Chapter4: Linear Transforms and Linear Operators

1) Linear Transformations

A mapping $T: V \rightarrow W$, is said to be a linear transformation under the following two conditions:

- $\forall\ u, v \in V,\ T(u+v) = T(u) + T(v)$ (rule 1)
- $\forall\ u \in V, \forall\ k \in \mathbb{R},\ T(ku) = kT(u)$ (rule 2) or if

$T(k_1 u + k_2 v) = k_1 T(u) + k_2 T(v)$ **with** $k_1, k_2 \in \mathbb{R}$

Here V = domain of T (could be $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots, or\ \mathbb{R}^n$) and W = co-domain of T .

Example 1.1: Show that $T: \mathbb{R} \rightarrow \mathbb{R}$ $such\ that\ T(x) = 10x$ is a linear transform

Answer : let $x, y \in \mathbb{R}$, $T(x+y) = 10(x+y) = 10x + 10y = T(x) + T(y)$, rule 1 is satisfied

For any scalar k , $T(kx) = 10(kx) = k(10x) = kT(x)$, rule 2 is satisfied.

So T is a linear transform

Example 1.2 : Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $such\ that\ T(\vec{u}) = T(\ (x, y)\ ) = (2x, y)$ is a linear transform

Answer : let $\vec{u}_1(x_1, y_1) \in \mathbb{R}^2$ and $\vec{u}_2(x_2, y_2) \in \mathbb{R}^2$.

$$T(\vec{u}_1 + \vec{u}_2) = T\big((x_1, y_1) + (x_2, y_2)\big) = T\big((x_1 + x_2, y_1 + y_2)\ \big) = \big(2(x_1 + x_2),\ y_1 + y_2\big)$$

$$T(\vec{u}_1 + \vec{u}_2) = \big(2x_1 + 2x_2,\ y_1 + y_2\big) = \big(2x_1,\ y_1\big) + \big(2x_2, y_2\big) = T(\vec{u}_1) + T(\vec{u}_2)$$, rule 1 satisfied

$$for\ k \in \mathbb{R},\ T(k\vec{u}_1) = T\big(\ k(x_1, y_1)\ \big) = T\big(\ (kx_1, ky_1)\ \big) = (2kx_1, ky_1) = k(2x_1, y_1) = kT\big((x_1, y_1)\big)$$

$T(k\vec{u}_1) = kT(\vec{u}_1)$ rule 2 is satisfied , therefore T is a linear transform.

Example 1.3: Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $such\ that\ T(\vec{u}) = A\vec{u}$ is a linear transform, with $A = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$

Answer: let $\vec{u}_1 \in \mathbb{R}^2$ and $\vec{u}_2 \in \mathbb{R}^2$, let $T(\vec{u}_1 + \vec{u}_2) = A(\vec{u}_1 + \vec{u}_2) = A\vec{u}_1 + A\vec{u}_2 = T(\vec{u}_1) + T(\vec{u}_2)$, rule 1

$for\ k \in \mathbb{R},\ T(k\vec{u}_1) = A(k\vec{u}_1) = k(A\vec{u}_1) = kT(\vec{u}_1)$ rule 2. We conclude that T is a linear transform.

Example 1.4: $T: \mathbb{R} \rightarrow \mathbb{R}$ $such\ that\ T(f) = \int f\,dx$

$$T(af + bg) = \int (af + bg)\,dx = \int (af)\,dx + \int (bg)\,dx = a\int (f)\,dx + b\int (g)\,dx$$

$$= aT(f) + bT(g) \ \Rightarrow \ T(f) = \int f\,dx \text{ is linear transform}$$

with $f\ and\ g$ being functions of x and $a\ and\ b$ some constants in $\mathbb{R}$

Example 1.5: $T: \mathbb{R} \rightarrow \mathbb{R}$ $such\ that\ T(f) = \dfrac{df}{dx}$

$$T(af + bg) = \frac{d}{dx}\big(af + bg\big) = \frac{d}{dx}\big(af\big) + \frac{d}{dx}\big(bg\big) = a\frac{df}{dx} + b\frac{dg}{dx}$$

$$= aT(f) + bT(g) \ \Rightarrow \ T(f) = \frac{df}{dx} \text{ is linear transform}$$

with $f\ and\ g$ being functions of x and $a\ and\ b$ some constants in $\mathbb{R}$

TODO➔ Go to Activity and solve question 1

2) Linear Transformation Matrix and Linear Operator

a. Linear Transform Matrix with respect to the Standard Basis

As a reminder, the standard basis S are :

$S_2 = \{\hat{e}_1, \hat{e}_2\} = \{(1,0),(0,1)\}$ $in\ \mathbb{R}^2$, same as $S_2 = \{\vec{i}, \vec{j}\}$

$S_3 = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{(1,0,0),(0,1,0),(0,0,1\}$ $in\ \mathbb{R}^3$, same as $S_3 = \{\vec{i}, \vec{j}, \vec{k}\}$

$S_4 = \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,01)\}$ $in\ \mathbb{R}^4$

For simplicity, we will keep our work on $\mathbb{R}^2$ *and* $\mathbb{R}^3$.

Let suppose we have the following Linear Transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix}$$

then a matrix representation of the Linear Transformation T with respect to the standard basis (S_3 in this case) is

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \vec{w} = T(\vec{u}) = [T]_S \cdot \vec{u}$$

where $[T]_S = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is called the standard matrix of the Linear Transformation T .

More generally :

if A is the matrix form of the linear transform T , that is $T(\vec{u}) = A \cdot \vec{u}$, compute the following

$T(\hat{e}_1) = A \cdot \hat{e}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $T(\hat{e}_2) = A \cdot \hat{e}_2 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $T(\hat{e}_3) = A \cdot \hat{e}_3 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and compute

the matrix $[T]_S$ column-wisely as $[T]_S = \left[T(\hat{e}_1) \mid T(\hat{e}_2) \mid T(\hat{e}_3)\right] = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Example 2.1: Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T(x, y, z) = (x - 2y + z,\ y\ ,\ x + 3y + z)$

Find the standard matrix of T

Answer:

Express T(x,y,z) in matrix form

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + z \\ y \\ x + 3y + z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The standard basis $in\ \mathbb{R}^3$ is $S=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ with $\hat{e}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\hat{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\hat{e}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$

We compute

$$T(\hat{e}_1)=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad,\quad T(\hat{e}_2)=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\3\end{pmatrix}$$

$$T(\hat{e}_3)=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix},\quad \text{finally}$$

$$[T]_S=\left[T(\hat{e}_1)\mid T(\hat{e}_2)\mid T(\hat{e}_3)\right]=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}$$

TIPS: You should notice that the answer is the same as expressing T in matrix form :

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-2y+z\\y\\x+3y+z\end{pmatrix}=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix},\ \text{so}\ [T]_S=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}$$

This is only true for the standard matrix computed with respect to the standard basis $S=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$

<u>Example 2.2</u> : Let $T:\mathbb{R}^2\to\mathbb{R}^2$ such $T(x,y)=(x-y,\ 2x+y)$

Find the standard matrix of T.

Answer :

Express T(x,y) in matrix form

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x-y\\2x+y\end{pmatrix}=\begin{pmatrix}1&-1\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

The standard basis $in\ \mathbb{R}^2$ is $S=\{\hat{e}_1,\hat{e}_2\}$ with $\hat{e}_1=\begin{pmatrix}1\\0\end{pmatrix},\ \hat{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$

We compute

$$T(\hat{e}_1)=\begin{pmatrix}1&-1\\2&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix},\quad T(\hat{e}_2)=\begin{pmatrix}1&-1\\2&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$$

finally

$$[T]_S=\left[T(\hat{e}_1)\mid T(\hat{e}_2)\right]=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$$

TIPS: You should notice that the answer is the same as expressing T in matrix:

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x-y\\2x+y\end{pmatrix}=\begin{pmatrix}1&-1\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\ \text{so}\ [T]_S=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$$

This is only true for the standard matrix computed with respect to the standard basis $S=\{\hat{e}_1,\hat{e}_2\}=\{(1,0),(0,1)\}$

TODO➔ Go to Activity and solve questions 2.1 and 2.2

b. Linear Transform Matrix with respect to any non-Standard Basis

Suppose we have the linear transformation $T: V \to W$ and $B_v = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for the Vector space V, and $B_w = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ a basis of vector space W; we want to find the matrix transformation of T relative to the bases B_v and B_w denoted $[T]_{B_W, B_V}$.

We compute the image of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by T and express the result as a linear combination of the vectors from $B_w = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$:

$$T(\vec{v}_1) = a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{B_W}$$

$$T(\vec{v}_2) = b_1\vec{w}_1 + b_2\vec{w}_2 + b_3\vec{w}_3 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{B_W}$$

$$T(\vec{v}_3) = c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{B_W}$$

So our matrix transformation of T relative to the bases B_v and B_w is :

$$[T]_{B_W, B_V} = \left[T(\vec{v}_1) \mid T(\vec{v}_2) \mid T(\vec{v}_3)\right] = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{and} \quad [\vec{u}]_{B_W} = [T]_{B_W, B_V} \bullet [\vec{u}]_{B_V}$$

Now how do we compute $a_i, b_i, \textit{and } c_i$ $(1 \le i \le 3)$ in $B_w = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$?

This will require solving 3 linear equations. We can simplify the process by using the following Algorithm:

a. Compute $T(\vec{v}_1) = \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}, T(\vec{v}_2) = \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} \textit{ and } T(\vec{v}_3) = \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix}$ in V

b. Construct the matrix $M_v = \left[T(\vec{v}_1) \mid T(\vec{v}_2) \mid T(\vec{v}_3)\right] = \begin{pmatrix} a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \\ a_3' & b_3' & c_3' \end{pmatrix}$

c. Construct the matrix $M_w = \left[\vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3\right] = \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{pmatrix}$

Assuming that $\vec{w}_1 = \begin{pmatrix} w_{11} \\ w_{12} \\ w_{13} \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} w_{21} \\ w_{22} \\ w_{23} \end{pmatrix} \textit{ and } \vec{w}_3 = \begin{pmatrix} w_{31} \\ w_{32} \\ w_{33} \end{pmatrix}$

d. Construct the augmented matrix $\left[M_w \mid M_v\right] = \left(\begin{array}{ccc|ccc} w_{11} & w_{21} & w_{31} & a_1' & b_1' & c_1' \\ w_{12} & w_{22} & w_{32} & a_2' & b_2' & c_2' \\ w_{13} & w_{23} & w_{33} & a_3' & b_3' & c_3' \end{array}\right)$

e. Convert $\left[M_w \mid M_v\right]$ to reduced row echelon form : $\left[I \mid [T]_{B_W,B_V}\right]$

This will convert M_w to I (identity matrix) and M_v to $[T]_{B_W,B_V}$.

The columns of $[T]_{B_W,B_V}$ are the coordinates of $T(\vec{v}_1),\ T(\vec{v}_2),\ and\ \ T(\vec{v}_3)$ in W

Example 2.3: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+y \end{pmatrix}$.

a) Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \{(2,1),(3,2)\}$ and $B_w = \{\vec{w}_1, \vec{w}_2\} = \{(1,1),(1,2)\}$

b) calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Answer: in matrix form we have $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

a) Compute $T(\vec{v}_1) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $T(\vec{v}_2) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

Construct $M_v = \left[T(\vec{v}_1) \mid T(\vec{v}_2)\right] = \begin{pmatrix} 1 & 1 \\ 5 & 8 \end{pmatrix}$

Construct $M_w = \left[\vec{w}_1 \mid \vec{w}_2\right] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Construct the augmented matrix $\left[M_w \mid M_v\right] = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 8 \end{array}\right)$

Convert $\left[M_w \mid M_v\right] = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 8 \end{array}\right)$ to reduced row echelon form

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 8 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 7 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -3 & -6 \\ 0 & 1 & 4 & 7 \end{array}\right)$$

We conclude that $[T]_{B_W,B_V} = \begin{pmatrix} -3 & -6 \\ 4 & 7 \end{pmatrix}$ and

This means that $[\vec{v}_1]_{B_W} = -3\vec{w}_1 + 4\vec{w}_2$ and $[\vec{v}_2]_{B_W} = -6\vec{w}_1 + 7\vec{w}_2$

b) $[\vec{u}]_{B_W} = [T]_{B_W,B_V} \bullet [\vec{u}]_{B_V} = \begin{pmatrix} -3 & -6 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -33 \\ 40 \end{pmatrix}$

Example 2.4: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T(x,\, y) = (x+y,\ x+2y)$. Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \{(1,1),(1,2)\}$ and $B_w = \{\vec{w}_1, \vec{w}_2\} = \{(1,2),(2,5)\}$

Answer:

Compute $T(\vec{v}_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $T(\vec{v}_2) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Construct $M_v = \left[T(\vec{v}_1) \mid T(\vec{v}_2)\right] = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

Construct $M_w = \left[\vec{w}_1 \mid \vec{w}_2\right] = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

Construct the augmented matrix $\left[M_w \mid M_v\right] = \left(\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 2 & 5 & 3 & 5 \end{array}\right)$

Convert $\left[M_w \mid M_v\right] = \left(\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 2 & 5 & 3 & 5 \end{array}\right)$ to reduced row echelon form

$$\left(\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 2 & 5 & 3 & 5 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 4 & 5 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

We conclude that $[T]_{B_W,B_V} = \begin{pmatrix} 4 & 5 \\ -1 & -1 \end{pmatrix}$ and

This means that $[\vec{v}_1]_{B_W} = 4\vec{w}_1 - \vec{w}_2$ and $[\vec{v}_2]_{B_W} = 5\vec{w}_1 - \vec{w}_2$

Example 2.5: Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ such $T(x,\, y) = (y, x+y, x-y)$.

Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \{(1,2),(2,5)\}$ for $\mathbb{R}^2$ and $B_w = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,2,1),(0,1,0),(2,0,3)\}$ for $\mathbb{R}^3$.

Answer:

Compute $T(\vec{v}_1) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $T(\vec{v}_2) = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix}$

Construct $M_v = \left[T(\vec{v}_1) \mid T(\vec{v}_2)\right] = \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ -1 & -3 \end{pmatrix}$

Construct $M_w = \left[\vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3\right] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

Construct the augmented matrix $\left[M_w \mid M_v\right]=\left(\begin{array}{ccc|cc}1&0&2&2&5\\2&1&0&3&7\\1&0&3&-1&-3\end{array}\right)$

Convert $\left[M_w \mid M_v\right]=\left(\begin{array}{ccc|cc}1&0&2&2&5\\2&1&0&3&7\\1&0&3&-1&-3\end{array}\right)$ to reduced row echelon form

$$\left(\begin{array}{ccc|cc}1&0&2&2&5\\2&1&0&3&7\\1&0&3&-1&-3\end{array}\right)\sim\left(\begin{array}{ccc|cc}1&0&2&2&5\\0&1&-4&-1&-3\\0&0&1&-3&-8\end{array}\right)\sim\left(\begin{array}{ccc|cc}1&0&0&8&21\\0&1&0&-13&-35\\0&0&1&-3&-8\end{array}\right)$$

We conclude that $[T]_{B_W,B_V}=\begin{pmatrix}8&21\\-13&-35\\-3&-8\end{pmatrix}$ and

This means that $[\vec{v}_1]_{B_W}=8\vec{w}_1-13\vec{w}_2-3\vec{w}_3$ and $[\vec{v}_2]_{B_W}=21\vec{w}_1-35\vec{w}_2-8\vec{w}_3$

TODO➔ Go to Activity and solve questions 3.1 and 3.2

3) <u>Kernel of linear transformation</u>

Suppose we have the linear transformation $T:V\to W$ such that $T(\vec{x})=A\vec{x}$, we define the kernel of T as the set of all vectors $\vec{x}$ such that $A\vec{x}=\vec{o}$ or $T(\vec{x})=\vec{o}$

We denote the kernel of T by $\ker(T)$ or $\ker(A)$.

Theorem 3.1:

$\ker(T)$ is a subspace of V.

$\dim(\ker(T))=nullity(T)$.

Example 3.1: Let $T:\mathbb{R}^2\to\mathbb{R}^2$ such $T(x,y)=(x+y,\,x+2y)$

Find Ker(T) and dim (Ker T)

Answer:

In matrix form , $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ and we solve $\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ using its augmented matrix $\left(\begin{array}{cc|c}1&1&0\\1&2&0\end{array}\right)$ and convert it in a row reduced echelon form:

$$\left(\begin{array}{cc|c}1&1&0\\1&2&0\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right) ➔ \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} ➔ \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\vec{o}$$

$Ker(T)=\vec{o}\ and\ \dim(\ker T)=0$

Example 3.2: Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T(x, y, z) = (x-2y+z,\ y\ , x+3y+z)$
Find Ker(T) and dim (Ker T)

Answer:

Express T(x,y,z) in matrix form $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2y+z \\ y \\ x+3y+z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Now solve $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, this implies constructing the augmented matrix

$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right)$ that will be converted in row reduced echelon form.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

now solve for x,y, and z

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ➔ $\begin{cases} x+z=0 \\ y=0 \end{cases}$ or $\begin{cases} x=-z \\ y=0 \end{cases}$ setting $z=t$ leads to

$\begin{cases} x=-t \\ y=0 \\ z=t \end{cases}$ so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Now we need to express the result in horizontal form (vector) :
$(x,y,z) = (-t,0,t) = t(-1,0,1)$ so $\ker(T) = span\{\,(-1,0,1)\,\}$ *and* $\dim(\ker T) = 1$

TODO➔ Go to Activity and solve question 4

4) <u>Range or image of a linear Transformation</u>

Suppose we have the linear transformation $T:V \to W$ such that $T(\vec{x}) = A\vec{x}$, we define the image(or range) of T as the set of all vectors $\vec{w} \in W$ such that $\vec{w} = A\vec{x}$ for $\vec{x} \in V$
We denote the image (or range) of T $Im(T)$ or $R(T)$.

The image of T is the span (linear combination) of the column space of A (colsp(A))

Theorem 4.1:

$Im(T)$ is a subspace of W.

$\dim(Im(T)) = colsp(T) = colsp(A)$.

$\dim(V) = \dim(Im\ T) + Ker(T) = colsp(A) + nullity(A)$

An alternative to compute $Im(T)$ is to calculate the image of the vector space V basis vectors image. Let's assume that a basis of V is the standard basis $S=\{\vec{i},\vec{j},\vec{k}\}=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

If $\vec{u}=(a,b,c)\in S$ then $\vec{u}=(a,b,c)=a\vec{i}+b\vec{j}+c\vec{k}$ and compute $T(\vec{u})=A\vec{u}$, that is $T(\vec{u})=A\vec{u}=A\left(a\vec{i}+b\vec{j}+c\vec{k}\right)=a\cdot A\vec{i}+b\cdot A\vec{j}+c\cdot A\vec{k}=aT(\vec{i})+bT(\vec{j})+cT(\vec{k})$

So we can see that $T(\vec{u})=a\cdot A\vec{i}+b\cdot A\vec{j}+c\cdot A\vec{k}=span\left\{A\vec{i},A\vec{j},A\vec{k}\right\}$ and

$Im(T)=$ $span\left\{A\vec{i},A\vec{j},A\vec{k}\right\}$

Example 4.1: Let $T:\mathbb{R}^3\to\mathbb{R}^3$ such $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-2y+z\\y\\x+3y+z\end{pmatrix}$

Find Im(T) and dim (Im T)

Answer:

Write T in matrix form , $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-2y+z\\y\\x+3y+z\end{pmatrix}=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

Since Im (T)= colsp(A), we Reduce $\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}$ in row echelon form

$\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\sim\begin{pmatrix}1&-2&1\\0&1&0\\0&5&0\end{pmatrix}\sim\begin{pmatrix}\underline{1}&-2&1\\0&\underline{1}&0\\0&0&0\end{pmatrix}$ with the underlined pivots in column 1 and 2

corresponding to $\begin{pmatrix}1\\0\\1\end{pmatrix}$ *and* $\begin{pmatrix}-2\\1\\3\end{pmatrix}$, $\text{Im}(T)=span\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}-2\\1\\3\end{pmatrix}\right\}$, dim(Im(T))=2

❖ **alternative method**: here again $A=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}$, compute :

$A\vec{i}=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $A\vec{j}=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\3\end{pmatrix}$, $A\vec{k}=\begin{pmatrix}1&-2&1\\0&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

So $\text{Im}(T)=span\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}-2\\1\\3\end{pmatrix}\right\}$ even tough $A\vec{i}=A\vec{k}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

Example 4.2: $T:\mathbb{R}^3 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ x+2y+z \end{pmatrix}$

Find Im(T) and dim (Im T).

Answer:

Write T in matrix form , $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Since Im (T)= colsp(A), we Reduce $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ in row echelon form:

$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 0 \end{pmatrix}$ with the underlined pivots in column 1 and 2

corresponding to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, so $\operatorname{Im}(T) = span\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ and dim(Im(T))=2

- ❖ Alternative method: here again $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$, and we compute

$$A\vec{i} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad A\vec{j} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\quad A\vec{k} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So again $\operatorname{Im}(T) = span\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ even though $A\vec{i} = A\vec{k} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Example 4.3: Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ -x \\ z \end{pmatrix}$

Find Im(T) and dim (Im T)

Answer : in matrix form we have $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ -x \\ z \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$T(\vec{i}) = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},\quad T(\vec{j}) = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},\quad T(\vec{k}) = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So $\mathrm{Im}(T) = span\left\{\begin{pmatrix}0\\-1\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and dim(Im(T))=3

TODO➔ Go to Activity and solve question 5

5) Geometry of linear Transformation

a) Homogeneous Components

The representation of an N-components position vector (vertex) by an (N+1)-components position vector is called homogeneous coordinate representation. The added element is a scale factor w.

So in general a homogeneous representation of vector $\vec{p} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ is $\vec{p}_h = \begin{bmatrix}wx\\wy\\wz\\w\end{bmatrix}$.

It is important to note that the point $w \bullet \vec{p}$ refers to the same point as $\vec{p}$ does that is

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}x/w\\y/w\\z/w\\1\end{bmatrix} \equiv \begin{bmatrix}x\\y\\z\end{bmatrix} \quad \text{with } w \neq 0.$$

We use w=*1* *for a vertex or position vector , w=0 for a vector.*

Example5.a.1: Find an homogeneous coordinate for the **vertex** $\vec{p} = \begin{pmatrix}2\\3\\4\end{pmatrix}$. **Answer**: $\begin{pmatrix}2\\3\\4\\1\end{pmatrix}$ choosing w=1

Example5.a.2: Find an homogeneous coordinate for the **vertex** $\vec{p} = \begin{pmatrix}4\\5\\7\end{pmatrix}$.**Answer**: $\begin{pmatrix}4\\5\\7\\1\end{pmatrix}$ choosing w=1

Example5.a.3: Find an homogeneous coordinate for the **vector** $\vec{p} = \begin{pmatrix}7\\8\\3\end{pmatrix}$. **Answer**: $\begin{pmatrix}7\\8\\3\\0\end{pmatrix}$ w=0 for vector

b) Affine Transformation

An affine Transformation maps a point $\vec{p}$ to a point $\vec{p}'$ such $\boxed{\vec{p}' = M \cdot \vec{p} + \vec{v}}$ where $\vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ is 3D vector and M a 3x3 Matrix. Introducing a new 4x4 homogeneous transformation matrix T, helps to represent the last Equation by a single matrix.

That is $\begin{bmatrix} \vec{p}' \\ 1 \end{bmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} M & \vec{v} \\ \vec{o}^t & 1 \end{bmatrix} \cdot \begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & a \\ m_{21} & m_{22} & m_{23} & b \\ m_{31} & m_{32} & m_{33} & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$

where $\vec{o}^t = [0\ 0\ 0]$ $\quad \vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{p}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ $\quad$ M= $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$

The 4x4 homogeneous transformation matrix T= $\begin{bmatrix} M & \vec{v} \\ \vec{o}^t & 1 \end{bmatrix}$ maps the homogeneous position vector $\begin{bmatrix} \vec{p} \\ 1 \end{bmatrix}$ to another homogeneous position vector $\begin{bmatrix} \vec{p}' \\ 1 \end{bmatrix}$.We describe an affine as a function as follows:

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\vec{p} \mapsto \vec{p}' = \begin{cases} x' = a_1x + b_1y + c_1z + e_1 \\ y' = a_2x + b_2y + c_2z + e_2 \\ z' = a_3x + b_3y + c_3z + e_3 \end{cases}$$ in matrix form it is $\vec{p}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = M \cdot \vec{p} + \vec{v}$

where the a's, b's and c's are constants and respectively coefficient of x , y and z ; $\vec{v} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$.

The homogeneous matrix of the affine transform using one matrix is :

$$T = \begin{bmatrix} M & \vec{v} \\ \vec{o}^t & 1 \end{bmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & e_1 \\ a_2 & b_2 & c_2 & e_2 \\ a_3 & b_3 & c_3 & e_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 5.b: Write the homogeneous matrix of the following affine transforms.

1) $\begin{cases} x' = x + 2y - z + 5 \\ y' = 2x + y + 3z + 2 \\ z' = x + 4y + z + 7 \end{cases}$

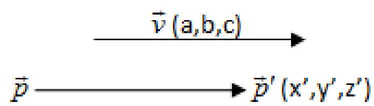
Answer: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$ ➔ $\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 2 & 1 & 3 & 2 \\ 1 & 4 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ **so finally our homogeneous matrix is** $\begin{pmatrix} 1 & 2 & -1 & 5 \\ 2 & 1 & 3 & 2 \\ 1 & 4 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2) $\begin{cases} x' = 4x+5y-2z+3 \\ y' = 2x+7y+3z+2 \\ z' = 9x+4y+6z+1 \end{cases}$

Answer: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 & 5 & -2 \\ 2 & 7 & 3 \\ 9 & 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 & -2 & 3 \\ 2 & 7 & 3 & 2 \\ 9 & 4 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ **so finally our homogeneous matrix is** $\begin{pmatrix} 4 & 5 & -2 & 3 \\ 2 & 7 & 3 & 2 \\ 9 & 4 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) <u>Translation</u>

In translation, an object initial position $\vec{p} = (x, y, z)$ is displaced a given distance and direction. If the displacement is given by the vector $\vec{v} = (a,b,c) = a\vec{i} + b\vec{j} + c\vec{k}$, then the object new position $\vec{p}' = (x', y', z')$ is defined by the relation $\overrightarrow{pp'} = \vec{v}$ $\Rightarrow$ or $\vec{p}' - \vec{p} = \vec{v}$ implying $\vec{p}' = \vec{p} + \vec{v}$.



$\vec{p}' = \vec{p} + \vec{v}$ is the vector equation of the translation and the algebraic equation is :

$$\begin{cases} x' = x + a \\ y' = y + b \\ z' = z + c \end{cases}$$

Translation is an affine transform since $\vec{p}' = \vec{p} + \vec{v} == I \cdot \vec{p} + \vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

with I= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So the homogeneous matrix transform is $\begin{bmatrix} \vec{p}' \\ 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v} \\ \vec{o}^t & 1 \end{bmatrix}\begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$

So we conclude that the translation matrix 3D space that moves a point $\vec{p}$ to new point $\vec{p}'$

in the direction of a vector $\vec{v}(a,b,c)$ is the 4x4 matrix $T_{\vec{v}(a,b,c)} = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This the standard matrix used in the right-handed system (OpenGL)

but $T_{\vec{v}(a,b,c)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$ in a left-handed system(DirectX).

The inverse of a translation is done by negating the direction vector $\vec{v}$: $\vec{p}' = \vec{p} - \vec{v}$

Its inverse is $T_{-\vec{v}(-a,-b,-c)} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $T_{-\vec{v}(-a,-b,-c)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{pmatrix}$ in DirectX.

In DirectX, we use D3DXMatrixTranslation(D3DXMATRIX* T, float a, float b, float c) to represent the translation transform .

In 2D space the translation matrix by a vector $\vec{v} = (a,b)$ is $T_{\vec{v}(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

The inverse transform is $T_{-\vec{v}(-a,-b)} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$

Example 5.c.1 : 2D Translation:

a)Write the translation matrix in the direction of $\vec{v}(5,4)$

$$T_{\vec{v}(5,4)} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

b) What is the image of $\vec{p} = (1,2,3)$ after the translation.

$$\vec{p}' = T_v \bullet \vec{p} = T_{\vec{v}(5,4)} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix}$$

c) What is the inverse matrix of the translation?

Use $-\vec{v} = (-5,-4)$ for the inverse matrix, therefore $T^{-1}_{\vec{v}(5,4)} = T_{-\vec{v}(-5,-4)} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

TODO➔ Go to Activity and solve question 6.

Example 5.c.2: 3D Translation

a)Write the translation matrix in the direction of $\vec{v}(2,4,3)$

$$T_{\vec{v}(2,4,3)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) What is the image of $\vec{p} = (1,2,3)$ after the translation.

$$\vec{p}' = T_v \bullet \vec{p} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \\ 1 \end{pmatrix}$$

TODO➔ Go to Activity and solve question 7

c) What is the inverse matrix of the translation?

Use $-\vec{v} = (-2,-4,-3)$ for the inverse matrix, therefore $T^{-1} = T_{-\vec{V}(-2,-4,-3)} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

TODO➔ Go to Activity and solve question 8

d) 2D Rotation

A θ° rotation of a point $\vec{p} = (x, y)$ about the origin $\vec{o}$ will yield a new point $\vec{p}' = (x', y')$. The two points are related by the following system of equations

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So the rotation of θ° about the origin $\vec{o}$ is $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Its inverse transform is

$$R^{-1}(\theta) = R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^t(\theta) \quad \text{and} \quad \det(R(\theta)) = 1$$

Example1:

a) What is the rotation matrix of 90 ?

$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with cos(90)=0 and sin(90)=1 we have $R(90) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) What is the image $\vec{p}'$ of $\vec{p} = (1,2)$ after it has been rotated by the above rotation?

$$\vec{p}' = R(90) \cdot \vec{p} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

c) What the inverse matrix?

Since $R^{-1}(\theta) = R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R^t(\theta)$, so we have

$$R^{-1}(90) = R^{t}(90) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

d) det (R)=1

e) 3D Rotations

*In 3D space, a r*otation is described by an angle of rotation θ°, a center of rotation $\vec{c} = \vec{o}$, and an axis of rotation (X-axis ,Y-axis or Z-axis) .

Z-axis Rotation

So a θ° rotation of a point $\vec{p} = (x, y, z)$ about the Z-axis will yield a new point $\vec{p}' = (x', y', z')$.

The two points are related by the following system of equations (Linear transformation)

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So the rotation of θ° about the Z-axis is $R_Z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (standard + OpenGL).

Its determinant is $\det\left(R_Z(\theta)\right) = 1$.

The inverse matrix is $R^{-1}{}_Z(\theta) = R^{t}{}_Z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (standard + OpenGL).

A rotation about the Z-axis in a left-handed system (DirectX) is $R_Z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

It's represented in DirectX by the function **D3DXMatrixRotationZ(D3DXMATRIX* R, float angle).**

Example: write the rotation matrix of 45 about the Z-axis and its inverse matrix.

Since $R_Z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we need to find $\cos(45) = \sin(45) = \frac{\sqrt{2}}{2}$

=> $R_Z(45) = \begin{pmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $R_z^{-1}(90) = R_z^{t}(90) \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Y-axis Rotation

The rotation of θ° about the Y-axis is $R_Y(\theta)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$ (standard +OpenGL).

Its determinant is $\det\left(R_Y(\theta)\right)=1$.

The inverse matrix is $R^{-1}{}_Y(\theta)=R^t{}_Y(\theta)\ =\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}$ (standard + OpenGL).

A rotation about the Y-axis in a left-handed system (DirectX) is $R_Y(\theta)=\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}$

it's represented in DirectX by the function **D3DXMatrixRotationY(D3DXMATRIX* R, float angle)** .

Example: write the rotation matrix of 90 about the Y-axis and its inverse matrix.

Since $R_Y(\theta)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$ we need to find cos(90)=0 and sin(90)=1

=> $R_Y(90)=\begin{pmatrix}\cos(90) & 0 & \sin(90)\\ 0 & 1 & 0\\ -\sin(90) & 0 & \cos(90)\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}$ and $R_Y^{-1}(90)=R_Y^t(90)\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$

X-axis Rotation

The rotation of θ° about the X-axis is $R_X(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$ (standard + OpenGL).

Its determinant is $\det\left(R_X(\theta)\right)=1$.

The inverse matrix is $R^{-1}{}_X(\theta)=R^t{}_X(\theta)\ =\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}$ (standard + OpenGL). det

A rotation about the X-axis in a left-handed system (DirectX) is $R_X(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}$

It's represented in DirectX by the function **D3DXMatrixRotationX(D3DXMATRIX* R, float angle).**

Note that, , and $\det(R_i(\theta))=1$ where i =X,Y,or Z

Example: write the rotation matrix of 30 about the X-axis and its inverse matrix.

Since $R_X(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$ we need to find cos(30)= $\sqrt{3}/2$ and sin(30)=1/2

$$=> \ R_X(30)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(30) & \sin(30)\\ 0 & \sin(30) & \cos(30)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{3}/2 & -1/2\\ 0 & 1/2 & \sqrt{3}/2\end{pmatrix} \text{ and } R_X^{-1}(30)=R_X^t(30)=\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{3}/2 & 1/2\\ 0 & -1/2 & \sqrt{3}/2\end{pmatrix}$$

TODO➔ Go to Activity and solve questions 9 and 10

Example:

a)Write the rotation matrix about the z-axis with angle 90

since cos(90)=0 and sin(90)=1 $R_z(90)=\begin{pmatrix}\cos(90) & -\sin(90) & 0\\ \sin(90) & \cos(90) & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

b)What is the image of $\vec{p}=(1,2,3)$ after the rotation.

$$\vec{p}'=R_z(\pi/2)\bullet\vec{p}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}-2\\ 1\\ 3\end{pmatrix}$$

TODO➔ Go to Activity and solve question 11

c)What is the inverse matrix of the rotation? $R^{-1}=R^T=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

f) <u>Scaling</u>

The process of scaling changes the dimension of an object. The scaling factor s determines whether the scaling is a magnification (s>1) or reduction (s<1)

if a point p(x,y,z) is scaled into a new point p'(x',y',z') and s_x, s_y, s_z are respectively the scaling factors on the X-,Y-, and Z-axis , then p and p' are related by the system of equation

$$\begin{cases}x'=s_x\bullet x\\ y'=s_y\bullet y\\ z'=s_z\bullet z\end{cases}\Rightarrow\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=\begin{pmatrix}s_x & 0 & 0\\ 0 & s_y & 0\\ 0 & 0 & s_z\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$,so the scaling matrix with scaling factors s_x, s_y, s_z is:

$$S_{s_x,s_y,s_z}=\begin{pmatrix}s_x & 0 & 0\\ 0 & s_y & 0\\ 0 & 0 & s_z\end{pmatrix}$$. Its inverse matrix is $$S^{-1}_{s_x,s_y,s_z}=S_{\frac{1}{s_x},\frac{1}{s_y},\frac{1}{s_z}}=\begin{pmatrix}\frac{1}{s_x} & 0 & 0\\ 0 & \frac{1}{s_y} & 0\\ 0 & 0 & \frac{1}{s_z}\end{pmatrix}$$

It's represented in DirectX by the function:

D3DXMatrixScaling(D3DXMATRIX* S, float s_x, float s_y, float s_z).

Example:

a)Write the scaling matrix with scaling factors s_x=2 , s_y=3 and s_z=5

from $S_{s_x,s_y,s_z} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$, $S_{2,3,5} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

TODO➔ Go to Activity and solve question 12

b)What is the image of $\vec{p} = (1,2,3)$ after it has been scaled by 2,3 and 5?

$$\vec{p}\,' = S_{2,3,5} \bullet \vec{p} == \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 15 \end{pmatrix}$$

TODO➔ Go to Activity and solve question 13

c)What is the inverse matrix of the scaling matrix?

From $S^{-1}_{s_x,s_y,s_z} = S_{\frac{1}{s_x},\frac{1}{s_y},\frac{1}{s_z}} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & \frac{1}{s_z} \end{pmatrix}$, $S^{-1}_{2,3,5} = S_{\frac{1}{2},\frac{1}{3},\frac{1}{5}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

TODO➔ Go to Activity and solve question 14

g) Arbitrary Axis Rotation (Optional)

A rotation of θ degree about any arbitrary axis L spanned by a vector $\hat{v}$(x,y,z) is represented in matrix form by the Rodriguez formula : $R_{\hat{v}}(\theta) = I + \sin(\theta) \bullet Skew(\hat{v}) + (1 - \cos(\theta)) \bullet Skew^2(\hat{v})$.

Where I=identity matrix= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Skew(\hat{v}) = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$ it's represented in DirectX by the function D3DXMatrixRotationAxis(D3DXMATRIX* R, D3DXVECTOR3* V, float angle).

** We use the Arbitrary rotation to align a vector onto another vector.

h) Orbiting (Optional)

Orbiting is a rotation about an Arbitrary Point in 2D or 3D Space

A rotation of θ° about an arbitrary point $\vec{p} = (x, y, z)$ in 3D is defined by the following transform:

W=$T_{-\vec{v}(x,y,z)} \bullet R(\theta) \bullet T_{\vec{v}(-x,-y,-z)}$ (Orbiting). The algorithm to orbiting is as follows:

Step 1: Translate the point $\vec{p} = (x, y, z)$ to the origin $\vec{o}$ by $\vec{v} = \overrightarrow{po} = \vec{o} - \vec{p} = -\vec{p} = (-x, -y, -z)$

We then construct the translation matrix $T_{\vec{v}(-x,-y,-z)} = \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Step 2: Rotate θ° about the arbitrary point $\vec{p} = (x, y, z)$ that is now at the origin $\vec{o}$.

If for instance the rotation is on Z-axis then $R_Z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Step 3: Translate the point $\vec{p} = (x, y, z)$ back by $-\vec{v} = (x, y, z)$

We then construct the translation matrix $T_{-\vec{v}(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Step 4: Construct the combination matrix :

$$T_{-\vec{v}(x,y,z)} \bullet R(\theta) \bullet T_{\vec{v}(-x,-y,-z)} = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6) linear Operators

A linear transformation $T: V \to W$ is a linear operator if the domain (V) is the same as the co-domain(W), That is if V=W.

Example 6.1:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ such $T(x, y, z) = (x - 2y + z,\ y,\ x + 3y + z)$ is a linear operator

$T: \mathbb{R}^2 \to \mathbb{R}^2$ such $T(x, y) = (x + y,\ y)$ is linear operator (same domain)

$T: \mathbb{R} \to \mathbb{R}$ such $T(x) = 2x$ is a linear operator (same domain)

$T: \mathbb{R}^3 \to \mathbb{R}^2$ such $T(x, y, z) = (x - y, y + z)$ is not a linear operator since $\mathbb{R}^3 \neq \mathbb{R}^2$

TODO➔ Go to Activity and solve question 15

a) Algebra of Linear operators

Let $T_1: \mathbb{R}^n \to \mathbb{R}^n$, $T_2: \mathbb{R}^n \to \mathbb{R}^n$ and $T_3: \mathbb{R}^n \to \mathbb{R}^n$ be 3 linear operators, then

- $T_1 + T_2 = T_2 + T_1$
- $T_1 \circ (T_2 + T_3) = T_1 \circ T_2 + T_1 \circ T_3$
- $T_1 \circ T_2 \neq T_2 \circ T_1$
- $T_1 \circ T_1^{-1} = T_{id}$ where T_{id} is the identity linear operator $T_{id}: \mathbb{R}^n \to \mathbb{R}^m$: $T_{id}(\vec{v}) = I_{id}\vec{v} = \vec{v}$

b) Composition of Linear operators

Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ be two linear operators with their respective standard matrices $[T_1]$ *and* $[T_2]$ then:

$T_2 \circ T_1 = T_2(T_1) : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation (linear operator if $m = n$) with $[T_2] \circ [T_1] = [T_2][T_1]$ as its standard matrix.

$T_2 \circ T_1$ is read "T_2 round T_1", "T_2 circle T_1", "T_2 of T_1"

Note that $T_2 \circ T_1 \neq T_1 \circ T_2$ **(not commutative)**

Example 6.b.1

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such as $T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such as $T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x+y \end{pmatrix}$

Find $T_2 \circ T_1$ *and* $T_1 \circ T_2$. Is $T_2 \circ T_1 = T_1 \circ T_2$.

Answer:

$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so $[T_1] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x+y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so $[T_2] = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$T_2 \circ T_1$ has matrix $[T_2] \circ [T_1] = [T_2][T_1] = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$T_2 \circ T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-y \\ 2x \end{pmatrix}$ so

$T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2 \circ T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-y \\ 2x \end{pmatrix}$

$T_1 \circ T_2$ has matrix $[T_1] \circ [T_2] = [T_1][T_2] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

$T_1 \circ T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ y \end{pmatrix}$ so

$T_1 \circ T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1 \circ T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ y \end{pmatrix}$

We can see that $T_2 \circ T_1 \neq T_1 \circ T_2$

TODO➔ Go to Activity and solve question 16

c) One-to-one Linear operator

A linear operator $T:\mathbb{R}^n \to \mathbb{R}^n$ is said to be one-to-one (injective) if T maps distinct elements of the domain ($\mathbb{R}^n$) to distinct elements of the co-domain ($\mathbb{R}^n$), that is given $\vec{a}, \vec{b} \in \mathbb{R}^n$, $T(\vec{a}) = T(\vec{b}) \Leftrightarrow \vec{a} = \vec{b}$

Theorem 6.c.1: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and [T] its standard matrix then the followings statements are equivalent:

- [T] is invertible
- T is one-to-one
- The range of [T] is the co-domain $\mathbb{R}^n$

Example 6.c.1 Show $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}$ is one-to-one.

The standard matrix of T is $[T] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$,

we compute its determinant $\det([T]) = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1=1$. Since $\det([T]) \neq 0$ ➔ [T] is invertible therefore the linear operator T is one-to-one and its range is its co-domain $\mathbb{R}^2$

Example 6.c.2 Is $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6x+3y \\ 4x+2y \end{pmatrix}$ one-to-one ?

The standard matrix of T is $[T] = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$,

we compute its determinant $\det([T]) = \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12-12=0$. Since $\det([T]) = 0$ ➔ [T] is not invertible therefore the linear operator T is not one-to-one

TODO➔ Go to Activity and solve question 17.

d) Inverse of a one-to-one Linear Operator

If $T:\mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear operator and [T] its standard matrix then [T] is invertible and the inverse of T is $T^{-1}:\mathbb{R}^n \to \mathbb{R}^n$ with standard matrix $[T^{-1}] = [T]^{-1}$

Example 6.d.1: Show that the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+2y \end{pmatrix}$

Is invertible and find $T^{-1}:\mathbb{R}^n \to \mathbb{R}^n$.

Compute $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $T^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Answer:

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x+2y\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow [T]=\begin{pmatrix}1&1\\1&2\end{pmatrix} \text{ and } \det([T])=\begin{vmatrix}1&1\\1&2\end{vmatrix}=2-1=1$$

$\det([T])\neq 0$ ➔ [T] is invertible and therefore the linear operator T is also invertible.

Finding $T^{-1}:\mathbb{R}^n\to\mathbb{R}^n$.

We compute the inverse of [T], $\left[T^{-1}\right]=[T]^{-1}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

$$T^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x-y\\-x+y\end{pmatrix} \Rightarrow T^{-1}:\mathbb{R}^n\to\mathbb{R}^n \text{ with } T^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x-y\\-x+y\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad T^{-1}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

Example 6.d.2: Show that the linear operator $T:\mathbb{R}^2\to\mathbb{R}^2$ such that $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-y\end{pmatrix}$

Is invertible and find $T^{-1}:\mathbb{R}^n\to\mathbb{R}^n$.

Compute $T\begin{pmatrix}1\\1\end{pmatrix}$ *and* $T^{-1}\begin{pmatrix}2\\0\end{pmatrix}$

Answer:

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-y\end{pmatrix}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow [T]=\begin{pmatrix}1&1\\1&-1\end{pmatrix} \text{ and } \det([T])=\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-1-1=-2$$

$\det([T])\neq 0$ ➔ [T] is invertible and therefore the linear operator T is also invertible.

Finding $T^{-1}:\mathbb{R}^n\to\mathbb{R}^n$.

We compute the inverse of [T], $\left[T^{-1}\right]=[T]^{-1}=\frac{1}{-2}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{-1}{2}\end{pmatrix}$

$$T^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{-1}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x}{2}+\frac{y}{2}\\\frac{x}{2}-\frac{y}{2}\end{pmatrix} \Rightarrow T^{-1}:\mathbb{R}^n\to\mathbb{R}^n \text{ with } T^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x}{2}+\frac{y}{2}\\\frac{x}{2}-\frac{y}{2}\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix} \quad \text{and} \quad T^{-1}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{-1}{2}\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

TODO➔ Go to Activity and solve question 18

e) Onto Linear Operator

A linear operator $T:\mathbb{R}^n \to \mathbb{R}^n$ is said to be onto (surjective) if for every element $\vec{b}$ of the co-domain ($\mathbb{R}^n$) there is a distinct element $\vec{a}$ of the domain ($\mathbb{R}^n$) such that $T(\vec{a})=\vec{b}$

If the matrix representation of the linear map is M, we need to solve $M\cdot\vec{a}=\vec{b}$.

If the augmented matrix $\left[M \,\vdots\, \vec{b}\right]$ is consistent, no zero-row in its reduced echelon form, then T is onto.

If the augmented matrix $\left[M \,\vdots\, \vec{b}\right]$ is not consistent, presence of a zero-row in its reduced echelon form then T is not onto.

Example: Show that the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ is onto

Let $M=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ be the matrix representation of the linear map T, and $\vec{b}=\begin{pmatrix} c \\ d \end{pmatrix}$

$\left[M \,\vdots\, \vec{b}\right] = \left(\begin{array}{cc:c} 1 & 1 & c \\ 1 & -1 & d \end{array}\right) \sim \left(\begin{array}{cc:c} 1 & 1 & c \\ 0 & -2 & d-c \end{array}\right)$, we can see that the equivalent row echelon form of $\left[M \,\vdots\, \vec{b}\right] = \left(\begin{array}{cc:c} 1 & 1 & c \\ 1 & -1 & d \end{array}\right)$ is consistent, no zero-row, therefore T is onto

Example: Show that the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix}$ is not onto

Let $M=\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ be the matrix representation of the linear map T, and $\vec{b}=\begin{pmatrix} c \\ d \end{pmatrix}$

$\left[M \,\vdots\, \vec{b}\right] = \left(\begin{array}{cc:c} 1 & 1 & c \\ 2 & 2 & d \end{array}\right) \sim \left(\begin{array}{cc:c} 1 & 1 & c \\ 0 & 0 & d-2c \end{array}\right)$, we can see that the equivalent row echelon form of $\left[M \,\vdots\, \vec{b}\right] = \left(\begin{array}{cc:c} 1 & 1 & c \\ 1 & -1 & d \end{array}\right)$ is **not** consistent, there is a zero-row, therefore T is not onto.

f) Isomorphism (optional)

g)