1. Lines in 3D Space

A line in 3D is defined as a parametric equation.

If $\vec{v}=(a,b,c)$ = line direction and $\vec{S}_0=(x_0,y_0,z_0)$ =source point (origin) , $\vec{S}=(x,y,z)$ = end point

It shows that $\overrightarrow{S_0S} // \vec{v} \Leftrightarrow \exists\, t\in\mathbb{R}^* \,/\, \overrightarrow{S_0S}=t\vec{v}$. $\overrightarrow{S_0S}=t\vec{v}$ => $\vec{S}=\vec{S}_0+t\vec{v}$ (line vector equation)

the parametric equation in term of t , we have : $\begin{cases} x=x_0+at \\ y=y_0+bt \quad with \quad t\in\mathbb{R}^* \\ z=z_0+ct \end{cases}$ (line parametric equation)

Example1.1: Write the equations of line passing through the point $\vec{s}_0$=(1,4,-6) and parallel to $\vec{v}$ =(2,3,1)

Using $\begin{cases} x=x_0+at \\ y=y_0+bt \quad with \quad t\in\mathbb{R}^* \\ z=z_0+ct \end{cases}$ where $\vec{S}_0=(x_0,y_0,z_0)$=(1,4,-6) and $\vec{v}=(a,b,c)$=(2,3,1), after plugging

in the values, we have the line equation $\begin{cases} x=1+2t \\ y=4+3t \quad with \quad t\in\mathbb{R}^* \\ z=6+t \end{cases}$

TODO➔ Go to Activity and Solve question 1

Example1.2: Find source point $\vec{S}_0=(x_0,y_0,z_0)$ and the direction $\vec{v}=(a,b,c)$ of the line $\begin{cases} x=7+5t \\ y=3+2t \quad with \quad t\in\mathbb{R}^* \\ z=4-t \end{cases}$

Comparing $\begin{cases} x=x_0+at \\ y=y_0+bt \\ z=z_0+ct \end{cases}$ to $\begin{cases} x=7+5t \\ y=3+2t \\ z=4-t \end{cases}$ we have x_0 =7, y_0=3, z_0=4 therefore $\vec{S}_0=(x_0,y_0,z_0)$=(7,3,4)

And a=5, b=2 , c= -1 therefore the direction of the line is $\vec{v}=(a,b,c)$=(5,2,-1)

TODO ➔ Go to Activity and Solve question 2

2. Rays in 3D Space

The equation of a ray is similar to the line equation.

Ray vector equation is $\vec{S}=\vec{S}_0+t\vec{v}$ with $t\geq 0$

The parametric equation is $\begin{cases} x=x_0+at \\ y=y_0+bt \quad with \quad t\geq 0 \\ z=z_0+ct \end{cases}$ **Note here that** : $t\geq 0$

Example2.1:Write the equations of ray passing through the point $\vec{s}_0$=(4,5,-3) and parallel to $\vec{v}$ =(11,2,9)

Using $\begin{cases} x=x_0+at \\ y=y_0+bt \quad with \quad t\geq 0 \\ z=z_0+ct \end{cases}$ where $\vec{S}_0=(x_0,y_0,z_0)$=(4,5,-3) and $\vec{v}=(a,b,c)$=(11,2,9), after plugging

In the values, we have the ray equation $\begin{cases} x=4+11t \\ y=5+2t \quad with\ t\geq 0 \\ z=-3+9t \end{cases}$

TODO ➔ Go to Activity and Solve questions 3 and 4

Example2.2: Find source point $\vec{S}_0=(x_0,y_0,z_0)$ and the direction $\vec{v}=(a,b,c)$ of the ray $\begin{cases} x=2+3t \\ y=7+2t & t\geq 0 \\ z=-1-2t \end{cases}$

Comparing $\begin{cases} x=x_0+at \\ y=y_0+bt \\ z=z_0+ct \end{cases}$ to $\begin{cases} x=2+3t \\ y=7+2t & t\geq 0 \\ z=-1-2t \end{cases}$ we have x_0 =2, y_0 =7, z_0 =-1 therefore $\vec{S}_0=(x_0,y_0,z_0)$=(2,7,1)

and a=3, b=2 , c= -2 therefore the direction of the ray is $\vec{v}=(a,b,c)$=(3,2,-2)

3. <u>Line Segments in 3D Space</u>

The equation of a line segment is as follow:

Line segment vector equation is $\vec{S}=\vec{S}_0+t\vec{v}$ with $0\leq t\leq 1$

The parametric equation is $\begin{cases} x=x_0+at \\ y=y_0+bt & with \quad 0\leq t\leq 1 \\ z=z_0+ct \end{cases}$

Example3.1: Write the equations of line segment passing through the point $\vec{s}_0$=(2,7,8) and parallel to $\vec{v}$=(5,9,3).

Using $\begin{cases} x=x_0+at \\ y=y_0+bt & with \quad 0\leq t\leq 1 \\ z=z_0+ct \end{cases}$ where $\vec{S}_0=(x_0,y_0,z_0)$=(2,7,8) and $\vec{v}=(a,b,c)$=(5,9,3), after plugging

In the values, we have the line segment equation $\begin{cases} x=2+5t \\ y=7+9t & with \quad 0\leq t\leq 1 \\ z=8+3t \end{cases}$

TODO ➔ Go to Activity and Solve question 5

Example3.2:Find source point $\vec{S}_0=(x_0,y_0,z_0)$ and the direction $\vec{v}=(a,b,c)$ of the segment $\begin{cases} x=\quad 3t \\ y=1+2t & 0\leq t\leq 1 \\ z=1-5t \end{cases}$

Comparing $\begin{cases} x=x_0+at \\ y=y_0+bt \\ z=z_0+ct \end{cases}$ to $\begin{cases} x=\quad 3t \\ y=1+2t & 0\leq t\leq 1 \\ z=1-5t \end{cases}$ we have x_0 =0, y_0 =1, z_0 =1 therefore $\vec{S}_0=(x_0,y_0,z_0)$=(0,1,1)

And a=3, b=2 , c= -5 therefore the direction of the line segment is $\vec{v}=(a,b,c)$=(3,2,-5)

TODO ➔ Go to Activity and Solve question 6

4. <u>Distance between a point and a Ray, or line segment</u> **(Optional)**

Let a ray be defined by its equation $\vec{S}=\vec{S}_0+t\vec{v}$, and the point $\vec{p}$ in the space. We first compute the vector $\vec{u}=\vec{p}-\vec{s}_o$ and $\vec{w}=proj_{\vec{v}}^{\vec{u}}=(\vec{u}\bullet\hat{v})\bullet\hat{v}$. Using the Pythagorean theorem , the distance between the line and the point is

$$d=\sqrt{\vec{u}\bullet\vec{u}-\hat{w}\bullet\hat{w}}=\sqrt{(\vec{p}-\vec{s}_o)\bullet(\vec{p}-\vec{s}_o)-[(\vec{u}\bullet\hat{v})\cdot\hat{v}]\bullet[(\vec{u}\bullet\hat{v})\cdot\hat{v}]}=\sqrt{(\vec{p}-\vec{s}_o)\bullet(\vec{p}-\vec{s}_o)-(\vec{u}\bullet\hat{v})\bullet(\vec{u}\bullet\hat{v})}$$

or better $\boxed{d^2=(\vec{p}-\vec{s}_o)\bullet(\vec{p}-\vec{s}_o)-(\vec{u}\bullet\hat{v})\bullet(\vec{u}\bullet\hat{v})}$ (to avoid the expensive square root math if used for collision).

where $\hat{v}$=normalized vector of $\vec{v}$.

5. <u>Properties of Lines , Rays and line segments</u>

Given two lines L : $\vec{S}=\vec{S}_0+t\vec{v}$ and L': $\vec{S}'=\vec{S}'_0+t\vec{v}'$

1) $L // L'$ *if only if* $\vec{v} // \vec{v}'$

2) $L \perp L'$ *if only if* $\vec{v}\perp\vec{v}'$

3) $Angle(L,L')=Angle(\vec{v},\vec{v}')=\cos^{-1}\left(\dfrac{\vec{v}\bullet\vec{v}'}{\|\vec{v}\|\bullet\|\vec{v}'\|}\right)$

Example5.1: **parallel rays**

Show that the 2 given rays ,L1 and L2 , are parallel where L1: $\begin{cases}x=5+t\\ y=3+2t & t\geq 0\\ z=2+3t\end{cases}$ **and L2:** $\begin{cases}x=3+3t\\ y=2+6t & t\geq 0\\ z=1+9t\end{cases}$

The 2 rays have respective direction $\vec{v}_1$=(1,2,3) for L1 and $\vec{v}_2$=(3,6,9) for L2.

$\vec{v}_2$=(3,6,9)=3(1,2,3)=3 $\vec{v}_1$. Since $\vec{v}_2$=3 $\vec{v}_1$ ➔ $\vec{v}_2 // \vec{v}_1$ ➔ L1//L2

TODO ➔ Go to Activity and Solve question 7

Example5.2: **perpendicular rays.**

Show that the 2 given rays ,L1 and L2 ,are perpendicular, L1: $\begin{cases}x=5+t\\ y=-3-2t & t\geq 0\\ z=t\end{cases}$ **and L2:** $\begin{cases}x=2\\ y=5+3t & t\geq 0\\ z=3+6t\end{cases}$

The 2 rays have respective direction $\vec{v}_1$=(1,-2,1) for L1 and $\vec{v}_2$=(0,3,6) for L2.

$\vec{v}_1\bullet\vec{v}_2=(1,-2,1)\bullet(0,3,6)=0-6+6=0$. Since $\vec{v}_1\bullet\vec{v}_2=0$ ➔ $\vec{v}_1\perp\vec{v}_2$ ➔ $L_1\perp L_2$

TODO ➔ Go to Activity and Solve question 8

Example5.3 angle between two rays

Find the angle between the two rays L1 and L2 , where L1: $\begin{cases}x=5+t\\ y=3+2t & t\geq 0\\ z=4+2t\end{cases}$ **and L2:** $\begin{cases}x=2+3t\\ y=5 & t\geq 0\\ z=3+4t\end{cases}$

The 2 rays have respective direction $\vec{v}_1$=(1,2,2) for L1 and $\vec{v}_2$=(3,0,4) for L2

$\vec{v}_1\bullet\vec{v}_2=(1,2,2)\bullet(3,0,4)=11$, $\|\vec{v}_1\|=3$, $\|\vec{v}_2\|=5$

$$\theta=Angle(L_1,L_2)=Angle(\vec{v}_1,\vec{v}_2)=\cos^{-1}\left(\frac{\vec{v}_1\bullet\vec{v}_2}{\|\vec{v}_1\|\bullet\|\vec{v}_2\|}\right)=\cos^{-1}\left(\frac{11}{(3)\cdot(5)}\right)=\cos^{-1}\left(\frac{11}{15}\right)=42.83^0$$

TODO ➔ Go to Activity and Solve question 9

6. Planes equation

Let $\vec{n}=(a,b,c)$ be the plane normal vector, and $\vec{p}_0=(x_0,y_0,z_0)$ a fixed point on the plane.

If $\vec{p}=(x,y,z)$ is an arbitrary point of the plane then $\vec{n}\perp\overrightarrow{P_0P} \Rightarrow \vec{n}\bullet\overrightarrow{P_0P}=0 \Rightarrow \vec{n}\bullet(\vec{p}-\vec{p}_0)=0$

or $\vec{n}\bullet\vec{p}-\vec{n}\bullet\vec{p}_0=0$. Now setting $d=-\vec{n}\bullet\vec{p}_0$, we finally have the vector equation

of the plane that is $\boxed{\vec{n}\bullet\vec{p}+d=0}$ (1) with $d=-\vec{n}\bullet\vec{p}_0$.

We can derive the analytical equation from (1); that is $\vec{n}\bullet\vec{p}+d=0 \rightarrow (a,b,c)\bullet(x,y,z)+d=0$

Or $\boxed{ax+by+cz+d=0}$ where $d=-\vec{n}\bullet\vec{p}_0=-(ax_0+by_0+cz_0)$.

Example6.1 :

Find the normal vector $\vec{n}$ of the plane 2x+4y-z+10=0 , its constant d , and a point on the plane $\vec{p}_0$

Answer➔ By comparing $ax+by+cz+d=0$ and **2x+4y-z+10=0 ,** we can see that a=2,b=4,c=-1 and d=10 so the plane normal vector is $\vec{n}=(2,4,-1)$ with d=10. Any point on the plane should have its coordinates satisfying **2x+4y-z+10=0 once plugged in.**

that is: 2(?) + 4(?) – (?) + 10 $\overset{?}{=}$ 0 where ? is x, y, or z is value to be plugged in.

we pick **,** $\vec{p}_0$=(-5,0,0) since 2(-5) + 4(0) - (0) +10=0 or,

$\vec{p}_0$=(-1,-2,0) since 2(-1) + 4(-2) - (0) +10=0 or,

$\vec{p}_0$=(0,0,10) since 2(0) + 4(0) - (10) +10=0. We stop here since there are many points. We only need to find one point.

TODO ➔ Go to Activity and Solve questions 10 and 11

Example6.2: **write the plane equation with normal vector $\vec{n}=(1,3,1)$ passing through $\vec{p}_0(1,1,1)$**

Answer➔ using $ax+by+cz+d=0$ since $\vec{n}=(a,b,c)=(1,3,1)$ we a=1 , b=3 , c=1 and

$d=-\vec{n}\cdot\vec{p}_0=-(1,3,1)\cdot(1,1,1)=-(1+3+1)=-5$, so $ax+by+cz+d=0$ becomes **x+3y+z-5=0**

Example6.3: **write the plane equation with normal vector $\vec{n}=(3,2,4)$ passing through $\vec{p}_0(0,2,1)$**

Answer➔ using $ax+by+cz+d=0$ since $\vec{n}=(a,b,c)=(3,2,4)$ we a=3 , b=2 , c= 4 and

$d=-\vec{n}\cdot\vec{p}_0=-(3,2,4)\cdot(0,2,1)=-(0+4+4)=-8$, so $ax+by+cz+d=0$ becomes 3**x+2y+4z-8=0**

TODO ➔ Go to Activity and Solve questions 12 and 13

7. Equation of Plane Passing Through Three Points(Triangle)

Let $\vec{v}_1,\vec{v}_2$ *and* $\vec{v}_3$ be 3 point in the space .the equation of a plane passing thru the points 3 $\vec{v}_1,\vec{v}_2$ *and* $\vec{v}_3$ is Computed as follows:

Find the normal vector $\vec{n}=\overrightarrow{v_1v_2}\times\overrightarrow{v_1v_3}=(\vec{v}_2-\vec{v}_1)\times(\vec{v}_3-\vec{v}_1)$

Find a reference point $\vec{p}_0=\vec{v}_1$ and compute $d=-\vec{n}\bullet\vec{p}_0$ to final get $\vec{n}\bullet\vec{p}+d=0$

Example7.1: Find the plane equation passing through $\vec{v}_1(3,2,1)$, $\vec{v}_2(1,3,2)$ *and* $\vec{v}_3(1,1,1)$

$\vec{v}_2-\vec{v}_1=(1,3,2)-(3,2,1)=(-2,1,1)$ $\quad$ $\vec{v}_3-\vec{v}_1=(1,1,1)-(3,2,1)=(-2,-1,0)$

$$\vec{n} = (\vec{v}_2 - \vec{v}_1) \times (\vec{v}_3 - \vec{v}_1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ -2 & -1 & 0 \\ + & - & + \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -2 & 1 \\ -2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -2 & 1 \\ -2 & -1 \end{vmatrix} \vec{k} = \quad 1$$

$\vec{n} = (0 - (-1))\vec{i} - (0 - (-2))\vec{j} + (2 - (-2))\vec{k} == \vec{i} - 2\vec{j} + 4\vec{k} = (1, -2, 4)$.

Take $\vec{p}_0 = \vec{v}_1 = (3,2,1)$ $\quad d = -\vec{n} \bullet \vec{p}_0 = -(1,-2,4)\bullet(3,2,1) = -(3-4+4) = -3$. So for any point $\vec{p} = (x,y,z)$
On the plane, the plane is obtained by computing $\vec{n} \bullet \vec{p} + d = 0$ **, that is (1,-2,4).(x,y,z) + (-3) =0 resulting in x + -2y + 4z -3=0**

8. Properties of the planes in 3D space

Given two planes P : $\vec{n} \bullet \vec{p} + d = 0$ and P': $\vec{n}' \bullet \vec{p} + d' = 0$

$P \,//\, P'$ *if only if* $\vec{n} \,//\, \vec{n}'$ *(parallel planes)*

$P \perp P'$ *if only if* $\vec{n} \perp \vec{n}'$ *(perpendicular planes)*

$$\theta = Angle(P, P') = Angle(\vec{n}, \vec{n}') = \cos^{-1}\left(\frac{\vec{n} \bullet \vec{n}'}{\|\vec{n}\| \bullet \|\vec{n}'\|} \right) \quad (angle\ between\ 2\ planes\ )$$

Example8.1: Parallel planes

Given the two planes P1: 2x +3y +4z = 7 and P2 : 6x +9y + 12z = 1 , show that they are parallel.

Answer: First find the planes normal vector. For plane P1 , $\vec{n}_1 = (2,3,4)$, and Plane P2 , $\vec{n}_2 = (6,9,12)$.

We check that $\vec{n}_2 = (6,9,12) = 3(2, 3, 4) = 3\,\vec{n}_2$.

We conclude: since $\vec{n}_2 = 3\,\vec{n}_2$ ➔ $\vec{n}_2 // \vec{n}_2$ ➔ P1//P2 .

TODO ➔ Go to Activity and Solve question 14

Example8.2: Perpendicular planes

Given the two planes P1: x +3y +4z = 7 and P2 : 6x +2y-3z=1 , show that they are perpendicular.

Answer: First find the planes normal vector. For plane P1 , $\vec{n}_1 = (1,3,4)$, and Plane P2 , $\vec{n}_2 = (6,2,-3)$.

We check that $\vec{n}_1 \bullet \vec{n}_2 = (1,3,4)\bullet(6,2,-3)$ **=6+6-12=0.**

We conclude: since $\vec{n}_1 \bullet \vec{n}_2 = 0$ ➔ $\vec{n}_2 \perp \vec{n}_2$ ➔ P1 $\perp$ P2.

TODO ➔ Go to Activity and Solve question 15

Example8.3: Angle between two planes

Given the two planes P1: x +2y +2z = 7 and P2 : 3x +4z = 1 ,find the angle between P1 and P2.

Answer: First find the planes normal vector. For plane P1 , $\vec{n}_1 = (1,2,2)$, and Plane P2 , $\vec{n}_2 = (3,0,4)$.

We calculate $\vec{n}_1 \bullet \vec{n}_2 = (1,2,2)\bullet(3,0,4) = 11$, $\|\vec{n}_1\| = 3$, $\|\vec{n}_2\| = 5$ and

$$\theta = Angle(P_1, P_2) = Angle(\vec{n}_1, \vec{n}_2) = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{\|\vec{n}_1\| \bullet \|\vec{n}_2\|} \right) = \cos^{-1}\left(\frac{11}{(3)\cdot(5)} \right) = \cos^{-1}\left(\frac{11}{15} \right) = 42.83^0$$

TODO ➔ Go to Activity and Solve question 16

9. <u>Distance between a point and a plane</u>

Let plane P: $\vec{n}\bullet\vec{p}+d=0$ and the point $\vec{p}_1=(x_1,y_1,z_1)$, then the distance between the plane and the point $\vec{p}_1$ is $D=\left|comp_{\vec{n}}^{\overrightarrow{p_0p_1}}\right|=\left|\dfrac{\vec{n}\bullet\overrightarrow{p_0p_1}}{\|\vec{n}\|}\right|=\dfrac{|\vec{n}\bullet\vec{p}_1+d|}{\|\vec{n}\|}=\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Example9.1: Find the distance D between the plane P: x + 2 y + 2z -13=0 and the point $\vec{p}_1$=(1,2,1).

Answer: here $\vec{n}=(a,b,c)=(1,2,2)$ d= -13 $\vec{p}_1$**=(1,2,1)** , $\|\vec{n}\|=\sqrt{1^2+2^2+2^2}=3$

$$|\vec{n}\bullet\vec{p}_1+d|=|(1,2,2)\bullet(1,2,1)-13|=|1+4+2-13|=|-6|=6 \qquad D=\frac{|\vec{n}\bullet\vec{p}_1+d|}{\|\vec{n}\|}=\frac{6}{3}=2$$

Example9.2: Find the distance D between the plane P: 3x - 4z + 13=0 and the point $\vec{p}_1$=(2,2,1).

Answer: here $\vec{n}=(a,b,c)=(3,0,-4)$ d= 13 $\vec{p}_1$**=(2,2,1)** , $\|\vec{n}\|=\sqrt{3^2+0^2+(-4)^2}=5$

$$|\vec{n}\bullet\vec{p}_1+d|=|(3,0,-4)\bullet(2,2,1)+13|=|6-4+13|=|15|=15 \qquad D=\frac{|\vec{n}\bullet\vec{p}_1+d|}{\|\vec{n}\|}=\frac{15}{5}=3$$

TODO ➔ Go to Activity and Solve question 17

10. <u>Half- Space Tests</u>

Given the plane P: $\vec{n}\bullet\vec{p}+d=0$ and the point $\vec{p}_1=(x_1,y_1,z_1)$, we want to verify if the point $\vec{p}_1=(x_1,y_1,z_1)$ is on the plane (coplanar), above the plane(positive half space), or behind the plane (negative half pace).

a) Coplanar test (is the point on the plane ?)
If $\vec{n}\bullet\vec{p}_1+d=0$ *or* $ax_1+by_1+cz_1+d=0$ then the point $\vec{p}_1$ is on the plane.
Example10.1: Show that the point $\vec{p}_1$=(3,0,1) is on the plane P: x + y + 2z -5=0
Answer: here $\vec{n}=(a,b,c)=(1,1,2)$ d= -5 $\vec{p}_1$**=(3,0,1).**
We calculate $\vec{n}\bullet\vec{p}_1+d=(1,1,2)\bullet(3,0,1)-5=3+0+2-5=0$
We conclude: since $\vec{n}\bullet\vec{p}_1+d=0$ ➔ $\vec{p}_1$ **is on the plane (coplanar)**

TODO ➔ Go to Activity and Solve question 18
Video:

b) Positive half-space test (Is the point above the plane ?)
If $\vec{n}\bullet\vec{p}_1+d>0$ *or* $ax_1+by_1+cz_1+d>0$ then the point $\vec{p}_1$ is above(on front of) the plane.
Example10.2: Show that the point $\vec{p}_1$=(2,2,1) is above the plane P: x + 2 y + 2z +3=0
Answer here $\vec{n}=(a,b,c)=(1,2,2)$ d= 3 $\vec{p}_1$**=(2,2,1).**
We calculate $\vec{n}\bullet\vec{p}_1+d=(1,2,2)\bullet(2,2,1)+3=2+4+2+3=11>0$
We conclude: since $\vec{n}\bullet\vec{p}_1+d>0$ ➔ $\vec{p}_1$ **is above the plane(positive half-space)**

TODO ➔ Go to Activity and Solve question 19

c) Negative half-space test (Is the point below the plane ?)
If $\vec{n} \bullet \vec{p}_1 + d < 0$ *or* $ax_1 + by_1 + cz_1 + d < 0$ then the point $\vec{p}_1$ is behind (below) the plane.

Example10.3: Show that the point $\vec{p}_1$=(2,2,1) is below the plane P: x + 2 y + 2z -10=0

Answer: here $\vec{n} = (a,b,c) = (1,2,2)$ d= -10 $\vec{p}_1$ **= (2,2,1).**

We calculate $\vec{n} \bullet \vec{p}_1 + d = (1,2,2) \bullet (2,2,1) - 10 = 2 + 4 + 2 - 10 = -2 < 0$

We conclude: since $\vec{n} \bullet \vec{p}_1 + d < 0$ ➔ $\vec{p}_1$ **is below the plane(negative half-space)**

TODO ➔ Go to Activity and Solve question 20

11. Projection of a point onto a Plane

Let $\vec{p}_1$ be a point in the positive half-space and $\vec{q}$ the point resulting from projecting $\vec{p}_1$ onto the plane with equation $\vec{n} \bullet \vec{p} + d = 0$ in the direction of a vector $\hat{v}$. It can be seen that $\overrightarrow{p_1 q} \,//\, \hat{v}$ implying that it exists $k \in \mathbb{R}^* \;/\; \overrightarrow{p_1 q} = k \cdot \hat{v}$ or $\vec{q} - \vec{p}_1 = k \cdot \hat{v}$ or $\vec{q} = \vec{p}_1 + k \cdot \hat{v}$.
Since $\vec{q}$ belongs to the plane (coplanar), we plug it into the plane equation to derive k as follows:
$\vec{n} \bullet \vec{q} + d = 0$ leading to $\vec{n} \bullet (\vec{p}_1 + k \cdot \hat{v}) + d = 0$ $\vec{n} \cdot \vec{p}_1 + k \cdot \vec{n} \cdot \hat{v} + d = 0$ ➔ $\vec{n} \bullet \hat{v} \cdot k = -\vec{n} \cdot \vec{p}_1 - d$

If $\vec{n} \bullet \vec{v} < 0$ with $\vec{n} \bullet \vec{q} + d > 0$ then $k = -\dfrac{\vec{n} \bullet \vec{p}_1 + d}{\vec{n} \bullet \hat{v}}$

And the projection equation $\vec{q} = \vec{p}_1 + k \cdot \hat{v}$ becomes $\boxed{\vec{q} = \vec{p}_1 - \dfrac{\vec{n} \bullet \vec{p}_1 + d}{\vec{n} \bullet \hat{v}} \cdot \hat{v}}$ with $d = -\vec{p}_0 \bullet \hat{n}$

If the projection vector is not normalized, that is using $\vec{v}$, then $\boxed{\vec{q} = \vec{p}_1 - \dfrac{\vec{n} \bullet \vec{p}_1 + d}{\vec{n} \bullet \vec{v}} \cdot \vec{v}}$

Example11.1 :Find the projected point $\vec{q}$ of $\vec{p}_1$ (1,3,-1) onto the plane x+y+z+5=0 in the direction of
a) the vector $\vec{v}$ (1,-2,-1)
b) the plane normal vector $\vec{n}$ (1,1,1)
Answer:

a) $\vec{n} = (1,1,1)$ d=5 $\vec{n} \bullet \vec{v} = (1,1,1) \bullet (1,-2,-1) = -2 < 0$ $\vec{n} \bullet \vec{p}_1 + d = (1,1,1) \bullet (1,3,-1) + 5 = 3 + 5 = 8 > 0$

$$\vec{q} = \vec{p}_1 - \frac{\vec{n} \bullet \vec{p}_1 + d}{\vec{n} \bullet \vec{v}} \cdot \vec{v} = = (1,3,-1) - \frac{8}{-2}(1,-2,-1)$$
$$= (1,3,-1) + 4(1,-2,-1) = (1,3,-1) + (4,-8,-4) = (5,-5,-5)$$

b) $\vec{n} = (1,1,1)$ d=5 here $\vec{v} = \vec{n}$ so $\vec{n} \bullet \vec{v} = \vec{n} \bullet \vec{n} = 3$ $\vec{n} \bullet \vec{p}_1 = (1,1,1) \bullet (1,3,-1) + d = 3 + 5 = 8$

$$\vec{q} = \vec{p}_1 - \frac{\vec{n} \bullet \vec{p}_1 + d}{\vec{n} \bullet \vec{n}} \cdot \vec{n} = = (1,3,-1) - \frac{8}{3}(1,1,1) = \left(\tfrac{-5}{3}, \tfrac{1}{3}, \tfrac{-11}{3}\right)$$

12. Parametric curves

13. Surfaces