Computational Engineering - Engr 8103
Problem Set #7

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Problem 1

(10 pts.) Consider the following transport PDE:

$$
\begin{aligned}
3u_t - u_x &= 0 \\
u(0,x) &= \frac{3-x}{1+x^2} \quad 0 \le x \le 3 \\
u(t,3) &= sin(t) \quad t \ge 0
\end{aligned}
$$

From the general transport equation

$$u(t,x) = f(x-ct)$$

Because:

$$
\begin{aligned}
u_t + cux &= 0 \\
u_t &= -cu_x \\
u_t &= -1/3u_x \\
c &= 1/3
\end{aligned}
$$

Therfore:

$$u(t,x) = f(x-ct) = f(x + \frac{1}{3}t)$$

From BC1:

$$
\begin{aligned}
u(0,x) = \frac{3-x}{1+x^2} = f(u) &= \frac{3-u}{1+u^2} \\
u(t,x) &= \frac{3-x-\frac{1}{3}t}{1+(x+\frac{1}{3}t)^2}
\end{aligned}
$$

From BC2:

$$u(t,3) = sin(t) = f(3+\frac{1}{3}t)u = 3 + 1/3t = t = 3(u-3) = f(u) = sin(3(u-3))$$

substituting in for u

$$
\begin{aligned}
u(t,3) = sin(t) &= f(3+\frac{1}{3}t) \\
u(t,x) &= sin(3(x+1/3t-3))
\end{aligned}
$$

Then rewrite solution

$$u(t,x) = u(t,x) = \frac{3 - x - \frac{1}{3}t}{1 + (x + \frac{1}{3}t)^2} \quad t \leq 3 - x$$
$$u(t,x) = sin(3(x + 1/3t - 3)) \quad t > 3 - x$$

for

$$0 \leq x \leq 3$$
$$t \geq 0$$

Problem 2

2. (20 pts.) A drug is administered to a patient through injection. The drug concentration in the blood stream changes through blood flow and diffusion according to the following PDE:

$$u_t = D_{u_{xx}} - Fu_x$$
$$u(0,x) = \frac{2x}{1 + x^4} \quad 0 \leq x \leq 20$$
$$u(t,0) = 0 \quad 0 \leq t \leq 3$$
$$u(t,20) = 0 \quad 0 \leq t \leq 3$$

C

$$u_t = \frac{u_k^{n+1} - u_k^n}{dt}$$
$$u_x = \frac{u_n^k - u_{k-1}^n}{dx}$$
$$u_{xx} = \frac{u_{k-1}^n - 2u_k^n + u_{k+1}^n}{dx^2}$$

Combining terms:

$$u_t = Du_{xx} - Fu_x = \frac{u_k^{n+1} - u_k^n}{dt}$$
$$= D\frac{(u_{k-1}^n - 2u_k^n + u_{k+1}^n)}{dx^2} - F\frac{(u_k^n - u_{k-1}^n)}{dx}$$
$$\therefore u_k^{n+1} = D\frac{(u_{k-1}^n - 2u_k^n + u_{k+1}^n)}{dx}dt + u_k^n$$

(b)(10 pts.) Write a Matlab code to solve this PDE using the discretization you developed in problem 1. Use D = 0.5, F = 2, dt = 0.02, dx = 0.2. Your code should plot the initial drug concentration and the drug concentrations after one, two and three seconds on the same figure. In other words, plot u(0, x), u(1, x), u(2, x) and u(3, x) vs x. Identify each plot using a legend. Include a hard copy of this figure with your HW solutions.

```
function drug1
    dt = 0.02;
    dx = 0.2;

    x = 0:dx:20;
    t = 0:dt:3;

    Tmax = length(t);
    Xmax = length(x);
```

```
    u(:,1) = 0;
    u(:,Xmax) = 0;

    u(1 ,:) = (2.*x)./(1 + (x.^4)); plot(x,u(1 ,:) , 'linewidth',2);
        axis ([0 10 0 1]);
    hold on;

    for n = 1:(Tmax - 1) for k = 2:(Xmax - 1)
         u(n+1,k) = ((0.5*dt/(dx^2))*(u(n,k-1)-(2*u(n,k))+u(n,k+1)) -
             (2*dt*(u(n,k) - u(n,k-1))/dx) + u(n,k);
    end
         graph = n*dt
         if (graph==1 || graph==2 || graph==3)
             axis ([0 10 0 1]);
             plot(x,u(n+1,:),'linewidth',2);
             hold on;
         end
    end
    title('u(t,x) vs x');
    legend( 'u(0,x)' , 'u(1,x)', 'u(2,x)' ,'u(3,x)')
```

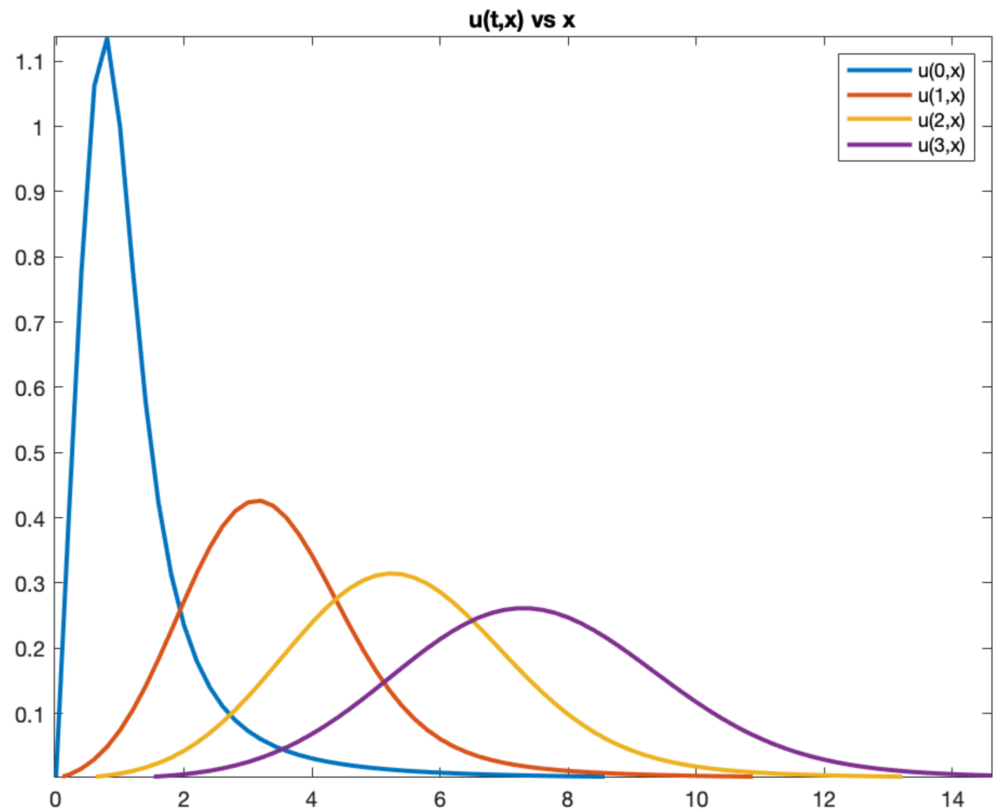

Figure 1: 2a

```
function drug2
    dt = 0.02;
```

```
dx = 0.2;

D = 1;
F = 1;


x = 0:dx:100;
t = 0:dt:50;

Tmax = length(t);
Xmax = length(x);

u(:,1) = 0;
u(:,Xmax) = 0;

u(1 ,:) = (2.*x)./(1 + (x.^4)); plot(x,u(1 ,:) , 'linewidth',2);
   axis ([0 10 0 2]);
hold on;

for n = 1:(Tmax - 1) for k = 2:(Xmax - 1)
    u(n+1,k) = ((D*dt/(dx^2))*(u(n,k-1)-(F*u(n,k))+u(n,k+1))) - (F*
       dt*(u(n,k) - u(n,k-1))/dx) + u(n,k);
end
    graph = n*dt
    if (graph==1 || graph==2 || graph==3)
        axis ([0 10 0 2]);
        plot(x,u(n+1,:),'linewidth',2);
        hold on;
    end
end
title('u(t,x) vs x');
legend( 'u(0,x)' , 'u(1,x)', 'u(2,x)' ,'u(3,x)')
```

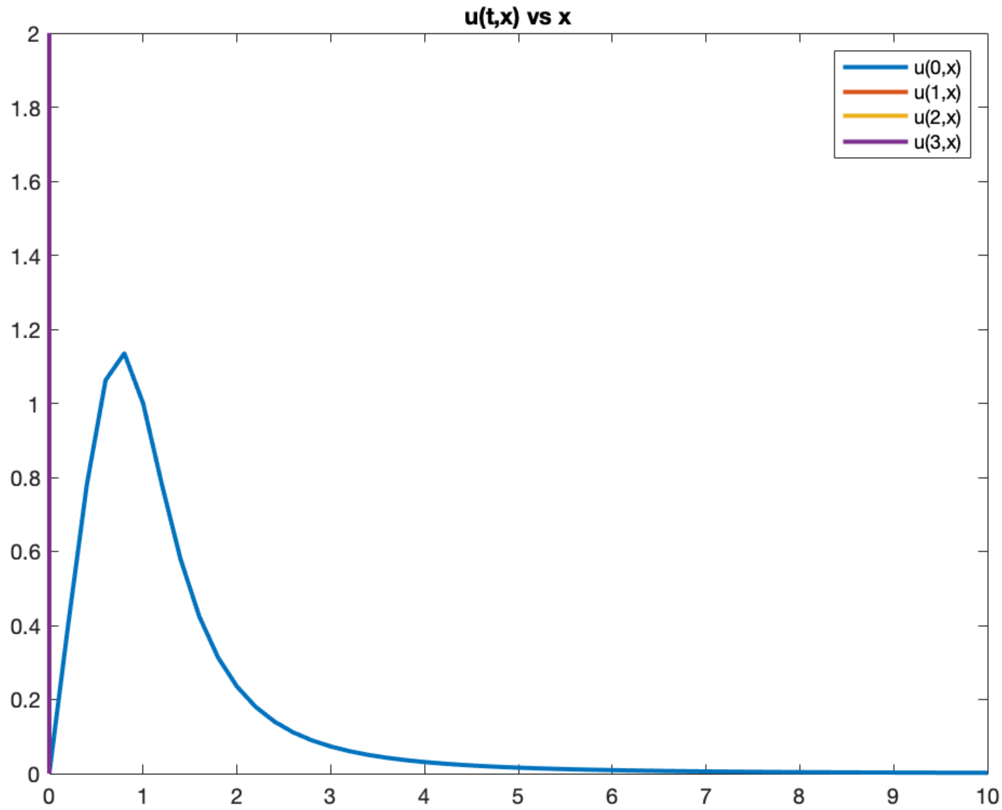

Figure 2: 2a