Computational Engineering - Engr 8103
Problem Set #4

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Question 1

(10 pts.) Derive the three point Gaussian quadrature formula. In other words, find coefficients A that make the following approximation exact for polynomials up to fifth degree.

$$\int_{-1}^{1} dx \simeq Af(-\alpha) + Bf(0) + Cf(\alpha)$$

Express its general form

$$\int_{a}^{b} dx \approx$$

General form for Gaussian quadrature is

$$\int_{a}^{b} dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Now you scale and translate the interval

$$\int_{a}^{b} f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx$$

Converting to the discretized formula

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)$$

Expanding terms

$$\int_{a}^{b} f(x)dx \approx= \frac{(b-a)}{2}\left(Af\left(\frac{(b-a)}{2}(-\alpha) + \frac{(a+b)}{2}\right) + Bf\left(\frac{(b-a)}{2})(0) + \frac{(a+b)}{2}\right) + Cf\left(\frac{(b-a)}{2}(\alpha) + \frac{(a+b)}{2}\right)\right)$$

After simplifying, you can generate a system of equations, this is done for all simple x functions that increase in degree up to $f(x) = x^5$

$$f(x) = 1, \quad \int_{-1}^{1} dx = A + B + C = 2$$
$$f(x) = x, \quad \int_{-1}^{1} xdx = C\alpha - A\alpha = 0$$
$$f(x) = x^2, \quad \int_{-1}^{1} x^2 dx = [\frac{x^3 3}{]}^{1}_{-1} = A\alpha^2 = C\alpha^2 = \frac{2}{3}$$
$$f(x) = x^3, \quad \int_{-1}^{1} x^3 dx = \frac{x^4}{4}]^1_{-1} = C\alpha^3 + A\alpha^3 = 0$$
$$f(x) = x^4, \quad \int_{1}^{-1} x^4 dx = [\frac{x^5}{5}]^1_{-1} = A\alpha^4 + C\alpha^4 = \frac{2}{5}$$
$$f(x) = x^5, \quad \int_{-1}^{1} x^5 dx = [\frac{x^6}{6}]^1_{-1} = C\alpha^5 - A\alpha^5 = 0$$

Now you can collect terms

$$C\alpha = A\alpha \rightarrow C = A$$
$$A\alpha^2 + C\alpha^2 = \frac{2}{3} \rightarrow A\alpha^2 + (A)\alpha^2 = \frac{2}{3} \rightarrow A = \frac{1}{3\alpha^2}$$
$$\text{therefore } A = \frac{1}{3\alpha^2} = C$$
$$A\alpha^4 + C\alpha^4 = \frac{2}{5} \rightarrow \frac{1}{3a^2}\alpha^4 = \frac{2}{5} \rightarrow \frac{\alpha^2}{3} + \frac{\alpha^2}{3} = \frac{2}{5} A\alpha^2 + C\alpha^2 = \frac{2}{3}$$
$$\rightarrow \frac{2\alpha^2}{3} = \frac{2}{5} \rightarrow 10\alpha^2 = 6 \rightarrow \alpha = \sqrt{\frac{3}{5}}$$
$$A = 3\alpha^2 \rightarrow A = \frac{1}{3\frac{3}{5}}^2 \alpha \rightarrow A = \frac{5}{9}$$
$$\text{subsititute} A = C = \frac{5}{9}$$
$$A + B + C = 2 \rightarrow \frac{5}{9} + B + (\frac{5}{9}) = 2$$
$$\therefore B = \frac{8}{9}$$

Substituting back into the original expanded sum:

$$\int_a^b \approx \frac{(b-a)}{2}(\frac{5}{9})f(\frac{a+b}{2} - \sqrt{\frac{3}{5}}\frac{(b-a)}{2}) + \frac{8}{9}f(\frac{(a+b)}{2}) + \frac{5}{9}f(\sqrt{\frac{3}{5}}\frac{(b-a)}{2} + \frac{a+b}{2}))$$

Question 2

(10 pts.) Derive a custom numerical integration formula of the form

$$\int_0^{pi} = f(x)dx \approx Af(0) + Bf(\frac{\pi}{2}) + Cf(\pi)$$

that is exact for all functions of type: $f(x) = a + bsinx + ccosx$

$$\int_0^{\pi} a + bsin(x) + ccos(x) = [ax]_0^{\pi} - [bcos(x)]_0^{\pi} + [csin(x)]_0^{\pi}$$
$$\therefore a\pi - b(-1-1) + c(0-0) = a\pi + 2b \approx Af(0) + Bf(\frac{\pi}{2}) + Cf(\pi)$$

Subsituting back into the

$$A(a + bsin((\theta)) + ccos((\theta))) + B(a + bsin((\pi)) + ccos((\pi))) + C(a + bsin((\pi)) + ccos((\pi))) \approx a\pi + 2b$$

simplifying

$$A(a + c) + B(a + b) + C(a - c) \approx a\pi + 2b$$

Notice $c(A - C) = 0 \Rightarrow A = C$
Also notice $bB = 2b, soB = 2$
Then, because $a(A + B + C) = \alpha\pi andA = C, A = C = \frac{\pi-2}{2}$

Finally, no change in variable is needed because the question explicitly asks for the interval of $[0, \pi]$, the formula is:

$$\int_0^{\pi} \approx \frac{\pi - 2}{2} f(0) + 2f(\frac{\pi}{2}) + \frac{\pi - 2}{2} f(\pi)$$

Question 3

(15 pts.) Consider the following integral

$$\int_0^{\pi} (3 + e^{\frac{-x}{2}} - sin(x) + 2cos(x))dx$$

(a) i.

$$f(x_0) = 3 + e^{\frac{0}{2}} - sin((0)) + 2cos((0)) = 6$$
$$2f(x_1) = 2(3 + e^{-\frac{\pi}{2}}) - sin(\frac{\pi}{2}) + 2cos(\frac{\pi}{2}) = 2e^{-\frac{\pi}{4}} + 4$$
$$f(x_2) = 3 + e^{-\frac{\pi}{2}} - sin(\pi) + 2cos(\pi) = e^{\frac{-\pi}{2}} + 1$$

So the integral estimation is 9.51883

ii. (2 pts.) Simpson's method using 3 function evaluations at $(0, \frac{\pi}{2}, \pi)$ Simpsons can be written as:

$$\int_b^a f(x)dx \approx \frac{b-a}{6} \left(f(a) + 4f\frac{(a+b)}{2} + f(b) \right)$$

This matches our size and bounds so

$$\frac{\pi}{6} \int_0^{\pi} f(x)dx \approx (f(0) + 4f(\frac{\pi}{2}) + f(\pi))$$

substitute in for $f(x)$:

$$f(0) = (3 + e^{\frac{-0}{2}} - sin(0) + 2cos(0)) = 6$$
$$4f(\frac{\pi}{2}) = 4(3 + e^{-(\pi/2)/2} - sin(\pi/2) + 2cos(\pi/2)) = 9.82375...$$
$$f(\pi) = (3 + e^{-(\pi)/2} - sin(\pi) + 2cos(\pi)) = 1.20787$$

So the integral estimate is ≈ 8.91774
iii. (2 pts.) Gaussian quadrature using 2 function evaluations The formula is:

$$\int_{-1}^{1} f(x)dx = Af(-\alpha) + Bf(\alpha)$$

$$f(x) = 1, \quad \int_{-1}^{1} dx = 2 = A + B$$
$$f(x) = x, \quad \int_{-1}^{1} xdx = 0 = B\alpha - A\alpha$$
$$f(x) = x^2, \quad \int_{-1}^{1} x^2 dx = \frac{2}{3} = A\alpha^2 + B\alpha^2$$

Notice, $A = B$ again, so $A = B = 1 \therefore A + B = 2$ Then substitute $A\alpha^2 + B\alpha^2 = \frac{2}{3}$ so $\alpha = \sqrt{\frac{1}{3}}$
Substituting

$$\int_0^{\pi} f(x)dx \approx \frac{\pi}{2}((1)f(\frac{\pi}{2} - \sqrt{\frac{1}{3}}\frac{\pi}{2}) + (1)f(\sqrt{\frac{1}{3}}\frac{\pi}{2} + \frac{\pi}{2}))) \Rightarrow \text{plug and chug} \Rightarrow 9.07112...$$

iv.(2 pts.) Gaussian quadrature using 3 function evaluations (Problem 1).
Now, $a = 0$ and $b = \pi$

$$\int_0^{\pi} f(x)dx \approx \frac{\pi}{2}\left(\frac{5}{9}f(\frac{a+b}{2} - \sqrt{\frac{3}{5}}\frac{b-a}{2}) + \frac{8}{9}f(\frac{\pi}{2}) + f(\sqrt{\frac{3}{5}}\frac{\pi}{2} + \frac{\pi}{2})\right)$$

Then, pluggin in known values into $f(x)$

$$\approx \pi(2.98165 + 2.18305 + 0.56971) \approx 9.00759...$$

v. (2 pts.) Gaussian quadrature using 4 function evaluations

$$\int_{-1}^{1} f(x)dx \approx \frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$$
$$+\frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}})$$
$$+\frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$$
$$+\frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}})$$

Adjusting for change of variable

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\frac{b-a}{2} +)$$
$$\frac{b-a}{2}\frac{18+\sqrt{30}}{36}f(\frac{b-a}{2} - \frac{b-a}{2}\sqrt{\frac{2}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}})$$
$$+\frac{b-a}{2}\frac{18+\sqrt{30}}{36}f(\frac{b-a}{2} + \frac{b-a}{2}\sqrt{\frac{2}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}})$$
$$+\frac{b-a}{2}\frac{18+\sqrt{30}}{36}f(\sqrt{\frac{2}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\frac{b-a}{2} + \frac{b-a}{2})$$

Subsituting the interval back in

$$\approx \frac{\pi}{2}(1.96446 + 2.44734 + 0.95885 + 0.37089) \approx 9.01878$$

vi. (2 pts.) Method you derived in Problem 2.

$$\int_0^{\pi} f(x)dx \approx \frac{\pi - 2}{2} f(0) + 2f(\frac{\pi}{2}) + \frac{\pi - 2}{2} f(\pi)$$

(b) (3 pts.) Take this interal and manually find its exact value. Find the absolute errors for each of the six methods above.

$$\int_0^3 (3 + e^{\frac{-x}{2}} - sinx + 2cosx)dx = \int_0^{\pi} 3dx + \int_0^{\pi} e^{\frac{-x}{2}} dx + \int_0^{\pi} sin(x)dx + \int_0^{\pi} 2cosxdx$$
$$= [3x]_0^{\pi} + [-2e^{\frac{-x}{2}}]_0^{\pi} + [cos(x)]_0^{\pi} + [2sin(x)]_0^{\pi} = 3\pi + (-2e^{\frac{-\pi}{2}} + 2) - (1 + 1) + (0 - 1) \approx 9.00901$$

The errors for the methods are in order as follows:
i. $|9.00901 - 9.51883| \approx 0.50982$
ii. $|9.00901 - 8.91774| \approx 0.09127$
iii.$|9.00901 - 9.07112| \approx 0.06211$
iv. $|9.00901 - 9.00759| \approx 0.00142$
v. $|9.00901 - 9.01878| \approx 0.00977$
vi. $|9.00901 - 9.02609| \approx 0.0170$