Chapter 6 : Collision Detection

1) Quadratic Equation

A quadratic equation is an equation which unknowns (solutions) satisfy the following equation: $ax^2 + bx + c = o$ where a ,b , and c are constant. A solution is computed by computing its discriminant $\Delta = b^2 - 4ac$.

If $\Delta > 0$ then $x_1 = \dfrac{-b+\sqrt{\Delta}}{2a}$ or $x_2 = \dfrac{-b-\sqrt{\Delta}}{2a}$ (two solutions are found)

If $\Delta = 0$ then $x = \dfrac{-b}{2a}$ (one solution)

If $\Delta < 0$ then $S_{\mathbb{R}} = \varnothing$ (no solutions)

The quadratic equation will serve as tool to compute the intersection points between a ray and a sphere.

Example1: Solve $x^2 - 4x + 3 = o$

Here a=1,b=-4,c=3 and $\Delta = b^2 - 4ac$ =(-4)² - 4(1)(3)=16-12=4; $\Delta > 0$ we have 2 solutions:

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} == \frac{-(-4)+\sqrt{4}}{2(1)} = \frac{4+2}{2} = 3 \quad , \quad x_2 = \frac{-b-\sqrt{\Delta}}{2a} == \frac{-(-4)-\sqrt{4}}{2(1)} = \frac{4-2}{2} = 1$$

Example2: $x^2 + x + 3 = o$

Here a=1, b=1 , c=3 and $\Delta = b^2 - 4ac$ =(1)² - 4(1)(3)=1-12= -11; since $\Delta < 0$ there is no solution for x.

Now if the coefficient b is even number , $b = 2b'$ **, then** $ax^2 + bx + c = o$ will have as discriminant $\Delta = b'^2 - ac$ where $b' = \dfrac{b}{2}$ **with solutions:**

$x_1 = \dfrac{-b'+\sqrt{\Delta}}{a}$ or $x_2 = \dfrac{-b'-\sqrt{\Delta}}{a}$ if $\Delta = b'^2 - ac > 0$, and

$x = \dfrac{-b'}{a}$ if $\Delta = b'^2 - ac = 0$

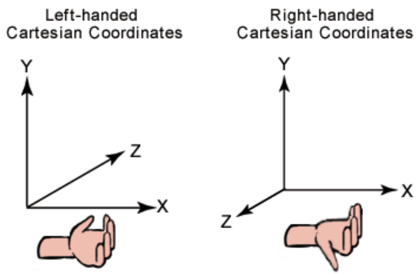
Example3 : Solve $x^2 - 4x + 3 = o$

Here a=1 c=3 , b=-4 is even so we get $b' = \dfrac{b}{2} = \dfrac{-4}{2} = -2$ **and** $\Delta = b'^2 - ac$ =(-2)² - (1)(3)=4-3=1

So $x_1 = \dfrac{-b'+\sqrt{\Delta}}{a} = \dfrac{-(-2)+\sqrt{1}}{1} = 3$ and $x_2 = \dfrac{-b'-\sqrt{\Delta}}{a} = \dfrac{-(-2)-\sqrt{1}}{1} = 2-1 = 1$

TODO ➔ Go to activity and solve question 1

2) Rectangular Cartesian Coordinate Systems and orthonormal basis



In this coordinates, the vectors that span the frame are the $\vec{i}$, $\vec{j}$, $\vec{k}$ vectors.The left-handed system is used in DirectX and the right-handed system in OpenGL. The right-handed system is the standard frame used in Math and in Physics. The frame $\{\vec{i},\vec{j},\vec{k}\}$ is said to be an orthonormal frame since $\vec{i} \perp \vec{j} \perp \vec{k}$ and $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$.

In general , a basis $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is said to be orthonormal basis if and only if :

- $\vec{e}_1 \perp \vec{e}_2, \vec{e}_1 \perp \vec{e}_3, \textit{and } \vec{e}_2 \perp \vec{e}_3$ or $\vec{e}_1 \bullet \vec{e}_2 = \vec{e}_1 \bullet \vec{e}_3 = \vec{e}_2 \bullet \vec{e}_3 = 0$
- $\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1$

Example 2.1: is $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), (0,1,0), \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\}$ an orthonormal basis ?

Answer :

$$\vec{e}_1 \bullet \vec{e}_2 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \bullet (0,1,0) = 0$$

$$\vec{e}_1 \bullet \vec{e}_3 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \bullet \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) = -1 + 1 = 0$$

$$\vec{e}_2 \bullet \vec{e}_3 = (0,1,0) \bullet \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) = 0$$

$$\|\vec{e}_1\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + 0 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1 \ , \ \|\vec{e}_2\| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + 0 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1 \ , \ \|\vec{e}_3\| = 1$$

So $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis.

Example 2.2: is $B = \{\vec{e}_1, \vec{e}_2\} = \{(1,1), (-1,1)\}$ an orthonormal basis ?

$\vec{e}_1 \bullet \vec{e}_2 = (1,1) \bullet (-1,1) = -1 + 1 = 0$ but $\|\vec{e}_1\| = \sqrt{2} \neq 1$ ➔ $B = \{\vec{e}_1, \vec{e}_2\}$ is not an orthonormal basis

Theorem 2.1 : if $B=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ is orthonormal basis , then $\det(\vec{e}_1,\vec{e}_2,\vec{e}_3)=\pm 1$ from the matrix $[\vec{e}_1\,|\vec{e}_2\,|\vec{e}_3]$

Example2.3 $B=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}=\left\{\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right),(0,1,0),\left(-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\}$

We write our basis vector as a column vector $B=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}=\left\{\begin{pmatrix}\frac{\sqrt{2}}{2}\\ 0\\ \frac{\sqrt{2}}{2}\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ 0\\ \frac{\sqrt{2}}{2}\end{pmatrix}\right\}$

$$M=[\vec{e}_1\,|\vec{e}_2\,|\vec{e}_3]=\begin{pmatrix}\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}\end{pmatrix} \Rightarrow \det(M)=\det[\vec{e}_1\,|\vec{e}_2\,|\vec{e}_3]=\begin{vmatrix}\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}\end{vmatrix}=1$$

3) World Space or Generalized Coordinate Coordinate System(Frame)

The world space is spanned by the $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$ and $\vec{k}=(0,0,1)$ vectors.

It is represented by the identity matrix $I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. It is a non- moving frame also called the generalized coordinate system in Physics.

4) Body Frame or local Space

The local space is also called the object space or body space. It is the coordinate system of a moving rigid body like a car where the car position is the center of the frame.

Their axes are spanned by 3 unit vectors $\hat{e}_1$ (Side or right vector), $\hat{e}_2$ (up vector) and $\hat{e}_3$ (look or forward vector) with origin $\vec{c}$ =body center of mass (position)

In body space (local space), $\hat{e}_1=(1,0,0)$ $\hat{e}_2=(0,1,0)$ $\hat{e}_3=(0,0,1)$ with $\vec{c}=(0,0,0)$ which is the body frame coordinate origin.

In world space, $\vec{c}=(x,y,z)$ represents the body position as defined in the world coordinate systems.

And $\hat{e}_1=(e_{1x},e_{1y},e_{1z})$ $\hat{e}_2=(e_{2x},e_{2y},e_{2z})$ $\hat{e}_3=(e_{3x},e_{3y},e_{3z})$

Example : see example Example 5.1

5) Rigid Body Orientation, Global Rotation and local Rotation

Let $\hat{e}_1 = \begin{pmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{pmatrix}$, $\hat{e}_2 = \begin{pmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{pmatrix}$, $\hat{e}_3 = \begin{pmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{pmatrix}$ represent respectively a rigid body local X-axis Y-axis and Z-axis direction vectors coordinate in world space, then the orientation matrix of the rigid body in the world space is $R = [\hat{e}_1 \,|\, \hat{e}_2 \,|\, \hat{e}_3] = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$.

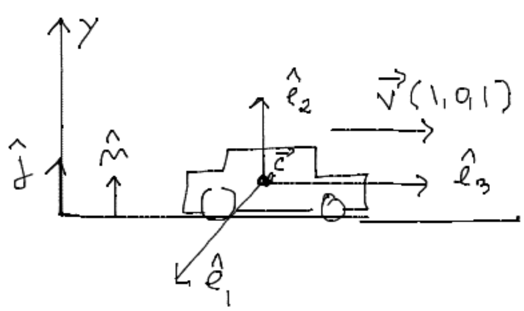
R is called a **global rotation** , denoted ${}^{G}R$.

Example 5.1: a car with position $\vec{c} = (2,6,3)$ is moving with velocity v(1,0,1)m/s on the ground that is a plane with normal vector $\hat{n} = \vec{j} = (0,1,0)$

a) What are its center, up , look and side vectors coordinate

- **In body frame coordinate ?**
- **In global frame coordinate ?**

b) What is the car orientation in the world coordinate?



Asnwer:

a)

In body space : $\hat{e}_1 = (1,0,0)$ $\hat{e}_2 = (0,1,0)$ $\hat{e}_3 = (0,0,1)$ with $\vec{c} = (0,0,0)$.

In world coordinate:

The forward vector $\hat{e}_3$ is in the direction of the velocity , $\hat{e}_3 = \hat{v} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

The up vector $\hat{e}_2$ is in the direction of $\hat{n} = \vec{j} = (0,1,0)$ ➔ $\hat{e}_2 = \hat{n} = (0,1,0)$

The side(right) $\hat{e}_1$ is a vector orthogonal to both $\hat{e}_2$ and $\hat{e}_3$, $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$

Car center coordinate in the world is its position $\vec{c} = (2,6,3)$

b) In world coordinate,

$$\hat{e}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \textbf{and} \quad \hat{e}_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$ **so the car orientation in the world coordinate is**

$$R = \left[\hat{e}_1 \,|\, \hat{e}_2 \,|\, \hat{e}_3\right] = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

local rotation

A local rotation is defined as the orientation of our rigid body as computed in the world and expressed in body frame coordinate. It is denoted ${}^{B}R$, ${}^{B}R = \left({}^{G}R\right)^{t}$ where the superscript B is for body space coordinate. **A local rotation is the transpose of the global rotation.**

Example 5.2: Write the global rotation matrix of 90 about the world z-axis, ${}^{G}R_z(\pi/2)$

Answer: ${}^{G}R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\theta = 90$ and cos(90)=0 sin(90)=1 we

$${}^{G}R_z(90) = \begin{pmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TO DO ➔ Go to activity and solve questions 2 and 3

Example 5.3: Write the local rotation matrix of 90 about the local Z-axis of a rigid body.

From example 5.2 , A global rotation matrix about the world z-axis is $^{G}R_z(\theta)=\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}$

And the local rotation matrix $^{B}R_Z(\theta)$ of θ^0 about the local Z-axis is just the inverse or the

transposed matrix of the global rotation: $^{B}R_Z(\theta)=\left(^{G}R_z(\theta)\right)^t$

since $^{G}R_z(90)=\begin{pmatrix}\cos(90) & -\sin(90) & 0\\ \sin(90) & \cos(90) & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ we have:

$$^{B}R_Z(90)=\left(^{G}R_z(90)\right)^t={}^{G}R_z(90)=\begin{pmatrix}\cos(90) & -\sin(90) & 0\\ \sin(90) & \cos(90) & 0\\ 0 & 0 & 1\end{pmatrix}^t=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}^t=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

TO DO ➔ Go to activity and solve questions 4 and 5

<u>Global rotation matrices versus local rotation matrices:</u>

Rotation of θ^o about the global z-axis $^{G}R_z(\theta)=\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}$

Rotation of θ^o about the local Z-axis $^{B}R_Z=\left(^{G}R_z\right)^t=\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}^t=\begin{pmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}$

Rotation of θ^o about the global y-axis $^{G}R_y(\theta)=\begin{pmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{pmatrix}$

Rotation of θ^o about the local Y-axis $^{B}R_Y=\left(^{G}R_Y\right)^t=\begin{pmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{pmatrix}^t=\begin{pmatrix}\cos(\theta) & 0 & -\sin(\theta)\\ 0 & 1 & 0\\ \sin(\theta) & 0 & \cos(\theta)\end{pmatrix}$

Rotation of θ^o about the global x-axis $^{G}R_x(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{pmatrix}$

Rotation of θ^o about the local X-axis $^{B}R_X=\left(^{G}R_x\right)^t=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{pmatrix}^t=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{pmatrix}$

6) Transforming a Vector from Local Space to World Space

Let $\vec{r}_o$ be a vector in a local space .If the local space orientation as defined in the world space is R,

then the transformation of $\vec{r}_o$ from body space to the world space is $\vec{r}=R\bullet\vec{r}_o$.

This also implies that $\vec{r}_o=R^t\bullet\vec{r}$ where $R^t={}^{B}R$.

Example 6.1: A vector coordinate in body frame coordinate (rigid body) is $\vec{r}_o$=(1,2,1),calculate the vector Coordinate in world space if the orientation of the rigid body is 90 about the world x-axis.

Answer: the orientation of the body about the world x-axis is

$$R_x(90)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(90) & -\sin(90)\\ 0 & \sin(90) & \cos(90)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} \text{ using } \vec{r}=R\bullet\vec{r}_o=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}=\begin{pmatrix}1\\ -1\\ 2\end{pmatrix}$$

Example 6.2: A vector coordinate in body frame coordinate (rigid body) is $\vec{r}_o$=(1,0,-3),calculate the vector coordinate in world space if the orientation of the rigid body is 90 about the **local X-axis.**

Answer: Here we need to be careful since the rotation is local about the X-axis , so we use

$$^{B}R_X=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{pmatrix} \textbf{ instead of } {}^{G}R_x(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{pmatrix}$$

So we compute **R=** $^{B}R_X=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(90) & \sin(90)\\ 0 & -\sin(90) & \cos(90)\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$

And $\vec{r} = R \bullet \vec{r}_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$

TO DO ➔ Go to activity and solve question 6

Transforming from a Vector from World coordinate to Local Coordinate

Let $\vec{r}$ be a vector in a global space (world coordinate) .If the local space(rigid body) orientation as defined in the world space is R, then the transformation of $\vec{r}$ from world space to local space is $\vec{r}_o = R^t \bullet \vec{r}$ where $R^t = {}^B R$.

Example 6.3: A vector Coordinate in world space is $\vec{r}$ =(1,2,3), calculate the vector coordinate in body frame coordinate (rigid body) $\vec{r}_o$ if the orientation of the rigid body is 90 about the **global y-axis.**

Answer:

First we calculate the orientation (global rotation) of the rigid body $R = {}^G R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$

$R = {}^G R_y(90) = \begin{pmatrix} \cos(90) & 0 & \sin(90) \\ 0 & 1 & 0 \\ -\sin(90) & 0 & \cos(90) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ then use $\vec{r}_o = R^t \bullet \vec{r}$.

$$\vec{r}_o = R^t \bullet \vec{r} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \text{ so } \vec{r}_o = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

TO DO ➔ Go to activity and solve question 7

7) Transforming a Point from Local Space to World Space

Given a rigid body with orientation R= $\begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$ and position $\vec{c}$ in world space, if a point $\vec{P}$ has coordinate $\vec{P}_{Local} = (l_x, l_y, l_z)$ in body(local) space and $\vec{P}_{World} = (w_x, w_y, w_z)$ in world space then R will transform $\vec{P}$ from local coordinate to world space coordinate as follows: $\boxed{\vec{P}_{World} = \vec{c} + R \cdot \vec{P}_{Local}}$.

Example 7.1: A box is located at $\vec{c} = (2,1,-2)$ with orientation of 45 degree about the global z-axis.

a) What is the box orientation in world space ?

b) Calculate the position $\vec{P}_{World}$ **of a point in world space** if the point coordinate in local space **is** $\vec{P}_{Local}\left(\sqrt{2},0,0\right)$

Answer :

a) $R = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\vec{P}_{World} = \vec{c} + R \cdot \vec{P}_{Local} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

Example 7.2: A box is located at $\vec{c} = (1,3,0)$ with orientation of 180 (π) degree about the local Y-axis.

a) What is the box orientation in world space ?

b) Calculate the position $\vec{P}_{World}$ **of a point in world space** if the point coordinate in local space **is** $\vec{P}_{Local}(10,0,5)$

Answer:

a) We compute the local rotation about the local Y-axis , then take the transpose to get the box orientation in the world.

$${}^{B}R_{Y} = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\pi) & 0 & -\sin(\pi) \\ 0 & 1 & 0 \\ \sin(\pi) & 0 & \cos(\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then the orientation in the world will be $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{t} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

b) $\vec{P}_{World} = \vec{c} + R \cdot \vec{P}_{Local} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -10 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ -5 \end{pmatrix}$

TO DO ➔ Go to activity and solve question 8

Transforming from world coordinate to local coordinate

If transforming the point from world coordinate to body coordinate , then from $\vec{P}_{World} = \vec{c} + R \cdot \vec{P}_{Local}$ we solve for $\vec{P}_{Local}$, that is $\vec{P}_{World} = \vec{c} + R \cdot \vec{P}_{Local}$ ➔ $\vec{P}_{World} - \vec{c} = R \cdot \vec{P}_{Local}$ ➔ $\vec{P}_{Local} = R^t \left(\vec{P}_{World} - \vec{c} \right)$.

Example 7.3: A box is located at $\vec{c} = (3,5,1)$ with orientation of 90 degree about the global z-axis. Calculate the position of a point in local space $\vec{P}_{Local}$ if the point coordinate in world space is $\vec{P}_{World} = (4,6,5)$

Answer: We use $\vec{P}_{Local} = R^t \left(\vec{P}_{World} - \vec{c} \right)$,

$$R = {}^G R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{P}_{Local} = R^t \left(\vec{P}_{World} - \vec{c} \right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^t \left(\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

Example 7.4 : A box is located at $\vec{c} = (0,0,0)$ with orientation of 45 degree about the global y-axis. Calculate the position of a point in local space $\vec{P}_{Local}$ if the point coordinate in world space is $\vec{P}_{World} = (1,2,0)$

Answer: We use $\vec{P}_{Local} = R^t \left(\vec{P}_{World} - \vec{c} \right)$,

$$R = {}^G R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{4}) & 0 & \sin(\frac{\pi}{4}) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{4}) & 0 & \cos(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\vec{P}_{Local} = R^t \left(\vec{P}_{World} - \vec{c} \right) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}^t \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 2 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

TO DO ➔ Go to activity and solve question 9

8) Basic Primitives

1) Planes Equation

The planes has equation $\vec{n} \bullet \vec{p} + d = 0$, with $d = -\vec{n} \bullet \vec{p}_o$. Where $\vec{p}_o(x_o, y_o, z_o)$ plane referenced point. Analytically we have ax + by + cz + d=0.

2) Ray and line segment Equation

The equation of a ray (segment) is $\vec{s}(t) = \vec{s}_o + \vec{v} \cdot t$ with t>0 for the ray and $0 \le t \le 1$ for the segment

3) A box is determined by its minimum point $\vec{b}_{\min} = (x_{\min}, y_{\min}, z_{\min})$ and maximum point $\vec{b}_{\max} = (x_{\max}, y_{\max}, z_{\max})$.

4) Sphere Equation

A sphere is defined as the set of points $\vec{p}(x, y, z)$ in the space which distance to a given fixed point $\vec{c}(a, b, c)$ (center) is a constant r (=radius). We write: $\left\|\overrightarrow{cp}\right\|$=r or $\overrightarrow{cp} \bullet \overrightarrow{cp}$=r² ; analytically this $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

Example 4: Find the center $\vec{c}$ and radius R of the sphere S: $(x-2)^2 + (y+1)^2 + (z+5)^2 = 9$

Answer :

Solving x-2=0 ➔ x=2, y+1=0➔ y=-1 , and z+ 5=0 ➔ z=-5

so the center is $\vec{c}(2, -1, -5)$ and radius $R = \sqrt{9} = 3$

TO DO ➔ Go to activity and solve question 10.1

Example 5: Find the center $\vec{c}$ and radius R of the sphere S: $x^2 + (y-10)^2 + (z+6)^2 = 4$

Answer :

Solving x=0 , y-10=0➔ y=10 , and z+6=0 ➔ z=-6

so the center is $\vec{c}(0, 10, -6)$ and radius $R = \sqrt{4} = 2$

TO DO ➔ Go to activity and solve question 10.2

Example 6: write the equation of a sphere with center $\vec{c}(1, -2, 3)$ and radius R=5

Answer: Using $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, a=1, b=-2 , c=3 r=5

We have $(x-1)^2 + (y-(-2))^2 + (z-3)^2 = 5^2$ or $(x-1)^2 + (y+2)^2 + (z-3)^2 = 25$

TO DO ➔ Go to activity and solve questions 10.3 and 10.

9) Separation Contact,Resting Contact and colliding contact

Let $\hat{n}$ be the normal vector of a plane (wall, floor), and $\vec{v}$ the velocity of a body.
We define the collision state of a body with respect to a plane normal vector $\vec{n}$ as either a separation contact ,a resting contact or a colliding contact.

If $\vec{v} \bullet \hat{n}$>0, then we have a separation , no collision: the body is moving away from the static surface.
If $\vec{v} \bullet \hat{n}$=0, then we have a resting contact, the body is moving parallel to the surface.
If $\vec{v} \bullet \hat{n}$<0, then we have colliding contact: the body is approaching the static surface (possibility of collision occurrence). We assume the body is in front (above) of the plane in all cases.

$\vec{v} \bullet \hat{n}$ represents the relative speed of the body as measured in the direction of the colliding plane normal vector $\vec{n}$.

Example 7: what is the collision state of a body moving with velocity $\vec{v}$ (1, 2, 0) m/s with respect to a plan with normal vector $\vec{n}$ (0,1,0)?

Answer: $\vec{v} \bullet \hat{n} = (1,2,0) \cdot (0,1,0) = 2 > 0$ ➔ separation contact (body moving away from plane)

Example8: what is the collision state of a body moving with velocity $\vec{v}$ (1, 2, 1) m/s with respect to a plan with normal vector $\vec{n}$ (-2,1,0)?

Answer: $\vec{v} \bullet \hat{n} = (1,2,1) \cdot (-2,1,0) = -2+2 = 0$ ➔ resting contact (body moving parallel to plane)

Example 9: what is the collision state of a body moving with velocity $\vec{v}$ (-3, 2, 1) m/s with respect to a plan with normal vector $\vec{n}$ (2,1,1)?

Answer: $\vec{v} \bullet \hat{n} = (-3,2,1) \cdot (2,1,1) = -6+2+1 = -3 < 0$ ➔ colliding contact (body approaching to plane)

TO DO ➔ Go to activity and solve question 11.1, 11.2, and 11.3 .

10) <u>Sphere-Sphere Collision Detection</u>

Given 2 spheres S1 and S2 : $\begin{cases} sphere1: [\ \vec{c}_1(x_1, y_1, z_1),\ r_1\ ] \\ sphere2: [\ \vec{c}_2(x_2, y_2, z_2), r_2] \end{cases}$ r=sphere radius ; $\vec{c}$ =sphere center

If $\|\vec{c}_1 - \vec{c}_2\| \le (r_1 + r_2)$ ➔ collision else if $\|\vec{c}_1 - \vec{c}_2\| > (r_1 + r_2)$ ➔ no collision

Example 10: **Given two spheres S1 : $(x+3)^2 + (y-5)^2 + (z+4)^2 = 25$ and S2: $(x+1)^2 + (y-2)^2 + (z-2)^2 = 16$**

a) Find the components , $\vec{c}_1$, of the center of the sphere S1.

b) Find the components , $\vec{c}_2$, of the center of the sphere S2.

c) Calculate the distance $\|\overrightarrow{c_1c_2}\|$ between the two centers.

d) Verify the possibility of a collision by computation

Answer:

a) $\vec{c}_1 = (-3,5,4)$ and radius $r_1 = \sqrt{25} = 5$

b) $\vec{c}_2 = (-1,2,2)$ and radius $r_2 = \sqrt{16} = 4$

c) $\overrightarrow{c_1c_2} = \vec{c}_2 - \vec{c}_1 = (-1,2,2) - (-3,5,-4) = (2,-3,6)$, so

$$\|\overrightarrow{c_1c_2}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$$

d) Since $\left\|\overrightarrow{c_1c_2}\right\| = 7$ and $r_1 + r_2 = 5 + 4 = 9$, we have $\left\|\overrightarrow{c_1c_2}\right\| \le r_1 + r_2$ **, therefore there is a collision**

Example 11: **Given two spheres S1 : $(x-1)^2 + (y+1)^2 + z^2 = 1$ and S2: $(x+1)^2 + (y-1)^2 + (z-1)^2 = 1$**

a) Find the components , $\vec{c}_1$, of the center of the sphere S1.

b) Find the components , $\vec{c}_2$, of the center of the sphere S2.

c) Calculate the distance $\left\|\overrightarrow{c_1c_2}\right\|$ between the two centers.

d) Verify the possibility of a collision by computation

Answer:

a) $\vec{c}_1 = (1,-1,0)$ and radius $r_1 = 1$

b) $\vec{c}_2 = (-1,1,1)$ and radius $r_2 = 1$

c) $\overrightarrow{c_1c_2} = \vec{c}_2 - \vec{c}_1 = (-1,1,1) - (1,-1,0) = (-2,2,1)$, so

$$\left\|\overrightarrow{c_1c_2}\right\| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

d) Since $\left\|\overrightarrow{c_1c_2}\right\| = 3$ and $r_1 + r_2 = 1 + 1 = 2$, we have $\left\|\overrightarrow{c_1c_2}\right\| > r_1 + r_2$ **, therefore there is no collision**

TO DO ➔ Go to activity and solve questions 12.1 , 12.2 , 12.3 , and 12.4 .

11) Sphere-Plane Collision

First use the half-space test to locate the sphere relative to plane (Is it behind , front or on the plane?)

- If $\hat{n}\bullet\vec{c} + d >$ r the sphere is in front of the normalized plane.

  Then if $\left|\hat{n}\bullet\vec{c} + d\right| >$ r OR $\vec{n} \bullet \vec{v} > 0$ ➔ no collision (moving away from plane)

  If $\left|\hat{n}\bullet\vec{c} + d\right| =$ r AND $\vec{n} \bullet \vec{v} < 0$ ➔ collision

  If $\left|\hat{n}\bullet\vec{c} + d\right| <$ r AND $\vec{n} \bullet \vec{v} < 0$ ➔ penetration

  If $\left|\hat{n}\bullet\vec{c} + d\right| =$r AND $\vec{n} \bullet \vec{v} = 0$ ➔ resting contact.

  Remember that r=sphere radius ; $\vec{c}$ =sphere center, $\vec{v}$ =sphere velocity

  We assume having a normalized plane, that is $\|\vec{n}\| = 1$

- If $\hat{n}\bullet\vec{c} + d <$ r the sphere is behind of the plane, then use -$\hat{n}$ and the same technique above.

Example 12: Verify the possibility of a collision by computation between the plane P with equation $2x + y + 3z - 6 = 0$ with reference point $\vec{p}_0(1, 1, 1)$ and the sphere S with equation $(x-1)^2 + (y-2)^2 + (z-3)^2 = 9$ moving at a constant velocity $\vec{v} = (1,2,1)m/s$.

Answer:

Find plane : $\vec{n} = (2,1,3)$ d= -6

Find sphere center: $\vec{c} = (1,2,3)$, **radius** r=3

Calculate: $\vec{n} \bullet \vec{c} + d = (2,1,3) \bullet (1,2,3) - 6 = 7$

since $\vec{n} \bullet \vec{c} + d \geq r$ **the sphere is above the plane**

$\vec{v} \bullet \vec{n} = (1,2,1) \cdot (2,1,3) = 2 + 2 + 3 = 7 > 0$ ➔ separation, no collision

Example 12: Verify the possibility of a collision by computation between the plane P with equation $2x + y + 2z - 5 = 0$ with reference point $\vec{p}_0(1, 1, 1)$ and the sphere S with equation $(x-3)^2 + (y-2)^2 + (z-5)^2 = 1$ moving at a constant velocity $\vec{v} = (1,-3,0)m/s$.

Answer:

Find plane : $\vec{n} = (2,1,2)$ ➔ $\hat{n} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ d=-5

Find sphere center: $\vec{c} = (3,2,5)$, **radius** r=1

Calculate: $\hat{n} \bullet \vec{c} + d = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \bullet (3,2,5) - 5 = 6 - 5 = 1$

since $\vec{n} \bullet \vec{c} + d \geq r$ **the sphere is above the plane**

$\vec{v} \bullet \vec{n} = (1,-3,0) \cdot (2,1,2) = 2 - 3 = -1 < 0$ ➔ colliding contact ,that is the ball is approaching the plane from above. Now we compute the distance between the ball and the plane: $|\hat{n} \bullet \vec{c} + d| = |1| = 1 = r$

So $|\hat{n} \bullet \vec{c} + d| = r$ and $\vec{v} \bullet \vec{n} < 0$ ➔ collision

12) <u>Ray-Plane Collision</u>

Given a ray L: $\vec{s}(t) = \vec{s}_o + \vec{v} \cdot t$ and a plane P : $\vec{n} \bullet \vec{p} + d = 0$

Use half space test to locate ray origin $\vec{s}_o$ (front or behind) relatively to the plane.

- If $\hat{n} \bullet \vec{s}_o + d > 0$ (front) and $\vec{n} \bullet \vec{v} > 0$ ➔ no collision (separation)
- If $\hat{n} \bullet \vec{s}_o + d > 0$ (front) and $\vec{n} \bullet \vec{v} < 0$ ➔ possible collision. Now compute $t = -\frac{\vec{n} \bullet \vec{s}_o + d}{\vec{n} \bullet \vec{v}}$

  if t>0 ➔ collision else if t<=0 ➔ no collision
- if $\hat{n} \bullet \vec{s}_o + d < 0$ (behind) and $\vec{n} \bullet \vec{v} < 0$ ➔ No collision (separation)

- if $\hat{n}\bullet\vec{s}_o + d < 0$ (behind) and $\vec{n}\bullet\vec{v} > 0$ ➔ possible collision, now compute $t = -\dfrac{\vec{n}\bullet\vec{s}_o + d}{\vec{n}\bullet\vec{v}}$

  if t >0 ➔ collision else if t<=0 ➔ no collision

- if $\vec{n}\bullet\vec{v} = 0$ ➔ no collision, the ray is moving parallel to the plane

Example 13.1 :

13) <u>Line Segment- Plane Collision</u>

Given a line segment L: $\vec{s}(t) = \vec{s}_o + \vec{v}\cdot t$ and a plane P : $\vec{n}\bullet\vec{p} + d = 0$

The line segment intersects the plane if $0 \le t \le 1$ where $t = -\dfrac{\vec{n}\bullet\vec{s}_o + d}{\vec{n}\bullet\vec{v}}$.

14) <u>Segment-Segment Intersection</u>

We define two line segments P: $\vec{p} = \vec{p}_1 + t_p(\vec{p}_2 - \vec{p}_1)$ and Q: $\vec{q} = \vec{q}_1 + t_q(\vec{q}_2 - \vec{q}_1)$ where $o \le t_p \le 1$ and $o \le t_q \le 1$. The segments intersect at a common point such that $\vec{p} = \vec{q}$ ➔ $\vec{p}_1 + t_p(\vec{p}_2 - \vec{p}_1) = \vec{q}_1 + t_q(\vec{q}_2 - \vec{q}_1)$.

By setting $\vec{u} = \vec{p}_2 - \vec{p}_1$ and $\vec{v} = \vec{q}_2 - \vec{q}_1$ our equation simplifies to $\vec{p}_1 + t_p\vec{u} = \vec{q}_1 + t_q\vec{v}$ (1) .We want to compute $o \le t_p \le 1$ and $o \le t_q \le 1$. $\vec{p}_1 + t_p\vec{u} = \vec{q}_1 + t_q\vec{v}$ ➔ $t_p\vec{u} = \vec{q}_1 - \vec{p}_1 + t_q\vec{v}$ (2) . We eliminate t_q by computing $\vec{w} = \vec{q}_2 - \vec{p}_2$ and $\vec{w}\times\vec{v}$ .Since $\vec{w}\times\vec{v} \perp \vec{v}$ ➔ $(\vec{w}\times\vec{v})\bullet\vec{v} = o$. By multiplying both sides of equation (2) by $\vec{w}\times\vec{v}$ we get $t_p\vec{u}\bullet(\vec{w}\times\vec{v}) = (\vec{w}\times\vec{v})\bullet(\vec{q}_1 - \vec{p}_1) + t_q(\vec{w}\times\vec{v})\bullet\vec{v}$ ➔

$$t_p = \frac{(\vec{w}\times\vec{v})\bullet(\vec{q}_1 - \vec{p}_1)}{(\vec{w}\times\vec{v})\bullet\vec{u}} = \frac{\left[(\vec{q}_2 - \vec{p}_2)\times(\vec{q}_2 - \vec{q}_1)\right]\bullet(\vec{q}_1 - \vec{p}_1)}{\left[(\vec{q}_2 - \vec{p}_2)\times(\vec{q}_2 - \vec{q}_1)\right]\bullet(\vec{p}_2 - \vec{p}_1)}.$$

If t_p<0 or t_p>1 ➔ No intersection (we stop), else if $o \le t_p \le 1$ we continue to compute t_q.

From $\vec{p}_1 + t_p\vec{u} = \vec{q}_1 + t_q\vec{v}$ we compute t_q that is $\vec{q}_1 + t_q\vec{v} = \vec{p}_1 + t_p\vec{u}$ ➔ $t_q\vec{v} = \vec{p}_1 - \vec{q}_1 + t_p\vec{u}$ (3).

We compute $\vec{w}\times\vec{u}$. $\vec{w}\times\vec{u} \perp \vec{u}$ ➔ $(\vec{w}\times\vec{u})\bullet\vec{u} = o$. By multiplying both sides of equation (3) by $\vec{w}\times\vec{u}$ we get ➔ $t_q\vec{v}\bullet(\vec{w}\times\vec{u}) = (\vec{w}\times\vec{u})\bullet(\vec{p}_1 - \vec{q}_1) + t_p(\vec{w}\times\vec{u})\bullet\vec{u} = (\vec{w}\times\vec{u})\bullet(\vec{p}_1 - \vec{q}_1)$ implying that

$$t_q = \frac{(\vec{w}\times\vec{u})\bullet(\vec{p}_1 - \vec{q}_1)}{(\vec{w}\times\vec{u})\bullet\vec{v}} = \frac{\left[(\vec{q}_2 - \vec{p}_2)\times(\vec{p}_2 - \vec{p}_1)\right]\bullet(\vec{p}_1 - \vec{q}_1)}{\left[(\vec{q}_2 - \vec{p}_2)\times(\vec{p}_2 - \vec{p}_1)\right]\bullet(\vec{q}_2 - \vec{q}_1)}$$

If t_q<0 or t_q>1 ➔ no intersection, else if $o \le t_q \le 1$ then we finally have an intersection.

15) Ray – Ray intersection

Same as Segment-Segment intersection but with $t_q \geq o$ and $t_p \geq o$.

16) Sphere –Ray Intersection

A sphere with center $\vec{c}$ has equation $(\vec{p}-\vec{c})\bullet(\vec{p}-\vec{c})=r^2$ where r=radius and $\vec{p}$ an arbitrary point of the sphere. An intersection of a sphere and ray with equation $\vec{s}(t)=\vec{s}_o+\vec{v}\cdot t$ is determined by replacing $\vec{p}$ by $\vec{s}(t)$ and computing for $t \geq 0$.Doing so leads to $(\vec{s}_o+\vec{v}\cdot t-\vec{c})\bullet(\vec{s}_o+\vec{v}\cdot t-\vec{c})=r^2$ or $(\vec{v}\cdot t+\vec{s}_o-\vec{c})\bullet(\vec{v}\cdot t+\vec{s}_o-\vec{c})=r^2$ ➔ $\boxed{\vec{v}\cdot\vec{v}\cdot t^2+2\vec{v}\cdot(\vec{s}_o-\vec{c})\cdot t+(\vec{s}_o-\vec{c})\bullet(\vec{s}_o-\vec{c})-r^2=0}$.

Setting $a=\vec{v}\cdot\vec{v}$, $b=\vec{v}\cdot(\vec{s}_o-\vec{c})$ and $c=(\vec{s}_o-\vec{c})\bullet(\vec{s}_o-\vec{c})-r^2$ our equation becomes $\boxed{a\cdot t^2+2b\cdot t+c=0}$.

We need to find t, so we compute the discriminant $\Delta=b^2-ac$.

- If Δ<0 ➔ no solution and no collision.
- If Δ=0 and $t=\dfrac{-b}{a}>0$ ➔ ray touches the sphere at one point.
- If Δ>0 then the ray may intersect the sphere in 2 points.

  We compute $t_1=\dfrac{-b+\sqrt{\Delta}}{a}$ and $t_2=\dfrac{-b-\sqrt{\Delta}}{a}$ or $t_1=-b+\sqrt{\Delta}$ and $t_2=-b-\sqrt{\Delta}$ if $\vec{v}$ is normalized.

  If t_1>0 and t_2>0 then the ray intersects the sphere in 2 points and the first contact occurs at $t=\min(t_1,t_2)$.

  If t_1<0 and t_2<0 ➔ no collision

  If t_1< 0 and t_2>0 ➔the intersection occurs from inside the sphere.

17) Point In Triangle

A point (vertex) $\vec{x}$ is inside a triangle with vertices $\vec{a}$, $\vec{b}$ and $\vec{c}$ if only and if there exist 3 real numbers $\lambda_1 \geq 0, \lambda_2 \geq 0$,and $\lambda_3 \geq 0$ such that $\boxed{\vec{x}=\lambda_1\cdot\vec{a}+\lambda_2\cdot\vec{b}+\lambda_3\cdot\vec{c}}$ where $\lambda_1+\lambda_2+\lambda_3=1$.

Setting λ_1=1-($\lambda_2+\lambda_3$)≥ 0 , the above equation becomes $\vec{x}=(1-\lambda_2-\lambda_3)\cdot\vec{a}+\lambda_2\cdot\vec{b}+\lambda_3\cdot\vec{c}$ or simply $\vec{x}=\vec{a}+\lambda_2\cdot\overrightarrow{ab}+\lambda_3\overrightarrow{ac}$.

So $\vec{x}$ is inside the triangle if $\boxed{\vec{x}=\vec{a}+\lambda_2\cdot\overrightarrow{ab}+\lambda_3\overrightarrow{ac}}$ with $\lambda_2 \geq 0$,and $\lambda_3 \geq 0$ and $(\lambda_2+\lambda_3) \leq 1$

18) Ray-Triangle Intersection

Here we consider a ray L with equation $\vec{s}(t)=\vec{s}_o+\vec{v}\cdot t$ with t > 0 and a triangle with vertices $\vec{a}$, $\vec{b}$ and $\vec{c}$.
Let α and β be the triangle barycentric coordinate to help express the coordinate of a point inside it.
So given a point $\vec{x}$ inside the triangle, $\vec{x}=\vec{a}+\alpha\cdot\overrightarrow{ab}+\beta\cdot\overrightarrow{ac}$ where $\alpha>0$, $\beta>0$, $\alpha+\beta<1$ and $\overrightarrow{ab}=\vec{b}-\vec{a}$ and $\overrightarrow{ac}=\vec{c}-\vec{a}$.
An intersection occurs if $\vec{x}\in$ L (ray); that is $\vec{x}=\vec{s}_o+\vec{v}\cdot t$ .Since $\vec{x}=\vec{a}+\alpha\cdot\overrightarrow{ab}+\beta\cdot\overrightarrow{ac}$ this leads us to $\vec{s}_o+\vec{v}\cdot t=\vec{a}+\alpha\cdot\overrightarrow{ab}+\beta\cdot\overrightarrow{ac}$ ➔ $\boxed{-\vec{v}\cdot t+\alpha\cdot\overrightarrow{ab}+\beta\cdot\overrightarrow{ac}=\vec{s}_o-\vec{a}}$ .That is solving an algebraic linear system of 3 equations with 3 unknowns (α,β,t) using the algebraic Cramer's method.
We compute the determinant of the system, $\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})=-\vec{v}\bullet(\overrightarrow{ab}\times\overrightarrow{ac})$.

- If $\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})=0$ ➔ there is not a solution : no contact

- If $\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})\neq 0$ then we compute

$$t=\frac{\det(\vec{s}_o-\vec{a},\overrightarrow{ab},\overrightarrow{ac})}{\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})}=\frac{(\vec{s}_o-\vec{a})\bullet(\overrightarrow{ab}\times\overrightarrow{ac})}{-\vec{v}\bullet(\overrightarrow{ab}\times\overrightarrow{ac})}\ ,\ \alpha=\frac{\det(-\vec{v},\vec{s}_o-\vec{a},\overrightarrow{ac})}{\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})}=\frac{\vec{v}\bullet\left[(\vec{s}_o-\vec{a})\times\overrightarrow{ac}\right]}{\vec{v}\bullet(\overrightarrow{ab}\times\overrightarrow{ac})}$$

and $\beta=\dfrac{\det(-\vec{v},\overrightarrow{ab},\vec{s}_o-\vec{a})}{\det(-\vec{v},\overrightarrow{ab},\overrightarrow{ac})}=\dfrac{\vec{v}\bullet\left[\overrightarrow{ab}\times(\vec{s}_o-\vec{a})\right]}{\vec{v}\bullet(\overrightarrow{ab}\times\overrightarrow{ac})}$. Now we have a collision if only and if $t\geq 0$,and $\alpha\geq 0$,and $\beta\geq 0$,and $\alpha+\beta\leq 1$

19) Box-Plane Intersection

Given a plane π: $\hat{n}\bullet\vec{p}+d=0$ and an OBB with vertices $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ as defined in local space,
We compute their respective coordinates , $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$ and $\vec{w}_4$ in world space as follows:
$\vec{w}_i=\vec{c}+R\cdot\vec{v}_i$ for $1\leq i\leq 4$ where $\vec{c}$ and R are respectively the box position and orientation in the world.
The box intersects the plane if $\boxed{\left|\hat{n}\bullet\vec{w}_i+d\right|=0}$ (or better if $\left|\hat{n}\bullet\vec{w}_i+d\right|\leq\varepsilon$ where ε=ZERO_TOLERANCE) for any $\vec{w}_i$.

20) Point In Box
21) Point In Sphere
22) Point In rectangle
23) Box-Sphere Intersection
24) Rectangle-Rectangle Intersection