Calculus Refresher

Intro:

This note is a calculus review , not a substitute to a full Calculus course.

It covers the necessary Calculus background for the Linear Algebra course.

1. Functions

A function is mapping between elements of the domain to the elements of the co-domain. We write $\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto y = f(x) \end{aligned}$ where x is the input argument and y is the output argument.

Example 1.1 : $\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto f(x) = x^3 + 2x + 1 \end{aligned}$

Find $f(-1), f(0), f(2)$

Answer:

$$f(-1) = (-1)^3 + 2(-1) + 1 = -1 - 2 + 1 = -2$$

$$f(0) = (0)^3 + 2(0) + 1 = 1$$

$$f(2) = (2)^3 + 2(2) + 1 = 8 + 4 + 1 = 13$$

A function can also have multiple variables as :

$$\begin{aligned} f &: \mathbb{R}^3 \to \mathbb{R} \\ (x,y,z) &\mapsto f(x,y,z) \end{aligned}$$

Example 1.2: given $\begin{aligned} f &: \mathbb{R}^3 \to \mathbb{R} \\ (x,y,z) &\mapsto f(x,y,z) = xy + z \end{aligned}$

Calculate $f(2,1,0)$, $f(-2,3,1)$

Answer:

$f(2,1,0)$ =(2)(1)+ 0=2

$f(-2,3,1)$ =(-2)(3) + 1=-6 +1=-5

Trigonometric Functions:

- $\cos(a+b) = \cos a \cdot \cos b - \sin \cdot a \sin b$
- $\cos(a-b) = \cos a \cdot \cos b + \sin \cdot a \sin b$
- $\sin(a+b) = \sin a \cdot \cos b + \cos \cdot a \sin b$
- $\sin(a-b) = \sin a \cdot \cos b - \cos \cdot a \sin b$
- $\cos(-x) = \cos x \quad and \quad \sin(-x) = -\sin x$

•

Cosine and sine of angles in radian and degree

θ	2π (0^0)	$\frac{\pi}{2}$ (90^0)	$\frac{\pi}{4}$ (45^0)	$\frac{\pi}{3}$ (60^0)	$\frac{\pi}{6}$ (30^0)	π (180^0)
$\cos\theta$	1	0	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	-1
$\sin\theta$	0	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

2. Limits.

Let's say we have a function $\begin{aligned} f:\mathbb{R}&\to\mathbb{R} \\ x&\mapsto f(x)\end{aligned}$ with input x and output f(x).

How will the output f(x) changes when its input changes from x to x_0 denoted $x \mapsto x_0$? obviously the output will change from f(x) to $f(x_0)$, that is $f(x)\mapsto f(x_0)$. So write that the function changes from $f(x)$ to $f(x_0)$ as its input changes from x to x_0 that is expressed in the following way as the limit : $\lim_{x\to x_0} f(x) = f(x_0)$

Example 2.1 : Find the following limits

a) $\lim_{x\to 1} x+2$

b) $\lim_{x\to 3} \frac{x+2}{x^2+1}$

c) $\lim_{x\to 2} \sqrt{x^3+1}$

Answer:

d) $\lim_{x\to 1} x+2 = 1+2 = 3$

e) $\lim_{x\to 3} \frac{x+2}{x^2+1} = \frac{3+2}{(3)^2+1} = \frac{5}{10} = \frac{1}{2}$

f) $\lim_{x\to 2} \sqrt{x^3+1} = \sqrt{2^3+1} = \sqrt{8+1} = 3$

3. Differentiation (derivative)

Let $x(t)$ be the position of a particle at time t, and the position $x(t+dt)$ at time $t+dt$ where dt is the change in the time. The distance traveled by the particle from time t to time $t+dt$ is distance= $x(t+dt)-x(t)$.

The rate of change of the distance with respect to the time t is:

$$\frac{x(t+dt)-x(t)}{(t+dt)-t}=\frac{x(t+dt)-x(t)}{dt}.$$

By taking the limit of the ratio $\frac{x(t+dt)-x(t)}{dt}$ when dt is close to zero, we define the derivative of the function position x with respect to the time t as the speed, and we write $\lim\limits_{dt\to 0}\frac{x(t+dt)-x(t)}{dt}=\frac{dx}{dt}=x'=speed$.

So differentiating a function is to compute its rate of change with respect to its variable.

Now we generalize for a function f with variable x, $f(x)$, its derivative is

$$\lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx}=\frac{df}{dx}=f'(x)$$

Example 3.1 : Given $f(x)=x^2+x$ calculate the first derivative $\frac{df}{dx}=f'(x)$

Answer :

$$\frac{df}{dx}=f'(x)=\lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx}=\lim_{dx\to 0}\frac{(x+dx)^2+(x+dx)-(x^2+x)}{dx}$$

$$=\lim_{dx\to 0}\frac{x^2+2xdx+dx^2+x+dx-x^2-x}{dx}=\lim_{dx\to 0}\frac{\cancel{x}^2+2xdx+dx^2+\cancel{x}+dx-\cancel{x}^2-\cancel{x}}{dx}$$

$$\lim_{dx\to 0}\frac{2xdx+dx^2+dx}{dx}=\lim_{dx\to 0}\frac{(2x+1+dx)dx}{dx}=\lim_{dx\to 0}(2x+1+dx)=2x+1$$

Derivative Rules

a) Power Rule

- If $f(x)=x^n$ then $\frac{df}{dx}=f'(x)=n\cdot x^{n-1}$
- Derivative of constant .

If $f(x)=c$ then $\frac{df}{dx}=f'(x)=0$ where c is constant

- Second , third …. Derivative

$$\frac{d^2f}{dx^2}=\frac{d}{dx}\left(\frac{df}{dx}\right) \quad \text{or} \quad \frac{d^2f}{dx^2}=f''(x)=(f')'$$

Example 3.2 :

- $f(x)=x^5 \Rightarrow f'(x)=5x^{5-1}=5x^4$

  $f''(x)=\left(f'\right)'=\left(5x^4\right)'==20x^3$
- $f(x)=3x^7 \Rightarrow f'(x)=(3)(7)x^{7-1}=21x^6$

  $f''(x)=\left(f'\right)'=\left(21x^6\right)'==126x^5$
- $f(x)=x^{10} \Rightarrow f'(x)=10x^{10-1}=10x^9$

  $f''(x)=\left(f'\right)'=\left(10x^9\right)'==90x^8$
- $f(x)=10 \Rightarrow f'(x)=0$ (constant)
-

b) Derivative of trigonometric functions

If $f(x)=\cos x$ then $\dfrac{df}{dx}=f'(x)=-\sin(x)$

If $f(x)=\sin(x)$ then $\dfrac{df}{dx}=f'(x)=\cos(x)$

c) Sum rules

$$\begin{cases} \dfrac{d}{dx}(f \pm g)=\dfrac{df}{dx} \pm \dfrac{dg}{dx} \\ (f \pm g)'=f' \pm g' \end{cases}$$

Example 3.3 :

- $(x^4+x^3+7+\cos(x)\ )'=(x^4)'+(x^3)'+(7)'+(\cos(x))'=4x^3+3x^2-\sin(x)$
- $(3x^5-x^3\ )'=3(x^5)'-(x^3)'=3(5)x^4-3x^2=15x^4-3x^2$

d) Product rule

$$\begin{cases} \dfrac{d}{dx}(fg)=\dfrac{df}{dx}g+f\dfrac{dg}{dx} \\ (fg)'=f'g+fg' \end{cases}$$

- $(x^7 \cos x)' = (x^7)' \cos x + x^7 (\cos x)' = 7x^6 \cos x + x^7(-\sin x) = 7x^6 \cos x - x^7 \sin x$

- $(\sin x \cdot \cos x)' = (\sin x)' \cdot \cos x + \sin x \cdot (\cos x)' = (\cos x)\cos x + \sin x(-\sin x)$
  $= \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$

4. **Integration**

Integration is the inverse of differentiation.

Given two functions $f(x)\ and\ F(x)$ such that $\dfrac{dF(x)}{dx} = f(x)$ then

$dF(x) = f(x)dx$ ➔ $\int dF(x) = \int f(x)dx$ ➔ $F(x) = \int f(x)dx + c$ where c is constant. We said that $F(x)$ is the anti-derivative of $f(x)$ or $F(x)$ is integral of $f(x)$.

Integration of elementary functions

- $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c$

  Example 4.1 $\int x^5 dx = \dfrac{x^{5+1}}{5+1} + c = \dfrac{x^6}{6} + c$

  Example 4.2: $\int 3x^2 dx = \dfrac{3x^{2+1}}{2+1} + c = \dfrac{3x^3}{3} + c = x^3 + c$

- $\int a\, dx = ax + c$ integral of constant function a

  Example 4.3 : $\int 3dx = 3x + c$

- $\int cox\, dx = \sin x + c$

- $\int \sin x\, dx = -\cos x + c$

Integration rules:

- $\int(\ f(x)+g(x)\ )dx=\int f(x)dx+\int g(x)dx$

  Example 4.4 : $\int(4x^3+cox)\,dx=\int 4x^3dx+\int coxdx=x^4-\sin x+c$
- $\int cf(x)\,dx=c\int f(x)dx$, c=constant

  Example 4.5: $\int 6x\,dx=6\int xdx=6\frac{x^{1+1}}{1+1}=\frac{6x^2}{2}=3x^2+c$

5. **Definite integration**

   Let $F(x)=\int f(x)\,dx+c$, we define the integral of f(x) over x where x takes its value from a to b as $\int_a^b f(x)dx=\left[F(x)\right]_a^b=F(b)-F(a)$

   Example 5.1
   - $\int_1^2 2xdx=\left[x^2\right]_1^2=(2)^2-(1)^2=4-1=3$
   - $\int_1^3(3x^2+2x)dx=\left[x^3+x^2\right]_1^3=(3^3+3^2)-(1^3+1^2)^2=36-2=34$
   - $\int_0^{\frac{\pi}{2}}\cos dx=\left[\sin\right]_0^{\frac{\pi}{2}}=(\sin\frac{\pi}{2})-\sin(0)^2=1-0=1$

6. **Partial Derivative**

   A partial derivative of a function of several variable is the derivative with respect to one of the variables while the others are held constant.

   let f(x,y) be a function of 2 variables, then the partial derivative with respect to x is $\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=f_x$, similarly a partial derivative with respect to y is

   $\frac{\partial f}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=f_y$

   For higher order and mixed derivatives we have .

   $\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=f_{xx}$, $\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=f_{yy}$

   $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\left(f_y\right)_x$ $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\left(f_x\right)_y$,

   $f_{xyz}=\left(f_{xy}\right)_z=\left(\left(f_x\right)_y\right)_z=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)=\frac{\partial^3 f}{\partial z\partial y\partial x}$

Note that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ if f(x,y) has continuous second partial derivatives.

Example 6.1: if $f(x,y)=4x^3y^2-3x^2 + y +5$,

$$\frac{\partial f}{\partial x} = 12x^2y^2-6x \quad \text{and} \quad \frac{\partial f}{\partial y} = 8x^3y + 1$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(8x^3y + 1) = 24x^2y$$

Example 6.2: $f(x,y,z) = x^4 y + z^3 y^2$, calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial z \partial y}$,

$\frac{\partial^3 f}{\partial x \partial y \partial z}$, $\frac{\partial^3 f}{\partial x \partial^2 z}$

- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4 y + z^3 y^2) = y\frac{\partial f}{\partial x}(x^4) + \frac{\partial f}{\partial x}(z^3 y^2) = y(4x^3) + (0) = 4x^3 y$

  Note that $z^3 y^2$ has no x (constant term) ➔ $\frac{\partial}{\partial x}(z^3 y^2) = 0$

- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4 y + z^3 y^2) = x^4\frac{\partial f}{\partial y}(y) + z^3\frac{\partial f}{\partial y}(y^2) = x^4(1) + z^3(2y) = x^4 + 2z^3 y$
- $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^4 y + z^3 y^2) = \frac{\partial f}{\partial z}(x^4 y) + y^2\frac{\partial f}{\partial z}(z^3) = (0) + y^2(3z^2) = 3y^2 z^2$

  Note that $x^4 y$ has no z (constant term) ➔ $\frac{\partial}{\partial z}(x^4 y) = 0$

- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^4 + 2z^3 y\right) = \frac{\partial}{\partial x}\left(x^4\right) + \frac{\partial}{\partial x}(2z^3 y) = 4x^3 + 0 = 4x^3$,

  $\frac{\partial}{\partial x}(2z^3 y) = 0$ no x present in the expression.

- $\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(3x^2 y^2\right)\right] = \frac{\partial}{\partial x}\left[6x^2 y\right] = 12xy$
- $\frac{\partial^3 f}{\partial z \partial^2 y}\ \frac{\partial^3 f}{\partial z \partial y \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(x^4 + 2z^3 y\right)\right] = \frac{\partial}{\partial z}\left[2z^3\right] = 6z^2$

7. ☺