Chapter 2 (IL) : Matrix Algebra, Basis and Dimension

1) Definition:

A matrix is a rectangular array of numbers

We write matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Example: $\mathrm{B} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ or $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ Notice the use of parenthesis ( ) and squared bracket [ ] to represent a matrix.

Wrong matrix representation: $A = \begin{Bmatrix} 1 & 0 \\ 5 & 3 \end{Bmatrix}$ **No curly bracket ;**

$A = \begin{vmatrix} 1 & 5 \\ 8 & 9 \end{vmatrix}$ **no absolute value sign** $A = \begin{matrix} 1 & 0 \\ 4 & 2 \end{matrix}$ **this is garbage, means nothing**

PYTHONIC : *a matrix in Python is create using:*

numpy.matrix( ) or numpy.mat( ) with numpy library
scipy.matrix() or scipy.mat() with scipy library.
sympy.Matrix() with sympy library with an uppercase M
syntax : matrix([[row1] , [row2] , [row3] ,…., [rowN]]dtype="data_type")
data_type= float,int,complex,…..

```python
#defining a matrix
import numpy  as np
import sympy  as sy
import scipy  as sp
#M=np.matrix([[1,2,3],[2,0,1] ,[1,1,1] ])
#creating a matrix using sympy
M= sy.Matrix([[1,2,3,2],[2,0,1,7] ,[1,1,1,3] ], dtype='float')
sy.pprint(M)


```

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Dimension of a matrix

The numbers in the array are called entries or elements of the matrix

If a matrix A has N rows and M columns , we said that the size or dimension or order of A is NxM (read N by M).

Example : $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ is a 3X3 matrix (3 rows and 3 columns)

$C = \begin{pmatrix} 1 & 0 & 5 & 4 \\ 2 & -1 & 5 & 2 \end{pmatrix}$ is a 2X4 matrix (2 row and 4 columns)

If N=M=n then we have a square matrix of order n , or a nxn(n by n) matrix.

So $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ is a square matrix of order 2

TODO➔ Go to Activity and solve question 1

PYTHONIC: In Python use **Matrix.shape** to get the matrix dimension.

```
#defining a matrix
import numpy  as np
import sympy  as sy
import scipy  as sp
M=sy.Matrix([ [1,2,3,9],[2,0,1,7] ], dtype='float')
#creating a matrix using sympy
sy.pprint(M)
print("Matrix dimension is: {0}".format( M.shape))

```

```
⎡1  2  3  9⎤
⎢          ⎥
⎣2  0  1  7⎦
Matrix dimension is: (2, 4)
```

Accessing values, rows and columns in a matrix

In matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, a_{11} is the value at entry point (1,1)

a_{23} is the value at entry point (2,3)

a_{11} , a_{22} , a_{33} are the matrix main diagonal elements.

$[a_{11}\ a_{12}\ a_{13}]$ is the first row; $[a_{21}\ a_{22}\ a_{23}]$ is the 2nd row ; $[a_{31}\ a_{32}\ a_{33}]$ is the 3rd row.

$[a_{11}\ a_{21}\ a_{31}]$ is the 1st column; $[a_{12}\ a_{22}\ a_{32}]$ is the 2nd ; $[a_{13}\ a_{23}\ a_{33}]$ is the 3rd column

In general a_{ij} is the value at entry point (i,j) where $1 \le i \le N$ and $1 \le j \le M$

Example:

$$A=\begin{pmatrix}1 & 2 & 5\\ 2 & 5 & 10\\ 6 & 1 & 0\end{pmatrix}$$ diagonal elements in A are 1 , 5 and 0

$a_{11}=1, a_{22}=5,\ a_{33}=0\ , a_{31}=6$, row1=[1 2 5] , row2=[2 5 10] , row3=[6 1 0]

row1=[1 2 5] , row2=[2 5 10] , row3=[6 1 0]

TODO➔ Go to Activity and solve question 2

PYTHONIC: Python matrix format is matrix[: , :] this means:
matrix[first_: last_row , first_column:last_column].
To get entry value at i,j : do matrix[i,j]
To get i^{th} row do : matrix[i , :]
To get j^{th} column do: matrix[: , j]
To get the diagonal :use numpy.diag(matrix), scipy.diag(matrix), sympy.diag(matrix)

```python
#getting a matrix row or column
import sympy  as sy
M=sy.Matrix([[1,2,3],[2,2,1] ,[1,2,3] ], dtype='float')
sy.pprint(M)
#getting value at entry 2,2
value=M[2,2]
sy.pprint("Value at entry 22 is {0}:".format(value) )
# getting the second row
row2=M[1,:]
print("second row is :")
sy.pprint(row2)
col3=M[:,2]
print("third column is:")
sy.pprint(col3)
#printing matrix diagonal using sympy.diag()
d=sp.diag(M)
print("Matrix diagonal is")
sy.pprint(d)

```

```
⎡1  2  3⎤
⎢       ⎥
⎢2  2  1⎥
⎢       ⎥
⎣1  2  3⎦
Value at entry 22 is 3:
second row is :
[2  2  1]
third column is:
⎡3⎤
⎢ ⎥
⎢1⎥
⎢ ⎥
⎣3⎦
Matrix diagonal is
[1 2 3]
```

2) Transpose of a matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we define the transpose of by $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

That is the row $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$ in A becomes the column $\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}$ in A^t

$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}$ in A becomes the column $\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$ in A^t

$\begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$ in A becomes the column $\begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}$ in A^t

Example : if $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ then $B^t = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 5 & 1 \\ 3 & 1 & 0 \end{pmatrix}$

If $C = \begin{pmatrix} 1 & 0 & 5 & 4 \\ 2 & -1 & 5 & 2 \end{pmatrix}$ then $C^t = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 5 & 5 \\ 4 & 2 \end{pmatrix}$

TODO➔ Go to Activity and solve questions 3.1 and 3.2

PYTHONIC: use the T for transpose as in M.T

```
#defining a matrix  and its order
import sympy  as sy
M=sy.Matrix([[1,2,3],[2,0,1] ,[1,1,1] ])
sy.pprint(M)
print("\n The transpose of Matrix M is: \n",end="")
# get the transpose of M using T.
transpose_M=M.T
sy.pprint(transpose_M)
```

```
⎡1  2  3⎤
⎢       ⎥
⎢2  0  1⎥
⎢       ⎥
⎣1  1  1⎦

 The transpose of Matrix M is:
⎡1  2  1⎤
⎢       ⎥
⎢2  0  1⎥
⎢       ⎥
⎣3  1  1⎦
```

3) <u>Columns and rows vectors</u>

The matrix coordinate a vector $\vec{a}=(a_1,a_2,a_3)$ is a 3x1 matrix , $\vec{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ called a column vector or a 1x3 matrix, $\vec{a}=(a_1\ a_2\ a_3)$ called a row vector. Note that $\vec{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}=(a_1\ a_2\ a_3)^t$

Example : write the matrix coordinate of $\vec{a}=(1,2,5)$ as a column vector and as a row vector

Ans: $\vec{a}=\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ column vector . $\vec{a}=(1\ 2\ 5)$ row vector **(no comas here !!)**

4) <u>Trace of a matrix</u>

Let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then the trace of A is Trace(A)= $a_{11}+a_{22}+a_{33}$.

The trace of matrix is the sum of the values on the main diagonal in a **square matrix**.

For a $n\times n$ matrix A, Trace(A)= $a_{11}+a_{22}+a_{33}+\cdots+a_{n-1,n-1}+a_{nn}=\sum_{i=j=1}^{n} a_{ij}$.

Example: find the trace in $A=\begin{pmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{pmatrix}$, $B=\begin{pmatrix} 2 & -1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $C=\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

Trace(A)= $a_{11}+a_{22}+a_{33}$ = 2 + 5 + 7 = 14 Trace(B)= 2 + 5 + 0 = 7 Trace(A)= 1 + 5 = 6

TODO➔ Go to Activity and solve question 4

5) <u>Property of matrix transpose</u>

a) $(A^t)^t=A$

b) $(A+B)^t=A^t+B^t$.

c) $(A\bullet B)^t=B^t\bullet A^t$

6) Special matrices

a) identity matrix I= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ similar to 1 in 1X4=4 , 1X10=10

b) lower and upper triangular matrix; A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ is upper B= $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is lower

Example: $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$ upper $\begin{pmatrix} 3 & 0 & 0 \\ 7 & 5 & 0 \\ 1 & 15 & 3 \end{pmatrix}$ lower

c) diagonal matrix A= $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$. Example : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

d) symmetric matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$. A is symmetric if and only if $A^t = A$

the numbers across the diagonal are the same.

Example: $\begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 6 \end{pmatrix}$ and $\begin{pmatrix} 0 & 15 & 20 \\ 15 & 7 & 10 \\ 20 & 10 & 6 \end{pmatrix}$ are symmetric matrix.

TODO➔ Go to Activity and solve question 5

e) skew symmetric matrix A= $\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$.

A is skew symmetric if and only if $A^t = -A$.

Example: $\begin{pmatrix} 0 & -4 & 7 \\ 4 & 0 & -3 \\ -7 & 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$ are skew symmetric matrix.

One can verify that $\begin{pmatrix} 0 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$

if $\vec{a}=(x,y,z)$ and $\vec{b}=(b_1,b_2,b_3)$ then $\vec{a}\times\vec{b}=skew(\vec{a})\bullet\vec{b}=\begin{pmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$

this is the matrix expression of the vector cross product

Examples: if $\vec{a}=(1,2,5)$ then $skew(\vec{a})=\begin{pmatrix}0 & -5 & 2\\ 5 & 0 & -1\\ -2 & 1 & 0\end{pmatrix}$

if $\vec{a}=(1,0,-3)$ then $skew(\vec{a})=\begin{pmatrix}0 & -(-3) & 0\\ -3 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 3 & 0\\ -3 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$

if $\vec{a}=(1,0,2)$ and $\vec{b}=(1,1,-1)$ then $\vec{a}\times\vec{b}=skew(\vec{a})\bullet\vec{b}=\begin{pmatrix}0 & -2 & 0\\ 2 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}=\begin{pmatrix}-2\\ 3\\ 1\end{pmatrix}$

how it works ? $\begin{bmatrix}0 & -2 & 0\end{bmatrix}\bullet\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}=(0\times1)+(-2\times1)+(0\times-1)=-2$

$$\begin{bmatrix}2 & 0 & -1\end{bmatrix}\bullet\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}=(2\times1)+(0\times1)+(-1\times-1)=2+1=3$$

$$\begin{bmatrix}0 & 1 & 0\end{bmatrix}\bullet\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}=(0\times1)+(1\times1)+(0\times-1)=1$$

And Also $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 2\\ 1 & 1 & -1\\ + & - & +\end{vmatrix}=\begin{vmatrix}0 & 2\\ 1 & -1\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}\hat{j}+\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}\hat{k}=(-2,3,1)$

7) Matrices Addition

Let A=$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ and B=$\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$ be 2 square matrices of order 3

Then C=A+B=$\begin{pmatrix}a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13}\\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23}\\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33}\end{pmatrix}$

Examples: Given $A=\begin{pmatrix}2 & -1 & 3\\ -1 & 5 & 1\\ 3 & 1 & 7\end{pmatrix}$, $B=\begin{pmatrix}2 & -1 & 3\\ 1 & 5 & 1\\ 2 & 1 & 0\end{pmatrix}$

A+B= $\begin{pmatrix}2 & -1 & 3\\ -1 & 5 & 1\\ 3 & 1 & 7\end{pmatrix}+\begin{pmatrix}2 & -1 & 3\\ 1 & 5 & 1\\ 2 & 1 & 0\end{pmatrix}=\begin{pmatrix}4 & -2 & 6\\ 0 & 10 & 2\\ 5 & 2 & 7\end{pmatrix}$

2A-3B= $2\begin{pmatrix}2 & -1 & 3\\ -1 & 5 & 1\\ 3 & 1 & 7\end{pmatrix}-3\begin{pmatrix}2 & -1 & 3\\ 1 & 5 & 1\\ 2 & 1 & 0\end{pmatrix}=\begin{pmatrix}4 & -2 & 6\\ -2 & 10 & 2\\ 6 & 2 & 14\end{pmatrix}+\begin{pmatrix}-6 & 3 & -9\\ -3 & -15 & -3\\ -6 & -3 & 0\end{pmatrix}=\begin{pmatrix}-2 & 1 & -3\\ -5 & -5 & -1\\ 0 & -1 & 14\end{pmatrix}$

TODO➔ Go to Activity and solve question 6

PYTHONIC: **Matrix addition/subtraction** is done using the overloaded + and - operators

$$M+N=\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}+\begin{pmatrix}1 & 5\\ 2 & -1\end{pmatrix}=\begin{pmatrix}2 & 7\\ 2 & 2\end{pmatrix}$$

```python

import sympy  as sy
M=sy.Matrix([[1,2],[0,3] ],dtype="float")
N=sy.Matrix([[1,5],[2,-1] ],dtype="float")
sy.pprint(M)
sy.pprint(N)
S=M+N
D=M-N
print("M+N: \n\n",end="")
sy.pprint(S)
print("M-N= \n")
sy.pprint(D)
```

```
⎡1  2⎤
⎢    ⎥
⎣0  3⎦
⎡1  5 ⎤
⎢     ⎥
⎣2  -1⎦
M+N:

⎡2  7⎤
⎢    ⎥
⎣2  2⎦
M-N=

⎡0   -3⎤
⎢      ⎥
⎣-2  4 ⎦
```

8) Properties of matrix addition and scalar multiplication

Suppose A and B are 2 matrices of order 3 and k1 and k2 are real number (scalar) then

a) $A+B=B+A$, commutative property

b) $(A+B)+C=A+(B+C)=A+B+C$, associative property

c) $K_1\cdot(A+B)=K_1\cdot A+K_1\cdot B$, distributive property

d) $(K_1+K_2)\cdot A=K_1\cdot A+K_2\cdot A$

9) **Matrix multiplication**

a)Matrix to matrix multiplication

Let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and B= $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ be 2 square matrices of order 3 then

$$C= A \bullet B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Example 9.1: let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

We want C=AB= $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$

Each row of A will dot multiply every column in B.

Step1: with 1st , $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$, row of A against all column vectors of B (dot multiplication).

$c_{11} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = (2\times2)+(1\times1)+(3\times2) = 11$, $c_{12} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (2\times0)+(1\times1)+(3\times1) = 4$

$c_{13} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = (2\times3)+(1\times1)+(3\times0) = 7$

Step2: with 2nd row of A , $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, against all column vectors of B (dot multiplication).

$c_{21} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = (0\times2)+(0\times1)+(1\times2) = 2$, $c_{22} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (0\times0)+(0\times1)+(1\times1) = 1$

$$c_{23}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\bullet\begin{bmatrix}3\\1\\0\end{bmatrix}=(0\times 3)+(0\times 1)+(1\times 0)=0$$

Step3: with 3rd row of A , $\begin{bmatrix}3 & 1 & 0\end{bmatrix}$, against all column vectors of B (dot multiplication).

$$c_{31}=\begin{bmatrix}3 & 1 & 0\end{bmatrix}\bullet\begin{bmatrix}2\\1\\2\end{bmatrix}=(3\times 2)+(1\times 1)+(0\times 2)=7 \quad , \quad c_{32}=\begin{bmatrix}3 & 1 & 0\end{bmatrix}\bullet\begin{bmatrix}0\\1\\1\end{bmatrix}=(3\times 0)+(1\times 1)+(0\times 1)=1$$

$$c_{33}=\begin{bmatrix}3 & 1 & 0\end{bmatrix}\bullet\begin{bmatrix}3\\1\\0\end{bmatrix}=(3\times 3)+(1\times 1)+(0\times 0)=10$$

So finally $C=\begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix}=A\cdot B=\begin{pmatrix}2 & 1 & 3\\ 0 & 0 & 1\\ 3 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}2 & 0 & 3\\ 1 & 1 & 1\\ 2 & 1 & 0\end{pmatrix}=\begin{pmatrix}11 & 4 & 7\\ 2 & 1 & 0\\ 7 & 1 & 10\end{pmatrix}$

Example 9.2: Let $A=\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$

Calculate $A\bullet B$ and $B\bullet A$. is $A\bullet B = B\bullet A$? (commutative)

$$A\cdot B=\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}=\begin{pmatrix}2*1+1*1 & 2*2+1*3\\ 0*1+5*1 & 0*2+5*3\end{pmatrix}=\begin{pmatrix}3 & 7\\ 5 & 15\end{pmatrix}$$

$$B\cdot A=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}=\begin{pmatrix}1*2+2*0 & 1*1+2*5\\ 1*2+3*0 & 1*1+3*5\end{pmatrix}=\begin{pmatrix}2 & 11\\ 2 & 16\end{pmatrix}, \quad \text{So } A\cdot B\neq B\cdot A$$

TODO➔ Go to Activity and solve question 7

PYTHONIC : The * operator is used to multiply 2 matrices.

```
import sympy  as sy
M=sy.Matrix([[1,2],[0,3] ],dtype="float")
N=sy.Matrix([[1,5],[2,-1] ],dtype="float")
sy.pprint(M)
sy.pprint(N)
mult=M*N
print("M*N: \n\n")
sy.pprint(mult)
```

```
⎡1  2⎤
⎢    ⎥
⎣0  3⎦
⎡1  5 ⎤
⎢     ⎥
⎣2  -1⎦
M*N:


⎡5  3 ⎤
⎢     ⎥
⎣6  -3⎦
```

Theorem: The multiplication of matrices is not commutative, that is: $A \bullet B \neq B \bullet A$

Theorem: let A be a $N \times P$ matrix and B a $Q \times R$, then $A \cdot B = A_{N\times P} \bullet B_{Q\times R}$ is possible if and only if $P = Q$ that is, if the number P of columns from A = number Q of rows from matrix B , and $A_{N\times P} \bullet B_{Q\times R} = C_{N\times R}$

Example: let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

$A \cdot B = A_{2X3} \cdot B_{2X2} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ not possible .there are 3 columns in A for 2 rows in B

$B \cdot A = B_{2X2} \cdot A_{2X3} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 1 \end{pmatrix}$ possible, there are 2 columns in B for 2 rows in A .

b) Matrix multiplication Property

Suppose A ,B and C are 3 matrices of order 3 ,then

b.1) $A \bullet B \neq B \bullet A$ not commutative

b.2) $A \bullet (B \bullet C) = (A \bullet B) \bullet C = A \bullet B \bullet C$ associative property

b.3) $A \bullet (B + C) = A \bullet B + A \bullet C$ distributive property

b.4) $\vec{a}^t \vec{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2$ **represents the dot product (outer product $\vec{a} \bullet \vec{a}$).**

b.5) Vector outer product $\vec{a}\vec{a}^t = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix}$ if $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

b.6) $\left(\vec{a} \bullet \vec{b}\right) \cdot \vec{a} = \left(\vec{a}\vec{a}^t\right) \cdot \vec{b}$ where $\vec{a}\vec{a}^t = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix}$

b.7) if $\vec{v} = M \cdot \vec{a}$ then $\vec{v} \bullet \vec{b} = \vec{v}^t \cdot \vec{b}$ = $\left(M \cdot \vec{a}\right)^t \cdot \vec{b} = \vec{a}^t \cdot M^t \cdot \vec{b}$

Not $\vec{v} \bullet \vec{b} =$ $\left(M \cdot \vec{a}\right) \bullet \vec{b}$, and if $M^t = M$, that is M being a symmetric matrix, then $\vec{a}^t \cdot M^t \cdot \vec{b} = \vec{a}^t \cdot M \cdot \vec{b}$

```python
import sympy  as sy
a=sy.Matrix([[1],[0],[2] ], dtype='float')
#creating a matrix using sympy
sy.pprint(a.T*a)
print("\n")
sy.pprint(a*a.T)
```

[5]

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

c) Vector- Matrix multiplication

Let M= $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ and $\vec{v}$ a vector with components x,y and z.

Then $M \cdot \vec{v} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m_{11} \cdot x + m_{12} \cdot y + m_{13} \cdot z \\ m_{21} \cdot x + m_{22} \cdot y + m_{23} \cdot z \\ m_{31} \cdot x + m_{32} \cdot y + m_{33} \cdot z \end{pmatrix}$ standard right-vector-matrix multiplication .

$\vec{v} \cdot M = \begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{pmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} = [w_1 \ w_2 \ w_3]$ is the left-vector-matrix multiplication

where the original matrix is transposed before the calculation. ,

with $w_1 = m_{11} \bullet x + m_{12} \bullet y + m_{13} \bullet z$, $w_2 = m_{21} \bullet x + m_{22} \bullet y + m_{23} \bullet z$, $w_3 = m_{31} \bullet x + m_{32} \bullet y + m_{33} \bullet z$

Ex: Calculate $M \cdot \vec{v}$ and $\vec{v} \cdot M$ if $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 1 & 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$M \cdot \vec{v} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1*1+2*2+3*3 \\ 2*1+0*2+5*3 \\ 1*1+1*2+3*3 \end{pmatrix} = \begin{pmatrix} 14 \\ 17 \\ 12 \end{pmatrix}$$

$$\vec{v} \cdot M = \vec{v} \cdot M = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 5 & 3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = (14\ 17\ 12)$$

$w_1 = 1*1+2*2+3*3 = 14$, $w_2 = 2*1+0*2+5*3 = 17$, $w_3 = 1*1+1*2+3*3 = 12$

TODO➔ Go to Activity and solve question 8

PYTHONIC: Use the * operator for matrix-vector multiplication

```python

import sympy  as sy
M=sy.Matrix( [ [1,2,3],[2,0,5],[1,1,3] ],dtype="float")
# column vector v=(1,2,3)
v=sy.Matrix([ [1],[2],[3] ],dtype="float")
sy.pprint(M)
sy.pprint(v)
u=M*v
print("M*v: \n\n")
sy.pprint(u)
```

```
⎡1  2  3⎤
⎢       ⎥
⎢2  0  5⎥
⎢       ⎥
⎣1  1  3⎦
⎡1⎤
⎢ ⎥
⎢2⎥
⎢ ⎥
⎣3⎦
M*v:

⎡14⎤
⎢  ⎥
⎢17⎥
⎢  ⎥
⎣12⎦
```

10) <u>System of Linear equations and Augmented Matrix.</u>

Let us consider the following system of linear equation

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{cases}$$

Expressed in matrix form we have $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ where $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

is called the coefficient matrix , and the corresponding augmented matrix is

$$\left(\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

Example 10.1: Find the coefficient and augmented matrix of $\begin{cases} x + y - z = 5 \\ 2x + 3y + z = 9 \\ x - y + 2z = 0 \end{cases}$

In matrix form we have $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 0 \end{pmatrix}$ where the coefficient matrix is $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

and the augmented matrix is $\left(\begin{array}{ccc:c} 1 & 1 & -1 & 5 \\ 2 & 3 & 1 & 9 \\ 1 & -1 & 2 & 0 \end{array}\right)$ or $\begin{pmatrix} 1 & 1 & -1 & 5 \\ 2 & 3 & 1 & 9 \\ 1 & -1 & 2 & 0 \end{pmatrix}$

TODO➔ Go to Activity and solve question 9

11) <u>row reductions matrix</u>

<u>Row Echelon form</u>

A matrix M is called an echelon matrix or it is said to be in row echelon form if it satisfies the following conditions:

1. All zero-rows are at the bottom of the matrix.
2. The first leading non-zero number(pivot) of a non-zero row is always to the right of the leading pivot of the row above it.

Example 11.1 $A = \begin{pmatrix} \underline{2} & 4 & 1 \\ 0 & \underline{5} & 1 \\ 0 & 0 & \underline{4} \end{pmatrix}$, $B = \begin{pmatrix} \underline{2} & 4 & 1 \\ 0 & \underline{5} & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} \underline{2} & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 6 \end{pmatrix}$, $D = \begin{pmatrix} 0 & \underline{1} & 2 \\ \underline{2} & 3 & 4 \\ 0 & 0 & \underline{1} \end{pmatrix}$

$E = \begin{pmatrix} 0 & \underline{2} & 3 & 0 & 7 \\ 0 & 0 & \underline{4} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ .A,B,E are in row echelon form with the pivots underlined in red. C is not since it fails condition 1, D is not since it fails condition 2.

D can be fixed if we swap the row1 and row2 to satisfy condition 2, $D = \begin{pmatrix} \underline{2} & 3 & 4 \\ 0 & \underline{1} & 2 \\ 0 & 0 & \underline{1} \end{pmatrix}$

TODO➔ Go to Activity and solve question 10

Computing a row echelon form:

First of all , we need to understand the following notations used in the process:

$R_i \leftrightarrow R_j$ means swap row i and row j.

$R_j + kR_i \to R_j$, $R_j \to R_j + kR_i$ *or* $R_j \leftarrow R_j + kR$ means *replace* R_j by $R_j + kR_i$

A row echelon form can be obtained by row operation using Gaussian elimination as illustrated in the example below:

Example 11.2: Reduce $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{pmatrix}$ in echelon form .

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{pmatrix} \xrightarrow[3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 6 & 10 \end{pmatrix} \xrightarrow{2R_2+R_3 \to R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{pmatrix}$$

So $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{pmatrix}$

NOTE : Finding the row echelon form of a matrix is not unique!

TODO➔ Go to Activity and solve question 11.

PYTHONIC: a matrix is converted in row echelon form by using matrix.echelon_form().

```
import sympy  as sy
M=sy.Matrix([ [1,2,3],[2,1,4],[-3,0,1]], dtype='float')
#reduced echelon form of M using  matrix.echelon_form()
sy.pprint(M)
e=M.echelon_form()
print("\n")
sy.pprint(e)
```

```
⎡1   2  3⎤
⎢        ⎥
⎢2   1  4⎥
⎢        ⎥
⎣-3  0  1⎦


⎡1  2    3 ⎤
⎢          ⎥
⎢0  -3  -2 ⎥
⎢          ⎥
⎣0  0   -18⎦
```

Reduced Row Echelon Form (rref) or row canonical form

A matrix M is in reduced row echelon form (row canonical form) if it satisfies the two conditions below:

1. M is in row echelon form
2. Every leading numbers (pivots) is 1 and is also the only non-zero entry in the column.

Example 11.3: $A = \begin{pmatrix} \underline{2} & 4 & 1 \\ 0 & \underline{5} & 1 \\ 0 & 0 & \underline{4} \end{pmatrix}$ is not in rref. The pivots (red underlined) are not 1.

$B = \begin{pmatrix} \underline{1} & 4 & 0 \\ 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \end{pmatrix}$ not in rref, the pivot 1 in row2 is not the only non-zero entry in the 2nd column(4 is above, it should be a zero above)

$C = \begin{pmatrix} \underline{1} & 3 & 0 & 8 & 0 \\ 0 & 0 & \underline{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \end{pmatrix}$ is in rref, it is in echelon form, all pivots are 1

All the pivots are the only non-zero entry in the columns.

TODO➔ Go to Activity and solve question 12

Computing a reduced row echelon form:

A row echelon form can be obtained by row operations using Gaussian elimination In two stages:

Stage 1: puts 0's below each pivot working from the top row to the bottom row.

Stage 2: puts 0's above each pivot working from the bottom row to the top row.

Example 11.4: Reduce $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{pmatrix}$ in reduced row echelon form (rref)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{pmatrix} \begin{matrix} \xrightarrow{-2R_1+R_2 \to R_2} \\ \xrightarrow{3R_1+R_3 \to R_3} \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 6 & 10 \end{pmatrix} \xrightarrow{2R_2+R_3 \to R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{pmatrix}$$

Now we reduce the leading number 6 in row3=[0 0 6] to 1 by dividing by 6 to get $A=\begin{pmatrix}1 & 2 & 3\\0 & -3 & -2\\0 & 0 & 1\end{pmatrix}$,from there we perform another Gaussian elimination working from the bottom row up (stage 2)

$$A=\begin{pmatrix}1 & 2 & 3\\0 & -3 & -2\\0 & 0 & 1\end{pmatrix}\xrightarrow[2R_3+R_2\to R_2]{-3R_3+R_1\to R_1}\begin{pmatrix}1 & 2 & 0\\0 & -3 & 0\\0 & 0 & 1\end{pmatrix}\xrightarrow{\frac{-1}{3}R_2\to R_2}\begin{pmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\xrightarrow{-2R_2+R_1\to R_1}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$ done!

Example 11.5: reduce $A=\begin{pmatrix}1 & 2 & -1 & 2\\2 & -3 & 4 & 8\\3 & 1 & 1 & 8\end{pmatrix}$ in row canonical form

$$A=\begin{pmatrix}1 & 2 & -1 & 2\\2 & -3 & 4 & 8\\3 & 1 & 1 & 8\end{pmatrix}\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\begin{pmatrix}1 & 2 & -1 & 2\\0 & -7 & 6 & 4\\0 & -5 & 4 & 2\end{pmatrix}\xrightarrow{-5R_2+7R_3\to R_3}\begin{pmatrix}1 & 2 & -1 & 2\\0 & -7 & 6 & 4\\0 & 0 & -2 & -6\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 2 & -1 & 2\\0 & -7 & 6 & 4\\0 & 0 & -2 & -6\end{pmatrix}\xrightarrow{\frac{-1}{2}R_3\to R_3}\begin{pmatrix}1 & 2 & -1 & 2\\0 & -7 & 6 & 4\\0 & 0 & 1 & 3\end{pmatrix}\xrightarrow[-6R_3+R_2\to R_2]{R_3+R_1\to R_1}\begin{pmatrix}1 & 2 & 0 & 5\\0 & -7 & 0 & -14\\0 & 0 & 1 & 3\end{pmatrix}$$

$$A\sim\begin{pmatrix}\underline{1} & 2 & 0 & 5\\0 & -\underline{7} & 0 & -14\\0 & 0 & \underline{1} & 3\end{pmatrix}\xrightarrow{\frac{-1}{7}R_2\to R_2}\begin{pmatrix}\underline{1} & 2 & 0 & 5\\0 & \underline{1} & 0 & 2\\0 & 0 & \underline{1} & 3\end{pmatrix}\xrightarrow{-2R_2+R_1\to R_1}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{pmatrix}$$

So $A\sim\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{pmatrix}$

TODO➔ Go to Activity and solve question 13

PYTHONIC: To put a matrix into reduced row echelon form, use `rref()`. `rref` returns a tuple of two elements. The first is the reduced row echelon form, and the second is a tuple of indices of the pivot columns.

```python
import sympy  as sy
M=sy.Matrix([ [1,2,-1,2],[2,-3,4,8],[3,1,1,8]], dtype='float')
#reduced row echelon form of M using  matrix.rref()
sy.pprint(M)
reduced_matrix , pivot_index=M.rref()
print("\n")
sy.pprint(reduced_matrix)
print("\n pivot index location:")
print(pivot_index)
```

```
⎡1  2   -1  2⎤
⎢            ⎥
⎢2  -3  4   8⎥
⎢            ⎥
⎣3  1   1   8⎦

⎡1  0  0  1⎤
⎢          ⎥
⎢0  1  0  2⎥
⎢          ⎥
⎣0  0  1  3⎦

 pivot index location:
(0, 1, 2)
```

Application to the solution of System of Linear Equations:

A system of linear equation can be solved using the reduced row echelon Form of its augmented matrix. We will illustrate the method by examples.

Example 11.6 : Solve $\begin{cases} x+2y=-4 \\ 3x+y=3 \end{cases}$ by row reduction

Express $\begin{cases} x+2y=-4 \\ 3x+y=3 \end{cases}$ in matrix form $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and its augmented matrix is $\left(\begin{array}{cc:c} 1 & 2 & -4 \\ 3 & 1 & 3 \end{array}\right)$, using an the RREF algorithm we get:

$$\left(\begin{array}{cc:c} 1 & 2 & -4 \\ 3 & 1 & 3 \end{array}\right) \xrightarrow{-3R_1+R_2\to R_2} \left(\begin{array}{cc:c} 1 & 2 & -4 \\ 0 & -5 & 15 \end{array}\right) \xrightarrow{\frac{-1}{5}R_2\to R_2} \left(\begin{array}{cc:c} 1 & 2 & -4 \\ 0 & 1 & -3 \end{array}\right)$$

$\left(\begin{array}{cc:c} 1 & 2 & -4 \\ 0 & 1 & -3 \end{array}\right) \xrightarrow{-2R_2+R_1\to R_1} \left(\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right)$, so $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ is equivalent

to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $or\ x=2\ and\ y=-3$

Example 11.7 : Solve $\begin{cases} x+\ \ y+2z=9 \\ 3x+2y-\ \ z=4 \\ 2x+3y-4z=-4 \end{cases}$ by row reduction

$\begin{cases} x+\ \ y+2z=9 \\ 3x+2y-\ \ z=4 \\ 2x+3y-4z=-4 \end{cases}$ ➔ $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 2 & 3 & -4 & -4 \end{array}\right)$, by reduced row echelon form,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 2 & 3 & -4 & -4 \end{array}\right) \xrightarrow[-2R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -7 & -23 \\ 0 & 1 & -8 & -22 \end{array}\right) \xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -7 & -23 \\ 0 & 0 & -15 & -45 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -7 & -23 \\ 0 & 0 & -15 & -45 \end{array}\right) \xrightarrow[\frac{-1}{15}R_3\to R_3]{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 7 & 23 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 7 & 23 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[-7R_3+R_2\to R_2]{-2R_3+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{-R_2+R_1\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Making x=1 , y=2 , z=3

TODO➔ Go to Activity and solve question 14

PYTHONIC: use the function **linsolve([list_of_equation],(variable_list))**

or **linsolve(augmented_matrix, (variable_list)) . so for** $\begin{cases} x+\ \ y+2z=9 \\ 3x+2y-\ \ z=4 \\ 2x+3y-4z=-4 \end{cases}$ we

will have it as $\begin{cases} x+\ \ y+2z-9=0 \\ 3x+2y-\ \ z-4=0 \\ 2x+3y-4z+4=0 \end{cases}$ and use

linsolve([x+y+2z-9 , 3x+2y-z-4 ,2x+3y-4z+4], (x,y,z))

```python
import sympy as sy
x,y,z=sy.symbols("x,y,z")
sy.linsolve([x+y+2*z-9,3*x+2*y-z-4,2*x+3*y-4*z+4],(x,y,z))
```

```
{(1, 2, 3)}
```

We can also use the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 2 & 3 & -4 & -4 \end{array}\right)$ form for a solution in Python (see code below) :

linsolve(Matrix(([1,1,2,9],[3,2,-1,4],[2,3,-4,-4])) ,(x,y,z)) or

get the augmented matrix as **M=sympy.Matrix([[1,1,2,9] , [3,2,-1,4] , [2,3,-4,-4]])** then **linsolve(M,(x,y,z))** that is neater. Similary the following functions will do the same thing, **sympy.solve_linear(M,x,y,z) , sympy.solve_linear_system(M,x,y,z)** Also we have to define x,y and z as symbols **x,y,z=sympy.Symbols("x,y,z").**

```
import sympy as sy
x,y,z=sy.symbols("x,y,z")
# create the augmented matrix of the system of linear equations
M=sy.Matrix([ [1,1,2,9],[3,2,-1,4],[2,3,-4,-4]])
answer=sy.linsolve(M,(x,y,z))
sy.pprint(answer)

```

```
{(1, 2, 3)}
```

12) <u>Rank of a matrix</u>

The rank of a $n \times m$ matrix A is the number of non-zero rows in row echelon form.
It is also the dimension of the vector space generated (spanned) by its columns or rows; that is the maximum number of linear independent rows or columns of A.

Note that:

$rank(A) = rank(A^T) = rank(AA^T) = rank(A^T A)$ for real matrices

$rank(A) \leq \min(n,m)$ where min is for minimum, A is an $n \times m$ matrix

$rank(A+B) \leq rank(A) + rank(B)$

Example 12.1 : Find the rank of $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

By row reduction $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & -2 & 3 \end{pmatrix} \xrightarrow{2R_2+R_3 \to R_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{pmatrix}$

There are 3 non-zero rows , so rank(A)=3 .

Example 12.2 Find the rank of $A=\begin{pmatrix}1&1&5\\1&2&3\\2&5&4\end{pmatrix}$

$$\begin{pmatrix}1&1&5\\1&2&3\\2&5&4\end{pmatrix}\xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2}\begin{pmatrix}1&1&5\\0&1&-2\\0&3&-6\end{pmatrix}\xrightarrow{-3R_2+R_3\to R_3}\begin{pmatrix}1&1&5\\0&1&-2\\0&0&0\end{pmatrix}$$

There are 2 non-zero rows , so rank(A)=2.
This means that the rows [1 1 5] and [1 2 3] are linearly independent or
The columns $[1\ 1\ 2]^T$ and $[1\ 2\ 5]^T$ are linearly independent.

TODO➔ Go to Activity and solve question

13) Linear dependence and independence using matrix Echelon Form

In this paragraph , we re-iterate what we covered in chapter1 on basis and linear dependence but using matrix and RREF to facilitate the calculation.

A non-empty set $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}=\left\{\begin{pmatrix}u_{11}\\ \vdots\\ u_{1n}\end{pmatrix},\begin{pmatrix}u_{21}\\ \vdots\\ u_{2n}\end{pmatrix},\begin{pmatrix}u_{31}\\ \vdots\\ u_{3n}\end{pmatrix},...,\begin{pmatrix}u_{n1}\\ \vdots\\ u_{nn}\end{pmatrix}\right\}$ in vector space $\mathbb{R}^n$ is linearly independent if the matrix $M=[\vec{u}_1\ \vec{u}_2\ \vec{u}_3...\vec{u}_n]=\begin{pmatrix}u_{11}&u_{21}&u_{31}&\cdots&u_{n1}\\ \vdots&\vdots&\vdots&\vdots&\vdots\\ u_{1n}&u_{2n}&u_{3n}&\cdots&u_{nn}\end{pmatrix}$

has no zero row, that is if ***rank(M)=n=dim(S)***, **n being the number of vectors in S.**
If ***rank(M)<n*** **then they are said to be linearly dependent**.

Example 13.1: Are $\vec{u}_1=(1,2,1),\ \vec{u}_2=(3,1,4)\ and\ \vec{u}_3=(-2,2,5)$ linear independent in $\mathbb{R}^3$?

As column vectors $\vec{u}_1=\begin{bmatrix}1\\2\\1\end{bmatrix},\vec{u}_2=\begin{bmatrix}3\\1\\4\end{bmatrix}\ and\ \vec{u}_3=\begin{bmatrix}-2\\2\\5\end{bmatrix}$

$M=[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]=\begin{pmatrix}u_{11}&u_{21}&u_{31}\\u_{13}&u_{22}&u_{32}\\u_{13}&u_{23}&u_{33}\end{pmatrix}=\begin{pmatrix}1&3&-2\\2&1&2\\1&4&5\end{pmatrix}$; reducing M in echelon form gives

$$\begin{pmatrix}1&3&-2\\2&1&2\\1&4&5\end{pmatrix}\xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\begin{pmatrix}1&3&-2\\0&-5&6\\0&1&7\end{pmatrix}\xrightarrow{R_2+5R_3\to R_3}\begin{pmatrix}1&3&-2\\0&-5&6\\0&0&41\end{pmatrix}$$

There are 3 non-zero rows in $\begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 6 \\ 0 & 0 & 41 \end{pmatrix}$, and n=3 vectors .

So rank(M)=n=3➔ $\vec{u}_1, \vec{u}_2$ *and* $\vec{u}_3$ are linearly Independent.

This is to make sure there are no zero rows like $[0\ 0\ 0]$ in the reduced matrix.

Example 13.2: Are $\vec{u}_1 = (1,2,5), \vec{u}_2 = (2,5,1)$ *and* $\vec{u}_3 = (1,5,2)$ linear independent in $\mathbb{R}^3$?

$$\text{Set } A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{pmatrix} \xrightarrow[-5R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -9 & -3 \end{pmatrix} \xrightarrow{9R_2+R_3 \to R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}$$

There are no zero row $[0\ 0\ 0]$, rank(A)=3=number of vectors ➔ they are independent

TODO➔ Go to Activity and solve question 16

Example 12.3: Are $\vec{u}_1 = (1,-1,1), \vec{u}_2 = (2,1,1)$ *and* $\vec{u}_3 = (3,0,2)$ linear independent in $\mathbb{R}^3$?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[-R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2+3R_3 \to R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Here we have a zero row $[0\ 0\ 0]$, $rank(A) = 2 < 3$ (3 vectors) so $\vec{u}_1, \vec{u}_2$ *and* $\vec{u}_3$ are not linear independent, but they are linearly dependent.

TODO➔ Go to Activity and solve question 17

14) <u>Basis Using Matrices Reduced Row Echelon Form(RREF)</u>

Let vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ *and* $\vec{c} = (c_1, c_2, c_3)$ with their corresponding column vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, they are said to be a basis to $\mathbb{R}^3$ if the matrix $M = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ can be put in reduced row echelon form with no zero-rows using the RREF algorithm.

Example 14.1: Do $\vec{u}_1 = (1,2,1), \vec{u}_2 = (3,1,4)\ and\ \vec{u}_3 = (-2,2,5)$ form a basis of $\mathbb{R}^3$.

From **example 13.1** , we had $M = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{13} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 2 \\ 1 & 4 & 5 \end{pmatrix}$ reduced

in echelon form $M = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 2 \\ 1 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 6 \\ 0 & 0 & 41 \end{pmatrix}$; now convert to reduced row echelon

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 6 \\ 0 & 0 & 41 \end{pmatrix} \xrightarrow{R_3/41} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-6R_3+R_2 \to R_2]{2R_3+R_1 \to R_1} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{R_2}{-5} \to R_2} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-3R_2+R_1 \to R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$, we have M in reduced row echelon form,

so $\vec{u}_1, \vec{u}_2\ and\ \vec{u}_3$ form a basis for $\mathbb{R}^3$.

Example 14.2: Do $\vec{u}_1 = (1,2,5),\ \vec{u}_2 = (2,5,1)\ and\ \vec{u}_3 = (1,5,2)$ form a basis of $\mathbb{R}^3$

From example 13.2 , we had $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3] = \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{13} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{pmatrix}$ reduced

in echelon form $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix} \xrightarrow{R_3/24} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-3R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_2+R_1 \to R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have M in reduced row echelon form, so $\vec{u}_1, \vec{u}_2\ and\ \vec{u}_3$ form a basis for $\mathbb{R}^3$.

Example 14.3 : Do $\vec{u}_1=(1,-1,1),\vec{u}_2=(2,1,1)\ and\ \vec{u}_3=(3,0,2)$ form a basis of $\mathbb{R}^3$

From example 13.2 , we had $A=[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]=\begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{13} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}=\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ reduced in echelon form $A=\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\sim\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ ➔ $\vec{u}_1,\vec{u}_2\ and\ \vec{u}_3$ are not linearly independent (zero-row presence), therefore not forming a basis for $\mathbb{R}^3$.

TODO➔ Go to Activity and solve question 18

15) Basis of a Matrix Row Space

Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then the row vectors of A are :

$\vec{r}_1=[a_{11}\ a_{12}\ a_{13}],\ \vec{r}_2=[a_{21}\ a_{22}\ a_{23}]\ ,\ \vec{r}_3=[a_{31}\ a_{32}\ a_{33}]$

Example 15.1 Find the row vectors of $A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{pmatrix}$

$\vec{r}_1=[1\ 2\ 1],\ \vec{r}_2=[2\ 5\ 5]\ ,\ \vec{r}_3=[5\ 1\ 2]$

Row Space Definition:

Let A be a $n\times m$ matrix, then the row space of A is the subspace of $\mathbb{R}^m$ spanned by its row vectors, denoted $rowsp(A)$.

Now, How to find the basis of row space of $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$?

Theorem : Given a matrix A and its row echelon form matrix E, then :

Row space of A= row space of E or $rowsp(A)=rowsp(E)$

Example 15.2 Find the basis of row space of $A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{pmatrix}$

From example 14.2 $A=\begin{pmatrix}1 & 2 & 1\\ 2 & 5 & 5\\ 5 & 1 & 2\end{pmatrix}$ has been reduced to $E=\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 3\\ 0 & 0 & 24\end{pmatrix}$

Since $rowsp(A)=rowsp(E)$, $\vec{r}_1=[1\ 2\ 1]$, $\vec{r}_2=[0\ 1\ 3]$, $\vec{r}_3=[0\ 0\ 24]$ form a basis of $rowsp(A)$ and dimension of $rowsp(A)$ $=3$ or $\dim(rowsp(A)\ )=3$

Example 15.3 Find the basis of row space of $A=\begin{pmatrix}1 & 2 & 3\\ -1 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}$

From example 14.2 $A=\begin{pmatrix}1 & 2 & 3\\ -1 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}$ has been reduced to $E=\begin{pmatrix}1 & 2 & 3\\ 0 & 3 & 3\\ 0 & 0 & 0\end{pmatrix}$

Since $rowsp(A)=rowsp(E)$,$\vec{r}_1=[1\ 2\ 3]$, $\vec{r}_2=[0\ 3\ 3]$ form a basis of $rowsp(A)$ and dimension of $rowsp(A)=2$ or $\dim(rowsp(A))=2$

TODO➔ Go to Activity and solve question 19

PYTHONIC : **use Matrix.rowspace() to get row space basis of matrix.**

```
import sympy as sy
x,y,z=sy.symbols("x,y,z")
# create the matrix
M=sy.Matrix([ [1,2,3],[-1,1,0],[1,1,2]])
# get the column space vectors using Matrix.rowspace()
rowsp_vectors=M.rowspace()
sy.pprint(rowsp_vectors)

```

```
[[1  2  3], [0  3  3]]
```

16) Basis of a Matrix Column Space

Let $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ then the column vectors of A are :

$\vec{c}_1=\begin{bmatrix}a_{11}\\ a_{21}\\ a_{31}\end{bmatrix}$ $\vec{c}_2=\begin{bmatrix}a_{12}\\ a_{22}\\ a_{32}\end{bmatrix}$, $\vec{c}_3=\begin{bmatrix}a_{13}\\ a_{23}\\ a_{33}\end{bmatrix}$

Example 16.1 Find the column vectors of $A=\begin{pmatrix}1 & 2 & 1\\ 2 & 5 & 5\\ 5 & 1 & 2\end{pmatrix}$

Answer : $\vec{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ $\vec{c}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $\vec{c}_3 = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

Column Space Definition:

Let A be a $n \times m$ matrix, then the column space of A is the subspace of $\mathbb{R}^n$ spanned By its column vectors, denoted $colsp(A)$.

Now, How to find the basis of column space of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$?

The theorem below will help us to find basis for the column space of a matrix in row echelon form.

THEOREM: *If a matrix A is in row echelon form, then the non-zero row vectors with the leading pivots(or leading 1's) form a basis for the row space of A, and the column vectors having the same leading* pivots (or leading 1's) *of the row vectors form a basis for the column space of R.*

Example 16.2: find the basis of column space of $A = \begin{pmatrix} \underline{1} & 2 & 3 \\ 0 & \underline{3} & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Since A is in echelon form with the red-underlined pivots 1 and 3, $\vec{r}_1 = (1\ 2\ 3)$, $\vec{r}_2 = (0\ 3\ 3)$ form A basis for the rows space of A ,and the leading pivots 1 and 3 are respectively in the first columns $\vec{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and second column $\vec{c}_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ of A therefore

$\vec{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{c}_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ form a basis of the column space of A.

Example 16.3: find the basis of column space of $A = \begin{pmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$

We reduce A in echelon form $A=\begin{pmatrix}1&1&2\\4&5&5\\5&8&1\\-1&-2&2\end{pmatrix}\begin{matrix}\xrightarrow{-4R_1+R_2}\\\xrightarrow{-5R_1+R_3}\\\xrightarrow{R_1+R_2}\end{matrix}\begin{pmatrix}1&1&2\\0&1&-3\\0&3&-9\\0&-1&4\end{pmatrix}$

$\begin{pmatrix}1&1&2\\0&1&-3\\0&3&-9\\0&-1&4\end{pmatrix}\begin{matrix}\xrightarrow{-3R_2+R_3}\\\xrightarrow{R_2+R_4}\end{matrix}\begin{pmatrix}1&1&2\\0&1&-3\\0&0&0\\0&0&1\end{pmatrix}\begin{matrix}\xrightarrow{R_3\leftarrow R_4}\\\xrightarrow{R_4\leftarrow R_3}\end{matrix}\begin{pmatrix}\underline{1}&1&2\\0&\underline{1}&-3\\0&0&\underline{1}\\0&0&0\end{pmatrix}$, the leading pivots 1,1,1

in the echelon matrix are in column $C_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, C_2=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ and $C_3=\begin{pmatrix}2\\-3\\1\\0\end{pmatrix}$, they are linearly

Independent and form a basis, therefore their corresponding column vectors in matrix A

$\vec{c}_1=\begin{pmatrix}1\\4\\5\\-1\end{pmatrix}$, $\vec{c}_2=\begin{pmatrix}1\\5\\8\\-2\end{pmatrix}$ and $\vec{c}_3=\begin{pmatrix}2\\5\\1\\2\end{pmatrix}$ form a basis for colspace(A)

Example 16.3: Given a matrix $A=\begin{pmatrix}1&1&-1\\2&3&-1\\3&1&-5\end{pmatrix}$

a) find the basis of row space of A and its dimension dim[rowsp(A)]
b) Calculate rank(A)
c) find the basis of column space of A
d) which vectors of colsp(A) is linearly depend?

Answer:

a)

Reduce A in echelon form $A=\begin{pmatrix}1&1&-1\\2&3&-1\\3&1&-5\end{pmatrix}\begin{matrix}\xrightarrow{-2R_1+R_2}\\\xrightarrow{-3R_1+R_3}\end{matrix}\begin{pmatrix}1&1&-1\\0&1&1\\0&-2&-2\end{pmatrix}$

$\begin{pmatrix}1&1&-1\\0&1&1\\0&-2&-2\end{pmatrix}\xrightarrow{2R_2+R_3}\begin{pmatrix}\underline{1}&1&-1\\0&\underline{1}&1\\0&0&0\end{pmatrix}$ **we have 2 non-zero rows therefore**

$\vec{r}_1=[1\ \ 1-1]$ *and* $\vec{r}_2=[0\ 1\ 1]$, $\{\vec{r}_1,\vec{r}_2\}$, **form a basis for rowsp(A)**

Since there are 2 vector in $\{\vec{r}_1,\vec{r}_2\}$**, dim[rowsp(A)]=2**

b) **we have 2 non-zero rows therefore rank(A)=2**

c) in $E=\begin{pmatrix} \underline{1} & 1 & -1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$ **the pivots 1 are in the first and second column, then**

$C_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $C_2=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **form a basis for colsp(E) , therefore their corresponding**

column vectors in A $\vec{c}_1=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\vec{c}_2=\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ **form a basis for colsp(A)**

So $colsp(A)=\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$

d) $c_1=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\vec{c}_2=\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ **are linearly independent.** $\vec{c}_3=\begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$ **is dependent.**

$\vec{c}_3=\begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$ **is a linear combination of** $c_1=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\vec{c}_2=\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

To express $\vec{c}_3=\begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$ **as a linear combination of** $\vec{c}_1=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\vec{c}_2=\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

we get a row reduced echelon form(RREF) of A starting with its echelon form

$E=\begin{pmatrix} \underline{1} & 1 & -1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2+R_1} \begin{pmatrix} \underline{1} & 0 & -2 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$ **, from there we can see that**

$-2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \vec{c}_3=-2\vec{c}_1+\vec{c}_2$

TODO➔ Go to Activity and solve question 20

PYTHONIC: use Matrix.columnspace() to get column space basis of matrix.

```
import sympy as sy
x,y,z=sy.symbols("x,y,z")
# create the matrix
M=sy.Matrix([ [1,1,-1],[2,3,-1],[3,1,-5]])
# get the column space vectors using Matrix.columnspace()
colsp_vectors=M.columnspace()
sy.pprint(colsp_vectors)
```

$$\left[\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right]$$

17) Basis of a Matrix Null Space

The null space of a $n \times m$ matrix A, denoted **null(A) , is the set of all solutions to the homogeneous equation** $A \cdot \vec{v} = \vec{0}$ that is $null(A) = \left\{ \vec{v} : \ \vec{v} \in \mathbb{R}^n \ and \ A \cdot \vec{v} = \vec{0} \right\}$

Example 17.1 Find the null space of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Given $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, Null(A) ➔ solving $A \cdot \vec{v} = \vec{0}$ ➔ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ➔ $\begin{cases} x + 2y = 0 \\ 3x + 4y = 0 \end{cases}$

This leads to $x = y = 0$ that is $null(A) = \vec{O}$

Example 17.2 Find a basis for the null space of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}$

Given $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, Null(A) ➔ solving $A \cdot \vec{v} = \vec{0}$ ➔ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We reduce the augmented matrix of A, $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right)$ in RREF.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

RREF(A)= $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+2y=0 \\ z=0 \end{cases}$$ from RREF(A) we can see that y is a free variable, so we set an arbitrary value to y , $y=t$ so that $x+2y=0$ becomes

$x+2t=0 \Rightarrow x=-2t$ so $\vec{v}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix}=t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ we finally have a basis of Null(A), that is $null(A)=\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$ and dimension of Null(A)=1 or $\dim(null(A))=1$

Example17.3: Find a basis for the null space of $A=\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

Given $\vec{v}=\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, Null(A) ➔ solving $A\cdot\vec{v}=\vec{0} \Rightarrow \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We reduce the augmented matrix of A , $\left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right)$ in RREF.

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{-1}{7}R_2} \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-4R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2/7 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

RREF(A)= $\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2/7 & 0 \\ 0 & 1 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ now compute $\begin{pmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ➔

$$\begin{cases} x + z - \frac{2}{7}w = 0 \\ y + z + \frac{4}{7}w = 0 \end{cases}$$; from RREF(A) the pivots are x ,y and the free variables are z and w

So we set $z = t \;\; and \;\; w = s$ leading to $\begin{cases} x + t - \frac{2}{7}s = 0 \\ y + t + \frac{4}{7}s = 0 \end{cases} \Rightarrow \begin{cases} x = -t + \frac{2}{7}s \\ y = -t - \frac{4}{7}s \\ z = t \\ w = s \end{cases}$ in

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -t + \frac{2}{7}s \\ -t - \frac{4}{7}s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{7}s \\ -\frac{4}{7}s \\ 0 \\ s \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} .$$

Since $\vec{v} = t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix}$, a basis of Null(A) is $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right\}$

and dimension of Null(A)=2 .

TODO➔ Go to Activity and solve question 21.

PYTHONIC: To find the nullspace of a matrix, use `nullspace()`. `nullspace` returns a `list` of column vectors that span the nullspace of the matrix.

```
import sympy as sy
# create the matrix
M=sy.Matrix([ [1,4,5,2],[2,1,3,0],[-1,3,2,2]])
# get the null space vectors using Matrix.nullpace()
nullsp_vectors=M.nullspace()
sy.pprint(nullsp_vectors)
```

```
⎡⎡-1⎤  ⎡2/7 ⎤⎤
⎢⎢  ⎥  ⎢    ⎥⎥
⎢⎢-1⎥  ⎢-4/7⎥⎥
⎢⎢  ⎥, ⎢    ⎥⎥
⎢⎢1 ⎥  ⎢ 0  ⎥⎥
⎢⎢  ⎥  ⎢    ⎥⎥
⎣⎣0 ⎦  ⎣ 1  ⎦⎦
```

18) **Coordinates**

If $B=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,.....,\vec{v}_n\}$ is basis of a vector space V, then any vector $\vec{u}\in V$ can be uniquely expressed as a linear combination of $\vec{v}_1,\vec{v}_2,\vec{v}_3,.....,\ and\ \vec{v}_n$ that is:
$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3+...+c_n\vec{v}_n$ with $c_i\in\mathbb{R}, 1\le i\le n$.

$c_1,c_2,c_3,...,\ and\ c_n$ are called the coordinate $\vec{u}$ with respect to $B=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,.....,\vec{v}_n\}$,

that is $[\vec{u}]_B=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$ or $[\vec{u}]_B=\begin{pmatrix}c_1 & c_2 & \cdots & c_n\end{pmatrix}$

Example 18.1: Finding coordinate of vector and matrix

a) coordinate of vector
the coordinate of $\vec{u}=2\vec{i}+3\vec{j}-\vec{k}$ with respect to the standard basis $S=\{\vec{i},\vec{j},\vec{k}\}$ is $\vec{u}=(2,3,-1)$

b) coordinate of a matrix
the coordinate of matrix $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ with respect to the basis
$B=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is $[A]_B=(1,2,3,4)$ since
$$A=\begin{pmatrix}1&2\\3&4\end{pmatrix}=1\begin{pmatrix}1&0\\0&0\end{pmatrix}+2\begin{pmatrix}0&1\\0&0\end{pmatrix}+3\begin{pmatrix}0&0\\1&0\end{pmatrix}+4\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

c) coordinate of a polynomial
the coordinate of a polynomial $p(x)=a_0+a_1x+a_2x^2$ with respect to the basis $B=\{1,x,x^2\}$ is $[p(x)]_B=(a_0,a_1,a_2)$ that is:
$p(x)=2+5x+x^2$ has coordinate $(2,5,1)$ in $B=\{1,x,x^2\}$
$p(x)=5-4x+7x^2+10x^3$ has coordinate $(5,-4,7,10)$ in $B=\{1,x,x^2,x^3\}$
TODO➔ Go to Activity and solve question 22

Example 18.2: Find the coordinate of $\vec{u}=(4,-3)$ with respect to $B=\{(2,1),(3,4)\}$
Answer: solve $\vec{u}=(4,-3)=c_1(2,1)+c_2(3,4)$ ➔ $(2c_1+3c_2,c_1+4c_2)=(4,-3)$
➔ $\begin{cases}2c_1+3c_2=4\\c_1+4c_2=-3\end{cases}$ ➔ $c_1=5,\ c_2=-2$ so $[\vec{u}]_B=(5,-2)$

19) Change of Basis

The standard basis of $\mathbb{R}^3$ is $S=\left\{\vec{i},\vec{j},\vec{k}\right\}$ with $\vec{i}=(1,0,0),\vec{j}=(0,1,0)$ *and* $\vec{k}=(0,0,1)$. Sometimes $S=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ with $\hat{e}_1=(1,0,0),\ \hat{e}_2=(0,1,0)$ *and* $\hat{e}_3=(0,0,1)$ is used as the standard basis. For the sake of simplicity, we will work on $\mathbb{R}^3$ and $\mathbb{R}^2$, but the concept can be extended to higher dimensional vector space ($\mathbb{R}^n$).

Change of Basis Using linear Combination of Vectors

Now let's assume that we have two bases $B=\{\vec{u}_1,\vec{u}_2\}$ and $B'=\{\vec{u}'_1,\vec{u}'_2\}$ in the same vector space $\mathbb{R}^2$, and a vector $\vec{v}$ with coordinate x *and* y in $B=\{\vec{u}_1,\vec{u}_2\}$, that is $[\vec{v}]_B=(x,y)=x\vec{u}_1+y\vec{u}_2$. We want to express the coordinates of $\vec{v}$ with respect to $B'=\{\vec{u}'_1,\vec{u}'_2\}$.Hence we need the transition matrix $M_{B'\leftarrow B}$ that will take us from the old basis $B=\{\vec{u}_1,\vec{u}_2\}$ to the new basis $B'=\{\vec{u}'_1,\vec{u}'_2\}$.
We want to express the coordinate of $\vec{u}_1$ *and* $\vec{u}_2$ in $B'=\{\vec{u}'_1,\vec{u}'_2\}$.

Let's say the coordinates of $\vec{u}_1$, *and* $\vec{u}_2$ in B' **are respectively** $[\vec{u}_1]_{B'}=(a,b)$ and $[\vec{u}_2]_{B'}=(c,d)$, then $[\vec{u}_1]_{B'}=a\vec{u}_1{}'+b\vec{u}_2{}'=\begin{bmatrix}a\\b\end{bmatrix}$ and $[\vec{u}_2]_{B'}=c\vec{u}_1{}'+d\vec{u}_2{}'=\begin{bmatrix}c\\d\end{bmatrix}$.
Since $[\vec{v}]_B=(x,y)=x\vec{u}_1+y\vec{u}_2$, we express its coordinate in B' to be
$\vec{v}=x\vec{u}_1+y\vec{u}_2=x(a\vec{u}_1{}'+b\vec{u}_2{}')+y(c\vec{u}_1{}'+d\vec{u}_2{}')=(ax+cy)\vec{u}_1{}'+(bx+dy)\vec{u}_2{}'$ so
$[\vec{v}]_{B'}=(ax+cy)\vec{u}_1{}'+(bx+dy)\vec{u}_2{}'$ or in matrix form $[\vec{v}]_{B'}=\begin{bmatrix}ax+cy\\bx+dy\end{bmatrix}=\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\Rightarrow$

$[\vec{v}]_{B'}=\begin{bmatrix}a&c\\b&d\end{bmatrix}[\vec{v}]_B$, or $[\vec{v}]_{B'}=M_{B'\leftarrow B}\bullet[\vec{v}]_B$ where $M_{B'\leftarrow B}=\begin{bmatrix}a&c\\b&d\end{bmatrix}$ is called the transition matrix (change-of-basis matrix) from basis $B=\{\vec{u}_1,\vec{u}_2\}$ to basis $B'=\{\vec{u}'_1,\vec{u}'_2\}$.
So to conclude, if the coordinates of the source basis ($B=\{\vec{u}_1,\vec{u}_2\}$ **) vectors** $\vec{u}_1$ **and** $\vec{u}_2$ **in the destination basis** $B'=\{\vec{u}'_1,\vec{u}'_2\}$ **are respectively** $[\vec{u}_1]_{B'}=\begin{bmatrix}a\\b\end{bmatrix}$ **and** $[\vec{u}_2]_{B'}=\begin{bmatrix}c\\d\end{bmatrix}$ **then the transition matrix from basis** $B=\{\vec{u}_1,\vec{u}_2\}$ **to basis** $B'=\{\vec{u}'_1,\vec{u}'_2\}$ **is**

$$M_{B'\leftarrow B}=\left[[\vec{u}_1]_{B'}\ \vdots\ [\vec{u}_2]_{B'}\right]=\begin{bmatrix}a&c\\b&d\end{bmatrix}$$

Example 19.1: Consider the basis $S=\{\vec{i},\vec{j}\}=\{(1,0),(0,1)\}$ and $B=\{\vec{u}_1,\vec{u}_2\}=\{(1,-1),(-3,4)\}$

a) Find the transition matrix from S to B , $M_{B\leftarrow S}$

b) If $\vec{v}=(3,2)=3\vec{i}+2\vec{j}$, calculate its coordinate in B, that is find $[\vec{v}]_B$

c) Find the transition matrix from B to S , $M_{S\leftarrow B}$

Answer:

a) We want to express $\vec{i}=(1,0)$ *and* $\vec{j}=(0,1)$ as vector in B with their unknown coordinates.

Working on $\vec{i}=(1,0)$:

if a *and* b are its coordinate in B then

$\vec{i}=(1,0)=a\vec{u}_1+b\vec{u}_2=a(1,-1)+b(-3,4)=(a-3b,-a+4b)$ ➔

$\begin{cases}a-3b=1\\-a+4b=0\end{cases}$ and $a=4,\ b=1$ ➔ $[\vec{i}]_B=\begin{bmatrix}4\\1\end{bmatrix}$

Working on $\vec{j}=(0,1)$: if c *and* d are its coordinate in B then

$\vec{j}=(0,1)=c\vec{u}_1+d\vec{u}_2=c(1,-1)+d(-3,4)=(c-3d,-c+4d)$ ➔

$\begin{cases}c-3d=0\\-c+4d=1\end{cases}$ and $c=3,\ d=1$ ➔ $[\vec{j}]_B=\begin{bmatrix}3\\1\end{bmatrix}$

transition matrix from S to B , $M_{B\leftarrow S}=\left[[\vec{i}]_B \mid [\vec{j}]_B\right]=\begin{bmatrix}4&3\\1&1\end{bmatrix}$

b) $[\vec{v}]_B=M_{B\leftarrow S}\cdot[\vec{v}]_S=\begin{pmatrix}4&3\\1&1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}18\\5\end{pmatrix}$

c) Find the transition matrix from B to S , $M_{S\leftarrow B}$ is the easy one to compute.

➔ $[\vec{u}_1]_B=(1,-1)=(1,0)+(0,-1)=(1,0)-(0,1)=\vec{i}-\vec{j}$ ➔ $[\vec{u}_1]_S=\begin{bmatrix}1\\-1\end{bmatrix}$

$[\vec{u}_2]_B=(-3,4)=(-3,0)+(0,4)=-3(1,0)+4(0,1)=-3\vec{i}+4\vec{j}$ ➔ $[\vec{u}_2]_S=\begin{bmatrix}-3\\4\end{bmatrix}$

transition matrix from B to S , $M_{S\leftarrow B}=\left[[\vec{u}_1]_S \mid [[\vec{u}_2]]_S\right]=\begin{bmatrix}1&-3\\-1&4\end{bmatrix}$

Example 19.2 Find the transition matrix from $B=\{\vec{u}_1,\vec{u}_2\}=\{(1,3),(-1,-1)\}$ to $B'=\{\vec{v}_1,\vec{v}_2\}=\{(3,2),(4,3)\}$

Answer:

We want to express $\vec{u}_1=(1,3),$ *and* $\vec{u}_2(-1,-1)$ as vector in B′ with their unknown coordinates.

Working on $\vec{u}_1=(1,3)$:

if a *and* b are the coordinate of $\vec{u}_1$ in B′ then

$\vec{u}_1=(1,3)=a\vec{v}_1+b\vec{v}_2=a(3,2)+b(4,3)=(3a+4b,2a+3b)$ ➔ $(3a+4b,2a+3b)=(1,3)$

$\begin{cases} 3a+4b=1 \\ 2a+3b=3 \end{cases}$ and $a=-9, b=7$ ➔ $[\vec{u}_1]_{B'}=\begin{bmatrix} -9 \\ 7 \end{bmatrix}$

Working on $\vec{u}_2=(-1,-1)$: if c *and* d are its coordinate in B′ then

$\vec{u}_2=(-1,-1)=c\vec{v}_1+d\vec{v}_2=c(3,2)+d(4,3)=(3c+4d,2c+3d)$, $(3c+4d,2c+3d)=(-1,-1)$

$\begin{cases} 3c+4d=-1 \\ 2c+3d=-1 \end{cases}$ and $c=1, d=-1$ ➔ $[\vec{u}_2]_{B'}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So the transition matrix from B to B′ , $M_{B'\leftarrow B}=\left[[\vec{u}_1]_{B'} \mid [\vec{u}_2]_{B'}\right]=\begin{bmatrix} -9 & 1 \\ 7 & -1 \end{bmatrix}$

Example 19.3 Find the transition matrix from basis $B=\{p_1,p_2\}=\{1+3x, x\}$ to basis $B'=\{q_1,q_2\}=\{3,2x\}$

Answer:

The vector coordinate for $q_1(x)=3$ and $q_2(x)=2x$ are $q_1=(3,0)$ and $q_1=(0,2)$

We want to express the coordinate of $p_1(x)=1+3x=(1,3)$ and $p_2(x)=x=(0,1)$ in B′.

Working on $p_1=(1,3)$:

if a *and* b are the coordinate of $p_1=(1,3)$ in B′ then

$p_1=(1,3)=aq_1+bq_2=a(3,0)+b(0,2)=(3a,2b)$ ➔ $(3a,2b)=(1,3)$

$\begin{cases} 3a=1 \\ 2b=3 \end{cases}$ and $a=\frac{1}{3},\ b=\frac{3}{2}$ ➔ $[p_1]_{B'}=\begin{bmatrix} \frac{1}{3} \\ \frac{3}{2} \end{bmatrix}$

Working on $p_2=(0,1)$:

if c *and* d are the coordinate of $p_2=(0,1)$ in B′ then

$p_2=(0,1)=cq_1+dq_2=c(3,0)+d(0,2)=(3c,2d)$ ➔ $(3c,2d)=(0,1)$

$\begin{cases} 3c=0 \\ 2d=1 \end{cases}$ and $c=0,\ d=\frac{1}{2}$ ➔ $[p_2]_{B'}=\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$

So the transition matrix from B to B′ , $M_{B'\leftarrow B}=\left[[p_1]_{B'} \mid [p_2]_{B'}\right]=\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

$[p_1]_{B'}=\left(\frac{1}{3},\frac{3}{2}\right)=\frac{1}{3}+\frac{3}{2}x$ and $[p_2]_{B'}=\left(0,\frac{1}{2}\right)=\frac{1}{2}x$

Using Matrix and its reduced row echelon form(RREF)

From above theory, we derived the transition matrix $M_{B'\leftarrow B}$ that will take us from the old basis $B=\{\vec{u}_1,\vec{u}_2\}$ to the new basis $B'=\{\vec{u}'_1,\vec{u}'_2\}$ with $[\vec{v}]_{B'}=M_{B'\leftarrow B}\bullet[\vec{v}]_B$

where $M_{B'\leftarrow B}=\left[B'|B\right]=[\vec{u}'_1\ \ \vec{u}'_2\ |\ \vec{u}_1\ \ \vec{u}_2]=\left(\begin{array}{cc|cc} a' & c' & a & c \\ b' & d' & b & d \end{array}\right)$ with $B=\{\vec{u}_1,\vec{u}_2\}=\left\{\begin{bmatrix} a \\ b \end{bmatrix},\begin{bmatrix} c \\ d \end{bmatrix}\right\}$

and $B'=\{\vec{u}'_1,\vec{u}'_2\}=\left\{\begin{bmatrix} a' \\ b' \end{bmatrix},\begin{bmatrix} c' \\ d' \end{bmatrix}\right\}$, from there we will RREF $M_{B'\leftarrow B}=[B'|B]$

$M_{B'\leftarrow B}=[I|M]$ (I is identity matrix) to finally get our transition matrix $M_{B'\leftarrow B}=M$.

Example 19.4: Consider the basis $S=\{\vec{i},\vec{j}\}=\{(1,0),(0,1)\}$ and $B=\{\vec{u}_1,\vec{u}_2\}=\{(1,-1),(-3,4)\}$

Find the transition matrix from S to B , $M_{B\leftarrow S}$.

Answer :

$$M_{B\leftarrow S}=\left[B|S\right]=[\vec{u}_1\ \ \vec{u}_2\ |\ \vec{i}\ \ \vec{j}]=\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right)$$

$$M_{B\leftarrow S}=\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right)\xrightarrow{R_1+R_2}\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)\xrightarrow{3R_2+R_1}\left(\begin{array}{cc|cc} 1 & 0 & 4 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Finally $M_{B\leftarrow S}=\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

Example 19.5 **Find the transition matrix from** $B=\{\vec{u}_1,\vec{u}_2\}=\{(1,3),(-1,-1)\}$ **to** $B'=\{\vec{v}_1,\vec{v}_2\}=\{(3,2),(4,3)\}$

Answer : **We make sure the vectors are column vectors in matrix M:**

$$\vec{u}_1=\begin{pmatrix} 1 \\ 3 \end{pmatrix},\vec{u}_2=\begin{pmatrix} -1 \\ -1 \end{pmatrix}\quad \textbf{and} \quad \vec{v}_1=\begin{pmatrix} 3 \\ 2 \end{pmatrix},\vec{v}_2=\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$M_{B'\leftarrow B}=\left[B'|B\right]=[\vec{v}_1\ \ \vec{v}_2|\vec{u}_1\ \ \vec{u}_2\ ]=\left(\begin{array}{cc|cc} 3 & 4 & 1 & -1 \\ 2 & 3 & 3 & -1 \end{array}\right)$$

$$rref(M_{B'\leftarrow B})=\left(\begin{array}{cc|cc} 3 & 4 & 1 & -1 \\ 2 & 3 & 3 & -1 \end{array}\right)\xrightarrow{-2R_1+3R_2}\left(\begin{array}{cc|cc} 3 & 4 & 1 & -1 \\ 0 & 1 & 7 & -1 \end{array}\right)\xrightarrow{-4R_2+R_1}\left(\begin{array}{cc|cc} 3 & 0 & -27 & 3 \\ 0 & 1 & 7 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 3 & 0 & -27 & 3 \\ 0 & 1 & 7 & -1 \end{array}\right)\xrightarrow{\frac{1}{3}R_1}\left(\begin{array}{cc|cc} 1 & 0 & -9 & 1 \\ 0 & 1 & 7 & -1 \end{array}\right)$$

Finally $M_{B'\leftarrow B}=\begin{pmatrix} -9 & 1 \\ 7 & -1 \end{pmatrix}$

TODO➔ Go to Activity and solve question 23.1 and 23.2