GEN 242: Linear Algebra

Chapter 7: Quaternions

Solutions Guide

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Table of Contents

Answers

1.A Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$:

1.A.1 $\tilde{q}_1 = [2,-1,-3,-1]$
$\tilde{q}_2 = [1,-1,0,-2]$

1.A.2 $N(q_1) = 15$
$N(q_2) = 6$

1.A.3 $q_1^{-1} = \left[\frac{2}{15}, -\frac{1}{15}, -\frac{1}{5}, -\frac{1}{15}\right]$
$q_2^{-1} = \left[\frac{1}{6}, -\frac{1}{6}, 0, -\frac{1}{3}\right]$

1.A.4 $q_1 + q_2 = [3,2,3,3]$

1.A.5 $q_1 \cdot q_2 = 5$

1.A.6 $q_1 q_2 = [-1,9,2,2]$

1.B Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$:

1.B.1 $\tilde{q}_1 = [3,-2,-1,-1]$
$\tilde{q}_2 = [2,-2,-1,0]$

1.B.2 $N(q_1) = 15$
$N(q_2) = 9$

1.B.3 $q_1^{-1} = \left[\frac{1}{5}, -\frac{2}{15}, -\frac{1}{15}, -\frac{1}{15}\right]$
$q_2^{-1} = \left[\frac{2}{9}, -\frac{2}{9}, -\frac{1}{9}, 0\right]$

1.B.4 $q_1 + q_2 = [5,4,2,1]$

1.B.5 $q_1 \cdot q_2 = 11$

1.B.6 $q_1 q_2 = [1,9,7,2]$

1.C Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$:

1.C.1 $\tilde{q}_1 = [-2,1,-1,-3]$
$\tilde{q}_2 = [5,0,0,-1]$

1.C.2 $N(q_1) = 15$
$N(q_2) = 26$

1.C.3 $q_1^{-1} = \left[-\frac{2}{15}, \frac{1}{15}, -\frac{1}{5}, -\frac{1}{5}\right]$
$q_2^{-1} = \left[\frac{5}{26}, 0,0, -\frac{1}{26}\right]$

1.C.4 $q_1 + q_2 = [3,-1,1,4]$

1.C.5 $q_1 \cdot q_2 = -7$

1.C.6 $q_1 q_2 = [-13,-4,6,13]$

1.D Given $q_1 = [2,-1,1,0]$ and $q_2 = [-1,3,-4,1]$:

1.D.1 $\tilde{q}_1 = [2,1,-1,0]$
$\tilde{q}_2 = [-1,-3,4,-1]$

1.D.2 $N(q_1) = 6$
$N(q_2) = 27$

1.D.3 $q_1^{-1} = \left[\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, 0\right]$
$q_2^{-1} = \left[-\frac{1}{27}, -\frac{1}{9}, \frac{4}{27}, -\frac{1}{27}\right]$

1.D.4 $q_1 + q_2 = [1,2,-3,1]$

1.D.5 $q_1 \cdot q_2 = -3$

1.D.6 $q_1 q_2 = [5,8,-8,3]$

2. Given $\theta_0 = 180°, \vec{v} = (0,0,1), \theta_1 = 120°, \vec{v}_1 = (1,1,-1)$:

2.a $q_0 = [0,0,0,1]$
$q_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$

2.b $q_0 + q_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$

2.c $\tilde{q}_0 = [0,0,0,-1]$

2.d $s = 2 \rightarrow \boxed{q_2 = [0,0,0,2]}$

2.e $N(q_2) = 4$

2.f $q_0 \cdot q_1 = -\frac{1}{2}$

2.g $q_0 q_1 = \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$

2.h $R_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.i $\vec{s} = (1,0,0) \rightarrow$

$\boxed{\vec{s}' = (-1,0,0)}$

3. Given $q_0 = [2,3,2,1], q_1 = [3,2,-2,0]$:

3.a $q_0 + q_1 = [5,5,0,1]$

3.b $q_o q_1 = [4,15,4,-7]$

3.c $q_0 \cdot q_1 = 8$

3.d $q_0^{-1} = \left[\frac{1}{9}, -\frac{1}{6}, -\frac{1}{9}, -\frac{1}{18}\right]$

3.e $\theta \approx 62.8°$

4. Given $q_0 = [2,0,-1,2], q_1 = [3,2,0,3]$:

4.a $q_0 + q_1 = [5,2,-1,5]$

4.b $q_0 q_1 = [0,1,1,14]$

4.c q$q_0 \cdot q_1 = 12$

4.d ${q_0}^{-1} = \left[\frac{2}{9}, 0, \frac{1}{9}, -\frac{2}{9}\right]$

4.e $\theta \approx 31.5°$

5. Given $\theta_0 = 90°, \vec{v}_0 = (0,1,0), \theta_1 = 180°, \vec{v}_1 = (1,1,0)$:

5.a $q_0 = \left[\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right]$

5.b $q_1 = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]$

5.c $\tilde{q}_1 = \left[0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]$

5.d $\theta = 60°$

6. Given $\theta_0 = 90°, \vec{v}_0 = (0,0,1), \theta_1 = 180°, \vec{v}_1 = (-1,0,1)$:

6.a $q_0 = \left[\frac{\sqrt{2}}{2}, 0,0, \frac{\sqrt{2}}{2}\right]$

6.b $q_1 = \left[0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$

6.c $\tilde{q}_1 = \left[0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]$

6.d $\theta = 60°$

Solutions

Problem 1

Problem 1.A

Given quaternions $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$:

1.A.1 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate $\tilde{q}_1$ and $\tilde{q}_2$.

$$q = [a, x, y, z] \rightarrow \tilde{q} = [a, -x, -y, -z]$$

$$\tilde{q}_1 = [(2), -(1), -(3), -(1)]$$

$$\boxed{\tilde{q}_1 = [2, -1, -3, -1]}$$

$$\tilde{q}_2 = [(1), -(1), -(0), -(2)]$$

$$\boxed{\tilde{q}_2 = [1, -1, 0, -2]}$$

1.A.2 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate $N(q_1)$ and $N(q_2)$.

$$q = [a, x, y, z] \rightarrow N(q) = a^2 + x^2 + y^2 + z^2$$

$$N(q_1) = (2)^2 + (1)^2 + (3)^2 + (1)^2$$

$$N(q_1) = 4 + 1 + 9 + 1$$

$$\boxed{N(q_1) = 15}$$

$$N(q_2) = (1)^2 + (1)^2 + (0)^2 + (2)^2$$

$$N(q_2) = 1 + 1 + 0 + 4$$

$$\boxed{N(q_2) = 6}$$

1.A.3 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate ${q_1}^{-1}$ and ${q_2}^{-1}$.

$$q^{-1} = \frac{\tilde{q}}{N(q)}$$

$${q_1}^{-1} = \frac{[(2), -(1), -(3), -(1)]}{(2)^2 + (1)^2 + (3)^2 + (1)^2}$$

$${q_1}^{-1} = \frac{[2, -1, -3, -1]}{4 + 1 + 9 + 1}$$

$${q_1}^{-1} = \frac{[2, -1, -3, -1]}{15}$$

$${q_1}^{-1} = \left[\frac{2}{15}, \frac{-1}{15}, \frac{-3}{15}, \frac{-1}{15}\right]$$

$$\boxed{{q_1}^{-1} = \left[\frac{2}{15}, -\frac{1}{15}, -\frac{1}{5}, -\frac{1}{15}\right]}$$

$${q_2}^{-1} = \frac{[(1), -(1), -(0), -(2)]}{(1)^2 + (1)^2 + (0)^2 + (2)^2}$$

$${q_2}^{-1} = \frac{[1, -1, 0, -2]}{1 + 1 + 0 + 4}$$

$${q_2}^{-1} = \frac{[1, -1, 0, -2]}{6}$$

$${q_2}^{-1} = \left[\frac{1}{6}, \frac{-1}{6}, \frac{0}{6}, \frac{-2}{6}\right]$$

$$\boxed{{q_2}^{-1} = \left[\frac{1}{6}, -\frac{1}{6}, 0, -\frac{1}{3}\right]}$$

1.A.4 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate $q_1 + q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2]$$

$$q_1 + q_2 = [a_1 + a_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$q_1 + q_2 = [a_1 + a_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$q_1 + q_2 = [(2) + (1), (1) + (1), (3) + (0), (1) + (2)]$$

$$q_1 + q_2 = [2 + 1, 1 + 1, 3 + 0, 1 + 2]$$

$$\boxed{q_1 + q_2 = [3,2,3,3]}$$

1.A.5 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate $q_1 \cdot q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2] \rightarrow q_1 \cdot q_2 = a_1a_2 + x_1x_2 + y_1y_2 + z_1z_2$$

$$q_1 \cdot q_2 = (2)(1) + (1)(1) + (3)(0) + (1)(2)$$

$$q_1 \cdot q_2 = 2 + 1 + 0 + 2$$

$$\boxed{q_1 \cdot q_2 = 5}$$

1.A.6 Given $q_1 = [2,1,3,1]$ and $q_2 = [1,1,0,2]$, calculate q_1q_2.

$$q_1q_2 = [a_1a_2 - \vec{v}_1 \cdot \vec{v}_2, a_1\vec{v}_2 + a_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$$

$$q_1q_2 = [(2)(1) - \vec{v}_1 \cdot \vec{v}_2, (2)(1,0,2) + (1)(1,3,1) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1q_2 = [2 - \vec{v}_1 \cdot \vec{v}_2, (2,0,4) + (1,3,1) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1q_2 = [2 - \vec{v}_1 \cdot \vec{v}_2, (3,3,5) + \vec{v}_1 \times \vec{v}_2]$$

$$\vec{v}_1 \cdot \vec{v}_2 = (v_1)_x(v_2)_x + (v_1)_y(v_2)_y + (v_1)_z(v_2)_z$$

$$\vec{v}_1 \cdot \vec{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

$$\vec{v}_1 \cdot \vec{v}_2 = (1)(1) + (3)(0) + (1)(2)$$

$$\vec{v}_1 \cdot \vec{v}_2 = 1 + 0 + 2$$

$$\vec{v}_1 \cdot \vec{v}_2 = 3$$

$$q_1q_2 = [2 - 3, (3,3,5) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1q_2 = [-1, (3,3,5) + \vec{v}_1 \times \vec{v}_2]$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} \hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = [(3)(2) - (0)(1)]\hat{\imath} - [(1)(2) - (1)(1)]\hat{\jmath} + [(1)(0) - (1)(3)]\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (6 - 0)\hat{\imath} - (2 - 1)\hat{\jmath} + (0 - 3)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (6)\hat{\imath} - (1)\hat{\jmath} + (-3)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (6, -1, -3)$$

$$q_1 q_2 = [-1, (3,3,5) + (6, -1, -3)]$$

$$q_1 q_2 = [-1, (9,2,2)]$$

$$\boxed{q_1 q_2 = [-1,9,2,2]}$$

Problem 1.B

Given quaternions $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$:

1.B.1 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate $\tilde{q}_1$ and $\tilde{q}_2$.

$$q = [a, x, y, z] \rightarrow \tilde{q} = [a, -x, -y, -z]$$

$$\tilde{q}_1 = [(3), -(2), -(1), -(1)]$$

$$\boxed{\tilde{q}_1 = [3, -2, -1, -1]}$$

$$\tilde{q}_2 = [(2), -(2), -(1), -(0)]$$

$$\boxed{\tilde{q}_2 = [2, -2, -1,0]}$$

1.B.2 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate $N(q_1)$ and $N(q_2)$.

$$q = [a, x, y, z] \rightarrow N(q) = a^2 + x^2 + y^2 + z^2$$

$$N(q_1) = (3)^2 + (2)^2 + (1)^2 + (1)^2$$

$$N(q_1) = 9 + 4 + 1 + 1$$

$$\boxed{N(q_1) = 15}$$

$$N(q_2) = (2)^2 + (2)^2 + (1)^2 + (0)^2$$

$$N(q_2) = 4 + 4 + 1 + 0$$

$$\boxed{N(q_2) = 9}$$

1.B.3 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate ${q_1}^{-1}$ and ${q_2}^{-1}$.

$$q^{-1} = \frac{\tilde{q}}{N(q)}$$

$${q_1}^{-1} = \frac{[(3), -(2), -(1), -(1)]}{(3)^2 + (2)^2 + (1)^2 + (1)^2}$$

$${q_1}^{-1} = \frac{[3, -2, -1, -1]}{9 + 4 + 1 + 1}$$

$${q_1}^{-1} = \frac{[3, -2, -1, -1]}{15}$$

$${q_1}^{-1} = \left[\frac{3}{15}, \frac{-2}{15}, \frac{-1}{15}, \frac{-1}{15}\right]$$

$$\boxed{{q_1}^{-1} = \left[\frac{1}{5}, -\frac{2}{15}, -\frac{1}{15}, -\frac{1}{15}\right]}$$

$${q_2}^{-1} = \frac{[(2), -(2), -(1), -(0)]}{(2)^2 + (2)^2 + (1)^2 + (0)^2}$$

$${q_2}^{-1} = \frac{[2, -2, -1, 0]}{4 + 4 + 1 + 0}$$

$${q_2}^{-1} = \frac{[2, -2, -1, 0]}{9}$$

$${q_2}^{-1} = \left[\frac{2}{9}, \frac{-2}{9}, \frac{-1}{9}, \frac{0}{9}\right]$$

$$\boxed{{q_2}^{-1} = \left[\frac{2}{9}, -\frac{2}{9}, -\frac{1}{9}, 0\right]}$$

1.B.4 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate $q_1 + q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2]$$

$$q_1 + q_2 = [a_1 + a_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$q_1 + q_2 = [(3) + (2), (2) + (2), (1) + (1), (1) + (0)]$$

$$q_1 + q_2 = [3 + 2, 2 + 2, 1 + 1, 1 + 0]$$

$$\boxed{q_1 + q_2 = [5,4,2,1]}$$

1.B.5 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate $q_1 \cdot q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2] \rightarrow q_1 \cdot q_2 = a_1 a_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$q_1 \cdot q_2 = (3)(2) + (2)(2) + (1)(1) + (1)(0)$$

$$q_1 \cdot q_2 = 6 + 4 + 1 + 0$$

$$\boxed{q_1 \cdot q_2 = 11}$$

1.B.6 Given $q_1 = [3,2,1,1]$ and $q_2 = [2,2,1,0]$, calculate $q_1 q_2$.

$$q_1 q_2 = [a_1 a_2 - \vec{v}_1 \cdot \vec{v}_2, a_1 \vec{v}_2 + a_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$$

$$q_1 q_2 = [(3)(2) - \vec{v}_1 \cdot \vec{v}_2, (3)(2,1,0) + (2)(2,1,1) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1 q_2 = [6 - \vec{v}_1 \cdot \vec{v}_2, (6,3,0) + (4,2,2) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1 q_2 = [6 - \vec{v}_1 \cdot \vec{v}_2, (10,5,2) + \vec{v}_1 \times \vec{v}_2]$$

$$\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\vec{v}_1 \cdot \vec{v}_2 = (2)(2) + (1)(1) + (1)(0)$$

$$\vec{v}_1 \cdot \vec{v}_2 = 4 + 1 + 0$$

$$\vec{v}_1 \cdot \vec{v}_2 = 5$$

$$q_1 q_2 = [6 - 5, (10,5,2) + \vec{v}_1 \times \vec{v}_2]$$

$$q_1 q_2 = [1, (10,5,2) + \vec{v}_1 \times \vec{v}_2]$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = [(1)(0) - (1)(1)]\hat{\imath} - [(2)(0) - (2)(1)]\hat{\jmath} + [(2)(1) - (2)(1)]\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (0-1)\hat{\imath} - (0-2)\hat{\jmath} + (2-2)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (-1)\hat{\imath} - (-2)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (-1,2,0)$$

$$q_1q_2 = [1,(10,5,2) + (-1,2,0)]$$

$$q_1q_2 = [1,(9,7,2)]$$

$$\boxed{q_1q_2 = [1,9,7,2]}$$

Problem 1.C

Given quaternions $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$:

1.C.1 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate $\tilde{q}_1$ and $\tilde{q}_2$.

$$q = [a,x,y,z] \rightarrow \tilde{q} = [a,-x,-y,-z]$$

$$\tilde{q}_1 = [(-2),-(-1),-(1),-(3)]$$

$$\boxed{\tilde{q}_1 = [-2,1,-1,-3]}$$

$$\tilde{q}_2 = [(5),-(0),-(0),-(1)]$$

$$\boxed{\tilde{q}_2 = [5,0,0,-1]}$$

1.C.2 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate $N(q_1)$ and $N(q_2)$.

$$q = [a,x,y,z] \rightarrow N(q) = a^2 + x^2 + y^2 + z^2$$

$$N(q_1) = (-2)^2 + (-1)^2 + (1)^2 + (3)^2$$

$$N(q_1) = 4 + 1 + 1 + 9$$

$$\boxed{N(q_1) = 15}$$

$$N(q_2) = (5)^2 + (0)^2 + (0)^2 + (1)^2$$

$$N(q_2) = 25 + 0 + 0 + 1$$

$$\boxed{N(q_2) = 26}$$

1.C.3 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate ${q_1}^{-1}$ and ${q_2}^{-1}$.

$$q^{-1} = \frac{\tilde{q}}{N(q)}$$

$${q_1}^{-1} = \frac{[(-2),-(-1),-(1),-(3)]}{(-2)^2+(-1)^2+(1)^2+(3)^2}$$

$${q_1}^{-1} = \frac{[-2,1,-1,-3]}{4+1+1+9}$$

$${q_1}^{-1} = \frac{[-2,1,-1,-3]}{15}$$

$${q_1}^{-1} = \left[\frac{-2}{15},\frac{1}{15},\frac{-1}{15},\frac{-3}{15}\right]$$

$$\boxed{{q_1}^{-1} = \left[-\frac{2}{15},\frac{1}{15},-\frac{1}{15},-\frac{1}{5}\right]}$$

$${q_2}^{-1} = \frac{[(5),-(0),-(0),-(1)]}{(5)^2+(0)^2+(0)^2+(1)^2}$$

$${q_2}^{-1} = \frac{[5,0,0,-1]}{25+0+0+1}$$

$${q_2}^{-1} = \frac{[5,0,0,-1]}{26}$$

$${q_2}^{-1} = \left[\frac{5}{26},\frac{0}{26},\frac{0}{26},\frac{-1}{26}\right]$$

$$\boxed{{q_2}^{-1} = \left[\frac{5}{26},0,0,-\frac{1}{26}\right]}$$

1.C.4 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate $q_1 + q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2]$$

$$q_1 + q_2 = [a_1 + a_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$q_1 + q_2 = [(-2)+(5),(-1)+(0),(1)+(0),(3)+(1)]$$

$$q_1 + q_2 = [-2+5,-1+0,1+0,3+1]$$

$$\boxed{q_1 + q_2 = [3,-1,1,4]}$$

1.C.5 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate $q_1 \cdot q_2$.

$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2] \rightarrow q_1 \cdot q_2 = a_1 a_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$

$q_1 \cdot q_2 = (-2)(5) + (-1)(0) + (1)(0) + (3)(1)$

$q_1 \cdot q_2 = -10 + 0 + 0 + 3$

$\boxed{q_1 \cdot q_2 = -7}$

1.C.6 Given $q_1 = [-2,-1,1,3]$ and $q_2 = [5,0,0,1]$, calculate $q_1 q_2$.

$q_1 q_2 = [a_1 a_2 - \vec{v}_1 \cdot \vec{v}_2, a_1 \vec{v}_2 + a_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$

$q_1 q_2 = [(-2)(5) - \vec{v}_1 \cdot \vec{v}_2, (-2)(0,0,1) + (5)(-1,1,3) + \vec{v}_1 \times \vec{v}_2]$

$q_1 q_2 = [-10 - \vec{v}_1 \cdot \vec{v}_2, (0,0,-2) + (-5,5,15) + \vec{v}_1 \times \vec{v}_2]$

$q_1 q_2 = [-10 - \vec{v}_1 \cdot \vec{v}_2, (-5,5,13) + \vec{v}_1 \times \vec{v}_2]$

$\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\vec{v}_1 \cdot \vec{v}_2 = (-1)(0) + (1)(0) + (3)(1)$

$\vec{v}_1 \cdot \vec{v}_2 = 0 + 0 + 3$

$\vec{v}_1 \cdot \vec{v}_2 = 3$

$q_1 q_2 = [-10 - 3, (-5,5,13) + \vec{v}_1 \times \vec{v}_2]$

$q_1 q_2 = [-13, (-5,5,13) + \vec{v}_1 \times \vec{v}_2]$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} \hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = [(1)(1) - (0)(3)]\hat{\imath} - [(-1)(1) - (0)(3)]\hat{\jmath}$$
$$+ [(-1)(0) - (1)(0)]\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1 - 0)\hat{\imath} - (-1 - 01)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1)\hat{\imath} - (-1)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1,1,0)$$

$$q_1 q_2 = [-13, (-5,5,13) + (1,1,0)]$$

$$q_1 q_2 = [-13, (-4,6,13)]$$

$$\boxed{q_1 q_2 = [-13, -4,6,13]}$$

Problem 1.D

Given quaternions $q_1 = [2, -1,1,0]$ and $q_2 = [-1,3, -4,1]$:

1.D.1 Given $q_1 = [2, -1,1,0]$ and $q_2 = [-1,3, -4,1]$, calculate $\tilde{q}_1$ and $\tilde{q}_2$.

$$q = [a, x, y, z] \rightarrow \tilde{q} = [a, -x, -y, -z]$$

$$\tilde{q}_1 = [(2), -(-1), -(1), -(0)]$$

$$\boxed{\tilde{q}_1 = [2,1, -1,0]}$$

$$\tilde{q}_2 = [(-1), -(3), -(-4), -(1)]$$

$$\boxed{\tilde{q}_2 = [-1, -3,4, -1]}$$

1.D.2 Given $q_1 = [2, -1,1,0]$ and $q_2 = [-1,3, -4,1]$, calculate $N(q_1)$ and $N(q_2)$.

$$q = [a, x, y, z] \rightarrow N(q) = a^2 + x^2 + y^2 + z^2$$

$$N(q_1) = (2)^2 + (-1)^2 + (1)^2 + (0)^2$$

$$N(q_1) = 4 + 1 + 1 + 0$$

$$\boxed{N(q_1) = 6}$$

$$N(q_2) = (-1)^2 + (3)^2 + (-4)^2 + (1)^2$$

$$N(q_2) = 1 + 9 + 16 + 1$$

$$\boxed{N(q_2) = 27}$$

1.D.3 Given $q_1 = [2,-1,1,0]$ and $q_2 = [-1,3,-4,1]$, calculate ${q_1}^{-1}$ and ${q_2}^{-1}$.

$$q^{-1} = \frac{\tilde{q}}{N(q)}$$

$${q_1}^{-1} = \frac{[(2),-(-1),-(1),-(0)]}{(2)^2+(-1)^2+(1)^2+(0)^2}$$

$${q_1}^{-1} = \frac{[2,1,-1,0]}{4+1+1+0}$$

$${q_1}^{-1} = \frac{[2,1,-1,0]}{6}$$

$${q_1}^{-1} = \left[\frac{2}{6},\frac{1}{6},\frac{-1}{6},\frac{0}{6}\right]$$

$$\boxed{{q_1}^{-1} = \left[\frac{1}{3},\frac{1}{6},-\frac{1}{6},0\right]}$$

$${q_2}^{-1} = \frac{[(-1),-(3),-(-4),-(1)]}{(-1)^2+(3)^2+(-4)^2+(1)^2}$$

$${q_2}^{-1} = \frac{[-1,-3,4,-1]}{1+9+16+1}$$

$${q_2}^{-1} = \frac{[-1,-3,4,-1]}{27}$$

$${q_2}^{-1} = \left[\frac{-1}{27},\frac{-3}{27},\frac{4}{27},\frac{-1}{27}\right]$$

$$\boxed{{q_2}^{-1} = \left[-\frac{1}{27},-\frac{1}{9},\frac{4}{27},-\frac{1}{27}\right]}$$

1.D.4 Given $q_1 = [2,-1,1,0]$ and $q_2 = [-1,3,-4,1]$, calculate $q_1 + q_2$.

$$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2]$$

$$q_1 + q_2 = [a_1 + a_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$q_1 + q_2 = [(2)+(-1),(-1)+(3),(1)+(-4),(0)+(1)]$$

$$q_1 + q_2 = [2-1,-1+3,1-4,0+1]$$

$$\boxed{q_1 + q_2 = [1,2,-3,1]}$$

1.D.5 Given $q_1 = [2, -1,1,0]$ and $q_2 = [-1,3, -4,1]$, calculate $q_1 \cdot q_2$.

$q_1 = [a_1, x_1, y_1, z_1], q_2 = [a_2, x_2, y_2, z_2] \rightarrow q_1 \cdot q_2 = a_1a_2 + x_1x_2 + y_1y_2 + z_1z_2$

$q_1 \cdot q_2 = (2)(-1) + (-1)(3) + (1)(-4) + (0)(1)$

$q_1 \cdot q_2 = -2 + 3 + (-4) + 0$

$\boxed{q_1 \cdot q_2 = -3}$

1.D.6 Given $q_1 = [2, -1,1,0]$ and $q_2 = [-1,3, -4,1]$, calculate q_1q_2.

$q_1q_2 = [a_1a_2 - \vec{v}_1 \cdot \vec{v}_2, a_1\vec{v}_2 + a_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$

$q_1q_2 = [(2)(-1) - \vec{v}_1 \cdot \vec{v}_2, (2)(3, -4,1) + (-1)(-1,1,0) + \vec{v}_1 \times \vec{v}_2]$

$q_1q_2 = [-2 - \vec{v}_1 \cdot \vec{v}_2, (6, -8,2) + (1, -1,0) + \vec{v}_1 \times \vec{v}_2]$

$q_1q_2 = [-2 - \vec{v}_1 \cdot \vec{v}_2, (7, -9,2) + \vec{v}_1 \times \vec{v}_2]$

$\vec{v}_1 \cdot \vec{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$

$\vec{v}_1 \cdot \vec{v}_2 = (-1)(3) + (1)(-4) + (0)(1)$

$\vec{v}_1 \cdot \vec{v}_2 = -3 + (-4) + 0$

$\vec{v}_1 \cdot \vec{v}_2 = -7$

$q_1q_2 = [-2 - (-7), (7, -9,2) + \vec{v}_1 \times \vec{v}_2]$

$q_1q_2 = [5, (7, -9,2) + \vec{v}_1 \times \vec{v}_2]$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ 3 & -4 & 1 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} \hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = [(1)(1) - (-4)(0)]\hat{\imath} - [(-1)(1) - (3)(0)]\hat{\jmath}$$
$$+ [(-1)(-4) - (3)(1)]\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1 - 0)\hat{\imath} - (-1 - 0)\hat{\jmath} + (4 - 3)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1)\hat{\imath} - (-1)\hat{\jmath} + (1)\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = (1,1,1)$$

$$q_1 q_2 = [5, (7, -9,2) + (1,1,1)]$$

$$q_1 q_2 = [5, (8, -8,3)]$$

$$\boxed{q_1 q_2 = [5,8, -8,3]}$$

Problem 2

Let q_0 be the quaternion representing a $180°$ rotation about the z-axis and let q_1 be the quaternion representing a $120°$ rotation about an axis parallel to vector $\vec{v}_1 = (1,1,-1)$.

Problem 2.a

Given $\theta_0 = 180°$, $\theta_1 = 120°$, $\vec{v}_0 = (0,01)$, and $\vec{v}_1 = (1,1,-1)$, construct q_0 and q_1.

$$q_0 = \cos\left(\frac{\theta_0}{2}\right) + (\hat{v}_0)_x \cdot \sin\left(\frac{\theta_0}{2}\right) \cdot \hat{\imath} + (\hat{v}_0)_y \cdot \sin\left(\frac{\theta_0}{2}\right) \cdot \hat{\jmath} + (\hat{v}_0)_z \cdot \sin\left(\frac{\theta_0}{2}\right) \cdot \hat{k}$$

$$q_0 = \cos\left(\frac{180°}{2}\right) + 0 \cdot \sin\left(\frac{180°}{2}\right) \cdot \hat{\imath} + 0 \cdot \sin\left(\frac{180°}{2}\right) \cdot \hat{\jmath} + 1 \cdot \sin\left(\frac{180°}{2}\right) \cdot \hat{k}$$

$$q_0 = \cos(90°) + 0\hat{\imath} + 0\hat{\jmath} + 1 \cdot \sin(90°) \cdot \hat{k}$$

Note that $\vec{v}_0 = (0,0,1) = \hat{v}$ already.

$$q_0 = 0 + 0\hat{\imath} + 0\hat{\jmath} + 1 \cdot 1 \cdot \hat{k}$$

$$q_0 = 0 + 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

$$\boxed{q_0 = [0,0,0,1]}$$

$$q_1 = \cos\left(\frac{\theta_1}{2}\right) + (\hat{v}_1)_x \cdot \sin\left(\frac{\theta_1}{2}\right) \cdot \hat{\imath} + (\hat{v}_1)_y \cdot \sin\left(\frac{\theta_1}{2}\right) \cdot \hat{\jmath} + (\hat{v}_1)_z \cdot \sin\left(\frac{\theta_1}{2}\right) \cdot \hat{k}$$

$$\hat{v}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\hat{v}_1 = \frac{\vec{v}_1}{\sqrt{{(v_1)_x}^2 + {(v_1)_y}^2 + {(v_1)_z}^2}}$$

$$\hat{v}_1 = \frac{(1,1,-1)}{\sqrt{(1)^2 + (1)^2 + (-1)^2}}$$

$$\hat{v}_1 = \frac{(1,1,-1)}{\sqrt{1+1+1}}$$

$$\hat{v}_1 = \frac{(1,1,-1)}{\sqrt{3}}$$

$$\hat{v}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$q_1 = \cos\left(\frac{120°}{2}\right) + \frac{1}{\sqrt{3}} \cdot \sin\left(\frac{120°}{2}\right) \cdot \hat{\imath} + \frac{1}{\sqrt{3}} \cdot \sin\left(\frac{120°}{2}\right) \cdot \hat{\jmath} + \left(-\frac{1}{\sqrt{3}}\right) \cdot \sin\left(\frac{120°}{2}\right) \cdot \hat{k}$$

$$q_1 = \cos(60°) + \frac{1}{\sqrt{3}}\sin(60°) \cdot \hat{\imath} + \frac{1}{\sqrt{3}}\sin(60°) \cdot \hat{\jmath} - \frac{1}{\sqrt{3}}\sin(60°) \cdot \hat{k}$$

$$q_1 = \frac{1}{2} + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\hat{\imath} + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\hat{\jmath} - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\hat{k}$$

$$q_1 = \frac{1}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} - \frac{1}{2}\hat{k}$$

$$\boxed{q_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]}$$

Problem 2.b

Using q_0 and q_1 found in 2.a, above, calculate $q_0 + q_1$.

$$q_0 + q_1 = [a_0 + a_1, x_0 + x_1, y_0 + y_1, z_0 + z_1]$$

$$q_0 + q_1 = \left[(0) + \left(\frac{1}{2}\right), (0) + \left(\frac{1}{2}\right), (0) + \left(\frac{1}{2}\right), (1) + \left(-\frac{1}{2}\right)\right]$$

$$q_0 + q_1 = \left[0 + \frac{1}{2}, 0 + \frac{1}{2}, 0 + \frac{1}{2}, 1 - \frac{1}{2}\right]$$

$$\boxed{q_0 + q_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]}$$

Problem 2.c

Using q_0 found in 2.a, above, determine the conjugate of q_0.

$$\tilde{q}_0 = [a, -x, -y, -z]$$

$$\tilde{q}_0 = [(0), -(0), -(0), -(1)]$$

$$\boxed{\tilde{q}_0 = [0,0,0,-1]}$$

Problem 2.d

Using q_0 found in 2.a, above, determine $q_2 = sq_0$ if $s = 2$.

$$q_2 = 2[0,0,0,1]$$

$$\boxed{q_2 = [0,0,0,2]}$$

Problem 2.e

Using q_2 found in 2.d, above, determine the norm of q_2.

$$N(q_2) = {a_2}^2 + {x_2}^2 + {y_2}^2 + {z_2}^2$$

$$N(q_2) = (0)^2 + (0)^2 + (0)^2 + (2)^2$$

$$N(q_2) = 0 + 0 + 0 + 4$$

$$\boxed{N(q_2) = 4}$$

Problem 2.f

Using q_0 and q_1 found in 2.a, above, calculate $q_0 \cdot q_1$.

$$q_0 \cdot q_1 = a_o a_1 + x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$q_0 \cdot q_1 = (0)\left(\frac{1}{2}\right) + (0)\left(\frac{1}{2}\right) + (0)\left(\frac{1}{2}\right) + (1)\left(-\frac{1}{2}\right)$$

$$q_0 \cdot q_1 = 0 + 0 + 0 + \left(-\frac{1}{2}\right)$$

$$\boxed{q_0 \cdot q_1 = -\frac{1}{2}}$$

Problem 2.g

Using q_0 and q_1 found in 2.a, above, calculate $q_0 q_1$.

$$q_0 q_1 = [a_0 a_1 - \vec{v}_0 \cdot \vec{v}_1, a_0 \vec{v}_1 + a_1 \vec{v}_0 + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = \left[(0)\left(\frac{1}{2}\right) - \vec{v}_0 \cdot \vec{v}_1, (0)\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) + \left(\frac{1}{2}\right)(0,0,1) + \vec{v}_0 \times \vec{v}_1\right]$$

$$q_0 q_1 = \left[0 - \vec{v}_0 \cdot \vec{v}_1, (0,0,0) + \left(0,0,\frac{1}{2}\right) + \vec{v}_0 \times \vec{v}_1\right]$$

$$q_0 q_1 = \left[0 - \vec{v}_0 \cdot \vec{v}_1, \left(0,0,\frac{1}{2}\right) + \vec{v}_0 \times \vec{v}_1\right]$$

$$\vec{v}_0 \cdot \vec{v}_1 = x_0x_1 + y_0y_1 + z_0z_1$$

$$\vec{v}_0 \cdot \vec{v}_1 = (0)\left(\frac{1}{2}\right) + (0)\left(\frac{1}{2}\right) + (1)\left(-\frac{1}{2}\right)$$

$$\vec{v}_0 \cdot \vec{v}_1 = 0 + 0 + \left(-\frac{1}{2}\right)$$

$$\vec{v}_0 \cdot \vec{v}_1 = -\frac{1}{2}$$

$$q_0q_1 = \left[0 - \left(-\frac{1}{2}\right), \left(0,0,\frac{1}{2}\right) + \vec{v}_0 \times \vec{v}_1\right]$$

$$q_0q_1 = \left[\frac{1}{2}, \left(0,0,\frac{1}{2}\right) + \vec{v}_0 \times \vec{v}_1\right]$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = \left[(0)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)(1)\right]\hat{\imath} - \left[(0)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)(1)\right]\hat{\jmath} + \left[(0)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)(0)\right]\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = \left(0 - \frac{1}{2}\right)\hat{\imath} - \left(0 - \frac{1}{2}\right)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = \left(-\frac{1}{2}\right)\hat{\imath} - \left(-\frac{1}{2}\right)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$q_0q_1 = \left[\frac{1}{2}, \left(0,0,\frac{1}{2}\right) + \left(-\frac{1}{2}, \frac{1}{2}, 0\right)\right]$$

$$q_0q_1 = \left[\frac{1}{2}, \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right]$$

$$\boxed{q_1q_2 = \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]}$$

Problem 2.h

Using q_0 found in 2.a, above, convert q_0 to a rotation matrix.

$$q = [a, x, y, z] \rightarrow R = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2az & 2xz + 2ay \\ 2xy + 2az & 1 - 2x^2 - 2z^2 & 2yz - 2ax \\ 2xz - 2ay & 2yz + 2ax & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

$$q_0 = [0,0,0,1]$$

$$R_0 = \begin{bmatrix} 1 - 2(0)^2 - 2(1)^2 & 2(0)(0) - 2(0)(1) & 2(0)(1) + 2(0)(0) \\ 2(0)(0) + 2(0)(1) & 1 - 2(0)^2 - 2(1)^2 & 2(0)(1) - 2(0)(0) \\ 2(0)(1) - 2(0)(0) & 2(0)(1) + 2(0)(0) & 1 - 2(0)^2 - 2(0)^2 \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 1 - 2(0) - 2(1) & 2(0)(0) - 2(0)(1) & 2(0)(1) + 2(0)(0) \\ 2(0)(0) + 2(0)(1) & 1 - 2(0) - 2(1) & 2(0)(1) - 2(0)(0) \\ 2(0)(1) - 2(0)(0) & 2(0)(1) + 2(0)(0) & 1 - 2(0) - 2(0) \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 1 - 0 - 2 & 0 - 0 & 0 + 0 \\ 0 + 0 & 1 - 0 - 2 & 0 - 0 \\ 0 - 0 & 0 + 0 & 1 - 0 - 0 \end{bmatrix}$$

$$\boxed{R_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem 2.i

Using q_0 found in 2.a, above, rotate the vector $\vec{s} = (1,0,0)$.

$$q_{s'} = q_0 q_s \tilde{q}_0$$

$q_0 = [0,0,0,1]$ From Part a, above.

$\tilde{q}_0 = [0,0,0,-1]$ From Part c, above.

$q_s = [0, \hat{s}] = [0,1,0,0]$ Note that $\vec{s} = \hat{s}$ already.

$$q_{s'} = [0,0,0,1][0,1,0,0]\tilde{q}_0$$

$$q_m q_n = [a_m a_n - \vec{v}_m \cdot \vec{v}_n, a_m \vec{v}_n + a_n \vec{v}_m + \vec{v}_m \times \vec{v}_n]$$

$$\vec{v}_q \times \vec{v}_s = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\vec{v}_q \times \vec{v}_s = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{k}$$

$$\vec{v}_q \times \vec{v}_s = [(0)(0) - (0)(1)]\hat{\imath} - [(0)(0) - (1)(1)]\hat{\jmath}$$

$$+ [(0)(0) - (1)(0)]\hat{k}$$

$$\vec{v}_q \times \vec{v}_s = (0 - 0)\hat{\imath} - (0 - 1)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\vec{v}_q \times \vec{v}_s = (0)\hat{\imath} - (-1)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_q \times \vec{v}_s = (0)\hat{\imath} + (1)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_q \times \vec{v}_s = (0,1,0)$$

$$q_{s'} = \big[(0)(0) - \big((0)(1) + (0)(0) + (0)(1)\big), (0)\vec{v}_m + (0)\vec{v}_n + (0,1,0)\big]\tilde{q}_0$$

$$q_{s'} = [0 - (0 + 0 + 0), 0 + 0 + (0,1,0)]\tilde{q}_0$$

$$q_{s'} = [0 - (0), (0,1,0)]\tilde{q}_0$$

$$q_{s'} = [0,0,1,0]\tilde{q}_0$$

$$q_{s'} = [0,0,1,0][0,0,0,-1]$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \hat{k}$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = [(1)(-1) - (0)(0)]\hat{\imath} - [(0)(-1) - (0)(0)]\hat{\jmath}$$

$$+ [(0)(0) - (0)(1)]\hat{k}$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = (-1 - 0)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = (-1)\hat{\imath} - (0)\hat{\jmath} + (0)\hat{k}$$

$$\vec{v}_{qs} \times \vec{v}_{\tilde{q}} = (-1,0,0)$$

$$q_{s'} = \left[(0)(0) - \left((0)(0) + (1)(0) + (0)(0)\right), (0)\vec{v}_n + (0)\vec{v}_m + (-1,0,0)\right]$$

$$q_{s'} = [0 - (0 + 0 + 0), 0 + 0 + (-1,0,0)]$$

$$q_{s'} = [0 - (0), (-1,0,0)]$$

$$q_{s'} = [0, (-1,0,0)]$$

$$q_{s'} = [0, \vec{v}_{s'}]$$

$$\boxed{\vec{v}_{s'} = (-1,0,0)}$$

Problem 3

Given quaternions $q_0 = [2,3,2,1]$ and $q_1 = [3,2,-2,0]$:

Problem 3.a

Given $q_0 = [2,3,2,1]$ and $q_1 = [3,2,-2,0]$, calculate $q_0 + q_1$.

$$q_0 + q_1 = [a_0 + a_1, x_0 + x_1, y_0 + y_1, z_0 + z_1]$$

$$q_0 + q_1 = [(2) + (3), (3) + (2), (2) + (-2), (1) + (0)]$$

$$\boxed{q_0 + q_1 = [5,5,0,1]}$$

Problem 3.b

Given $q_0 = [2,3,2,1]$ and $q_1 = [3,2,-2,0]$, calculate $q_0 q_1$.

$$q_0 q_1 = [a_0 a_1 - \vec{v}_0 \cdot \vec{v}_1, a_0 \vec{v}_1 + a_1 \vec{v}_0 + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [(2)(3) - \vec{v}_0 \cdot \vec{v}_1, (2)(2,-2,0) + (3)(3,2,1) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [6 - \vec{v}_0 \cdot \vec{v}_1, (4,-4,0) + (9,6,3) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [6 - \vec{v}_0 \cdot \vec{v}_1, (13,2,3) + \vec{v}_0 \times \vec{v}_1]$$

$$\vec{v}_0 \cdot \vec{v}_1 = x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$\vec{v}_0 \cdot \vec{v}_1 = (3)(2) + (2)(-2) + (1)(0)$$

$$\vec{v}_0 \cdot \vec{v}_1 = 6 + (-4) + (0)$$

$$\vec{v}_0 \cdot \vec{v}_1 = 2$$

$$q_0q_1 = [6 - 2, (13,2,3) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0q_1 = [4, (13,2,3) + \vec{v}_0 \times \vec{v}_1]$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 2 & 1 \\ 2 & -2 & 0 \end{vmatrix}$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} \hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = [(2)(0) - (-2)(1)]\hat{\imath} - [(3)(0) - (2)(1)]\hat{\jmath} + [(3)(-2) - (2)(2)]\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = [0 - (-2)]\hat{\imath} - (0 - 2)\hat{\jmath} + (-6 - 4)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (2)\hat{\imath} - (-2)\hat{\jmath} + (-10)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (2)\hat{\imath} + (2)\hat{\jmath} + (-10)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (2,2,-10)$$

$$q_0q_1 = [4, (13,2,3) + (2,2,-10)]$$

$$q_0q_1 = [4, (15,4,-7)]$$

$$\boxed{q_1q_2 = [4,15,4,-7]}$$

Problem 3.c

Given $q_0 = [2,3,2,1]$ and $q_1 = [3,2,-2,0]$, calculate $q_0 \cdot q_1$.

$$q_0 \cdot q_1 = a_o a_1 + x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$q_0 \cdot q_1 = (2)(3) + (3)(2) + (2)(-2) + (1)(0)$$

$$q_0 \cdot q_1 = 6 + 6 + (-4) + 0$$

$$\boxed{q_0 \cdot q_1 = 8}$$

Problem 3.d

Given $q_0 = [2,3,2,1]$, determine the inverse of q_0.

$$q_0^{-1} = \frac{\tilde{q}_0}{N(q_0)}$$

$$\tilde{q}_0 = [a_0, -x_0, -y_0, -z_0]$$

$$\tilde{q}_0 = [(2), -(3), -(2), -(1)]$$

$$\tilde{q}_0 = [2, -3, -2, -1]$$

$$N(q_0) = a_0^2 + x_0^2 + y_0^2 + z_0^2$$

$$N(q_0) = (2)^2 + (3)^2 + (2)^2 + (1)^2$$

$$N(q_0) = 4 + 9 + 4 + 1$$

$$N(q_0) = 18$$

$$q_0^{-1} = \frac{[2, -3, -2, -1]}{18}$$

$$q_0^{-1} = \left[\frac{2}{18}, \frac{-3}{18}, \frac{-2}{18}, \frac{-1}{18}\right]$$

$$\boxed{q_0^{-1} = \left[\frac{1}{9}, -\frac{1}{6}, -\frac{1}{9}, -\frac{1}{18}\right]}$$

Problem 3.e

Given $q_0 = [2,3,2,1]$ and $q_1 = [3,2,-2,0]$, calculate the angle between q_0 and q_1.

$$q_0 \cdot q_1 = \sqrt{N(q_0)} \cdot \sqrt{N(q_1)} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}$$

$$\theta = \cos^{-1}\left(\frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}\right)$$

$$q_0 \cdot q_1 = 8$$

From Part c, above.

$$N(q_0) = 18$$

From Part d, above.

$$N(q_1) = {a_1}^2 + {x_1}^2 + {y_1}^2 + {z_1}^2$$

$$N(q_1) = (3)^2 + (2)^2 + (-2)^2 + (0)^2$$

$$N(q_1) = 9 + 4 + 4 + 0$$

$$N(q_1) = 17$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{18} \cdot \sqrt{17}}\right)$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{306}}\right)$$

$$\boxed{\theta \approx 62.8°}$$

Problem 4

Given quaternions $q_0 = [2,0,-1,2]$ and $q_1 = [3,2,0,3]$:

Problem 4.a

Given $q_0 = [2,0,-1,2]$ and $q_1 = [3,2,0,3]$, calculate $q_0 + q_1$.

$$q_0 + q_1 = [a_0 + a_1, x_0 + x_1, y_0 + y_1, z_0 + z_1]$$

$$q_0 + q_1 = [(2) + (3), (0) + (2), (-1) + (0), (2) + (3)]$$

$$\boxed{q_0 + q_1 = [5,2,-1,5]}$$

Problem 4.b

Given $q_0 = [2,0,-1,2]$ and $q_1 = [3,2,0,3]$, calculate $q_0 q_1$.

$$q_0 q_1 = [a_0 a_1 - \vec{v}_0 \cdot \vec{v}_1, a_0 \vec{v}_1 + a_1 \vec{v}_0 + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [(2)(3) - \vec{v}_0 \cdot \vec{v}_1, (2)(2,0,3) + (3)(0,-1,2) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [6 - \vec{v}_0 \cdot \vec{v}_1, (4,0,6) + (0,-3,6) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [6 - \vec{v}_0 \cdot \vec{v}_1, (4,-3,12) + \vec{v}_0 \times \vec{v}_1]$$

$$\vec{v}_0 \cdot \vec{v}_1 = x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$\vec{v}_0 \cdot \vec{v}_1 = (0)(2) + (-1)(0) + (2)(3)$$

$$\vec{v}_0 \cdot \vec{v}_1 = 0 + 0 + 6$$

$$\vec{v}_0 \cdot \vec{v}_1 = 6$$

$$q_0 q_1 = [6 - 6, (4,-3,12) + \vec{v}_0 \times \vec{v}_1]$$

$$q_0 q_1 = [0, (4,-3,12) + \vec{v}_0 \times \vec{v}_1]$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -1 & 2 \\ 2 & 0 & 3 \end{vmatrix}$$

$$\vec{v}_0 \times \vec{v}_1 = \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} \hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = [(-1)(3) - (0)(2)]\hat{\imath} - [(0)(3) - (2)(2)]\hat{\jmath} + [(0)(0) - (2)(-1)]\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (-3 - 0)\hat{\imath} - (0 - 4)\hat{\jmath} + [0 - (-2)]\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (-3)\hat{\imath} - (-4)\hat{\jmath} + (2)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (-3)\hat{\imath} + (4)\hat{\jmath} + (2)\hat{k}$$

$$\vec{v}_0 \times \vec{v}_1 = (-3,4,2)$$

$$q_0 q_1 = [0, (4, -3,12) + (-3,4,2)]$$

$$q_0 q_1 = [0, (1,1,14)]$$

$$\boxed{q_1 q_2 = [0,1,1,14]}$$

Problem 4.c

Given $q_0 = [2,0,-1,2]$ and $q_1 = [3,2,0,3]$, calculate $q_0 \cdot q_1$.

$$q_0 \cdot q_1 = a_o a_1 + x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$q_0 \cdot q_1 = (2)(3) + (0)(2) + (-1)(0) + (2)(3)$$

$$q_0 \cdot q_1 = 6 + 0 + 0 + 6$$

$$\boxed{q_0 \cdot q_1 = 12}$$

Problem 4.d

Given $q_0 = [2,0,-1,2]$, determine the inverse of q_0.

$$q_0{}^{-1} = \frac{\tilde{q}_0}{N(q_0)}$$

$$\tilde{q}_0 = [a_0, -x_0, -y_0, -z_0]$$

$$\tilde{q}_0 = [(2), -(0), -(-1), -(2)]$$

$$\tilde{q}_0 = [2,0,1,-2]$$

$$N(q_0) = a_0{}^2 + x_0{}^2 + y_0{}^2 + z_0{}^2$$

$$N(q_0) = (2)^2 + (0)^2 + (-1)^2 + (2)^2$$

$$N(q_0) = 4 + 0 + 1 + 4$$

$$N(q_0) = 9$$

$$q_0{}^{-1} = \frac{[2,0,1,-2]}{9}$$

$$q_0{}^{-1} = \left[\frac{2}{9}, \frac{0}{9}, \frac{1}{9}, \frac{-2}{9}\right]$$

$$\boxed{q_0{}^{-1} = \left[\frac{2}{9}, 0, \frac{1}{9}, -\frac{2}{9}\right]}$$

Problem 4.e

Given $q_0 = [2,0,-1,2]$ and $q_1 = [3,2,0,3]$, calculate the angle between q_0 and q_1.

$$q_0 \cdot q_1 = \sqrt{N(q_0)} \cdot \sqrt{N(q_1)} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}$$

$$\theta = \cos^{-1}\left(\frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}\right)$$

$$q_0 \cdot q_1 = 12$$ From Part c, above.

$$N(q_0) = 9$$ From Part d, above.

$$N(q_1) = {a_1}^2 + {x_1}^2 + {y_1}^2 + {z_1}^2$$

$$N(q_1) = (3)^2 + (2)^2 + (0)^2 + (3)^2$$

$$N(q_1) = 9 + 4 + 0 + 9$$

$$N(q_1) = 22$$

$$\theta = \cos^{-1}\left(\frac{12}{\sqrt{9} \cdot \sqrt{22}}\right)$$

$$\theta = \cos^{-1}\left(\frac{12}{3\sqrt{22}}\right)$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{22}}\right)$$

$$\boxed{\theta \approx 31.5°}$$

Problem 5

Let q_0 be the quaternion representing a $90°$ rotation about the y-axis and let q_1 be the quaternion representing a $180°$ rotation about an axis parallel to vector $\vec{v} = (1,1,0)$.

Problem 5.a

Given $\theta_0 = 90°$ and $\vec{v}_0 = \hat{j}$, construct q_0.

$$q_0 = \left[\cos\left(\frac{\theta_0}{2}\right), \hat{j} \cdot \sin\left(\frac{\theta_0}{2}\right)\right]$$

$$q_0 = \left[\cos\left(\frac{90°}{2}\right), (0,1,0) \cdot \sin\left(\frac{90°}{2}\right)\right]$$

$$q_0 = [\cos(45°), (0,1,0) \cdot \sin(45°)]$$

$$q_0 = \left[\frac{\sqrt{2}}{2}, (0,1,0) \cdot \frac{\sqrt{2}}{2}\right]$$

$$\boxed{q_0 = \left[\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right]}$$

Problem 5.b

Given $\theta_1 = 180°$ and $\vec{v}_1 = (1,1,0)$, construct q_1.

$$q_1 = \left[\cos\left(\frac{\theta_1}{2}\right), \hat{v} \cdot \sin\left(\frac{\theta_1}{2}\right)\right]$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

$$\|\vec{v}\| = \sqrt{(1)^2 + (1)^2 + (0)^2}$$

$$\|\vec{v}\| = \sqrt{1 + 1 + 0}$$

$$\|\vec{v}\| = \sqrt{2}$$

$$\hat{v} = \frac{(1,1,0)}{\sqrt{2}}$$

$$\hat{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{0}{\sqrt{2}}\right)$$

$$\hat{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$q_1 = \left[\cos\left(\frac{180°}{2}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \sin\left(\frac{180°}{2}\right)\right]$$

$$q_1 = \left[\cos(90°), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \sin(90°)\right]$$

$$q_1 = \left[0, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot 1\right]$$

$$q_1 = \left[0, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]$$

$$\boxed{q_1 = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]}$$

Problem 5.c

Using q_1 from 5.b, above, determine the conjugate of q_1.

$$\tilde{q}_1 = [a_1, -x_1, -y_1, -z_1]$$

$$\tilde{q}_1 = \left[(0), -\left(\frac{1}{\sqrt{2}}\right), -\left(\frac{1}{\sqrt{2}}\right), -(0)\right]$$

$$\boxed{\tilde{q}_1 = \left[0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]}$$

Problem 5.d

Using q_0 from 5.a, above, and q_1 from 5.b, above, calculate the angle between q_0 and q_1.

$$q_0 \cdot q_1 = \sqrt{N(q_0)} \cdot \sqrt{N(q_1)} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}$$

$$\theta = \cos^{-1}\left(\frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}\right)$$

$$q_0 \cdot q_1 = a_0 a_1 + x_0 x_1 + y_0 y_1 + z_0 z_1$$

$$q_0 \cdot q_1 = \left(\frac{\sqrt{2}}{2}\right)(0) + (0)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + (0)(0)$$

$$q_0 \cdot q_1 = 0 + 0 + \frac{1}{2} + 0$$

$$q_0 \cdot q_1 = \frac{1}{2}$$

$$N(q_0) = {a_0}^2 + {x_0}^2 + {y_0}^2 + {z_0}^2$$

$$N(q_0) = \left(\frac{\sqrt{2}}{2}\right)^2 + (0)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + (0)^2$$

$$N(q_0) = \frac{2}{4} + 0 + \frac{2}{4} + 0$$

$$N(q_1) = 1$$

$$N(q_1) = {a_1}^2 + {x_1}^2 + {y_1}^2 + {z_1}^2$$

$$N(q_1) = (0)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2$$

$$N(q_1) = 0 + \frac{1}{2} + \frac{1}{2} + 0$$

$$N(q_1) = 1$$

$$\theta = \cos^{-1}\left(\frac{1/2}{\sqrt{1}\cdot\sqrt{1}}\right)$$

$$\theta = \cos^{-1}\left(\frac{1/2}{1\cdot 1}\right)$$

$$\theta = \cos^{-1}\left(\frac{1/2}{1}\right)$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = 60°}$$

Problem 6

Let q_0 be the quaternion representing a $90°$ rotation about the z-axis and let q_1 be the quaternion representing a $180°$ rotation about an axis parallel to vector $\vec{v} = (-1,0,1)$.

Problem 6.a

Given $\theta_0 = 90°$ and $\vec{v}_1 = \hat{k}$, construct q_0.

$$q_0 = \left[\cos\left(\frac{\theta_0}{2}\right), \hat{k} \cdot \sin\left(\frac{\theta_0}{2}\right)\right]$$

$$q_0 = \left[\cos\left(\frac{90°}{2}\right), (0,0,1) \cdot \sin\left(\frac{90°}{2}\right)\right]$$

$$q_0 = [\cos(45°), (0,0,1) \cdot \sin(45°)]$$

$$q_0 = \left[\frac{\sqrt{2}}{2}, (0,0,1) \cdot \frac{\sqrt{2}}{2}\right]$$

$$\boxed{q_0 = \left[\frac{\sqrt{2}}{2}, 0,0, \frac{\sqrt{2}}{2}\right]}$$

Problem 6.b

Given $\theta_1 = 180°$ and $\vec{v}_1 = (-1,0,1)$, construct q_1.

$$q_1 = \left[\cos\left(\frac{\theta_1}{2}\right), \hat{v} \cdot \sin\left(\frac{\theta_1}{2}\right)\right]$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (0)^2 + (1)^2}$$

$$\|\vec{v}\| = \sqrt{1 + 0 + 1}$$

$$\|\vec{v}\| = \sqrt{2}$$

$$\hat{v} = \frac{(-1,0,1)}{\sqrt{2}}$$

$$\hat{v} = \left(\frac{-1}{\sqrt{2}}, \frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\hat{v} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$q_1 = \left[\cos\left(\frac{180°}{2}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \sin\left(\frac{180°}{2}\right)\right]$$

$$q_1 = \left[\cos(90°), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \sin(90°)\right]$$

$$q_1 = \left[0, \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot 1\right]$$

$$q_1 = \left[0, \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right]$$

$$\boxed{q_1 = \left[0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]}$$

Problem 6.c

Using q_1 from 6.b, above, determine the conjugate of q_1.

$$\tilde{q}_1 = [a_1, -x_1, -y_1, -z_1]$$

$$\tilde{q}_1 = \left[(0), -\left(-\frac{1}{\sqrt{2}}\right), -(0), -\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$\boxed{\tilde{q}_1 = \left[0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]}$$

Problem 6.d

Using q_0 from 6.a, above, and q_1 from 6.b, above, calculate the angle between q_0 and q_1.

$$q_0 \cdot q_1 = \sqrt{N(q_0)} \cdot \sqrt{N(q_1)} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}$$

$$\theta = \cos^{-1}\left(\frac{q_0 \cdot q_1}{\sqrt{N(q_0)} \cdot \sqrt{N(q_1)}}\right)$$

$$q_0 \cdot q_1 = a_0a_1 + x_0x_1 + y_0y_1 + z_0z_1$$

$$q_0 \cdot q_1 = \left(\frac{\sqrt{2}}{2}\right)(0) + (0)\left(-\frac{1}{\sqrt{2}}\right) + (0)(0) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$q_0 \cdot q_1 = 0 + 0 + 0 + \frac{1}{2}$$

$$q_0 \cdot q_1 = \frac{1}{2}$$

$$N(q_0) = {a_0}^2 + {x_0}^2 + {y_0}^2 + {z_0}^2$$

$$N(q_0) = \left(\frac{\sqrt{2}}{2}\right)^2 + (0)^2 + (0)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$$

$$N(q_0) = \frac{2}{4} + 0 + 0 + \frac{2}{4}$$

$$N(q_1) = 1$$

$$N(q_1) = {a_1}^2 + {x_1}^2 + {y_1}^2 + {z_1}^2$$

$$N(q_1) = (0)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$N(q_1) = 0 + \frac{1}{2} + 0 + \frac{1}{2}$$

$$N(q_1) = 1$$

$$\theta = \cos^{-1}\left(\frac{1/2}{\sqrt{1}\cdot\sqrt{1}}\right)$$

$$\theta = \cos^{-1}\left(\frac{1/2}{1\cdot 1}\right)$$

$$\theta = \cos^{-1}\left(\frac{1/2}{1}\right)$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = 60°}$$

END