Problem Set 3: Disks

1. Derive the spectral slope of a flat disk. Use the following assumptions:

$$T = T_0 \left(\frac{r}{r_0}\right)^{-q} \tag{1}$$

where T_0, r_0, and q are constants, and T_0 is the temperature of the disk at r_0

(a) Use the equation for the flux density from a disk inclined at angle θ to the line of sight (θ is sometimes written as i when we talk of $v \sin i$, but i is an inconvenient variable to use in handwriting).

$$F_\nu = \frac{2\pi \cos\theta}{d^2} \int_{R_{min}}^{R_{max}} B_\nu(T(r)) r dr \tag{2}$$

to determine F_ν in terms of constants of the system (T_0, R_0, $\cos\theta$) and ν. In this equation, d is the distance to the object, R_{inner} and R_{outer} are the inner & outer disk radii, and $B(T)$ is the Planck function.

You can solve this by changing variables and coming up with an integral whose value you may not be able to evaluate, but which is independent of frequency (and therefore can be treated as a constant).

(b) Determine the spectral index α_{IR} as a function of the power-law index q. What is α_{IR} for a passively-heated flat disk, where $T \propto r^{-3/4}$?

Unfortunately, the term *spectral index* is highly overloaded, and there are two definitions in common use by different wavelength astronomers:

$$F_\nu = F_{\nu,0} \left(\frac{\nu}{\nu_0}\right)^{-\alpha_{radio}} \tag{3}$$

$$\alpha_{radio} = -\frac{d \log F_\nu}{d \log \nu} \tag{4}$$

$\alpha_{IR} = \alpha_{radio} - 1$

$$\nu F_\nu = \nu F_{\nu,0} \left(\frac{\nu}{\nu_0}\right)^{1-\alpha_{IR}} \tag{5}$$

$$\nu F_\nu = F_{\nu,0} \left(\frac{\nu}{\nu_0}\right)^{-\alpha_{IR}} \tag{6}$$

$$\alpha_{IR} = -\frac{d\nu \log F_\nu}{d \log \nu} = -1 - \frac{d \log F_\nu}{d \log \nu} \tag{7}$$

2. Write down the radiative transfer equation of a two layer disk, the two layers being a dense inner layer centered on the mid-plane of the disk and the outer layer being a thinner disk atmosphere.

   Consider each layer as an infinite slab of constant thickness. Consider only radiative transfer perpendicular to the slab (reducing this to a 1-D problem).

   First, write the equation for the intensity of light emitted perpendicular to the slabs in terms of the temperature of the inner slab T_2, outer slab temperature T_1, the opacity per mass κ_ν, the density of the outer slab ρ_1, and the thickness of the outer slab, h_1. Assume that the inner slab has $\tau_2 >> 1$. In which of the following cases do you see the silicate features in emission, in absorption, or no silicate feature? (recall that the "silicate features" are around $\lambda \sim 10\mu$m)

$$T1 > T2, \quad \tau_1 << 1, \quad \tau_2 >> 1 \tag{8}$$
$$T1 > T2, \quad \tau_1 >> 1, \quad \tau_2 >> 1 \tag{9}$$
$$T1 < T2, \quad \tau_1 << 1, \quad \tau_2 >> 1 \tag{10}$$

3. **Self-Similar Viscous Disks.**
   Consider a protostellar disk orbiting a star, governed by the usual viscous evolution equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{\varpi}\frac{\partial}{\partial \varpi}\left[\varpi^{1/2}\frac{\partial}{\partial \varpi}\left(\nu \Sigma \varpi^{1/2}\right)\right],$$

   where Σ is the surface density, ϖ is the radius in cylindrical coordinates, and ν is the viscosity. Suppose that the viscosity is linearly proportional to the radius, $\nu = \nu_1(\varpi/\varpi_1)$.

   (a) Non-dimensionalize the evolution equation by making a change of variables to the dimensionless position, time, and surface density $x = \varpi/\varpi_1$, $T = t/t_s$, $S = \Sigma/\Sigma_1$, where $t_s = \varpi_1^2/(3\nu_1)$.

   (b) Use your non-dimensionalized equation to show that

$$\Sigma = \left(\frac{C}{3\pi\nu_1}\right)\frac{e^{-x/T}}{xT^{3/2}}$$

   is a solution of the equation for an arbitrary constant C.

   (c) Calculate the total mass in the disk in terms of C, t_s, and t, and calculate the time rate of change of this mass. Based on your result, give a physical interpretation of what the constant C means. (Hint: what units does C have?)

   (d) Plot S versus x at $T = 1, 1.5, 2$, and 4. Give a physical interpretation of the results.

4. **BONUS: Toomre Instability.**
Chapter 10 discusses the Toomre instability as a potentially important factor in driving star formation. It may also be relevant to determining the maximum masses of molecular clouds. In this problem we will calculate the stability condition and related quantities. Consider a uniform, infinitely thin disk of surface density Σ occupying the $z = 0$ plane. The disk has a flat rotation curve with velocity v_R, so the angular velocity is $\Omega = \Omega \hat{\mathbf{e}}_z$, with $\Omega = v_R/r$ at a distance r from the disk center. The velocity of the fluid in the $z = 0$ plane is $\mathbf{v}$ and its vertically-integrated pressure is $\Pi = \int_{-\infty}^{\infty} P\, dz = \Sigma c_s^2$.

   (a) Consider a coordinate system co-rotating with the disk, centered at a distance R from the disk center, oriented so that the x direction is radially outward and the y direction is in the direction of rotation. In this frame, the vertically-integrated equations of motion and the Poisson equation are

$$
\begin{aligned}
\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) &= 0 \\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\frac{\nabla \Pi}{\Sigma} - \nabla \phi - 2\Omega \times \mathbf{v} + \Omega^2 (x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y) \\
\nabla^2 \phi &= 4\pi G \Sigma \delta(z).
\end{aligned}
$$

   The last two terms in the second equation are the Coriolis and centrifugal force terms. We wish to perform a stability analysis of these equations. Consider a solution (Σ_0, ϕ_0) to these equations in which the gas is in equilibrium (i.e., $\mathbf{v} = 0$), and add a small perturbation: $\Sigma = \Sigma_0 + \epsilon \Sigma_1$, $\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1$, $\phi = \phi_0 + \epsilon \phi_1$, where $\epsilon \ll 1$. Derive the perturbed equations by substituting these values of Σ, $\mathbf{v}$, and ϕ into the equations of motion and keeping all the terms that are linear in ϵ.

   (b) The perturbed equations can be solved by Fourier analysis. Consider a trial value of Σ_1 described by a single Fourier mode $\Sigma_1 = \Sigma_a \exp[i(kx - \omega t)]$, where we choose to orient our coordinate system so that the wave vector $\mathbf{k}$ for this mode is in the x direction. As an *ansatz* for ϕ_1, we will look for a solution of the form $\phi_1 = \phi_a \exp[i(kx - \omega t) - |kz|]$. (One can show that the solution must take this form, but we will not do so here.) Derive the relationship between ϕ_a and Σ_a.

   (c) Now try a similar single-Fourier mode form for the perturbed velocity: $\mathbf{v}_1 = (v_{ax}\hat{\mathbf{e}}_x + v_{ay}\hat{\mathbf{e}}_y) \exp[i(kx - \omega t)]$. Derive three equations relating the unknowns Σ_a, v_{ax}, and v_{ay}. You will find it useful to expand Ω in a Taylor series around the origin of your coordinate system, i.e., write $\Omega = \Omega_0 + (d\Omega/dx)_0 x$, where $\Omega_0 = v_R/R$ and $(d\Omega/dx)_0 = -\Omega_0/R$.

(d) Show that these equations have non-trivial solutions only if

$$\omega^2 = 2\Omega_0^2 - 2\pi G\Sigma_0|k| + k^2 c_s^2.$$

This is the dispersion relation for our rotating thin disk.

(e) Solutions with $\omega^2 > 0$ correspond to oscillations, while those with $\omega^2 < 0$ correspond to pairs of modes, one of which decays with time and one of which grows. We refer to the growing modes as unstable, since in the linear regime they become arbitrarily large. Show that an unstable mode exists if $Q < 1$, where

$$Q = \frac{\sqrt{2}\Omega_0 c_s}{\pi G\Sigma_0}.$$

is called the Toomre parameter. Note that this stability condition refers only to axisymmetric modes in infinitely thin disks; non-axisymmetric instabilities in finite thickness disks usually appear around $Q \approx 1.5$.

(f) When an unstable mode exists, we define the Toomre wave number k_T as the wave number that corresponds to mode for which the instability grows fastest. Calculate k_T and the corresponding Toomre wavelength, $\lambda_T = 2\pi/k_T$.

(g) The Toomre mass, defined as $M_T = \lambda_T^2\Sigma_0$, is the characteristic mass of an unstable fragment produced by Toomre instability. Compute M_T, and evaluate it for $Q = 1$, $\Sigma_0 = 12$ pc^{-2} and $c_s = 6$ km s^{-1}, typical values for the atomic ISM in the solar neighborhood. Compare the mass you find to the maximum molecular cloud mass observed in the Milky Way as reported by Rosolowsky (2005, *PASP*, 117, 1403).

5. BONUS: **A Simple T Tauri Disk Model.**
In this problem we will construct a simple model of a T Tauri star disk in terms of a few parameters: the midplane density and temperature ρ_m and T_m, the surface temperature T_s, the angular velocity Ω, and the specific opacity of the disk material κ. We assume that the disk is very geometrically thin and optically thick, and that it is in thermal and mechanical equilibrium.

(a) Assume that the disk radiates as a blackbody at temperature T_s. Show that the surface and midplane temperatures are related approximately by

$$T_m \approx \left(\frac{3}{8}\kappa\Sigma\right)^{1/4} T_s$$

where Σ is the disk surface density.

For an optically thick atmosphere, we use the diffusion approximation for energy transport:

$$F = \frac{c}{3\kappa\rho}\frac{d}{dz}E = \frac{4\sigma_{SB}}{3\kappa\rho}\frac{d}{dz}T^4$$

In thermal equilibrium, the flux is constant with position.

You can assume $T_m >> T_s$ to solve for the flux at the midplane. The surface flux per unit area is given by the Stephan-Boltzmann law.

(b) Suppose the disk is characterized by a standard α model, meaning that the viscosity $\nu = \alpha c_s H$, where H is the scale height and c_s is the sound speed. For such a disk the rate per unit area of the disk surface (counting each side separately) at which energy is released by viscous dissipation is $F_d = (9/8)\nu\Sigma\Omega^2$. Derive an estimate for the midplane temperature T_m in terms of Σ, Ω, and α.

(c) Calculate the cooling time of the disk in terms of the orbital period. Should the behavior of the disk be closer to isothermal or adiabatic?

(d) Consider a disk with a mass of 0.03 orbiting a 1 star, which has $\kappa = 3$ cm^2 g^{-1} and $\alpha = 0.01$. The disk runs from 1 to 20 AU, and the surface density varies with radius ϖ as ϖ^{-1}. Use your model to express ρ_m, T_m, and T_s as functions of the radius, normalized to 1 AU; i.e., derive results of the form $\rho_m = \rho_0(\varpi/\mathrm{AU})^p$ for each of the quantities listed. Is your numerical model disk gravitationally unstable (i.e., $Q < 1$) anywhere?