Chapter7:Quaternion

1) **Definition**

A quaternion is a super set of the complex numbers. We write quaternion

q=[a , x , y , z]= $a + x\vec{i} + y\vec{j} + z\vec{k} = a + \vec{v} = [a, \vec{v}]$, $where\ \vec{v} = (x, y, z)\ is\ the\ vector\ part$

and **a** is the real part of the quaternion with $\vec{i}^2 = \vec{i}\vec{i} = -1$, $\vec{j}^2 = \vec{j}\vec{j} = -1$, $\vec{k}^2 = \vec{k}\vec{k} = -1$, $\vec{i}\vec{j}\vec{k} = -1$

$and\ \ \vec{i}\vec{j} = \vec{i} \times \vec{j} = \vec{k}$, $\vec{j}\vec{k} = \vec{j} \times \vec{k} = \vec{i}$, $\vec{k}\vec{i} = \vec{k} \times \vec{i} = \vec{j}$.

Example 1.1: Find the real part and vector part of the quaternion q=[5 , 3,1,7]
Answer: real part a=5 and vector part $\vec{v} = (3,1,7)$

Example 1.2: Write the quaternion with real part 5 and vector part $\vec{v}(2,6,-4)$.
Answer : q=[5,2,6,-4] or $q = 5 + 2\vec{i} + 6\vec{j} - 4\vec{k}$ or $q = 5 + \vec{v}$ or $q = [5, \vec{v}]$ with $\vec{v} = (2,6,-4)$

Example 1.3: Write the quaternion with real part 10 and vector part $\vec{v}(1,2,3)$.
Answer : q=[10,1,2,3] or $q = 10 + \vec{i} + 2\vec{j} + 3\vec{k}$ or $q = 10 + \vec{v}$ or $q = [10, \vec{v}]$ with $\vec{v} = (1,2,3)$

TODO➔ Go to activity and solve questions 1.1 , 1.2 and 1.3

2) **Norm of a quaternion and normalized quaternion**

The **norm** of a quaternion q=[a , x , y , z] is N(q)= $a^2 + x^2 + y^2 + z^2$.
That is N(q)= $q \cdot \tilde{q}$,and we denote the **length or magnitude** of q to be
L(q)= $\sqrt{N(q)} = \sqrt{a^2 + x^2 + y^2 + z^2}$. Note that L(q) and N(q) are different term

Example 2.1: Calculate the norm of the quaternion q=[1,0,2,-3]
Answer: N(q) = $1^2 + 0^2 + 2^2 + (-3)^2$ =1+0+4+9= 14

Example 2.2: Calculate the length of the quaternion q=[1,0,1,-1]
Answer: N(q) = $1^2 + 0^2 + 1^2 + (-1)^2$ =1+0+1+1= 3 and L(q)= $\sqrt{N(q)} = \sqrt{3}$

Example 2.3: Calculate the length of the quaternion q=[1,0,2,2]
Answer: N(q) = $1^2 + 0^2 + 2^2 + 2^2$ =1+0+4+4= 9 and L(q)= $\sqrt{N(q)} = \sqrt{9} = 3$

TODO➔ Go to activity and solve question 2

Normalizing a quaternion

If N(q) $\neq$ 0 then the normalized quaternion of q is $q_n = \dfrac{q}{\sqrt{N(q)}} = \dfrac{q}{L(q)}$

Example 2.4: Normalize the quaternion q=[2,0,2,-1]

Answer : N(q) = $2^2 + 0^2 + 2^2 + (-1)^2$ = 4+4+1=9 and L(q)=$\sqrt{N(q)} = \sqrt{9} = 3$

So $q_n = \dfrac{q}{\sqrt{N(q)}} = \dfrac{q}{L(q)} == \dfrac{q}{3} = \left[\dfrac{2}{3}, 0, \dfrac{2}{3}, \dfrac{-1}{3}\right]$

Example 2.5: Normalize the quaternion q=[1,1,2,-1]

Answer : N(q) = $1^2 + 1^2 + 2^2 + (-1)^2$ = 1+1+4+1=7 and L(q)=$\sqrt{N(q)} = \sqrt{7}$

So $q_n = \dfrac{q}{\sqrt{N(q)}} = \dfrac{q}{L(q)} == \dfrac{q}{\sqrt{7}} = \left[\dfrac{1}{\sqrt{7}}, \dfrac{1}{\sqrt{7}}, \dfrac{2}{\sqrt{7}}, \dfrac{-1}{\sqrt{7}}\right]$

TODO➔ Go to activity and solve question 3

3) Conjugate of a quaternion

The conjugate of q=[a , x , y , z] is $\tilde{q} = [a, -x, -y, -z]$.

Example 3.1: Find the conjugate of q=[2 , 3,-5,7]

Answer ➔ $\tilde{q}$ **=[2 , -3 , 5 , -7]** .

Example 3.2: Find the conjugate of q=[-2 , 5,9,-1]

Answer ➔ $\tilde{q}$ **=[-2 , -5,-9,1]** .

Example 3.3: Find the conjugate of q=[-3 , -10,6,-4]

Answer ➔ $\tilde{q}$ **=[-3,10 ,-6,4]**.

TODO ➔ Go to activity and solve question 4

4) Inverse of a quaternion

Given a quaternion q=[a , x , y , z] such as $N(q) \neq 0$, then its inverse is $q^{-1} = \frac{\tilde{q}}{N(q)}$

Example 4.1: Find the inverse of q=[1 ,0 , -1, 1]

Answer ➔ N(q)=3 , $\tilde{q}$=[1 , 0 , 1 , -1] , $q^{-1} = \frac{\tilde{q}}{N(q)} = \frac{\tilde{q}}{3} = \left[\frac{1}{3}, 0, \frac{1}{3}, -\frac{1}{3}\right]$

Example 4.2: Find the inverse of q=[1 ,2 , 2, 1]

Answer ➔ N(q)=10 , $\tilde{q}$=[1 , -2 , -2 , -1] , $q^{-1} = \frac{\tilde{q}}{N(q)} = \frac{\tilde{q}}{10} = \left[\frac{1}{10}, \frac{-2}{10}, \frac{-2}{10}, \frac{-1}{10}\right] = \left[\frac{1}{10}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{10}\right]$

TODO ➔ Go to activity and solve questions 5 and 6

5) Quaternion Algebra

We consider here two quaternion $q_1 = [a_1, x_1, y_1, z_1]$ *and* $q_2 = [a_2, x_2, y_2, z_2]$.

a) the dot product of two quaternions

The dot product of q_1 and q_2 is $q_1 \bullet q_2 = [a_1, x_1, y_1, z_1] \bullet [a_2, x_2, y_2, z_2] = a_1a_2 + x_1x_2 + y_1y_2 + z_1z_2$.

Note that $q_1 \bullet q_2 = q_2 \bullet q_1$ (commutative)

Example 5.a.1 : **Calculate** $q_1 \bullet q_2$ **and** $q_3 \bullet q_2$.

if $q_1 = [2,1,0,-1]$ **,** $q_2 = [3,1,-2,1]$**, and** $q_3 = [2,0,-3,1]$

Answer: $q_1 \bullet q_2$=$[2,1,0,-1]$ •$[3,1,-2,1]$= (2)(3) + (1)(1) +(0)(-2)+(-1)(1)=6

$q_3 \bullet q_2$=$[2,0,-3,1]$ •$[3,1,-2,1]$= (2)(3) + (0)(1) +(-3)(-2)+(1)(1)=1

TODO➔ Go to activity and solve questions 7 and 8

b) Addition of two quaternion

$q_1 + q_2 = q_2 + q_1 = [a_1, x_1, y_1, z_1] + [a_2, x_2, y_2, z_2] = [a_1 + a_2 , x_1 + x_2 , y_1 + y_2 , z_1 + z_2]$

Example 5.b.1: Calculate $q_1 + q_2$ **if** $q_1 = [2,1,0,-1]$ **and** $q_2 = [3,1,-2,1]$

Answer : $q_1 + q_2$ **= [2,1,0,-1] + [3,1,-2,1] = [5 , 2 , -2 , 0]**

Example-5.b.2: Calculate $q_1 + q_2$ **if** $q_1 = [2,1,0,-1]$ **and** $q_2 = [3,1,-2,1]$

Answer : $3q_1 - 2q_2$**= 3[2,1,0,-1] -2 [3,1,-2,1]=[0,1,4,-5]**

TODO➔ Go to activity and solve question 9.1 and 9.2

c) <u>Multiplication of two quaternions</u>

The multiplication of q_1 and q_2 is:

$q_1q_2 = [a_1, x_1, y_1, z_1][a_2, x_2, y_2, z_2] = [a_1, \vec{v}_1][a_2, \vec{v}_2] = [a_1a_2 - \vec{v}_1 \bullet \vec{v}_2, a_1\vec{v}_2 + a_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$,

where $\vec{v}_1 = (x_1, y_1, z_1)$ *and* $\vec{v}_2 = (x_2, y_2, z_2)$ **.note that** $q_1q_2 \neq q_2q_1$.

Example 5.c.1 : If $q_1 = [2,1,0,1]$ **and** $q_2 = [3,1,2,1]$ **calculate** q_1q_2

Using $q_1q_2 = [a_1, \vec{v}_1][a_2, \vec{v}_2] = [a_1a_2 - \vec{v}_1 \bullet \vec{v}_2, a_1\vec{v}_2 + a_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$

with a_1=2 , $\vec{v}_1 = (1,0,1)$, a_2=3 , $\vec{v}_2 = (1,2,1)$ we have:

q_1q_2=$[2, \vec{v}_1]\ [3, \vec{v}_2]$= $[(2)(3) - \vec{v}_1 \bullet \vec{v}_2\ ,\ 2\vec{v}_2 + 3\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$,

q_1q_2=$[6 - \vec{v}_1 \bullet \vec{v}_2\ ,\ 2\vec{v}_2 + 3\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$, $3\vec{v}_1 = 3(1,0,1) = (3,0,3)$, $2\vec{v}_2 = 2(1,2,1) = (2,4,2)$

$\vec{v}_1 \bullet \vec{v}_2 = (1,0,1) \bullet (1,2,1) = (1)\cdot(1) + (0)\cdot(2) + (1)\cdot(1) = 2$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 2 & 1 \\ + & - & + \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\vec{k} =$$

$(0-2)\vec{i} - (1-1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{k} = (-2,0,2)$

q_1q_2=$[6 - \vec{v}_1 \bullet \vec{v}_2\ ,\ 2\vec{v}_2 + 3\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$=$[6-2, (2,4,2)+(3,0,3)+(-2,0,2)]$=[4,3,4,7]

Example 5.c.2: If $q_1 = [5,1,1,3]$ and $q_2 = [2,2,0,1]$ calculate q_1q_2

Using $q_1q_2 = [a_1, \vec{v}_1][a_2, \vec{v}_2] = [a_1a_2 - \vec{v}_1 \bullet \vec{v}_2, a_1\vec{v}_2 + a_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$

with a_1=5 , $\vec{v}_1 = (1,1,3)$, a_2=2 , $\vec{v}_2 = (2,0,1)$ we have:

q_1q_2=$[5, \vec{v}_1]\ [2, \vec{v}_2]$= $[(5)(2) - \vec{v}_1 \bullet \vec{v}_2\ ,\ 5\vec{v}_2 + 2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$,

q_1q_2=$[10 - \vec{v}_1 \bullet \vec{v}_2\ ,\ 5\vec{v}_2 + 2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$, $2\vec{v}_1 = 2(1,1,3) = (2,2,6)$, $5\vec{v}_2 = 5(2,0,1) = (10,0,5)$

$\vec{v}_1 \bullet \vec{v}_2 = (1,1,3) \bullet (2,0,1) = (1)\cdot(2) + (1)\cdot(0) + (3)\cdot(1) = 5$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ 2 & 0 & 1 \\ + & - & + \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\vec{k} =$$

$= (1-0)\vec{i} - (1-6)\vec{j} + (0-2)\vec{k} = \vec{i} + 5\vec{j} - 2\vec{k} = (1,5,-2)$, finally

q_1q_2=$[10 - \vec{v}_1 \bullet \vec{v}_2\ ,\ 5\vec{v}_2 + 2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$=$[10-5, (10,0,5)+(2,2,6)+(1,5,-2)]$=[5,(13,7,9)]

That is q_1q_2=[5,13,7,9]

TODO➔ Go to activity and solve questions 10.1 and 10.2

6) Unit quaternion and Rotation Quaternion

A special unit quaternion is q=[1,0,0,0]=[1, $\vec{o}$] (identity quaternion).
A normalized quaternion is a quaternion whose norm is 1
Note that q$\tilde{q}$=[1 , 0 , 0 ,0]

A rotation of θ degree about any arbitrary axis of rotation spanned by a vector $\hat{v}$ is represented in matrix form by the Rodriguez formula :

$R_{\hat{v}}(\theta) = I + \sin(\theta) \bullet Skew(\hat{v}) + (1 - \cos(\theta)) \bullet Skew^2(\hat{v})$. I=identity matrix

The quaternion representation of this matrix is :

$$q = \left[\cos\left(\frac{\theta}{2}\right), \hat{v} \cdot \sin\left(\frac{\theta}{2}\right)\right] = \cos\left(\frac{\theta}{2}\right) + v_x \cdot \sin\left(\frac{\theta}{2}\right)\vec{i} + v_y \cdot \sin\left(\frac{\theta}{2}\right)\vec{j} + v_z \cdot \sin\left(\frac{\theta}{2}\right)\vec{k}$$

where $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$

Example 6.1: write the quaternion of a rotation of $\theta = 90$ about the vector $\vec{v} = (1,1,1)$

Answer ➔: $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(1,0,1)}{\sqrt{2}} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $\frac{\theta}{2} = 45$ ➔ $\sin(\frac{\theta}{2}) = \cos(\frac{\theta}{2}) = \frac{\sqrt{2}}{2}$

So $q = [\cos(45), \hat{v}\sin(45)] = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right] = \left[\frac{\sqrt{2}}{2}, \frac{1}{2}, 0, \frac{1}{2}\right]$

Example 6.2: write the quaternion of a rotation of $\theta = 60$ about the z-axis

Answer :

The direction of the z-axis is $\vec{v} = \hat{k} = (0,0,1)$ **which is normalized,** $\hat{v} = (0,0,1)$

$\frac{\theta}{2} = \frac{60}{2} = 30$ and $\hat{v} = (0,0,1)$ with $\cos(30) = \frac{\sqrt{3}}{2}$, $\sin(30) = \frac{1}{2}$ we have :

$$q = [\cos(30), \hat{v}\sin(30)] = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\hat{v}\right] = \left[\frac{\sqrt{3}}{2}, \frac{1}{2}(0,0,1)\right] = \left[\frac{\sqrt{3}}{2}, 0, 0, \frac{1}{2}\right]$$

Example 6.3: write the quaternion of a rotation of $\theta = 180$ about the vector $\vec{v} = (1,2,2)$

Answer :

$\frac{\theta}{2} = \frac{180}{2} = 90$ $\vec{v} = (1,2,2)$ **is not** normalized , so get $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(1,2,2)}{3} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\cos(90) = 0$, $\sin(90) = 1$ in $q = \left[\cos\left(\frac{\theta}{2}\right), \hat{v}\sin\left(\frac{\theta}{2}\right)\right] = [\cos(90), \hat{v}\sin(90)]$ leads to

$$q=[0,(1)\hat{v}]=\left[0,\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)\right]=\left[0,\frac{1}{3},\frac{2}{3},\frac{2}{3}\right]$$

Note : The inverse of a unit or rotation quaternion q is $q^{-1}=\dfrac{\tilde{q}}{N(q)}=\tilde{q}$ since N(q)=1

TODO➔ Go to activity and solve questions 11.1 , 11.2 , and 11.3

7) Angle between two Normalized (rotation) quaternion

If q_1 and q_2 are two **normalized quaternions**, then the angle between them is such that

$$q_1 \bullet q_2=\sqrt{N(q_1)}\bullet\sqrt{N(q_2)}\bullet\cos\theta \Rightarrow \theta=\cos^{-1}\left(\frac{q_1 \bullet q_2}{\sqrt{N(q_1)}\bullet\sqrt{N(q_2)}}\right)=\cos^{-1}(q_1 \bullet q_2)$$

since q_1 and q_2 are two **normalized quaternions making** $N(q_1)=N(q_2)=1$

Example 7.1 : Calculate the angle between $q_1=\left[\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},0,-\frac{1}{\sqrt{6}}\right]$ **and** $q_2=\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]$

Answer: $q_1 \bullet q_2=\left[\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},0,-\frac{1}{\sqrt{6}}\right]\bullet\left[\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]=\frac{1}{\sqrt{6}}+\frac{1}{2\sqrt{6}}-\frac{1}{2\sqrt{6}}=\frac{1}{\sqrt{6}}$

So $\theta=\cos^{-1}(q_1 \bullet q_2)=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\approx 65.90^0$.

Example 7.2 : Calculate the angle between $q_1=\left[\frac{\sqrt{2}}{2},\frac{1}{2},0,\frac{1}{2}\right]$ **and** $q_2\left[\frac{\sqrt{3}}{2},0,\frac{1}{2},0\right]$

Answer:

$$q_1 \bullet q_2=\left[\frac{\sqrt{2}}{2},\frac{1}{2},0,\frac{1}{2}\right]\bullet\left[\frac{\sqrt{3}}{2},0,\frac{1}{2},0\right]=\frac{\sqrt{6}}{4}$$

So $\theta=\cos^{-1}(q_1 \bullet q_2)=\cos^{-1}\left(\frac{\sqrt{6}}{4}\right)\approx 52.24$.

Example 7.3 : Calculate the angle between $q_1 == \left[\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right]$ **and** $q_2 = [0,0,-1,0]$

Answer :

$$q_1 \bullet q_2 = q_1 == \left[\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right] \bullet [0,0,-1,0] = -\frac{\sqrt{2}}{2}$$

So $\theta = \cos^{-1}(q_1 \bullet q_2) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135$

TODO ➔ Go to activity and solve questions 12 and 13

8) Exponential Expression of a Rotation Quaternion

A rotation quaternion $q = \cos\left(\frac{\theta}{2}\right) + \hat{v} \bullet \sin\left(\frac{\theta}{2}\right)$ can be expressed as $q = e^{\frac{\theta}{2}\hat{v}}$ e=2.7

With the angle in radian.

Example 8.1: A quaternion of angle $\theta = 90$ about $\vec{v} = (1,2,2)$.

Answer: $\theta = 90$ ➔ $\frac{\theta}{2} = 45$ and $\hat{v} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, $\theta = 90$ is $\frac{\pi}{2}$ radian

So $q = \cos\left(\frac{\theta}{2}\right) + \hat{v} \bullet \sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\hat{v}$ and $q = e^{\frac{\theta}{2}\hat{v}} = e^{\frac{\pi}{4}\hat{v}}$

9) Quaternion representation of a vector

a. Vector to quaternion

Given a vector $\vec{v} = (x, y, z)$, its quaternion representation is $q_v = [0, \vec{v}] = [0, x, y, z]$

Example 9a.1 : write the quaternion representation of the vector v(1,3,-5) $\vec{v} = (1,3,-5)$

Answer ➔ q=[0,1,3,-5]

Example 9a.2 : write the quaternion representation of the vector $\vec{k} = (0,0,1)$

Answer ➔ q=[0,0,0,1]

TODO ➔ Go to activity and solve question 14

b. Rotating a vector by a quaternion

Given a rotation quaternion q=[w , x , y , z] and a vector $\vec{v}$, the image of $\vec{v}$ ($\vec{v}'$) after a rotation by q is the vector part of $q_{v'} = q[0,\vec{v}]\tilde{q}$

Example: **Let the quaternion q represent a 90° rotation of a body about the x-axis. Rotate the vector $\vec{u}$ = <0, 1, 2> by q.**

10) Quaternion to rotation matrix transform

The rotation matrix corresponding to the quaternion q = [w, **v**] = [w, x, y, z] is

$$R = \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}$$