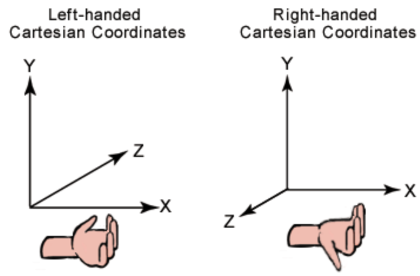
Chapter1 : Vectors in 3D Space and Vector Space

1) Vectors in 3D Space

Rectangular Cartesian Coordinate Systems



In this coordinates, the vectors that span the frame axes(X,Y,Z) are the $\vec{i}$, $\vec{j}$, $\vec{k}$ vectors. The left-handed system is used in DirectX and the right-handed system in OpenGL. The right-handed system is the standard frame used in Math and in Physics. The frame $\{\vec{i},\vec{j},\vec{k}\}$ is said to be an orthonormal frame since $\vec{i}\perp\vec{j}\perp\vec{k}$ and $\|\vec{i}\|=\|\vec{j}\|=\|\vec{k}\|=1$.

A vector is a mathematical entity having a direction and a magnitude .
We write vector a , as $\vec{a}$(x,y,z).
x=1st component, y=2nd component and z=3rd .Three very important vectors are the ones that spanned
the world space: $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$ and $\vec{k}=(0,0,1)$.
Every vector in the world space can be expressed as a linear combination of $\vec{i},\vec{j}$ and $\vec{k}$; that is $\vec{a}=(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$.
Example 1.1: $\vec{a}=(2,-3,5)=2\vec{i}-3\vec{j}+5\vec{k}$.
PYTHONIC: How to define a vector in Python with **numpy, scipy** and **sympy**

First import **numpy** , **scipy or sympy** libraries, then use the **array(data,dtype)** function
for **numpy** , **scipy and Array(data)** for **sympy** (notice the upper case A).
dtype is for data type(float, integer(int),complex,..)

```
import numpy as np
a=np.array([2,-3,5],dtype=float)
print("numpy vector a=",a)
#when using scipy library, do:
import scipy as sp
b=sp.array([4,-3,0],dtype=float)
print("scipy vector b=",b)
import sympy as sy
c=sy.Array([1,2,3])
print("sympy vector c=",c)

```

```
numpy vector a= [ 2. -3.  5.]
scipy vector b= [ 4. -3.  0.]
sympy vector c= [1, 2, 3]
```

Also sympy uses another approach through its Physics vector library by importing a reference frame:
from sympy.physics.vector import ReferenceFrame

The reference frame in math is the frame $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ equivalent to $\{\vec{i}, \vec{j}, \vec{k}\}$
As an example $\vec{v} = (1,2,-3) = \hat{e}_x + 2\hat{e}_y - 3\hat{e}_z$ same as $\vec{v} = \vec{i} + 2\vec{j} - 3\vec{k}$ will be will be in python v=e.x + 2*e.y -3*e.z. (see example code below).

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
v=e.x + 2*e.y-3*e.z
sy.pprint(v)

```

```
e_x + 2 e_y + -3 e_z
```

2) <u>Vector Norm</u>

Given a vector $\vec{a} = (a_1, a_2, a_3)$, we define its norm(length or magnitude) as $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Note that $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$

Example 2.1: Calculate the norm of $\vec{a} = (3,0,-4)$.

Answer: $\|\vec{a}\| = \sqrt{3^2 + 0^2 + (-4)^2} = \sqrt{9+0+16} = \sqrt{25} = 5$

Example 2.2 Calculate the norm of $\vec{a} = (2,-1,1)$.

Answer: $\|\vec{a}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

PYTHONIC: How to compute a vector length in Python?
Python uses **numpy.linalg.norm() or scipy.linalg.norm()** where **linalg** is the linear algebra package.
Sympy will use **magnitude()** ***from sympy.physics.vector import ReferenceFrame.***

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
v=e.x + 2*e.y-2*e.z
sy.pprint(v)
m=v.magnitude()
print("from sympy ,magnitude of v=",v,"is", m)
import numpy as np
a=np.array([3,0,4])
m=np.linalg.norm(a)
print("from numpy ,magnitude of a=",a,"is", m)
import scipy as sp
b=sp.array([1,1,-1])
m=sp.linalg.norm(b)
print("from scipy ,magnitude of b=",b,"is", m)
```

```
e_x + 2 e_y + -2 e_z
from sympy ,magnitude of v= e.x + 2*e.y - 2*e.z is 3
from numpy ,magnitude of a= [3 0 4] is 5.0
from scipy ,magnitude of b= [ 1  1 -1] is 1.7320508075688772
```

3) Normalized Vectors and Units Vectors.

The normalized Vector of $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|}$. A unit vector is a normalized vector. $\|\hat{a}\| = 1$ always.

Example 3.1: Normalize $\vec{a} = (1,0,-1)$.

Answer: $\|\vec{a}\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$ ➔ $\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{(1,0,-1)}{\sqrt{2}} = (\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}})$.

Example 3.2: Normalize $\vec{u} = (3,1,2)$

Answer: $\|\vec{u}\| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{u}}{\sqrt{14}} = \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

TODO: Go to Activity and solve question 1

PYTHONIC: How do you normalize a vector in python?

Python uses **normalize() from** ***sympy.physics.vector(see code below)***

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
a= e.x + e.z
v=a.normalize()
sy.pprint(v)
```

```
√2       √2
── e_x + ── e_z
2        2
```

Vector Direction

The direction of a vector $\vec{a}$ is its normalized vector $\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$

Example 3.3 : What is the direction of a car moving with velocity $\vec{v} = (1,0,1)\ m/s$

Answer : $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(1,0,1)}{\sqrt{1^2+0+1^2}} = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

4) Parallel and Collinear Vectors

$\vec{a} // \vec{b} \Leftrightarrow \exists k \in \mathbb{R}^* / \vec{a} = k\vec{b}$ or $\vec{b} = k\vec{a}$.

Note : // symbol means " parallel to"

$\exists$ symbol means " there is" . / for such that

$\in$ symbol means "element of ", $\mathbb{R}^*$=all real number without zero

Example 4.1: Show that $\vec{a} = (2,3,1)$ and $\vec{b} = (6,9,3)$ are collinear or parallel.

$\vec{b} = (6,9,3) = 3(2,3,1) = 3\vec{a}$ .Since $\vec{b} = 3\vec{a} \Rightarrow \vec{a} // \vec{b}$

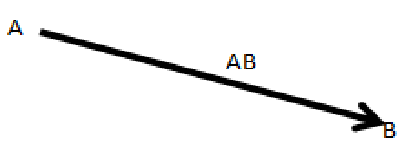
Example 4.2 Show that $\vec{a} = (5,15,-10)$ and $\vec{b} = (1,3,-2)$ are collinear or parallel

$\vec{a} = (5,15,-10)$ **=5(1,3,-2)=**$5\vec{b}$ **.** $\vec{a} = 5\vec{b} \Rightarrow \vec{a} // \vec{b}$

TODO: Go to Activity and solve question 2

5) Building a Vector From Two Given Points(Vertices)

Given 2 points(vertices) $\vec{A}$ and $\vec{B}$, we compute the $\overrightarrow{AB}$ = $\vec{B}$ - $\vec{A}$.



Here $\vec{A}$=vector origin , $\vec{B}$=vector head (terminal point)

The opposite of vector $\overrightarrow{AB}$ is $-\overrightarrow{AB} = \overrightarrow{BA}$

Note that : $\overrightarrow{AB}$**=-**$\overrightarrow{BA}$**.**

$\vec{A}$ **and** $\vec{B}$ **are not vectors but vertices or position vectors**

Distance between two points: The distance between $\vec{A}$ and $\vec{B}$ is $\|\overrightarrow{AB}\| = \|\vec{B} - \vec{A}\|$

Example 5.1: Given two vertices $\vec{A}$=(2,3,1) and $\vec{B}$=(5,3,5) calculate

1) Vector $\overrightarrow{AB}$

   $\overrightarrow{AB} = \vec{B} - \vec{A}$=(5,3,5)-(2,3,1)=(3,0,4).

2) The opposite of vector $\overrightarrow{AB}$

   The opposite of $\overrightarrow{AB}$ is $\overrightarrow{BA}$=-$\overrightarrow{AB}$ that is $\overrightarrow{BA} = \vec{A} - \vec{B} = (2,3,1) - (5,3,5) = (-3,0,-4)$

3) The distance between $\vec{A}$ and $\vec{B}$.

   Since $\overrightarrow{AB}$=(3,0,4) then the distance is $\|\overrightarrow{AB}\| = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{9+16} = 5$

TODO: Go to Activity and solve question 3

6) Vectors Algebra

Let $\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be 2 vectors in 3D space.

a. Vectors Addition

$\vec{a} + \vec{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Example 6.1: $\vec{a} = (2,-1,5)$ *and* $\vec{b} = (1,3,1)$

- $\vec{a} + \vec{b}$=(2,-1,5) + (1,3,1)=(3,2,6).
- $2\vec{a} - 3\vec{b}$= 2(2,-1,5) - 3(1,3,1)=(4,-2,10) + (-3,-9,-3)=(1,-11,7).

Theorem

1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutativity)

2) $\vec{a} - \vec{a} = \vec{o}$

3) $\vec{a} + \vec{o} = \vec{a}$

4) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + \vec{b} + \vec{c}$

5) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ where $k \in \mathbb{R}$.

TODO: Go to Activity and solve question 4

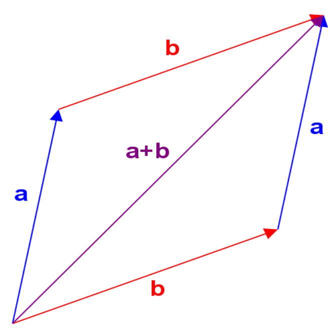
PYTHONIC: How to add vector in python

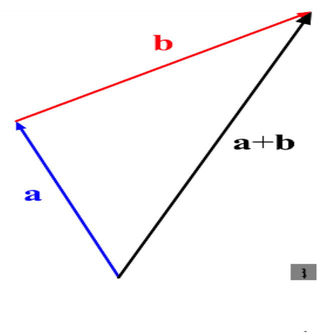
```python
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
a= 2*e.x - e.y+ 5*e.z
b= e.x +3*e.y+ e.z
s=2*a-3*b
print("vector sum in sympy, 2a-3b =",end=" ")
sy.pprint(s)
# using numpy and scipy
import numpy as np
a=np.array([2,-1,5])
b=np.array([1,3,1])
s=2*a-3*b
print("vector sum in scipy, 2a-3b=",s)


```

```
vector sum in sympy, 2a-3b = e_x + -11 e_y + 7 e_z
vector sum in scipy, 2a-3b= [  1 -11   7]
```

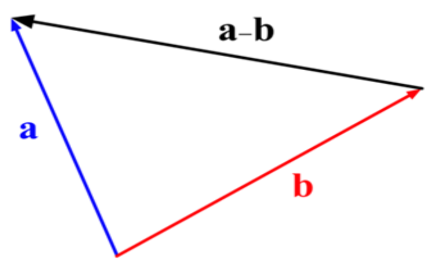
How to Add Vectors Geometrically?

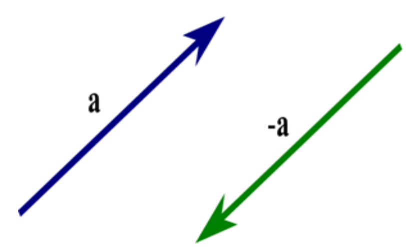
Given two vectors $\vec{a}$ and $\vec{b}$, their sum is $\vec{s} = \vec{a} + \vec{b}$ as illustrated below.





Their difference is $\vec{d} = \vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$ the opposite of $\vec{a}$ is $-\vec{a}$





b. The Dot Product of 2 Vectors

The dot product of $\vec{a}$ *and* $\vec{b}$ is $\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos(\theta)$ where $\theta = Angle(\vec{a}, \vec{b})$

Theorem: $\vec{a} \bullet \vec{b} = (a_1, a_2, a_3) \bullet (b_1, b_2, b_3) = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$

Example 6.2 : Calculate $\vec{a} \bullet \vec{b}$ if $\vec{a} = (2, -1, 5)$, $\vec{b} = (1, 3, 1)$, *and* $\vec{c} = (1, 0, 3)$.

Ans: $\vec{a} \bullet \vec{b} = (2, -1, 5) \bullet (1, 3, 1) = (2)(1) + (-1)(3) + (5)(1) = 2 - 3 + 5 = 4$

$\vec{a} \bullet \vec{c} = (2, -1, 5) \bullet (1, 0, 3) = (2)(1) + (-1)(0) + (5)(3) = 2 + 15 = 17$

$\vec{b} \bullet \vec{c} = (1, 3, 1) \bullet (1, 0, 3) = (1) \cdot (1) + (3) \cdot (0) + (1) \cdot (3) = 1 + 3 = 4$

Note that $\boxed{\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}}$ **and** $\boxed{\|\vec{a}\|^2 = \vec{a} \bullet \vec{a}}$

TODO: Go to Activity and solve question 5

PYTHONIC: How to dot multiply in Python

Python uses **numpy.dot()** or **scipy.dot(),** and **dot()** from **sympy.physics.vector.**

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
a= 2*e.x - e.y+ 5*e.z
b= e.x +3*e.y+ e.z
d=a.dot(b) # use physics.vector  dot()
print("dot product of a and b in sympy, a*b =",end=" ")
sy.pprint(d)
# using numpy.dot() and scipy.dot()
import numpy as np
a=np.array([2,-1,5])
b=np.array([1,3,1])
d=np.dot(a,b)
print("dot product in scipy, a*b=",d)
```

```
dot product of a and b in sympy, a*b = 4
dot product in scipy, a*b= 4
```

c. Angle Between 2 Vectors

The angle θ between $\vec{a}$ *and* $\vec{b}$ is $\cos\theta = \dfrac{\vec{a}\bullet\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$ or $\theta = \cos^{-1}\left(\dfrac{\vec{a}\bullet\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}\right)$

Example 6.3 :

a) Find the angle between $\vec{a} = (1,0,1)$ *and* $\vec{b} = (2,0,0)$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\bullet\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}\right) = \cos^{-1}\left(\frac{(1,0,1)\bullet(2,0,0)}{\sqrt{1^2+0+1^2}\cdot\sqrt{2^2+0+0}}\right) = \cos^{-1}\left(\frac{2}{2\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$$

b) Find the angle between $\vec{a} = (1,3,1)$ *and* $\vec{b} = (2,-1,1)$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\bullet\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}\right) = \cos^{-1}\left(\frac{(1,3,1)\bullet(2,-1,1)}{\sqrt{1^2+3^2+1^2}\cdot\sqrt{2^2+1+1}}\right) = \cos^{-1}\left(\frac{2-3+1}{\sqrt{11}\cdot\sqrt{6}}\right) = \cos^{-1}\left(\frac{0}{\sqrt{66}}\right) = \cos^{-1}(0) = 90^\circ$$

TODO: Go to Activity and solve question 6

d. Type of Angles

Given two vectors $\vec{a}\, and\, \vec{b}$, the type of angle between the two vectors is defined as follows:

assuming $0 \le \theta \le \pi (180^o)$

$\theta =$ acute if only if $\vec{a} \bullet \vec{b} > 0$

$\theta =$ obtuse if only if $\vec{a} \bullet \vec{b} < 0$

$\theta =$ right if only if $\vec{a} \bullet \vec{b} = 0$.

If $\theta > 180$ ➔ $\theta =$ reflex angle

Example 6.4: find the type of angle between $\vec{a} = (-1,1,2), \vec{b} = (2,1,-1), \vec{c} = (0,1,1)$

Ans:

$$\vec{a} \bullet \vec{b} = (-1,1,2) \bullet (2,1,-1) = -2+1-2 = -3 < 0 \Rightarrow \theta = obtuse$$

$$\vec{a} \bullet \vec{c} = (-1,1,2) \bullet (0,1,1) = 0+1+2 = 3 > 0 \Rightarrow \theta = acute.$$

$$\vec{b} \bullet \vec{c} = (2,1,-1) \bullet (0,1,1) = 0+1-1 = 0 \Rightarrow \theta = Right\ angle$$

TODO: Go to Activity and solve question 7

Theorem(Orthogonal or Perpendicular vectors)

$$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \bullet \vec{b} = 0 \qquad (The\ \ symbol \perp means\ \ orthogonal\ or\ \ perpendicular)$$

Example 6.5: show that $\vec{a} = (2,3,2)\ \ and\ \ \vec{b} = (2,2,-5)$ are orthogonal

$$\vec{a} \bullet \vec{b} = (2,3,2) \bullet (2,2,-5) = (2)(2)+(3)(2)+(2)(-5) = 4+6-10 = 0$$

Since $\vec{a} \bullet \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

TODO: Go to Activity and solve question 8

e. Components of a vector $\vec{a}$ onto a vector $\vec{b}$

The component of $\vec{a}$ onto $\vec{b}$ is the scalar $Comp_{\vec{b}}^{\vec{a}} = \dfrac{\vec{a} \bullet \vec{b}}{\|\vec{b}\|} = \vec{a} \bullet \hat{b}$

f. Projection of a vector $\vec{a}$ onto a vector $\vec{b}$

The Projection of $\vec{a}$ onto $\vec{b}$ is the vector $\Pr oj_{\vec{b}}^{\vec{a}} = Comp_{\vec{b}}^{\vec{a}} \bullet \hat{b}$

Example 6.7: given $\vec{a} = (1,0,1)\ and\ \vec{b} = (1,1,1)$ calculate $Comp_{\vec{b}}^{\vec{a}}\ and\ \Pr oj_{\vec{b}}^{\vec{a}}$.

Answer : $Comp_{\vec{b}}^{\vec{a}} = \dfrac{\vec{a} \bullet \vec{b}}{\|\vec{b}\|} = \dfrac{(1,0,1) \bullet (1,1,1)}{\sqrt{1^2+1^2+1^2}} = \dfrac{1+0+1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

$$\Pr oj_{\vec{b}}^{\vec{a}} = Comp_{\vec{b}}^{\vec{a}} \bullet \hat{b} = \frac{2}{\sqrt{3}} \hat{b} = \frac{2}{\sqrt{3}} \left(\frac{\vec{b}}{\sqrt{3}} \right) = \frac{2}{3} \vec{b} = \frac{2}{3}(1,1,1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right)$$

TODO: Go to Activity and solve question 9

g. <u>The Perpendicular Vector of $\vec{a}$ to a vector $\vec{b}$</u>

The perpendicular vector of $\vec{a}$ to a vector $\vec{b}$ is $\vec{a}_{\perp} = perp_{\vec{b}}^{\vec{a}}$ such that

$\vec{a} = proj_{\vec{b}}^{\vec{a}} + \vec{a}_{\perp}$ ➔ $\vec{a}_{\perp} = \vec{a} - proj_{\vec{b}}^{\vec{a}} = \vec{a} - (\vec{a}\bullet\hat{b})\hat{b}$

Example 6.8: Find $\vec{a}_{\perp}$ if $\vec{a} = (1,0,1)$ *and* $\vec{b} = (1,1,1)$

$$\vec{a}_{\perp} = \vec{a} - proj_{\vec{b}}^{\vec{a}} = \vec{a} - (\vec{a}\bullet\hat{b})\hat{b} = (1,0,1) - \frac{2}{3}(1,1,1) = (1,0,1) - \left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right) = \left(\frac{1}{3},\frac{-2}{3},\frac{1}{3}\right)$$

h. <u>The Cross Product of 2 Vectors</u>

h.1) <u>Definition</u>

The cross product of $\vec{a}$ *and* $\vec{b}$ is a vector $\vec{c} = \vec{a}\times\vec{b} = (\|\vec{a}\|\cdot\|\vec{b}\| Sin\theta)\cdot\hat{d}$

where $\theta = Angle(\vec{a},\vec{b})$ such that $\vec{a}\perp\vec{c}$ *and* $\vec{b}\perp\vec{c}$.

h.2) <u>The Cross Product of $\vec{i}$, $\vec{j}$ and $\vec{k}$</u>

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{o}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{o}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{o}$

h.3) <u>Properties of the Cross Product</u>

- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ *i.e* $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ not commutative.
- $\vec{a}\times\vec{a} = \vec{o}$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- $\vec{a}\bullet(\vec{a}\times\vec{b}) = 0$
- $\vec{b}\bullet(\vec{a}\times\vec{b}) = 0$
- $(\vec{a}\times\vec{b})\bullet\vec{c} = \vec{a}\bullet(\vec{b}\times\vec{c})$ triple scalar product
- $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ bac-cab rule
- If $\vec{a}\neq\vec{o}$ and $\vec{b}\neq\vec{o}$ then $\vec{a}\times\vec{b} = \vec{o} \Leftrightarrow \vec{a}\,//\,\vec{b}$

Example: $\vec{i}\times(\vec{k}\times\vec{i}) = \vec{i}\times(\vec{j}) = \vec{k}$; $\vec{i}\times(\vec{k}\times\vec{j}) = \vec{i}\times(-\vec{i}) = -\vec{i}\times\vec{i} = \vec{o}$

$\vec{i}\cdot(\vec{j}\times\vec{j}) = \vec{i}\cdot(\vec{o}) = 0$; $\vec{i}\times(2\vec{i}\times\vec{j}) = \vec{i}\times(2\vec{k}) = 2(\vec{i}\times\vec{k}) = 2(-\vec{j}) = -2\vec{j}$

$\vec{i}\times(\vec{k}+3\vec{j}) = \vec{i}\times\vec{k} + \vec{i}\times(3\vec{j}) = -\vec{j} + 3(\vec{i}\times\vec{j}) = -\vec{j} + 3\vec{k}$;

$\vec{j}\cdot(\vec{j}\times\vec{k}) = \vec{j}\cdot(\vec{i}) = \vec{j}\cdot\vec{i} = 0$;

h.4) <u>The Pseudo-Determinant Method</u>

Let $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$ then

$$\vec{c}=(c_1,c_2,c_3)=\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ {\scriptstyle +} & {\scriptstyle -} & {\scriptstyle +}\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

$$=(a_2\cdot b_3-b_2\cdot a_3)\vec{i}-(a_1\cdot b_3-b_1\cdot a_3)\vec{j}+(a_1\cdot b_2-b_1\cdot a_2)\vec{k}$$

So $\vec{c}=(c_1,c_2,c_3)=(a_2\cdot b_3-b_2\cdot a_3\ ,\ b_1\cdot a_3-a_1\cdot b_3\ ,\ a_1\cdot b_2-b_1\cdot a_2)$

Example 6.9: Given $\vec{a}=(-1,0,1)\ and\ \vec{b}=(1,2,3)$ calculate $\vec{c}=\vec{a}\times\vec{b}$.

Ans : $$\vec{c}=\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 1\\ 1 & 2 & 3\\ + & - & +\end{vmatrix}=\begin{vmatrix}0 & 1\\ 2 & 3\end{vmatrix}\vec{i}-\begin{vmatrix}-1 & 1\\ 1 & 3\end{vmatrix}\vec{j}+\begin{vmatrix}-1 & 0\\ 1 & 2\end{vmatrix}\vec{k}=(0-2)\vec{i}-(-3-1)\vec{j}+(-2-0)\vec{k}$$

$$=-2\vec{i}+4\vec{j}-2\vec{k}=(-2,4,-2)$$

Example 6.10:

Answer: Given $\vec{v}_1=(1,0,1)$ and $\vec{v}_2=(1,2,1)$, Calculate $\vec{v}_1\times\vec{v}_2$

$$\vec{v}_1\times\vec{v}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 1\\ 1 & 2 & 1\\ + & - & +\end{vmatrix}=\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix}\vec{i}-\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}\vec{j}+\begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix}\vec{k}=(0-2)\vec{i}-(1-1)\vec{j}+(2-0)\vec{k}=-2\vec{i}+2\vec{k}=(-2,0,2)$$

TODO: Go to Activity and solve question 10

Video :

PYTHONIC: How to cross multiply in Python

Python uses **numpy.cross(), scipy.cross()** and in sympy uses **cross( )** or **^ for cross product**

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy import init_printing
init_printing(use_latex=True)
e=ReferenceFrame('e')
a= -1*e.x - 0*e.y+ 1*e.z
b= 1*e.x +2*e.y+ 3*e.z
c=a^b # use physics.vector  cross() or ^,
print("cross product of a and b in sympy using ^, axb =",end=" ")
sy.pprint(c)
print("cross product of b and a in sympy using cross(), bxa =",end=" ")
c=b.cross(a)  #   remember  axb=-bxa
sy.pprint(c)
# using numpy.cross() and scipy.cross()
import numpy as np
a=np.array([-1,0,1])
b=np.array([1,2,3])
c=np.cross(a,b)
print("cross product in scipy, axb=",c)
```

```
cross product of a and b in sympy using ^, axb = -2 e_x + 4 e_y + -2 e_z
cross product of b and a in sympy using cross(), bxa = 2 e_x + -4 e_y + 2 e_z
cross product in scipy, axb= [-2  4 -2]
```

7) Vectors Calculus(basic)

a. Vector differentiation

Given a vector $\vec{u}(x,y,z) = (\ x(t),\ y(t),\ z(t)\ ) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, the derivation of $\vec{u}$ with respect to t is a vector:

$$\frac{d\vec{u}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} + \dot{z}(t)\vec{k}\,.$$

Example 7.1: Given $\vec{v} = \left(\ t^4,\ 2t^6,\ 5\right)$ calculate $\frac{d\vec{v}}{dt}$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\ t^4,\ 2t^6,\ 5\right) = \left(\ \frac{d}{dt}(t^4), \frac{d}{dt}(2t^6), \frac{d}{dt}(5)\right) = \left(\ 4t^3,\ 12t^5,\ 0\right)$$

Example 7.2: Given $\vec{u} = 3t^2\vec{i} + t^3\vec{j} - 2t^5\vec{k}$ calculate $\frac{d\vec{u}}{dt}$

$$\frac{d\vec{u}}{dt} = \frac{d(3t^2)}{dt}\vec{i} + \frac{d(t^3)}{dt}\vec{j} - \frac{d(2t^5)}{dt}\vec{k} = (6t)\vec{i} + (\ 3t^2)\vec{j} - (10t^4)\vec{k} = (6t\ ,\ 3t^2, -10t^4\ )$$

TODO: Go to Activity and solve question 11

Video :

PYTHONIC:

how do you differentiate a vector with parameter t(time-derivative)?

Python uses the function dt(frame) from **sympy.physics.vector** for this purpose as shown below:

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy.interactive import init_printing
init_printing(use_latex=True)
#define the symbole(variable)
t=sy.Symbol('t')
e=ReferenceFrame('e')
# our vector u=(3t^2,t^3,-2t^5)
u=(3*t**2)*e.x +(t**3)*e.y + (-2*t**5)*e.z
print("vector u(t)")
sy.pprint(u)
print("time-derivative of u is:")
# time-derive of u , dudt , using dt( ) with reference frame e
dudt=u.dt(e)
sy.pprint(dudt)

```

```
vector u(t)
   2       3          5
3⋅t  e_x + t  e_y + -2⋅t  e_z
time-derivative of u is:
             2          4
6⋅t e x + 3⋅t  e y + -10⋅t  e z
```

If $\vec{a}$ and $\vec{b}$ are differentiable vectors, then

- $\frac{d}{dt}(\vec{a}\pm\vec{b})=\frac{d\vec{a}}{dt}\pm\frac{d\vec{b}}{dt}$
- $\frac{d}{dt}(\vec{a}\bullet\vec{b})=\frac{d\vec{a}}{dt}\bullet\vec{b}+\vec{a}\bullet\frac{d\vec{b}}{dt}$
- $\frac{d}{dt}(\vec{a}\times\vec{b})=\frac{d\vec{a}}{dt}\times\vec{b}+\vec{a}\times\frac{d\vec{b}}{dt}$

b. Partial Derivatives of Vectors

Given a vector $\vec{u}=(u_x,u_y,u_z)$,the partial derivative of u with respect to x, y and z are

$$\frac{\partial\vec{u}}{\partial x}=\left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial x},\frac{\partial u_z}{\partial x}\right)=\frac{\partial u_x}{\partial x}\vec{i}+\frac{\partial u_y}{\partial x}\vec{j}+\frac{\partial u_z}{\partial x}\vec{k}\quad;\quad\frac{\partial\vec{u}}{\partial y}=\left(\frac{\partial u_x}{\partial y},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}\right)=\frac{\partial u_x}{\partial y}\vec{i}+\frac{\partial u_y}{\partial y}\vec{j}+\frac{\partial u_z}{\partial y}\vec{k}$$

$$\frac{\partial\vec{u}}{\partial z}=\left(\frac{\partial u_x}{\partial z},\frac{\partial u_y}{\partial z},\frac{\partial u_z}{\partial z}\right)=\frac{\partial u_x}{\partial z}\vec{i}+\frac{\partial u_y}{\partial z}\vec{j}+\frac{\partial u_z}{\partial z}\vec{k}\quad\text{and}\quad\frac{\partial^2\vec{u}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\vec{u}}{\partial x}\right),\quad\frac{\partial^2\vec{u}}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\vec{u}}{\partial y}\right)$$

Example 7.3: Given $\vec{u}(x,y,z)=(u_x,u_y,u_z)=(xy^2,z,xz^2)$ find $\frac{\partial\vec{u}}{\partial x},\frac{\partial\vec{u}}{\partial y},\frac{\partial\vec{u}}{\partial z}\;\frac{\partial^2\vec{u}}{\partial x\partial y}$

$$\frac{\partial\vec{u}}{\partial x}=\left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial x},\frac{\partial u_z}{\partial x}\right)=\left(y^2,0,z^2\right)\quad ;\quad \frac{\partial\vec{u}}{\partial y}=\left(\frac{\partial u_x}{\partial y},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}\right)=\left(2xy,0,0\right)$$

$$\frac{\partial\vec{u}}{\partial z}=\left(\frac{\partial u_x}{\partial z},\frac{\partial u_y}{\partial z},\frac{\partial u_z}{\partial z}\right)=\left(0,1,2xz\right)\quad ,\quad \frac{\partial^2\vec{u}}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\vec{u}}{\partial y}\right)=\frac{\partial}{\partial x}\left(2xy,0,0\right)=(2y,0,0)$$

TODO: Go to Activity and solve question 12

Video :

PYTHONIC: How to derive a vector in python

Python uses **sympy.physics.vector.diff(derivative_argument,frame) as illustrated below**

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
from sympy.interactive import init_printing
init_printing(order='lex')
x,y=sy.symbols('x y')
# define your reference i(1,0,0) ,j(0,1,0) ,k(0,0,1) in math
e=ReferenceFrame('e')
u=(x*y**2)*e.x + (z)*e.y +(x*z**2)*e.z
print("vector u :")
sy.pprint(u)
print(" first derivative of u with respect to x:")
dudx=u.diff(x,e)
sy.pprint(dudx)
print(" first derivative of u with respect to y:")
dudy=u.diff(y,e)
sy.pprint(dudy)
print(" first derivative of u with respect to z:")
dudz=u.diff(z,e)
sy.pprint(dudz)
print(" second  derivative of u with respect to y and then x:")
ddudxdy=(dudy.diff(x,e))
sy.pprint(ddudxdy)
```

```
vector u :
   2                  2
x⋅y  e_x + z e_y + x⋅z  e_z
 first derivative of u with respect to x:
 2        2
y  e_x + z  e_z
 first derivative of u with respect to y:
2⋅x⋅y e_x
 first derivative of u with respect to z:
e_y + 2⋅x⋅z e_z
 second  derivative of u with respect to y and then x:
2⋅y e_x
```

c. Integration of Vectors

Given a vector $\vec{u}(x,y,z) = (x(t), y(t), z(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, the integration of $\vec{u}$ over $a \le t \le b$ is a vector .

$$\vec{U} = \int_a^b \vec{u}\cdot dt = \int_a^b \left(x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\right)dt = \left(\int_a^b x(t)dt\right)\vec{i} + \left(\int_a^b y(t)dt\right)\vec{j} + \left(\int_a^b z(t)dt\right)\vec{k}\ .$$

Example 7.4: Given $\vec{u} = 3t^2\vec{i} + 2t\vec{j} + 5\vec{k}$ calculate $\vec{U} = \int_1^2 \vec{u}\,\mathrm{dt}$

$$\vec{U} = \int_1^2 (3t^2\vec{i} + 2t\vec{j} + 5\vec{k})\,\mathrm{dt} = \left[\int_1^2 \left(3t^2dt\right)\right]\vec{i} + \left[\int_1^2 \left(2tdt\right)\right]\vec{j} + \left[\int_1^2 (5dt)\right]\vec{k} = \left[t^3\right]_1^2\vec{i} + \left[t^2\right]_1^2\vec{j} + [5t]_1^2\vec{k}$$

$$= \left[(2)^3 - (1)^3\right]\vec{i} + \left[(2)^2 - (1)^2\right]\vec{j} + 5[(2-1)]\vec{k} = 7\vec{i} + 3\vec{j} + 5\vec{k}$$

TODO: Go to Activity and solve question 13

Video :

PYTHONIC: How to integrate a vector

Python uses **sympy.integrate(f,(x, a, b))** to integrate each components of the vector :

```
import sympy as sy
from sympy.physics.vector import ReferenceFrame
sy.init_printing(use_latex=True)
#define the symbole(variable) t
t=sy.Symbol('t')
# define frame of reference
e=ReferenceFrame('e')
# our vector u=(3t^2,t^3,-2t^5)
u=(3*t**2)*e.x +(t**3)*e.y + (-2*t**5)*e.z
print("vector u(t)")
sy.pprint(u)
print("Integral of u from 1 to 2  is:")
# intgerate each components using sympy.integrate( f,(x,a,b))
U=sy.integrate(3*t**2, (t,1,2))*e.x +  sy.integrate(2*t,(t,1,2))*e.y  +  sy.integrate(5,(t,1,2))*e.z
sy.pprint(U)
u
U
```

```
vector u(t)
   2       3         5
3⋅t  e_x + t  e_y + -2⋅t  e_z
Integral of u from 1 to 2  is:
7 e_x + 3 e_y + 5 e_z
```

$$3t^2\hat{\mathbf{e}}_{\mathbf{x}} + t^3\hat{\mathbf{e}}_{\mathbf{y}} - 2t^5\hat{\mathbf{e}}_{\mathbf{z}}$$

$$7\hat{\mathbf{e}}_{\mathbf{x}} + 3\hat{\mathbf{e}}_{\mathbf{y}} + 5\hat{\mathbf{e}}_{\mathbf{z}}$$

8) <u>System of linear Equations and Gaussian Elimination</u>

a) Systems of Linear Equations

A system of linear equations is a list of linear equations with the same unknowns in each equation. It has the following form:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

or in curly bracket

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{cases}$$

where a_{ij} are the coefficient the unknown x_j and b_i is the equation constant

<u>**Example 9.1**</u>:
$$\begin{cases} x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 3 \\ 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & = & 10 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ 4x_1 & + & 6x_2 & + & x_3 & - & x_4 & = & -5 \end{cases}$$

homogeneous system of linear equations

if $b_i = 0$, we have an homogeneous system of linear equations like

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 0 \end{cases}$$

Example 9.2 homogeneous system of linear equations $\begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$

We will sometimes also use x ,y , z, and t as the unknowns in the system.

Example 9.3: $\begin{cases} x + y - z = 5 \\ 2x + 3y + z = 9 \\ x - y + 2z = 0 \end{cases}$ instead of $\begin{cases} x_1 + x_2 - x_3 = 5 \\ 2x_1 + 3x_2 + x_3 = 9 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$

$\begin{cases} x_1 + x_2 - x_3 = 5 \\ 2x_1 + 3x_2 + x_3 = 9 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$ is not homogeneous , $\begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$ is homogeneous.

TODO: Go to Activity and solve question 14

Video :

b) Solving a System of Linear Equation with Gaussian Elimination

Given the equation $\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$, we want to reduce it into a triangular form or echelon form as

$$\begin{array}{ccccccccc} c_{11}x_1 & + & c_{12}x_2 & + & \cdots & + & c_{1n}x_n & = & d_1 \\ & & c_{22}x_2 & + & \cdots & + & c_{2n}x_n & = & d_2 \\ & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & c_{nn}x_n & = & d_n \end{array}$$

Then use a back-substitution to solve the unknowns starting from $x_n = \dfrac{d_n}{c_{nn}}$

In the echelon form equation, the leading unknowns also called pivots are:
x_1 for the equation 1, and the others $x_2, x_3, x_4, \ldots.,$ and x_n are free variables.
x_2 is the pivot for the equation 2, and the others $x_3, x_4, x_5, \ldots.,$ and x_n are free variables.
…..
x_n for the n^{th} equation .

Example 9.4: in $\begin{cases} x_1 - 2x_2 + x_3 - x_4 + x_5 = 8 \\ x_3 - x_4 + 2x_5 = 5 \\ x_4 - x_5 = 1 \end{cases}$

Leading unknowns(pivots): $x_1, x_3,$ and x_4 . free variables : x_2 and x_5.

In $\begin{cases} 2x + y + z + t = 4 \\ y + z - t = 3 \end{cases}$ Leading unknowns(pivots): x and y, free variables : z and t

TODO: Go to Activity and solve question 15

Video :

Now we will explain the Gaussians elimination through examples:

Solving a 2 equations with 2 unknowns:

Solve $\begin{cases} x + 3y = 9 \\ 2x - y = 4 \end{cases}$

Using E1 and E2 to represent equation 1 and 2 $\begin{cases} x + 3y = 9 \ (E_1) \\ 2x - y = 4 \ (E_2) \end{cases}$,

we want to eliminate the unknown x from E2 that is $E_2 \leftarrow -2E_1 + E_2$

$$\begin{array}{l} x + 3y = 9 \\ 2x - y = 4 \end{array} \xrightarrow{-2E_1 + E_2} \begin{array}{l} x + 3y = 9 \\ -7y = -14 \end{array}$$

$E_2 \leftarrow -2E_1 + E_2$ **or** $-2E_1 + E_2 \rightarrow E_2$ **means "replace** E_2 by $-2E_1 + E_2$**"**

That is $E_2 \leftarrow -2E_1 + E_2$: $$\begin{array}{r} -2x - 6y = -18 \\ + \quad 2x - y = 4 \\ \hline -7y = -14 \end{array}$$ ➔ y=2.

Now back substitute y=2 in equation E2 $2x - y = 4$ **gives** $2x - 2 = 4$ ➔ **x=3.** So the final answer **is x=3 and y=2**

<u>Example 9.5</u> :

Solving a 3 equations with 3 unknowns using Gaussian elimination:

Solve $\begin{cases} x+2y-z=2 \\ 2x+y+2z=10 \\ 4x-y+3z=11 \end{cases}$

STEP 1

$$\begin{cases} x+2y-z\ \ =2 & (E_1) \\ 2x+y+2z=10 & (E_2) \\ 4x-y+3z=11 & (E_3) \end{cases} \Rightarrow \begin{matrix} x+2y-z\ \ =2 & (E_1) \\ 2x+y+2z=10 & (E_2) \\ 4x-y+3z=11 & (E_3) \end{matrix} \xrightarrow{-2E_1+E_2\to E_2} \begin{matrix} x+2y-z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ 4x-y+3z=11 & (E_3) \end{matrix}$$

That is $-2E_1+E_2 \to E_2$ is

$$\begin{array}{r} -2x-4y+2z=-4 \\ +\quad 2x+y+2z=10 \\ \hline -3y+4z=6 \end{array}$$

STEP 2

$$\begin{matrix} x+2y-\ z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ 4x-y+3z=11 & (E_3) \end{matrix} \xrightarrow{-4E_1+E_3\to E_3} \begin{matrix} x+2y-\ z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ -9y+7z=3 & (E_3) \end{matrix}$$

The system is not in echelon form yet, so proceed to step 3 to eliminate y from E3 using E2

STEP3

$$\begin{matrix} x+2y-\ z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ -9y+7z=3 & (E_3) \end{matrix} \xrightarrow{-3E_2+E_3\to E_3} \begin{matrix} x+2y-\ z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ -5z=-15 & (E_3) \end{matrix}$$

That is $E_3 \leftarrow -3E_2+E_3$:

$$\begin{array}{r} 9y-12z=-18 \\ -9y+7z\ =\ 3 \\ \hline -5z=-15 \end{array} ,$$

now we have an echelon form as

$$\begin{matrix} x+2y-\ z\ \ =2 & (E_1) \\ -3y+4z=6 & (E_2) \\ -5z=-15 & (E_3) \end{matrix}$$

; using back substitution , we solve for z in $-5z=-15$

➔ z=3 ; substitute z=3 in E2 gives $-3y+4(3)=6$ or y=2;

finally substitute y=2, z=3 in E1 gives

$x+2(2)-(3)=2$ leading to x+1=2 and x=1.

We finally have **x=1,y=2,z=3 as solutions**

TODO: Go to Activity and solve question 16

Video :

<u>Pythonic</u>: How to do it in Python

Python uses sympy.solve(Eq(expression,value) ,[x,y,z]) for this purpose. **Eq** for **Equation, [x,y,z]** the vector solution of the system of equations. See code below**.**

```python
import sympy as sy
from sympy import Eq as Eq
from sympy import symbols
#initialize the pretty printing
from sympy.interactive import init_printing
init_printing(use_latex=True)

#define the symbols(equations unknowns)
x,y,z=symbols("x y z")
# we want to solve x+2y-z=2, 2x+y+2z=10, 4x-y+3z=11
answer=sy.solve([Eq(x+2*y-z,2),Eq(2*x+y+2*z,10),Eq(4*x-y+3*z,11)])
print(answer)
# solve 2x+4y=1, and x-3y=7  without using Eq
answer=sy.solve([2*x+4*y-1, x-3*y-7],[x,y])
print("\n")
sy.pprint(answer)
```

```
{x: 1, y: 2, z: 3}
```

$$\left\{x: \frac{31}{10}, \; y: \frac{-13}{10}\right\}$$

c) <u>Consistence and Inconsistence of System of linear Equations</u>

A system of linear equations is inconsistent if its row echelon form

$$\begin{array}{ccccccccc} c_{11}x_1 & + & c_{12}x_2 & + & \cdots & + & c_{1n}x_n & = & d_1 \\ & & c_{22}x_2 & + & \cdots & + & c_{2n}x_n & = & d_2 \\ & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & c_{nn}x_n & = & d_n \end{array}$$

has no solution, and it is consistent if it has one or more solutions.

A characteristic of inconsistent equation is when one of its rows is of the form $0 \cdot x = d$, *where* $d \neq 0$.

<u>Example 9.6</u>:

$$\begin{cases} x+2y+z=4 \\ \quad y+z=3 \\ \quad 0z=2 \; (inconsistence) \end{cases} \qquad \begin{cases} x+2y+z=4 \\ \quad y+z=3 \\ \quad z=2 \; \; (consistence) \end{cases}$$

$$\begin{cases} x+y+z=4 \\ \quad y+z=3 \\ \quad 0z=3 \; \; (inconsistence) \end{cases}$$

TODO: Go to Activity and solve question 17 and 18

Video :

9) Vector Space and Subspace in $\mathbb{R}^n$

a) Vector Space in $\mathbb{R}^n$

A vector space is a non- empty set V , with the addition (+) and scalar multiplication $(\cdot)$ operations over the field $\mathbb{R}$ satisfying the following properties:

- Closure property of addition : if u and v are in V then $u+v$ is in V
- Commutativity of addition : $u+v=v+u$
- $w+(u+v)=(w+u)+v$
- For u in V there is **0** in V such that $u+0=0+u=u$
- For u in V there is $-u$ in V that such that $u+(-u)=(-u)+u=0$
- Closure property of scalar multiplication : if u is in V then $a\cdot u$ is in V for $a\in\mathbb{R}$
- $a(u+v)=au+av$ for $a\in\mathbb{R}$
- $(a+b)u=au+bu$ for $a\in\mathbb{R}$ and $b\in\mathbb{R}$
- $(ab)u=a(bu)$ for $a\in\mathbb{R}$ and $b\in\mathbb{R}$
- $1\cdot u=u$ for $1\in\mathbb{R}$

b) Subspace in $\mathbb{R}^n$

A subspace W of a vector space V is called a subspace of V if W is itself a vector space under the addition (+) and scalar Multiplication $(\cdot)$ defined in V .

W (non-empty) is a subspace of V if and only if :

1) u and v are in W then $u+v$ is in W **(rule 1)**
2) u is in W then $a\cdot u$ is in W for $a\in\mathbb{R}$ **(rule 2)**

Example 10.1: Is $W = \{\ (2x,3y)\ \ : x \in \mathbb{R}\ and\ \ y \in \mathbb{R}\ \}$ a subspace of $\mathbb{R}^2$?

We show W is not empty since (0,0)=(2*0,3*0) is part of W.

rule 1: let $u \in W$ ➔ $u = (2a_1, 3b_1)$ with $a_1 \in \mathbb{R}, b_1 \in \mathbb{R}$,

$v \in W$ ➔ $v = (2a_2, 3b_2)$ with $a_2 \in \mathbb{R}, b_2 \in \mathbb{R}$ then

$u + v = (2a_1, 3b_1) + (2a_2, 3b_2) = (2a_1 + 2a_2, 3b_1 + 3b_2) = (2(a_1 + a_2), 3(b_1 + b_2))$

So $u + v \in W$ ($u + v$ is part of W) , **rule 1 is satisfied**

rule 2: let $u \in W$ that is $u = (2a_1, 3b_1)$ and $k \in \mathbb{R}$ then

$ku = k(2a_1, 3b_1) = (2ka_1, 3kb_1)$

$ku = (2(ka_1), 3(kb_1)) \in W$, so rule 2 is satisfied .

we conclude W is subspace of $\mathbb{R}^2$

Note : all you need to do, is to verify if the pattern is the same under addition (+) and scalar multiplication ($\bullet$)

Example 10.2: Is $W = \{\ a + bx + cx^2 : x \in \mathbb{R}\ \}$ a subspace of $\mathrm{P}_2(x)$ (2^{nd} order polynomial) ?

W is not empty since $0 = 0 + 0x + 0x^2 \in \mathrm{P}_2(x)$

Let $p \in W$ ➔ $p(x) = a_1 + b_1x + c_1x^2$ and let $q \in W$ ➔ $q(x) = a_2 + b_2x + c_2x^2$

$p(x) + q(x) = (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) = a_1 + a_2 + (b_1 + b_2)x + (c_1 + c_2)x^2$

This implies that $p(x) + q(x) \in W$ **, rule 1 is satisfied**

$k \cdot p(x) = k(a_1 + b_1x + c_1x^2) = ka_2 + (kb_2)x + (kc_2)x^2 \ \in W$

Rule 2 is satisfied therefore $W\ is\ subspace\ of\ P_2(x)$

Example10.3: Is $W = \{\ (2x,1)\ \ : x \in \mathbb{R}\ and\ \ y \in \mathbb{R}\ \}$ a subspace of $\mathbb{R}^2$

let $u \in W$ ➔ $u = (2a_1, 1)$, $v \in W$ ➔ $v = (2a_2, 1)$ then

$u + v = (2a_1, 1) + (2a_2, 1) = (2a_1 + 2a_2, 2)$

$= (2(a_1 + a_2), 2) \notin W$

So $u + v \notin W$ ($u + v$ is not part of W) , **rule 1 is not satisfied**

$W\ is\ not\ a\ subspace\ of\ \mathbb{R}^2$

d)Linear Combination and spanning set

Let V be a vector space over $\mathbb{R}$,denoted $\mathrm{V}(\mathbb{R})$.
A vector $\vec{v}$ in V is a linear combination of $\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n$ if there exist some scalars $c_1,c_2,c_3,.....,c_n$ in $\mathbb{R}$ such that
$\vec{v}=c_1\vec{u}_1+c_2\vec{u}_2+c_3\vec{u}_3+....+c_n\vec{u}_n$.
$\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n$ is a span of V if every vector $\vec{w}$ of V can be expressed as a linear combination of $\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n$, that is
$\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2+c_3\vec{u}_3+....+c_n\vec{u}_n$:
We say that $\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}$ is a spanning set of V.

<u>Example 10.3</u>: Express $\vec{a}=(8,13)\in\mathbb{R}^2$ as a linear combination of $\vec{b}=(1,2)$ and $\vec{c}=(2,3)$

We need to find x *and* y such that $\vec{a}=x\vec{b}+y\vec{c}$ ➔ $(8,13)=x(1,2)+y(2,3)$

$(8,13)=(x,2x)+(2y,3y)=(x+2y,2x+3y)$ ➔ $\begin{cases}x+2y=8 & (E_1)\\ 2x+3y=13 & (E_2)\end{cases}$

We solve the system using Gaussian elimination,

$$\begin{cases}x+2y=8\\ 2x+3y=13\end{cases}\xrightarrow{-2E_1+E_2\to E_2}\begin{cases}x+2y=8\\ \quad -y=-3\end{cases}$$ ➔

So $y=3$ in $x+2y=8$ gives $x+2(3)=8$, that is $x=2$ so
$\vec{a}=2\vec{b}+3\vec{c}$ that is $(8,13)=2(1,2)+3(2,3)$
So $\vec{b}=(1,2)$ and $\vec{c}=(2,3)$ **span** $\mathbb{R}^2$ **or** $Span(\mathbb{R}^2)=\{\vec{a},\vec{b}\}$

<u>Example 10.4</u> :

Express $\vec{w}=(3,-5,0)$ as a linear combination of
$\vec{a}=(1,2,1)$ $\vec{b}=(2,3,1)$ *and* $\vec{c}=(4,1,1)$
we want to find x , y *and* z such that $\vec{w}=x\vec{a}+y\vec{b}+z\vec{c}$.

$\vec{w}=x\vec{a}+y\vec{b}+z\vec{c}$ ➔ $(3,-5,0)=x(1,2,1)+y(2,3,1)+z(4,1,1)$ ➔

$(x,2x,x)+(2y,3y,y)+(4z,z,z)=(3,-5,0)$ ➔ $\begin{cases}x+2y+4z=3 & (E_1)\\ 2x+3y+z=-5 & (E_2)\\ x+y+z=0 & (E_3)\end{cases}$

Solve using Gaussian elimination ,

$$\begin{cases} x+2y+4z=3 & (E_1) \\ 2x+3y+z=-5 & (E_2) \\ x+\ \ y+z=0 & (E_3) \end{cases} \begin{matrix} \\ \xrightarrow{-2E_1+E_2\to E_2} \\ \xrightarrow{-E_1+E_3\to E_3} \end{matrix} \begin{cases} x+2y+4z=3 & (E_1) \\ \quad -y-7z=-11 & (E_2) \\ \quad -y-3z=-3 & (E_3) \end{cases}$$

$$\begin{cases} x+2y+4z=3 & (E_1) \\ \quad -y-7z=-11 & (E_2) \\ \quad -y-3z=-3 & (E_3) \end{cases} \begin{matrix} \\ \\ \xrightarrow{-E_2+E_3\to E_3} \end{matrix} \begin{cases} x+2y+4z=3 & (E_1) \to x=1 \\ \quad -y-7z=-11 & (E_2) \to y=-3 \\ \quad\quad 4z=8 & (E_3) \to z=2 \end{cases} \Rightarrow$$

z=2,y=-3 ,x=1

So $\vec{w}=\vec{a}-3\vec{b}+2\vec{c}$.

We also conclude that $\vec{a}=(1,2,1)$, $\vec{b}=(2,3,1)$ *and* $\vec{c}=(4,1,1)$ span $\mathbb{R}^3$

or $Span(\mathbb{R}^3)=\{\vec{a}\ ,\vec{b}\ ,\vec{c}\ \}$

Example 10.5 : Express the polynomial $p=-9+9x+2x^2$ as a linear combination of $p_1=1+2x+x^2$, $p_2=2-2x+3x^2$ and $p_3=-3+x+x^2$

We want to find a,b *and* c such that $p=ap_1+bp_2+cp_3$.

This will lead to :

$$-9+9x+2x^2=a+2ax+ax^2+2b-2bx+3bx^2-3c+cx+cx^2 \Rightarrow$$

$$-9+9x+2x^2=a+2b-3c+2ax-2bx+cx+ax^2+3bx^2+cx^2 \Rightarrow$$

$$-9+9x+2x^2=a+2b-3c+(2a-2b+c)x+(a+3b+c)x^2 \Rightarrow$$

$$\begin{cases} a+2b-3c=-9 & (E_1) \\ 2a-2b+c=9 & (E_2) \\ a+3b+c=2 & (E_3) \end{cases}$$ Solve using Gaussian elimination,

$$\begin{cases} a+2b-3c=-9 & (E_1) \\ 2a-2b+c=9 & (E_2) \\ a+3b+c=2 & (E_3) \end{cases} \begin{matrix} \\ \xrightarrow{-2E_1+E_2\to E_2} \\ \xrightarrow{-E_1+E_3\to E_3} \end{matrix} \begin{cases} a+2b-3c=-9 & (E_1) \\ \quad -6b+7c=27 & (E_2) \\ \quad\quad b+4c=11 & (E_3) \end{cases} \Rightarrow$$

$$\begin{cases} a+2b-3c=-9 & (E_1) \\ \quad -6b+7c=27 & (E_2) \\ \quad\quad b+4c=11 & (E_3) \end{cases} \begin{matrix} \\ \\ \xrightarrow{E_2+6E_3\to E_3} \end{matrix} \begin{cases} a+2b-3c=-9 & (E_1) \to a=2 \\ \quad -6b+7c=27 & (E_2) \to b=-1 \\ \quad\quad 31c=93 & (E_3) \to c=3 \end{cases}$$

So a=2 , b=-1 and c=3 we finally have $p=2p_1-p_2+3p_3$

TODO: Go to Activity and solve question 19

Video :

e) **Linear independence**

A non-empty set $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}$ in vector space V is **linearly independent** if and only if the equation $c_1\vec{u}_1+c_2\vec{u}_2+c_3\vec{u}_3+....+c_n\vec{u}_n=\vec{o}$ has only solution $c_1=c_2=c_3=.....=c_n=0$ (all zero coefficients)

Example 10.6: show that $\vec{u}_1=(2,3)$ and $\vec{u}_2=(4,0)$ are linear independent

We do $c_1\vec{u}_1+c_2\vec{u}_2=\vec{o}$ ➔ $c_1(2,3)+c_2(4,0)=\vec{o}$ ➔ $(2c_1,3c_1)+(4c_2,0)=(0,0)$

➔ $(2c_1+4c_2,3c_1)=(0,0)$ ➔ $\begin{cases}2c_1+4c_2=0\\3c_1=0\end{cases}$ ➔ $c_1=0 \;\; and \;\; c_2=0$

Since $c_1=c_2=0$, we conclude that $\vec{u}_1=(2,3)$ and $\vec{u}_2=(4,0)$ are linearly independent.

Example10.7: show that $s=\{p_0,p_1,p_2\}=\{1,x,x^2\}$ are linear independent

Set $ap_0+bp_1+cp_2=0$ ➔ $a1+bx+cx^2=0$ or $a+bx+cx^2=0$ ➔ $a=b=c=0$

$a=b=c=0$ implies that $\{p_0,p_1,p_2\}=\{1,x,x^2\}$ are linear independent

TODO: Go to Activity and solve question 20

Video :

Example 10.8: Show that $p_1=t+1$, $p_2=t-1$, $p_3=t^2-2t+1$ are linear independent.

Set $ap_1+bp_2+cp_3=0$ ➔ $a(t+1)+b(t-1)+c(t^2-2t+1)=0$ ➔

$at+a+bt-b+ct^2-2ct+c=0$ ➔ $ct^2+(a+b-2c)t+a-b+c=0$ ➔

$\begin{cases}c=0\\a+b-2c=0\\a-b+c=0\end{cases}$ ➔ $\begin{cases}c=0\\a+b=0\\a-b=0\end{cases}$ ➔ $a=b=c=0$. Therefore p_1 , p_2 *and* p_3 are linear independent.

f) **Linear dependence**

$S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}$ is said to be **linearly dependent** if there exist some scalars $c_1,c_2,c_3,.....,c_n$ **not all zero** such that $c_1\vec{u}_1+c_2\vec{u}_2+c_3\vec{u}_3+....+c_n\vec{u}_n=\vec{o}$

Example10.9: : if $\vec{u}_1=(3,2,3)$,$\vec{u}_2=(2,4,1)$, and $\vec{u}_3=(0,-8,-3)$ then the vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3$ are linear dependent since $2\vec{u}_1-3\vec{u}_2+\vec{u}_3=\vec{o}$,
that is $2(3,2,3)-3(2,4,1)+(0,8,-3)=(0,0,0)$

Example 10.10: Show $\vec{u}_1=(3,2)$ *and* $\vec{u}_2=(15,10)$ are linear dependent
$\vec{u}_2=(15,10)=5(3,2)=5\vec{u}_1$, since $\vec{u}_2=5\vec{u}_1$ ➔ $\vec{u}_1$ *and* $\vec{u}_2$ are linear dependent.

TODO: Go to Activity and solve question 21

Video :

g)Basis of a Vector Space

A set $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}$ of vectors is a basis of the vector space V if :

- S is linearly independent
- $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,.......,\vec{u}_n\}$ spans V

Example 10.11: show that $\vec{i}=(1,0,0),\vec{j}=(0,1,0)$ *and* $\vec{k}=(0,0,1)$ form a basis for $\mathbb{R}^3$

Show that they are linearly independent: $x,y,z\in\mathbb{R}$ such that
$x\vec{i}+y\vec{j}+z\vec{k}=\vec{0}$ ➔ $x=y=z=0$.
$x(1,0,0)+y(0,1,0)+z(0,0,1)=(0,0,0)\Rightarrow(x,y,z)=(0,0,0)$

So $\vec{i},\vec{j}$ *and* $\vec{k}$ are linearly independent

Show that the set of vector $\{\vec{i},\vec{j},\vec{k}\}$ spans $\mathbb{R}^3$,
that is any arbitrary vector $\vec{v}=(a,b,c)$ in $\mathbb{R}^3$ can be expressed as a linear combination of $\vec{i},\vec{j}$ *and* $\vec{k}$.

That is to say we can find some $x,y,z\in\mathbb{R}$ such that :
$x\vec{i}+y\vec{j}+z\vec{k}=\vec{v}$ ➔ $x(1,0,0)+y(0,1,0)+z(0,0,1)=(a,b,c)\Rightarrow(x,y,z)=(a,b,c)$
or $x=a,\ y=b,\ z=c$; verify : $\vec{v}=a\vec{i}+b\vec{j}+c\vec{k}$

$\vec{i},\vec{j}$ *and* $\vec{k}$ are linearly independent and span $\mathbb{R}^3$, therefore they form a basis for $\mathbb{R}^3$.

TODO: Go to Activity and solve question 22

Video :

h)Dimension of a Vector Space

The dimension of a vector space V, denoted by ***dim(V)***, is the number of vectors in the basis of V.

Example 10.12 :

dim(R²)=2 , dim(R³)=3 , dim(R⁴)=4 , ..., ***dim(Rⁿ)=n***

If $E=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{u}_4,\vec{u}_5\}$ dim(E) = 5.

dim(Pₙ)=n+1, standard basis of polynomial $p_n(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

If $p(x)=3+5x+x^2+7x^4$ then dim(P)=4+1=5

TODO: Go to Activity and solve question 23

Video :

10) Inner Product Space

An inner product is the generalization of the dot product .
When used in vector space, it represents the vector outer product with the result as a scalar .

For a real vector space ($\mathbb{R}^n$), an inner product $\langle\cdot,\cdot\rangle$ satisfies the following conditions:

Let $\vec{u},\vec{v}$ and $\vec{w}$ and a scalar $k\in\mathbb{R}$ then

- $\langle\vec{u},\vec{v}\rangle=\langle\vec{v},\vec{u}\rangle$ symmetry
- $\langle\vec{u},\vec{u}\rangle\geq 0$ positive definite property
- $\langle\vec{w},\vec{u}+\vec{v}\rangle=\langle\vec{w},\vec{u}\rangle+\langle\vec{w},\vec{v}\rangle$
- $\langle\vec{u},k\vec{v}\rangle=\langle k\vec{u},\vec{v}\rangle=k\langle\vec{u},\vec{v}\rangle$ the last 2 properties combine to the linear property.

Note that the norm of a vector $\vec{v}$ **is** $\|\vec{v}\|=\sqrt{\langle\vec{v},\vec{v}\rangle}$ and $\|\vec{v}\|^2=\langle\vec{v},\vec{v}\rangle$

The normalized vector of $\vec{v}$ **is** $\hat{v}=\dfrac{\vec{v}}{\|\vec{v}\|}=\dfrac{\vec{v}}{\sqrt{\langle\vec{v},\vec{v}\rangle}}$

Examples of Inner Product Spaces:

1. The Real Number $\mathbb{R}$

   $\langle x,y\rangle=xy$ where $x\in\mathbb{R}, y\in\mathbb{R}$

   Example 11.1:

   $\langle 4,3\rangle=(4)(3)=12$, $\langle xy,x\rangle=(xy)(x)=x^2y$

2. Euclidian n-Space $\mathbb{R}^3$

   Given $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$

   $\langle\vec{a},\vec{b}\rangle=\vec{a}\cdot\vec{b}=(a_1,a_2,a_3)\cdot(b_1,b_2,b_3)=a_1\cdot b_1+a_2\cdot b_2+a_3\cdot b_3$

$$\left\langle \vec{a},\vec{b} \right\rangle = \vec{a}^T\vec{b} = (a_1\ a_2\ a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \quad \text{if} \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Example 11.2: Given $\vec{a} = (1,0,2)$, $\vec{b} = (3,1,-1)$, $\vec{c} = (1,-1,1)$

$$\left\langle \vec{a},\vec{c} \right\rangle = (1,0,2) \bullet (1,-1,1) = (1)(1) + (0)(-1) + (2)(1) = 1 + 2 = 3$$

$$\left\langle \vec{b},\vec{c} \right\rangle = (3,1,-1) \bullet (1,-1,1) = (3)(1) + (1)(-1) + (-1)(1) = 3 - 2 = 1$$

$$\left\langle 2\vec{a} + 3\vec{b},\vec{c} \right\rangle = \left\langle 2\vec{a},\vec{c} \right\rangle + \left\langle 3\vec{b},\vec{c} \right\rangle = 2\left\langle \vec{a},\vec{c} \right\rangle + 3\left\langle \vec{b},\vec{c} \right\rangle = (2)(3) + 3(1) = 9$$

$$\|\vec{c}\| = \sqrt{\langle \vec{c},\vec{c} \rangle} = \sqrt{(1,-1,1)\cdot(1,-1,1)} = \sqrt{1+1+1} = \sqrt{3}$$

$$\hat{c} = \frac{\vec{c}}{\|\vec{c}\|} = \frac{\vec{c}}{\sqrt{\langle \vec{c},\vec{c} \rangle}} = \frac{(1,-1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

TODO: Go to Activity and solve question 24.1

Video :

3. Function Space on a closed interval $[a,b]$ ($C[a,b]$ in $\mathbb{R}$)

Given two functions $f(x)$ *and* $g(x)$ *continuous on* $[a,b]$, their inner product on $C[a,b]$ can be defined as

$$\langle f,g \rangle = \int_a^b f(x)g(x)dx .$$

$f(x)$ *and* $g(x)$ *are said to be orthognal on* $[a,b]$ if $\langle f,g \rangle = \int_a^b f(x)g(x)dx = 0$

$$\|f\|^2 = \langle f,f \rangle = \int_a^b f(x)f(x)dx = \int_a^b (f(x))^2\, dx$$

$\|f\| = \sqrt{\langle f,f \rangle} = \sqrt{\int_a^b [f(x)]^2\, dx}$ and normalized f is $\hat{f} = \dfrac{f}{\|f\|} = \dfrac{f}{\sqrt{\langle f,f \rangle}}$

Example 11.3

given $f(x) = 3x$ *and* $g(x) = 4x^2$ **with inner product** $\langle f,g \rangle = \int_{-1}^1 f(x)g(x)dx$

Find $\langle f,g \rangle$, $\|f\|$ and compute normalized f that is $\hat{f}$

Answer:

$$\langle f,g \rangle = \int_{-1}^1 f(x)g(x)dx = \int_{-1}^1 (3x)(4x^2)dx = \int_{-1}^1 12x^3 dx = 3\left[x^4\right]_{-1}^1 = 0$$

So $f(x)$ *and* $g(x)$ *are orthognal on* $[-1,1]$.

$$\|f\|^2 = \langle f,f \rangle = \langle 3x,3x \rangle = \int_{-1}^1 9x^2 dx = \left[\frac{9x^3}{3}\right]_{-1}^1 = 3\left[x^3\right]_{-1}^1 = 3\left[(1)^3 - (-1)^3\right] = 6$$

Then $\|f\| = \sqrt{6}$ and $\hat{f} = \frac{f}{\|f\|} = \frac{f}{\sqrt{\langle f, f \rangle}} = \frac{3x}{\sqrt{6}}$

Example 11.4

given $f(x) = 2x \ and \ g(x) = 3$ **with inner product** $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$

Find $\langle f, g \rangle$, $\|f\|$ and compute normalized f that is $\hat{f}$

Answer:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx = \int_0^1 (2x)(3)dx = \int_0^1 6xdx = 3\left[x^2\right]_0^1 = 3$$

$$\|f\|^2 = \langle f, f \rangle = \langle 2x, 2x \rangle = \int_0^1 4x^2 dx = \left[\frac{4x^3}{3}\right]_0^1 = \frac{4}{3}\left[x^3\right]_0^1 = \frac{4}{3}$$

Then $\|f\| = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$ and $\hat{f} = \frac{f}{\|f\|} = \frac{f}{\sqrt{\langle f, f \rangle}} = \frac{2x}{\frac{2}{\sqrt{3}}} = \frac{2\sqrt{3}}{2}x = x\sqrt{3}$

TODO: Go to Activity and solve question 24.2

Video :

4. Polynomial Space P_n

If $p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and $p = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$ then an inner on P_n can be defined as

$$\langle p, q \rangle = \langle (a_0, a_1, \cdots, a_n), (b_0, b_1, \cdots, b_n) \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

Example 11.5

Given $p = 2 + 3x + x^2 + 5x^3$ and $q = 1 + 6x - 3x^2 + x^3$ compute $\langle p, q \rangle$

$$\langle p, q \rangle = \langle (2,3,1,5), (1,6,-3,1) \rangle = 2*1 + 3*6 + 1*(-3) + 5*1 = 2 + 18 - 3 + 5 = 22$$

5. Matrix Space $M_{n \times m}$

Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ are two matrices in $M_{2\times 2}$, then an inner product on $M_{2\times 2}$ can be defined as

$$\langle A, B \rangle = trace\left(A^T \cdot B\right) = a_{11}b_{11} + a_{21}b_{12} + a_{12}b_{21} + a_{22}b_{22}$$

Note that $\langle A, B \rangle = \langle B, A \rangle = trace\left(B^T \cdot A\right)$

Example 11.6

Given $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}5&1\\0&3\end{pmatrix}$

$\langle A,B\rangle=(1)(5)+(2)(1)+(3)(0)+(4)(3)=19$

6.

Theorem (Cauchy-Schwarz inequality):

If $\vec{u}$ and $\vec{v}$ are vectors in real inner product space , then $|\langle\vec{u},\vec{v}\rangle|\le\|\vec{u}\|\cdot\|\vec{v}\|$

11) Orthonormal Basis and Gram-Schmidt Process (optional)

1. Orthogonal and orthonormal Bases

A set of vectors $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ in an inner product space is called an orthogonal set if $\vec{u}_1\perp\vec{u}_2\perp\vec{u}_3$ that is $\langle\vec{u}_1,\vec{u}_2\rangle=\langle\vec{u}_1,\vec{u}_3\rangle=\langle\vec{u}_2,\vec{u}_3\rangle=0$.

Example 11.1: given $\vec{u}_1=(0,1,0),\vec{u}_2=(1,0,1)$, *and* $\vec{u}_3=(1,0,-1)$

$\langle\vec{u}_1,\vec{u}_2\rangle=(0,1,0)\bullet(1,0,1)=0$, $\langle\vec{u}_1,\vec{u}_3\rangle=(0,1,0)\bullet(1,0,-1)=0$, $\langle\vec{u}_2,\vec{u}_3\rangle=(1,0,1)\bullet(1,0,-1)=0$.

$\langle\vec{u}_1,\vec{u}_2\rangle=\langle\vec{u}_1,\vec{u}_3\rangle=\langle\vec{u}_2,\vec{u}_3\rangle=0$ ➔ $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ orthogonal set.

A set of vectors $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ in an inner product space is called an orthonormal set if $\vec{u}_1\perp\vec{u}_2\perp\vec{u}_3$, that is $\langle\vec{u}_1,\vec{u}_2\rangle=\langle\vec{u}_1,\vec{u}_3\rangle=\langle\vec{u}_2,\vec{u}_3\rangle=0$, and $\|\vec{u}_1\|=\|\vec{u}_2\|=\|\vec{u}_3\|=1$.

Example 11.2: given $\vec{u}_1=(0,1,0),\vec{u}_2=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$, *and* $\vec{u}_3=\left(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)$

We can see that $\langle\vec{u}_1,\vec{u}_2\rangle=\langle\vec{u}_1,\vec{u}_3\rangle=\langle\vec{u}_2,\vec{u}_3\rangle=0$ and $\|\vec{u}_1\|=\|\vec{u}_2\|=\|\vec{u}_3\|=1$

$\|\vec{u}_3\|=\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+0^2+\left(\frac{-1}{\sqrt{2}}\right)^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1$ so $S=\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ is an orthonormal set.

2. Coordinates relative to orthonormal bases.

Theorem 11.1: If $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an orthonormal basis for an inner product space V and $\vec{u}$ is any vector in V , then the coordinate of $\vec{u}$ relative to S is $[\vec{u}]_S=\langle\vec{u},\vec{v}_1\rangle\vec{v}_1+\langle\vec{u},\vec{v}_2\rangle\vec{v}_2+\langle\vec{u},\vec{v}_3\rangle\vec{v}_3$.

Example 11.3: given $\vec{v}_1=(0,1,0),\ \vec{v}_2=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\ and\ \vec{v}_3=\left(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)$,

What are the coordinate of $\vec{u}=(1,1,1)$ relative to $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$

Answer :

$$\langle\vec{u},\vec{v}_1\rangle=(1,1,1)\bullet(0,1,0)=1$$

$$\langle\vec{u},\vec{v}_2\rangle=(1,1,1)\bullet\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

$$\langle\vec{u},\vec{v}_3\rangle=(1,1,1)\bullet\left(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$[\vec{u}]_S=\langle\vec{u},\vec{v}_1\rangle\vec{v}_1+\langle\vec{u},\vec{v}_2\rangle\vec{v}_2+\langle\vec{u},\vec{v}_3\rangle\vec{v}_3=1\cdot\vec{v}_1+\sqrt{2}\cdot\vec{v}_2+0.\vec{v}_3.$$

So the coordinate of $\vec{u}=(1,1,1)$ with respect to S is $[\vec{u}]_S=\left(1,\sqrt{2},0\right)$

3. Coordinates relatives to orthogonal bases.

Theorem 11.2: If $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an orthogonal basis for an inner product space V. and $\vec{u}$ is any vector in V, then the coordinate of $\vec{u}$ relative to S is $[\vec{u}]_S=\frac{\langle\vec{u},\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1+\frac{\langle\vec{u},\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2+\frac{\langle\vec{u},\vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$.

Example 11.3: given $\vec{v}_1=(3,4)\ and\ \vec{v}_2=(-4,3)$,

What are the coordinate of $\vec{u}=(1,2)$ relative to $S=\{\vec{v}_1,\vec{v}_3\}$

Answer :

$$\langle\vec{u},\vec{v}_1\rangle=(1,2)\cdot(3,4)=3+8=11$$

$$\langle\vec{u},\vec{v}_2\rangle=(1,2)\bullet(-4,3)=-4+6=2$$

$\|\vec{v}_1\|=\|\vec{v}_2\|=5$, means that S is not an orthonormal set but orthogonal , so

$$[\vec{u}]_S=\frac{\langle\vec{u},\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1+\frac{\langle\vec{u},\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{11}{25}\vec{v}_1+\frac{2}{25}\vec{v}_2.$$

So the coordinate of $\vec{u}=(1,2)$ with respect to S is $[\vec{u}]_S=\left(\frac{11}{25},\frac{2}{25}\right)$

4. Orthogonal projection

Let W be a subspace of a finite real inner product space V ($\mathbb{R}^3$).
If $\vec{v}_1, \vec{v}_2, and\ \vec{v}_3$ form an orthonormal basis for W, and $\vec{u}$ is any vector in V, then the orthogonal projection of $\vec{u}$ on W is $proj_W \vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{u}, \vec{v}_3 \rangle \vec{v}_3$
if $\vec{v}_1, \vec{v}_2, and\ \vec{v}_3$ form an orthogonal basis for W, and $\vec{u}$ is any vector in V, then

$$proj_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|} \vec{v}_2 + \frac{\langle \vec{u}, \vec{v}_3 \rangle}{\|\vec{v}_3\|} \vec{v}_3$$

Example 11.4: Let W be a subspace of a finite inner product space $\mathbb{R}^2$
If W is spanned by the orthogonal vectors $\vec{v}_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$ and $\vec{v}_2 = \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \right)$,

Then calculate $proj_W \vec{u}$ where $\vec{u} = (1,2)$

Answer: we use $proj_W \vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2$ since $\vec{v}_1$ and $\vec{v}_2$ are orthonormal vectors.

$$proj_W \vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 = proj_W \vec{u} = \frac{3\sqrt{2}}{2} \vec{v}_1 - \frac{\sqrt{2}}{2} \vec{v}_2$$

$$proj_W \vec{u} = \frac{3\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) - \frac{\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \right) = \left(\frac{6}{2}, \frac{6}{2} \right) + \left(\frac{-2}{2}, \frac{2}{2} \right) = (2,4)$$

5. Finding orthogonal and orthonormal bases (Gram-Schmidt process)

Let's consider an arbitrary basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, we want to construct an orthonormalized basis $B' = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ from using the Gram-Schmidt orthonornalization algorithm.

We proceed as follows:

Step1 : set $\vec{e}_1 = \vec{v}_1$ then $B' = \{\vec{e}_1\}$

Step2 : compute $\vec{e}_2 = \vec{v}_2 - \mathrm{Proj}_{B'} \vec{v}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \cdot \vec{e}_1$ $\quad B' = \{\vec{e}_1, \vec{e}_2\}$

Step3 : $\vec{e}_3 = \vec{v}_3 - \mathrm{Proj}_{B'} \vec{v}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \cdot \vec{e}_1 - \frac{\langle \vec{v}_3, \vec{e}_2 \rangle}{\|\vec{e}_2\|^2} \cdot \vec{e}_2$

We finally have $B' = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ that is now an orthogonal frame.
By normalizing $\vec{e}_1, \vec{e}_2, and\ \vec{e}_3$, we get $\hat{e}_1, \hat{e}_2,\ and\ \hat{e}_3$, we finally obtain $B' = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ that is an orthonormalized basis.

Example 11.5: use the Gram-Schmidt algorithm to orthonormalize the basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (1,0,1)$ and $\vec{v}_3 = (0,1,1)$

Answer:

Step 1: $\vec{e}_1 = \vec{v}_1 = (1,1,0)$ ➔ , $B' = \{\vec{e}_1\} = \{(1,1,0)\}$

Step 2 : $\vec{e}_2 = \vec{v}_2 - \mathrm{Proj}_{B'}^{\vec{v}_2} = \vec{v}_2 - \dfrac{\langle \vec{v}_2, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \cdot \vec{e}_1 = (1,0,1) - \dfrac{1}{2}(1,1,0) = \left(\dfrac{1}{2}, -\dfrac{1}{2}, 1\right)$

$$B' = \{\vec{e}_1, \vec{e}_2\} = \left\{(1,1,0), \left(\frac{1}{2}, -\frac{1}{2}, 1\right)\right\}$$

Step 3: $\vec{e}_3 = \vec{v}_3 - \mathrm{Proj}_{B'}^{\vec{v}_3} = \vec{v}_3 - \dfrac{\langle \vec{v}_3, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2} \cdot \vec{e}_1 - \dfrac{\langle \vec{v}_3, \vec{e}_2 \rangle}{\|\vec{e}_2\|^2} \cdot \vec{e}_2 = (0,1,1) - \dfrac{1}{2}(1,1,0) - \dfrac{\frac{1}{2}}{\frac{3}{2}}\left(\dfrac{1}{2}, -\dfrac{1}{2}, 1\right)$

$$\vec{e}_3 = (0,1,1) - \frac{1}{2}(1,1,0) - \frac{1}{3}\left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Now we normalize the 3 vectors to get :

$$\hat{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \hat{e}_2 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \quad \hat{e}_3 = \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

$B' = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

12) ☺