**Problem Set 3: IMF**

1. The IMF is the function describing how many stars are born at each mass.
$$\xi \equiv \frac{dN}{dM} \tag{1}$$
There are several common parametrizations. We adopt $M_{max} = 120\ \mathrm{M}_\odot$ as the default for all.

   (a) Salpeter (you will find different $M_{min}$ in use in different locations in the literature; we adopt 0.03 here to simplify comparisons):
$$\xi = M^{-2.35}, M_{min} = 0.03\mathrm{M}_\odot \tag{2}$$

   (b) Kroupa (eqn 2 of 2001MNRAS.322..231K):
$$\xi = M^{-\alpha}\begin{cases}\alpha = 0.3, & 0.01 \le M/\mathrm{M}_\odot < 0.08\\ \alpha = 1.3, & 0.08 \le M/\mathrm{M}_\odot < 0.5\\ \alpha = 2.3, & 0.5 \le M/\mathrm{M}_\odot < M_{max}\end{cases}$$

   (c) Chabrier (Eqn 18 of 2003PASP..115..763C):
$$\begin{cases}\xi(\log M) \equiv \frac{dN}{d\log M} = A\exp\left[-(\log M - \log M_c)^2/2\sigma^2\right] & \\ A = 0.086, M_c = 0.22, \sigma = 0.57 & M \le 1\mathrm{M}_\odot\\ \xi(\log M) = AM^{-\Gamma} & \\ \Gamma = 1.3, A = 4.43 & M > \mathrm{M}_\odot\end{cases}$$

   Plot these IMFs.

   You will perform calculations using these equations to infer properties of stellar populations. For each distribution, compute:

   (a) What is the average mass?

   (b) What is the average mass of stars with $M > 8\mathrm{M}_\odot$?

   (c) What is the ratio of the number of high-mass to low-mass stars? (use M=8 $\mathrm{M}_\odot$ as the dividing mass)

   (d) What is the ratio of the *mass* of high-mass to low-mass stars?

   (e) Do these numbers change if you change $M_{max}$ to 100 $\mathrm{M}_\odot$? To 1000 $\mathrm{M}_\odot$?

   (f) Do these numbers change if you change $M_{min}$ to 0.03 $\mathrm{M}_\odot$? To 0.3 $\mathrm{M}_\odot$?

   (g) For a cluster of 1000 stars, how many would you expect to be $M > 8\mathrm{M}_\odot$ (able to go supernova)?

(h) In an 'optimal distribution function' (ODF), the cluster mass to maximum star mass is fixed by defining $\int_{M_{max,cl}}^{M_{max}} \xi dM = 1$, where $M_{max}$ is the maximum possible mass for a star and $M_{max,cl}$ is the most massive star in the cluster. From this definition, determine how many stars must be in a cluster to form one 10 $\mathrm{M}_\odot$ star or one 100 $\mathrm{M}_\odot$ star.

(i) What *mass* of cluster is required to produce a star of that mass?

(j) In a probability distribution function, there is only a fix likelihood of forming a star of a given mass. What is the minimum cluster mass required to have a $> 95\%$ ($> 63.21\%$) probability of forming at least one $\geq 100\ \mathrm{M}_\odot$ star?
Recall that the likelihood of rolling at least one six after 100 rolls is equal to one minus the likelihood of rolling *no* sixes in 100 rolls i.e., $P(\geq 1$⚅$) = 1 - \left(\frac{5}{6}\right)^{100}$.

(k) Compare the results from the ODF and the PDF for the presence of a $M > 100\mathrm{M}_\odot$ star. How can you interpret the difference?

(l) What is the expected light-to-mass ratio of each distribution assuming $L = \mathrm{L}_\odot \left(\frac{M}{\mathrm{M}_\odot}\right)^3$?

(m) The IMF is the *system* IMF. If we are primarily interested in the luminosity of a system, splitting the mass between multiple stars can make a big difference. If we assume every star system consists of an equal-mass binary, what is the effect on the L/M ratio? Is this a reasonable approximation to the multiplicity fraction?

(n) For an ODF, the effective maximum stellar mass can be smaller. If all star-forming events in a galaxy occur in Taurus-like star-forming regions, with $M_* = 100\mathrm{M}_\odot$, what is the maximum mass? What is the resulting L/M ratio? Recall that this is the *maximum* the L/M will be in such a galaxy.

You may choose to use external resources, like the `imf` package, but it's a good idea to try to solve some of these - at least the simple Salpeter IMF - by hand.

Answers:

1. (a) Salpeter mean mass: 0.11
ChabrierPowerLaw mean mass: 0.67
Kroupa mean mass: 0.43

(b) Salpeter Mean M(M>8) = 19.40
ChabrierPowerLaw Mean M(M>8) = 19.87
Kroupa Mean M(M>8) = 19.87

(c) Salpeter N(M>8) / N(tot) = 0.0001, N(M>8) / N(low) = 0.0005
ChabrierPowerLaw N(M>8) / N(tot) = 0.0079, N(M>8) / N(low) = 0.0085
Kroupa N(M>8) / N(tot) = 0.0040, N(M>8) / N(low) = 0.0046

(d) Salpeter M(M>8) / M(tot) = 0.0917, M(M>8) / M(low) = 0.1010
ChabrierPowerLaw M(M>8) / M(tot) = 0.2491, M(M>8) / M(low) = 0.3318
Kroupa M(M>8) / M(tot) = 0.2094, M(M>8) / M(low) = 0.2648

(e) Mmax = 100
Salpeter mean mass: 0.11
Salpeter M(M>8) / M(tot) = 0.0882, M(M>8) / M(low) = 0.0968
Salpeter N(M>8) / N(tot) = 0.0001, N(M>8) / N(low) = 0.0005
ChabrierPowerLaw mean mass: 0.66
ChabrierPowerLaw M(M>8) / M(tot) = 0.2406, M(M>8) / M(low) = 0.3169
ChabrierPowerLaw N(M>8) / N(tot) = 0.0079, N(M>8) / N(low) = 0.0084
Kroupa mean mass: 0.43
Kroupa M(M>8) / M(tot) = 0.2019, M(M>8) / M(low) = 0.2530
Kroupa N(M>8) / N(tot) = 0.0039, N(M>8) / N(low) = 0.0046
Mmax = 1000
Salpeter mean mass: 0.11
Salpeter M(M>8) / M(tot) = 0.1185, M(M>8) / M(low) = 0.1345
Salpeter N(M>8) / N(tot) = 0.0001, N(M>8) / N(low) = 0.0005
ChabrierPowerLaw mean mass: 0.73
ChabrierPowerLaw M(M>8) / M(tot) = 0.3133, M(M>8) / M(low) = 0.4563
ChabrierPowerLaw N(M>8) / N(tot) = 0.0081, N(M>8) / N(low) = 0.0087
Kroupa mean mass: 0.47
Kroupa M(M>8) / M(tot) = 0.2670, M(M>8) / M(low) = 0.3643
Kroupa N(M>8) / N(tot) = 0.0041, N(M>8) / N(low) = 0.0047

(f) Mmax = 120, Mmin=0.03
Salpeter mean mass: 0.11
Salpeter M(M>8) / M(tot) = 0.0917, M(M>8) / M(low) = 0.1010
Salpeter N(M>8) / N(tot) = 0.0005, N(M>8) / N(low) = 0.0005
ChabrierPowerLaw mean mass: 0.67
ChabrierPowerLaw M(M>8) / M(tot) = 0.2491, M(M>8) / M(low) = 0.3318
ChabrierPowerLaw N(M>8) / N(tot) = 0.0084, N(M>8) / N(low) = 0.0085
Kroupa mean mass: 0.43
Kroupa M(M>8) / M(tot) = 0.2094, M(M>8) / M(low) = 0.2648

Kroupa N(M>8) / N(tot) = 0.0046, N(M>8) / N(low) = 0.0046
Mmax = 120, Mmin=0.3
Salpeter mean mass: 1.02
Salpeter M(M>8) / M(tot) = 0.2212, M(M>8) / M(low) = 0.2841
Salpeter N(M>8) / N(tot) = 0.0116, N(M>8) / N(low) = 0.0117
ChabrierPowerLaw mean mass: 1.37
ChabrierPowerLaw M(M>8) / M(tot) = 0.2817, M(M>8) / M(low) = 0.3922
ChabrierPowerLaw N(M>8) / N(tot) = 0.0194, N(M>8) / N(low) = 0.0198
Kroupa mean mass: 1.18
Kroupa M(M>8) / M(tot) = 0.2588, M(M>8) / M(low) = 0.3491
Kroupa N(M>8) / N(tot) = 0.0046, N(M>8) / N(low) = 0.0156

(g) Salpeter NSN = N(M>8) = 11.4
ChabrierPowerLaw NSN = N(M>8) = 14.2
Kroupa NSN = N(M>8) = 13.0

(h) salpeter: nstars to make 10 = 117
chabrier: nstars to make 10 = 172
kroupa: nstars to make 10 = 295
salpeter: nstars to make 100 = 11666
chabrier: nstars to make 100 = 15631
kroupa: nstars to make 100 = 26818

(i) Salpeter: mass(100Msun) = 1.2e+06, N(100msun) = 1151273
ChabrierPowerLaw: mass(100Msun) = 2.1e+04, N(100msun) = 15482
Kroupa: mass(100Msun) = 3.6e+04, N(100msun) = 30855

(j) ODF Salpeter P(>100) = 0.999999131396621, N* = 1/(1-P(>100)=1151273
ODF Salpeter: M = 1.3e+05
PDF Salpeter: Min mass for >1 100 Msun star at 95%: 3.8e+05 nstars=3448905
PDF Salpeter: Min mass for >1 100 Msun star at 62.3%: 1.3e+05 nstars=1151208
ODF ChabrierPowerLaw P(>100) = 0.999935410117752, N* = 1/(1-P(>100)=15482
ODF ChabrierPowerLaw: M = 1e+04
PDF ChabrierPowerLaw: Min mass for >1 100 Msun star at 95%: 3.1e+04 nstars=46379
PDF ChabrierPowerLaw: Min mass for >1 100 Msun star at 62.3%: 1e+04 nstars=15480
ODF Kroupa P(>100) = 0.9999675911858097, N* = 1/(1-P(>100)=30855
ODF Kroupa: M = 1.3e+04
PDF Kroupa: Min mass for >1 100 Msun star at 95%: 4e+04 nstars=92434

PDF Kroupa: Min mass for >1 100 Msun star at 62.3%: 1.3e+04
nstars=30853

(k) Answer the question from the above data

(l) Salpeter: L/M = 427.7
ChabrierPowerLaw: L/M = 571.2
Kroupa: L/M = 524.6

(m) Salpeter: L/M = 106.9
ChabrierPowerLaw: L/M = 142.8
Kroupa: L/M = 131.1

(n) Salpeter: Mmax = 1.0 L/M = 0.3
ChabrierPowerLaw: Mmax = 5.3 L/M = 4.3
Kroupa: Mmax = 3.5 L/M = 2.1