GEN 242: Linear Algebra

Chapter 8:
Vectors/Matrix Differential Calculus

Solutions Guide

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2. $\vec{p} = \left(1, \sqrt{3}\right)_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(\frac{\pi}{3}, 2\right)_{\text{polar}}}$

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Converting from Polar to Rectangular Coordinates

4. $\vec{p} = \left(\frac{7\pi}{4}, 3\sqrt{2}\right)_{\text{polar}} \rightarrow \boxed{\vec{p} = (3, -3)_{\text{rect}}}$

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7. $\vec{p} = \left(3, \frac{\pi}{2}, 1\right)_{\text{cyl}} \rightarrow \boxed{\vec{p} = (0,3,1)_{\text{rect}}}$

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9. $\vec{p} = \left(1, \frac{\pi}{4}, 5\right)_{\text{cyl}} \rightarrow \boxed{\vec{p} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5\right)_{\text{rect}}}$

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10. $\vec{p} = (1,1,1)_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(\sqrt{2}, \frac{\pi}{4}, 1\right)_{\text{cyl}}}$

11. $\vec{p} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 5\right)_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(1, \frac{\pi}{3}, 5\right)_{\text{cyl}}}$

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Converting from Spherical to Rectangular Coordinates

13. $\vec{p} = \left(1, \frac{\pi}{4}, \pi\right)_{\text{sphere}} \rightarrow \boxed{\vec{p} = (0,0,-1)_{\text{rect}}}$

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Converting from Rectangular to Spherical Coordinates

16. $\vec{p} = (1,1,\sqrt{2})_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)_{\text{sphere}}}$

17. $\vec{p} = \left(\frac{\sqrt{3}}{4}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(\frac{\sqrt{19}}{4}, 0.65, 0.86\right)_{\text{sphere}}}$

18. $\vec{p} = (1,1,0)_{\text{rect}} \rightarrow \boxed{\vec{p} = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)_{\text{sphere}}}$

Converting from Spherical to Cylindrical Coordinates

19. $\vec{p} = \left(4, \frac{\pi}{4}, \frac{\pi}{3}\right)_{\text{sphere}} \rightarrow \boxed{\vec{p} = \left(2\sqrt{2}, \frac{\pi}{3}, 2\sqrt{2}\right)_{\text{cyl}}}$

20. $\vec{p} = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)_{\text{sphere}} \rightarrow \boxed{\vec{p} = \left(2, \frac{\pi}{3}, 2\right)_{\text{cyl}}}$

21. $\vec{p} = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)_{\text{sphere}} \rightarrow \boxed{\vec{p} = \left(\sqrt{2}, \frac{\pi}{4}, 0\right)_{\text{cyl}}}$

Converting from Cylindrical to Spherical Coordinates

22. $\vec{p} = \left(1, \frac{\pi}{2}, 1\right)_{\text{cyl}} \rightarrow \boxed{\vec{p} = \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)_{\text{sphere}}}$

23. $\vec{p} = \left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)_{\text{cyl}} \rightarrow \boxed{\vec{p} = \left(2\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{2}\right)_{\text{sphere}}}$

24. $\vec{p} = \left(1, \frac{\pi}{4}, 5\right)_{\text{cyl}} \rightarrow \boxed{\vec{p} \approx \left(\sqrt{26}, 0.20, \frac{\pi}{4}\right)_{\text{sphere}}}$

Gradient of a Scalar Field

25. $f(x, y, z) = x^2y + xz + y^2 \rightarrow \boxed{\overrightarrow{\text{grad}}(f) = (2xy + z, x^2 + 2y, x)}$

26. $f(x, y, z) = x^2 + y^2 + z^2 + 2 \rightarrow \boxed{\overrightarrow{\text{grad}}(f) = (2x, 2y, 2z)}$

27. $f(x, y, z) = x + 3y + 5z + 2 \rightarrow \boxed{\overrightarrow{\text{grad}}(f) = (1,3,5)}$

Curl of a Vector Field

28. $\vec{u} = (x^2y)\hat{\imath} + (yz)\hat{\jmath} - (z^2)\hat{k} \rightarrow \boxed{\text{curl}(\vec{u}) = -y\hat{\imath} - x^2\hat{k} = (-y, 0, -x^2)}$

29. $\vec{u} = (x^2)\hat{\imath} + (z^2)\hat{\jmath} - (xy^3)\hat{k} \rightarrow \boxed{\text{curl}(\vec{u}) = -2xy^2\hat{\imath} + y^3\hat{\jmath} = (-2xy^2, y^3, 0)}$

30. $\vec{u} = (x)\hat{\imath} + (z)\hat{\jmath} - (x)\hat{k} \rightarrow \boxed{\text{curl}(\vec{u}) = -\hat{\imath} + \hat{\jmath} = (-1,1,0)}$

Divergence of a Vector Field

31. $\vec{u} = x^2 y\hat{\imath} + zy\hat{\jmath} - z^2\hat{k} \rightarrow \boxed{\text{div}\,\vec{u} = 2xy - z}$

32. $\vec{u} = x^2\hat{\imath} + z^2\hat{\jmath} - xy^3\hat{k} \rightarrow \boxed{\text{div}\,\vec{u} = 2x}$

33. $\vec{u} = x\hat{\imath} + z\hat{\jmath} - x\hat{k} \rightarrow \boxed{\text{div}\,\vec{u} = 1}$

Laplacian of a Scalar Field

34. $f(x, y, z) = x^2 y + xz + y^2 \rightarrow \boxed{\vec{\nabla}^2 f = 2y + 2}$

35. $f(x, y, z) = x^2 + y^2 + z^2 + 2 \rightarrow \boxed{\vec{\nabla}^2 f = 6}$

36. $f(x, y, z) = zx + 3x^3y^2 + 2xz^2 \rightarrow \boxed{\vec{\nabla}^2 f = 18xy^2 + 6x^3 + 4x}$

Laplacian of a Vector Field

37. $\vec{u} = (3x^2y, zy^2, 3z^2) \rightarrow \boxed{\vec{\nabla}^2\vec{u} = (6y, 2z, 6)}$

38. $\vec{u} = (x^2 + y, x + zy^2, z^2) \rightarrow \boxed{\vec{\nabla}^2\vec{u} = (2,2z, 2)}$

39. Duplicate of #37.

Derivative of a Vector with Respect to a Vector

40. $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{cases} w_1 = 2x + 3y \\ w_2 = 7x + 5y \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \boxed{\frac{\partial\vec{w}}{\partial\vec{u}} = \begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix}}$

41. $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{cases} w_1 = x - y + z \\ w_2 = x + 2y - z \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \boxed{\frac{\partial\vec{w}}{\partial\vec{u}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}}$

42. $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \begin{cases} w_1 = xy + z \\ w_2 = x - y^2 + z \\ w_3 = 2x + y + xz \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \boxed{\frac{\partial\vec{w}}{\partial\vec{u}} = \begin{bmatrix} y & x & 1 \\ 1 & -2y & 1 \\ 2 + z & 1 & x \end{bmatrix}}$

Derivative of a Scalar with Respect to a Vector

43. $s(\vec{x}) = (x_1 + 1)^2 + {x_2}^2 + (x_3 + 2)^2, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \boxed{\frac{\partial s}{\partial \vec{x}} = \begin{bmatrix} 2x_1 + 2 \\ 2x_2 \\ 2x_3 + 4 \end{bmatrix}}$

44. $f(\vec{x}) = x + y + z, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \boxed{\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

45. $g(\vec{x}) = x + xy + z^2, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \boxed{\frac{\partial g}{\partial \vec{x}} = \begin{bmatrix} 1 + y \\ x \\ 2z \end{bmatrix}}$

Quadric Forms

46. $f(x, y) = x^2 + 6xy + 2y^2 \rightarrow \boxed{\begin{array}{c} f(x, y) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \cdot \vec{x} \\ \frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 2x + 6y \\ 6x + 4y \end{bmatrix} \end{array}}$

47. $f(x, y) = 5x^2 + 2xy + 2y^2 \rightarrow \boxed{\begin{array}{c} f(x, y) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \cdot \vec{x} \\ \frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 10x + 2y \\ 2x + 4y \end{bmatrix} \end{array}}$

48. $f(x, y) = 3x^2 + 8xy + 6xz + y^2 + 6yz + 3z^2 \rightarrow \boxed{\begin{array}{c} f(x, y, z) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix} \cdot \vec{x} \\ \frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 6x + 8y + 6z \\ 8x + 2y + 6z \\ 6x + 6y + 6z \end{bmatrix} \end{array}}$

Solutions

Rectangular to Polar Coordinates Conversion

Problem 1

Change $\vec{p} = \left(3, \sqrt{3}\right)$ from rectangular coordinates to polar coordinates.

$$\vec{p}_{\text{rect}} = \left(3, \sqrt{3}\right) = (x, y)$$

$$x = 3 \qquad\qquad y = \sqrt{3}$$

$$\vec{p}_{\text{polar}} = (\theta, r)$$

$$y = \sqrt{3} \rightarrow y \geq 0$$

$$\therefore \theta = \cos^{-1}\left(\frac{x}{r}\right)$$

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

$$\theta = \frac{\pi}{6}$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(3)^2 + \left(\sqrt{3}\right)^2}$$

$$r = \sqrt{9 + 3}$$

$$r = \sqrt{12} = 2\sqrt{3}$$

$$\boxed{\vec{p}_{\text{polar}} = \left(\frac{\pi}{6}, 2\sqrt{3}\right)}$$

Problem 2

Change $\vec{p} = \left(1, \sqrt{3}\right)$ from rectangular coordinates to polar coordinates.

$$\vec{p}_{\text{rect}} = \left(1, \sqrt{3}\right) = (x, y)$$

$$x = 1 \qquad\qquad y = \sqrt{3}$$

$$\vec{p}_{\text{polar}} = (r, \theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2}$$

$$r = \sqrt{1 + 3}$$

$$r = \sqrt{4} = 2$$

$$y = \sqrt{3} \rightarrow y \geq 0$$

$$\theta = \cos^{-1}\left(\frac{x}{r}\right)$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

$$\boxed{\vec{p}_{\text{polar}} = \left(\frac{\pi}{3}, 2\right)}$$

Problem 3

Change $\vec{p} = (-1, -1)$ from rectangular coordinates to polar coordinates.

$$\vec{p}_{\text{rect}} = (-1, -1) = (x, y)$$

$$x = -1 \qquad\qquad y = -1$$

$$\vec{p}_{\text{polar}} = (\theta, r)$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-1)^2 + (-1)^2}$$

$$r = \sqrt{1 + 1}$$

$$r = \sqrt{2}$$

$$y = -1 \rightarrow y < 0$$

$$\theta = -\cos^{-1}\left(\frac{x}{r}\right)$$

$$\theta = -\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$\theta = -\frac{3\pi}{4}$$

$$\boxed{\vec{p}_{\text{polar}} = \left(-\frac{3\pi}{4}, \sqrt{2}\right)}$$

Polar to Rectangular Coordinates Conversion

Problem 4

Change $\vec{p} = \left(\frac{7\pi}{4}, 3\sqrt{2}\right)$ from polar coordinates to rectangular coordinates.

$$\vec{p}_{\text{polar}} = \left(\frac{7\pi}{4}, 3\sqrt{2}\right) = (\theta, r)$$

$$r = 3\sqrt{2}$$

$$\theta = \frac{7\pi}{4}$$

$$\vec{p}_{\text{rect}} = (x, y)$$

$$x = r\cos(\theta)$$

$$x = 3\sqrt{2} \cdot \cos\left(\frac{7\pi}{4}\right)$$

$$x = 3\sqrt{2} \cdot \frac{\sqrt{2}}{2}$$

$$x = 3$$

$$y = r\sin(\theta)$$

$$y = 3\sqrt{2} \cdot \sin\left(\frac{7\pi}{4}\right)$$

$$y = 3\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right)$$

$$y = -3$$

$$\boxed{\vec{p}_{\text{rect}} = (3, -3)}$$

Problem 5

Change $\vec{p} = \left(\frac{\pi}{4}, 4\right)$ from polar coordinates to rectangular coordinates.

$$\vec{p}_{\text{polar}} = \left(\frac{\pi}{4}, 4\right) = (\theta, r)$$

$$r = 4$$

$$\theta = \frac{\pi}{4}$$

$$\vec{p}_{\text{rect}} = (x, y)$$

$$x = r\cos(\theta)$$

$$x = 4 \cdot \cos\left(\frac{\pi}{4}\right)$$

$$x = 4 \cdot \frac{\sqrt{2}}{2}$$

$$x = 2\sqrt{2}$$

$$y = 4 \cdot \sin\left(\frac{\pi}{4}\right)$$

$$y = 4 \cdot \frac{\sqrt{2}}{2}$$

$$y = 2\sqrt{2}$$

$$y = r\sin(\theta)$$

$$\boxed{\vec{p}_{\text{rect}} = \left(2\sqrt{2}, 2\sqrt{2}\right)}$$

Problem 6

Change $\vec{p} = \left(\frac{\pi}{3}, 1\right)$ from polar coordinates to rectangular coordinates.

$$\vec{p}_{\text{polar}} = \left(\frac{\pi}{3}, 1\right) = (\theta, r)$$

$$r = 1 \qquad \theta = \frac{\pi}{3}$$

$$\vec{p}_{\text{rect}} = (x, y)$$

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$x = 1 \cdot \cos\left(\frac{\pi}{3}\right) \qquad y = 1 \cdot \sin\left(\frac{\pi}{3}\right)$$

$$x = 1 \cdot \frac{1}{2} \qquad y = 1 \cdot \frac{\sqrt{3}}{2}$$

$$x = \frac{1}{2} \qquad y = \frac{\sqrt{3}}{2}$$

$$\boxed{\vec{p}_{\text{rect}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$$

Cylindrical to Rectangular Coordinates Conversion

Problem 7

Change $\vec{p} = \left(3, \frac{\pi}{2}, 1\right)$ from cylindrical coordinates to rectangular coordinates.

$$\vec{p}_{\mathrm{cyl}} = \left(3, \frac{\pi}{2}, 1\right) = (\rho, \phi, z)$$

$$\rho = 3 \qquad \phi = \frac{\pi}{2} \qquad z = 1$$

$$\vec{p}_{\mathrm{rect}} = (x, y, z)$$

$$x = \rho\cos(\phi) \qquad y = \rho\sin(\phi) \qquad z = z$$

$$x = 3\cos\left(\frac{\pi}{2}\right) \qquad y = 3\sin\left(\frac{\pi}{2}\right) \qquad z = 1$$

$$x = 3 \cdot 0 \qquad y = 3 \cdot 1$$

$$x = 0 \qquad y = 3$$

$$\boxed{\vec{p}_{\mathrm{rect}} = (0,3,1)}$$

Problem 8

Change $\vec{p} = \left(4, \frac{\pi}{6}, 2\right)$ from cylindrical coordinates to rectangular coordinates.

$$\vec{p}_{\text{cyl}} = \left(4, \frac{\pi}{6}, 2\right) = (\rho, \phi, z)$$

$$\rho = 4 \qquad \phi = \frac{\pi}{6} \qquad z = 2$$

$$\vec{p}_{\text{rect}} = (x, y, z)$$

$$x = \rho\cos(\phi) \qquad y = \rho\sin(\phi) \qquad z = z$$

$$x = 4\cos\left(\frac{\pi}{6}\right) \qquad y = 4\sin\left(\frac{\pi}{6}\right) \qquad z = 2$$

$$x = 4 \cdot \frac{\sqrt{3}}{2} \qquad y = 4 \cdot \frac{1}{2}$$

$$x = 2\sqrt{3} \qquad y = 2$$

$$\boxed{\vec{p}_{\text{rect}} = \left(2\sqrt{3}, 2, 2\right)}$$

Problem 9

Change $\vec{p} = \left(1, \frac{\pi}{4}, 5\right)$ from cylindrical coordinates to rectangular coordinates.

$$\vec{p}_{\text{cyl}} = \left(1, \frac{\pi}{4}, 5\right) = (\rho, \phi, z)$$

$$\rho = 1 \qquad \phi = \frac{\pi}{4} \qquad z = 5$$

$$\vec{p}_{\text{rect}} = (x, y, z)$$

$$x = \rho\cos(\phi) \qquad y = \rho\sin(\phi) \qquad z = z$$

$$x = 1\cos\left(\frac{\pi}{4}\right) \qquad y = 1\sin\left(\frac{\pi}{4}\right) \qquad z = 5$$

$$x = 1 \cdot \frac{\sqrt{2}}{2} \qquad y = 1 \cdot \frac{\sqrt{2}}{2}$$

$$x = \frac{\sqrt{2}}{2} \qquad y = \frac{\sqrt{2}}{2}$$

$$\boxed{\vec{p}_{\text{rect}} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5\right)}$$

Rectangular to Cylindrical Coordinates Conversion

Problem 10

Change $\vec{p} = (1,1,1)$ from rectangular coordinates to cylindrical coordinates.

$$\vec{p}_{\text{rect}} = (1,1,1) = (x, y, z)$$

$$x = 1 \qquad y = 1 \qquad z = 1$$

$$\vec{p}_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\rho = \sqrt{(1)^2 + (1)^2}$$

$$\rho = \sqrt{1 + 1}$$

$$\rho = \sqrt{2}$$

$$x = 1 \rightarrow x \geq 0$$

$$\phi = \sin^{-1}\left(\frac{y}{\rho}\right)$$

$$\phi = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\phi = \frac{\pi}{4}$$

$$z = z$$

$$z = 1$$

$$\boxed{\vec{p}_{\text{cyl}} = \left(\sqrt{2}, \frac{\pi}{4}, 1\right)}$$

Problem 11

Change $\vec{p} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 5\right)$ from rectangular coordinates to cylindrical coordinates.

$$\vec{p}_{\text{rect}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 5\right) = (x, y, z)$$

$$x = \frac{1}{2} \qquad y = \frac{\sqrt{3}}{2} \qquad z = 5$$

$$\vec{p}_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\rho = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$\rho = \sqrt{1} = 1$$

$$x = \frac{1}{2} \rightarrow x \geq 0$$

$$\phi = \sin^{-1}\left(\frac{y}{\rho}\right)$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{3}/2}{1}\right)$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\phi = \frac{\pi}{3}$$

$$z = z$$

$$z = 5$$

$$\boxed{\vec{p}_{\text{cyl}} = \left(1, \frac{\pi}{3}, 5\right)}$$

Problem 12

Change $\vec{p} = \left(\sqrt{2}, \sqrt{2}, 3\right)$ from rectangular coordinates to cylindrical coordinates.

$$\vec{p}_{\text{rect}} = \left(\sqrt{2}, \sqrt{2}, 3\right) = (x, y, z)$$

$$x = \sqrt{2} \qquad y = \sqrt{2} \qquad z = 3$$

$$\vec{p}_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\rho = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2}$$

$$\rho = \sqrt{2 + 2}$$

$$\rho = \sqrt{4} = 2$$

$$x = \sqrt{2} \rightarrow x \geq 0$$

$$\phi = \sin^{-1}\left(\frac{y}{\rho}\right)$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\phi = \frac{\pi}{4}$$

$$z = z$$

$$z = 3$$

$$\boxed{\vec{p}_{\text{cyl}} = \left(2, \frac{\pi}{4}, 3\right)}$$

Spherical to Rectangular Coordinates Conversion

Problem 13

Change $\vec{p} = \left(1, \frac{\pi}{4}, \pi\right)$ from spherical coordinates to rectangular coordinates.

$$\vec{p}_{\text{sphere}} = \left(1, \frac{\pi}{4}, \pi\right) = (r, \theta, \phi)$$

$$r = 1 \qquad \phi = \frac{\pi}{4} \qquad \theta = \pi$$

$$\vec{p}_{\text{rect}} = (x, y, z)$$

$$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta)$$

$$x = 1\sin(\pi)\cos\left(\frac{\pi}{4}\right) \qquad y = 1\sin(\pi)\sin\left(\frac{\pi}{4}\right) \qquad z = 1\cos(\pi)$$

$$z = 1 \cdot (-1)$$

$$x = 1 \cdot 0 \cdot \frac{\sqrt{2}}{2} \qquad y = 1 \cdot 0 \cdot \frac{\sqrt{2}}{2}$$

$$z = -1$$

$$x = 0 \qquad y = 0$$

$$\boxed{\vec{p}_{\text{rect}} = (0,0,-1)}$$

Problem 14

Change $\vec{p} = \left(3, \frac{\pi}{3}, \frac{\pi}{4}\right)$ from spherical coordinates to rectangular coordinates.

$$\vec{p}_{\text{sphere}} = \left(3, \frac{\pi}{3}, \frac{\pi}{4}\right) = (r, \theta, \phi)$$

$$r = 3 \qquad \theta = \frac{\pi}{3} \qquad \phi = \frac{\pi}{4}$$

$$\vec{p}_{\text{rect}} = (x, y, z)$$

$$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta)$$

$$x = 3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) \qquad y = 3\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \qquad z = 3\cos\left(\frac{\pi}{3}\right)$$

$$x = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \qquad y = 3 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \qquad z = 3 \cdot \frac{1}{2}$$

$$x = \frac{3\sqrt{6}}{4} \qquad y = \frac{3\sqrt{6}}{4} \qquad z = \frac{3}{2}$$

$$\boxed{\vec{p}_{\text{rect}} = \left(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{6}}{4}, \frac{3}{2}\right)}$$

Problem 15

Change $\vec{p} = \left(5, \frac{\pi}{2}, \pi\right)$ from spherical coordinates to rectangular coordinates.

$$\vec{p}_{\text{sphere}} = \left(5, \frac{\pi}{2}, \pi\right) = (r, \theta, \phi)$$

$$r = 5 \qquad \theta = \frac{\pi}{2} \qquad \phi = \pi$$

$$\vec{p}_{\text{rect}} = (x, y, z)$$

$$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta)$$

$$x = 5\sin\left(\frac{\pi}{2}\right)\cos(\pi) \qquad y = 5\sin\left(\frac{\pi}{2}\right)\sin(\pi) \qquad z = 5\cos\left(\frac{\pi}{2}\right)$$

$$x = 5 \cdot 1 \cdot 0 \qquad y = 5 \cdot 1 \cdot (-1) \qquad z = 5 \cdot (0)$$

$$x = 0 \qquad y = -5 \qquad z = 0$$

$$\boxed{\vec{p}_{\text{rect}} = (0, -5, 0)}$$

Rectangular to Spherical Coordinates Conversion

Problem 16

Change $\vec{p} = \left(1,1,\sqrt{2}\right)$ from rectangular coordinates to spherical coordinates.

$$\vec{p}_{\text{rect}} = \left(1,1,\sqrt{2}\right) = (x, y, z)$$

$$x = 1 \qquad y = 1 \qquad z = \sqrt{2}$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r = \sqrt{(1)^2 + (1)^2 + \left(\sqrt{2}\right)^2}$$

$$r = \sqrt{1 + 1 + 2}$$

$$r = \sqrt{4}$$

$$r = 2$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{1}{1}\right)$$

$$\phi = \tan^{-1}(1)$$

$$\phi = \frac{\pi}{4}$$

$$\boxed{\vec{p}_{\text{sphere}} = \left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)}$$

Problem 17

Change $\vec{p} = \left(\frac{\sqrt{3}}{4}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ from rectangular coordinates to spherical coordinates.

$$\vec{p}_{\text{rect}} = \left(\frac{\sqrt{3}}{4}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (x, y, z)$$

$$x = \frac{\sqrt{3}}{4} \qquad y = \frac{1}{2} \qquad z = \frac{\sqrt{3}}{2}$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{3}{16} + \frac{1}{4} + \frac{3}{4}}$$

$$r = \sqrt{\frac{3}{16} + \frac{4}{16} + \frac{12}{16}}$$

$$r = \sqrt{\frac{19}{16}}$$

$$r = \frac{\sqrt{19}}{4}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}/2}{\sqrt{19}/4}\right)$$

$$\theta = \cos^{-1}\left(\frac{2\sqrt{57}}{19}\right)$$

$$\theta \approx 0.65$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/4}\right)$$

$$\phi = \tan^{-1}\left(\frac{2\sqrt{3}}{3}\right)$$

$$\phi \approx 0.86$$

$$\boxed{\vec{p}_{\text{sphere}} \approx \left(\frac{\sqrt{19}}{4}, 0.65, 0.86\right)}$$

Problem 18

Change $\vec{p} = (1,1,0)$ from rectangular coordinates to spherical coordinates.

$$\vec{p}_{\text{rect}} = (1,1,0) = (x, y, z)$$

$$x = 1 \qquad y = 1 \qquad z = 0$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r = \sqrt{(1)^2 + (1)^2 + (0)^2}$$

$$r = \sqrt{1 + 1 + 0}$$

$$r = \sqrt{2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\theta = \cos^{-1}\left(\frac{0}{\sqrt{2}}\right)$$

$$\theta = \cos^{-1}(0)$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{1}{1}\right)$$

$$\phi = \tan^{-1}(1)$$

$$\phi = \frac{\pi}{4}$$

$$\boxed{\vec{p}_{\text{sphere}} = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)}$$

Spherical to Cylindrical Coordinates Conversion

Problem 19

Change $\vec{p} = \left(4, \frac{\pi}{4}, \frac{\pi}{3}\right)$ from spherical coordinates to cylindrical coordinates.

$$\vec{p}_{\text{sphere}} = \left(4, \frac{\pi}{4}, \frac{\pi}{3}\right) = (r, \theta, \phi)$$

$$r = 4 \qquad \theta = \frac{\pi}{4} \qquad \phi = \frac{\pi}{3}$$

$$p_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = r\sin(\theta) \qquad \phi = \phi \qquad z = r\cos(\theta)$$

$$\rho = (4)\sin\left(\frac{\pi}{4}\right) \qquad \phi = \frac{\pi}{3} \qquad z = (4)\cos\left(\frac{\pi}{4}\right)$$

$$\rho = (4)\left(\frac{\sqrt{2}}{2}\right) \qquad z = (4)\left(\frac{\sqrt{2}}{2}\right)$$

$$\rho = 2\sqrt{2} \qquad z = 2\sqrt{2}$$

$$\boxed{p_{\text{cyl}} = \left(2\sqrt{2}, \frac{\pi}{3}, 2\sqrt{2}\right)}$$

Problem 20

Change $\vec{p} = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$ from spherical coordinates to cylindrical coordinates.

$$\vec{p}_{\text{sphere}} = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right) = (r, \theta, \phi)$$

$$r = 2\sqrt{2} \qquad \theta = \frac{\pi}{4} \qquad \phi = \frac{\pi}{3}$$

$$p_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = r\sin(\theta) \qquad \phi = \phi \qquad z = r\cos(\theta)$$

$$\rho = (2\sqrt{2})\sin\left(\frac{\pi}{4}\right) \qquad \phi = \frac{\pi}{3} \qquad z = \left(2\sqrt{2}\right)\cos\left(\frac{\pi}{4}\right)$$

$$\rho = \left(2\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}\right) \qquad z = \left(2\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\rho = 2 \qquad z = 2$$

$$\boxed{p_{\text{cyl}} = \left(2, \frac{\pi}{3}, 2\right)}$$

Problem 21

Change $\vec{p} = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$ from spherical coordinates to cylindrical coordinates.

$$\vec{p}_{\text{sphere}} = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right) = (r, \theta, \phi)$$

$$r = \sqrt{2} \qquad \theta = \frac{\pi}{2} \qquad \phi = \frac{\pi}{4}$$

$$p_{\text{cyl}} = (\rho, \phi, z)$$

$$\rho = r\sin(\theta) \qquad \phi = \phi \qquad z = r\cos(\theta)$$

$$\rho = (\sqrt{2})\sin\left(\frac{\pi}{2}\right) \qquad \phi = \frac{\pi}{4} \qquad z = \left(\sqrt{2}\right)\cos\left(\frac{\pi}{2}\right)$$

$$\rho = \left(\sqrt{2}\right)(1) \qquad z = \left(2\sqrt{2}\right)(0)$$

$$\rho = \sqrt{2} \qquad z = 0$$

$$\boxed{p_{\text{cyl}} = \left(\sqrt{2}, \frac{\pi}{4}, 0\right)}$$

Cylindrical to Spherical Coordinates Conversion

Problem 22

Change $\vec{p} = \left(1, \frac{\pi}{2}, 1\right)$ from cylindrical coordinates to spherical coordinates.

$$\vec{p}_{\text{cyl}} = \left(1, \frac{\pi}{2}, 1\right) = (\rho, \phi, z)$$

$$\rho = 1 \qquad \phi = \frac{\pi}{2} \qquad z = 1$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{\rho^2 + z^2}$$

$$r = \sqrt{(1)^2 + (1)^2}$$

$$r = \sqrt{1 + 1}$$

$$r = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$\theta = \tan^{-1}(1)$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \phi$$

$$\phi = \frac{\pi}{2}$$

$$\boxed{\vec{p}_{\text{sphere}} = \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)}$$

Problem 23

Change $\vec{p} = \left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical coordinates to spherical coordinates.

$$\vec{p}_{\text{cyl}} = \left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right) = (\rho, \phi, z)$$

$$\rho = \sqrt{6} \qquad \phi = \frac{\pi}{4} \qquad z = \sqrt{2}$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{\rho^2 + z^2}$$

$$r = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2}$$

$$r = \sqrt{6 + 2}$$

$$r = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{2}}\right)$$

$$\theta = \tan^{-1}\left(\sqrt{3}\right)$$

$$\theta = \frac{\pi}{3}$$

$$\phi = \phi$$

$$\phi = \frac{\pi}{2}$$

$$\boxed{\vec{p}_{\text{sphere}} = \left(2\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{2}\right)}$$

Problem 24

Change $\vec{p} = \left(1, \frac{\pi}{4}, 5\right)$ from cylindrical coordinates to spherical coordinates.

$$\vec{p}_{\text{cyl}} = \left(1, \frac{\pi}{4}, 5\right) = (\rho, \phi, z)$$

$$\rho = 1 \qquad \phi = \frac{\pi}{4} \qquad z = 5$$

$$\vec{p}_{\text{sphere}} = (r, \theta, \phi)$$

$$r = \sqrt{\rho^2 + z^2}$$

$$r = \sqrt{(1)^2 + (5)^2}$$

$$r = \sqrt{1 + 25}$$

$$r = \sqrt{26}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{5}\right)$$

$$\theta \approx 0.20$$

$$\phi = \phi$$

$$\phi = \frac{\pi}{4}$$

$$\boxed{\vec{p}_{\text{sphere}} \approx \left(\sqrt{26}, 0.20, \frac{\pi}{4}\right)}$$

Gradient of a Scalar Field

Problem 25

Given the scalar field $f(x, y, z) = x^2y + xz + y^2$, calculate $\overrightarrow{\text{grad}}(f) = \vec{\nabla}f$.

$$\overrightarrow{\text{grad}}(f) = \vec{\nabla}f$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial y}\hat{k}$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial}{\partial x}(x^2y + xz + y^2) \cdot \hat{\imath} + \frac{\partial}{\partial y}(x^2y + xz + y^2) \cdot \hat{\jmath} + \frac{\partial}{\partial z}(x^2y + xz + y^2) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = (2xy + z + 0) \cdot \hat{\imath} + (x^2 + 0 + 2y) \cdot \hat{\jmath} + (0 + x + 0) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = (2xy + z)\hat{\imath} + (x^2 + 2y)\hat{\jmath} + x\hat{k}$$

$$\boxed{\overrightarrow{\text{grad}}(f) = (2xy + z, x^2 + 2y, x)}$$

Problem 26

Given the scalar field $f(x, y, z) = x^2 + y^2 + z^2 + 2$, calculate $\overrightarrow{\text{grad}}(f) = \vec{\nabla}f$.

$$\overrightarrow{\text{grad}}(f) = \vec{\nabla}f$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial y}\hat{k}$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 + 2) \cdot \hat{\imath} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2 + 2) \cdot \hat{\jmath}$$

$$+ \frac{\partial}{\partial z}(x^2 + y^2 + z^2 + 2) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = (2x + 0 + 0 + 0) \cdot \hat{\imath} + (0 + 2y + 0 + 0) \cdot \hat{\jmath} + (0 + 0 + 2z + 0) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\boxed{\overrightarrow{\text{grad}}(f) = (2x, 2y, 2z)}$$

Problem 27

Given the scalar field $f(x,y,z) = x + 3y + 5z + 2$, calculate $\overrightarrow{\text{grad}}(f) = \vec{\nabla} f$.

$$\overrightarrow{\text{grad}}(f) = \vec{\nabla} f$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial y}\hat{k}$$

$$\overrightarrow{\text{grad}}(f) = \frac{\partial}{\partial x}(x + 3y + 5z + 2) \cdot \hat{\imath} + \frac{\partial}{\partial y}(x + 3y + 5z + 2) \cdot \hat{\jmath}$$

$$+ \frac{\partial}{\partial z}(x + 3y + 5z + 2) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = (1 + 0 + 0 + 0) \cdot \hat{\imath} + (0 + 3 + 0 + 0) \cdot \hat{\jmath} + (0 + 0 + 5 + 0) \cdot \hat{k}$$

$$\overrightarrow{\text{grad}}(f) = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$$

$$\boxed{\overrightarrow{\text{grad}}(f) = (1,3,5)}$$

Curl of a Vector Field

Problem 28

Given the vector field $\vec{u} = \left(u_x, u_y, u_z\right) = (x^2y)\hat{\imath} + (yz)\hat{\jmath} - (z^2)\hat{k}$, calculate

$\text{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$.

$$\text{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$$

$$\text{curl}(\vec{u}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$\text{curl}(\vec{u}) = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_y & u_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ u_x & u_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ u_x & u_y \end{vmatrix} \hat{k}$$

$$\text{curl}(\vec{u}) = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{k}$$

$$\text{curl}(\vec{u}) = \left[\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(yz)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial z}(x^2y)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(x^2y)\right]\hat{k}$$

$$\text{curl}(\vec{u}) = (0 - y)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - x^2)\hat{k}$$

$$\text{curl}(\vec{u}) = (-y)\hat{\imath} - (0)\hat{\jmath} + (-x^2)\hat{k}$$

$$\boxed{\text{curl}(\vec{u}) = (-y, 0, -x^2)}$$

Problem 29

Given the vector field $\vec{u} = \left(u_x, u_y, u_z\right) = (x^2)\hat{\imath} + (z^2)\hat{\jmath} - (xy^3)\hat{k}$, calculate

$\mathrm{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$.

$$\vec{u} = (x^2)\hat{\imath} + (z^2)\hat{\jmath} + (-xy^3)\hat{k}$$

$$\mathrm{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$$

$$\mathrm{curl}(\vec{u}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$\mathrm{curl}(\vec{u}) = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_y & u_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ u_x & u_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ u_x & u_y \end{vmatrix} \hat{k}$$

$$\mathrm{curl}(\vec{u}) = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{k}$$

$$\mathrm{curl}(\vec{u}) = \left[\frac{\partial}{\partial y}(-xy^3) - \frac{\partial}{\partial z}(z^2)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(-xy^3) - \frac{\partial}{\partial z}(x^2)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial y}(x^2)\right]\hat{k}$$

$$\mathrm{curl}(\vec{u}) = (-2xy^2 - 0)\hat{\imath} - (-y^3 - 0)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\mathrm{curl}(\vec{u}) = (-2xy^2)\hat{\imath} - (-y^3)\hat{\jmath} + (0)\hat{k}$$

$$\mathrm{curl}(\vec{u}) = (-2xy^2)\hat{\imath} + (y^3)\hat{\jmath} + (0)\hat{k}$$

$$\boxed{\mathrm{curl}(\vec{u}) = (-2xy^2, y^3, 0)}$$

Problem 30

Given the vector field $\vec{u} = (u_x, u_y, u_z) = (x)\hat{\imath} + (z)\hat{\jmath} - (x)\hat{k}$, calculate $\text{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$.

$$\vec{u} = (x)\hat{\imath} + (z)\hat{\jmath} + (-x)\hat{k}$$

$$\text{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$$

$$\text{curl}(\vec{u}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$\text{curl}(\vec{u}) = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_y & u_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ u_x & u_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u_x & u_y \end{vmatrix} \hat{k}$$

$$\text{curl}(\vec{u}) = \left(\frac{\partial}{\partial y} u_z - \frac{\partial}{\partial z} u_y\right)\hat{\imath} - \left(\frac{\partial}{\partial x} u_z - \frac{\partial}{\partial z} u_x\right)\hat{\jmath} + \left(\frac{\partial}{\partial x} u_y - \frac{\partial}{\partial y} u_x\right)\hat{k}$$

$$\text{curl}(\vec{u}) = \left[\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(z)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(x)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(x)\right]\hat{k}$$

$$\text{curl}(\vec{u}) = (0 - 1)\hat{\imath} - (-1 - 0)\hat{\jmath} + (0 - 0)\hat{k}$$

$$\text{curl}(\vec{u}) = (-1)\hat{\imath} - (-1)\hat{\jmath} + (0)\hat{k}$$

$$\text{curl}(\vec{u}) = (-1)\hat{\imath} + (1)\hat{\jmath} + (0)\hat{k}$$

$$\boxed{\text{curl}(\vec{u}) = (-1,1,0)}$$

Computing the Divergence of a Vector Field

Problem 31

Given the vector field $\vec{u} = (u_x, u_y, u_z) = x^2y\hat{\imath} + zy\hat{\jmath} - z^2\hat{k}$, calculate $\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$.

$$\vec{u} = (x^2y)\hat{\imath} + (zy)\hat{\jmath} + (-z^2)\hat{k}$$

$$\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$$

$$\operatorname{div}\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{div}\vec{u} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(zy) + \frac{\partial}{\partial z}(-z^2)$$

$$\operatorname{div}\vec{u} = 2xy + z + (-2z)$$

$$\boxed{\operatorname{div}\vec{u} = 2xy - z}$$

Problem 32

Given the vector field $\vec{u} = (u_x, u_y, u_z) = x^2\hat{\imath} + z^2\hat{\jmath} - xy^3\hat{k}$, calculate $\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$.

$$\vec{u} = (x^2)\hat{\imath} + (z^2)\hat{\jmath} + (-xy^3)\hat{k}$$

$$\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$$

$$\operatorname{div}\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{div}\vec{u} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(-xy^3)$$

$$\operatorname{div}\vec{u} = 2x + 0 + 0$$

$$\boxed{\operatorname{div}\vec{u} = 2x}$$

Problem 33

Given the vector field $\vec{u} = \left(u_x, u_y, u_z\right) = x\hat{\imath} + z\hat{\jmath} - x\hat{k}$, calculate $\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$.

$$\vec{u} = (x)\hat{\imath} + (z)\hat{\jmath} + (-x)\hat{k}$$

$$\operatorname{div}\vec{u} = \vec{\nabla}\cdot\vec{u}$$

$$\operatorname{div}\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{div}\vec{u} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(-x)$$

$$\operatorname{div}\vec{u} = 1 + 0 + 0$$

$$\boxed{\operatorname{div}\vec{u} = 1}$$

Computing the Laplacian of a Scalar Field

Problem 34

Given the scalar field $f(x, y, z) = x^2y + xz + y^2$, compute the Laplacian

$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(f)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2y + xz + y^2)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy + z + 0)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy + z)$$

$$\frac{\partial^2 f}{\partial x^2} = 2y + 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(x^2y + xz + y^2)\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 0 + 2y)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2y)$$

$$\frac{\partial^2 f}{\partial y^2} = 0 + 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(x^2y + xz + y^2)\right]$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(0 + x + 0)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(x)$$

$$\frac{\partial^2 f}{\partial z^2} = 0$$

$$\vec{\nabla}^2 f = 2y + 2 + 0$$

$$\boxed{\vec{\nabla}^2 f = 2y + 2}$$

Problem 35

Given the scalar field $f(x,y,z) = x^2 + y^2 + z^2 + 2$, compute the Laplacian

$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2 + y^2 + z^2 + 2)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x + 0 + 0 + 0)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x)$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(x^2 + y^2 + z^2 + 2)\right]$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(0 + 0 + 2z + 0)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(2z)$$

$$\frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(x^2 + y^2 + z^2 + 2)\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(0 + 2y + 0 + 0)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2y)$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\vec{\nabla}^2 f = 2 + 2 + 2$$

$$\boxed{\vec{\nabla}^2 f = 6}$$

Problem 36

Given the scalar field $f(x,y,z) = xz + 3x^3y^2 + 2xz^2$, compute the Laplacian

$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(xz + 3x^3y^2 + 2xz^2)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(z + 9x^2y^2 + 2z^2)$$

$$\frac{\partial^2 f}{\partial x^2} = 0 + 18xy^2 + 0$$

$$\frac{\partial^2 f}{\partial x^2} = 18xy^2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(xz + 3x^3y^2 + 2xz^2)\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(0 + 6x^3y + 0)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(6x^3y)$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^3$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(xz + 3x^3y^2 + 2xz^2)\right]$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(x + 0 + 4xz)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(x + 4xz)$$

$$\frac{\partial^2 f}{\partial z^2} = 0 + 4x$$

$$\frac{\partial^2 f}{\partial z^2} = 4x$$

$$\boxed{\vec{\nabla}^2 f = 18xy^2 + 6x^3 + 4x}$$

Computing the Laplacian of a Vector Field

Problem 37

Given the vector field $\vec{u} = (u_x, u_y, u_z) = (3x^2y, y^2z, 3z^2)$, find the Laplacian of $\vec{u}$, $\vec{\nabla}^2\vec{u}$.

$$\vec{\nabla}^2\vec{u} = (\vec{\nabla}^2 u_x)\hat{\imath} + (\vec{\nabla}^2 u_y)\hat{\jmath} + (\vec{\nabla}^2 u_z)\hat{k}$$

$$\vec{\nabla}^2 u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}$$

$$\vec{\nabla}^2 u_x = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(3x^2y)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(3x^2y)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(3x^2y)\right]$$

$$\vec{\nabla}^2 u_x = \frac{\partial}{\partial x}(6xy) + \frac{\partial}{\partial y}(3x^2) + \frac{\partial}{\partial z}(0)$$

$$\vec{\nabla}^2 u_x = 6y + 0 + 0$$

$$\vec{\nabla}^2 u_x = 6y$$

$$\vec{\nabla}^2 u_y = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}$$

$$\vec{\nabla}^2 u_y = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(y^2z)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(y^2z)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(y^2z)\right]$$

$$\vec{\nabla}^2 u_y = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(y^2)$$

$$\vec{\nabla}^2 u_y = 0 + 2z + 0$$

$$\vec{\nabla}^2 u_y = 2z$$

$$\vec{\nabla}^2 u_z = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}$$

$$\vec{\nabla}^2 u_z = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(3z^2)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(3z^2)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(3z^2)\right]$$

$$\vec{\nabla}^2 u_z = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(6z)$$

$$\vec{\nabla}^2 u_z = 0 + 0 + 6$$

$$\vec{\nabla}^2 u_z = 6$$

$$\vec{\nabla}^2\vec{u} = (6y)\hat{\imath} + (2z)\hat{\jmath} + (6)\hat{k}$$

$$\boxed{\vec{\nabla}^2\vec{u} = (6y, 2z, 6)}$$

Problem 38

Given the vector field $\vec{u} = \left(u_x, u_y, u_z\right) = (x^2 + y, x + y^2 z, z^2)$, find the Laplacian of $\vec{u}$, $\vec{\nabla}^2\vec{u}$.

$$\vec{\nabla}^2\vec{u} = \left(\vec{\nabla}^2 u_x\right)\hat{\imath} + \left(\vec{\nabla}^2 u_y\right)\hat{\jmath} + \left(\vec{\nabla}^2 u_z\right)\hat{k}$$

$$\vec{\nabla}^2 u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}$$

$$\vec{\nabla}^2 u_x = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x^2 + y)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(x^2 + y)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(x^2 + y)\right]$$

$$\vec{\nabla}^2 u_x = \frac{\partial}{\partial x}(2x + 0) + \frac{\partial}{\partial y}(0 + 1) + \frac{\partial}{\partial z}(0 + 0)$$

$$\vec{\nabla}^2 u_x = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(0)$$

$$\vec{\nabla}^2 u_x = 2 + 0 + 0$$

$$\vec{\nabla}^2 u_x = 2$$

$$\vec{\nabla}^2 u_y = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}$$

$$\vec{\nabla}^2 u_y = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(x + y^2 z)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(x + y^2 z)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(x + y^2 z)\right]$$

$$\vec{\nabla}^2 u_y = \frac{\partial}{\partial x}(1 + 0) + \frac{\partial}{\partial y}(0 + 2yz) + \frac{\partial}{\partial z}(0 + y^2)$$

$$\vec{\nabla}^2 u_y = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(y^2)$$

$$\vec{\nabla}^2 u_y = 0 + 2z + 0$$

$$\vec{\nabla}^2 u_y = 2z$$

$$\vec{\nabla}^2 u_z = \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}$$

$$\vec{\nabla}^2 u_z = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(z^2)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(z^2)\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(z^2)\right]$$

$$\vec{\nabla}^2 u_z = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(2z)$$

$$\vec{\nabla}^2 u_z = 0 + 0 + 2$$

$$\vec{\nabla}^2 u_z = 2$$

$$\vec{\nabla}^2 \vec{u} = (2)\hat{\imath} + (2z)\hat{\jmath} + (2)\hat{k}$$

$$\boxed{\vec{\nabla}^2 \vec{u} = (2, 2z, 2)}$$

Problem 39

Given the vector field $\vec{u}\left(u_x, u_y, u_z\right) = (3x^2y, y^2z, 3z^2)$, find the Laplacian of $\vec{u}$, $\vec{\nabla}^2\vec{u}$.

[duplicate to #37]

Derivative of a Vector with Respect to a Vector

Problem 40

Given $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{cases} w_1 = 2x + 3y \\ w_2 = 7x + 5y \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, calculate $\frac{\partial \vec{w}}{\partial \vec{u}}$.

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \dfrac{\partial w_1}{\partial u_1} & \dfrac{\partial w_1}{\partial u_2} \\ \dfrac{\partial w_2}{\partial u_1} & \dfrac{\partial w_2}{\partial u_2} \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \dfrac{\partial}{\partial x}(2x + 3y) & \dfrac{\partial}{\partial y}(2x + 3y) \\ \dfrac{\partial}{\partial x}(7x + 5y) & \dfrac{\partial}{\partial y}(7x + 5y) \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} 2 + 0 & 0 + 3 \\ 7 + 0 & 0 + 5 \end{bmatrix}$$

$$\boxed{\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix}}$$

Problem 41

Given $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{cases} w_1 = x - y + z \\ w_2 = x + 2y - z \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, calculate $\frac{\partial \vec{w}}{\partial \vec{u}}$.

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \dfrac{\partial w_1}{\partial u_1} & \dfrac{\partial w_1}{\partial u_2} & \dfrac{\partial w_1}{\partial u_3} \\ \dfrac{\partial w_2}{\partial u_1} & \dfrac{\partial w_2}{\partial u_2} & \dfrac{\partial w_2}{\partial u_3} \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \dfrac{\partial}{\partial x}(x - y + z) & \dfrac{\partial}{\partial y}(x - y + z) & \dfrac{\partial}{\partial z}(x - y + z) \\ \dfrac{\partial}{\partial x}(x + 2y - z) & \dfrac{\partial}{\partial y}(x + 2y - z) & \dfrac{\partial}{\partial z}(x + 2y - z) \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} 1 - 0 + 0 & 0 - 1 + 0 & 0 - 0 + 1 \\ 1 + 0 - 0 & 0 + 2 - 0 & 0 + 0 - 1 \end{bmatrix}$$

$$\boxed{\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}}$$

Problem 42

Given $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \begin{cases} w_1 = xy + z \\ w_2 = x - y^2 + z \\ w_3 = 2x + y + xz \end{cases}, \vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, calculate $\frac{\partial \vec{w}}{\partial \vec{u}}$.

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial w_1}{\partial u_1} & \frac{\partial w_1}{\partial u_2} & \frac{\partial w_1}{\partial u_3} \\ \frac{\partial w_2}{\partial u_1} & \frac{\partial w_2}{\partial u_2} & \frac{\partial w_2}{\partial u_3} \\ \frac{\partial w_3}{\partial u_1} & \frac{\partial w_3}{\partial u_2} & \frac{\partial w_3}{\partial u_3} \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial}{\partial x}(xy + z) & \frac{\partial}{\partial y}(xy + z) & \frac{\partial}{\partial z}(xy + z) \\ \frac{\partial}{\partial x}(x - y^2 + z) & \frac{\partial}{\partial y}(x - y^2 + z) & \frac{\partial}{\partial z}(x - y^2 + z) \\ \frac{\partial}{\partial x}(2x + y + xz) & \frac{\partial}{\partial y}(2x + y + xz) & \frac{\partial}{\partial z}(2x + y + xz) \end{bmatrix}$$

$$\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} y + 0 & x + 0 & 0 + 1 \\ 1 - 0 + 0 & 0 - 2y + 0 & 0 - 0 + 1 \\ 2 + 0 + z & 0 + 1 + 0 & 0 + 0 + x \end{bmatrix}$$

$$\boxed{\frac{\partial \vec{w}}{\partial \vec{u}} = \begin{bmatrix} y & x & 1 \\ 1 & -2y & 1 \\ 2 + z & 1 & x \end{bmatrix}}$$

Derivative of a Scalar with Respect to a Vector (Jacobian)

Problem 43

Given $s = s(\vec{x}) = (x_1 + 1)^2 + {x_2}^2 + (x_3 + 2)^2, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, calculate $\frac{\partial s}{\partial x}$.

$$\frac{\partial s}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial s}{\partial x_1} & \frac{\partial s}{\partial x_2} & \frac{\partial s}{\partial x_3} \end{bmatrix}$$

$$\frac{\partial s}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1}[(x_1 + 1)^2 + {x_2}^2 + (x_3 + 2)^2] & \frac{\partial}{\partial x_2}[(x_1 + 1)^2 + {x_2}^2 + (x_3 + 2)^2] & \frac{\partial}{\partial x_3}[(x_1 + 1)^2 + {x_2}^2 + (x_3 + 2)^2] \end{bmatrix}$$

$$\frac{\partial s}{\partial \vec{x}} = [2(x_1 + 1) + 0 + 0 \quad 0 + 2x_2 + 0 \quad 0 + 0 + 2(x_3 + 2)]$$

$$\boxed{\frac{\partial s}{\partial \vec{x}} = [2x_1 + 2 \quad 2x_2 \quad 2x_3 + 4]}$$

Problem 44

Given $f = f(\vec{x}) = x + y + z, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, calculate $\frac{\partial f}{\partial x}$.

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial}{\partial x}(x + y + z) & \frac{\partial}{\partial y}(x + y + z) & \frac{\partial}{\partial z}(x + y + z) \end{bmatrix}$$

$$\frac{\partial f}{\partial \vec{x}} = [1 + 0 + 0 \quad 0 + 1 + 0 \quad 0 + 0 + 1]$$

$$\boxed{\frac{\partial f}{\partial \vec{x}} = [1 \quad 1 \quad 1]}$$

Problem 45

Given $g = g(\vec{x}) = x + xy + z^2, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, calculate $\frac{\partial g}{\partial x}$.

$$\frac{\partial g}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix}$$

$$\frac{\partial g}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial}{\partial x}(x + xy + z^2) & \frac{\partial}{\partial y}(x + xy + z^2) & \frac{\partial}{\partial z}(x + xy + z^2) \end{bmatrix}$$

$$\frac{\partial g}{\partial \vec{x}} = [1 + y + 0 \quad 0 + x + 0 \quad 0 + 0 + 2z]$$

$$\boxed{\frac{\partial g}{\partial \vec{x}} = [1 + y \quad x \quad 2z]}$$

Quadric Forms

Problem 46

Given the quadric form $f(x, y) = x^2 + 6xy + 2y^2$:

a. Express in matrix form, $f(\vec{x}) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$.

$$f(x, y) = a_{11}x^2 + a_{12}xy + a_{21}xy + a_{22}y^2$$

$$f(x, y) = (1)x^2 + \left(\frac{6}{2}\right)xy + \left(\frac{6}{2}\right)xy + (2)y^2$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$f(x, y) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$$

$$\boxed{f(x, y) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \cdot \vec{x}}$$

b. Calculate $\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$.

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\vec{x}^{\mathrm{T}} A^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} (2)(x)(1) + (2)(y)(3) \\ (2)(x)(3) + (2)(y)(2) \end{bmatrix}$$

$$\boxed{\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 2x + 6y \\ 6x + 4y \end{bmatrix}}$$

Problem 47

Given the quadric form $f(x,y) = 5x^2 + 2xy + 2y^2$:

a. Express in matrix form, $f(\vec{x}) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$.

$$f(x,y) = a_{11}x^2 + a_{12}xy + a_{21}xy + a_{22}y^2$$

$$f(x,y) = (5)x^2 + \left(\frac{2}{2}\right)xy + \left(\frac{2}{2}\right)xy + (2)y^2$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

$$f(x,y) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$$

$$\boxed{f(x,y) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}}$$

b. Calculate $\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$.

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\vec{x}^{\mathrm{T}} A^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} (2)(x)(5) + (2)(y)(1) \\ (2)(x)(1) + (2)(y)(2) \end{bmatrix}$$

$$\boxed{\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 10x + 2y \\ 2x + 4y \end{bmatrix}}$$

Problem 48

Given the quadric form $f(x,y) = 3x^2 + 8xy + 6xz + y^2 + 6yz + 3z^2$:

a. Express in matrix form, $f(\vec{x}) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$.

$$f(x,y,z) = a_{11}x^2 + a_{12}xy + a_{21}xy + a_{13}xz + a_{31}xz + a_{22}y^2 + a_{23}yz + a_{32}yz + a_{33}z^2$$

$$f(x,y,z) = (3)x^2 + \left(\frac{8}{2}\right)xy + \left(\frac{8}{2}\right)xy + \left(\frac{6}{2}\right)xz + \left(\frac{6}{2}\right)xz + (1)y^2 + \left(\frac{6}{2}\right)yz + \left(\frac{6}{2}\right)yz + (3)z^2$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$f(x,y,z) = \vec{x}^{\mathrm{T}} \cdot A \cdot \vec{x}$$

$$\boxed{f(x,y,z) = \vec{x}^{\mathrm{T}} \cdot \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix} \cdot \vec{x}}$$

b. Calculate $\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$.

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(\vec{x}^{\mathrm{T}} A \vec{x})$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\vec{x}^{\mathrm{T}} A^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}^{\mathrm{T}}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = 2\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} (2)(x)(3) + (2)(y)(4) + (2)(z)(3) \\ (2)(x)(4) + (2)(y)(1) + (2)(z)(3) \\ (2)(x)(3) + (2)(y)(3) + (2)(z)(3) \end{bmatrix}$$

$$\boxed{\frac{\partial f(\vec{x})}{\partial \vec{x}} = \begin{bmatrix} 6x + 8y + 6z \\ 8x + 2y + 6z \\ 6x + 6y + 6z \end{bmatrix}}$$

END