GEN 242: Linear Algebra

Chapter 3: Determinants & Eigen Space

Solutions Guide

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Answers

Determinant of a Matrix

1.a $\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4$

1.b $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 22$

1.c $\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix} = 52$

1.d $\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 0$

1.e $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = -65$

1.f $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = -4$

1.g $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = -5$

1.h $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = -9$

Determinant of a Triangular/Diagonal Matrix

2.a $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 5 & 7 & 9 & 2 \end{vmatrix} = 30$

2.b $\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14$

2.c $\begin{vmatrix} 2 & 1 & 3 & 5 & 7 \\ 0 & 3 & 7 & 11 & 2 \\ 0 & 0 & 4 & 9 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 96$

Minor Determinant and Cofactor

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$:

3.a $M_{1,1} = 7$

3.b $M_{2,1} = 5$

3.c $M_{3,2} = 1$

3.d $C_{1,2} = 2$

3.e $C_{3,3} = 2$

3.f $C_{1,1} = 7$

Inverse of a 2x2 Matrix

4.a $\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$

4.b $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

4.c $\begin{bmatrix} 4 & 8 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{4} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$

4.d $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Inverse of a 3x3 Matrix

5.a

$$\begin{vmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{vmatrix} = -1 \neq 0 \rightarrow \exists \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}^{-1}$$

$$\text{Adj}\left(\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}\right) = \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$$

5.b

$$\begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix} = -6 \neq 0 \rightarrow \exists \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}^{-1}$$

$$\text{Adj}\left(\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}\right) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$$

5.c

$$\begin{vmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 4 \neq 0 \rightarrow \exists \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$$

$$\text{Adj}\left(\begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

5.d

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix} = -1 \neq 0 \rightarrow \exists \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}^{-1}$$

$$\text{Adj}\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 7 & -5 & -4 \\ 2 & -2 & -1 \\ -4 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & 5 & 4 \\ -2 & 2 & 1 \\ 4 & -3 & -2 \end{bmatrix}$$

Solving Two Linear Equations with Two Unknowns Using Matrices

6.a $\begin{cases} 5x + 7y = 3 \\ 2x + 4y = 1 \end{cases} \rightarrow \begin{cases} x = \frac{5}{6} \\ y = -\frac{1}{6} \end{cases}$

6.b $\begin{matrix} 3x - 2y = 7 \\ -5x + 6y = -5 \end{matrix} \rightarrow \begin{cases} x = 4 \\ y = \frac{5}{2} \end{cases}$

6.c $\begin{cases} 4x + y = 6 \\ 5x + 2y = 7 \end{cases} \rightarrow \begin{cases} x = \frac{5}{3} \\ y = -\frac{2}{3} \end{cases}$

Matrix Inverse Using Reduced Row-Echelon Form

7.a $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

7.b $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

7.c $\begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix}$

7.d $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

7.e $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$

7.f $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$

7.g $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

7.h $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 5 \\ 4 & 11 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}$

Least-Square Approximation Method

These questions are incomplete.

Systems of Equations with Cramer's Method

10.a $\begin{cases} 5x + 7y = 3 \\ 2x + 4y = 1 \end{cases} \rightarrow \begin{cases} x = \frac{5}{6} \\ y = -\frac{1}{6} \end{cases}$

10.b $\begin{cases} 3x - 2y = 7 \\ -5x + 6y = -5 \end{cases} \rightarrow \begin{cases} x = 4 \\ y = \frac{5}{2} \end{cases}$

10.c $\begin{cases} 4x + y = 6 \\ 5x + 2y = 7 \end{cases} \rightarrow \begin{cases} x = \frac{5}{3} \\ y = -\frac{2}{3} \end{cases}$

10.d $\begin{cases} 2x + y = 4 \\ -3x + z = -8 \\ y + 2z = -3 \end{cases} \rightarrow \begin{cases} x = \frac{9}{4} \\ y = -\frac{1}{2} \\ z = -\frac{5}{4} \end{cases}$

10.e $\begin{cases} 2x + y + z = 4 \\ -x + 2z = 2 \\ 3x + y + 3z = -2 \end{cases} \rightarrow \begin{cases} x = -4 \\ y = 13 \\ z = -1 \end{cases}$

10.f $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases} \rightarrow \begin{cases} x = -\frac{144}{55} \\ y = -\frac{61}{55} \\ z = \frac{46}{11} \end{cases}$

Linear Independence of Vectors Using Determinants

11.a $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3 \neq 0 \rightarrow$ linearly independent

11.b $\begin{vmatrix} 5 & 0 \\ 0 & 1 \end{vmatrix} = 5 \neq 0 \rightarrow$ linearly independent

11.c $\begin{vmatrix} 1 & 4 \\ -1 & 5 \end{vmatrix} = 9 \neq 0 \rightarrow$ linearly independent

11.d $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0 \rightarrow$ linearly independent

11.e $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24 \neq 0 \rightarrow$ linearly independent

11.f $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{vmatrix} = -2 \neq 0 \rightarrow$ linearly independent

Basis of a Vector Space Using Determinants

12.a $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3 \neq 0 \rightarrow$ linearly independent → basis for $\mathbb{R}^2$

12.b $\begin{vmatrix} 5 & 0 \\ 0 & 1 \end{vmatrix} = 5 \neq 0 \rightarrow$ linearly independent → basis for $\mathbb{R}^2$

12.c $\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1 \neq 0 \rightarrow$ linearly independent → basis for $\mathbb{R}^2$

12.d $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = 0 \rightarrow$ linearly dependent → **not** a basis for $\mathbb{R}^2$

13.a $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0 \rightarrow$ linearly independent $\rightarrow$ basis for $\mathbb{R}^3$

13.b $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24 \neq 0 \rightarrow$ linearly independent $\rightarrow$ basis for $\mathbb{R}^3$

13.c $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2 \neq 0 \rightarrow$ linearly independent $\rightarrow$ basis for $\mathbb{R}^3$

13.d $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -1 \neq 0 \rightarrow$ linearly independent $\rightarrow$ basis for $\mathbb{R}^3$

Characteristic Equation

14.a $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow \lambda^2 - 5\lambda + 6 = 0$

14.b $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \lambda^2 - 3\lambda + 2 = 0$

14.c $\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} \rightarrow \lambda^2 + \lambda - 6 = 0$

14.d $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \rightarrow \lambda^2 - 5\lambda + 6 = 0$

14.e $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

14.f $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix} \rightarrow -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$

14.g $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow -\lambda^3 + 9\lambda^2 - 26\lambda + 24 = 0$

Eigen Values and Eigen Vectors

15.a $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 2, \lambda_2 = 3 \\ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}$

15.b $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 1, \lambda_2 = 2 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}$

15.c $\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 2, \lambda_2 = -3 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix} \end{matrix}$

15.d $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 3, \lambda_2 = 2 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \end{matrix}$

15.e $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$

15.f $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 1, \lambda_2 = 2 \\ \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \\ 1 \end{bmatrix} \end{matrix}$

15.g $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2 \\ \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$

Cayley-Hamilton Theorem

16.a $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow$ satisfies characteristic equation

16.b $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$ satisfies characteristic equation

16.c $\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} \rightarrow$ does **not** satisfy characteristic equation

16.d $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \rightarrow$ satisfies characteristic equation

Solutions

Matrix Determinant

Problem 1

$$\det(M_{2x2}) = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{21}m_{12}$$

$$\det(M_{3x3}) = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} m_{11} - \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} m_{12} + \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} m_{13}$$

1.a Evaluate $\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = (1)(6) - (2)(5)$$

$$\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = 6 - 10$$

$$\boxed{\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4}$$

1.b Evaluate $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$.

$$\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = (3)(4) - (-2)(5)$$

$$\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 12 - (-10)$$

$$\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 12 + 10$$

$$\boxed{\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 22}$$

1.c Evaluate $\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix}$.

$$\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (-7)(6)$$

$$\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix} = 10 - (-42)$$

$$\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix} = 10 + 42$$

$$\boxed{\begin{vmatrix} -5 & 6 \\ -7 & -2 \end{vmatrix} = 52}$$

1.d Evaluate $\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = (2)(3) - (1)(6)$$

$$\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 6 - 6$$

$$\boxed{\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 0}$$

1.e Evaluate $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 5 & -7 \\ 6 & 2 \end{vmatrix}(-2) - \begin{vmatrix} 3 & -7 \\ 1 & 2 \end{vmatrix}(1) + \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix}(4)$$

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = [(5)(2) - (6)(-7)](-2) - [(3)(2) - (1)(-7)](1) + [(3)(6) - (1)(5)](4)$$

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = [10 - (-42)](-2) - [6 - (-7)](1) + (18 - 5)(4)$$

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = (10 + 42)(-2) - (6 + 7)(1) + (18 - 5)(4)$$

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = (52)(-2) - (13)(1) + (13)(4)$$

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = -104 - 13 + 52$$

$$\boxed{\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = -65}$$

1.f Evaluate $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}$.

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix}(-1) - \begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix}(1) + \begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix}(2)$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = [(0)(2) - (7)(-5)](-1) - [(3)(2) - (1)(-5)](1) + [(3)(7) - (1)(0)](2)$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = [0 - (-35)](-1) - [6 - (-5)](1) + (21 - 0)(2)$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = (0 + 35)(-1) - (6 + 5)(1) + (21 - 0)(2)$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = (35)(-1) - (11)(1) + (21)(2)$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = -35 - 11 + 42$$

$$\boxed{\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = -4}$$

1.g Evaluate $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$.

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}(2) - \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}(-4) + \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}(3)$$

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = [(1)(-1) - (4)(2)](2) - [(3)(-1) - (1)(2)](-4) + [(3)(4) - (1)(1)](3)$$

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = (-1 - 8)(2) - (-3 - 2)(-4) + (12 - 1)(3)$$

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = (-9)(2) - (-5)(-4) + (11)(3)$$

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = -18 - 20 + 33$$

$$\boxed{\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = -5}$$

1.h Evaluate $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix}(2) - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}(0) + \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}(1)$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = [(1)(0) - (4)(1)](2) - [(0)(0) - (1)(1)](0) + [(0)(4) - (1)(1)](1)$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = (0 - 4)(2) - (0 - 1)(0) + (0 - 1)(1)$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = (-4)(2) - (-1)(0) + (-1)(1)$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = -8 - 0 + (-1)$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = -8 - 0 - 1$$

$$\boxed{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = -9}$$

Triangular/Diagonal-Matrix Determinant

Problem 2

The determinant of a triangular or a diagonal matrix is the product of the entries on the matrix's main diagonal.

2.a Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 5 & 7 & 9 & 2 \end{bmatrix}$.

All the entries above the main diagonal are zeroes, so this matrix is triangular.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 5 & 7 & 9 & 2 \end{vmatrix} = (1)(5)(3)(2)$$

$$\boxed{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 5 & 7 & 9 & 2 \end{vmatrix} = 30}$$

2.b Find the determinant of $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

All the entries off the main diagonal are zeroes, so this matrix is diagonal.

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1)(7)(-2)(1)$$

$$\boxed{\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14}$$

2.c Find the determinant of $\begin{bmatrix} 2 & 1 & 3 & 5 & 7 \\ 0 & 3 & 7 & 11 & 2 \\ 0 & 0 & 4 & 9 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

All the entries below the main diagonal are zeroes, so this matrix is triangular.

$$\begin{vmatrix} 2 & 1 & 3 & 5 & 7 \\ 0 & 3 & 7 & 11 & 2 \\ 0 & 0 & 4 & 9 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = (2)(3)(4)(4)(1)$$

$$\boxed{\begin{vmatrix} 2 & 1 & 3 & 5 & 7 \\ 0 & 3 & 7 & 11 & 2 \\ 0 & 0 & 4 & 9 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 96}$$

Problem 3

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$, calculate the following cofactors and minor determinants:

The minor determinant of (i, j) – written $M_{i,j}$ is the determinant of the matrix formed when row i and column j are crossed out.

The cofactor of (i, j) is the minor determinant of (i, j) possibly negated as per: $C_{i,j} = (-1)^{i+j} \cdot M_{i,j}$

3.a $M_{1,1}$

$$M_{1,1} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{1,1} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$M_{1,1} = (2)(4) - (1)(1)$$

$$M_{1,1} = 8 - 1$$

$$\boxed{M_{1,1} = 7}$$

3.b $M_{2,1}$

$$M_{2,1} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{2,1} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$M_{2,1} = (2)(4) - (1)(3)$$

$$M_{2,1} = 8 - 3$$

$$\boxed{M_{2,1} = 5}$$

3.c $M_{3,2}$

$$M_{3,2} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$M_{3,2} = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$M_{3,2} = (1)(1) - (0)(3)$$

$$M_{3,2} = 1 - 0$$

$$\boxed{M_{3,2} = 1}$$

3.d $C_{1,2}$

$$C_{1,2} = (-1)^{1+2} \cdot M_{1,2}$$

$$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix}$$

$$C_{1,2} = (-1)^{1+2} \cdot [(0)(4) - (2)(1)]$$

$$C_{1,2} = (-1)^{3} \cdot (0 - 2)$$

$$C_{1,2} = -1 \cdot (-2)$$

$$\boxed{C_{1,2} = 2}$$

3.e $C_{3,3}$

$$C_{3,3} = (-1)^{3+3} \cdot M_{3,3}$$

$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$C_{3,3} = (-1)^{3+3} \cdot [(1)(2) - (0)(2)]$$

$$C_{3,3} = (-1)^{6} \cdot (2 - 0)$$

$$C_{3,3} = 1 \cdot 2$$

$$\boxed{C_{3,3} = 2}$$

3.f $C_{1,1}$

$C_{1,1} = (-1)^{1+1} \cdot M_{1,1}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot [(2)(4) - (1)(1)]$

$C_{1,1} = (-1)^2 \cdot (8 - 1)$

$C_{1,1} = 1 \cdot 7$

$\boxed{C_{1,1} = 7}$

Inverse of a 2x2 matrix

Problem 4

A square matrix is invertible only if its determinant is not zero.

$$M^{-1} = \frac{\text{Adj}(M)}{\det(M)}$$

$\text{Adj}(M)$, or the adjoint of the matrix, is the transpose of the matrix of cofactors of the original matrix. 2x2 matrices are special cases, where:

$$\text{Adj}(M_{2x2}) = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$

4.a Show that $A = \begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$ is invertible and find its inverse.

$\det(A) = \begin{vmatrix} 7 & 5 \\ 3 & 2 \end{vmatrix}$

$\det(A) = (7)(2) - (3)(5)$

$\det(A) = 14 - 15$

$\det(A) = -1$

$\det(A) \neq 0$

$\exists A^{-1}$

$A^{-1} = \frac{\text{Adj}(A)}{\det(A)}$

$\text{Adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$\text{Adj}(A) = \begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}$

$A^{-1} = \frac{\begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}}{-1}$

$A^{-1} = \begin{bmatrix} {}^{2}/_{-1} & {}^{-5}/_{-1} \\ {}^{-3}/_{-1} & {}^{7}/_{-1} \end{bmatrix}$

$\boxed{A^{-1} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}}$

4.b Show that $B = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is invertible and find its inverse.

$$\det(B) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$
$$\det(B) = (1)(3) - (2)(4)$$
$$\det(B) = 3 - 8$$
$$\det(B) = -5$$
$$\det(B) \neq 0$$
$$\exists B^{-1}$$

$$B^{-1} = \frac{\text{Adj}(B)}{\det(B)}$$
$$\text{Adj}(B) = \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}$$
$$\text{Adj}(B) = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$$
$$B^{-1} = \frac{\begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}}{-5}$$
$$B^{-1} = \begin{bmatrix} 3/-5 & -4/-5 \\ -2/-5 & 1/-5 \end{bmatrix}$$
$$\boxed{B^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}}$$

4.c Show that $C = \begin{bmatrix} 4 & 8 \\ 2 & 3 \end{bmatrix}$ is invertible and find its inverse.

$$\det(C) = \begin{vmatrix} 4 & 8 \\ 2 & 3 \end{vmatrix}$$
$$\det(C) = (4)(3) - (2)(8)$$
$$\det(C) = 12 - 16$$
$$\det(C) = -4$$
$$\det(C) \neq 0$$
$$\exists C^{-1}$$

$$C^{-1} = \frac{\text{Adj}(C)}{\det(C)}$$
$$\text{Adj}(C) = \begin{bmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{bmatrix}$$
$$\text{Adj}(C) = \begin{bmatrix} 3 & -8 \\ -2 & 4 \end{bmatrix}$$
$$C^{-1} = \frac{\begin{bmatrix} 3 & -8 \\ -2 & 4 \end{bmatrix}}{-4}$$
$$C^{-1} = \begin{bmatrix} 3/-4 & -8/-4 \\ -2/-4 & 4/-4 \end{bmatrix}$$
$$\boxed{C^{-1} = \begin{bmatrix} -\frac{3}{4} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}}$$

4.d Show that $D = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible and find its inverse.

$\det(D) = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$

$\det(D) = (2)(3) - (1)(5)$

$\det(D) = 6 - 5$

$\det(D) = 1$

$\det(D) \neq 0$

$\exists D^{-1}$

$D^{-1} = \dfrac{\mathrm{Adj}(D)}{\det(D)}$

$\mathrm{Adj}(D) = \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}$

$\mathrm{Adj}(D) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$D^{-1} = \dfrac{\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}}{1}$

$\boxed{D^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}}$

Inverse of a 3x3 matrix

Problem 5

5.a Show that $M = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ is invertible; calculate its adjoint and inverse matrices.

$\det(M) = \begin{vmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{vmatrix}$

$\det(M) = \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix}(2) - \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix}(5) + \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix}(5)$

$\det(M) = [(-1)(3) - (4)(0)](2) - [(-1)(3) - (2)(0)](5) + [(-1)(4) - (2)(-1)](5)$

$\det(M) = (-3 - 0)(2) - (-3 - 0)(5) + [-4 - (-2)](5)$

$\det(M) = (-3 - 0)(2) - (-3 - 0)(5) + (-4 + 2)(5)$

$\det(M) = (-3)(2) - (-3)(5) + (-2)(5)$

$\det(M) = -6 - (-15) + (-10)$

$\det(M) = -6 + 15 - 10$

$\det(M) = -1$

$\det(M) \neq 0$

$\exists M^{-1}$

$$\text{Adj}(M) = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}^T$$

$C_{1,1} = (-1)^{1+1} \cdot M_{1,1}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot [(-1)(3) - (4)(0)]$

$C_{1,1} = (-1)^2 \cdot (-3 - 0)$

$C_{1,1} = 1 \cdot (-3)$

$C_{1,1} = -3$

$C_{2,1} = (-1)^{2+1} \cdot M_{2,1}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot [(5)(3) - (4)(5)]$

$C_{2,1} = (-1)^3 \cdot (15 - 20)$

$C_{2,1} = -1 \cdot (-5)$

$C_{2,1} = 5$

$C_{1,2} = (-1)^{1+2} \cdot M_{1,2}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot [(-1)(3) - (2)(0)]$

$C_{1,2} = (-1)^3 \cdot (-3 - 0)$

$C_{1,2} = -1 \cdot (-3)$

$C_{1,2} = 3$

$C_{2,2} = (-1)^{2+2} \cdot M_{2,2}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot [(2)(3) - (2)(5)]$

$C_{2,2} = (-1)^4 \cdot (6 - 10)$

$C_{2,2} = 1 \cdot (-4)$

$C_{2,2} = -4$

$C_{1,3} = (-1)^{1+3} \cdot M_{13}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot [(-1)(4) - (2)(-1)]$

$C_{1,3} = (-1)^4 \cdot [-4 - (-2)]$

$C_{1,3} = 1 \cdot (-4 + 2)$

$C_{1,3} = 1 \cdot (-2)$

$C_{1,3} = -2$

$C_{2,3} = (-1)^{2+3} \cdot M_{2,3}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot [(2)(4) - (2)(5)]$

$C_{2,3} = (-1)^5 \cdot (8 - 10)$

$C_{2,3} = -1 \cdot (-2)$

$C_{2,3} = 2$

$$C_{3,1} = (-1)^{3+1} \cdot M_{3,1}$$
$$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix}$$
$$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix}$$
$$C_{3,1} = (-1)^{3+1} \cdot [(5)(0) - (-1)(5)]$$
$$C_{3,1} = (-1)^{4} \cdot [0 - (-5)]$$
$$C_{3,1} = 1 \cdot (0 + 5)$$
$$C_{3,1} = 1 \cdot 5$$
$$C_{3,1} = 5$$

$$C_{3,3} = (-1)^{3+3} \cdot M_{3,3}$$
$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$$
$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix}$$
$$C_{3,3} = (-1)^{3+3} \cdot [(2)(-1) - (-1)(5)]$$
$$C_{3,3} = (-1)^{6} \cdot [-2 - (-5)]$$
$$C_{3,3} = 1 \cdot (-2 + 5)$$
$$C_{3,3} = 1 \cdot 3$$
$$C_{3,3} = 3$$

$$C_{3,2} = (-1)^{3+2} \cdot M_{3,2}$$
$$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix}$$
$$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix}$$
$$C_{3,2} = (-1)^{3+2} \cdot [(2)(0) - (-1)(5)]$$
$$C_{3,2} = (-1)^{5} \cdot [0 - (-5)]$$
$$C_{3,2} = -1 \cdot (0 + 5)$$
$$C_{3,2} = -1 \cdot 5$$
$$C_{3,2} = -5$$

$$\text{Adj}(M) = \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^{T}$$
$$\text{Adj}(M) = \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}(M)}{\det(M)}$$

$$M^{-1} = \frac{\begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}}{-1}$$

$$M^{-1} = \begin{bmatrix} {}^{-3}/_{-1} & {}^{5}/_{-1} & {}^{5}/_{-1} \\ {}^{3}/_{-1} & {}^{-4}/_{-1} & {}^{-5}/_{-1} \\ {}^{-2}/_{-1} & {}^{2}/_{-1} & {}^{3}/_{-1} \end{bmatrix}$$

$$\boxed{M^{-1} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}}$$

Check:

$$\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} (2)(3)+(5)(-3)+(5)(2) & (2)(-5)+(5)(4)+(5)(-2) & (2)(-5)+(5)(5)+(5)(-3) \\ (-1)(3)+(-1)(-3)+(0)(2) & (-1)(-5)+(-1)(4)+(0)(-2) & (-1)(-5)+(-1)(5)+(0)(-3) \\ (2)(3)+(4)(-3)+(3)(2) & (2)(-5)+(4)(4)+(3)(-2) & (2)(-5)+(4)(5)+(3)(-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

5.b Show that $M = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$ is invertible; calculate its adjoint and inverse matrices.

$$\det(M) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix}$$

$$\det(M) = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix}(2) - \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix}(0) + \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix}(3)$$

$$\det(M) = [(3)(-4) - (0)(2)](2) - [(0)(-4) - (-2)(2)](0) + [(0)(0) - (-2)(3)](3)$$

$$\det(M) = (-12 - 0)(2) - [-4 - (-4)](0) + [0 - (-6)](3)$$

$$\det(M) = (-12 - 0)(2) - (-4 + 4)(0) + (0 + 6)(3)$$

$$\det(M) = (-12)(2) - (0)(0) + (6)(3)$$

$$\det(M) = -24 - 0 + 18$$

$$\det(M) = -6$$

$$\det(M) \neq 0$$

$$\exists M^{-1}$$

$$\text{Adj}(M) = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}^T$$

$$C_{1,1} = (-1)^{1+1} \cdot M_{1,1}$$
$$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}$$
$$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix}$$
$$C_{1,1} = (-1)^{1+1} \cdot [(3)(-4) - (0)(2)]$$
$$C_{1,1} = (-1)^2 \cdot (-12 - 0)$$
$$C_{1,1} = 1 \cdot (-12)$$
$$C_{1,1} = -12$$

$$C_{2,1} = (-1)^{2+1} \cdot M_{2,1}$$
$$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix}$$
$$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} 0 & 3 \\ 0 & -4 \end{vmatrix}$$
$$C_{2,1} = (-1)^{2+1} \cdot [(0)(-4) - (0)(3)]$$
$$C_{2,1} = (-1)^3 \cdot (0 - 0)$$
$$C_{2,1} = -1 \cdot 0$$
$$C_{2,1} = 0$$

$$C_{1,2} = (-1)^{1+2} \cdot M_{12}$$
$$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix}$$
$$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix}$$
$$C_{1,2} = (-1)^{1+2} \cdot [(0)(-4) - (-2)(2)]$$
$$C_{1,2} = (-1)^3 \cdot [0 - (-4)]$$
$$C_{1,2} = (-1)^3 \cdot (0 + 4)$$
$$C_{1,2} = -1 \cdot (4)$$
$$C_{1,2} = -4$$

$$C_{2,2} = (-1)^{2+2} \cdot M_{2,2}$$
$$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix}$$
$$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix}$$
$$C_{2,2} = (-1)^{2+2} \cdot [(2)(-4) - (-2)(3)]$$
$$C_{2,2} = (-1)^4 \cdot [-8 - (-6)]$$
$$C_{2,2} = (-1)^4 \cdot (-8 + 6)$$
$$C_{2,2} = 1 \cdot (-2)$$
$$C_{2,2} = -2$$

$$C_{1,3} = (-1)^{1+3} \cdot M_{1,3}$$
$$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$
$$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix}$$
$$C_{1,3} = (-1)^{1+3} \cdot [(0)(0) - (-2)(3)]$$
$$C_{1,3} = (-1)^4 \cdot [0 - (-6)]$$
$$C_{1,3} = 1 \cdot (0 + 6)$$
$$C_{1,3} = 1 \cdot 6$$
$$C_{1,3} = 6$$

$$C_{2,3} = (-1)^{2+3} \cdot M_{2,3}$$
$$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$$
$$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix}$$
$$C_{2,3} = (-1)^{2+3} \cdot [(2)(0) - (-2)(0)]$$
$$C_{2,3} = (-1)^5 \cdot (0 - 0)$$
$$C_{2,3} = -1 \cdot 0$$
$$C_{2,3} = 0$$

$$C_{3,1} = (-1)^{3+1} \cdot M_{3,1}$$
$$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix}$$
$$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix}$$
$$C_{3,1} = (-1)^{3+1} \cdot [(0)(2) - (3)(3)]$$
$$C_{3,1} = (-1)^{4} \cdot (0 - 9)$$
$$C_{3,1} = 1 \cdot (-9)$$
$$C_{3,1} = -9$$

$$C_{3,3} = (-1)^{3+3} \cdot M_{3,3}$$
$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$$
$$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$
$$C_{3,3} = (-1)^{3+3} \cdot [(2)(3) - (0)(0)]$$
$$C_{3,3} = (-1)^{6} \cdot (6 - 0)$$
$$C_{3,3} = 1 \cdot 6$$
$$C_{3,3} = 6$$

$$C_{3,2} = (-1)^{3+2} \cdot M_{3,2}$$
$$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix}$$
$$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$
$$C_{3,2} = (-1)^{3+2} \cdot [(2)(2) - (0)(3)]$$
$$C_{3,2} = (-1)^{5} \cdot (4 - 0)$$
$$C_{3,2} = -1 \cdot 4$$
$$C_{3,2} = -4$$

$$\text{Adj}(M) = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}^T$$
$$\text{Adj}(M) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}(B)}{\det(B)}$$
$$M^{-1} = \frac{\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}}{-6}$$
$$M^{-1} = \begin{bmatrix} {}^{-12}/_{-6} & {}^{0}/_{-6} & {}^{-9}/_{-6} \\ {}^{-4}/_{-6} & {}^{-2}/_{-6} & {}^{-4}/_{-6} \\ {}^{6}/_{-6} & {}^{0}/_{-6} & {}^{6}/_{-6} \end{bmatrix}$$
$$\boxed{M^{-1} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}}$$

Check:

$$\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} (2)(3) + (5)(-3) + (5)(2) & (2)(-5) + (5)(4) + (5)(-2) & (2)(-5) + (5)(5) + (5)(-3) \\ (-1)(3) + (-1)(-3) + (0)(2) & (-1)(-5) + (-1)(4) + (0)(-2) & (-1)(-5) + (-1)(5) + (0)(-3) \\ (2)(3) + (4)(-3) + (3)(2) & (2)(-5) + (4)(4) + (3)(-2) & (2)(-5) + (4)(5) + (3)(-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.c Show that $M = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible; calculate its adjoint and inverse matrices.

$\det(M) = \begin{vmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix}$

All entries below the main diagonal are zeroes, so this is a triangular matrix.

$\det(M) = (2)(1)(2)$

$\det(M) = 4$

$\det(M) \neq 0$

$\exists M^{-1}$

Alternate:

$\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}(2) - \begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix}(-3) + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}(5)$

$[(1)(2) - (0)(-3)](2) - [(0)(2) - (0)(3)](-3) + [(0)(0) - (0)(1)](5)$

$(2 - 0)(2) - (0 - 0)(-3) + (0 - 0)(5)$

$(2)(2) - (0)(-3) + (0)(5)$

4

$\operatorname{Adj}(M) = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}^T$

$C_{1,1} = (-1)^{1+1} \cdot M_{1,1}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot [(1)(2) - (0)(-3)]$

$C_{1,1} = (-1)^2 \cdot (2 - 0)$

$C_{1,1} = 1 \cdot 2$

$C_{1,1} = 2$

$C_{1,3} = (-1)^{1+3} \cdot M_{1,3}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot [(0)(0) - (0)(1)]$

$C_{1,3} = (-1)^4 \cdot (0 - 0)$

$C_{1,3} = 1 \cdot 0$

$C_{1,3} = 0$

$C_{1,2} = (-1)^{1+2} \cdot M_{1,2}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot [(0)(2) - (0)(-3)]$

$C_{1,2} = (-1)^3 \cdot (0 - 0)$

$C_{1,2} = -1 \cdot 0$

$C_{1,2} = 0$

$C_{2,1} = (-1)^{2+1} \cdot M_{2,1}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot [(-3)(2) - (0)(5)]$

$C_{2,1} = (-1)^3 \cdot (-6 - 0)$

$C_{2,1} = -1 \cdot (-6)$

$C_{2,1} = 6$

$C_{2,2} = (-1)^{2+2} \cdot M_{2,2}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot [(2)(2) - (0)(5)]$

$C_{2,2} = (-1)^{4} \cdot (4 - 0)$

$C_{2,2} = 1 \cdot 4$

$C_{2,2} = 4$

$C_{3,2} = (-1)^{3+2} \cdot M_{3,2}$

$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix}$

$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix}$

$C_{3,2} = (-1)^{3+2} \cdot [(2)(-3) - (0)(5)]$

$C_{3,2} = (-1)^{5} \cdot (-6 - 0)$

$C_{3,2} = -1 \cdot (-6)$

$C_{3,2} = 6$

$C_{2,3} = (-1)^{2+3} \cdot M_{2,3}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot [(2)(0) - (0)(-3)]$

$C_{2,3} = (-1)^{5} \cdot (0 - 0)$

$C_{2,3} = -1 \cdot 0$

$C_{2,3} = 0$

$C_{3,3} = (-1)^{3+3} \cdot M_{3,3}$

$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$

$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix}$

$C_{3,3} = (-1)^{3+3} \cdot [(2)(1) - (0)(-3)]$

$C_{3,3} = (-1)^{6} \cdot (2 - 0)$

$C_{3,3} = 1 \cdot 2$

$C_{3,3} = 2$

$C_{3,1} = (-1)^{3+1} \cdot M_{3,1}$

$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix}$

$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix}$

$C_{3,1} = (-1)^{3+1} \cdot [(-3)(-3) - (1)(5)]$

$C_{3,1} = (-1)^{4} \cdot (9 - 5)$

$C_{3,1} = 1 \cdot 4$

$C_{3,1} = 4$

$$\mathrm{Adj}(M) = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 4 & 6 & 2 \end{bmatrix}^{T}$$

$$\mathrm{Adj}(M) = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}(M)}{\det(M)}$$

$$M^{-1} = \frac{\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}}{4}$$

$$M^{-1} = \begin{bmatrix} {}^{2}/_{4} & {}^{6}/_{4} & {}^{4}/_{4} \\ {}^{0}/_{4} & {}^{4}/_{4} & {}^{6}/_{4} \\ {}^{0}/_{4} & {}^{0}/_{4} & {}^{2}/_{4} \end{bmatrix}$$

$$\boxed{M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}$$

5.d Show that $M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ is invertible; calculate its adjoint and inverse matrices.

$$\det(M) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix}$$

$$\det(M) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}(1) - \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix}(2) + \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix}(3)$$

$$\det(M) = [(2)(4) - (1)(1)](1) - [(0)(4) - (2)(1)](2) + [(0)(1) - (2)(2)](3)$$

$$\det(M) = (8 - 1)(1) - (0 - 2)(2) + (0 - 4)(3)$$

$$\det(M) = (7)(1) - (-2)(2) + (-4)(3)$$

$$\det(M) = 7 - (-4) + (-12)$$

$$\det(M) = 7 + 4 - 12$$

$$\det(M) = -1$$

$$\det(M) \neq 0$$

$$\exists M^{-1}$$

$$\text{Adj}(M) = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}^T$$

$C_{1,1} = (-1)^{1+1} \cdot M_{1,1}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$

$C_{1,1} = (-1)^{1+1} \cdot [(2)(4) - (1)(1)]$

$C_{1,1} = (-1)^2 \cdot (8 - 1)$

$C_{1,1} = 1 \cdot 7$

$C_{1,1} = 7$

$C_{2,1} = (-1)^{2+1} \cdot M_{2,1}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$

$C_{2,1} = (-1)^{2+1} \cdot [(2)(4) - (1)(3)]$

$C_{2,1} = (-1)^3 \cdot (8 - 3)$

$C_{2,1} = -1 \cdot 5$

$C_{2,1} = -5$

$C_{1,2} = (-1)^{1+2} \cdot M_{1,2}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix}$

$C_{1,2} = (-1)^{1+2} \cdot [(0)(4) - (2)(1)]$

$C_{1,2} = (-1)^3 \cdot (0 - 2)$

$C_{1,2} = -1 \cdot (-2)$

$C_{1,2} = 2$

$C_{2,2} = (-1)^{2+2} \cdot M_{2,2}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

$C_{2,2} = (-1)^{2+2} \cdot [(1)(4) - (2)(3)]$

$C_{2,2} = (-1)^4 \cdot (4 - 6)$

$C_{2,2} = 1 \cdot (-2)$

$C_{2,2} = -2$

$C_{1,3} = (-1)^{1+3} \cdot M_{1,3}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix}$

$C_{1,3} = (-1)^{1+3} \cdot [(0)(1) - (2)(2)]$

$C_{1,3} = (-1)^4 \cdot (0 - 4)$

$C_{1,3} = 1 \cdot (-4)$

$C_{1,3} = -4$

$C_{2,3} = (-1)^{2+3} \cdot M_{2,3}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

$C_{2,3} = (-1)^{2+3} \cdot [(1)(1) - (2)(2)]$

$C_{2,3} = (-1)^5 \cdot (1 - 4)$

$C_{2,3} = -1 \cdot (-3)$

$C_{2,3} = 3$

$C_{3,1} = (-1)^{3+1} \cdot M_{3,1}$

$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix}$

$C_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$

$C_{3,1} = (-1)^{3+1} \cdot [(2)(1) - (2)(3)]$

$C_{3,1} = (-1)^{4} \cdot (2 - 6)$

$C_{3,1} = 1 \cdot (-4)$

$C_{3,1} = -4$

$C_{3,3} = (-1)^{3+3} \cdot M_{3,3}$

$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$

$C_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$

$C_{3,3} = (-1)^{3+3} \cdot [(1)(2) - (0)(2)]$

$C_{3,3} = (-1)^{6} \cdot (2 - 0)$

$C_{3,3} = 1 \cdot 2$

$C_{3,3} = 2$

$C_{3,2} = (-1)^{3+2} \cdot M_{3,2}$

$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix}$

$C_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$

$C_{3,2} = (-1)^{3+2} \cdot [(1)(1) - (0)(3)]$

$C_{3,2} = (-1)^{5} \cdot (1 - 0)$

$C_{3,2} = -1 \cdot 1$

$C_{3,2} = -1$

$$\mathrm{Adj}(M) = \begin{bmatrix} 7 & 2 & -4 \\ -5 & -2 & 3 \\ -4 & -1 & 2 \end{bmatrix}^T$$

$$\mathrm{Adj}(M) = \begin{bmatrix} 7 & -5 & -4 \\ 2 & -2 & -1 \\ -4 & 3 & 2 \end{bmatrix}$$

$$M^{-1} = \frac{\text{Adj}(M)}{\det(M)}$$

$$M^{-1} = \frac{\begin{bmatrix} 7 & -5 & -4 \\ 2 & -2 & -1 \\ -4 & 3 & 2 \end{bmatrix}}{-1}$$

$$M^{-1} = \begin{bmatrix} 7/_{-1} & -5/_{-1} & -4/_{-1} \\ 2/_{-1} & -2/_{-1} & -1/_{-1} \\ -4/_{-1} & 3/_{-1} & 2/_{-1} \end{bmatrix}$$

$$\boxed{M^{-1} = \begin{bmatrix} -7 & 5 & 4 \\ -2 & 2 & 1 \\ 4 & -3 & -2 \end{bmatrix}}$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -7 & 5 & 4 \\ -2 & 2 & 1 \\ 4 & -3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} (1)(-7)+(2)(-2)+(3)(4) & (1)(5)+(2)(2)+(3)(-3) & (1)(4)+(2)(1)+(3)(-2) \\ (0)(-7)+(2)(-2)+(1)(4) & (0)(5)+(2)(2)+(1)(-3) & (0)(4)+(2)(1)+(1)(-2) \\ (2)(-7)+(1)(-2)+(4)(4) & (2)(5)+(1)(2)+(4)(-3) & (2)(4)+(1)(1)+(4)(-2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Two Linear Equations with Two Unknowns Using Matrices

Problem 6

For each of the following systems of linear equations, express them in matrix form and solve them using the inverse matrix methods.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} e \\ f \end{bmatrix}$$

6.a Solve $\begin{cases} 5x + 7y = 3 \\ 2x + 4y = 1 \end{cases}$ using the inverse matrix.

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{\operatorname{Adj}\left(\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}\right)}{\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix}}$$

$$\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} = (5)(4) - (2)(7)$$

$$\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} = 20 - 14$$

$$\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} = 6$$

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 4 & -7 \\ -2 & 5 \end{bmatrix}}{6}$$

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4/6 & -7/6 \\ -2/6 & 5/6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{6} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{6} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{3}\right)(3) + \left(-\frac{7}{6}\right)(1) \\ \left(-\frac{1}{3}\right)(3) + \left(\frac{5}{6}\right)(1) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 + \left(-\frac{7}{6}\right) \\ -1 + \frac{5}{6} \end{bmatrix}$$

$$\boxed{\begin{cases} x = \frac{5}{6} \\ y = -\frac{1}{6} \end{cases}}$$

Check:

$5x + 7y = 3$

$5\left(\frac{5}{6}\right) + 7\left(-\frac{1}{6}\right) = 3$

$3 = 3$

$2x + 4y = 1$

$2\left(\frac{5}{6}\right) + 4\left(-\frac{1}{6}\right) = 1$

$1 = 1$

6.b Solve $\begin{cases} 3x - 2y = 7 \\ -5x + 6y = -5 \end{cases}$ using the inverse matrix.

$$\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}^{-1} = \frac{\operatorname{Adj}\left(\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}\right)}{\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix}}$$

$$\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = (3)(6) - (-5)(-2)$$

$$\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18 - 10$$

$$\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 8$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}}{8}$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 6/8 & 2/8 \\ 5/8 & 3/8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{5}{8} & \frac{3}{8} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{5}{8} & \frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{4}\right)(7) + \left(\frac{1}{4}\right)(-5) \\ \left(\frac{5}{8}\right)(7) + \left(\frac{3}{8}\right)(-5) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{21}{4} + \left(-\frac{5}{4}\right) \\ \frac{35}{8} + \left(-\frac{15}{8}\right) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{16}{4} \\ \frac{20}{8} \end{bmatrix}$$

$$\boxed{\begin{cases} x = 4 \\ y = \frac{5}{2} \end{cases}}$$

Check:

$3x - 2y = 7$

$3(4) - 2\left(\frac{5}{2}\right) = 7$

$7 = 7$

$-5x + 6y = -5$

$-5(4) + 6\left(\frac{5}{2}\right) = -5$

$-5 = -5$

6.c Solve $\begin{cases} 4x + y = 6 \\ 5x + 2y = 7 \end{cases}$ using the inverse matrix.

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{Adj\left(\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}\right)}{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}}$$

$$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = (4)(2) - (5)(1)$$

$$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 8 - 5$$

$$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 3$$

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 2 & -1 \\ -5 & 4 \end{bmatrix}}{3}$$

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -5/3 & 4/3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{4}{3} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{3}\right)(6) + \left(-\frac{1}{3}\right)(7) \\ \left(-\frac{5}{3}\right)(6) + \left(\frac{4}{3}\right)(7) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 + \left(-\frac{7}{3}\right) \\ -10 + \frac{28}{3} \end{bmatrix}$$

$$\boxed{\begin{cases} x = \frac{5}{3} \\ y = -\frac{2}{3} \end{cases}}$$

Check:

$4x + y = 6$

$4\left(\frac{5}{3}\right) + \left(-\frac{2}{3}\right) = 6$

$6 = 6$

$5x + 2y = 7$

$5\left(\frac{5}{3}\right) + 2\left(-\frac{2}{3}\right) = 7$

$7 = 7$

Matrix Inverse by Reduced Row-Echelon Form

Problem 7

For each of the following matrices, computer its inverse using reduced row-echelon form:

> Create the augmented matrix $[M|I]$, then convert to reduced row-echelon form. The right-hand matrix will be the inverse of the original matrix.

7.a Given $M = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, compute its inverse.

$[M|I]$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2-2(1) & 3-2(4) & 0-2(1) & 1-2(0) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right] \xrightarrow{r_2/-5} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ {}^{0}/_{-5} & {}^{-5}/_{-5} & {}^{-2}/_{-5} & {}^{1}/_{-5} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right] \xrightarrow{R_1 - 4r_2} \left[\begin{array}{cc|cc} 1-4(0) & 4-4(1) & 1-4\left(\frac{2}{5}\right) & 0-4\left(-\frac{1}{5}\right) \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)\left(-\frac{3}{5}\right) + (4)\left(\frac{2}{5}\right) & (1)\left(\frac{4}{5}\right) + (4)\left(-\frac{1}{5}\right) \\ (2)\left(-\frac{3}{5}\right) + (3)\left(\frac{2}{5}\right) & (2)\left(\frac{4}{5}\right) + (3)\left(-\frac{1}{5}\right) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

7.b Given $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2-2(1) & 3-2(2) & 0-2(1) & 1-2(0) \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{r_2/-1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0/_{-1} & -1/_{-1} & -2/_{-1} & 1/_{-1} \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cc|cc} 1-2(0) & 2-2(1) & 1-2(2) & 0-2(-1) \\ 0 & 1 & 2 & -1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$[I|M^{-1}]$

$\boxed{M^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}}$

Check:

$M \cdot M^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$M \cdot M^{-1} = \begin{bmatrix} (1)(-3)+(2)(2) & (1)(2)+(2)(-1) \\ (2)(-3)+(3)(2) & (2)(2)+(3)(-1) \end{bmatrix}$

$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

7.c Given $M = \begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \xrightarrow{r_2-5r_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5-5(1) & 4-5(1) & 0-5(1) & 1-5(0) \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 1 \end{array}\right] \xrightarrow{r_2/-1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0/_{-1} & -1/_{-1} & -5/_{-1} & 1/_{-1} \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & -1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & -1 \end{array}\right] \xrightarrow{r_1-r_2} \left[\begin{array}{cc|cc} 1-0 & 1-1 & 1-5 & 0-(-1) \\ 0 & 1 & 5 & -1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & -4 & 1 \\ 0 & 1 & 5 & -1 \end{array}\right]$

$[I|M^{-1}]$

$\boxed{M^{-1} = \begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix}}$

Check:

$M \cdot M^{-1} = \begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix}$

$M \cdot M^{-1} = \begin{bmatrix} (1)(-4)+(1)(5) & (1)(1)+(1)(-1) \\ (5)(-4)+(4)(5) & (5)(1)+(4)(-1) \end{bmatrix}$

$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7.d Given $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\xrightarrow{r_3 - r_1}]{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2-2(1) & 0-2(-1) & 3-2(2) & 0-2(1) & 1-2(0) & 0-2(0) \\ 1-1 & 0-(-1) & 1-2 & 0-1 & 0-0 & 1-0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\xrightarrow{r_2 - 2r_3}]{r_1 + r_3} \left[\begin{array}{ccc|ccc} 1+0 & -1+1 & 2+(-1) & 1+(-1) & 0+0 & 0+1 \\ 0-2(0) & 2-2(1) & -1-2(-1) & -2-2(-1) & 1-2(0) & 0-2(1) \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\xrightarrow{r_3 + r_2}]{r_1 - r_2} \left[\begin{array}{ccc|ccc} 1-0 & 0-0 & 1-1 & 0-0 & 0-1 & 1-(-2) \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0+0 & 1+0 & -1+0 & -1+0 & 0+1 & 1+(-2) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & 1 & -1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 3 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)(0)+(-1)(-1)+(2)(0) & (1)(-1)+(-1)(1)+(2)(1) & (1)(3)+(-1)(-1)+(2)(-2) \\ (2)(0)+(0)(-1)+(3)(0) & (2)(-1)+(0)(1)+(3)(1) & (2)(3)+(0)(-1)+(3)(-2) \\ (1)(0)+(0)(-1)+(1)(0) & (1)(-1)+(0)(1)+(1)(1) & (1)(3)+(0)(-1)+(1)(-2) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

7.e Given $M = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2-3r_1} \\ \xrightarrow{r_3-r_1} \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3-3(1) & -1-3(-1) & 5-3(2) & 0-3(1) & 1-3(0) & 0-3(0) \\ 1-1 & 0-(-1) & 1-2 & 0-1 & 0-0 & 1-0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1+r_3} \\ \xrightarrow{r_2-2r_3} \end{array} \left[\begin{array}{ccc|ccc} 1+0 & -1+1 & 2+(-1) & 1+(-1) & 0+0 & 0+1 \\ 0-2(0) & 2-2(1) & -1-2(-1) & -3-2(-1) & 1-2(0) & 0-2(1) \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1-r_2} \\ \\ \xrightarrow{r_3+r_2} \end{array} \left[\begin{array}{ccc|ccc} 1-0 & 0-0 & 1-1 & 0-(-1) & 0-1 & 1-(-2) \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0+0 & 1+0 & -1+0 & -1+(-1) & 0+1 & 1+(-2) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -2 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -2 & 1 & -1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)(1)+(-1)(-2)+(2)(-1) & (1)(-1)+(-1)(1)+(2)(1) & (1)(3)+(-1)(-1)+(2)(-2) \\ (3)(1)+(-1)(-2)+(5)(1) & (3)(-1)+(-1)(1)+(5)(1) & (3)(3)+(-1)(-1)+(5)(-2) \\ (1)(1)+(0)(-2)+(1)(-1) & (1)(-1)+(0)(1)+(1)(1) & (1)(3)+(0)(-1)+(1)(-2) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

7.f Given $M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3-2r_1]{r_2-r_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1-1 & 1-(-1) & 1-0 & 0-1 & 1-0 & 0-0 \\ 2-2(1) & 1-2(-1) & 1-2(0) & 0-2(1) & 0-2(0) & 1-2(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{r_2/2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0/2 & 2/2 & 1/2 & -1/2 & 1/2 & 0/2 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow[r_3-3r_2]{r_1+r_2} \left[\begin{array}{ccc|ccc} 1+0 & -1+1 & 0+\frac{1}{2} & 1+\left(-\frac{1}{2}\right) & 0+\frac{1}{2} & 0+0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0-3(0) & 3-3(1) & 1-3\left(\frac{1}{2}\right) & -2-3\left(-\frac{1}{2}\right) & 0-3\left(\frac{1}{2}\right) & 1-3(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow[r_2+r_3]{r_1+r_3} \left[\begin{array}{ccc|ccc} 1+0 & 0+0 & \frac{1}{2}+\left(-\frac{1}{2}\right) & \frac{1}{2}+\left(-\frac{1}{2}\right) & \frac{1}{2}+\left(-\frac{3}{2}\right) & 0+1 \\ 0+0 & 1+0 & \frac{1}{2}+\left(-\frac{1}{2}\right) & -\frac{1}{2}+\left(-\frac{1}{2}\right) & \frac{1}{2}+\left(-\frac{3}{2}\right) & 0+1 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{-2r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ -2(0) & -2(0) & -2\left(-\frac{1}{2}\right) & -2\left(-\frac{1}{2}\right) & -2\left(-\frac{3}{2}\right) & -2(1) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 3 & -2 \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)(0)+(-1)(-1)+(2)(1) & (1)(-1)+(-1)(-1)+(0)(3) & (1)(1)+(-1)(1)+(0)(-2) \\ (1)(0)+(1)(-1)+(1)(1) & (1)(-1)+(1)(-1)+(1)(3) & (1)(1)+(1)(1)+(1)(-2) \\ (2)(0)+(1)(1)+(1)(1) & (2)(-1)+(1)(-1)+(1)(3) & (2)(1)+(1)(1)+(1)(-2) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

7.g Given $M = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - 3r_1} \\ \xrightarrow{r_3 - 4r_1} \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3-3(1) & 4-3(1) & 5-3(2) & 0-3(1) & 1-3(0) & 0-3(0) \\ 4-4(1) & 5-4(1) & 6-4(2) & 0-4(1) & 0-4(0) & 1-4(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 1 & -2 & -4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 1 & -2 & -4 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 - r_2} \\ \\ \xrightarrow{r_3 - r_2} \end{array} \left[\begin{array}{ccc|ccc} 1-0 & 1-1 & 2-(-1) & 1-(-3) & 0-1 & 0-0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0-0 & 1-1 & -2-(-1) & -4-(-3) & 0-1 & 1-0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -1 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -1 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 + 3r_3} \\ \xrightarrow{r_2 - r_3} \end{array} \left[\begin{array}{ccc|ccc} 1+3(0) & 0+3(0) & 3+3(-1) & 4+3(-1) & -1+3(-1) & 0+3(1) \\ 0-0 & 1-0 & -1-(-1) & -3-(-1) & 1-(-1) & 0-1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ -0 & -0 & -(-1) & -(-1) & -(-1) & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)(1)+(1)(-2)+(2)(1) & (1)(-4)+(1)(2)+(2)(1) & (1)(3)+(1)(-1)+(2)(-1) \\ (3)(1)+(4)(-2)+(5)(1) & (3)(-4)+(4)(2)+(5)(1) & (3)(3)+(4)(-1)+(5)(-1) \\ (4)(1)+(5)(-2)+(6)(1) & (4)(-4)+(5)(2)+(6)(1) & (4)(3)+(5)(-1)+(6)(-1) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

7.h Given $M = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 5 \\ 4 & 11 & 6 \end{bmatrix}$, calculate its inverse.

$[M|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 9 & 5 & 0 & 1 & 0 \\ 4 & 11 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 9 & 5 & 0 & 1 & 0 \\ 4 & 11 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2-3r_1} \\ \xrightarrow{r_3-4r_1} \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3-3(1) & 9-3(3) & 5-3(2) & 0-3(1) & 1-3(0) & 0-3(0) \\ 4-4(1) & 11-4(3) & 6-4(2) & 0-4(1) & 0-4(0) & 1-4(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0 & -1 & -2 & -4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0 & -1 & -2 & -4 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{r_1+2r_2} \\ \xrightarrow{r_3-2r_2} \end{array} \left[\begin{array}{ccc|ccc} 1+2(0) & 3+2(0) & 2+2(-1) & 1+2(-3) & 0+2(1) & 0+2(0) \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0-2(0) & -1-2(0) & -2-2(-1) & -4-2(-3) & 0-2(1) & 1-2(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 2 & 0 \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0 & -1 & 0 & 2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 2 & 0 \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0 & -1 & 0 & 2 & -2 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{-r_2} \\ \xrightarrow{-r_3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 2 & 0 \\ -(0) & -(0) & -(-1) & -(-3) & -(1) & -(0) \\ -(0) & -(-1) & -(0) & -(2) & -(-2) & -(1) \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 2 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 2 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \end{array}\right] \xrightarrow{r_1-3r_3} \left[\begin{array}{ccc|ccc} 1-3(0) & 3-3(1) & 0-3(0) & 1 & 2-3(2) & 0-3(-1) \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & -1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 3 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 3 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 3 & -1 & 0 \end{array}\right]$$

$[I|M^{-1}]$

$$\boxed{M^{-1} = \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}}$$

Check:

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 5 \\ 4 & 11 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 & 3 \\ -2 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} (1)(1)+(3)(-2)+(2)(3) & (1)(-4)+(3)(2)+(2)(-1) & (1)(3)+(3)(-1)+(2)(0) \\ (3)(1)+(9)(-2)+(5)(3) & (3)(-4)+(9)(2)+(5)(-1) & (3)(3)+(9)(-1)+(5)(0) \\ (4)(1)+(11)(-2)+(6)(3) & (4)(-4)+(11)(2)+(6)(-1) & (4)(3)+(11)(-1)+(6)(0) \end{bmatrix}$$

$$M \cdot M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Least Square Approximation Method

Problem 8

Blank.

Problem 9

Blank.

Systems of Equations with Cramer's Method

Problem 10

Solve the following systems of linear equations using Cramer's method:

Create a matrix of coefficients (Δ) and find its determinant. For each variable, create a new matrix (Δ_i) from the matrix of coefficients by replacing the i column with the constants from the original equations and find its determinant. The solution value of each variable is found by dividing the determinant of the variable-specific matrix by the determinant of the matrix of coefficients.

$$i = \frac{\det(\Delta_i)}{\det(\Delta)}$$

10.a Solve $\begin{cases} 5x + 7y = 3 \\ 2x + 4y = 1 \end{cases}$ using Cramer's method.

$\Delta = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}$

$\det(\Delta) = \begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix}$

$\det(\Delta) = (5)(4) - (2)(7)$

$\det(\Delta) = 20 - 14$

$\det(\Delta) = 6$

$\Delta_x = \begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix}$

$\det(\Delta_x) = \begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix}$

$\det(\Delta_x) = (3)(4) - (1)(7)$

$\det(\Delta_x) = 12 - 7$

$\det(\Delta_x) = 5$

$x = \dfrac{\det(\Delta_x)}{\det(\Delta)}$

$x = \dfrac{5}{6}$

$\Delta_y = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

$\det(\Delta_y) = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}$

$\det(\Delta_y) = (5)(1) - (2)(3)$

$\det(\Delta_y) = 5 - 6$

$\det(\Delta_y) = -1$

$y = \dfrac{\det(\Delta_y)}{\det(\Delta)}$

$y = \dfrac{-1}{6}$

$$\boxed{\begin{cases} x = \frac{5}{6} \\ y = -\frac{1}{6} \end{cases}}$$

Check:

$5\left(\frac{5}{6}\right) + 7\left(-\frac{1}{6}\right) = 3$
$3 = 3$

$2\left(\frac{5}{6}\right) + 4\left(-\frac{1}{6}\right) = 1$
$1 = 1$

10.b Solve $\begin{cases} 3x - 2y = 7 \\ -5x + 6y = -5 \end{cases}$ using Cramer's method.

$\Delta = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$

$\det(\Delta) = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix}$

$\det(\Delta) = (3)(6) - (-5)(-2)$

$\det(\Delta) = 18 - 10$

$\det(\Delta) = 8$

$\Delta_x = \begin{bmatrix} 7 & -2 \\ -5 & 6 \end{bmatrix}$

$\det(\Delta_x) = \begin{vmatrix} 7 & -2 \\ -5 & 6 \end{vmatrix}$

$\det(\Delta_x) = (7)(6) - (-5)(-2)$

$\det(\Delta_x) = 42 - 10$

$\det(\Delta_x) = 32$

$x = \dfrac{\det(\Delta_x)}{\det(\Delta)}$

$x = \dfrac{32}{8}$

$x = 4$

$\Delta_y = \begin{bmatrix} 3 & 7 \\ -5 & -5 \end{bmatrix}$

$\det(\Delta_y) = \begin{vmatrix} 3 & 7 \\ -5 & -5 \end{vmatrix}$

$\det(\Delta_y) = (3)(-5) - (-5)(7)$

$\det(\Delta_y) = -15 - (-35)$

$\det(\Delta_y) = 20$

$y = \dfrac{\det(\Delta_y)}{\det(\Delta)}$

$y = \dfrac{20}{8}$

$y = \dfrac{5}{2}$

$\boxed{\begin{cases} x = 4 \\ y = \frac{5}{2} \end{cases}}$

Check:

$3(4) - 2\left(\dfrac{5}{2}\right) = 7$

$7 = 7$

$-5(4) + 6\left(\dfrac{5}{2}\right) = -5$

$-5 = -5$

10.c Solve $\begin{cases} 4x + y = 6 \\ 5x + 2y = 7 \end{cases}$ using Cramer's method.

$\Delta = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}$

$\det(\Delta) = \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}$

$\det(\Delta) = (4)(2) - (5)(1)$

$\det(\Delta) = 8 - 5$

$\det(\Delta) = 3$

$\Delta_x = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}$

$\det(\Delta_x) = \begin{vmatrix} 6 & 1 \\ 7 & 2 \end{vmatrix}$

$\det(\Delta_x) = (6)(2) - (7)(1)$

$\det(\Delta_x) = 12 - 7$

$\det(\Delta_x) = 5$

$x = \dfrac{\det(\Delta_x)}{\det(\Delta)}$

$x = \dfrac{5}{3}$

$\Delta_y = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$

$\det(\Delta_y) = \begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix}$

$\det(\Delta_y) = (4)(7) - (5)(6)$

$\det(\Delta_y) = 28 - 30$

$\det(\Delta_y) = -2$

$y = \dfrac{\det(\Delta_y)}{\det(\Delta)}$

$y = \dfrac{-2}{3}$

$$\boxed{\begin{cases} x = \dfrac{5}{3} \\ y = -\dfrac{2}{3} \end{cases}}$$

Check:

$4\left(\frac{5}{3}\right) + \left(-\frac{2}{3}\right) = 6$

$6 = 6$

$5\left(\frac{5}{3}\right) + 2\left(-\frac{2}{3}\right) = 7$

$7 = 7$

10.d Solve $\begin{cases} 2x + y = 4 \\ -3x + z = -8 \\ y + 2z = -3 \end{cases}$ using Cramer's method.

$$\Delta = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\det(\Delta) = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\det(\Delta) = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}(2) - \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix}(1) + \begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix}(0)$$

$$\det(\Delta) = [(0)(2) - (1)(1)](2) - [(-3)(2) - (0)(1)](1) + [(-3)(1) - (0)(0)](0)$$

$$\det(\Delta) = (0 - 1)(2) - (-6 - 0)(1) + (-3 - 0)(0)$$

$$\det(\Delta) = (-1)(2) - (-6)(1) + (-3)(0)$$

$$\det(\Delta) = -2 - (-6) + 0$$

$$\det(\Delta) = 4$$

$$\Delta_x = \begin{bmatrix} 4 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\det(\Delta_x) = \begin{vmatrix} 4 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$

$$\det(\Delta_x) = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}(4) - \begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix}(1) + \begin{vmatrix} -8 & 0 \\ -3 & 1 \end{vmatrix}(0)$$

$$\det(\Delta_x) = [(0)(2) - (1)(1)](4) - [(-8)(2) - (-3)(1)](1) + [(-8)(1) - (-3)(0)](0)$$

$$\det(\Delta_x) = (0 - 1)(4) - [-16 - (-3)](1) + (-8 - 0)(0)$$

$$\det(\Delta_x) = (-1)(4) - (-13)(1) + (-8)(0)$$

$$\det(\Delta_x) = -4 - (-13) + 0$$

$$\det(\Delta_x) = 9$$

$$x = \frac{\det(\Delta_x)}{\det(\Delta)}$$

$$x = \frac{9}{4}$$

$$\Delta_y = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\det(\Delta_y) = \begin{vmatrix} 2 & 4 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix}$$

$$\det(\Delta_y) = \begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix}(2) - \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix}(4) + \begin{vmatrix} -3 & -8 \\ 0 & -3 \end{vmatrix}(0)$$

$$\det(\Delta_y) = [(-8)(2) - (-3)(1)](2) - [(-3)(2) - (0)(1)](4) + [(-3)(-3) - (0)(-8)](0)$$

$$\det(\Delta_y) = [-16 - (-3)](2) - (-6 - 0)(4) + (9 - 0)(0)$$

$$\det(\Delta_y) = (-13)(2) - (-6)(4) + (9)(0)$$

$$\det(\Delta_y) = -26 - (-24) + 0$$

$$\det(\Delta_y) = -2$$

$$y = \frac{\det(\Delta_y)}{\det(\Delta)}$$

$$y = \frac{-2}{4}$$

$$y = -\frac{1}{2}$$

$$\Delta_z = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} 2 & 1 & 4 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} 0 & -8 \\ 1 & -3 \end{vmatrix}(2) - \begin{vmatrix} -3 & -8 \\ 0 & -3 \end{vmatrix}(1) + \begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix}(4)$$

$$\det(\Delta_z) = [(0)(-3) - (1)(-8)](2) - [(-3)(-3) - (0)(-8)](1) + [(-3)(1) - (0)(0)](1)$$

$$\det(\Delta_z) = [0 - (-8)](2) - (9 - 0)(1) + (-3 - 0)(4)$$

$$\det(\Delta_z) = (8)(2) - (9)(1) + (-3)(4)$$

$$\det(\Delta_z) = 16 - 9 + (-12)$$

$$\det(\Delta_z) = -5$$

$$z = \frac{\det(\Delta_z)}{\det(\Delta)}$$

$$z = \frac{-5}{4}$$

$$\boxed{\begin{cases} x = \dfrac{9}{4} \\ y = -\dfrac{1}{2} \\ z = -\dfrac{5}{4} \end{cases}}$$

Check:

$2\left(\frac{9}{4}\right) + \left(-\frac{1}{2}\right) = 4$ $\quad$ $-3\left(\frac{9}{4}\right) + \left(-\frac{5}{4}\right) = -8$ $\quad$ $\left(-\frac{1}{2}\right) + 2\left(-\frac{5}{4}\right) = -3$

$4 = 4$ $\quad$ $-8 = -8$ $\quad$ $-3 = -3$

10.e Solve $\begin{cases} 2x + y + z = 4 \\ -x + 2z = 2 \\ 3x + y + 3z = -2 \end{cases}$ using Cramer's method.

$$\Delta = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$\det(\Delta) = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix}$$

$$\det(\Delta) = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}(2) - \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}(1) + \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix}(1)$$

$$\det(\Delta) = [(0)(3) - (1)(2)](2) - [(-1)(3) - (3)(2)](1) + [(-1)(1) - (3)(0)](1)$$

$$\det(\Delta) = (0 - 2)(2) - (-3 - 6)(1) + (-1 - 0)(1)$$

$$\det(\Delta) = (-2)(2) - (-9)(1) + (-1)(1)$$

$$\det(\Delta) = -4 - (-9) + (-1)$$

$$\det(\Delta) = 4$$

$$\Delta_x = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\det(\Delta_x) = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix}$$

$$\det(\Delta_x) = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}(4) - \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix}(1) + \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}(1)$$

$$\det(\Delta_x) = [(0)(3) - (1)(2)](4) - [(2)(3) - (-2)(2)](1) + [(2)(1) - (-2)(0)](1)$$

$$\det(\Delta_x) = (0 - 2)(4) - [6 - (-4)](1) + (2 - 0)(1)$$

$$\det(\Delta_x) = (-2)(4) - (10)(1) + (2)(1)$$

$$\det(\Delta_x) = -8 - 10 + 2$$

$$\det(\Delta_x) = -16$$

$$x = \frac{\det(\Delta_x)}{\det(\Delta)}$$

$$x = \frac{-16}{4}$$

$$x = -4$$

$$\Delta_y = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix}$$

$$\det(\Delta_y) = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix}$$

$$\det(\Delta_y) = \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix}(2) - \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}(4) + \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}(1)$$

$$\det(\Delta_y) = [(2)(3) - (-2)(2)](2) - [(-1)(3) - (3)(2)](4) + [(-1)(-2) - (3)(2)](1)$$

$$\det(\Delta_y) = [6 - (-4)](2) - (-3 - 6)(4) + (2 - 6)(1)$$

$$\det(\Delta_y) = (10)(2) - (-9)(4) + (-4)(1)$$

$$\det(\Delta_y) = 20 - (-36) + (-4)$$

$$\det(\Delta_y) = 52$$

$$y = \frac{\det(\Delta_y)}{\det(\Delta)}$$

$$y = \frac{52}{4}$$

$$y = 13$$

$$\Delta_z = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}(2) - \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}(1) + \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix}(4)$$

$$\det(\Delta_z) = [(0)(-2) - (1)(2)](2) - [(-1)(-2) - (3)(2)](1) + [(-1)(1) - (3)(0)](4)$$

$$\det(\Delta_z) = (0 - 2)(2) - (2 - 6)(1) + (-1 - 0)(4)$$

$$\det(\Delta_z) = (-2)(2) - (-4)(1) + (-1)(4)$$

$$\det(\Delta_z) = -4 - (-4) + (-4)$$

$$\det(\Delta_z) = -4$$

$$z = \frac{\det(\Delta_z)}{\det(\Delta)}$$

$$z = \frac{-4}{4}$$

$$z = -1$$

$$\boxed{\begin{cases} x = -4 \\ y = 13 \\ z = -1 \end{cases}}$$

Check:

$2(-4) + (13) + (-1) = 4$ $\quad$ $-(-4) + 0(13) + 2(-1) = 2$ $\quad$ $3(-4) + (13) + 3(-1) = -2$

$4 = 4$ $\quad$ $2 = 2$ $\quad$ $-2 = -2$

10.f Solve $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$ using Cramer's method.

$$\Delta=\begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\det(\Delta)=\begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix}$$

$$\det(\Delta)=\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}(1)-\begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix}(-4)+\begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix}(1)$$

$$\det(\Delta)=[(-1)(-3)-(2)(2)](1)-[(4)(-3)-(2)(2)](-4)+[(4)(2)-(2)(-1)](1)$$

$$\det(\Delta)=(3-4)(1)-(-12-4)(-4)+[8-(-2)](1)$$

$$\det(\Delta)=(-1)(1)-(-16)(-4)+(10)(1)$$

$$\det(\Delta)=-1-64+10$$

$$\det(\Delta)=-55$$

$$\Delta_x=\begin{bmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{bmatrix}$$

$$\det(\Delta_x)=\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}$$

$$\det(\Delta_x)=\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}(6)-\begin{vmatrix} -1 & 2 \\ -20 & -3 \end{vmatrix}(-4)+\begin{vmatrix} -1 & -1 \\ -20 & 2 \end{vmatrix}(1)$$

$$\det(\Delta_x)=[(-1)(-3)-(2)(2)](6)-[(-1)(-3)-(-20)(2)](-4)$$
$$+[(-1)(2)-(-20)(-1)](1)$$

$$\det(\Delta_x)=(3-4)(6)-[3-(-40)](-4)+(-2-20)(1)$$

$$\det(\Delta_x)=(-1)(6)-(43)(-4)+(-22)(1)$$

$$\det(\Delta_x)=-6-(-172)+(-22)$$

$$\det(\Delta_x)=144$$

$$x=\frac{\det(\Delta_x)}{\det(\Delta)}$$

$$x=\frac{144}{-55}$$

$$\Delta_y=\begin{bmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{bmatrix}$$

$$\det(\Delta_y)=\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}$$

$$\det(\Delta_y) = \begin{vmatrix} -1 & 2 \\ -20 & -3 \end{vmatrix}(1) - \begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix}(6) + \begin{vmatrix} 4 & -1 \\ 2 & -20 \end{vmatrix}(1)$$

$$\det(\Delta_y) = [(-1)(-3) - (-20)(2)](1) - [(4)(-3) - (2)(2)](6) + [(4)(-20) - (2)(-1)](1)$$

$$\det(\Delta_y) = [3 - (-40)](1) - (-12 - 4)(6) + [-80 - (-2)](1)$$

$$\det(\Delta_y) = (43)(1) - (-16)(6) + (-78)(1)$$

$$\det(\Delta_y) = 43 - (-96) + (-78)$$

$$\det(\Delta_y) = 61$$

$$y = \frac{\det(\Delta_y)}{\det(\Delta)}$$

$$y = \frac{61}{-55}$$

$$\Delta_z = \begin{bmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{bmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix}$$

$$\det(\Delta_z) = \begin{vmatrix} -1 & -1 \\ 2 & -20 \end{vmatrix}(1) - \begin{vmatrix} 4 & -1 \\ 2 & -20 \end{vmatrix}(-4) + \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix}(6)$$

$$\det(\Delta_z) = [(-1)(-20) - (2)(-1)](1) - [(4)(-20) - (2)(-1)](-4) + [(4)(2) - (2)(-1)](6)$$

$$\det(\Delta_z) = [20 - (-2)](1) - [-80 - (-2)](-4) + [8 - (-2)](6)$$

$$\det(\Delta_z) = (22)(1) - (-78)(-4) + (10)(6)$$

$$\det(\Delta_z) = 22 - 312 + 60$$

$$\det(\Delta_z) = -230$$

$$z = \frac{\det(\Delta_z)}{\det(\Delta)}$$

$$z = \frac{-230}{-55}$$

$$z = \frac{46}{11}$$

$$\boxed{\begin{cases} x = -\dfrac{144}{55} \\ y = -\dfrac{61}{55} \\ z = \dfrac{46}{11} \end{cases}}$$

Check:

$\left(-\frac{144}{55}\right) - 4\left(-\frac{61}{55}\right) + \left(\frac{46}{11}\right) = 6$ $\quad 4\left(-\frac{144}{55}\right) - \left(-\frac{61}{55}\right) + 2\left(\frac{46}{11}\right) = -1$ $\quad 2\left(-\frac{144}{55}\right) + 2\left(-\frac{61}{55}\right) - 3\left(\frac{46}{11}\right) = -20$

$6 = 6$ $\quad -1 = -1$ $\quad -20 = -20$

Linear Independence of Vector Using Determinant

Problem 11

For each of the following vector sets, determine if it is linearly dependent or linearly independent:

> Form a matrix using the vector set in column form. Take the determinant of that matrix. If the determinant equals zero, the set is linearly dependent, else it is linearly independent.

11.a Are $\vec{v}_1 = (2,1)$ and $\vec{v}_2 = (5,4)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2]$

$V = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$

$\det(V) = v_{11}v_{22} - v_{21}v_{12}$

$\det(V) = (2)(4) - (1)(5)$

$\det(V) = 8 - 5$

$\det(V) = 3$

$\det V \neq 0$

Linearly independent.

11.b Are $v_1 = (5,0)$ and $\vec{v}_2 = (0,1)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2]$

$V = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 5 & 0 \\ 0 & 1 \end{vmatrix}$

$\det(V) = v_{11}v_{22} - v_{21}v_{12}$

$\det(V) = (5)(1) - (0)(0)$

$\det(V) = 5 - 0$

$\det(V) = 5$

$\det(V) \neq 0$

Linearly independent.

11.c Are $\vec{v}_1 = (1, -1)$ and $\vec{v}_2 = (4,5)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2]$

$V = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 1 & 4 \\ -1 & 5 \end{vmatrix}$

$\det(V) = v_{11}v_{22} - v_{21}v_{12}$

$\det(V) = (1)(5) - (-1)(4)$

$\det(V) = 5 - (-4)$

$\det(V) = 9$

$\det(V) \neq 0$

Linearly independent.

11.d Are $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,2,1)$, and $\vec{v}_3 = (0,0,1)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$

$V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}$

$\det(V) = M_{1,1} \cdot v_{11} - M_{1,2} \cdot v_{12} + M_{1,3} \cdot d_{13}$

$\det(V) = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}(1) - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}(0) + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}(0)$

$\det(V) = [(2)(1) - (1)(0)](1) - [(1)(1) - (0)(0)](0) + [(1)(1) - (0)(2)](0)$

$\det(V) = (2 - 0)(1) - (1 - 0)(0) + (1 - 0)(0)$

$\det(V) = (2)(1) - (1)(0) + (1)(0)$

$\det(V) = 2 - 0 + 0$

$\det(V) = 2$

$\det(V) \neq 0$

Linearly independent.

Alternate:

V is a triangular matrix. Determinant of triangular matrix is equal to product of entries on main diagonal.

$\det(V) = (1)(2)(1) = 2$

11.e Are $\vec{v}_1 = (4,0,0)$, $\vec{v}_2 = (0,2,0)$, and $\vec{v}_3 = (0,0,3)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$

$$V = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(V) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$\det(V) = M_{1,1} \cdot v_{11} - M_{1,2} \cdot v_{12} + M_{1,3} \cdot v_{13}$

$\det(V) = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}(4) - \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix}(0) + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}(0)$

$\det(V) = [(2)(3) - (0)(0)](4) - [(0)(3) - (0)(0)](0) + [(0)(0) - (0)(2)](0)$

$\det(V) = (6 - 0)(4) - (0 - 0)(0) + (0 - 0)(0)$

$\det(V) = (6)(4) - (0)(0) + (0)(0)$

$\det(V) = 24 - 0 + 0$

$\det(V) = 24$

$\det(V) \neq 0$

Linearly independent.

> Alternate:
>
> V is a diagonal matrix. Determinant of diagonal matrix is equal to product of entries on main diagonal.
>
> $\det(F) = (4)(2)(3) = 24$

11.f Are $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,2,1)$, and $\vec{v}_3 = (1,3,1)$ linearly dependent or independent?

$V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\det(V) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{vmatrix}$$

$\det(V) = M_{1,1} \cdot v_{11} - M_{1,2} \cdot v_{12} + M_{1,3} \cdot v_{13}$

$\det(V) = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}(1) - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}(0) + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}(1)$

$\det(V) = [(2)(1) - (2)(3)](1) - [(1)(1) - (0)(3)](0) + [(1)(2) - (0)(2)](1)$

$\det(V) = (2 - 6)(1) - (1 - 0)(0) + (2 - 0)(1)$

$\det(V) = (-4)(1) - (1)(0) + (2)(1)$

$\det(V) = -4 - 0 + 2$

$\det(V) = -2$

$\det(V) \neq 0$

Linearly independent.

Basis of a Vector Space Using Determinant

Problem 12

For each of the following vector pairs, determine if the pair forms a basis for $\mathbb{R}^2$:

Form a matrix using the vector set in column form. Take the determinant of that matrix. If the determinant equals zero, the set is linearly dependent, else it is linearly independent. Because any two linearly independent 2D vectors form a basis for $\mathbb{R}^2$, if the set is linearly independent, it forms a basis for $\mathbb{R}^2$.

12.a Do $\vec{v}_1 = (2,1)$ and $\vec{v}_2 = (5,4)$ form a basis for $\mathbb{R}^2$?

$V = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$

$\det(V) = 3$

Proved in previous problem.

$\det(V) \neq 0$

Linearly independent.

v_1 and $\vec{v}_2$ form a basis for $\mathbb{R}^2$.

12.b Do $\vec{v}_1 = (5,0)$ and $\vec{v}_2 = (0,1)$ form a basis for $\mathbb{R}^2$?

$V = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(V) = 5$

Proved in previous problem.

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$ and $\vec{v}_2$ form a basis for $\mathbb{R}^2$.

12.c Do $\vec{v}_1 = (5,2)$ and $\vec{v}_2 = (2,1)$ form a basis for $\mathbb{R}^2$?

$V = [\vec{v}_1 \quad \vec{v}_2]$

$V = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix}$

$\det(V) = v_{11}v_{22} - v_{21}v_{12}$

$\det(V) = (5)(1) - (2)(2)$

$\det(V) = 5 - 4$

$\det(V) = 1$

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$ and $\vec{v}_2$ form a basis for $\mathbb{R}^2$.

12.d Do $\vec{v}_1 = (1,2)$ and $\vec{v}_2 = (4,8)$ form a basis for $\mathbb{R}^2$?

$V = [\vec{v}_1 \quad \vec{v}_2]$

$V = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$

$\det(V) = v_{11}v_{22} - v_{21}v_{12}$

$\det(V) = (1)(8) - (2)(4)$

$\det(V) = 8 - 8$

$\det(V) = 0$

Linearly dependent.

$\vec{d}_1$ and $\vec{d}_2$ do **not** form a basis for $\mathbb{R}^2$.

Alternate:

$\vec{v}_2 = 4(1,2)$

$\vec{v}_2 = 4\vec{v}_1$

$\vec{v}_2 = k\vec{v}_1, k \in \mathbb{R}$

Linearly dependent.

Problem 13

For each of the following vector sets, determine if the set forms a basis for $\mathbb{R}^3$:

Form a matrix using the vector set in column form. Take the determinant of that matrix. If the determinant equals zero, the set is linearly dependent, else it is linearly independent. Because any three linearly independent 3D vectors form a basis for $\mathbb{R}^3$, if the set is linearly independent, it forms a basis for $\mathbb{R}^3$.

13.a Do $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,2,1)$, and $\vec{v}_3 = (0,0,1)$ form a basis for $\mathbb{R}^3$?

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\det(V) = 2$

Proved in Problem 11.d, above.

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form a basis for $\mathbb{R}^3$.

13.b Do $\vec{v}_1 = (4,0,0)$, $\vec{v}_2 = (0,2,0)$, and $\vec{v}_3 = (0,0,3)$ form a basis for $\mathbb{R}^3$?

$$V = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\det(V) = 24$

Proved in Problem 11.e, above.

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ for a basis for $\mathbb{R}^3$.

13.c Do $\vec{v}_1 = (1,1,0)$, $v_2 = (0,2,1)$, and $\vec{v}_3 = (1,3,1)$ form a basis for $\mathbb{R}^3$?

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$\det(V) = -2$

Proved in Problem 11.f, above.

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ for a basis for $\mathbb{R}^3$.

13.d Do $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,2,1)$, and $\vec{v}_3 = (0,3,1)$ form a basis for $\mathbb{R}^3$?

$V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$

$V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

$\det(V) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$

$\det(V) = M_{1,1} \cdot v_{11} - M_{1,2} \cdot v_{12} + M_{1,3} \cdot v_{13}$

$\det(V) = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}(1) - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}(0) + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}(0)$

$\det(V) = [(2)(1) - (1)(3)](1) - [(1)(1) - (0)(3)](0) + [(1)(2) - (0)(2)](0)$

$\det(V) = (2 - 3)(1) - (1 - 0)(0) + (2 - 0)(0)$

$\det(V) = (-1)(1) - (1)(0) + (2)(0)$

$\det(V) = -1 - 0 + 0$

$\det(V) = -1$

$\det(V) \neq 0$

Linearly independent.

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ for a basis for $\mathbb{R}^3$.

Characteristic Equation

Problem 14

For the following matrices, find the characteristic equation:

Characteristic equation: $\det(M = \lambda I) = 0, \lambda \in \mathbb{R}$.

14.a Find the characteristic equation of $M = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

$\det(M - \lambda I) = 0$

$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} 2 - \lambda & 0 \\ 1 & 3 - \lambda \end{bmatrix}\right) = 0$

$\begin{vmatrix} 2 - \lambda & 0 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$(M - \lambda I)_{11}(M - \lambda I)_{22} - (M - \lambda I)_{21}(M - \lambda I)_{12} = 0$

$(2 - \lambda)(3 - \lambda) - (1)(0) = 0$

$(2 - \lambda)(3 - \lambda) - 0 = 0$

$(2 - \lambda)(3 - \lambda) = 0$

$(2 - \lambda)(3) - (2 - \lambda)(\lambda) = 0$

$(6 - 3\lambda) - (2\lambda - \lambda^2) = 0$

$6 - 3\lambda - 2\lambda + \lambda^2 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$\boxed{\lambda^2 - 5\lambda + 6 = 0}$

14.b Find the characteristic equation of $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(M - \lambda I)_{11}(M - \lambda I)_{22} - (M - \lambda I)_{21}(M - \lambda I)_{12} = 0$$

$$(1-\lambda)(2-\lambda) - (0)(0) = 0$$

$$(1-\lambda)(2-\lambda) - 0 = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$(1-\lambda)(2) - (1-\lambda)(\lambda) = 0$$

$$(2-2\lambda) - (\lambda - \lambda^2) = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 = 0$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\boxed{\lambda^2 - 3\lambda + 2 = 0}$$

14.c Find the characteristic equation of $M = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}$.

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ 0 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 0 & -3-\lambda \end{vmatrix} = 0$$

$$(M - \lambda I)_{11}(M - \lambda I)_{22} - (M - \lambda I)_{21}(M - \lambda I)_{12} = 0$$

$$(2-\lambda)(-3-\lambda) - (0)(3) = 0$$

$$(2-\lambda)(-3-\lambda) - 0 = 0$$

$$(2-\lambda)(-3-\lambda) = 0$$

$$(2-\lambda)(-3) - (2-\lambda)(\lambda) = 0$$

$$(-6 + 3\lambda) - (2\lambda - \lambda^2) = 0$$

$$-6 + 3\lambda - 2\lambda + \lambda^2 = 0$$

$$\lambda^2 + 3\lambda - 2\lambda - 6 = 0$$

$$\boxed{\lambda^2 + \lambda - 6 = 0}$$

14.d Find the characteristic equation of $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(M - \lambda I)_{11}(M - \lambda I)_{22} - (M - \lambda I)_{21}(M - \lambda I)_{12} = 0$$

$$(4-\lambda)(1-\lambda) - (2)(-1) = 0$$

$$(4-\lambda)(1-\lambda) - (-2) = 0$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$(4-\lambda)(1) - (4-\lambda)(\lambda) + 2 = 0$$

$$(4-\lambda) - (4\lambda - \lambda^2) + 2 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - \lambda - 4\lambda + 4 + 2 = 0$$

$$\boxed{\lambda^2 - 5\lambda + 6 = 0}$$

14.e Find the characteristic equation of $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$M_{1,1} \cdot (M - \lambda I)_{11} - M_{1,2} \cdot (M - \lambda I)_{12} + M_{1,3} \cdot (M - \lambda I)_{13} = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}(1-\lambda) - \begin{vmatrix} 0 & 1 \\ 0 & 3-\lambda \end{vmatrix}(1) + \begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}(1) = 0$$

$$[(2-\lambda)(3-\lambda) - (0)(1)](1-\lambda) - [(0)(3-\lambda) - (0)(1)](1) + [(0)(0) - (0)(2-\lambda)](1) = 0$$

$$\{[(2-\lambda)(3) - (2-\lambda)(\lambda)] - 0\}(1-\lambda) - (0-0)(1) + (0-0)(1) = 0$$

$$\{[(6-3\lambda) - (2\lambda - \lambda^2)] - 0\}(1-\lambda) - (0)(1) + (0)(1) = 0$$

$$[(6 - 3\lambda - 2\lambda + \lambda^2) - 0](1-\lambda) - 0 + 0 = 0$$

$$[(\lambda^2 - 3\lambda - 2\lambda + 6) - 0](1-\lambda) = 0$$

$$(\lambda^2 - 3\lambda - 2\lambda + 6 - 0)(1-\lambda) = 0$$

$$(\lambda^2 - 5\lambda + 6)(1-\lambda) = 0$$

$$(\lambda^2 - 5\lambda + 6)(1) - (\lambda^2 - 5\lambda + 6)(\lambda) = 0$$

$$(\lambda^2 - 5\lambda + 6) - (\lambda^3 - 5\lambda^2 + 6\lambda) = 0$$

$$\lambda^2 - 5\lambda + 6 - \lambda^3 + 5\lambda^2 - 6\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 5\lambda^2 - 5\lambda - 6\lambda + 6 = 0$$

$$\boxed{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$$

Alternate:

$M - \lambda I$ is a triangular matrix.
Determinant of a triangular matrix is equal to the product of entries on main diagonal.

$$\det(E - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$$\det(E - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

14.f Find the characteristic equation of $M = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$.

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 4 & 5 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$M_{1,1} \cdot (M - \lambda I)_{11} - M_{1,2} \cdot (M - \lambda I)_{12} + M_{1,3} \cdot (M - \lambda I)_{13} = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 5 & 1-\lambda \end{vmatrix}(2-\lambda) - \begin{vmatrix} 1 & 1 \\ 4 & 1-\lambda \end{vmatrix}(0) + \begin{vmatrix} 1 & 1-\lambda \\ 4 & 5 \end{vmatrix}(0) = 0$$

$$[(1-\lambda)(1-\lambda) - (5)(0)](2-\lambda) - [(1)(1-\lambda) - (4)(1)](0) + [(1)(5) - (4)(1-\lambda)](0) = 0$$

$$\{[(1-\lambda)(1) - (1-\lambda)(\lambda)] - 0\}(2-\lambda) - 0 + 0 = 0$$

$$\{[(1-\lambda) - (\lambda - \lambda^2)] - 0\}(2-\lambda) = 0$$

$$[(1 - \lambda - \lambda + \lambda^2) - 0](2-\lambda) = 0$$

$$[(\lambda^2 - \lambda - \lambda + 1) - 0](2-\lambda) = 0$$

$$[(\lambda^2 - 2\lambda + 1) - 0](2-\lambda) = 0$$

$$(\lambda^2 - 2\lambda + 1 - 0)(2-\lambda) = 0$$

$$(\lambda^2 - 2\lambda + 1)(2-\lambda) = 0$$

$$(\lambda^2 - 2\lambda + 1)(2) - (\lambda^2 - 2\lambda + 1)(\lambda) = 0$$

$$(2\lambda^2 - 4\lambda + 2) - (\lambda^3 - 2\lambda^2 + \lambda) = 0$$

$$2\lambda^2 - 4\lambda + 2 - \lambda^3 + 2\lambda^2 - \lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 2\lambda^2 - 4\lambda - \lambda + 2 = 0$$

$$\boxed{-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0}$$

Alternate:

$M - \lambda I$ is a triangular matrix.
Determinant of a triangular matrix is equal to the product of entries on main diagonal.

$$\det(F - \lambda I) = (2-\lambda)(1-\lambda)(1-\lambda)$$

$$\det(F - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 26$$

14.f Find the characteristic equation of $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$\det(M - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

Alternate:

$M - \lambda I$ is a diagonal matrix.
Determinant of a diagonal matrix is equal to the product of entries on main diagonal.

$$\det(G - \lambda I) = (2-\lambda)(4-\lambda)(3-\lambda)$$

$$\det(G - \lambda I) = -\lambda^3 + 9\lambda^2 - 26\lambda + 24$$

$$M_{1,1} \cdot (M - \lambda I)_{11} - M_{1,2} \cdot (M - \lambda I)_{12} + M_{1,3} \cdot (M - \lambda I)_{13} = 0$$

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix}(2-\lambda) - \begin{vmatrix} 0 & 0 \\ 0 & 3-\lambda \end{vmatrix}(0) + \begin{vmatrix} 0 & 4-\lambda \\ 0 & 0 \end{vmatrix}(0) = 0$$

$$[(4-\lambda)(3-\lambda) - (0)(0)](2-\lambda) - 0 + 0 = 0$$

$$\{[(4-\lambda)(3) - (4-\lambda)(\lambda)] - 0\}(2-\lambda) = 0$$

$$[(12 - 3\lambda) - (4\lambda - \lambda^2)](2-\lambda) = 0$$

$$(12 - 3\lambda - 4\lambda + \lambda^2)(2-\lambda) = 0$$

$$(\lambda^2 - 3\lambda - 4\lambda + 12)(2-\lambda) = 0$$

$$(\lambda^2 - 7\lambda + 12)(2-\lambda) = 0$$

$$(\lambda^2 - 7\lambda + 12)(2) - (\lambda^2 - 7\lambda + 12)(\lambda) = 0$$

$$(2\lambda^2 - 14\lambda + 24) - (\lambda^3 - 7\lambda^2 + 12\lambda) = 0$$

$$2\lambda^2 - 14\lambda + 24 - \lambda^3 + 7\lambda^2 - 12\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 7\lambda^2 - 14\lambda - 12\lambda + 24 = 0$$

$$\boxed{-\lambda^3 + 9\lambda^2 - 26\lambda + 24 = 0}$$

Eigen Values and Eigen Vectors

Problem 15

For each of the following matrices, find the eigen values and eigen vectors:

Set up the matrix's characteristic equation: $\det(M = \lambda I) = 0, \lambda \in \mathbb{R}$. Solve for λ to find the eigen values.

Create new matrices $M - \lambda I$ for each eigen value. Multiply each by an unknown vector and set equal to the zero vector. Solve for the unknown vectors to find the eigen vectors.

15.a Given $M = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, find its eigen values and eigen vectors.

$(2 - \lambda)(3 - \lambda) = 0$

Found in Problem 14.a, above.

$2 - \lambda_1 = 0$

$\boxed{\lambda_1 = 2}$

$3 - \lambda_2 = 0$

$\boxed{\lambda_2 = 3}$

$(M - \lambda_1 I) \cdot \vec{v}_1 = \vec{0}$

$\begin{bmatrix} 2-(2) & 0 \\ 1 & 3-(2) \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{1,y}$ is the only free variable. Assign it an arbitrary value of $t: t \in \mathbb{R}$.

$v_{1,y} = t$

$v_{1,x} + v_{1,y} = 0$

$v_{1,x} = -v_{1,y}$

$v_{1,x} = -t$

$\vec{v}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\boxed{\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$(M - \lambda_2 I) \cdot \vec{v}_2 = \vec{0}$

$\begin{bmatrix} 2-(3) & 0 \\ 1 & 3-(3) \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{r_1+r_2} \left[\begin{array}{cc|c} -1+1 & 0+0 & 0+0 \\ 1 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{2,x} = 0$, while v_2, y may have any real value. Set $v_{2,y} = t : t \in \mathbb{R}$.

$\vec{v}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$\lambda_1 = 2, \lambda_2 = 3$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

15.b Given $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find its eigen values and eigen vectors.

$(1 - \lambda)(2 - \lambda) = 0$

Found in Problem 14.b, above.

$1 - \lambda_1 = 0$

$\boxed{\lambda_1 = 1}$

$2 - \lambda_2 = 0$

$\boxed{\lambda_2 = 2}$

$(M - \lambda_1 I) \cdot \vec{v}_1 = \vec{0}$

$\begin{bmatrix} 1-(1) & 0 \\ 0 & 2-(1) \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_{1,y} = 0$. $v_{1,x}$ may have an real value. Set $v_{1,x} = t : t \in \mathbb{R}$.

$\vec{v}_1 = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$(M - \lambda_2 I) \cdot \vec{v}_2 = \vec{0}$

$\begin{bmatrix} 1-(2) & 0 \\ 0 & 2-(2) \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_1} \left[\begin{array}{cc|c} -(-1) & -(0) & -(0) \\ 1 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{2,x} = 0$, while v_2, y may have any real value. Set $v_{2,y} = t : t \in \mathbb{R}$.

$\vec{v}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$\lambda_1 = 1, \lambda_2 = 2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

15.c Given $M = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}$, find its eigen values and eigen vectors.

$(2 - \lambda)(-3 - \lambda) = 0$

Found in Problem 14.c, above.

$2 - \lambda_1 = 0$ $\quad$ $-3 - \lambda_2 = 0$

$\boxed{\lambda_1 = 2}$ $\quad$ $\boxed{\lambda_2 = -3}$

$(M - \lambda_1 I) \cdot \vec{v}_1 = \vec{0}$

$\begin{bmatrix} 2-(2) & 3 \\ 0 & -3-(2) \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 3 \\ 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -5 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -5 & 0 \end{array}\right] \xrightarrow{r_1/3} \left[\begin{array}{cc|c} {}^0/_3 & {}^3/_3 & {}^0/_3 \\ 0 & -5 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \xrightarrow{r_2+5r_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0+5(0) & -5+5(1) & 0+5(0) \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{1,y} = 0$. $v_{1,x}$ may have an real value. Set $v_{1,x} = t : t \in \mathbb{R}$.

$\vec{v}_1 = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$(M - \lambda_2 I) \cdot \vec{v}_2 = \vec{0}$

$\begin{bmatrix} 2-(-3) & 3 \\ 0 & -3-(-3) \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 5 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 5 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1/5} \left[\begin{array}{cc|c} {}^5/_5 & {}^3/_5 & {}^0/_5 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{2,y}$ is the only free variable. Set $v_{2,y} = t: t \in \mathbb{R}$.

$$v_{2,x} + \frac{3}{5}v_{2,y} = 0$$

$$v_{2,x} + \frac{3}{5}t = 0$$

$$v_{2,x} = -\frac{3}{5}t$$

$$\vec{v}_2 = \begin{bmatrix} -\frac{3}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{array}{l} \lambda_1 = 2, \lambda_2 = -3 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix} \end{array}}$$

15.d Given $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, find its eigen values and eigen vectors.

$$\lambda^2 - 5\lambda + 6 = 0$$

Found in Problem 14.d, above.

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda - 3 = 0 \qquad\qquad \lambda - 2 = 0$$

$$\boxed{\lambda_1 = 3} \qquad\qquad \boxed{\lambda_1 = 2}$$

$$(M - \lambda_1 I) \cdot \vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} 4-(3) & -1 \\ 2 & 1-(3) \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2-2(1) & -2-2(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_{1,y}$ is the only free variable. Set $v_{1,y} = t: t \in \mathbb{R}$.

$$v_{1,y} = t$$

$$v_{1,x} - v_{1,y} = 0$$

$v_{1,x} - t = 0$

$v_{1,x} = t$

$\vec{v}_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$(M - \lambda_2 I) \cdot \vec{v}_2 = \vec{0}$

$\begin{bmatrix} 4-(2) & -1 \\ 2 & 1-(2) \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2-(2) & -1-(-1) & 0-0 \end{array}\right]$

$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1/2} \left[\begin{array}{cc|c} 2/2 & -1/2 & 0/2 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

Reduced row-echelon form.

$v_{2,y}$ is the only free variable. Set $v_{2,y} = t : t \in \mathbb{R}$.

$v_{2,y} = t$

$v_{2,x} - \frac{1}{2} v_{2,y} = 0$

$v_{2,x} - \frac{1}{2} t = 0$

$v_{2,x} = \frac{1}{2} t$

$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\boxed{\vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}$

$\lambda_1 = 3, \lambda_2 = 2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

15.e Given $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, find its eigen values and eigen vectors.

$(3 - \lambda)(2 - \lambda(1 - \lambda) = 0$

Found in Problem 14.e, above.

$\lambda_{1,2}{}^2 - 5\lambda_{1,2} + 6 = 0$

$\boxed{\lambda_1 = 3}$ $\boxed{\lambda_2 = 2}$ $1 - \lambda_3 = 0$

$\boxed{\lambda_3 = 1}$

Roots found in Problem 13d, above.

$[M - \lambda_1 I] \cdot \vec{v}_1 = \vec{0}$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ 0 & 0 & 3-3 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -(0) & -(-1) & -(1) & -(0) \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} -2-0 & 1-1 & 1-(-1) & 0-0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1/-2} \left[\begin{array}{ccc|c} {}^{-2}/_{-2} & {}^{0}/_{-2} & {}^{2}/_{-2} & {}^{0}/_{-2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{1,z}$ is the only free variable. Set $v_{1,z} = t: t \in \mathbb{R}$.

$v_{1,y} = t$

$v_{1,x} - v_{1,z} = 0$

$v_{1,x} - t = 0$

$v_{1,x} = t$

$v_{1,y} - v_{1,z} = 0$

$v_{1,y} - t = 0$

$v_{1,y} = t$

$$\vec{v}_1 = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$[M - \lambda_2 I] \cdot \vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 3-2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_1 - r_2} \\ \\ \xrightarrow{r_3 - r_2} \end{array} \left[\begin{array}{ccc|c} -1-0 & 1-0 & 1-1 & 0-0 \\ 0 & 0 & 1 & 0 \\ 0-0 & 0-0 & 1-1 & 0-0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_1} \left[\begin{array}{ccc|c} -(-1) & -(1) & -(0) & -(0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{2,y}$ is the only free variable. Set $v_{2,y} = t: t \in \mathbb{R}$.

$v_{2,y} = t$

$v_{2,x} - v_{2,y} = 0$

$v_{2,x} - t = 0$

$v_{2,x} = t$

$v_{2,z} = 0$

$$\vec{v}_2 = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$[M - \lambda_3 I] \cdot \vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ 0 & 0 & 3-1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \underset{\xrightarrow{r_3/2}}{\xrightarrow{r_1 - r_2}} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0-0 & 1-1 & 1-1 & 0-0 \\ 0/2 & 0/2 & 2/2 & 0/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{\xrightarrow{r_2 \leftrightarrow r_3}}{\xrightarrow{r_1 - r_3}} \left[\begin{array}{ccc|c} 0-0 & 1-0 & 1-1 & 0-0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{3,x}$ is the only free variable. Set $v_{1,x} = t : t \in \mathbb{R}$

$v_{1,x} = t$ $v_{1,y} = 0$ $v_{1,z} = 0$

$$\vec{v}_3 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{\begin{matrix} \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1 \\ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix}}$$

15.f Given $M = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$, find its eigen values and eigen vectors.

$(1 - \lambda)(1 - \lambda)(2 - \lambda) = 0$

Found in Problem 14.f, above.

$1 - \lambda_1 = 0$ $2 - \lambda_2 = 0$

$\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 2}$

$[M - \lambda_1 I] \cdot \vec{v}_1 = \vec{0}$

$$\begin{bmatrix} 2-1 & 0 & 0 \\ 1 & 1-1 & 0 \\ 4 & 5 & 1-1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 4 & 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \xrightarrow{r_3 - 4r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1-1 & 0-0 & 0-0 & 0-0 \\ 4-4(1) & 5-4(0) & 0-4(0) & 0-4(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{1,z}$ is the only free variable. Set $v_{1,z} = t: t \in \mathbb{R}$.

$v_{1,z} = t$

$v_{1,x} = 0$

$v_{1,y} = 0$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$[M - \lambda_2 I] \cdot \vec{v}_2 = \vec{0}$

$$\begin{bmatrix} 2-2 & 0 & 0 \\ 1 & 1-2 & 0 \\ 4 & 5 & 1-2 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \\ v_{2,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 4 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,x} \\ v_{2,y} \\ v_{2,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & 5 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & 5 & -1 & 0 \end{array}\right] \xrightarrow{r_3 - 4r_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 4-4(1) & 5-4(-1) & -1-4(0) & 0-4(0) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 9 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 9 & -1 & 0 \end{array}\right] \xrightarrow{r_3/9} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0/9 & 10/9 & -1/9 & 0/9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right] \xrightarrow{r_2 + r_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1+0 & -1+1 & 0+\left(-\frac{1}{9}\right) & 0+0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 1 & 0 & -\frac{1}{9} & 0 \\ & & & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{2,z}$ is the only free variable. Set $v_{2,z} = t : t \in \mathbb{R}$.

$v_{2,z} = t$

$$v_{2,x} - \frac{1}{9} v_{2,z} = 0$$

$$v_{2,x} - \frac{1}{9} t = 0$$

$$v_{2,x} = \frac{1}{9} t$$

$$v_{2,y} - \frac{1}{9} v_{2,z} = 0$$

$$v_{2,y} - \frac{1}{9} t = 0$$

$$v_{2,y} = \frac{1}{9} t$$

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{9} t \\ \frac{1}{9} t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \\ 1 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{array}{l} \lambda_1 = 1, \lambda_2 = 2 \\ \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \\ 1 \end{bmatrix} \end{array}}$$

15.g Given $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find its eigen values and eigen vectors.

$(4-\lambda)(3-\lambda)(2-\lambda) = 0$

Found in Problem 12f, above.

$4-\lambda_1 = 0$ $\quad$ $3-\lambda_2 = 0$ $\quad$ $2-\lambda_3 = 0$

$\boxed{\lambda_1 = 4}$ $\quad$ $\boxed{\lambda_2 = 3}$ $\quad$ $\boxed{\lambda_3 = 2}$

$[M-\lambda_1 I]\cdot\vec{v}_1 = \vec{0}$

$$\begin{bmatrix} 2-4 & 0 & 0 \\ 0 & 4-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix}\cdot\begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_1/-2} \\ \\ \xrightarrow{r_3/-1} \end{array} \left[\begin{array}{ccc|c} {}^{-2}/_{-2} & {}^{0}/_{-2} & {}^{0}/_{-2} & {}^{0}/_{-2} \\ 0 & 0 & 0 & 0 \\ {}^{0}/_{-1} & {}^{0}/_{-1} & {}^{-1}/_{-1} & {}^{0}/_{-1} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_{1,y}$ is the only free variable. Set $v_{1,y} = t : t \in \mathbb{R}$.

$v_{1,y} = t$ $\quad$ $v_{1,x} = 0$ $\quad$ $v_{1,z} = 0$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$[M - \lambda_2 I] \cdot \vec{v}_2 = \vec{0}$

$$\begin{bmatrix} 2-3 & 0 & 0 \\ 0 & 4-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_1} \left[\begin{array}{ccc|c} -(-1) & -(0) & -(0) & -(0) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_{2,z}$ is the only fr

Reduced row-echelon form.

$v_{2,z} = t$

$v_{2,x} = 0$

$v_{2,y} = 0$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$[M - \lambda_3 I] \cdot \vec{v}_3 = \vec{0}$

$$\begin{bmatrix} 2-2 & 0 & 0 \\ 0 & 4-2 & 0 \\ 0 & 0 & 3-2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_2/2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ {}^{0}/_{0} & {}^{2}/_{0} & {}^{0}/_{0} & {}^{0}/_{0} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form.

$v_{3,x}$ is the only free variable. Set $v_{1,x} = t : t \in \mathbb{R}$

$v_{3,x} = t$ $\quad v_{3,y} = 0$ $\quad v_{3,z} = 0$

$$\vec{v}_3 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\boxed{\begin{array}{l} \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2 \\ \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array}}$$

Cayley-Hamilton Theorem

Problem 16

For each of the following matrices, determine if it satisfies its characteristic equation:

Substitute the matrix into its characteristic equation. If the equation is valid, the matrix satisfies its characteristic equation.

16.a Determine if $M = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ satisfies its characteristic equation.

$\lambda^2 - 5\lambda + 6 = 0$

Characteristic equation, found in Problem 14.a, above.

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} (2)(2) + (0)(1) & (2)(0) + (0)(3) \\ (1)(2) + (3)(1) & (1)(0) + (3)(3) \end{bmatrix} - \begin{bmatrix} 5(2) & 5(0) \\ 5(1) & 5(3) \end{bmatrix} + \begin{bmatrix} 6(1) & 6(0) \\ 6(0) & 6(1) \end{bmatrix} = 0$$

$$\begin{bmatrix} 4+0 & 0+0 \\ 2+3 & 0+9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 5 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 0 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 5 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 - 10 + 6 & 0 - 0 + 0 \\ 5 - 5 + 0 & 9 - 15 + 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$0 = 0$

M satisfies its characteristic equation.

16.b Determine if $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ satisfies its characteristic equation.

$\lambda^2 - 3\lambda + 2 = 0$

Characteristic equation, found in Problem 14.b, above.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 - 3\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} (1)(1) + (0)(0) & (1)(0) + (0)(2) \\ (0)(1) + (2)(0) & (0)(0) + (2)(2) \end{bmatrix} - \begin{bmatrix} 3(1) & 3(0) \\ 3(0) & 3(2) \end{bmatrix} + \begin{bmatrix} 2(1) & 2(0) \\ 2(0) & 2(1) \end{bmatrix} = 0$$

$$\begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-3+2 & 0-0+0 \\ 0-0+0 & 4-6+2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$0 = 0$$

M satisfies its characteristic equation.

16.c Determine if $M = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}$ satisfies its characteristic equation.

$$\lambda^2 + \lambda + 2 = 0$$

$$\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}^2 + \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} (2)(2)+(3)(0) & (2)(3)+(3)(-3) \\ (0)(2)+(-3)(0) & (0)(3)+(-3)(-3) \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2(1) & 2(0) \\ 2(0) & 2(1) \end{bmatrix} = 0$$

$$\begin{bmatrix} 4+0 & 6+(-9) \\ 0+0 & 0+9 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -3 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4+2+2 & -3+3+0 \\ 0+0+0 & 9+(-3)+2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix} = 0$$

M does **not** satisfy its characteristic equation.

16.d Determine if $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ satisfies its characteristic equation.

$$\lambda^2 - 5\lambda + 6 = 0$$

Characteristic equation, found in Problem 14.d, above.

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^2 - 5\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} (4)(4)+(-1)(2) & (4)(-1)+(-1)(1) \\ (2)(4)+(1)(2) & (2)(-1)+(1)(1) \end{bmatrix} - \begin{bmatrix} 5(4) & 5(-1) \\ 5(2) & 5(1) \end{bmatrix} + \begin{bmatrix} 6(1) & 6(0) \\ 6(0) & 6(1) \end{bmatrix} = 0$$

$$\begin{bmatrix} 16+(-2) & -4+(-1) \\ 8+2 & -2+1 \end{bmatrix} - \begin{bmatrix} 20 & -5 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 14 & -5 \\ 10 & -1 \end{bmatrix} - \begin{bmatrix} 20 & -5 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 14 - 20 + 6 & -5 - (-5) + 0 \\ 10 - 10 + 0 & -1 - 5 + 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$0 = 0$$

M satisfies its characteristic equation.

END