$$\text{Computational Engineering - Engr 8103}$$

# Computational Engineering - Engr 8103
# Problem Set #9


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## Question 1

(5 pts.) We use Von-Neumann stability analysis to make sure that the solution un does not increase in size as n increases. We do this by making sure that $\|g(dt, dx, \theta)\| \leq 1$ for all $\theta$ where

$$\hat{u}^{n+1}(\theta) = g(dt, dx, \theta)\hat{u}^n(\theta)$$

and therefore

$$\|\hat{u}^{n+1}(\theta)\| = \|g(dt, dx, \theta)\hat{u}^n(\theta)\| \stackrel{(*)}{=} \|g(dt, dx, \theta)\|\|u^n(\theta)\| \leq \|\hat{u}^n(\theta)\|$$

For this to work, we still need to show $(*)$. In other words, for complex numbers $z_1$ and $z_2$, show that the following is always true.

$$\|z_1 z_2\| = \|z_1\|\|z_2\|$$

Assume

$$z_1 = (a + ib)$$
$$z_2 = (c + id)$$
$$\|z_1\| = \sqrt{a + ib} = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}$$
$$\|z_2\| = \sqrt{c + id} = \sqrt{(c + id)(c - id)} = \sqrt{c^2 + d^2}$$
$$\therefore \|z_1\|\|z_1\| = \sqrt{(a^2 + b^2)(c^2 + d^2)}$$
$$z_1 z_1 = (a + ib)(c + id) = (ac - db) + i(ad + bc)$$
$$\|z_1 z_1\| = \sqrt{\underbrace{(ac - db)}_{\gamma} + \underbrace{i(ad + bc)}_{\beta}} = \sqrt{\gamma^2 + \beta^2}$$
$$\|z_1 z_1\| = \sqrt{(ac - db)^2 + (ad - bc)^2} = \sqrt{(ac)^2 + (db)^2 + (ad)^2 + (bc)^2 - 2adbc + 2adbc}$$
$$\|z_1 z_1\| = \sqrt{(ac)^2 + (db)^2 + (ad)^2 + (bc)^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)}$$
$$\therefore \|z_1 z_1\| = \|z_1\|\|z_1\|$$

So as long as $\|g(dt, dx, \theta)\| \leq 1$ for all $\theta$ then

$$\|g(dt, dx, \theta)\hat{u}_n(\theta)\| = \|g(dt, dx, \theta)\|\|u^n(\theta)\| \leq \|\hat{u}^n(\theta)\|$$

which yeilds stability by Von-Neumann analysis

# Question 2

(10 pts.) Determine the conditions for stability in terms of dt and dx using Von-Neumann stability analysis if **forward time, centered space** discretization used for the following PDE:

$$u_t - 4u_x = 0$$

Using $F_tC_x$ discretization

$$\begin{aligned} u_t - 4u_x \simeq \frac{U_k^{n+1} - U_k^n}{dt} + 4\frac{U_{k+1}^n - U_{k-1}^n}{2dx} = 0 \\ \frac{U_k^{n+1}}{dt} = \frac{U_k^n}{dt} - 2\frac{U_{k+1}^n - U_{k-1}^n}{dx} \end{aligned}$$

From Von-Nuemman stability analysis we know $U_k^n = \hat{U}^n e^{jk\theta}$ therefore

$$\begin{aligned} \frac{\hat{U}^{n+1}e^{jk\theta}}{dt} = \frac{\hat{U}^n e^{jk\theta}}{dt} + 2\frac{\hat{U}^n e^{(k+1)j\theta} - \hat{U}^n e^{(k-1)j\theta}}{dx} \\ \frac{\hat{U}^{n+1}e^{jk\theta}}{dt} = \hat{U}^n e^{jk\theta}\left[\frac{1}{dt} + 2\frac{e^{j\theta} - 2e^{-j\theta}}{dx}\right] \\ \hat{U}^{n+1} = \hat{U}^n \underbrace{\left[1 + \underbrace{2\frac{dt}{dx}}_{\text{L}}\left(e^{j\theta} - e^{-j\theta}\right)\right]}_{\text{B}} \end{aligned}$$

For stability $B \leq 1$ so

$$\begin{aligned} B &= 1 + Le^{j\theta} - Le^{-j\theta} \\ &= \|1 + Le^{j\theta} - Le^{-j\theta}\| \leq 1 \\ &= \|1 + L\left[cos(\theta) + jsin(\theta) + cos(\theta) - jsin(\theta)\right]\| \\ &= \|1 + 2Lsin(\theta)j\| \leq 1 = \sqrt{1 + 4L^2sin^2(\theta)} \leq 1 \\ &= 1 + 4L^2sin^2(\theta) \leq 1 \\ &= 4L^2sin^2(\theta) \leq 0 \\ &\Rightarrow \mid L \mid = \mid \frac{2dt}{dx} \mid \leq 0 \end{aligned}$$

$$\begin{cases} L \text{ or } sin(\theta) \text{ must be} = 0 \text{ (or both)} \\ \text{if } sin(\theta) = 0 \text{ then } dt, dx \in \mathbb{R} \text{ but } dx \neq 0 \\ \text{if } sin(\theta) \neq 0 \text{ then } dt = 0 \text{ and } dx \in \mathbb{R} \text{ but } dx \neq 0 \end{cases}$$

# Question 3

(10 pts.) Determine the conditions for stability in terms of dt and dx using Von-Neumann stability analysis if **backward time, centered space** discretization used for the same PDE as in the previous problem.

Using $B_tC_x$ discretization

$$u_t - 4u_x \simeq \frac{U_k^n - U_k^{n-1}}{dt} + 4\frac{U_{k+1}^n - U_{k-1}^n}{2dx} = 0$$
$$\frac{U_k^{n-1}}{dt} = \frac{U_k^n}{dt} + 2\frac{U_{k+1}^n - U_{k-1}^n}{dx}$$
$$\textit{Let } n \to n+1$$
$$\frac{U_k^n}{dt} = \frac{U_k^{n+1}}{dt} + 2\frac{U_{k+1}^{n+1} - U_{k-1}^{n+1}}{dx}$$
$$\mathcal{F} : \hat{U}^n = \hat{U}^{n+1}\underbrace{\left[1 + 2\underbrace{\frac{dt}{dx}}_{\text{L}}\left(e^{j\theta} - e^{-j\theta}\right)\right]}_{\text{B}}$$

Since this analysis requires $\|\hat{U}^{n+1}(\theta)\| \leq \|\hat{U}^n(\theta)\|$ and $\|\frac{1}{A}\| = \frac{1}{\|A\|}$ where $A$ is some function

$$B = 1 + Le^{j\theta} - Le^{-j\theta}$$
$$= \|1 + Le^{j\theta} - Le^{-j\theta}\| \leq 1$$
$$= \|1 + L\left[cos(\theta) + jsin(\theta) + cos(\theta) - jsin(\theta)\right]\|$$
$$= \|1 + 2Lsin(\theta)j\| \leq 1 = \sqrt{1 + 4L^2sin^2(\theta)} \leq 1$$
$$= \frac{1}{1 + 4L^2sin^2(\theta)} \leq 1$$
$$= 4L^2sin^2(\theta) \geq 0$$
$$\Rightarrow \mid L \mid = \mid \frac{2dt}{dx} \mid \geq 0$$
$$\therefore dt \in \mathbb{R}, \text{ and } dx \in \mathbb{R}, \text{ but } dx \neq 0$$

# Question 4

(10 pts.) Determine the conditions for stability in terms of $dt$ and $dx$ using Von-Neumann stability analysis if **Crank-Nicholson** discretization with $s = \frac{2}{3}$ is used for the following PDE:

$$u_t = 2u_x$$

For your reference, this discretization is

$$\frac{u_k^{n+1}}{dt} = 2\left[\frac{2}{3}\frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{dx^2} + \frac{1}{3}\frac{u^nk + 1 - 2u_k^n + u_{k-1}^n}{dx^2}\right]$$

Discretization using Von-Nuemman stability analysis. I call $u = U$ The is setup in the form

$$\hat{U}^{n+1}e^{jk\theta}\left[1 - \frac{4}{3}\underbrace{\frac{dt}{(dx)^2}}_{\text{L}}B^{n+1}\right] = \hat{U}^n e^{jk\theta}\left[1 + \frac{2}{3}\underbrace{\frac{dt}{(dx)^2}}_{\text{L}}A^n\right]$$
$$\hat{U}^{n+1}e^{jk\theta}\left[1 - \frac{4}{3}L(e^{j\theta} - 2 + e^{-j\theta})\right] = \hat{U}^n e^{jk\theta}\left[1 + \frac{2}{3}L(e^{j\theta} - 2 + e^{-j\theta})\right]$$

Therefore

$$\|\frac{\left[1+\frac{2}{3}L(e^{j\theta}-2+e^{-j\theta})\right]}{\left[1-\frac{4}{3}L(e^{j\theta}-2+e^{-j\theta})\right]}\|\leq 1$$

$$\Rightarrow (1+\frac{2}{3}L(2cos(\theta)-2)^2=(1-\frac{4}{3}L(2cos(\theta)-2))^2$$

$$1+\frac{2}{3}L\leq 1-\frac{4}{3}L$$

$$0\leq -2L$$

$$\begin{cases} dt\leq 0\\ dx\neq 0\end{cases}$$