim(\text{rowsp}(A)) = 2}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 2}$

Basis of a Matrix Column Space

Problem 27

To find the basis of a matrix's column space, find the matrix's row-echelon form. Note which columns have pivots. The corresponding columns in the original matrix are vectors forming a basis for the original matrix's column space The number of vectors in the basis is the dimension of the matrix's column space. The number of non-zero rows in the matrix's row-echelon form is the matrix's rank.

27.a Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$, find the basis of its column space, the dimension of its column space, and its rank.

$$A \sim \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{1} & -6 \\ 0 & 0 & \mathbf{-40} \end{bmatrix}$$

Row-echelon form, found in #26.a.

The row echelon form has three pivots, one in each column. Therefore, all three columns in A form a basis of A's column space.

A basis of A's row space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}}$.

The dimension of matrix A's column space is the number of vectors in the column space's basis.

$\boxed{\dim(\text{colsp}(A)) = 3}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 3}$

27.b Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, find the basis of its column space, the dimension of its column space, and its rank.

$$A \sim \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & \mathbf{-4} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form, found in #26.b.

The row echelon form has two pivots, in the first two columns. Therefore, the first two columns in A form a basis of A's column space.

A basis of A's column space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}}$.

The dimension of matrix A's column space is the number of vectors in the column space's basis.

$\boxed{\dim(\text{colsp}(A)) = 2}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 2}$

27.c Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 1 \\ 3 & -4 & 3 \end{bmatrix}$, find the basis of its column space, the dimension of its column space, and its rank.

$$A \sim \begin{bmatrix} \mathbf{1} & -1 & 2 \\ 0 & \mathbf{-1} & -3 \\ 0 & 0 & \mathbf{-27} \end{bmatrix}$$

Row-echelon form, found in #26.c.

The row echelon form has three pivots, one in each column. Therefore, all three columns in A form a basis of A's column space.

A basis of A's column space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}}$.

The dimension of matrix A's column space is the number of vectors in the column space's basis.

$\boxed{\dim(\text{colsp}(A)) = 3}$

The number of non-zero rows in the row-echelon form of matrix A is A's rank.

$\boxed{\text{rank}(A) = 2}$

27.d Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}$, find the basis of its column space, the dimension of its column space, and its rank.

$$A \sim \begin{bmatrix} \mathbf{1} & 1 & -1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form, found in #26.d.

The row echelon form has two pivots, in the first two columns. Therefore, the first two columns in A form a basis of A's column space.

A basis of A's column space is $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}}$.

The dimension of matrix A's column space is the number of vectors in the column space's basis.

$\boxed{\dim(\text{colsp}(A)) = 2}$

The number of non-zero rows in the row-echelon form of matrix $A$is A's rank.

$\boxed{\text{rank}(A) = 2}$

Basis of a Matrix Null Space

Problem 28

A matrix's null space is the set of all vectors that will multiply with the matrix to produce $\vec{0}$. Set up an equation with an arbitrary vector. Create an equivalent augmented matrix. Find the augmented matrix's reduced row-echelon form to solve for the vector set. In each solution, set the free variables to arbitrary values and solve for the remaining (leading) variable.

28.a Given matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, find a basis for the matrix's null space and the null space's dimension.

$$A \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 - 3(1) & 5 - 3(2) & 0 - 3(0) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{r_2/-1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0/_{-1} & -1/_{-1} & 0/_{-1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{cc|c} 1 - 2(0) & 2 - 2(0) & 0 - 2(0) \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The basis of $\mathrm{Null}(A)$ is $\boxed{\{\vec{0}\}}$.

$$\boxed{\dim(\mathrm{Null}(A)) = 0}$$

28.b Given matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, find a basis for the matrix's null space and the null space's dimension.

$A \cdot \vec{v} = \vec{0}$

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 - 2(1) & 6 - 2(3) & 0 - 2(0) \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} x + 3y = 0 \\ \quad 0 = 0 \end{cases}$

$x + 3t = 0$

$x = -3t$

$v = \begin{bmatrix} -3t \\ t \end{bmatrix}$

$v = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

The basis of $\text{Null}(A)$ is $\boxed{\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}}$.

$\boxed{\dim(\text{Null}(A)) = 1}$

28.c Given matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$, find a basis for the matrix's null space and the null space's dimension.

$$A \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - r_1} \\ \xrightarrow{r_3 - 2r_1} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1-1 & 2-1 & 0-0 & 0-0 \\ 2-2(1) & 3-2(1) & 1-2(0) & 0-2(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 - r_2} \\ \\ \xrightarrow{r_3 - r_2} \end{array} \left[\begin{array}{ccc|c} 1-0 & 1-1 & 0-0 & 0-0 \\ 0 & 1 & 0 & 0 \\ 0-0 & 1-1 & 1-0 & 0-0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

$$\vec{v} = \vec{0}$$

The basis of $\text{Null}(A)$ is $\boxed{\{\vec{0}\}}$.

$$\boxed{\dim(\text{Null}(A)) = 0}$$

28.d Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix}$, find a basis for the matrix's null space and the null space's dimension.

$$A \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 3 & 8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 3 & 8 & 0 \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1-1 & 2-2 & 5-3 & 0-0 \\ 2-2(1) & 3-2(2) & 8-2(3) & 0-2(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow[r_3/2]{r_2/-1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ {}^{0}\!/\!_{-1} & {}^{-1}\!/\!_{-1} & {}^{2}\!/\!_{-1} & {}^{0}\!/\!_{-1} \\ {}^{0}\!/\!_{2} & {}^{0}\!/\!_{2} & {}^{2}\!/\!_{2} & {}^{0}\!/\!_{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_2 + 2r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0+2(0) & 1+2(0) & -2+2(1) & 0+2(0) \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 - 2r_2 - 3r_3} \left[\begin{array}{ccc|c} 1-2(0)-3(0) & 2-2(1)-3(0) & 3-2(0)-3(1) & 0-2(0)-3(0) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases}x=0\\y=0\\z=0\end{cases}$$

$$\vec{v}=\vec{0}$$

The basis of $\text{Null}(A)$ is $\boxed{\{\,\vec{0}\,\}}$.

$$\boxed{\dim\big(\text{Null}(A)\big)=0}$$

28.e Given matrix $A=\begin{bmatrix}1&5&3\\2&5&1\end{bmatrix}$, find a basis for the matrix's null space and the null space's dimension.

$$A\cdot\vec{v}=\vec{0}$$

$$\begin{bmatrix}1&5&3\\2&5&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&5&3&0\\2&5&1&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&5&3&0\\2&5&1&0\end{array}\right]\xrightarrow{r_2-2r_1}\left[\begin{array}{ccc|c}1&5&3&0\\2-2(1)&5-2(5)&1-2(3)&0-2(0)\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&5&3&0\\2-2&5-10&1-6&0-0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&5&3&0\\0&-5&-5&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&5&3&0\\0&-5&-5&0\end{array}\right]\xrightarrow{r_1+r_2}\left[\begin{array}{ccc|c}1+0&5+(-5)&3+(-5)&0+0\\0&-5&-5&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&0&-2&0\\0&-5&-5&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&0&-2&0\\0&-5&-5&0\end{array}\right]\xrightarrow{r_2/(-5)}\left[\begin{array}{ccc|c}1&0&-2&0\\{}^{0}/_{-5}&{}^{-5}/_{-5}&{}^{-5}/_{-5}&{}^{0}/_{-5}\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&0&-2&0\\0&1&1&0\end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x - 2z = 0 \\ y + z = 0 \end{cases}$$

x and y are pivots; z is the only free variable.

Assign an arbitrary value to z: $z = t$

$x - 2(t) = 0$	$y + (t) = 0$
$x - 2t = 0$	$y + t = 0$
$x = 2t$	$y = -t$

$$\vec{v} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathrm{Null}(A) = \mathrm{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$

The basis of $\mathrm{Null}(A)$ contains only one vector, $\boxed{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}$.

$$\boxed{\dim\big(\mathrm{Null}(A)\big) = 1}$$

Coordinate of a Vector and Matrix

Problem 29

29.a Find the coordinates of $\vec{u} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ with respect to the basis $B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

By inspection, $\boxed{[\vec{u}]_B = (2,3,-1)}$.

29.b Find the coordinates of $\vec{v} = \hat{\imath} + \hat{\jmath} - \hat{k}$ with respect to the basis $B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

By inspection, $\boxed{[\vec{v}]_B = (1,1,-1)}$.

29.c Find the coordinates of $\vec{w} = 5\hat{\imath} - \hat{k}$ with respect to the basis $B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

By inspection, $\boxed{[\vec{v}]_B = (5,0,-1)}$.

Problem 30

30.a Find the coordinates of matrix $A = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$ with respect to $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

By inspection, $\boxed{[A]_B = (2,2,4,3)}$.

30.b Find the coordinates of matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ with respect to $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

By inspection, $\boxed{[A]_B = (1,2,1,0)}$.

30.c Find the coordinates of matrix $A = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$ with respect to $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

By inspection, $\boxed{[A]_B = (0,4,2,1)}$.

30.d Find the coordinates of matrix $A = \begin{bmatrix} 3 & -7 \\ 2 & 4 \end{bmatrix}$ with respect to $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

By inspection, $\boxed{[A]_B = (3,-7,2,4)}$.

Problem 31

31.a Given polynomial $p(x) = 5 - 4x + 7x^2 + 10x^3$, find its coordinates with respect to base $B = \{1, x, x^2, x^3\}$

By inspection, $\boxed{[p(x)]_B = (5,-4,7,10)}$.

31.b Given polynomial $p(x) = -x + 3x^2$, find its coordinates with respect to base $B = \{1, x, x^2, x^3\}$

By inspection, $\boxed{[p(x)]_B = (0,-1,3,0)}$.

31.c Given polynomial $p(x) = -x + 3x^2$, find its coordinates with respect to base $B = \{1, x, x^2\}$

By inspection, $\boxed{[p(x)]_B = (0,-1,3)}$.

31.d Given polynomial $p(x) = 2 - x + 7x^2$, find its coordinates with respect to base $B = \{1, x, x^2\}$

By inspection, $\boxed{[p(x)]_B = (2,-1,7)}$.

Problem 32

32.a Calculate the coordinates of vector $\vec{u} = (2,-3)$ with respect to basis $B = \{(1,1),(3,4)\}$.

$$\vec{u} = c_1(1,1) + c_2(3,4)$$

$$(2,-3) = c_1(1,1) + c_2(3,4)$$

$$\begin{cases} c_1 + 3c_2 = 2 \\ c_1 + 4c_2 = -3 \end{cases}$$

$$\begin{cases} c_1 + 3c_2 = 2 \\ c_1 + 4c_2 = -3 \end{cases} \xrightarrow{E_2 - E_1} \begin{array}{r} c_1 + 4c_2 = -3 \\ -c_1 - 3c_2 = -2 \\ \hline c_2 = -5 \end{array}$$

$$\begin{cases} c_1 + 3c_2 = 2 \\ c_2 = -5 \end{cases}$$

$$\begin{cases} c_1 + 3c_2 = 2 \\ c_2 = -5 \end{cases} \xrightarrow{E_1 - 3E_1} \begin{array}{r} c_1 + 3c_2 = 2 \\ -3c_2 = -3(-5) \\ \hline c_1 = 17 \end{array}$$

$$\begin{cases} c_1 = 17 \\ c_2 = -5 \end{cases}$$

$$\boxed{[\vec{u}]_B = (17,-5)}$$

32.b Calculate the coordinates of vector $\vec{u} = (8,7)$ with respect to basis $B = \{(1,2),(2,1)\}$.

$$\vec{u} = c_1(1,2) + c_2(2,1)$$

$$(8,7) = c_1(1,2) + c_2(2,1)$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ 2c_1 + c_2 = 7 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ 2c_1 + c_2 = 7 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{r} 2c_1 + c_2 = 7 \\ -2c_1 - 2(2c_2) = -2(8) \\ \hline -3c_2 = -9 \end{array}$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ -3c_2 = -9 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ -3c_2 = -9 \end{cases} \xrightarrow{E_2/-3} \begin{cases} c_1 + 2c_2 = 8 \\ {}^{-3c_2}/_{-3} = {}^{-9}/_{-3} \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ c_2 = 3 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = 8 \\ c_2 = 3 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{r} c_1 + 2c_2 = 8 \\ -2c_2 = -2(3) \\ \hline c_1 = 2 \end{array}$$

$$\begin{cases} c_1 = -9 \\ c_2 = 3 \end{cases}$$

$$\boxed{[\vec{u}]_B = (2,3)}$$

32.c Calculate the coordinates of vector $\vec{u} = (-3,1)$ with respect to basis $B = \{(1,3),(2,1)\}$.

$\vec{u} = c_1(1,3) + c_2(2,1)$

$(-3,1) = c_1(1,3) + c_2(2,1)$

$$\begin{cases} c_1 + 2c_2 = -3 \\ 3c_1 + c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = -3 \\ 3c_1 + c_2 = 1 \end{cases} \xrightarrow{E_2 - 3E_1} \begin{array}{c} 3c_1 + c_2 = 1 \\ -3c_1 - 3(2c_2) = -3(-3) \\ \hline -5c_2 = 10 \end{array}$$

$$\begin{cases} c_1 + 2c_2 = -3 \\ -5c_2 = 10 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = -3 \\ -5c_2 = 10 \end{cases} \xrightarrow{E_2/-5} \begin{cases} c_1 + 2c_2 = -3 \\ {}^{-5c_2}/_{-5} = {}^{10}/_{-5} \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = -3 \\ c_2 = -2 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = -3 \\ c_2 = -2 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{c} c_1 + 2c_2 = -3 \\ -2c_2 = -2(-2) \\ \hline c_1 = 1 \end{array}$$

$$\begin{cases} c_1 = 1 \\ c_2 = -2 \end{cases}$$

$$\boxed{[\vec{u}]_B = (1,-2)}$$

32.d Calculate the coordinates of vector $\vec{u} = (1,2)$ with respect to basis $B = \{(1,1),(3,4)\}$.

$\vec{u} = c_1(1,1) + c_2(3,4)$

$(1,2) = c_1(1,1) + c_2(3,4)$

$$\begin{cases} c_1 + 3c_2 = 1 \\ c_1 + 4c_2 = 2 \end{cases}$$

$$\begin{cases} c_1 + 3c_2 = 1 \\ c_1 + 4c_2 = 2 \end{cases} \xrightarrow{E_2 - E_1} \begin{array}{c} c_1 + 4c_2 = 2 \\ -c_1 - 3c_2 = -1 \\ \hline c_2 = 1 \end{array}$$

$$\begin{cases} c_1 + 3c_2 = 1 \\ c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 + 3c_2 = 1 \\ c_2 = 1 \end{cases} \xrightarrow{E_1 - 3E_2} \begin{array}{c} c_1 + 3c_2 = 1 \\ -3c_2 = -3(1) \\ \hline c_1 = -2 \end{array}$$

$$\begin{cases} c_1 = -2 \\ c_2 = 1 \end{cases}$$

$$\boxed{[\vec{u}]_B = (-2,1)}$$

Change of Basis and Transition Matrix

Problem 33

Given the bases $S = \{\hat{\imath}, \hat{\jmath}\}$ and $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,2), (2,5)\}$...

33.a Find the transition matrix from S to B, $M_{B \leftarrow S}$.

$$\hat{\imath} = a\vec{u}_1 + b\vec{u}_2$$

$$(1,0) = a(1,2) + b(2,5)$$

$$\begin{cases} a + 2b = 1 \\ 2a + 5b = 0 \end{cases}$$

$$\begin{cases} a + 2b = 1 \\ 2a + 5b = 0 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{c} 2a + 5b = 0 \\ \underline{-2a - 2(2b) = -2(1)} \\ b = -2 \end{array}$$

$$\begin{cases} a + 2b = 1 \\ b = -2 \end{cases}$$

$$\begin{cases} a + 2b = 1 \\ b = -2 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{c} a + 2b = 1 \\ \underline{-2b = -2(-2)} \\ a = 5 \end{array}$$

$$\begin{cases} a = 5 \\ b = -2 \end{cases}$$

$$[\hat{\imath}]_B = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\hat{\jmath} = c\vec{u}_1 + d\vec{u}_2$$

$$(0,1) = c(1,2) + d(2,5)$$

$$\begin{cases} c + 2d = 0 \\ 2c + 5d = 1 \end{cases}$$

$$\begin{cases} c + 2d = 0 \\ 2c + 5d = 1 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{c} 2c + 5d = 1 \\ \underline{-2c - 2(2d) = -2(0)} \\ d = 1 \end{array}$$

$$\begin{cases} c + 2d = 0 \\ d = 1 \end{cases}$$

$$\begin{cases} c + 2d = 0 \\ d = 1 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{c} c + 2d = 0 \\ \underline{-2d = -2(1)} \\ c = -2 \end{array}$$

$$\begin{cases} c = -2 \\ d = 1 \end{cases}$$

$$[\hat{\jmath}]_B = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$M_{B\leftarrow S} = [[\hat{\imath}]_B \quad [\hat{\jmath}]_B]$

$M_{B\leftarrow S} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\boxed{M_{B\leftarrow S} = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}}$

33.b Find the coordinate of $\vec{v} = (1,2) = \hat{\imath} + 2\hat{\jmath}$ in B, $[\vec{v}]_B$.

$[\vec{v}]_B = M_{B\leftarrow S} \cdot \vec{v}$

$[\vec{v}]_B = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} (5)(1) + (-2)(2) \\ (2)(1) + (1)(2) \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} 5 + (-4) \\ 2 + 2 \end{bmatrix}$

$\boxed{[\vec{v}]_B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$

33.c Find the transition matrix from B to S, a.k.a. $M_{S\leftarrow B}$.

$\vec{u}_1 = a\hat{\imath} + b\hat{\jmath}$

$(1,2) = a(1,0) + b(0,1)$

By inspection, $\begin{cases} a = 1 \\ b = 2 \end{cases}$.

$[\vec{u}_1] = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{u}_1$

$\vec{u}_2 = c\hat{\imath} + d\hat{\jmath}$

$(2,5) = c(1,0) + d(0,1)$

By inspection, $\begin{cases} c = 2 \\ d = 5 \end{cases}$.

$[\vec{u}_2]_S = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \vec{u}_2$

$M_{S\leftarrow B} = [[\vec{u}_1]_S \quad [\vec{u}_2]_S]$

$\boxed{M_{S\leftarrow B} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}}$

Problem 34

Given the bases $S = \{\hat{\imath}, \hat{\jmath}\}$ and $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3),(1,4)\}$...

34.a Find the transition matrix from S to B, $M_{B\leftarrow S}$.

$$\hat{\imath} = a\vec{u}_1 + b\vec{u}_2$$

$$(1,0) = a(1,3) + b(1,4)$$

$$\begin{cases} a + b = 1 \\ 3a + 4b = 0 \end{cases}$$

$$\begin{cases} a + b = 1 \\ 3a + 4b = 0 \end{cases} \xrightarrow{E_2 - 3E_1} \begin{array}{r} 3a + 4b = 0 \\ -3a - 3b = -3(1) \\ \hline b = -3 \end{array}$$

$$\begin{cases} a + b = 1 \\ b = -3 \end{cases}$$

$$\begin{cases} a + b = 1 \\ b = -3 \end{cases} \xrightarrow{E_1 - E_2} \begin{array}{r} a + b = 1 \\ -b = -(-3) \\ \hline a = 4 \end{array}$$

$$\begin{cases} a = 4 \\ b = -3 \end{cases}$$

$$[\hat{\imath}]_B = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\hat{\jmath} = c\vec{u}_1 + d\vec{u}_2$$

$$(0,1) = c(1,3) + d(1,4)$$

$$\begin{cases} c + d = 0 \\ 3c + 4d = 1 \end{cases}$$

$$\begin{cases} c + d = 0 \\ 3c + 4d = 1 \end{cases} \xrightarrow{E_2 - 3E_1} \begin{array}{r} 3c + 4d = 1 \\ -3c - 3d = -3(0) \\ \hline d = 1 \end{array}$$

$$\begin{cases} c + d = 0 \\ d = 1 \end{cases}$$

$$\begin{cases} c + d = 0 \\ d = 1 \end{cases} \xrightarrow{E_1 - E_2} \begin{array}{r} c + d = 0 \\ -d = -(1) \\ \hline c = -1 \end{array}$$

$$\begin{cases} c = -1 \\ d = 1 \end{cases}$$

$$[\hat{\jmath}]_B = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$M_{B\leftarrow S} = [[\hat{\imath}]_B \quad [\hat{\jmath}]_B]$

$M_{B\leftarrow S} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\boxed{M_{B\leftarrow S} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}}$

34.b Find the coordinate of $\vec{v} = (1,2) = \hat{\imath} + 2\hat{\jmath}$ in B, a.k.a. $[\vec{v}]_B$.

$[\vec{v}]_B = M_{B\leftarrow S} \cdot \vec{v}$

$[\vec{v}]_B = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} (4)(1) + (-1)(2) \\ (-3)(1) + (1)(2) \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} 4 + (-2) \\ -3 + 2 \end{bmatrix}$

$\boxed{[\vec{v}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

34.c Find the transition matrix from B to S, a.k.a. $M_{S\leftarrow B}$.

$\vec{u}_1 = a\hat{\imath} + b\hat{\jmath}$

$(1,3) = a(1,0) + b(0,1)$

By inspection, $\begin{cases} a = 1 \\ b = 3 \end{cases}$.

$[\vec{u}_1] = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{u}_1$

$\vec{u}_2 = c\hat{\imath} + d\hat{\jmath}$

$(1,4) = c(1,0) + d(0,1)$

By inspection, $\begin{cases} c = 1 \\ d = 4 \end{cases}$.

$[\vec{u}_2]_S = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \vec{u}_2$

$M_{S\leftarrow B} = [[\vec{u}_1]_S \quad [\vec{u}_2]_S]$

$\boxed{M_{S\leftarrow B} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}}$

Problem 35

Given the bases $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3),(1,4)\}$ and $B' = \{\vec{v}_1, \vec{v}_2\} = \{(1,2),(2,5)\}$…

35.a The transition matrix from B to B', $M_{B' \leftarrow B}$.

$$\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2$$

$$(1,3) = a(1,2) + b(2,5)$$

$$\begin{cases} a + 2b = 1 \\ 2a + 5b = 3 \end{cases}$$

$$\begin{cases} a + 2b = 1 \\ 2a + 5b = 3 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{c} 2a + 5b = 3 \\ \underline{-2a - 2(2b) = -2(1)} \\ b = 1 \end{array}$$

$$\begin{cases} a + 2b = 1 \\ b = 1 \end{cases}$$

$$\begin{cases} a + 2b = 1 \\ b = 1 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{c} a + 2b = 1 \\ \underline{-2b = -2(1)} \\ a = -1 \end{array}$$

$$\begin{cases} a = -1 \\ b = 1 \end{cases}$$

$$[\vec{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = c\vec{v}_1 + d\vec{v}_2$$

$$(1,4) = c(1,2) + d(2,5)$$

$$\begin{cases} c + 2d = 1 \\ 2c + 5d = 4 \end{cases}$$

$$\begin{cases} c + 2d = 1 \\ 2c + 5d = 4 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{c} 2c + 5d = 4 \\ \underline{-2c - 2(2d) = -2(1)} \\ d = 2 \end{array}$$

$$\begin{cases} c + 2d = 1 \\ d = 2 \end{cases}$$

$$\begin{cases} c + 2d = 1 \\ d = 2 \end{cases} \xrightarrow{E_1 - 2E_2} \begin{array}{c} c + 2d = 1 \\ \underline{-2d = -2(2)} \\ c = -3 \end{array}$$

$$\begin{cases} c = -3 \\ d = 2 \end{cases}$$

$$[\vec{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$M_{B'\leftarrow B} = [[\vec{u}_1]_{B'} \quad [\vec{u}_2]_{B'}]$

$M_{B'\leftarrow B} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\boxed{M_{B'\leftarrow B} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}}$

Alternate:

$[M_{B'}|M_B]$

$[\vec{v}_1 \quad \vec{v}_2|\vec{u}_1 \quad \vec{u}_2]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 4 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 4 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2-2(1) & 5-2(2) & 3-2(1) & 4-2(1) \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cc|cc} 1-2(0) & 2-2(1) & 1-2(1) & 1-2(2) \\ 0 & 1 & 1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$\boxed{M_{B'\leftarrow B} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}}$

35.b Find the coordinate of $[\vec{v}]_B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ in B'.

$[\vec{v}]_{B'} = M_{B'\leftarrow B} \cdot [\vec{v}]_B$

$[\vec{v}]_{B'} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$[\vec{v}]_{B'} = \begin{bmatrix} (-1)(2) + (-3)(5) \\ (1)(2) + (2)(5) \end{bmatrix}$

$[\vec{v}]_{B'} = \begin{bmatrix} -2 + (-15) \\ 2 + 10 \end{bmatrix}$

$\boxed{[\vec{v}]_{B'} = \begin{bmatrix} -17 \\ 12 \end{bmatrix}}$

Problem 36

Given the bases $B = \{\vec{u}_1, \vec{u}_2\} = \{(1,3), (1,4)\}$ and $B' = \{\vec{v}_1, \vec{v}_2\} = \{(1,2), (1,1)\}$...

36.a Find the transition matrix from B to B', $M_{B'\leftarrow B}$.

$\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2$

$(1,3) = a(1,2) + b(1,1)$

$\begin{cases} a + b = 1 \\ 2a + b = 3 \end{cases}$

$\begin{cases} a + b = 1 \\ 2a + b = 3 \end{cases} \xrightarrow{E_2-2E_1} \begin{array}{r} 2a + b = 3 \\ -2a - 2b = -2(1) \\ \hline -b = 1 \end{array}$

$\begin{cases} a + b = 1 \\ -b = 1 \end{cases}$

$\begin{cases} a + b = 1 \\ -b = 1 \end{cases} \xrightarrow{E_2/-1} \begin{cases} a + b = 1 \\ {-b}/{-1} = {1}/{-1} \end{cases}$

$\begin{cases} a + b = 1 \\ b = -1 \end{cases}$

$\begin{cases} a + b = 1 \\ b = -1 \end{cases} \xrightarrow{E_1-E_2} \begin{array}{r} a + b = 1 \\ -b = -(-1) \\ \hline a = 2 \end{array}$

$\begin{cases} a = 2 \\ b = -1 \end{cases}$

$$[\vec{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\vec{u}_2 = c\vec{v}_1 + d\vec{v}_2$

$(1,4) = c(1,2) + d(1,1)$

$$\begin{cases} c + d = 1 \\ 2c + d = 4 \end{cases}$$

$$\begin{cases} c + d = 1 \\ 2c + d = 4 \end{cases} \xrightarrow{E_2 - 2E_1} \begin{array}{c} 2c + d = 4 \\ \underline{-2c - 2d = -2(1)} \\ -d = 2 \end{array}$$

$$\begin{cases} c + d = 1 \\ -d = 2 \end{cases} \xrightarrow{E_2/-1} \begin{cases} c + d = 1 \\ {-d}/_{-1} = {2}/_{-1} \end{cases}$$

$$\begin{cases} c + d = 1 \\ d = -2 \end{cases}$$

$$\begin{cases} c + d = 1 \\ d = -2 \end{cases} \xrightarrow{E_1 - E_2} \begin{array}{c} c + d = 1 \\ \underline{-d = -(-2)} \\ c = 3 \end{array}$$

$$\begin{cases} c = 3 \\ d = -2 \end{cases}$$

$$[\vec{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$M_{B' \leftarrow B} = [[\vec{u}_1]_{B'} \quad [\vec{u}_2]_{B'}]$$

$$M_{B' \leftarrow B} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\boxed{M_{B' \leftarrow B} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}}$$

Alternate:

$[M_{B'} | M_B]$

$[\vec{v}_1 \quad \vec{v}_2 | \vec{u}_1 \quad \vec{u}_2]$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 - 2(1) & 1 - 2(1) & 3 - 2(1) & 4 - 2(1) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow{r_2/-1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ {0}/_{-1} & {-1}/_{-1} & {1}/_{-1} & {2}/_{-1} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|cc} 1 - 0 & 1 - 1 & 1 - (-1) & 1 - (-2) \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\boxed{M_{B' \leftarrow B} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}}$$

36.b Find the coordinate of $[\vec{v}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in B'.

$$[[\vec{v}]_B]_{B'} = M_{B' \leftarrow B} \cdot [\vec{v}]_B$$

$$[[\vec{v}]_B]_{B'} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[[\vec{v}]_B]_{B'} = \begin{bmatrix} (2)(3) + (3)(1) \\ (-1)(3) + (-2)(1) \end{bmatrix}$$

$$[[\vec{v}]_B]_{B'} = \begin{bmatrix} 6 + 3 \\ -3 + (-2) \end{bmatrix}$$

$$\boxed{[[\vec{v}]_B]_{B'} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}}$$

END