GEN 242: Linear Algebra

Chapter 4: Linear Transformations

Solutions Guide

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Answers

Linear Transformation

Plain text.

1. $T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 5x$ Linear.
2. $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = xy$ Not linear.
3. $T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = x^3$ Not linear.
4. $T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 0$ Linear.
5. $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x + y$ Linear.
6. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(\vec{x}) = A\vec{x}$ Linear.

Linear Transformation Matrix – Standard Matrix, Standard Basis

7. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 5y \\ x + 6y \end{bmatrix} \to [T] = \begin{bmatrix} 2 & -5 \\ 1 & 6 \end{bmatrix}$
8. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 10x - 5y \\ y \end{bmatrix} \to [T] = \begin{bmatrix} 10 & -5 \\ 0 & 1 \end{bmatrix}$
9. $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 5y + z \\ x + 6y - z \\ x + y + z \end{bmatrix} \to [T] = \begin{bmatrix} 2 & -5 & 1 \\ 1 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix}$
10. $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + z \\ x + y - z \\ x + y + z \end{bmatrix} \to [T] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Linear Transformation Matrix – Standard Matrix, Non-Standard Basis

11. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x + y, x - 2y)$:
    a. $[T]_{B_W=\{(1,1),(1,2)\},B_V=\{(2,1),(3,2)\}} = \begin{bmatrix} 6 & 11 \\ -3 & -6 \end{bmatrix}$
    b. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_V} \to [\vec{u}]_{B_W} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}_{B_W}$
12. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - 2y \end{bmatrix}$:
    a. $[T]_{B_W=\{(1,1),(4,5)\},B_V=\{(2,1),(3,2)\}} = \begin{bmatrix} 15 & 29 \\ -3 & -6 \end{bmatrix}$
    b. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B_V} \to [\vec{u}]_{B_W} = \begin{bmatrix} 102 \\ -21 \end{bmatrix}_{B_W}$
13. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$:
    a. $[T]_{B_W=\{(1,1),(2,3)\},B_V=\{(1,0),(0,2)\}} = \begin{bmatrix} -1 & 10 \\ 1 & -6 \end{bmatrix}$
    b. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B_V} \to [\vec{u}]_{B_W} = \begin{bmatrix} 29 \\ -17 \end{bmatrix}_{B_W}$

14. Given $T: \mathbb{R}^2 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (x, x+y, y)$:

$$[T]_{B_W=\{(1,2,1),(0,1,0),(2,0,3)\},B_V=\{(1,2),(1,1)\}} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 6 \\ 0 & -1 & 5 \end{bmatrix}$$

Kernel of Linear Transformation

15. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x - y, 2x + y)$:
    a. $\text{Ker}(T) = \vec{0}$
    b. $\dim(\text{Ker}(T)) = 0$
16. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x + y, x)$:
    a. $\text{Ker}(T) = \vec{0}$
    b. $\dim(\text{Ker}(T)) = 0$
17. Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x - y$:
    a. $\text{Ker}(T) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
    b. $\dim(\text{Ker}(T)) = 1$
18. Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x, y, z) = (x - y + z, x + 4y + 1, y)$:
    a. $\text{Ker}([T]) = \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$
    b. $\dim(\text{Ker}(T)) = 0$

Range/Image of a Linear Transformation

19. Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ -6x + 3y \end{bmatrix}$:
    a. $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 2 \\ -6 \end{bmatrix}\right\}$
    b. $\dim(\text{Im}(T))$
20. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 2x + 6y \end{bmatrix}$:
    a. $\text{Im}(T) = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}\right\}$
    b. $\dim(\text{Im}(T)) = 2$
21. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ x + 4y \end{bmatrix}$:
    a. $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$
    b. $\dim(\text{Im}(T)) = 2$
22. Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ x + 4y + z \\ x + 3y + z \end{bmatrix}$:
    a. $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}\right\}$
    b. $\dim(\text{Im}(T)) = 2$

2D/3D Geometric Transformation

23. Given $\vec{p} = (1,1,-1)$:
    a. $\vec{v} = (3,2,4) \to \vec{p}' = (4,3,3)$
    b. $R_y\left(\frac{\pi}{4}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, R_y^{-1}\left(\frac{\pi}{4}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$
    c. $\theta = 45° \to \vec{p}' = \left(0,1,-\sqrt{2}\right)$
    d. $S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \to \vec{p}' = (2,10,-5)$
24. Given $\vec{p} = (2,-4)$:
    a. $R_z(30°) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
    b. $\vec{p} = (-1,2) \to \vec{p}' = \left(2+\sqrt{3}, 1-2\sqrt{3}, 0\right)$
25. $\theta = 30°, \vec{p} = (2,-4) \to [W] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2}+\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1+\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$
26. Given Δabc where $\vec{a} = (0,0), \vec{b} = (1,1), \vec{c} = (5,2)$ and $\theta = 45°$:
    a. $\Delta a'b'c', \begin{cases} \vec{a}' = (0,0) \\ \vec{b}' = \left(0,\sqrt{2}\right) \\ \vec{c}' = \left(3\sqrt{2}, 7\sqrt{2}\right) \end{cases}$
    b. $\Delta a'b'c', \begin{cases} \vec{a}' = \left(1, 1-\sqrt{2}\right) \\ \vec{b}' = (1,1) \\ \vec{c}' = \left(1+\dfrac{3\sqrt{2}}{2}, 1+\dfrac{5\sqrt{2}}{2}\right) \end{cases}$

27. Given Δabc where $\vec{a} = (1,0,2), \vec{b} = (-1,3,1), \vec{c} = (5,2,-1)$:
    a. $\Delta_{a'b'c'}, \begin{cases} \vec{a}' = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right) \\ \vec{b}' = \left(-2\sqrt{2}, \sqrt{2}, 1\right) \\ \vec{c}' = \left(\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}, -1\right) \end{cases}$
    b. $\Delta_{a'b'c'}, \begin{cases} \vec{a}' = \left(\dfrac{5\sqrt{2}}{2}-1, 3-\dfrac{\sqrt{2}}{2}, 2\right) \\ \vec{b}' = (-1,3,1) \\ \vec{c}' = \left(\dfrac{7\sqrt{2}}{2}-1, \dfrac{5\sqrt{2}}{2}+3, -1\right) \end{cases}$

28. Scaling transformation matrices:
    a. $[S] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
    b. $[S] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$
    c. $[S] = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$
29. $\vec{p} = (1,-1) \to [W] = \begin{bmatrix} s_x & 0 & -s_x+1 \\ 0 & s_y & s_y-1 \\ 0 & 0 & 1 \end{bmatrix}$

30. $\Delta abc, \begin{cases} \vec{a} = (0,0) \\ \vec{b} = (1,1) \\ \vec{c} = (5,2) \end{cases} \to \Delta a'b'c', \begin{cases} \vec{a} = (-5,-2) \\ \vec{b} = (-3,0) \\ \vec{c} = (5,2) \end{cases}$

31. Question is blank.

32. $\vec{p} = (1,0,1), \theta_Z = 45°, \theta_X = 90° \to \vec{p}' = \left(\sqrt{2},0,0\right)$

33. $\vec{u} = (1,0,1) \to [R_{\hat{u}}(45°)] = \begin{bmatrix} \frac{4+\sqrt{2}}{4} & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\ \frac{1}{2} & \frac{2+\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{2-\sqrt{2}}{4} & \frac{1}{2} & \frac{4+\sqrt{2}}{4} \end{bmatrix}$

34. $\vec{u} = (3,0,4) \to [R_{\hat{u}}(180°)] = \begin{bmatrix} -\frac{7}{25} & 0 & -\frac{24}{25} \\ 0 & \frac{11}{25} & 0 \\ \frac{24}{25} & 0 & \frac{43}{25} \end{bmatrix}$

35. $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Linear Operators

36. Given the following linear transformations:
   a. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x,y) = (x-2y, y, x+3y)$ is a linear operator.
   b. $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x,y,z) = (x-2y-z, y, x+y+z)$ is a linear operator.
   c. $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x,y) = x-2y$ is ***not*** a linear operator.
   d. $T: \mathbb{R}^2 \to \mathbb{R}^3$, such that $T(x,y) = (x-y, y, x+y)$ is ***not*** a linear operator.
   e. $T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 2x$ is a linear operator.

Composition of Linear Operators

37. Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x+y \end{bmatrix}$:
   a. $T_2 \circ T_1 = \begin{bmatrix} 5x+y \\ 2x+y \end{bmatrix}$.
   b. $T_1 \circ T_2 = \begin{bmatrix} 4x+5y \\ x+2y \end{bmatrix}$
   c. $T_2 \circ T_1 \neq T_1 \circ T_2$

38. Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x+y \\ y \end{bmatrix}$:
    a. $T_2 \circ T_1 = \begin{bmatrix} 6x+y \\ x+y \end{bmatrix}$
    b. $T_1 \circ T_2 = \begin{bmatrix} 5x+y \\ 5x+2y \end{bmatrix}$
39. Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -x+y \end{bmatrix}$:
    a. $T_2 \circ T_1 = \begin{bmatrix} 3x+2y \\ -3x+2y \end{bmatrix}$
    b. $T_1 \circ T_2 = \begin{bmatrix} 3x+3y \\ -2x+2y \end{bmatrix}$
40. Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$:
    a. $T_2 \circ T_1 = \begin{bmatrix} x-y \\ 2x-y \end{bmatrix}$
    b. $T_1 \circ T_2 = \begin{bmatrix} x+y \\ -2x-y \end{bmatrix}$

One-To-One Linear Operators

41. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ is one-to-one.
42. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$ is one-to-one.
43. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x+y \end{bmatrix}$ is ***not*** one-to-one.
44. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 6x+3y \end{bmatrix}$ is ***not*** one-to-one.
45. $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y+3z \\ z \\ 2z \end{bmatrix}$ is ***not*** one-to-one.

Inverse of a One-To-One Linear Operator

46. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix} \to T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$.
47. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} \to T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{1}{2}y \end{bmatrix}$.
48. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x+y \end{bmatrix} \to T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ -x+2y \end{bmatrix}$.
49. $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x+2y \end{bmatrix} \to T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-3y \\ -x+2y \end{bmatrix}$

Solutions

Linear Transformation

For each the following transformations, determine if it is a linear transformation:

A transformation is linear if it satisfies two tests:

- $T(u+v) = T(u) + T(v)$
- $T(ku) = kT(u)$

Problem 1

$T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 5x$

Addition test:

Assume $u, v \in \mathbb{R}$.

$T(u) = 5(u)$

$T(u) = 5u$

$T(v) = 5(v)$

$T(v) = 5v$

$T(u) + T(v) = (5u) + (5v)$

$T(u) + T(v) = 5u + 5v$

$T(u+v) = 5(u+v)$

$T(u+v) = 5u + 5v$

$T(u+v) = T(u) + T(v)$

T satisfies the addition test.

Multiplication test:

$v \cdot T(u) = v \cdot (5u)$

$v \cdot T(u) = 5uv$

$T(vu) = 5(vu)$

$T(vu) = 5uv$

$T(vu) = v \cdot T(u)$

T satisfies the multiplication test.

Since T satisfies both tests for linearity, T is a linear transformation.

Problem 2

$T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = xy$

Addition test:

Assume $s, t, u, v \in \mathbb{R}$.

$T(s, t) = (s)(t)$

$T(s, t) = st$

$T(u, v) = (u)(v)$

$T(u, v) = uv$

$T(s, t) + T(u, v) = (st) + (uv)$

$T(s, t) + T(u, v) = st + u5v$

$T(s + u, t + v) = (s + u)(t + v)$

$T(s + u, t + v) = s(t + v) + u(t + v)$

$T(s + u, t + v) = st + sv + ut + uv$

$T(s + u, t + v) \neq T(s, t) + T(u, v)$

T **fails** the addition test.

T is **not** a linear transformation.

Problem 3

$T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = x^3$

Addition test:

Assume $u, v \in \mathbb{R}$.

$T(u) = (u)^3$ $T(v) = (v)^3$

$T(u) = u^3$ $T(v) = v^3$

$T(u) + T(v) = (u^3) + (v^3)$

$T(u) + T(v) = u^3 + v^3$

$T(u + v) = (u + v)^3$

$T(u + v) = (u + v)(u + v)(u + v)$

$T(u + v) = [u(u + v) + v(u + v)](u + v)$

$T(u + v) = (u^2 + uv + uv + v^2)(u + v)$

$T(u + v) = (u^2 + 2uv + v^2)(u + v)$

$T(u + v) = (u^2 + 2uv + v^2)(u) + (u^2 + 2uv + v^2)(v)$

$T(u + v) = u^3 + 2u^2v + uv^2 + u^2v + 2uv^2 + v^3$

$T(u + v) = u^3 + 2u^2v + u^2v + uv^2 + 2uv^2 + v^3$

$T(u + v) = u^3 + 3u^2v + 3uv^2 + v^3$

$T(u + v) \neq T(u) + T(v)$

T **fails** the addition test.

T is **not** a linear transformation.

Problem 4

$T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 0$

Addition test:

Assume $u, v \in \mathbb{R}$.

$T(u) = 0$ $\quad$ $T(v) = 0$

$T(u) + T(v) = (0) + (0)$ $\quad$ $T(u + v) = 0$

$T(u) + T(v) = 0$ $\quad$ $T(u + v) = T(u) + T(v)$

T satisfies the addition test.

Multiplication test:

$v \cdot T(u) = v \cdot (0)$ $\quad$ $T(vu) = 0$

$v \cdot T(u) = 0$ $\quad$ $T(vu) = v \cdot T(u)$

T satisfies the multiplication test.

Since T satisfies both tests for linearity, T is a linear transformation.

Problem 5

$T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x + y$

Addition test:

Assume $s, t, u, v \in \mathbb{R}$.

$T(s,t) = (s) + (t)$

$T(s,t) = s + t$

$T(u,v) = (u) + (v)$

$T(u,v) = u + v$

$T(s,t) + T(u,v) = (s + t) + (u + v)$

$T(s,t) + T(u,v) = s + t + u + v$

$T(s + u, t + v) = (s + u) + (t + v)$

$T(s + u, t + v) = s + u + t + v$

$T(s + u, t + v) = s + t + u + v$

$T(s + u, t + v) = T(s,t) + T(u,v)$

T satisfies the addition test.

Multiplication test:

$v \cdot T(s,t) = v(s + t)$

$v \cdot T(s,t) = sv + tv$

$T(vs, vt) = (vs) + (vt)$

$T(vs, vt) = sv + tv$

$T(vs, vt) = v \cdot T(u)$

T satisfies the multiplication test.

Since T satisfies both tests for linearity, T is a linear transformation.

Problem 6

$T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(\vec{x}) = A\vec{x}$ (A being a 2×2 matrix and $\vec{x}$ being a 2×1 column vector).

Addition test:

Assume $\vec{u}, \vec{v} \in \mathbb{R}^2$.

$$T(\vec{u}) = A \cdot (\vec{u}) \qquad T(\vec{v}) = A \cdot (\vec{v})$$

$$T(\vec{u}) = A \cdot \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad T(\vec{v}) = A \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$T(\vec{u}) = \begin{bmatrix} a_{11}u_x + a_{12}u_y \\ a_{21}u_x + a_{22}u_y \end{bmatrix} \qquad T(\vec{v}) = \begin{bmatrix} a_{11}v_x + a_{12}v_y \\ a_{21}v_x + a_{22}v_y \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} a_{11}u_x + a_{12}u_y \\ a_{21}u_x + a_{22}u_y \end{bmatrix} + \begin{bmatrix} a_{11}v_x + a_{12}v_y \\ a_{21}v_x + a_{22}v_y \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} a_{11}u_x + a_{12}u_y + a_{11}v_x + a_{12}v_y \\ a_{21}u_x + a_{22}u_y + a_{21}v_x + a_{22}v_y \end{bmatrix}$$

$$T(\vec{u} + \vec{v}) = A \cdot (\vec{u} + \vec{v})$$

$$T(\vec{u} + \vec{v}) = A \cdot \begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{bmatrix} a_{11}(u_x + v_x) + a_{12}(u_y + v_y) \\ a_{21}(u_x + v_x) + a_{22}(u_y + v_y) \end{bmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{bmatrix} a_{11}u_x + a_{11}v_x + a_{12}u_y + a_{12}v_y \\ a_{21}u_x + a_{21}v_x + a_{22}u_y + a_{22}v_y \end{bmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{bmatrix} a_{11}u_x + a_{12}u_y + a_{11}v_x + a_{12}v_y \\ a_{21}u_x + a_{22}u_y + a_{21}v_x + a_{22}v_y \end{bmatrix}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

T satisfies the addition test.

Multiplication test:

Assume $k \in \mathbb{R}$.

$$k \cdot T(\vec{u}) = k \cdot \begin{bmatrix} a_{11}u_x + a_{12}u_y \\ a_{21}u_x + a_{22}u_y \end{bmatrix}$$

$$k \cdot T(\vec{u}) = \begin{bmatrix} k\left(a_{11}u_x + a_{12}u_y\right) \\ k\left(a_{21}u_x + a_{22}u_y\right) \end{bmatrix}$$

$$k \cdot T(\vec{u}) = \begin{bmatrix} a_{11}ku_x + a_{12}ku_y \\ a_{21}ku_x + a_{22}ku_y \end{bmatrix}$$

$$T(k \cdot \vec{u}) = A \cdot (k \cdot \vec{u})$$

$$T(k \cdot \vec{u}) = A \cdot \left(k \cdot \begin{bmatrix} u_x \\ u_y \end{bmatrix}\right)$$

$$T(k \cdot \vec{u}) = A \cdot \begin{bmatrix} ku_x \\ ku_y \end{bmatrix}$$

$$T(k \cdot \vec{u}) = \begin{bmatrix} a_{11}ku_x + a_{12}ku_y \\ a_{21}ku_x + a_{22}ku_y \end{bmatrix}$$

$$T(k \cdot \vec{u}) = k \cdot T(\vec{u})$$

T satisfies the multiplication test.

Since T satisfies both tests for linearity, T is a linear transformation.

Linear Transformation Matrix – Standard Matrix, Standard Basis

Problem 7

Find the standard matrix of $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 5y \\ x + 6y \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 5y \\ x + 6y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -5 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{[T] = \begin{bmatrix} 2 & -5 \\ 1 & 6 \end{bmatrix}}$$

Problem 8

Find the standard matrix of $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 10x - 5y \\ y \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 10x - 5y \\ 0x + y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 10 & -5 \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{[T] = \begin{bmatrix} 10 & -5 \\ 0 & 1 \end{bmatrix}}$$

Problem 9

Find the standard matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 5y + z \\ x + 6y - z \\ x + y + z \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & -5 & 1 \\ 1 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{[T] = \begin{bmatrix} 2 & -5 & 1 \\ 1 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix}}$$

Problem 10

Find the standard matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ x+y-z \\ x+y+z \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+0y+z \\ x+y-z \\ x+y+z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{[T] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}}$$

Linear Transformation Matrix – Standard Matrix, Non-Standard Basis

Problem 11

11.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x + y, x - 2y)$, find the matrix of T relative to the bases $B_V = \{\vec{v}_1, \vec{v}_2\} = \{(2,1), (3,2)\}$ and $B_W = \{\vec{w}_1, \vec{w}_2\} = \{(1,1), (1,2)\}$.

Transformation of basis vectors for B_V:

$$T(\vec{v}_1) = T(2,1) \qquad T(\vec{v}_2) = T(3,2)$$
$$T(\vec{v}_1) = \big((2) + (1), (2) - 2(1)\big) \qquad T(\vec{v}_2) = \big((3) + (2), (3) - 2(2)\big)$$
$$T(\vec{v}_1) = (2 + 1, 2 - 2) \qquad T(\vec{v}_2) = (3 + 2, 3 - 4)$$
$$T(\vec{v}_1) = (3,0) \qquad T(\vec{v}_2) = (5, -1)$$

Basis matrices:

$$M_{T(B_V)} = [T(\vec{v}_1) \quad T(\vec{v}_2)] \qquad M_{B_W} = [\vec{w}_1 \quad \vec{w}_2]$$
$$M_{T(B_V)} = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} \qquad M_{B_W} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Augmented matrix:

$$\left[M_{B_W} \middle| M_{T(B_V)}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 1 & 2 & 0 & -1 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 1 & 2 & 0 & -1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 1 - 1 & 2 - 1 & 0 - 3 & -1 - 5 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 0 & 1 & -3 & -6 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 0 & 1 & -3 & -6 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|cc} 1 - 0 & 1 - 1 & 3 - (-3) & 5 - (-6) \\ 0 & 1 & -3 & -6 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & 11 \\ 0 & 1 & -3 & -6 \end{array}\right]$$
$$\left[I \middle| [T]_{B_W, B_V}\right]$$
$$\boxed{[T]_{B_W, B_V} = \begin{bmatrix} 6 & 11 \\ -3 & -6 \end{bmatrix}}$$

11.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x + y, x - 2y)$, $B_V = \{\vec{v}_1, \vec{v}_2\} = \{(2,1), (3,2)\}$, and $B_W = \{\vec{w}_1, \vec{w}_2\} = \{(1,1), (1,2)\}$, calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_V}$.

$$[\vec{u}]_{B_W} = [T]_{B_W,B_V} \cdot [\vec{u}]_{B_V}$$

$$[\vec{u}]_{B_W} = \begin{bmatrix} 6 & 11 \\ -3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$[T]_{B_W,B_V}$ from 11.a, above.

$$[\vec{u}]_{B_W} = \begin{bmatrix} (6)(1) + (11)(-1) \\ (-3)(1) + (-6)(-1) \end{bmatrix}_{B_W}$$

$$[\vec{u}]_{B_W} = \begin{bmatrix} 6 + (-11) \\ -3 + 6 \end{bmatrix}_{B_W}$$

$$\boxed{[\vec{u}]_{B_W} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}_{B_W}}$$

Problem 12

12.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-2y \end{bmatrix}$, find the matrix of T relative to the bases $B_V = \{\vec{v}_1, \vec{v}_2\} = \{(2,1),(3,2)\}$ and $B_W = \{\vec{w}_1, \vec{w}_2\} = \{(1,1),(4,5)\}$.

Basis matrices:

$$M_{T(B_V)} = [T(\vec{v}_1) \quad T(\vec{v}_2)]$$

$$M_{T(B_V)} = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$$

$M_{T(B_V)}$ from in 11.a, above.

$$M_{B_W} = [\vec{w}_1 \quad \vec{w}_2]$$

$$M_{B_W} = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$$

Augmented matrix:

$$\left[M_{B_W} \middle| M_{T(B_V)}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 3 & 5 \\ 1 & 5 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 3 & 5 \\ 1 & 5 & 0 & -1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|cc} 1 & 4 & 3 & 5 \\ 1-1 & 5-4 & 0-3 & -1-5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 3 & 5 \\ 0 & 1 & -3 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 3 & 5 \\ 0 & 1 & -3 & -6 \end{array}\right] \xrightarrow{r_1 - 4r_2} \left[\begin{array}{cc|cc} 1-4(0) & 4-4(1) & 3-4(-3) & 5-4(-6) \\ 0 & 1 & -3 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1-0 & 4-4 & 3+12 & 5+24 \\ 0 & 1 & -3 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 15 & 29 \\ 0 & 1 & -3 & -6 \end{array}\right]$$

$$\left[I \middle| [T]_{B_W,B_V}\right]$$

$$\boxed{[T]_{B_W,B_V} = \begin{bmatrix} 15 & 29 \\ -3 & -6 \end{bmatrix}}$$

12.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-2y \end{bmatrix}$, calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B_V}$.

$[\vec{u}]_{B_W} = [T]_{B_W,B_V} \cdot [\vec{u}]_{B_V}$

$[\vec{u}]_{B_W} = \begin{bmatrix} 15 & 29 \\ -3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B_V}$

$[T]_{B_W,B_V}$ from 12.a, above.

$[\vec{u}]_{B_W} = \begin{bmatrix} (15)(1) + (29)(3) \\ (-3)(1) + (-6)(3) \end{bmatrix}_{B_W}$

$[\vec{u}]_{B_W} = \begin{bmatrix} 15 + 87 \\ -3 + (-18) \end{bmatrix}_{B_W}$

$\boxed{[\vec{u}]_{B_W} = \begin{bmatrix} 102 \\ -21 \end{bmatrix}_{B_W}}$

Problem 13

13.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$, find the matrix of T relative to the bases $B_V = \{\vec{v}_1, \vec{v}_2\} = \{(1,0),(0,2)\}$ and $B_W = \{\vec{w}_1, \vec{w}_2\} = \{(1,2),(2,3)\}$.

Transformation of basis vectors for B_V:

$$T(\vec{v}_1) = T(1,0) \qquad T(\vec{v}_2) = T(0,2)$$

$$T(\vec{v}_1) = \begin{bmatrix} (1)-(0) \\ (1)+(0) \end{bmatrix} \qquad T(\vec{v}_2) = \begin{bmatrix} (0)-(2) \\ (0)+(2) \end{bmatrix}$$

$$T(\vec{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad T(\vec{v}_2) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Basis matrices:

$$M_{T(B_V)} = [T(\vec{v}_1) \quad T(\vec{v}_2)] \qquad M_{B_W} = [\vec{w}_1 \quad \vec{w}_2]$$

$$M_{T(B_V)} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \qquad M_{B_W} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Augmented matrix:

$$\left[M_{B_W} \middle| M_{T(B_V)}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 2 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 2-2(1) & 3-2(2) & 1-2(1) & 2-2(-2) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 2-2 & 3-4 & 1-2 & 2+4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 6 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ -(0) & -(-1) & -(-1) & -(6) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & 1 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & -2 \\ 0 & 1 & 1 & -6 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cc|cc} 1-2(0) & 2-2(1) & 1-2(1) & -2-2(-6) \\ 0 & 1 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1-0 & 2-2 & 1-2 & -2+12 \\ 0 & 1 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 10 \\ 0 & 1 & 1 & -6 \end{array}\right]$$

$$\left[I \middle| [T]_{B_W,B_V}\right]$$

$$\boxed{[T]_{B_W,B_V} = \begin{bmatrix} -1 & 10 \\ 1 & -6 \end{bmatrix}}$$

13.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$, calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$[\vec{u}]_{B_W} = [T]_{B_W,B_V} \cdot [\vec{u}]_{B_V}$$

$$[\vec{u}]_{B_W} = \begin{bmatrix} -1 & 10 \\ 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{B_V}$$

$[T]_{B_W,B_V}$ from 13.a, above.

$$[\vec{u}]_{B_W} = \begin{bmatrix} (-1)(1) + (10)(3) \\ (1)(1) + (-6)(3) \end{bmatrix}_{B_W}$$

$$[\vec{u}]_{B_W} = \begin{bmatrix} -1 + 30 \\ 1 + (-18) \end{bmatrix}_{B_W}$$

$$\boxed{[\vec{u}]_{B_W} = \begin{bmatrix} 29 \\ -17 \end{bmatrix}_{B_W}}$$

Problem 14

Given $T: \mathbb{R}^2 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (x, x+y, y)$ and the bases $B_V = \{\vec{v}_1, \vec{v}_2\} = \{(1,2),(1,1)\}$ for $\mathbb{R}^2$ and $B_W = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,2,1),(0,1,0),(2,0,3)\}$ for $\mathbb{R}^3$, find the matrix of T relative to B_V and B_W.

Transformation of basis vectors for B_V:

$$T(\vec{v}_1) = T(1,2)$$
$$T(\vec{v}_1) = \big((1),(1)+(2),(2)\big)$$
$$T(\vec{v}_1) = (1,3,2)$$
$$T(\vec{v}_2) = T(1,1)$$
$$T(\vec{v}_2) = \big((1),(1)+(1),(1)\big)$$
$$T(\vec{v}_2) = (1,2,1)$$

Basis matrices:

$$M_{T(B_V)} = [T(\vec{v}_1) \quad T(\vec{v}_2)] \qquad M_{B_W} = [\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3]$$

$$M_{T(B_V)} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 1 \end{bmatrix} \qquad M_{B_W} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Augmented matrix:

$$\left[M_{B_W} \middle| M_{T(B_V)}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 3 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 3 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 & 3 \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 3-3(1) & 2-3(1) & 2-3(1) & 1-3(0) & 0-3(2) \\ 2-2(1) & 1-2(1) & 1-2(1) & 0-2(0) & 3-2(2) \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 3-3 & 2-3 & 2-3 & 1-0 & 0-6 \\ 2-2 & 1-2 & 1-2 & 0-0 & 3-4 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & -6 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & -6 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 2 \\ -(0) & -(-1) & -(-1) & -(1) & -(-6) \\ 0 & -1 & -1 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc}1 & 1 & 1 & 0 & 2\\0 & 1 & 1 & -1 & 6\\0 & -1 & -1 & 0 & -1\end{array}\right]$$

$$\left[\begin{array}{cc|ccc}1 & 1 & 1 & 0 & 2\\0 & 1 & 1 & -1 & 6\\0 & -1 & -1 & 0 & -1\end{array}\right] \xrightarrow{r_3+r_2} \left[\begin{array}{cc|ccc}1 & 1 & 1 & 0 & 2\\0 & 1 & 1 & -1 & 6\\0+0 & -1+1 & -1+1 & 0+(-1) & -1+6\end{array}\right]$$

$$\left[\begin{array}{cc|ccc}1 & 1 & 1 & 0 & 2\\0 & 1 & 1 & -1 & 6\\0 & 0 & 0 & -1 & 5\end{array}\right]$$

$$\left[I \middle| [T]_{B_W,B_V}\right]$$

$$\boxed{[T]_{B_W,B_V} = \begin{bmatrix}1 & 0 & 2\\1 & -1 & 6\\0 & -1 & 5\end{bmatrix}}$$

Kernel of Linear Transformation

Problem 15

15.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x - y, 2x + y)$, find $\mathrm{Ker}(T)$.

$$T(x,y) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(x,y) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$[T] \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 - 2(1) & 1 - 2(-1) & 0 - 2(0) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 - 2 & 1 + 2 & 0 - 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right] \xrightarrow{r_2/3} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ {}^{0}\!/_{3} & {}^{3}\!/_{3} & {}^{0}\!/_{3} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{cc|c} 1 + 0 & -1 + 1 & 0 + 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} v_x = 0 \\ v_y = 0 \end{cases} \to \vec{v} = \vec{0}$$

$$\mathrm{Ker}([T]) = \vec{0}$$

$$\boxed{\mathrm{Ker}(T) = \vec{0}}$$

15.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x - y, 2x + y)$, find $\dim\big(\mathrm{Ker}(T)\big)$.

$\mathrm{Ker}(T)$ is a single vector (see 15.a, above). Countable sets have no dimension, so $\boxed{\dim\big(\mathrm{Ker}(T)\big) = 0}$.

Problem 16

16.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x,y) = (x+y, x)$, find $\mathrm{Ker}(T)$.

$$T(x,y) = (x+y, x+0y)$$

$$T(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(x,y) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[T] \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1-1 & 0-1 & 0-0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{ccc} 1 & 1 & 0 \\ -(0) & -1 & -(0) \end{array}\right.$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cc|c} 1-0 & 1-1 & 0-0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} v_x = 0 \\ v_y = 0 \end{cases} \to \vec{v} = \vec{0}$$

$$\mathrm{Ker}([T]) = \vec{0}$$

$$\boxed{\mathrm{Ker}(T) = \vec{0}}$$

16.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x,y) = (x+y, x)$, find $\dim\big(\mathrm{Ker}(T)\big)$.

$\mathrm{Ker}(T)$ is a single vector (see 16.a, above). Countable sets have no dimension, so $\boxed{\dim\big(\mathrm{Ker}(T)\big) = 0}$.

Problem 17

17.a Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x - y$, find $\text{Ker}(T)$.

$$T = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$[T] \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} = 0$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \end{array}\right]$$

v_y is a free variable. Set $v_y = t, t \in \mathbb{R}$.

$$v_x - v_y = 0$$

$$v_x - t = 0$$

$$v_x = t$$

$$\vec{v} = (t, t) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\text{Ker}([T]) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\boxed{\text{Ker}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$$

17.b Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x - y$, find $\dim\big(\text{Ker}(T)\big)$.

The basis of $\text{Ker}(T)$ contains one non-zero vector (see 17.a, above), so $\boxed{\dim\big(\text{Ker}(T)\big) = 1}$.

Problem 18

18.a Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x, y, z) = (x - y + z, x + 4y + 1, y)$, find $\text{Ker}(T)$.

$$T(x,y,z) = \begin{bmatrix} x - y + z + 0 \\ x + 4y + 0z + 1 \\ 0x + y + 0z + 0 \end{bmatrix}$$

$$T(x,y,z) = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$T(x,y,z) = [T] \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[T] \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 1-1 & 4-(-1) & 0-1 & 1-0 & 0-0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 + r_3} \\ \xrightarrow{r_2 - 5r_3} \end{array} \left[\begin{array}{cccc|c} 1+0 & -1+1 & 1+0 & 0+0 & 0+0 \\ 0-5(0) & 5-5(1) & -1-5(0) & 1-5(0) & 0-5(0) \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1+0 & -1+1 & 1+0 & 0+0 & 0+0 \\ 0-0 & 5-5 & -1-0 & 1-0 & 0-0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{-r_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -(0) & -(0) & -(-1) & -(1) & -(0) \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{r_1 - r_3} \left[\begin{array}{cccc|c} 1-0 & 0-0 & 1-1 & 0-(-1) & 0-0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{cases} v_x + 1 = 0 \\ v_y = 0 \\ v_z - 1 = 0 \end{cases} \rightarrow \begin{cases} v_x = -1 \\ v_y = 0 \\ v_z = 1 \end{cases} \rightarrow \vec{v} = (-1,0,1)$$

$$\boxed{\operatorname{Ker}([T]) = \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

18.b Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x,y,z) = (x - y + z, x + 4y + 1, y)$, find $\dim\big(\operatorname{Ker}(T)\big)$.

$\operatorname{Ker}(T)$ is a single vector (see 18.a, above). Countable sets have no dimension, so $\boxed{\dim\big(\operatorname{Ker}(T)\big) = 0}$.

Range/Image of a Linear Transformation

The image of a transformation is the span of the transformation's standard matrix's column space.

Problem 19

19.a Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ -6x + 3y \end{bmatrix}$, find $\operatorname{Im}(T)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \xrightarrow{r_2 + 3r_1} \begin{bmatrix} 2 & -1 \\ -6 + 3(2) & 3 + 3(-1) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -6 + 6 & 3 + (-3) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_1/2} \begin{bmatrix} 2/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Reduced row-echelon form.

The RREF has only one pivot, in the first column. Therefore, a basis of $[T]$'s column space is the first column in $[T]$.

$$\operatorname{colsp}([T]) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -6 \end{bmatrix}\right\}$$

$$\boxed{\operatorname{Im}(T) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -6 \end{bmatrix}\right\}}$$

19.b Given $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ -6x + 3y \end{bmatrix}$, find $\dim(\operatorname{Im}(T))$.

The basis of $\operatorname{Im}(T)$ has one vector (see 19.a, above), so $\boxed{\dim(\operatorname{Im}(T)) = 1}$.

Problem 20

20.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+6y \end{bmatrix}$, find $\mathrm{Im}(T)$.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$[T] = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 \\ 2-2(1) & 6-2(1) \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 2-2 & 6-2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \xrightarrow{r_2/4} \begin{bmatrix} 1 & 1 \\ 0/4 & 4/4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1-0 & 1-1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Reduced row-echelon form.

The RREF has two pivots, one in each column. Therefore, a basis of $[T]$'s column space is the first two columns of $[T]$.

$\mathrm{colsp}([T]) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}\right\}$

$\boxed{\mathrm{Im}(T) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}\right\}}$

20.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+6y \end{bmatrix}$, find $\dim(\mathrm{Im}(T))$.

The basis of $\mathrm{Im}(T)$ contains two vectors (see 20.a, above), so $\boxed{\dim(\mathrm{Im}(T)) = 2}$.

Problem 21

21.a Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ x + 4y \end{bmatrix}$, find $\operatorname{Im}(T)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 3 \\ 1-1 & 4-3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \begin{bmatrix} 1-3(0) & 3-3(1) \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-0 & 3-3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reduced row-echelon form.

The RREF has two pivots, one in each column. Therefore, a basis of $[T]$'s column space is the first two columns of $[T]$.

$$\operatorname{colsp}([T]) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$$

$$\boxed{\operatorname{Im}(T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}}$$

21.b Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ x + 4y \end{bmatrix}$, find $\dim(\operatorname{Im}(T))$.

The basis of $\operatorname{Im}(T)$ contains two vectors, so $\boxed{\dim(\operatorname{Im}(T)) = 2}$.

Problem 22

22.a Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ x + 4y + z \\ x + 3y + z \end{bmatrix}$, find $\mathrm{Im}(T)$.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix} \underset{\xrightarrow{r_3 - r_1}}{\xrightarrow{r_2 - r_1}} \begin{bmatrix} 1 & -2 & 1 \\ 1-1 & 4-(-2) & 1-1 \\ 1-1 & 3-(-2) & 1-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 6 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 6 & 0 \\ 0 & 5 & 0 \end{bmatrix} \xrightarrow{r_2/6} \begin{bmatrix} 1 & -2 & 1 \\ 0/6 & 6/6 & 0/6 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \underset{\xrightarrow{r_3 - 5(r_2)}}{\xrightarrow{r_1 + 2(r_2)}} \begin{bmatrix} 1+2(0) & -2+2(1) & 1+2(0) \\ 0 & 1 & 0 \\ 0-5(0) & 5-5(1) & 0-5(0) \end{bmatrix}$$

$$\begin{bmatrix} 1+0 & -2+2 & 1+0 \\ 0 & 1 & 0 \\ 0-0 & 5-5 & 0-0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced row-echelon form.

The RREF has pivots in the first two columns, so a basis of $[T]$'s column space is composed of the first two columns of $[T]$.

$$\mathrm{colsp}([T]) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}\right\}$$

$$\boxed{\mathrm{Im}(T) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}\right\}}$$

22.b Given $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ x + 4y + z \\ x + 3y + z \end{bmatrix}$, find $\dim(\mathrm{Im}(T))$.

The basis of $Im(T)$ contains two vectors (see 21.a, above), so $\boxed{\dim(\mathrm{Im}(T)) = 2}$.

2D/3D Geometric Transformation

Problem 23

23.a Given point $\vec{p} = (1,1,-1)$, calculate the image of $\vec{p}$ after a translation by vector $\vec{v} = (3,2,4)$.

$$\vec{p}'_{\mathrm{a}} = T_{\vec{v}} \cdot \vec{p}$$

$$\vec{p}'_{\mathrm{a}} = \begin{bmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$\vec{p}'_{\mathrm{a}} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{p}'_{\mathrm{a}} = \begin{bmatrix} (1)(1) + (0)(1) + (0)(-1) + (3)(1) \\ (0)(1) + (1)(1) + (0)(-1) + (2)(1) \\ (0)(1) + (0)(1) + (1)(-1) + (4)(1) \\ (0)(1) + (0)(1) + (0)(-1) + (1)(1) \end{bmatrix}$$

$$\vec{p}'_{\mathrm{a}} = \begin{bmatrix} 1 + 0 + 0 + 3 \\ 0 + 1 + 0 + 2 \\ 0 + 0 + (-1) + 4 \\ 0 + 0 + 0 + 1 \end{bmatrix}$$

$$\vec{p}'_{a} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{p}'_{\mathrm{a}} = (4,3,3)}$$

23.b Find the rotation matrix $R_y\left(\frac{\pi}{4}\right)$ and its inverse.

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_y\left(\frac{\pi}{4}\right) = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & 0 & \sin\left(\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ -\sin\left(\frac{\pi}{4}\right) & 0 & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$$

$$\boxed{R_y\left(\frac{\pi}{4}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}}$$

$$R_y^{-1}(\theta) = R_y^{t}(\theta)$$

$$R_y^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_y^{-1}\left(\frac{\pi}{4}\right) = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & 0 & -\sin\left(\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{\pi}{4}\right) & 0 & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$$

$$\boxed{R_y^{-1}\left(\frac{\pi}{4}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}}$$

23.c Given point $\vec{p} = (1,1,-1)$, calculate the image of $\vec{p}$ after a $45°$ rotation about the y-axis.

$$R_y(45°) = R_y\left(\frac{\pi}{4}\right)$$

$$R_y(45°) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Found in 23.b, above.

$$\vec{p}_c' = R_y(45°) \cdot \vec{p}$$

$$\vec{p}_c' = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{p}_c' = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + (0)(1) + \left(\frac{\sqrt{2}}{2}\right)(-1) \\ (0)(1) + (1)(1) + (0)(-1) \\ \left(-\frac{\sqrt{2}}{2}\right)(1) + (0)(1) + \left(\frac{\sqrt{2}}{2}\right)(-1) \end{bmatrix}$$

$$\vec{p}_c' = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + \left(-\frac{\sqrt{2}}{2}\right) \\ 0 + 1 + 0 \\ -\frac{\sqrt{2}}{2} + 0 + \left(-\frac{\sqrt{2}}{2}\right) \end{bmatrix}$$

$$\vec{p}_c' = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

$$\boxed{\vec{p}_c' = \left(0,1,-\sqrt{2}\right)}$$

23.d Given point $\vec{p} = (1,1,-1)$, calculate the image of $\vec{p}$ after a scaling transformation where $s_x = 2, s_y = 10, s_z = 5$.

$$\vec{p}'_{\mathrm{d}} = S_{s_x,s_y,s_z} \cdot \vec{p}$$

$$\vec{p}'_{\mathrm{d}} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\vec{p}'_{\mathrm{d}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} (2)(1) + (0)(1) + (0)(-1) \\ (0)(1) + (10)(1) + (0)(-1) \\ (0)(1) + (0)(1) + (5)(-1) \end{bmatrix}$$

$$\vec{p}'_{\mathrm{d}} = \begin{bmatrix} 2 + 0 + 0 \\ 0 + 10 + 0 \\ 0 + 0 + (-5) \end{bmatrix}$$

$$\vec{p}'_{\mathrm{d}} = \begin{bmatrix} 2 \\ 10 \\ -5 \end{bmatrix}$$

$$\boxed{\vec{p}'_{\mathrm{d}} = (2,10,-5)}$$

Problem 24

24.a Given point $\vec{p} = (2, -4)$, find the transformation that represents an object's rotation of $30°$ about the origin.

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z(30°) = \begin{bmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{R_z(30°) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

24.b Given point $\vec{p} = (2, -4)$, what are the new coordinates of the point $\vec{p}$ after this rotation?

$$\vec{p}' = R_z(30°) \cdot \vec{p}$$

Assume a z-value for point $\vec{p}$.

$$\vec{p} = (2, -4, 0)$$

$$\vec{p}' = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

$R_z(30°)$ found in 24.a, above.

$$\vec{p}' = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right)(2) + \left(-\frac{1}{2}\right)(-4) + (0)(0) \\ \left(\frac{1}{2}\right)(2) + \left(\frac{\sqrt{3}}{2}\right)(-4) + (0)(0) \\ (0)(2) + (0)(-4) + (1)(0) \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} \sqrt{3} + 2 + 0 \\ 1 + \left(-2\sqrt{3}\right) + 0 \\ 0 + 0 + 0 \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} 2 + \sqrt{3} \\ 1 - 2\sqrt{3} \\ 0 \end{bmatrix}$$

$$\boxed{\vec{p}' = \left(2 + \sqrt{3}, 1 - 2\sqrt{3}, 0\right)}$$

Problem 25

Write the transformation matrix that rotates an object $60°$ about a fixed center of rotation $\vec{p} = (-1,2)$.

We will require a translation transformation to shift the center of rotation to the origin, then a rotation transformation, and then an inverse translation transformation to return the center of rotation to its original position.

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & -(-1) \\ 0 & 1 & -(2) \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_z(60°)] = \begin{bmatrix} \cos(60°) & -\sin(60°) & 0 \\ \sin(60°) & \cos(60°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_z(60°)] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{p}}] = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{p}}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = [T_{\vec{p}}] \cdot [R_z(60°)] \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W]=\begin{bmatrix}(1)\left(\frac{1}{2}\right)+(0)\left(\frac{\sqrt{3}}{2}\right)+(-1)(0) & (1)\left(-\frac{\sqrt{3}}{2}\right)+(0)\left(\frac{1}{2}\right)+(-1)(0) & (1)(0)+(0)(0)+(-1)(1)\\ (0)\left(\frac{1}{2}\right)+(1)\left(\frac{\sqrt{3}}{2}\right)+(2)(0) & (0)\left(-\frac{\sqrt{3}}{2}\right)+(1)\left(\frac{1}{2}\right)+(2)(0) & (0)(0)+(1)(0)+(2)(1)\\ (0)\left(\frac{1}{2}\right)+(0)\left(\frac{\sqrt{3}}{2}\right)+(1)(0) & (0)\left(-\frac{\sqrt{3}}{2}\right)+(0)\left(\frac{1}{2}\right)+(1)(0) & (0)(0)+(0)(0)+(1)(1)\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

$$[W]=\begin{bmatrix}\frac{1}{2}+0+0 & -\frac{\sqrt{3}}{2}+0+0 & 0+0+(-1)\\ 0+\frac{\sqrt{3}}{2}+0 & 0+\frac{1}{2}+0 & 0+0+2\\ 0+0+0 & 0+0+0 & 0+0+1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

$$[W]=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1\\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

$$[W]=\begin{bmatrix}\left(\frac{1}{2}\right)(1)+\left(-\frac{\sqrt{3}}{2}\right)(0)+(-1)(0) & \left(\frac{1}{2}\right)(0)+\left(-\frac{\sqrt{3}}{2}\right)(1)+(-1)(0) & \left(\frac{1}{2}\right)(1)+\left(-\frac{\sqrt{3}}{2}\right)(-2)+(-1)(1)\\ \left(\frac{\sqrt{3}}{2}\right)(1)+\left(\frac{1}{2}\right)(0)+(2)(0) & \left(\frac{\sqrt{3}}{2}\right)(0)+\left(\frac{1}{2}\right)(1)+(2)(0) & \left(\frac{\sqrt{3}}{2}\right)(1)+\left(\frac{1}{2}\right)(-2)+(2)(1)\\ (0)(1)+(0)(0)+(1)(0) & (0)(0)+(0)(1)+(1)(0) & (0)(1)+(0)(-2)+(1)(1)\end{bmatrix}$$

$$[W]=\begin{bmatrix}\frac{1}{2}+0+0 & 0+\left(-\frac{\sqrt{3}}{2}\right)+0 & \frac{1}{2}+\sqrt{3}+(-1)\\ \frac{\sqrt{3}}{2}+0+0 & 0+\frac{1}{2}+0 & \frac{\sqrt{3}}{2}+(-1)+2\\ 0+0+0 & 0+0+0 & 0+0+1\end{bmatrix}$$

$$\boxed{[W]=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2}+\sqrt{3}\\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1+\frac{\sqrt{3}}{2}\\ 0 & 0 & 1\end{bmatrix}}$$

Problem 26

26.a Perform a $45°$ rotation of triangle Δabc where $\vec{a} = (0,0), \vec{b} = (1,1), \vec{c} = (5,2)$ about the origin.

$$[R(45°)] = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix}$$

$$[R(45°)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [R(45°)] \cdot [\Delta_{abc}]$$

$$[\Delta_{a'b'c'}] = [R(45°)] \cdot [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(0) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(-\frac{\sqrt{2}}{2}\right)(2) \\ \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(\frac{\sqrt{2}}{2}\right)(2) \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} 0+0 & \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) & \frac{5\sqrt{2}}{2} + \left(-\sqrt{2}\right) \\ 0+0 & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} + \sqrt{2} \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} 0 & 0 & \frac{5\sqrt{2}}{2} + \left(-\frac{2\sqrt{2}}{2}\right) \\ 0 & \sqrt{2} & \frac{5\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [\vec{a}' \quad \vec{b}' \quad \vec{c}']$$

$$\boxed{\Delta a'b'c', \begin{cases} \vec{a}' = (0,0) \\ \vec{b}' = \left(0, \sqrt{2}\right) \\ \vec{c}' = \left(3\sqrt{2}, 7\sqrt{2}\right) \end{cases}}$$

26.b Perform a $45°$ rotation of triangle Δabc where $\vec{a} = (0,0), \vec{b} = (1,1), \vec{c} = (5,2)$ about $\vec{d} = (1,1)$.

$$[R_z(45°)] = \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_z(45°)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{d}}] = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{d}}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{d}}] = \begin{bmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{d}}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = [T_{\vec{d}}] \cdot [R_z(45°)] \cdot [T_{-\vec{d}}]$$

$$[W] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{d}}]$$

$$[W] = \begin{bmatrix} (1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (1)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (1)(0) + (0)(0) + (1)(1) \\ (0)\left(\frac{\sqrt{2}}{2}\right) + (1)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (0)\left(-\frac{\sqrt{2}}{2}\right) + (1)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (0)(0) + (1)(0) + (1)(1) \\ (0)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (0)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) & (0)(0) + (0)(0) + (1)(1) \end{bmatrix} \cdot [T_{-\vec{d}}]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 & -\frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 1 \\ 0 + \frac{\sqrt{2}}{2} + 0 & 0 + \frac{\sqrt{2}}{2} + 0 & 0 + 0 + 1 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix} \cdot [T_{-\vec{d}}]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{d}}]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (1)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(1) + (1)(0) & \left(\frac{\sqrt{2}}{2}\right)(-1) + \left(-\frac{\sqrt{2}}{2}\right)(-1) + (1)(1) \\ \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(0) + (1)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(1) + (1)(0) & \left(\frac{\sqrt{2}}{2}\right)(-1) + \left(\frac{\sqrt{2}}{2}\right)(-1) + (1)(1) \\ (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(1) + (1)(0) & (0)(-1) + (0)(-1) + (1)(1) \end{bmatrix}$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 & 0 + \left(-\frac{\sqrt{2}}{2}\right) + 0 & \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + 1 \\ \frac{\sqrt{2}}{2} + 0 + 0 & 0 + \frac{\sqrt{2}}{2} + 0 & -\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) + 1 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{2} + 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [W] \cdot [\Delta_{abc}]$$

$$[\Delta_{a'b'c'}] = [W] \cdot [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{2} + 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (1)(1) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(1) + (1)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(-\frac{\sqrt{2}}{2}\right)(2) + (1)(1) \\ \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\sqrt{2}+1\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\sqrt{2}+1\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(\frac{\sqrt{2}}{2}\right)(2) + \left(-\sqrt{2}+1\right)(1) \\ (0)(0) + (0)(0) + (1)(1) & (0)(1) + (0)(1) + (1)(1) & (0)(5) + (0)(2) + (1)(1) \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} 0+0+1 & \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) + 1 & \frac{5\sqrt{2}}{2} + \left(-\frac{2\sqrt{2}}{2}\right) + 1 \\ 0+0+\left(1-\sqrt{2}\right) & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \left(1-\sqrt{2}\right) & \frac{5\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} + \left(1-\sqrt{2}\right) \\ 0+0+1 & 0+0+1 & 0+0+1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} 1 & 1 & 1 + \frac{3\sqrt{2}}{2} \\ 1-\sqrt{2} & 1 & 1 + \frac{5\sqrt{2}}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [\vec{a}' \quad \vec{b}' \quad \vec{c}']$$

$$\boxed{\Delta a'b'c', \begin{cases} \vec{a}' = \left(1, 1-\sqrt{2}\right) \\ \vec{b}' = (1,1) \\ \vec{c}' = \left(1 + \frac{3\sqrt{2}}{2}, 1 + \frac{5\sqrt{2}}{2}\right) \end{cases}}$$

Problem 27

27.a Perform a $45°$ rotation of triangle Δabc where $\vec{a} = (1,0,2), \vec{b} = (-1,3,1)$, and $\vec{c} = (5,2,-1)$ about the z-axis.

$$[R_z(45°)] = \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_z(45°)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [R_z(45°)] \cdot [\Delta_{abc}]$$

$$[\Delta_{a'b'c'}] = [R_z(45°)] \cdot [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (0)(2) & \left(\frac{\sqrt{2}}{2}\right)(-1) + \left(-\frac{\sqrt{2}}{2}\right)(3) + (0)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(-\frac{\sqrt{2}}{2}\right)(2) + (0)(-1) \\ \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(0) + (0)(2) & \left(\frac{\sqrt{2}}{2}\right)(-1) + \left(\frac{\sqrt{2}}{2}\right)(3) + (0)(1) & \left(\frac{\sqrt{2}}{2}\right)(5) + \left(\frac{\sqrt{2}}{2}\right)(2) + (0)(-1) \\ (0)(1) + (0)(0) + (1)(2) & (0)(-1) + (0)(3) + (1)(1) & (0)(5) + (0)(2) + (1)(-1) \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 & -\frac{\sqrt{2}}{2} + \left(-\frac{3\sqrt{2}}{2}\right) + 0 & \frac{5\sqrt{2}}{2} + \left(-\sqrt{2}\right) + 0 \\ \frac{\sqrt{2}}{2} + 0 + 0 & -\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + 0 & \frac{5\sqrt{2}}{2} + \sqrt{2} + 0 \\ 0 + 0 + 2 & 0 + 0 + 1 & 0 + 0 + (-1) \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{4\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{2\sqrt{2}}{2} & \frac{7\sqrt{2}}{2} \\ 2 & 1 & -1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -2\sqrt{2} & \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 2 & 1 & -1 \end{bmatrix}$$

$$[\Delta_{a'b'c'}] = [\vec{a}' \quad \vec{b}' \quad \vec{c}']$$

$$\boxed{\Delta_{a'b'c'}, \begin{cases} \vec{a}' = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right) \\ \vec{b}' = \left(-2\sqrt{2}, \sqrt{2}, 1\right) \\ \vec{c}' = \left(\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}, -1\right) \end{cases}}$$

27.b Perform a $45°$ rotation of triangle Δabc where $\vec{a} = (1,0,2), \vec{b} = (-1,3,1)$, and $\vec{c} = (5,2,-1)$ about the z-axis by keeping $\vec{d} = (-1,3,1)$ fixed.

$$[R_z(45°)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{d}}] = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{d}}] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{d}}] = \begin{bmatrix} 1 & 0 & 0 & -d_x \\ 0 & 1 & 0 & -d_y \\ 0 & 0 & 1 & -d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{d}}] = \begin{bmatrix} 1 & 0 & 0 & -(-1) \\ 0 & 1 & 0 & -(3) \\ 0 & 0 & 1 & -(1) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{d}}] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[W] = \left[T_{\vec{d}}\right] \cdot [R_z(45°)] \cdot \left[T_{-\vec{d}}\right]$$

$$[W] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \left[T_{-\vec{d}}\right]$$

$$[W] = \begin{bmatrix} (1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (-1)(0) & (1)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (-1)(0) & (1)(0) + (0)(0) + (0)(1) + (-1)(0) & (1)(0) + (0)(0) + (0)(0) + (-1)(1) \\ (0)\left(\frac{\sqrt{2}}{2}\right) + (1)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (3)(0) & (0)\left(-\frac{\sqrt{2}}{2}\right) + (1)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (3)(0) & (0)(0) + (1)(0) + (0)(1) + (3)(0) & (0)(0) + (1)(0) + (0)(0) + (3)(1) \\ (0)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) + (1)(0) & (0)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (1)(0) + (1)(0) & (0)(0) + (0)(0) + (1)(1) + (1)(0) & (0)(0) + (0)(0) + (1)(0) + (1)(1) \\ (0)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (1)(0) & (0)\left(-\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + (0)(0) + (1)(0) & (0)(0) + (0)(0) + (0)(1) + (1)(0) & (0)(0) + (0)(0) + (0)(0) + (1)(1) \end{bmatrix} \cdot \left[T_{-\vec{d}}\right]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 + 0 & -\frac{\sqrt{2}}{2} + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + (-1) \\ 0 + \frac{\sqrt{2}}{2} + 0 + 0 & 0 + \frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 3 \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 1 + 0 & 0 + 0 + 0 + 1 \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 1 \end{bmatrix} \cdot \left[T_{-\vec{d}}\right]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \left[T_{-\vec{d}}\right]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (0)(0) + (-1)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(1) + (0)(0) + (-1)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (0)(1) + (-1)(0) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(-3) + (0)(-1) + (-1)(1) \\ \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(0) + (0)(0) + (3)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(1) + (0)(0) + (3)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) + (0)(1) + (3)(0) & \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(-3) + (0)(-1) + (3)(1) \\ (0)(1) + (0)(0) + (1)(0) + (1)(0) & (0)(0) + (0)(1) + (1)(0) + (1)(0) & (0)(0) + (0)(0) + (1)(1) + (1)(0) & (0)(1) + (0)(-3) + (1)(-1) + (1)(1) \\ (0)(0) + (0)(0) + (0)(0) + (0)(1) & (0)(0) + (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(0) + (0)(1) + (1)(0) & (0)(1) + (0)(-3) + (0)(-1) + (1)(1) \end{bmatrix}$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 + 0 & 0 - \frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 0 + 0 & \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + 0 + (-1) \\ \frac{\sqrt{2}}{2} + 0 + 0 + 0 & 0 + \frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 0 + 0 & \frac{\sqrt{2}}{2} + \left(-\frac{3\sqrt{2}}{2}\right) + 0 + 3 \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 1 + 0 & 0 + 0 + (-1) + 1 \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 1 \end{bmatrix}$$

$$[W]=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{4\sqrt{2}}{2}-1\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{2\sqrt{2}}{2}+3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[W]=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2}-1\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 3-\sqrt{2}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[\Delta_{a'b'c'}]=[W]\cdot[\Delta_{abc}]$$

$$[\Delta_{a'b'c'}]=[W]\cdot[\vec{a}\quad\vec{b}\quad\vec{c}]$$

$$[\Delta_{a'b'c'}]=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2}-1\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 3-\sqrt{2}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & -1 & 5\\ 0 & 3 & 2\\ 2 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}$$

$$[\Delta_{a'b'c'}]=\begin{bmatrix}\left(\frac{\sqrt{2}}{2}\right)(1)+\left(-\frac{\sqrt{2}}{2}\right)(0)+(0)(2)+\left(2\sqrt{2}-1\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(-1)+\left(-\frac{\sqrt{2}}{2}\right)(3)+(0)(1)+\left(2\sqrt{2}-1\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(5)+\left(-\frac{\sqrt{2}}{2}\right)(2)+(0)(-1)+\left(2\sqrt{2}-1\right)(1)\\ \left(\frac{\sqrt{2}}{2}\right)(1)+\left(\frac{\sqrt{2}}{2}\right)(0)+(0)(2)+\left(3-\sqrt{2}\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(-1)+\left(\frac{\sqrt{2}}{2}\right)(3)+(0)(1)+\left(3-\sqrt{2}\right)(1) & \left(\frac{\sqrt{2}}{2}\right)(5)+\left(\frac{\sqrt{2}}{2}\right)(2)+(0)(-1)+\left(3-\sqrt{2}\right)(1)\\ (0)(1)+(0)(0)+(1)(2)+(0)(1) & (0)(-1)+(0)(3)+(1)(1)+(0)(1) & (0)(5)+(0)(2)+(1)(-1)+(0)(1)\\ (0)(1)+(0)(0)+(0)(2)+(1)(1) & (0)(-1)+(0)(3)+(0)(1)+(1)(1) & (0)(5)+(0)(2)+(0)(-1)+(1)(1)\end{bmatrix}$$

$$[\Delta_{a'b'c'}]=\begin{bmatrix}\frac{\sqrt{2}}{2}+0+0+\left(2\sqrt{2}-1\right) & -\frac{\sqrt{2}}{2}+\left(-\frac{3\sqrt{2}}{2}\right)+0+\left(2\sqrt{2}-1\right) & \frac{5\sqrt{2}}{2}+\left(-\frac{2\sqrt{2}}{2}\right)+0+\left(2\sqrt{2}-1\right)\\ \frac{\sqrt{2}}{2}+0+0+\left(3-\sqrt{2}\right) & -\frac{\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}+0+\left(3-\sqrt{2}\right) & \frac{5\sqrt{2}}{2}+\frac{2\sqrt{2}}{2}+0+\left(3-\sqrt{2}\right)\\ 0+0+2+0 & 0+0+1+0 & 0+0+(-1)+0\\ 0+0+0+1 & 0+0+0+1 & 0+0+0+1\end{bmatrix}$$

$$[\Delta_{a'b'c'}]=\begin{bmatrix}\frac{5\sqrt{2}}{2}-1 & -1 & \frac{7\sqrt{2}}{2}-1\\ 3-\frac{\sqrt{2}}{2} & 3 & \frac{5\sqrt{2}}{2}+3\\ 2 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}$$

$$\boxed{\Delta_{a'b'c'},\begin{cases}\vec{a}'=\left(\frac{5\sqrt{2}}{2}-1, 3-\frac{\sqrt{2}}{2}, 2\right)\\ \vec{b}'=(-1,3,1)\\ \vec{c}'=\left(\frac{7\sqrt{2}}{2}-1, \frac{5\sqrt{2}}{2}+3, -1\right)\end{cases}}$$

Problem 28

28.a Find the transformation that scales (with respect to the origin) by 3 units in the x-direction.

$$[S] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\boxed{[S] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}$$

28.b Find the transformation that scales (with respect to the origin) by 4 units in the y-direction.

$$[S] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\boxed{[S] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}$$

28.c Find the transformation that scales (with respect to the origin) simultaneously by 3 units in the x-direction and by 4 units in the y-direction.

$$[S] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\boxed{[S] = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}}$$

Problem 29

Write the transformation for scaling with respect to fixed point $\vec{p} = (1, -1)$.

$$[T_{\vec{p}}] = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{p}}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & -(1) \\ 0 & 1 & -(-1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{p}}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[S] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = [T_{\vec{p}}] \cdot [S] \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} (1)(s_x) + (0)(0) + (1)(0) & (1)(0) + (0)(s_y) + (1)(0) & (1)(0) + (0)(0) + (1)(1) \\ (0)(s_x) + (1)(0) + (-1)(0) & (0)(0) + (1)(s_y) + (-1)(0) & (0)(0) + (1)(0) + (-1)(1) \\ (0)(s_x) + (0)(0) + (1)(0) & (0)(0) + (0)(s_y) + (1)(0) & (0)(0) + (0)(0) + (1)(1) \end{bmatrix} \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} s_x + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \\ 0 + 0 + 0 & 0 + s_y + 0 & 0 + 0 + (-1) \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix} \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} s_x & 0 & 1 \\ 0 & s_y & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{p}}]$$

$$[W] = \begin{bmatrix} s_x & 0 & 1 \\ 0 & s_y & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} (s_x)(1) + (0)(0) + (1)(0) & (s_x)(0) + (0)(1) + (1)(0) & (s_x)(-1) + (0)(1) + (1)(1) \\ (0)(1) + (s_y)(0) + (-1)(0) & (0)(0) + (s_y)(1) + (-1)(0) & (0)(-1) + (s_y)(1) + (-1)(1) \\ (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(1) + (0)(0) & (0)(-1) + (0)(1) + (1)(1) \end{bmatrix}$$

$$[W] = \begin{bmatrix} s_x + 0 + 0 & 0 + 0 + 0 & -s_x + 0 + 1 \\ 0 + 0 + 0 & 0 + s_y + 0 & 0 + s_y + (-1) \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix}$$

$$\boxed{[W] = \begin{bmatrix} s_x & 0 & -s_x + 1 \\ 0 & s_y & s_y - 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem 30

Magnify the triangle with vertices $\vec{a} = (0,0), \vec{b} = (1,1), \vec{c} = (5,2)$ to twice its size by keeping $\vec{c} = (5,2)$ fixed.

$$[T_{\vec{c}}] = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{\vec{c}}] = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{c}}] = \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{c}}] = \begin{bmatrix} 1 & 0 & -(5) \\ 0 & 1 & -(2) \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{-\vec{c}}] = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[S] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[S] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = [T_{\vec{c}}] \cdot [S] \cdot [T_{-\vec{c}}]$$

$$[W] = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{c}}]$$

$$[W] = \begin{bmatrix} (1)(2)+(0)(0)+(0)(0) & (1)(0)+(0)(2)+(5)(0) & (1)(0)+(0)(0)+(5)(1) \\ (0)(2)+(1)(0)+(2)(0) & (0)(0)+(1)(2)+(2)(0) & (0)(0)+(1)(0)+(2)(1) \\ (0)(2)+(0)(0)+(1)(0) & (0)(0)+(0)(2)+(1)(0) & (0)(0)+(0)(0)+(1)(1) \end{bmatrix} \cdot [T_{-\vec{c}}]$$

$$[W] = \begin{bmatrix} 2+0+0 & 0+0+0 & 0+0+5 \\ 0+0+0 & 0+2+0 & 0+0+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} \cdot [T_{-\vec{c}}]$$

$$[W] = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot [T_{-\vec{c}}]$$

$$[W] = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} (2)(1)+(0)(0)+(5)(0) & (2)(0)+(0)(1)+(5)(0) & (2)(-5)+(0)(-2)+(5)(1) \\ (0)(1)+(2)(0)+(2)(0) & (0)(0)+(2)(1)+(2)(0) & (0)(-5)+(2)(-2)+(2)(1) \\ (0)(1)+(0)(0)+(1)(0) & (0)(0)+(0)(1)+(1)(0) & (0)(-5)+(0)(-2)+(1)(1) \end{bmatrix}$$

$$[W] = \begin{bmatrix} 2+0+0 & 0+0+0 & -10+0+5 \\ 0+0+0 & 0+2+0 & 0+(-4)+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$[W] = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\Delta a'b'c'] = [W] \cdot [\Delta abc]$$

$$[\Delta a'b'c'] = [W] \cdot [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$[\Delta a'b'c'] = [W] \cdot \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix}$$

$$[\Delta a'b'c'] = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[\Delta a'b'c'] = \begin{bmatrix} (2)(0)+(0)(0)+(-5)(1) & (2)(1)+(0)(1)+(-5)(1) & (2)(5)+(0)(2)+(-5)(1) \\ (0)(0)+(2)(0)+(-2)(1) & (0)(1)+(2)(1)+(-2)(1) & (0)(5)+(2)(2)+(-2)(1) \\ (0)(0)+(0)(0)+(1)(1) & (0)(1)+(0)(1)+(1)(1) & (0)(5)+(0)(2)+(1)(1) \end{bmatrix}$$

$$[\Delta a'b'c'] = \begin{bmatrix} 0+0+(-5) & 2+0+(-5) & 10+0+(-5) \\ 0+0+(-2) & 0+2+(-2) & 0+4+(-2) \\ 0+0+1 & 0+0+1 & 0+0+1 \end{bmatrix}$$

$$[\Delta a'b'c'] = \begin{bmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boxed{\Delta a'b'c', \begin{cases} \vec{a} = (-5,-2) \\ \vec{b} = (-3,0) \\ \vec{c} = (5,2) \end{cases}}$$

Problem 31

Blank.

Problem 32

Calculate the image of $\vec{p} = (1,0,1)$ after a $45°$ rotation about the z-axis, followed by a $90°$ rotation about the x-axis.

$$[R_z(\theta)] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [R_x(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$[R_z(45°)] = \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [R_x(90°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90°) & -\sin(90°) \\ 0 & \sin(90°) & \cos(90°) \end{bmatrix}$$

$$[R_z(45°)] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [R_x(90°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[W] = [R_x(90°)] \cdot [R_z(45°)]$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[W] = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (0)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(0) + (0)(1) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(-\frac{\sqrt{2}}{2}\right)(-1) + (0)(0) \\ \left(\frac{\sqrt{2}}{2}\right)(1) + \left(\frac{\sqrt{2}}{2}\right)(0) + (0)(0) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) + (0)(1) & \left(\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(-1) + (0)(0) \\ (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(0) + (1)(1) & (0)(0) + (0)(-1) + (1)(0) \end{bmatrix}$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 0 & 0 + \frac{\sqrt{2}}{2} + 0 \\ \frac{\sqrt{2}}{2} + 0 + 0 & 0 + 0 + 0 & 0 + \left(-\frac{\sqrt{2}}{2}\right) + 0 \\ 0 + 0 + 0 & 0 + 0 + 1 & 0 + 0 + 0 \end{bmatrix}$$

$$[W] = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{p}' = [W] \cdot \vec{p}$$

$$\vec{p}' = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(1) + (0)(0) + \left(\frac{\sqrt{2}}{2}\right)(1) \\ \left(\frac{\sqrt{2}}{2}\right)(1) + (0)(0) + \left(-\frac{\sqrt{2}}{2}\right)(1) \\ (0)(1) + (1)(0) + (0)(1) \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} \frac{\sqrt{2}}{2} + 0 + \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} + 0 + \left(-\frac{\sqrt{2}}{2}\right) \\ 0 + 0 + 0 \end{bmatrix}$$

$$\vec{p}' = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{p}' = \left(\sqrt{2}, 0,0\right)}$$

Problem 33

Write the matrix transformation of a $45°$ rotation about an arbitrary axis parallel to the direction $\vec{u} = (1,0,1)$.

$$[R_{\hat{u}}(\theta)] = I + \sin(\theta) \cdot skew(\hat{u}) + [1 - \cos(\theta)] \cdot skew^2(\hat{u})$$

$$[R_{\hat{u}}(45°)] = I + \sin(45°) \cdot skew(\hat{u}) + [1 - \cos(45°)] \cdot skew^2(\hat{u})$$

$$[R_{\hat{u}}(45°)] = I + \frac{\sqrt{2}}{2} \cdot skew(\hat{u}) + \left(1 - \frac{\sqrt{2}}{2}\right) \cdot skew^2(\hat{u})$$

$$[R_{\hat{u}}(45°)] = I + \frac{\sqrt{2}}{2} \cdot skew(\hat{u}) + \left(-\frac{\sqrt{2}}{2}\right) \cdot skew^2(\hat{u})$$

$$skew(\hat{u}) = \begin{bmatrix} 0 & -\hat{u}_z & \hat{u}_y \\ \hat{u}_z & 0 & -\hat{u}_x \\ -\hat{u}_y & \hat{u}_x & 0 \end{bmatrix}$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\|\vec{u}\| = \sqrt{(u_x)^2 + \left(u_y\right)^2 + (u_z)^2}$$
$$\|\vec{u}\| = \sqrt{(1)^2 + (0)^2 + (1)^2}$$
$$\|\vec{u}\| = \sqrt{1 + 0 + 1}$$
$$\|\vec{u}\| = \sqrt{2}$$

$$\hat{u} = \frac{(1,0,1)}{\sqrt{2}}$$
$$\hat{u} = \left(\frac{1}{\sqrt{2}}, \frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$\hat{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$skew(\hat{u}) = \begin{bmatrix} 0 & -\left(\frac{1}{\sqrt{2}}\right) & (0) \\ \left(\frac{1}{\sqrt{2}}\right) & 0 & -\left(\frac{1}{\sqrt{2}}\right) \\ -(0) & \left(\frac{1}{\sqrt{2}}\right) & 0 \end{bmatrix}$$
$$skew(\hat{u}) = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} (0)(0) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + (0)(0) & (0)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)(0) + (0)\left(\frac{1}{\sqrt{2}}\right) & (0)(0) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + (0)(0) \\ \left(\frac{1}{\sqrt{2}}\right)(0) + (0)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)(0) & \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + (0)(0) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) & \left(\frac{1}{\sqrt{2}}\right)(0) + (0)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)(0) \\ (0)(0) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + (0)(0) & (0)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)(0) + (0)\left(\frac{1}{\sqrt{2}}\right) & (0)(0) + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + (0)(0) \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} 0 + \left(-\frac{1}{2}\right) + 0 & 0 + 0 + 0 & 0 + \frac{1}{2} + 0 \\ 0 + 0 + 0 & -\frac{1}{2} + 0 + \left(-\frac{1}{2}\right) & 0 + 0 + 0 \\ 0 + \frac{1}{2} + 0 & 0 + 0 + 0 & 0 + \left(-\frac{1}{2}\right) + 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$[R_{\hat{u}}(45°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\sqrt{2}}{2} \cdot \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + \left(-\frac{\sqrt{2}}{2}\right) \cdot \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$[R_{\hat{u}}(45°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)(0) & \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(\frac{\sqrt{2}}{2}\right)(0) \\ \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) & \left(\frac{\sqrt{2}}{2}\right)(0) & \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ \left(\frac{\sqrt{2}}{2}\right)(0) & \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) & \left(\frac{\sqrt{2}}{2}\right)(0) \end{bmatrix}$$

$$+ \begin{bmatrix} \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) & \left(-\frac{\sqrt{2}}{2}\right)(0) & \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ \left(-\frac{\sqrt{2}}{2}\right)(0) & \left(-\frac{\sqrt{2}}{2}\right)(-1) & \left(-\frac{\sqrt{2}}{2}\right)(0) \\ \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) & \left(-\frac{\sqrt{2}}{2}\right)(0) & \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) \end{bmatrix}$$

$$[R_{\hat{u}}(45°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} \end{bmatrix}$$

$$[R_{\hat{u}}(45°)] = \begin{bmatrix} 1 + 0 + \frac{\sqrt{2}}{4} & 0 + \left(-\frac{1}{2}\right) + 0 & 0 + 0 + \left(-\frac{\sqrt{2}}{4}\right) \\ 0 + \frac{1}{2} + 0 & 1 + 0 + \frac{\sqrt{2}}{2} & 0 + \left(-\frac{1}{2}\right) + 0 \\ 0 + \frac{1}{2} - \frac{\sqrt{2}}{4} & 0 + \frac{1}{2} + 0 & 1 + 0 + \frac{\sqrt{2}}{4} \end{bmatrix}$$

$$\boxed{[R_{\hat{u}}(45°)] = \begin{bmatrix} \frac{4+\sqrt{2}}{4} & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\ \frac{1}{2} & \frac{2+\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{2-\sqrt{2}}{4} & \frac{1}{2} & \frac{4+\sqrt{2}}{4} \end{bmatrix}}$$

Problem 34

Write the transformation matrix of a $180°$ rotation about an arbitrary axis parallel to the direction $\vec{u} = (3,0,4)$.

$[R_{\hat{u}}(\theta)] = I + \sin(\theta) \cdot skew(\hat{u}) + [1 - \cos(\theta)] \cdot skew^2(\hat{u})$

$[R_{\hat{u}}(180°)] = I + \sin(180°) \cdot skew(\hat{u}) + [1 - \cos(180°)] \cdot skew^2(\hat{u})$

$[R_{\hat{u}}(180°)] = I + (0) \cdot skew(\hat{u}) + [1 - (-1)] \cdot skew^2(\hat{u})$

$[R_{\hat{u}}(180°)] = I + 0 + 2 \cdot skew^2(\hat{u})$

$[R_{\hat{u}}(180°)] = I + 2 \cdot skew^2(\hat{u})$

$$skew(\hat{u}) = \begin{bmatrix} 0 & -\hat{u}_z & \hat{u}_y \\ \hat{u}_z & 0 & -\hat{u}_x \\ -\hat{u}_y & \hat{u}_x & 0 \end{bmatrix}$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\|\vec{u}\| = \sqrt{(u_x)^2 + (u_y)^2 + (u_z)^2}$$

$$\|\vec{u}\| = \sqrt{(3)^2 + (0)^2 + (4)^2}$$

$$\|\vec{u}\| = \sqrt{9 + 0 + 16}$$

$$\|\vec{u}\| = \sqrt{25}$$

$$\|\vec{u}\| = 5$$

$$\hat{u} = \frac{(3,0,4)}{5}$$

$$\hat{u} = \left(\frac{3}{5}, \frac{0}{5}, \frac{4}{5}\right)$$

$$\hat{u} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

$$skew(\hat{u}) = \begin{bmatrix} 0 & -\left(\frac{4}{5}\right) & (0) \\ \left(\frac{4}{5}\right) & 0 & -\left(\frac{3}{5}\right) \\ -(0) & \left(\frac{3}{5}\right) & 0 \end{bmatrix}$$

$$skew(\hat{u}) = \begin{bmatrix} 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & \frac{3}{5} & 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & \frac{3}{5} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & \frac{3}{5} & 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} (0)(0) + \left(-\frac{4}{5}\right)\left(\frac{4}{5}\right) + (0)(0) & (0)\left(-\frac{4}{5}\right) + \left(-\frac{4}{5}\right)(0) + (0)\left(\frac{3}{5}\right) & (0)(0) + \left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) + (0)(0) \\ \left(\frac{4}{5}\right)(0) + (0)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)(0) & \left(\frac{4}{5}\right)\left(-\frac{4}{5}\right) + (0)(0) + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) & \left(\frac{4}{5}\right)(0) + (0)\left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)(0) \\ (0)(0) + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + (0)(0) & (0)\left(-\frac{4}{5}\right) + \left(\frac{3}{5}\right)(0) + (0)\left(\frac{3}{5}\right) & (0)(0) + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) + (0)(0) \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} 0 + \left(-\frac{16}{25}\right) + 0 & 0 + 0 + 0 & 0 + \left(-\frac{12}{25}\right) + 0 \\ 0 + 0 + 0 & \left(-\frac{16}{25}\right) + 0 + \frac{9}{25} & 0 + 0 + 0 \\ 0 + \frac{12}{25} + 0 & 0 + 0 + 0 & 0 + \frac{9}{25} + 0 \end{bmatrix}$$

$$skew^2(\hat{u}) = \begin{bmatrix} -\frac{16}{25} & 0 & -\frac{12}{25} \\ 0 & -\frac{7}{25} & 0 \\ \frac{12}{25} & 0 & \frac{9}{25} \end{bmatrix}$$

$$[R_{\hat{u}}(180°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{16}{25} & 0 & -\frac{12}{25} \\ 0 & -\frac{7}{25} & 0 \\ \frac{12}{25} & 0 & \frac{9}{25} \end{bmatrix}$$

$$[R_{\hat{u}}(180°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} (2)\left(-\frac{16}{25}\right) & (2)(0) & (2)\left(-\frac{12}{25}\right) \\ (2)(0) & (2)\left(-\frac{7}{25}\right) & (2)(0) \\ (2)\left(\frac{12}{25}\right) & (2)(0) & (2)\left(\frac{9}{25}\right) \end{bmatrix}$$

$$[R_{\hat{u}}(180°)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{32}{25} & 0 & -\frac{24}{25} \\ 0 & -\frac{14}{25} & 0 \\ \frac{24}{25} & 0 & \frac{18}{25} \end{bmatrix}$$

$$[R_{\hat{u}}(180°)] = \begin{bmatrix} 1 + \left(-\frac{32}{25}\right) & 0 + 0 & 0 + \left(-\frac{24}{25}\right) \\ 0 + 0 & 1 + \left(-\frac{14}{25}\right) & 0 + 0 \\ 0 + \frac{24}{25} & 0 + 0 & 1 + \frac{18}{25} \end{bmatrix}$$

$$\boxed{[R_{\hat{u}}(180°)] = \begin{bmatrix} -\frac{7}{25} & 0 & -\frac{24}{25} \\ 0 & \frac{11}{25} & 0 \\ \frac{24}{25} & 0 & \frac{43}{25} \end{bmatrix}}$$

Problem 35

Calculate the inverse of the rotation matrix $[R] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ without using its adjoint matrix.

The inverse of a rotation matrix is the rotation matrix for the negative angle. That is, $[R(\theta)]^{-1} = [R(-\theta)]$. In practice, the inverse of a rotation matrix is its transpose:

$$[R(\theta)]^{-1} = [R(\theta)]^T$$

$$[R] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R]^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\boxed{[R]^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Linear Operators

A linear operator is a linear transformation with a domain equal to the co-domain.

Problem 36

Which of the following linear transformation is a linear operator?

36.a $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x - 2y, y, x + 3y)$

$\mathbb{R}^2 = \mathbb{R}^2$, so T is a linear operator.

36.b $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x, y, z) = (x - 2y - z, y, x + y + z)$

$\mathbb{R}^3 = \mathbb{R}^3$, so T is a linear operator.

36.c $T: \mathbb{R}^2 \to \mathbb{R}$, such that $T(x, y) = x - 2y$

$\mathbb{R}^2 \neq \mathbb{R}$, so T is ***not*** a linear operator.

36.d $T: \mathbb{R}^2 \to \mathbb{R}^3$, such that $T(x, y) = (x - y, y, x + y)$

$\mathbb{R}^2 \neq \mathbb{R}^3$, so T is ***not*** a linear operator.

36.e $T: \mathbb{R} \to \mathbb{R}$, such that $T(x) = 2x$

$\mathbb{R} = \mathbb{R}$, so T is a linear operator.

Composition of Linear Operators

Problem 37

37.a Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + y \end{bmatrix}$, find $T_2 \circ T_1$.

$$T_2 \circ T_1 \sim [T_2] \cdot [T_1]$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$
$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_1] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T_1] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + y \end{bmatrix}$$
$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T_2] = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} (2)(1) + (3)(1) & (2)(2) + (3)(-1) \\ (1)(1) + (1)(1) & (1)(2) + (1)(-1) \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} 2 + 3 & 4 + (-3) \\ 1 + 1 & 2 + (-1) \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$$
$$\boxed{T_2 \circ T_1 = \begin{bmatrix} 5x + y \\ 2x + y \end{bmatrix}}$$

37.b Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + y \end{bmatrix}$, find $T_1 \circ T_2$.

$T_1 \circ T_2 \sim [T_1] \cdot [T_2]$

$T_1 \circ T_2 \sim \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$[T_1]$ and $[T_2]$ from 37.a, above.

$T_1 \circ T_2 \sim \begin{bmatrix} (1)(2) + (2)(1) & (1)(3) + (2)(1) \\ (1)(2) + (-1)(1) & (1)(3) + (-1)(1) \end{bmatrix}$

$T_1 \circ T_2 \sim \begin{bmatrix} 2 + 2 & 3 + 2 \\ 2 + (-1) & 3 + (-1) \end{bmatrix}$

$T_1 \circ T_2 \sim \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$

$\boxed{T_1 \circ T_2 = \begin{bmatrix} 4x + 5y \\ x + 2y \end{bmatrix}}$

37.c Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + y \end{bmatrix}$, is $T_2 \circ T_1 = T_1 \circ T_2$?

$\begin{bmatrix} 5x + y \\ 2x + y \end{bmatrix} \neq \begin{bmatrix} 4x + 5y \\ x + 2y \end{bmatrix}$

$\boxed{T_2 \circ T_1 \neq T_1 \circ T_2}$

Problem 38

38.a Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x+y \\ y \end{bmatrix}$, find $T_2 \circ T_1$.

$$T_2 \circ T_1 \sim [T_2] \cdot [T_1]$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$$
$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_1] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T_1] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x+y \\ y \end{bmatrix}$$
$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T_2] = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} (5)(1)+(1)(1) & (5)(0)+(1)(1) \\ (0)(1)+(1)(1) & (0)(0)+(1)(1) \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} 5+1 & 0+1 \\ 0+1 & 0+1 \end{bmatrix}$$
$$T_2 \circ T_1 \sim \begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\boxed{T_2 \circ T_1 = \begin{bmatrix} 6x+y \\ x+y \end{bmatrix}}$$

38.b Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x+y \\ y \end{bmatrix}$, find $T_1 \circ T_2$.

$$T_1 \circ T_2 \sim [T_1] \cdot [T_2]$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$$

$\boxed{[T_1] \text{ and } [T_2] \text{ from 37.a, above.}}$

$$T_1 \circ T_2 \sim \begin{bmatrix} (1)(5)+(0)(0) & (1)(1)+(0)(1) \\ (1)(5)+(1)(0) & (1)(1)+(1)(1) \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 5+0 & 1+0 \\ 5+0 & 1+1 \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 5 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\boxed{T_1 \circ T_2 = \begin{bmatrix} 5x+y \\ 5x+2y \end{bmatrix}}$$

Problem 39

39.a Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -x+y \end{bmatrix}$, find $T_2 \circ T_1$.

$$T_2 \circ T_1 \sim [T_2] \cdot [T_1]$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_1] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -x+y \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} (1)(3)+(1)(0) & (1)(0)+(1)(2) \\ (-1)(3)+(1)(0) & (-1)(0)+(1)(2) \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 3+0 & 0+2 \\ -3+0 & 0+2 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 3 & 2 \\ -3 & 2 \end{bmatrix}$$

$$\boxed{T_2 \circ T_1 = \begin{bmatrix} 3x+2y \\ -3x+2y \end{bmatrix}}$$

39.b Given $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ -x + y \end{bmatrix}$, find $T_1 \circ T_2$.

$$T_1 \circ T_2 \sim [T_1] \cdot [T_2]$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} (3)(1) + (0)(-1) & (3)(1) + (0)(1) \\ (0)(1) + (2)(-1) & (0)(1) + (2)(1) \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 3 + 0 & 3 + 0 \\ 0 + (-2) & 0 + 2 \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 3 & 3 \\ -2 & 2 \end{bmatrix}$$

$$\boxed{T_1 \circ T_2 = \begin{bmatrix} 3x + 3y \\ -2x + 2y \end{bmatrix}}$$

Problem 40

40.a Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$, find $T_2 \circ T_1$.

$$T_2 \circ T_1 \sim [T_2] \cdot [T_1]$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} \qquad T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_1] \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [T_2] = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} (1)(1)+(1)(0) & (1)(0)+(1)(-1) \\ (2)(1)+(1)(0) & (2)(0)+(1)(-1) \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 1+0 & 0+(-1) \\ 2+0 & 0+(-1) \end{bmatrix}$$

$$T_2 \circ T_1 \sim \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\boxed{T_2 \circ T_1 = \begin{bmatrix} x-y \\ 2x-y \end{bmatrix}}$$

40.b Given $T_1: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$ and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$, find $T_1 \circ T_2$.

$$T_1 \circ T_2 \sim [T_1] \cdot [T_2]$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} (1)(1)+(0)(2) & (1)(1)+(0)(1) \\ (0)(1)+(-1)(2) & (0)(1)+(-1)(1) \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 1+0 & 1+0 \\ 0+(-2) & 0+(-1) \end{bmatrix}$$

$$T_1 \circ T_2 \sim \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\boxed{T_1 \circ T_2 = \begin{bmatrix} x+y \\ -2x-y \end{bmatrix}}$$

One-To-One Linear Operators

A linear operator is one-to-one if its standard matrix is invertible. A matrix is invertible if its determinant is not zero.

Problem 41

Is $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ a one-to-one linear operator?

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0x + 1y \\ 1x + 0y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det([T]) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$\det([T]) = (0)(0) - (1)(1)$

$\det([T]) = 0 - 1$

$\det([T]) = -1$

$\det([T]) \neq 0 \to \exists [T]^{-1}$

T is a one-to-one linear operator.

Problem 42

Is $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$ a one-to-one linear operator?

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\det([T]) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$\det([T]) = (1)(-1) - (1)(1)$

$\det([T]) = -1 - 1$

$\det([T]) = -2$

$\det([T]) \neq 0 \to \exists [T]^{-1}$

T is a one-to-one linear operator.

Problem 43

Is $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x+y \end{bmatrix}$ a one-to-one linear operator?

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0x + 0y \\ 1x + 1y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$[T] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$\det([T]) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}$

$\det([T]) = (0)(1) - (1)(0)$

$\det([T]) = 0 - 0$

$\det([T]) = 0 \to \nexists [T]^{-1}$

T is ***not*** a one-to-one linear operator.

Problem 44

Is $T_a: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 6x + 3y \end{bmatrix}$ a one-to-one linear operator?

$\mathrm{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$[T] = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$\det([T]) = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}$

$\det([T]) = (2)(3) - (6)(1)$

$\det([T]) = 6 - 6$

$\det([T]) = 0 \to \nexists [T]^{-1}$

T is ***not*** a one-to-one linear operator.

Problem 45

Is $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z \\ z \\ 2z \end{bmatrix}$ a one-to-one linear operator?

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 0y + 1z \\ 0x + 0y + 2z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det([T]) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\det([T]) = \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}(1) - \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}(2) + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(3)$$

$$\det([T]) = [(0)(2) - (0)(1)](1) - [(0)(2) - (0)(1)](2) + [(0)(0) - (0)(0)](3)$$

$$\det([T]) = (0 - 0)(1) - (0 - 0)(2) + (0 - 0)(3)$$

$$\det([T]) = (0)(1) - (0)(2) + (0)(3)$$

$$\det([T]) = 0 - 0 + 0$$

$$\det([T]) = 0 \rightarrow \nexists [T]^{-1}$$

T is ***not*** a one-to-one linear operator.

Inverse of a One-To-One Linear Operator

Problem 46

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$, verify that it is invertible and compute its inverse.

$$\det([T]) = -1 \to \det([T]) \neq 0 \to \exists [T]^{-1} \to \exists T^{-1}$$

Proved in 41, above.

$$[T]^{-1} = \frac{\mathrm{Adj}([T])}{\det([T])}$$

$$[T]^{-1} = \frac{\begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix}}{-1}$$

$$[T]^{-1} = \begin{bmatrix} t_{22}/_{-1} & -t_{12}/_{-1} \\ -t_{21}/_{-1} & t_{11}/_{-1} \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} -t_{22} & t_{12} \\ t_{21} & -t_{11} \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Found in 41, above.

$$[T]^{-1} = \begin{bmatrix} -(0) & (1) \\ (1) & -(0) \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}}$$

Check:

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} (0)(0) + (1)(1) & (0)(1) + (1)(0) \\ (1)(0) + (0)(1) & (1)(1) + (0)(0) \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 0 + 1 & 0 + 0 \\ 0 + 0 & 1 + 0 \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Problem 47

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$, verify that it is invertible and compute its inverse.

$$\det([T]) = -2 \to \det([T]) \neq 0 \to \exists [T]^{-1} \to \exists T^{-1}$$

Proved in 42, above.

$$[T]^{-1} = \frac{\mathrm{Adj}([T])}{\det([T])}$$

$$[T]^{-1} = \frac{\begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix}}{-2}$$

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Found in 42, above.

$$[T]^{-1} = \frac{\begin{bmatrix} (-1) & -(1) \\ -(1) & (1) \end{bmatrix}}{-2}$$

$$[T]^{-1} = \frac{\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}}{-2}$$

$$[T]^{-1} = \begin{bmatrix} {}^{-1}\!/_{-2} & {}^{-1}\!/_{-2} \\ {}^{-1}\!/_{-2} & {}^{1}\!/_{-2} \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\boxed{T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{1}{2}y \end{bmatrix}}$$

Check:

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) & (1)\left(\frac{1}{2}\right) + (1)\left(-\frac{1}{2}\right) \\ (1)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) & (1)\left(\frac{1}{2}\right) + (-1)\left(-\frac{1}{2}\right) \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \left(-\frac{1}{2}\right) \\ \frac{1}{2} + \left(-\frac{1}{2}\right) & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Problem 48

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x + y \end{bmatrix}$, verify that it is invertible and compute its inverse.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x + y \end{bmatrix}$$
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [\mathrm{T}] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det([T]) = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$
$$\det([T]) = (2)(1) - (1)(1)$$
$$\det([T]) = 2 - 1$$
$$\det([T]) = 1$$
$$\det([T]) = 1$$
$$\det([T]) \neq 0 \to \exists [T]^{-1} \to \exists T^{-1}$$

$$[T]^{-1} = \frac{\mathrm{Adj}([T])}{\det([T])}$$

$$\mathrm{Adj}([T]) = \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix}$$
$$\mathrm{Adj}([T]) = \begin{bmatrix} (1) & -(1) \\ -(1) & (2) \end{bmatrix}$$
$$\mathrm{Adj}([T]) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[T]^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}{1}$$
$$[T]^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$\boxed{T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ -x + 2y \end{bmatrix}}$$

Check:

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} (2)(1) + (1)(-1) & (2)(-1) + (1)(2) \\ (1)(1) + (1)(-1) & (1)(-1) + (1)(2) \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} 2 + (-1) & -2 + 2 \\ 1 + (-1) & -1 + 2 \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Problem 49

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + 2y \end{bmatrix}$, verify that it is invertible and compute its inverse.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T] \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
$$[T] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\det([T]) = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$
$$\det([T]) = (2)(2) - (1)(3)$$
$$\det([T]) = 4 - 3$$
$$\det([T]) = 1$$
$$\det([T]) \neq 0 \to \exists [T]^{-1} \to \exists T^{-1}$$

$$[T]^{-1} = \frac{\mathrm{Adj}([T])}{\det([\mathrm{T}])}$$

$$\mathrm{Adj}([T]) = \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix}$$
$$\mathrm{Adj}([T]) = \begin{bmatrix} (2) & -(3) \\ -(1) & (2) \end{bmatrix}$$
$$\mathrm{Adj}([T]) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$[T]^{-1} = \frac{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}{1}$$
$$[T]^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$
$$\boxed{T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -x + 2y \end{bmatrix}}$$

Check:

$$[T] \cdot [T]^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} (2)(2) + (3)(-1) & (2)(-3) + (3)(2) \\ (1)(2) + (2)(-1) & (1)(-3) + (2)(2) \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} 4 + (-3) & -6 + 6 \\ 2 + (-2) & -3 + 4 \end{bmatrix}$$
$$[T] \cdot [T]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$