HomeworkTwo

Tuesday, March 1, 2022 4:58 PM

1. Derive spectral slope of flat disk

$$T = T_0 \left(\frac{r}{r_0}\right)^{-q}$$

a) $$F_\nu = \frac{2\pi\cos\theta}{d^2}\int_{R_{min}}^{R_{max}} B_\nu(T(r))\, r dr$$

$B(T)$ = planck function

$$B(T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1}$$

$$B(r) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT_0 (r/r_0)^{q}} - 1\right)^{-1}$$

$d\ln(x) = \frac{1}{x}dx$

$$F_\nu = \frac{4\pi h\nu^3\cos\theta}{c^2 d^2}\int_{R_{min}}^{R_{max}}\left[\exp\left(\frac{h\nu}{kT_0}\left(\frac{r}{r_0}\right)^q\right) - 1\right]^{-1} r dr$$

$$u = \frac{h\nu}{kT_0}\left(\frac{r}{r_0}\right)^q \qquad du = \frac{h\nu}{kT_0}\frac{1}{r_0}q\left(\frac{r}{r_0}\right)^{q-1}dr \Rightarrow dr = \frac{kT_0 r_0}{h\nu q}\left(\frac{r_0}{r}\right)^{q-1}du$$

$$\frac{kT_0}{h\nu}r_0^q\, u = r^q \Rightarrow r = \left(\frac{kT_0}{h\nu}\right)^{1/q} r_0\, u^{1/q}$$

$$F_\nu = \frac{4\pi h\nu^3\cos\theta}{c^2 d^2}\left(\frac{kT_0}{h\nu}\right)^{1/q} r_0 \frac{kT_0 r_0}{h\nu q}(r_0)^{q-1}\int r^{q-1} u^{1/q}\frac{1}{e^u - 1}du$$

$$F_\nu = \frac{4\pi \nu^2\cos\theta}{c^2 d^2}\left(\frac{kT_0}{h\nu}\right)^{1/q}\frac{kT_0 r_0^{q+1}}{q}\int\left(\left(\frac{kT_0}{h\nu}\right)^{\frac{q-1}{q}} r_0^{q-1} u^{\frac{q-1}{q}} u^{1/q}\right)\left(\frac{1}{e^u-1}\right)du$$

$$\frac{q-1}{q} + \frac{1}{q} = \frac{q\cancel{-1+1}}{q} = 1$$

$$F_\nu = \frac{4\pi \nu^2\cos\theta}{c^2 d^2}\left(\frac{kT_0}{h\nu}\right)^{1/q}\frac{kT_0 r_0^{q+1}}{q}\left(\frac{kT_0}{h\nu}\right)^{\frac{q-1}{q}} r_0^{q-1}\int\frac{u}{e^u-1}du$$

$$F_\nu = \frac{4\pi\cos\theta}{c^2 d^2}\frac{\nu^2}{\nu^{1/q}}\left(\frac{kT_0}{h}\right)\left(\frac{kT_0}{q}\right) r_0^{2q}\int\frac{u}{e^u-1}du$$

$$F_\nu = \frac{4\pi\cos\theta}{c^2 d^2}\left(\frac{k^2 T_0^2}{hq}\right) r_0^{2q}\, \nu^{2-1/q}\int\frac{u}{e^u-1}du = C\nu^{2-1/q}$$

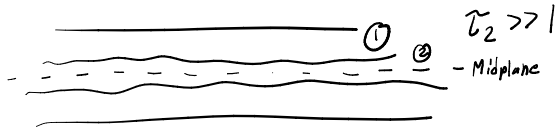
b) $$\alpha_{IR} = -1 - \frac{d\log F_\nu}{d\log\nu} \qquad d\log x = \frac{1}{x}dx$$

$$\frac{d\log F_\nu}{d\log\nu} = \frac{\frac{1}{F_\nu}\cdot d(F_\nu)}{\frac{1}{\nu}\cdot d\nu} = \frac{\nu}{F_\nu}\frac{\partial F_\nu}{\partial\nu} = \frac{1}{\cancel{C}\cancel{\nu^{1-1/q}}}\cancel{C}\cdot(2-1/q)\cancel{\nu^{1-1/q}} = 2 - 1/q$$

$$\alpha_{IR} = -1 - 2 + 1/q = \boxed{-3 + 1/q = \alpha_{IR}}$$

$$\alpha_{IR} = -1 - 2 + 1/q = \boxed{-3 + 1/q = \alpha_{IR}}$$

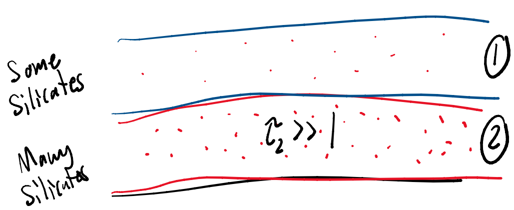
2.



$$I_\nu = B_\nu(T_1)(1 - e^{-\kappa_\nu \rho_1 h_1}) + B_\nu(T_2)(1 - \cancel{e^{-\kappa_\nu \rho_2 h_2}})$$

$$I_\nu = B_\nu(T_2) + B_\nu(T_1)(1 - e^{-\kappa_\nu \rho_1 h_1})$$

$$B_\nu(T_2) \longrightarrow B_\nu(T_1)$$



(8)	(9)	(10)
$T_1 > T_2$	$T_1 > T_2$	$T_1 < T_2$
$\tau_1 << 1$	$\tau_1 >> 1$	$\tau << 1$
optically thin	optically thick	optically thin
Emission Lines	Nothing	Absorption

3. $$\frac{\partial \Sigma}{\partial t} = \frac{3}{\omega} \frac{\partial}{\partial \omega}\left[\omega^{1/2} \frac{\partial}{\partial \omega}\left(\nu \Sigma \omega^{1/2}\right)\right]$$

$$V = v_1 \left(\frac{\omega}{\omega_1}\right)$$

a) $x = \frac{\omega}{\omega_1}$, $T = \frac{t}{t_s}$, $S = \frac{\Sigma}{\Sigma_1}$, $t_s = \frac{\omega_1^2}{3v_1}$

$$\frac{\partial \Sigma}{\partial t} = \frac{3v_1}{\omega} \frac{\partial}{\partial \omega}\left[\omega^{1/2} \frac{\partial}{\partial \omega}\left(\frac{\omega^{3/2}}{\omega_1} \Sigma\right)\right]$$

$$\frac{\partial \Sigma}{\partial t} = \frac{3v_1}{x\omega_1} \frac{\partial}{\partial \omega}\left[(x\omega_1)^{1/2} \frac{\partial}{\partial \omega}\left(\frac{(x\omega_1)^{3/2}}{\omega_1} S \Sigma_1\right)\right]$$

$$dx = \frac{1}{\omega_1} d\omega \Rightarrow \frac{1}{\omega_1}\frac{\partial}{\partial x} = \frac{\partial}{\partial \omega}, \quad \frac{d\Sigma}{dt} = \Sigma_1 \frac{dS}{dt}$$

$$\cancel{\Sigma_1} \frac{\partial S}{\partial t} = \frac{3v_1}{x\omega_1^2} \frac{\partial}{\partial x}\left[\frac{(x\omega_1)^{1/2}}{\omega_1} \frac{\partial}{\partial x}\left(\frac{(x\omega_1)^{3/2}}{\omega_1} S \cancel{\Sigma_1}\right)\right]$$

$$dT = \frac{dt}{t_s} \Rightarrow dt = t_s \, dT$$

$$\frac{\partial S}{\partial t} = \frac{1}{x t_s} \frac{\partial}{\partial x}\left[x^{1/2} \frac{\partial}{\partial x}\left(x^{3/2} S\right)\right]$$

$$\frac{\partial t}{} \quad x t_s \, \partial x \, [$$

$$\frac{\partial S}{\partial T} = \frac{1}{x}\frac{\partial}{\partial x}\left[x^{1/2}\frac{\partial}{\partial x}\left(x^{3/2} S\right)\right]$$

b) $\Sigma = \left(\frac{C}{3\pi\nu_1}\right)\frac{e^{-x/T}}{xT^{3/2}}$ $\qquad S = \frac{\Sigma}{\Sigma_1} \Rightarrow \Sigma = S\Sigma_1$

$$S = \left(\frac{C}{3\pi\nu_1}\right)\frac{e^{-x/T}}{xT^{3/2}}\frac{1}{\Sigma_1}$$

$$= \frac{1}{x}\frac{\partial}{\partial x}\left[x^{1/2}\frac{\partial}{\partial x}\left(x^{3/2}\frac{C}{3\pi\nu_1}\frac{e^{-x/T}}{xT^{3/2}}\frac{1}{\Sigma_1}\right)\right]$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\frac{\partial}{\partial x}\left[x^{1/2}\frac{\partial}{\partial x}\left(\frac{x^{1/2}}{T^{3/2}}e^{-x/T}\right)\right]$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\frac{\partial}{\partial x}\left[x^{1/2}\left(\frac{1}{2}\frac{x^{-1/2}}{T^{3/2}}e^{-x/T} - \frac{x^{1/2}}{T^{5/2}}e^{-x/T}\right)\right]$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\frac{\partial}{\partial x}\left[\frac{1}{2}T^{-3/2}e^{-x/T} - \frac{x}{T^{5/2}}e^{-x/T}\right]$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\left[-\frac{1}{2}T^{-5/2}e^{-x/T} + \frac{x}{T^{7/2}}e^{-x/T} - \frac{1}{T^{5/2}}e^{-x/T}\right]$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\left[\frac{-3}{2x}T^{-5/2}e^{-x/T} + \frac{1}{T^{7/2}}e^{-x/T}\right] =$$

$$\frac{\partial S}{\partial T} = \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\frac{\partial}{\partial T}\left(T^{-3/2}e^{-x/T}\right)$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\frac{1}{x}\left(-\frac{3}{2}T^{-5/2}e^{-x/T} + xT^{-7/2}e^{-x/T}\right)$$

$$= \frac{C}{3\pi\nu_1\Sigma_1}\left(-\frac{3}{2x}T^{-5/2}e^{-x/T} + T^{-7/2}e^{-x/T}\right)$$

c) Calculate the total mass in the disk in terms of C, t_s & t. Calculate the time rate of change. Physical interpretation of C? [units of C=?]

$$\frac{\partial S}{\partial T} = \frac{1}{x}\frac{\partial}{\partial x}\left[x^{1/2}\frac{\partial}{\partial x}\left(x^{3/2} S\right)\right]$$

$x = \frac{\omega}{\omega_1} \Rightarrow \omega = x\omega_1 \quad dx = \frac{d\omega}{\omega_1} \Rightarrow d\omega = \omega_1 dx$

$$M = \int_0^\infty 2\pi\omega\,\Sigma\,d\omega$$

$$\Sigma = \left(\frac{C}{3\pi\nu_1}\right)\frac{e^{-x/T}}{xT^{3/2}}$$

$$M = \int_0^\infty 2\pi x\omega_1\Sigma\,\omega_1\,dx = 2\pi\omega_1^2\int_0^\infty x\Sigma\,dx$$

$$= \frac{2\pi\omega_1^2 C}{3\pi\nu_1 T^{3/2}}\int_0^\infty e^{-x/T}dx = \frac{2\omega_1^2 C}{3\nu_1 T^{1/2}} = 2Ct_s\left(\frac{t}{t_s}\right)^{-1/2}$$

$T = t/t_s$

$t_s = \frac{\omega_1^2}{3\nu_1}$

$$= \frac{2\cancel{\pi}\,\omega_1^2 C}{3\cancel{\pi}\,v_1 T^{3/2}} \underbrace{\int_0^\infty e^{-x/T}\,dx}_{T} = \frac{2\omega_1 C}{3 v_1 T^{1/2}} = 2Ct_s\left(\frac{}{t_s}\right) \qquad t_s = \frac{\omega_1^2}{3v_1}$$

$$M = 2Ct_s\left(\frac{t}{t_s}\right)^{-1/2} \qquad \frac{dM}{dt} = 2Ct_s^{3/2}\frac{d}{dt}\left(t^{-1/2}\right) = \cancel{2}Ct_s^{3/2}\left(\frac{-1}{\cancel{2}}t^{-3/2}\right)$$

units of C = mass/time $\qquad \frac{dM}{dt} = -C\left(\frac{t}{t_s}\right)^{-3/2}$

The constant C is related to the accretion rate of mass onto the protostar.

d) On Jupyter Notebook