ENGR 8103: Problem Set #5

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Problem 1

If $T = A(h) + a_1 h^{1/2} + a_2 h^{2/2} + a_3 h^{3/2}$, then what combination of $A(h)$ and $A(h/2)$ should give an accurate estimate of T ?

$$T = A(h) + a_1 h^{1/2} + a_2 h^{2/2} + a_3 h^{3/2} + \cdots \tag{1}$$

$$T = A(h/2) + \frac{a_1 h^{1/2}}{2^{1/2}} + \frac{a_2 h^1}{2} + \frac{a_3 h^{3/2}}{2^{3/2}} + \cdots \tag{2}$$

Combine (1) and equation (2) as follows $T - \frac{-T}{\sqrt{2}}$ this yields the resultant equation:

$$T - \frac{T}{\sqrt{2}} = A(h/2) - \frac{A(h)}{\sqrt{2}} + \underbrace{\frac{a_2 h^1}{2} + \frac{a_3 h^{3/2}}{2^3/2} - \frac{a_3 h^{3/2}}{\sqrt{2}} + \cdots}_{\text{Error (E)}} \tag{3}$$

Therefore the combination of $A(h) and A(h/2)$ is as follows:

$$T = \frac{A(h/2) - \frac{A(h)}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} + E \tag{4}$$

Problem 2

Consider a second order approximation $A(h)$ to T such that

$$T = A(h) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots \tag{5}$$

a) Find a sixth order approximation to T using $A(h), A(2h), A(3h)$
Using step size $2h$

$$-T = A(2h) + 4a_2 h^2 + 16a_4 h^4 + 32a_6 h^6 + \cdots \tag{6}$$

Combining equation 5 and 6.

$$-T = A(2h) + 4a_2 h^2 + 16a_4 h^4 + 32a_6 h^6 + \cdots \tag{7}$$

$$4T = 4A(h) + 4a_2 h^2 + 4a_4 h^4 + 4a_6 h^6 + \cdots \tag{8}$$

$$\approx 3T = 4A(h) - A(2h) - 12a_4 h^4 - \frac{60a_6}{h^6} 3 \cdots \tag{9}$$

$$T = \frac{4A(h) - A(2h)}{3} - 4a_4 h^4 - 20a_6 h^6 + \cdots \tag{10}$$

Now coarse correction using step size $3h$

$$T = A(3h) + 9a_2h^2 + 81a_4h^4 + 729a_6h^6 + \cdots \tag{11}$$

Combining equation 12 and 5

$$9T = 9A(h) + 9a_2h^2 + 9a_4h^4 + 9a_6h^6 + \cdots \tag{12}$$

$$-T = A(3h) + 9a_2h^2 + 81a_4h^4 + 729a_6h^6 + \cdots \tag{13}$$

$$\approx 9T - T = 9A(h) - A(3h) - 72a_4h^4 - 720a_6h^6 + \cdots \tag{14}$$

This yields...

$$T = \frac{9A(h) - A(3h)}{8} - 9a_4h^4 - 90a_6h^6 + \cdots \tag{15}$$

Cancel out high order term using equation 8 and equation 15

$$(4T = 4A(h) + 4a_2h^2 + 4a_4h^4 + 4a_6h^6 + \cdots) * 9 \tag{16}$$

$$(T = \frac{9A(h) - A(3h)}{8} - 9a_4h^4 - 90a_6h^6 + \cdots) * -4 \tag{17}$$

Subtract the equations

$$5T = \frac{3A(h)}{2} - \frac{3A(2h)}{5} - \frac{A(3h)}{10} + 180a_6h^6 \tag{18}$$

Solving gives you a 6th approximation of of T

$$T = \frac{3A(h)}{2} - \frac{3A(2h)}{5} - \frac{A(3h)}{10} + 180a_6h^6 \tag{19}$$

b) Applying part a)

$$T = \underbrace{\frac{3A(h)}{2} - \frac{3A(2h)}{5} - \frac{A(3h)}{10} + 180a_6h^6 + \cdots}_{\text{A(h)}} \tag{20}$$

$$A(h) = \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] + \mathcal{O}(h^6) \tag{21}$$

By finding the common denominator of all of the fractional componenents in equation 20 you can solve the following

$$f''(x) = \frac{1}{36h^2}[49f(x+h)+49f(x+2h)+49f(x+3h)-294f(x)+49f(x-h)+49f(x-2h)+49f(x-3h)]+\mathcal{O}(h^6) \tag{22}$$

Problem 3

(a): Consider the following IVP ?

$$x' = 1 + t + x^2 \tag{23}$$

$$x(0) = 1 \tag{24}$$

```
%
% Euler's method
% Function provided Dr. Caner for Comp Engr 8103
%

function Euler

f = @(t,x) 1 + t + x^2;
```

```
time = 4;
h = 1;
t(1) = 0;
x(1) = 1;

for i=1:time
  x(i+1) = x(i) + h*f(t(i),x(i));
  t(i+1) = t(i) + h;
  if (i == 1)
      fprintf('index:%i %i\n', 0 , 1)
  end
  fprintf('index:%i %i\n',i,x(i+1))
end

plot(t,x,'.-');
savefig('./latex/docs/euler.fig')

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Ouput:

index:0 1
index:1 3
index:2 14
index:3 213
index:4 45586

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```

From Euler's method $x(2) = 14$

(b): Use second-order Taylor Series method to estimate $x(2)$ with step size $h = 1$

```
%
% Taylor's series second order method
% Allen Spain
%

function Taylor

f = @(t,x) 1 + t + x^2;

ff = @(t,x) 1 + 2*x + 2*x*t + 2*x^3;

time = 4;
h = 1;
t(1) = 0;
x(1) = 1;

for i=1:time
  x(i+1) = x(i) + h*f(t(i),x(i)) + h^2*ff(t(i),x(i))/2;
  t(i+1) = t(i) + h;
  if (i == 1)
      fprintf('index:%i %i\n', 0 , 1)
  end
  fprintf('index:%i %i\n',i,x(i+1))
end
```

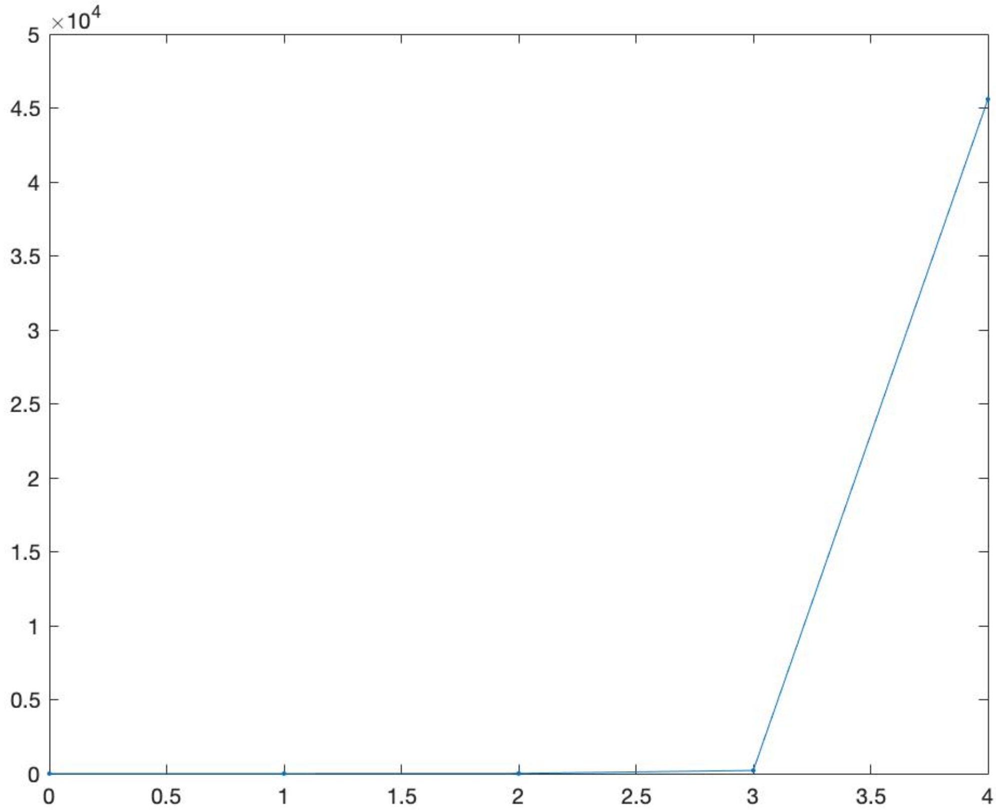

Figure 1: Euler's method for approximating x

```
plot(t,x,'.-');
savefig('./latex/docs/euler.fig')

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Ouput:

index:0 1
index:1 5.500000e+00
index:2 2.156250e+02
index:3 1.007266e+07
index:4 1.021957e+21

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```

Using Taylor series second order approximation $x(2) \approx 215$

(c): Use third-order Taylor Series method to estimate $x(2)$ with step size $h = 1$ $x(2) \approx 399410$

```
%
% Taylor's third order method for
% Allen Spain
%

function Taylor

f = @(t,x) 1 + t + x^2;

ff = @(t,x) 2*x*t + 2*x*t^2  + 12* x^2 + 2*x^3 + 2*x^2*t + 12 * x^4;

time = 4;
h = 1;
t(1) = 0;
x(1) = 1;

for i=1:time
  x(i+1) = x(i) + h*f(t(i),x(i)) + h^2*ff(t(i),x(i))/2;
  t(i+1) = t(i) + h;
  if (i == 1)
      fprintf('index:%i %i\n', 0 , 1)
  end
  fprintf('index:%i %i\n',i,x(i+1))
end

plot(t,x,'.-');
savefig('./latex/docs/euler.fig')

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Ouput:

index:0 1
index:1 3
index:2 14
index:3 213
index:4 45586

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```