Homework Two

Monday, October 3, 2022 6:26 PM

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1. $\rho_0 = \frac{3}{4\pi}\frac{M}{R^3}$ (uniform density)

a) $\frac{dP(r)}{dr} = \frac{-GM(r)\rho(r)}{r^2}$

$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$

$M(r) = \int_0^r 4\pi r^2 \rho_0 dr = \frac{4}{3}\pi r^3 \rho_0$

$\frac{dP(r)}{dr} = \frac{-4\pi}{3}\frac{G\rho_0^2 r^{\cancel{3}}}{\cancel{r^2}}$

$P(r) = -\frac{4\pi G}{3}\rho_0^2 \int_r^R r\,dr \quad \frac{r^2}{2}$

$$\boxed{P(r) = \frac{2\pi}{3}G\rho_0^2(R^2 - r^2)}$$

b) $P(0) = \frac{2\pi}{3}G\rho_0^2 R^2$

$= \frac{\cancel{2\pi}}{\cancel{3}}G\frac{3\cancel{9}}{\cancel{16}\pi}\frac{M^2}{R^{\cancel{6}4}}\cancel{R^2}$ (8)

$$\boxed{P_c = \frac{3}{8\pi}\frac{M^2}{R^4},\ k = \frac{3}{8\pi}}$$

c) $P_c = \frac{3}{8\pi}G\frac{M^2}{R^4}$

$k_B = 1.38\times10^{-16}$

$M_\odot = 2\times10^{33}$ g

$R_\odot = 6.957\times10^{10}$ cm

$G = 6.67\times10^{-8}$ [cgs]

$$\boxed{P_c = 1.36\times10^{15}\ \text{dyn/cm}^2}$$

$\rho_c = \frac{3}{4\pi}\frac{M}{R^3}$

$$\boxed{\rho_c = 1.42\ \text{g/cm}^3}$$

$P_c = \frac{\rho_c k T_c}{\mu m_H}$ $\qquad m_H = 1.67\times10^{-27}$ g

$\mu = 2X + \frac{3}{4}Y + \frac{1}{2}Z = 0 + \frac{3}{4}(0.36) + \frac{1}{2}(0.64)$

$\mu^{-1} = 0.59 \Rightarrow \mu = 1.695$

$\bar{\mu} = 0.59 \Rightarrow \mu = 1.695$

$$T_c = \frac{\mu m_H P_c}{\rho_c k_B} = \frac{\mu m_H}{k_B} \frac{3}{8\pi} G \frac{M^2}{R^4} \frac{3}{4\pi} \frac{M}{R^3}$$

$$\boxed{T_c = 19{,}664.78\ K}$$

vs 1.3×10^7 K for hydrogen burning

Too far below burning temp

2. $P = K\rho^2 \quad P(z=0) = P_0$

$\rho(z=0) = \rho_0$

a) find density in terms of ρ_0 $\quad P = K\rho^2$

$$\frac{dP}{dz} = -\cancel{\rho(z)} g = 2K\cancel{\rho}\frac{d\rho}{dz} \qquad d\rho = \frac{-g}{2K}\int dz$$

$$\frac{d\rho}{dz} = \frac{-g}{2K} \qquad \rho(z) = \frac{-g}{2K} z + C$$

$$\rho(z=0) = \rho_0 = 0 + C \Rightarrow \rho_0 = C$$

$$\rho(z_*) = 0 = \frac{-g}{2K} z_* + \rho_0 \qquad \boxed{\rho(z) = \frac{-g}{2K} z + \rho_0}$$

$$\boxed{\frac{2\rho_0 K}{g} = z_*}$$

b) Given $\tau = 1$ between $0 < z < z_*$

$$\tau = \int_0^{z_*} K_c \rho\, dz = K_c \int_0^{z_*} \left(\frac{-g}{2K} z + \rho_0\right) dz$$

$$= K_c \left[\frac{-g}{4K} z^2 + \rho_0 z\right]\Big|_0^{z_*}$$

$$\tau = K_c \left[\frac{-g}{4K} z_*^2 + \rho_0 z_*\right] = 1$$

$$\boxed{K_c = \left[\frac{-g}{4K} z_*^2 + \rho_0 z_*\right]^{-1}}$$

$$\boxed{\tau = \int \frac{\frac{-g}{2K} z + \rho_0}{\frac{-g}{4K} z_*^2 + \rho_0 z_*} dz}$$

c) $P = \frac{\rho k_B T}{m} \qquad PV = nkT$

assume hydrostatic eq.

$\overline{\qquad}$
m assume hydrostatic eq.

$$\cancel{\frac{dP}{dz} = -\rho g = \frac{dP}{dz} = \frac{k_B}{m}\left(\frac{d\rho}{dz}T + \rho\frac{dT}{dz}\right)}$$

$$P = \frac{\rho k_B T}{m} = K\rho^2 \Rightarrow T = \frac{mK\rho}{k_B}$$

$$T(z) = \frac{mK}{k_B}\left(\frac{-g}{2K}z + \rho_0\right) \Rightarrow \boxed{T(z) = \frac{m}{k_B}\left(\frac{-g}{2}z + \rho_0 K\right)}$$

$$B_\nu(\nu, T) = \frac{2\nu^2 k_B T}{c^2} \text{ so } B_\nu = \frac{2\nu^2}{c^2}(-gz + 2\rho_0 K)$$

so $a = 4\rho_0 K \frac{\nu^2}{c^2}$ and $b = -2g\frac{\nu^2}{c^2}$

$$\beta = \frac{P_{gas}}{P_{tot}} = \text{const.}$$

3. $U = \int (u_{gas} + u_{rad})\, dm$

$$E = U + \Omega = \frac{-\beta}{2-\beta}U = \frac{\beta}{2}\Omega$$

Implications for $\beta = 0$ and $\beta = 1$?
(rad. dominated) (gas dominated)

E = total energy
Ω = total grav. potential energy
U = total thermal energy

a) Virial theorem

$$E = U + \Omega = \frac{-\beta}{2-\beta}U = \frac{\beta}{2}\Omega$$

$\beta = 0$

$$\frac{-0}{2-\beta}U = \frac{0}{2}\Omega$$

$$U = \Omega\ ?$$

No grav. potential, unstable

$\beta = 1$

$$\frac{-1}{2-1}U = \frac{1}{2}\Omega$$

$$-U = \frac{1}{2}\Omega$$

Star in equilibrium, Virial Theorem holds

b) Stability of ★'s, contraction & expansion

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$\beta = 0$ unstable → no const. total energy, will expand & contract

$\beta = 1$ stable → const. energy, will not expand or contract