HomeworkOne_handwritten

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1. $\frac{1}{2}\ddot{I} = 2(T - T_s) + W$

$T_s = 4\pi R^3 P_s$ $\quad W = a\frac{GM^2}{R}$

$T = \frac{3}{2} M c_s^2$ $\quad \rho = \text{const.}$

$\ddot{I} = 0$ $\quad a = \frac{-3}{5}$ for uniform cloud

$$0 = 2\left[\frac{3}{2}Mc_s^2 - 4\pi R^3 P_s\right] - \frac{3}{5}\frac{GM^2}{R}$$

$$0 = 3Mc_s^2 - 8\pi R^3 P_s - \frac{3}{5}\frac{GM^2}{R}$$

$$8\pi R^3 P_s = 3Mc_s^2 - \frac{3}{5}\frac{GM^2}{R}$$

$$P_s = \frac{3M}{8\pi}\left(\frac{c_s^2}{R^3} - \frac{1}{5}\frac{GM}{R^4}\right)$$

2. $\frac{dP_s}{dR} = 0 = \frac{d}{dR}\left[c_s^2 R^{-3} - \frac{1}{5}GMR^{-4}\right]$

$$0 = -3c_s^2 R^{-4} + \frac{4}{5}GMR^{-5}$$

$$3c_s^2 R^{-4} = \frac{4}{5}GMR^{-5}$$

$$3c_s^2 R = \frac{4}{5}GM$$

$$R = \frac{4}{15}\frac{GM}{c_s^2}$$

$$P_s = \frac{3}{8}\frac{M}{\pi}\left[c_s^2\left(\frac{15c_s^2}{4GM}\right)^3 - \frac{1}{5}GM\left(\frac{15c_s^2}{4GM}\right)^4\right]$$

$$P_{s,max} = 1.57\frac{c_s^8}{G^3M^2}$$

3. $-\frac{1}{\rho}\frac{d}{dr}P = \frac{d}{dr}\phi$

$P = \rho c_s^2$ $\quad \phi =$ grav. potential

$\rho_c = \rho(r=0)$

$\phi = 0$ @ $r = 0$

$$\frac{d}{dx}(\ln y(x)) = \frac{1}{y}\frac{dy}{dx}$$

$$-\frac{1}{\rho}\frac{d}{dr}(P) = \frac{d}{dr}(\phi)$$

$$\underbrace{-\frac{1}{\rho}\frac{d}{dr}(\rho c_s^2)}dr = \frac{d}{dr}(\phi)\,dr$$

$$\hookrightarrow -c_s^2\frac{1}{\rho}\frac{d\rho}{dr} = c_s^2\frac{d}{dr}(\ln\rho(r)) = \frac{d}{dr}(\phi)dr$$

$$\hookrightarrow -c_s^2 \frac{1}{\rho}\frac{d\rho}{dr} = c_s^2 \frac{d}{dr}(\ln \rho(r)) = \frac{d}{dr}(\phi) dr$$

$$-c_s^2 \int \frac{d}{dr}(\ln \rho(r))\, dr = \int \frac{d}{dr}(\phi)\, dr$$

$$-c_s^2 \ln(\rho(r)) + A = \phi \qquad \hookrightarrow \phi = c_s^2 \ln\left(\frac{-\rho(r)}{\rho_c}\right)$$

$$-c_s^2 \ln(\rho_c) + A = 0 \qquad \frac{\phi}{c_s^2} = \ln\left(\frac{-\rho}{\rho_c}\right)$$

$$A = +c_s^2 \ln(\rho_c) \qquad \frac{\rho}{\rho_c} = e^{\phi/c_s^2} \Rightarrow \boxed{\rho = \rho_c e^{-\phi/c_s^2}}$$

4. $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G\rho \qquad \psi = \phi/c_s^2$

$$\xi = \frac{r}{r_0} \text{ or } r = \xi r_0 \qquad \frac{d\psi}{dr} = \frac{d\psi}{d\xi}\frac{d\xi}{dr}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G \rho_c e^{-\phi/c_s^2}$$

$$\frac{d\xi}{dr} = \frac{d}{dr}\left(\frac{r}{r_0}\right) = \frac{1}{r_0}$$

$$\frac{c_s^2}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = 4\pi G\rho_c e^{-\psi}$$

$$\frac{c_s^2}{\xi^2 r_0^2}\frac{1}{\cancel{r_0}}\frac{d}{d\xi}\left(\xi^2 \cancel{r_0^2} \cdot \frac{1}{\cancel{r_0}}\frac{d\psi}{d\xi}\right) = 4\pi G\rho_c e^{-\psi} \qquad \frac{c_s^2}{r_0^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = 4\pi G\rho_c e^{-\psi}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = \frac{4\pi G\rho_c r_0^2}{c_s^2} e^{-\psi} \longrightarrow r_0^2 = \frac{c_s^2}{4\pi G\rho_c}$$

$$\therefore \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = e^{-\psi}} \quad \checkmark$$

5. $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = e^{-\psi}$

a) $\psi' = \frac{d\psi}{d\xi}$

$$\frac{1}{\xi^2}\frac{d}{\xi}\left(\xi^2 \psi'\right) = e^{-\psi}$$

$$\frac{1}{\xi^2}\frac{d}{\xi}\left(\xi^2\psi'\right) = e$$

$$\frac{1}{\xi^2}\left(2\xi\psi' + \xi^2\frac{d\psi'}{d\xi}\right) = e^{-\psi}$$

$$\frac{d\psi'}{d\xi} = e^{-\psi} - \frac{2}{\xi}\psi' \quad \Big| \quad \underline{\psi = 0} \quad \underline{\frac{d\psi}{d\xi} = 0} \quad \xi = 0 = \frac{r}{r_0}$$

$$\psi(\xi) = \cancel{a_0} + \cancel{a_1\xi} + a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)$$

$$\psi(\xi) = a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)$$

$$\underline{\psi' = \frac{d\psi}{d\xi}} = 2a_2\xi + 3a_3\xi^2 + \mathcal{O}'(3)$$

b) $$\frac{d\psi'}{d\xi} = e^{a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)} - \frac{2}{\xi}\left[2a_2\xi + 3a_3\xi^2 + \mathcal{O}(3)\right]$$

$$= e^{a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)} - 4a_2 - 6a_3\xi + \mathcal{O}(2)$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$e^{a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)} = 1 + a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4) + \left(a_2\xi^2 + a_3\xi^3 + \mathcal{O}(4)\right)^2$$

$$\frac{d\psi'}{d\xi} = 1 + \cancel{a_2\xi^2} + \cancel{a_3\xi^3} - 4a_2 - 6a_3\xi + \cancel{\mathcal{O}(2)}$$

$$\frac{d\psi'}{d\xi} = 1 - 4a_2 - 6a_3\xi + \mathcal{O}(2)$$

$$\frac{d\psi'}{d\xi} = \frac{d}{d\xi}\left(\overbrace{2a_2\xi + 3a_3\xi^2 + \mathcal{O}(3)}\right)$$

$$= 2a_2 + 6a_3\xi + \mathcal{O}(2)$$

$$2a_2 + 6a_3\xi + \mathcal{O}(2) = 1 - 4a_2 - 6a_3\xi + \mathcal{O}(2)$$

$$6a_2 + 12a_3\xi + \mathcal{O}(2) = 1 + \mathcal{O}(2)$$

$$a_2 = 1/6 \qquad a_3 = 0$$

$a_2 = 1/6 \qquad a_3 = 0$

$\therefore \; \psi(\xi) = \dfrac{\xi^2}{6} + O(4)$

so,

$\psi' = \dfrac{d\psi}{d\xi} = \dfrac{\xi}{3} + O(3)$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = e^{-\psi}$$

6. Mass enclosed:

$M = 4\pi \int_0^R \rho r^2 dr \Rightarrow M = \dfrac{c_s^4}{\sqrt{4\pi G^3 P_s}}\left(e^{-\psi/2}\xi^2\dfrac{d\psi}{d\xi}\right)\Big|_{\xi_s}$ $\qquad r_0^2 = \dfrac{c_s^2}{4\pi G\rho_c}$

$\xi_s = \dfrac{R}{r_0}$, $dR = r_0 d\xi_s$ $\qquad \dfrac{P}{P_c} = \dfrac{\rho}{\rho_c} = e^{-\psi}$ $\qquad \underline{P_s = \rho_s c_s^2}$ $\qquad \psi = \phi/c^2$ $\qquad \dfrac{P}{P_c} = \dfrac{\rho}{\rho_c} = e^{-\psi}$

$R = \xi_s r_0$ $\qquad \rho_c = \rho e^{\psi}$

$M = 4\pi \int_0^{\xi_s r_0} \rho \xi^2 r_0^3 d\xi = 4\pi\rho_c r_0^3 \int_0^{\xi_s} e^{-\psi}\xi^2 d\xi$ $\qquad r_0 = \dfrac{c_s}{\sqrt{4\pi G\rho_c}}$

$e^{-\psi}\xi^2 = \dfrac{d}{d\xi}\left(\xi^2\dfrac{d\psi}{d\xi}\right)$ $\qquad c_s = \sqrt{\dfrac{P_s}{\rho_s}}$

$M = 4\pi\rho_c r_0^3 \int_0^{\xi_s} \dfrac{d}{d\xi}\left(\xi^2\dfrac{d\psi}{d\xi}\right)d\xi = 4\pi\rho_c r_0^3\left(\xi^2\dfrac{d\psi}{d\xi}\right)\Big|_{\xi_s}$

$= \dfrac{\cancel{4\pi}\rho_c c_s^3}{(4\pi G\rho_c)^{3/2}}\left(\xi^2\dfrac{d\psi}{d\xi}\right)\Big|_{\xi_s} \cdot \dfrac{c_s}{\sqrt{P_s/\rho_s}} = \dfrac{c_s^4}{\sqrt{4\pi G^3\rho_c P_s/\rho_s}}\left(\xi^2\dfrac{d\psi}{d\xi}\right)\Big|_{\xi_s}$

$M = \dfrac{c_s^4}{\sqrt{4\pi G^3 \rho e^{\psi} P_s/\rho_s}}\left(\xi^2\dfrac{d\psi}{d\xi}\right)\Big|_{\xi_s} \Rightarrow \boxed{M = \dfrac{c_s^4}{\sqrt{4\pi G^3 P_s \rho/\rho_s}}\left(e^{-\psi/2}\xi^2\dfrac{d\psi}{d\xi}\right)}$

☆ 7. $m = \dfrac{M\sqrt{G^3 P_s}}{c_s^4}$ $\qquad$ coding

$P_s = \dfrac{1}{G^3}\dfrac{m^2 c_s^8}{M^2}$

☆ 8. $P_{s,max} = \dfrac{1}{G^3}\dfrac{m_{max}^2 c_s^8}{M^2}$ $\qquad P_s = \rho_s c_s^2$ $\qquad P_s/k_B = 3\times10^5\ K\ cm^{-3}$

$T = 10K$, $\mu = 2.3$

8. $P_{s,max} = \frac{1}{G^3} \frac{m_{max}\, c_s}{M^2}$

$T = 10K,\ \mu = 2.3$

$c_s = \sqrt{k_B T/m}$

9. $M_{BE} = m_{max} \frac{c_s^4}{\sqrt{G^3 P_s}} = 1.18 \frac{c_s^4}{\sqrt{G^3 P_s}}$

$M_{BE} = 1.18 \frac{(k_B T/m)^2}{\sqrt{G^3 P_s}}$

$P_s/k_B = 10^7\, K\, cm^{-3}$
$T = 50K$

10. $r_0 = \frac{c_s}{\sqrt{4\pi G_0 \rho_c}}$

11. coding

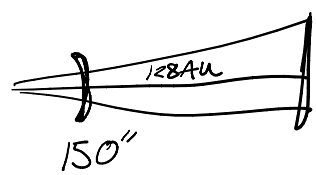
12. coding

13. refers to 7

$B68 = BE$ sphere
$\xi_{max} = 6.9$

Is the core stable? Reference part 7 for max stable dimensionless mass

14. B68 $r = 150''$ @ $d = 128$ AU $T = 10.5K$



Size?

Central density? $\xi = r/r_0$ $(\xi_{max} = r_{max}/r_0)$

15. Compare surface pressure

$r_0 = \frac{c_s}{\sqrt{4\pi G_0 \rho_c}}$ $\rho/\rho_c = 0.06$

$r_0^2\, 4\pi G \rho_c = c_s^2$

$\rho_c = \frac{c_s^2}{4\pi G r_0^2}$