Linear Algebra Linear Transformations (Practice)

Linear Transformation

1) Is $T:\mathbb{R}\to\mathbb{R}$ *, such that* $T(x)=5x$ a linear transformation ?
2) Is $T:\mathbb{R}^2\to\mathbb{R}$ *, such that* $T(x,y)=xy$ a linear transformation ?
3) Is $T:\mathbb{R}\to\mathbb{R}$ *, such that* $T(x)=x^3$ a linear transformation ?
4) Is $T:\mathbb{R}\to\mathbb{R}$ *, such that* $T(x)=0$ a linear transformation ?
5) Is $T:\mathbb{R}^2\to\mathbb{R}$ *, such that* $T(x,y)=x+y$ a linear transformation ?
6) Is $T:\mathbb{R}^2\to\mathbb{R}^2$ *, such that* $T(\vec{x})=A\vec{x}$ a linear transformation ?

   A *is* $n\times n$ *matrix and* $\vec{x}$ *is a* $n\times 1$ *column vector*

Linear Transformation matrix from standard and non-standard basis

Standard Matrix from Standard basis:

Find the standard matrix of the following linear transformation s

7) $T:\mathbb{R}^2\to\mathbb{R}^2$ *, such that* $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x-5y\\x+6y\end{pmatrix}$
8) $T:\mathbb{R}^2\to\mathbb{R}^2$ *, such that* $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10x-5y\\y\end{pmatrix}$
9) $T:\mathbb{R}^3\to\mathbb{R}^3$ *, such that* $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x-5y+z\\x+6y-z\\x+y+z\end{pmatrix}$
10) $T:\mathbb{R}^3\to\mathbb{R}^3$ *, such that* $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+z\\x+y-z\\x+y+z\end{pmatrix}$

Matrix from Non-Standard Basis:

11) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ such $T(x,\ y)=(x+y,\ x-2y)$.

    a) Find the matrix of T relative to the basis $B_v=\{\vec{v}_1,\vec{v}_2\}=\{(2,1),(3,2)\}$ and $B_w=\{\vec{w}_1,\vec{w}_2\}=\{(1,1),(1,2)\}$

    b) calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V}=\begin{pmatrix}1\\-1\end{pmatrix}$

12) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \end{pmatrix}$.

a) Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ and

$B_w = \{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$

b) calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

13) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$.

a) Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ and

$B_w = \{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

b) calculate $[\vec{u}]_{B_W}$ if $[\vec{u}]_{B_V} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

14) Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ such $T(x, y) = (x, x+y, y)$.

Find the matrix of T relative to the basis $B_v = \{\vec{v}_1, \vec{v}_2\} = \{(1,2),(1,1)\}$ for $\mathbb{R}^2$ and $B_w = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,2,1),(0,1,0),(2,0,3)\}$ for $\mathbb{R}^3$.

Kernel of linear transformation

15) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T(x, y) = (x-y,\ 2x+y)$

Find Ker(T) and dim (Ker T)

16) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T(x, y) = (x+y,\ x)$

Find Ker(T) and dim (Ker T)

17) Let $T:\mathbb{R}^2 \to \mathbb{R}$ such $T(x, y) = x-y$

Find Ker(T) and dim (Ker T)

18) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T(x, y, z) = (x-y+z,\ x+4y+1,\ y)$

Find Ker(T) and dim (Ker T)

Range or image of a linear transformation

19) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ -6x+3y \end{pmatrix}$

Find Im(T) and dim (Im T)

20) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x+6y \end{pmatrix}$

Find Im(T) and dim (Im T)

21) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ x+4y \end{pmatrix}$

Find Im(T) and dim (Im T)

22) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2y+z \\ x+4y+z \\ x+3y+z \end{pmatrix}$

Find Im(T) and dim (Im T)

2D/3D Geometric Transformation

23) Given the point $\vec{p}$ (1,1,-1) .
   a) Calculate the image of $\vec{p}$ after a translation by a vector $\vec{v} = (3,2,4)$.
   b) Find the rotation of 45º about the Y-axis $R_Y\left(\frac{\pi}{4}\right)$, and its inverse $R_Y^{-1}\left(\frac{\pi}{4}\right)$
   c) Calculate the image of $\vec{p}$ after a 45º rotation about the Y-axis
   d) Calculate the image of $\vec{p}$ after a scaling transform where $s_x = 2, s_y = 10, s_z = 5$

24) Find the transformation that represents a rotation of an object by 30º about the origin . What are the new coordinates of the point $\vec{p}$ (2,-4) after the rotation ?

25) Write the transformation that rotates an object 60 degrees about a fixed center of rotation $\vec{p}$ (-1,2) .

26) Perform a 45º rotation of triangle $\triangle abc$ where $\vec{a}$=(0,0) , $\vec{b}$=(1,1) , $\vec{c}$=(5,2)
   a) About the origin.
   b) About $\vec{b}$ (1,1).

27) Perform a 45º rotation of triangle $\triangle abc$ where $\vec{a}$=(1,0,2) , $\vec{b}$=(-1,3,1) , $\vec{c}$=(5,2-1)
   a) About the z-axis.
   b) About the z-axis by keeping $\vec{b}$ (-1,3,1) fixed.

28) Find the transformation that scales (with respect to the origin) by

a) 3 units in the x-direction

b) 4 units in the y-direction

c) simultaneously 3 units in the x-direction and 4 units the y-direction .

29) Write the transform for scaling with respect to a fixed point $\vec{p}$ (1,-1).

30) Magnify the triangle with vertices $\vec{a}$ (0,0) , $\vec{b}$ (1,1), and $\vec{c}$ (5,2) to twice its size by keeping $\vec{c}$ (5,2) fixed

31)

32) Calculate the image of $\vec{p}$ (1,0,1) after a 45º rotation about the Z-axis followed by a 90º rotation about the X-axis.

33) Write the matrix transform of a 45º rotation about an arbitrary axis parallel to $\vec{u} = (1,0,1)$ direction.

Remember : $R_{\vec{V}}(\theta) = I + \sin\theta \bullet skew(\hat{v}) + (1-\cos\theta) \bullet skew^2(\hat{v})$

$$where\ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}\ \ skew(\hat{v}) = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \quad \text{if}\ \ v=(x,y,z)$$

34) Write the matrix transform of a 180º rotation about an arbitrary axis parallel to $\vec{u} = (3,0,4)$ direction.

35) Calculate the inverse of the following rotation matrix $R = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ without its adjoint matrix.

Linear Operators

36) Which of the following linear transformations is a linear operator ?

a) $T:\mathbb{R}^2 \to \mathbb{R}^2$ such $T(x, y) = (x-2y,\ y\ , x+3y)$

b) $T:\mathbb{R}^3 \to \mathbb{R}^3$ such $T(x, y, z) = (x-2y-z,\ y\ , x+y+z)$ operator

c) $T:\mathbb{R}^2 \to \mathbb{R}$ such $T(x, y) = x-2y$

d) $T:\mathbb{R}^2 \to \mathbb{R}^3$ such $T(x, y) = (x-y,\ y\ , x+y)$

e) $T:\mathbb{R} \to \mathbb{R}$ such $T(x) = 2x$

Composition of Linear operators

37) Let $T_1:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y\\x-y\end{pmatrix}$ and $T_2:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+3y\\x+y\end{pmatrix}$

Find $T_2\circ T_1$ *and* $T_1\circ T_2$. Is $T_2\circ T_1=T_1\circ T_2$.

38) Let $T_1:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\x+y\end{pmatrix}$ and $T_2:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5x+y\\y\end{pmatrix}$

Find $T_2\circ T_1$ *and* $T_1\circ T_2$.

39) Let $T_1:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x\\2y\end{pmatrix}$ and $T_2:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\-x+y\end{pmatrix}$

Find $T_2\circ T_1$ *and* $T_1\circ T_2$.

40) Let $T_1:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}$ and $T_2:\mathbb{R}^2\to\mathbb{R}^2$ such as $T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\2x+y\end{pmatrix}$

Find $T_2\circ T_1$ *and* $T_1\circ T_2$.

One-to-one Linear operator

In each part determine whether the linear operator is one-to-one

41) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$

42) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-y\end{pmatrix}$

43) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\x+y\end{pmatrix}$

44) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y\\6x+3y\end{pmatrix}$

45) $T:\mathbb{R}^3\to\mathbb{R}^3$ such as $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+2y+3z\\z\\2z\end{pmatrix}$

Inverse of a one-to-one Linear Operator

In each part verify if the linear operator T is invertible, and compute the its inverse $T^{-1}:\mathbb{R}^2\to\mathbb{R}^2$

46) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$

47) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-y\end{pmatrix}$

48) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y\\x+y\end{pmatrix}$

49) $T:\mathbb{R}^2\to\mathbb{R}^2$ such as $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+3y\\x+2y\end{pmatrix}$