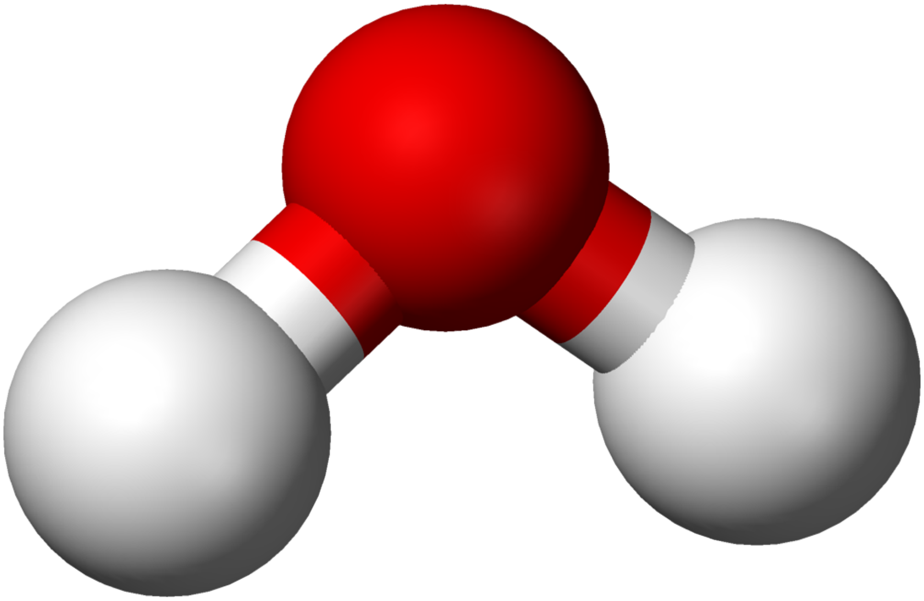

Representation Theory Notes and Exercises

With 0 Figures

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TODOs

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⟨⟨ Proper Exercise Header ⟩⟩

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Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. I make 0 claims that my writing is original, and anything that is well written most likely is a transcription from Serre. Occasionally, other sources may be used, such as the set of notes by Charles Rezk created for Math 427 [Rez20]. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

Chapter 1

Generalities on Linear Representations

Unless otherwise specified, V will denote a vector space, usually over the field $\mathbb{C}$. We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

Definition 1.1 (Linear Representation, Representation Space): Let G be a group with identity e. A *linear representation* of G in V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will frequently, and often interchangeably, write $\rho_s := \rho(s)$. Given ρ, we will say that V is a *representation space* or *representation* of G.

Definition 1.2 (Degree): Let $\rho : G \to V$ be a representation of G in a vector space V. Then the *degree* of ρ is $\dim(V)$.

Let $\rho : G \to V$ be a representation of G in a vector space V with $n := \dim(V)$. Fix a basis (e_j) of V. Then since each ρ_s is an invertible linear transformation of V, we may define an $n \times n$ matrix $R_s \equiv (r_{ij}(s))$ where each $r_{ij}(s)$ is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^{n} r_{ij}(s) e_i.$$

Definition 1.3 (Matrix of a Representation): We call $R_s = (r_{ij}(s))$ above the *matrix of* ρ_s with respect to the basis (e_j).

Note that R_s satisfies the following:

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^{n} r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two $n \times n$ matrices A, A' are *similar* if there exists an invertible matrix T such that $TA = A'T$. We may extend this notion to representations.

Definition 1.4 (Similar/Isomorphic Representations): Let ρ and ρ' be two representations of the same group G in vector spaces V and V' respectively. We say ρ and ρ' are *similar* or *isomorphic* if there exists an isomorphism $\tau : V \to V'$ such that for all $s \in G$, τ satisfies $\tau \circ \rho(s) = \rho'(s) \circ \tau$. If R_s, R'_s are the corresponding matrices then this is equivalent to saying there exists an invertible matrix T such that $TR_s = R'_s T$ for all $s \in G$.

Note that if ρ and ρ' are isomorphic, then they must have the same degree.

We now give some examples of these things.

Example 1.5 (Unit/Trivial Representation): Let G be a finite group. Representations of degree 1 must be of the form $\rho : G \to \mathbb{C}^\times$. Since elements s of G are of finite order, $\rho(s)$ must also be of finite order. Thus, for all $s \in G$, $\rho(s)$ is a root of unity. If we take $\rho(s) = 1$ for all $s \in G$, we obtain the *unit* or *trivial* representation of G. This also means that $R_s = 1$ for all s.

Example 1.6 (Regular Representation): Let g be the order of G, and let V be a vector space of dimension g with a basis $(e_t)_{t \in G}$. For each $s \in G$, define ρ_s as the linear map $\rho_s : V \to V$ such that $\rho_s(e_t) = e_{st}$. This is a linear representation of G called the *regular* representation of G. Since for each $s \in G$, $e_s = \rho_s(e_1)$ and thus the images of e_1 form a basis of V. Conversely, let W be a representation of G with a vector w satisfying the collection of all $\rho_s(w)$, $s \in G$, forms a basis of W. Then W is isomorphic to the regular representation of G by the isomorphism $\tau(e_s) = \rho_s(w)$.

For example, let $G = \mathbb{Z}_3$ and $V = \mathbb{C}^3$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$. Then for example, $\rho_0, \rho_1, \rho_2 : \mathbb{C}^3 \to \mathbb{C}^3$ are the linear maps such that

$$\begin{array}{lll}
\rho_0(e_0) = e_{0+0} = e_0 & \rho_0(e_1) = e_{0+1} = e_1 & \rho_0(e_2) = e_{0+2} = e_2 \\
\rho_1(e_0) = e_{1+0} = e_1 & \rho_1(e_1) = e_{1+1} = e_2 & \rho_1(e_2) = e_{1+2} = e_0 \\
\rho_2(e_0) = e_{2+0} = e_2 & \rho_2(e_1) = e_{2+1} = e_0 & \rho_2(e_2) = e_{2+2} = e_1
\end{array}$$

With this, the matrix representations of ρ_0, ρ_1 and ρ_2 is similarly straightforward:

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 1.7 (Permutation Representation): We may generalize the regular representation to any group action $G \curvearrowright X$, X a finite set. Recall that for such an action, the map $x \mapsto sx$ for each $s \in G$ is a permutation $X \longleftrightarrow X$. Let V be a vector space with dimension the size of X, and so a basis $(e_x)_{x \in X}$. Define a representation ρ of G by defining ρ_s as the linear map sending $e_x \mapsto e_{sx}$. This representation is known as the *Permutation* representation of G associated with X. If we consider $X = [n]$ and $G = S_n$, then take $V = \mathbb{C}^n$ as our vector space and e_i as the standard basis vector. Then $\rho_\sigma(e_j) = e_{\sigma_j}$. Thus for each $\sigma \in S_n$, we have that $R_\sigma = (r_{ij}(\sigma))$ where entry $r_{ij}(\sigma) = 1$ if $i = \sigma(j)$ and 0 otherwise.

Definition 1.8 (Stable/Invariant Subspaces, Subrepresentation): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation and $W \subseteq V$ a subspace of V. We say that W is *stable* under the action of G if $x \in W$ implies that $\rho_s(x) \in W$ for all $s \in G$., Thus, the restriction $\rho_s^W := \rho_s \mid_W$ is an isomorphism of W onto itself. Restrictions satisfy the property that $\rho_s^W \circ \rho_t^W = \rho_{st}^W$. Thus, $\rho^W : G \to \mathrm{GL}(W)$ is a linear representation of G in W and we say that W is a *subrepresentation* of V.

Example 1.9 (Subrepresentations of the Regular Representation): Let G be a group. Recall the regular representation V given in Example 1.6. Let W be the 1 dimensional subspace of V generated by the element $x = \sum_{s \in G} e_s$. Then note that $\rho_s(x) = x$ for all $s \in G$ and thus W is a subrepresentation of V. Furthermore, this is isomorphic to the unit representation Example 1.5 with $\tau : C^\times \to W$ such that $\tau(1) = x$. For example, let $G = \mathbb{Z}_3$ and $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ the representation given in Example 1.6. Then $x = (1,1,1)$ and for example we have that

$$\rho_1(x) = \rho_1(e_0) + \rho_1(e_1) + \rho_1(e_2) = e_1 + e_2 + e_0 = x.$$

Theorem 1.10: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V and let W be a subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.

Proof: Let W' be an arbitrary complement of W in V, and let $p : V \to W$ be the projection. Then we form the average p^0 of conjugates of p by elements in G:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Since $p : V \to W$ and ρ_t preserves W, we have that p^0 maps V onto W. Furthermore, note that ρ_t^{-1} also preserves W.

Thus we have that

$$(p \circ \rho_t^{-1})(x) = \rho_t^{-1}(x), \qquad (\rho_t \circ p \circ \rho_t^{-1})(x) = x, \qquad p^0(x) = x.$$

Thus, p^0 is a projection of V onto W, corresponding to some complement W^0 of W. Moreover, we have that $\rho_s \circ p^0 = p^0 \circ \rho_s$ for all $s \in G$ because

$$\rho_s \circ p^0 \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \circ \rho_t \circ p \circ \rho_t^{-1} \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \circ p \rho_{st}^{-1} = p^0.$$

Now suppose that $x \in W^0$ and $s \in G$, we have that $p^0(x) = 0$ and hence $(p^0 \circ \rho_s)(x) = (\rho_s \circ p^0)(x) = 0$, meaning that $\rho_s(x) \in W^0$. This, W^0 is stable under G. □

Suppose that V had an inner product $\langle x, y \rangle$, and furthermore suppose this inner product was invariant under G meaning that for all $s \in G$, $\langle \rho_s(x), \rho_s(y) \rangle = \langle x, y \rangle$. We may also reduce to this case by replacing $\langle x, y \rangle$ with $\sum_{t \in G} \langle \rho_t(x), \rho_t(y) \rangle$. With this, the orthogonal complement $W^\perp$ of W in V is a complement of W stable under G. Note that the invariance of $\langle x, y \rangle$ means that if (e_i) is an orthonormal basis of V, then R_s is a unitary matrix.

Using the notation of Theorem 1.10, let $x \in V$ and w, w^0 be the projections of x on W and W^0 respectively. Thus for all $s \in G$, $\rho_s(x) = \rho_s(w) + \rho_s(w^0)$. Since W and W^0 are stable under G, we have that $\rho_s(w) \in W$ and $\rho_s(w^0) \in W^0$. This means that $\rho_s(w)$ and $\rho_s(w^0)$ are the projections of $\rho_s(x)$ and in turn the representations of W and W^0 determine the representations of V.

Definition 1.11 (Direct Sum of Representations): Given the above, we write $V = W \oplus W^0$ as the *direct sum* of W and W^0. We identify elements $v \in V$ as pairs (w, w^0) given by their projections.

If the representations W and W^0 are given in matrices R_s and R_s^0, then the matrix form of the representation V is given by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

Similar results hold for arbitrarily many, but finite, direct sums of representations.

The above is a discussion on how to compose two or more representations into one larger representation. The natural question is then about the opposite.

Definition 1.12 (Irreducible/Simple Representations): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Then this representation is *irreducible* or *simple* if V has no subspaces stable under G besides 0 and V itself.

By Theorem 1.10, this is equivalent to saying that V is not the direct sum of two representations besides $V = 0 \oplus V$. A representation of degree 1 is reducible. We may use irreducible representations to construct other ones via the direct sum.

Theorem 1.13: Every representation is a direct sum of irreducible representations.

Proof: Let V be a linear representation of G. We induct on $\dim(V)$. If $\dim(V) = 0$, then $V = 0$ which is the direct sum of an empty family of irreducible representations. So suppose that
$dim(V) \geq 1$. If V is irreducible, then we are done. Otherwise, there exists a subspace $W \subsetneq V$ stable under G and by Theorem 1.10 a stable complement W^0 such that $V = W \oplus W^0$. By assumption, $W \neq 0 \neq W^0$ and so $\dim(W) < V$ and $\dim(W^0) < \dim(V)$. By induction, we have obtained a decomposition of V into irreducibles. $\square$

Example 1.14 (Decomposition of Representation of $\mathbb{Z}_3$ into Irreducibles): Recall from Example 1.6 the regular representation $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$ and

$$\begin{array}{lll} \rho_0(e_0) = e_0 & \rho_0(e_1) = e_1 & \rho_0(e_2) = e_2 \\ \rho_1(e_0) = e_1 & \rho_1(e_1) = e_2 & \rho_1(e_2) = e_0 \\ \rho_2(e_0) = e_2 & \rho_2(e_1) = e_0 & \rho_2(e_2) = e_1 \end{array}$$

Our goal will be to decompose ρ into $\rho^1 \oplus \rho^2 \oplus \rho^3$. We aim to find the elements fixed by $\mathbb{Z}_3$. Note that if an element is fixed by 1, the generator of $\mathbb{Z}_3$, then it is fixed by all of $\mathbb{Z}_3$. We want to find 1-dimensional $\mathbb{Z}_3$-invarient subspaces of $\mathbb{C}^3$. This is equivalent to finding the eigenvalues of the matrix

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues their eigenvectors of R_1 are as follows, note the inclusion of the trivial subrepresentation from Example 1.9:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}, v_2 = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \qquad \lambda_3 = \frac{-1 + i\sqrt{3}}{2}, v_3 = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Thus $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$ where $V_i := \mathrm{span}(v_i)$. Note that there are only 3 morphisms $\mathbb{Z}_3 \to \mathbb{C}^\times$ mapping 1 to 1, ω, or ω^2 where ω is a cube root of unity. Thus ρ^1, ρ^2, and ρ^3 must correspond to these morphisms **⟨⟨ but which ones ⟩⟩** .

A natural question is if such a decomposition $V = W_1 \oplus \cdots \oplus W_k$ is unique. However, suppose that ρ is the trivial representation (Example 1.5). Then each component of the decomposition is a line and we can decompose a vector space into the direct sum of lines in a number of ways. However, we will see that the number of W_i that are isomorphic to a given irreducible representation does not depend on the choice of decomposition.

Definition 1.15 (Tensor/Kronekcer Product of Representations): Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two representations of a group G. We construct a representation $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ such that

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \circ \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

The existence and uniqueness of ρ follow immediately from the existence and uniqueness of the tensor product. We write $\rho_s \equiv \rho_s^1 \otimes \rho_s^2$ as the *tensor product* of the given representations.

Recall that if (e_{i_1}) and (e_{i_2}) be bases of V_1 and V_2 respectively, then $(e_{i_1} \otimes e_{i_2})$ is a basis of $V_1 \otimes V_2$. If $(r_{i_1 j_1}(s))$ and $(r_{i_2 j_2}(s))$ are the matrices of ρ_s^1 and ρ_s^2 respectively satisfying

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1} e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2} e_{i_2}$$

then the matrix of ρ_s is $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$ satisfying

$$\rho_s(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \otimes e_{i_2}.$$

Note that the tensor product of two irreducible representations is not in general irreducible.

Example 1.16 (Tensor Product of Two Irreducible Representations that is not Irreducible): Consider $G = \mathbb{Z}/4\mathbb{Z}$. Consider the representation $\rho : G \to \mathrm{GL}(\mathbb{R}^2)$ such that

$$\rho_1 = M := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since M does not have 1 as an eigenvalue, there are no $\mathbb{Z}/4\mathbb{Z}$-invariant subspaces. Thus, ρ is irreducible.

Now consider $\rho' = \rho \otimes \rho$. Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$, and so $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ a basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$. We have that

$$\begin{aligned}
\rho_1'(e_1 \otimes e_1) &= Me_1 \otimes Me_1 = e_2 \otimes e_2, \\
\rho_1'(e_1 \otimes e_2) &= Me_1 \otimes Me_2 = e_2 \otimes -e_1, \\
\rho_1'(e_2 \otimes e_1) &= Me_2 \otimes Me_1 = -e_1 \otimes e_2, \\
\rho_1'(e_2 \otimes e_2) &= Me_2 \otimes Me_2 = e_2 \otimes e_2.
\end{aligned}$$

Thus the matrix of ρ_1' is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that the subspace generated by $(e_1 \otimes e_1, e_2 \otimes e_2)$ is a $\mathbb{Z}/4\mathbb{Z}$-invariant subspace.

We now consider the special case of $V \otimes V$. Let (e_i) be a basis of V and define an automorphism θ of $V \otimes V$ such that $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Then note that $\theta^2 \equiv \mathrm{id}_{V \otimes V}$. We may decompose $V \otimes V$ into the direct sum

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Here, $\mathrm{Sym}^2(V)$ is the set of $z \in V \otimes V$ such that $\theta(z) = z$ and $\mathrm{Alt}^2(V)$ is the set of $z \in V \otimes V$ where $\theta(z) = -z$. These have bases $(e_i \otimes e_j + e_j \otimes e_i)_{i \leq j}$ and $(e_i \otimes e_j - e_j \otimes e_i)_{i<j}$ respectively. As such, $\dim(\mathrm{Sym}^2(V)) = \frac{n(n+1)}{2}$ and $\dim(\mathrm{Alt}^2(V)) = \frac{n(n-1)}{2}$ where $n \coloneqq \dim(V)$.

Definition 1.17 (Symmetric Square, Alternating Square): These subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ of $V \otimes V$ are respectively called the *symmetric square* and *alternative square* of the given representation.

Chapter 2

Character Theory

Definition 2.1 (Character): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of a finite group G in V. Then the *character* χ_ρ of ρ is the function

$$\chi_\rho(s) := \mathrm{Tr}(R_s) \equiv \mathrm{Tr}(\rho_s).$$

for each $s \in G$.

Proposition 2.2: If χ is the character of a representation ρ of degree n then

1. $\chi(e) = 1$;
2. $\chi(s^{-1}) = \chi(s)^*$, the complex conjugate of $\chi(s)$,
3. $\chi(tst^{-1}) = \chi(s)$.

Proof: The first is immediate since ρ_1 is the identity matrix I and $\mathrm{Tr}(I) = n$. Then recall that we may choose our basis to be orthonormal, and as such ρ_s is a unitary matrix. Thus, each eigenvalue $\lambda_1, \ldots, \lambda_n$ has an absolute value equal to 1. Thus

$$\chi(s)^* = \mathrm{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \mathrm{Tr}(\rho_s)^{-1} = \mathrm{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

This uses the fact that the conjugate of the sum is the sum of the conjugates as well as the fact that the eigenvalues of R_s^{-1} are the inverses of the eigenvalues of R_s. Finally, letting $u = ts$ and $v = t^{-1}$ allows us to write $\chi(tst^{-1}) = \chi(s)$ as $\chi(uv) = \chi(vu)$ which is immediate since for any complex matrices A, B we have that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. $\square$

Proposition 2.3: Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two linear representations with characters χ_1 and χ_2 respectively. Then

1. The character χ of the direct sum representation $V_1 \oplus V_2$ is $\chi_1 + \chi_2$.
2. The character ψ of the tensor product representation $V_1 \otimes V_2$ is $\chi_1 \cdot \chi_2$.

Proof: Let R_s^1, R_s^2 be the matrix forms of ρ_s^1 and ρ_s^2 respectively. Then the matrix form R_s of the representation of $V_1 \oplus V_2$ is given by

$$R_s = \begin{pmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{pmatrix}$$

and thus $\mathrm{Tr}(R_s) = \mathrm{Tr}\left(R_s^1\right) + \mathrm{Tr}\left(R_s^2\right)$. Let (e_{i_1}) and (e_{i_2}) be bases for V_1 and V_2. Then we have that

$$\psi(s) = \sum_{i_1 i_2} r_{i_1 i_1}(s) \cdot r_{i_2 i_2}(s) = \left(\sum_{i_1} r_{i_1 i_1}(s) \right) \cdot \left(\sum_{i_2} r_{i_2 i_2}(s) \right) = \chi_1(s) \cdot \chi_2(s).$$

□

Proposition 2.4: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G with character χ. Let χ_σ^2 be the character of $\mathrm{Sym}^2(V)$ and χ_α^2 be the character of $\mathrm{Alt}^2(V)$ from Definition 1.17. Then

$$\chi_\sigma^2(s) = \frac{1}{2}\left(\chi(s)^2 + \chi(s^2)\right)$$
$$\chi_\alpha^2(s) = \frac{1}{2}\left(\chi(s)^2 - \chi(s^2)\right)$$

which directly implies that $\chi_\sigma^2 + \chi_\alpha^2 = \chi$.

Proof: Let $s \in G$ and (e_i) a basis of V consisting solely of eigenvectors for ρ_s. Then $\rho_s(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$. Thus

$$\chi(s) = \sum \lambda_i \qquad\qquad \chi(s^2) = \sum \lambda_i^2.$$

We also have that

$$(\rho_s \otimes \rho_s)(e_i \otimes e_j + e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i)$$
$$(\rho_s \otimes \rho_s)(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

which yields that

$$\chi_\sigma^2(s) = \sum_{i \le j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\sum \lambda_i\right)^2 + \frac{1}{2}\sum \lambda_i^2 \chi_\alpha^2(s) \quad = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\sum \lambda_i\right)^2 - \frac{1}{2}\sum \lambda_i^2.$$

The proposition then directly follows. Note that the equality $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$ directly reflects the fact that $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$. $\square$

Proposition 2.5 (Schur's Lemma): Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $f : V_1 \to V_2$ be a linear map such that $f \circ \rho_s^1 = \rho_s^2 \circ f$ for all $s \in G$. Then

1. If ρ^1 and ρ^2 are not isomorphic, then $f = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$ then f is a *homothety*, a scalar multiple of the identity.

Proof: The case of $f = 0$ is trivial, so suppose that $f \neq 0$. Let $W_1 = \ker(f)$ and $W_2 = \mathrm{im}(f)$. Then for $x \in W_1$ we have that $f(\rho_s^1(x)) = \rho_s^2(f(x)) = 0$ which means that $\rho_s^1(x) \in W_1$. Thus W_1 is stable under G and irreducibility of V_1 combined with the assumption that $f \neq 0$ implies that $W_1 = 0$. Similarly, we have that for $f(x) \in W_2$, we have that $\rho_s^2(f(x)) = f(\rho_s^1(x)) \in W_2$, so $\rho_s^2(f(x)) \in W_2$. Thus W_2 is also stable under G meaning that by a similar argument, $W_2 = V_2$. Since $\ker(f) = 0$ and $\mathrm{im}(f) = V_2$, we must have that f is an isomorphism $V_1 \to V_2$. This proves the first claim.

Now suppose that $V_1 = V_2$, $\rho^1 = \rho^2$, and that λ is some eigenvalue of f. Let $f' = f - \lambda$. Since λ is an eigenvalue, then $\ker(f') \neq 0$. However, we also have that $f' \circ \rho_s^1 = \rho_s^2 \circ f'$. The first part of this proof shows that this implies that $f' = 0$. Thus, $f = \lambda$ and f is a homothety. $\square$

Corollary 2.6: Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $h \colon V_1 \to V_2$ and define h^0 such that

$$h^0 = \frac{1}{|G|} \sum_{t \in G} (\rho_t^2)^{-1} \circ h \circ \rho_t^1.$$

Then

1. If ρ^1 and ρ^2 are not isomorphic, then $h^0 = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$, then h^0 is a homothety of ratio $\frac{1}{n}\mathrm{Tr}(h)$, with $n = \dim(V_1)$.

Proof: First for $s \in G$ we have that

$$(\rho_s^2)^{-1} \circ h^0 \circ \rho_s^1 = \frac{1}{|G|} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} \circ h \circ \rho_t^1 \rho_s^1. \qquad = \frac{1}{|G|} \sum_{t \in G} (\rho_{ts}^2)^{-1} \circ h \circ \rho_{ts}^1 = h^0.$$

Thus, Proposition 2.5 applies to h^0 and in the first case $h^0 = 0$ and in the second h^0 is a homothety of scalar λ. Moreover we have that

$$n \cdot \lambda = \mathrm{Tr}(\lambda) = \frac{1}{|G|} \sum_{t \in G} \mathrm{Tr}\left((\rho_t^1)^{-1} \circ h \circ \rho_t^1\right) = \mathrm{Tr}(h).$$

Thus, $\lambda = \frac{1}{n}\mathrm{Tr}(h)$. $\square$

Consider Corollary 2.6 in matrix form where $\rho_s^1 = (r_{i_1 j_1}(s))$ and $\rho_s^2 = (r_{i_2 j_2}(s))$. Then our linear map h is given by the matrix $(x_{i_2 i_1})$ and similarly h^0 is given by the matrix $(x^0_{i_2 i_1})$. Then by definition of h^0 we have that

$$x^0_{i_2 i_1} = \frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t).$$

In case (1) of Corollary 2.6, the right hand side must vanish and so all coefficients of 0:

Corollary 2.7: In case (1) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = 0$$

for all i_1, j_1, i_2, j_2.

We now consider case (2) of Corollary 2.6 in the matrix form. We have that $h^0 = \lambda$, with $\lambda = \frac{1}{n} \operatorname{Tr}(h)$, meaning that $x^0_{i_2 i_1} = \lambda \delta_{i_2 i_1}$. That is, $\lambda = \frac{1}{n} \sum \delta_{i_2 i_1} \cdot x_{i_2 i_1}$. This we have that

$$\frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t) = \frac{1}{n} \sum_{j_1, j_2} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1} \cdot x_{j_2 j_1}.$$

Equating coefficients of the $x_{j_2 j_1}$ yields the following corollary:

Corollary 2.8: In case (2) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

We may generalize these notions to further functions. Let ϕ, ψ be complex valued functions on G. Define

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{t \in G} \phi(t^{-1}) \cdot \psi(t) = \frac{1}{|G|} \sum_{t \in G} \phi(t) \cdot \psi(t^{-1}).$$

Then $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$ and $\langle \phi, \psi \rangle$ is linear in ϕ and in ψ. Thus, Corollary 2.7 and Corollary 2.8 respectively become

$$\left\langle r_{i_2 j_2}, r_{j_1 i_1} \right\rangle = 0 \qquad\qquad \left\langle r_{i_2 j_2}, r_{j_1 i_1} \right\rangle = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

If the matrices $(r_{ij}(t))$ are unitary, realized by a suitable choice of basis, then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and Corollary 2.7 and Corollary 2.8 are just *orthogonality relations*.

Definition 2.9 (Scalar Product): If ϕ, ψ are two complex valued functions on G, then let

$$(\phi \mid \psi) := \frac{1}{|G|} \sum_{t \in G} \phi(t)\psi(t)^*.$$

This is a *scalar product*. It is linear in ϕ, semilinear in ψ, and $(\phi \mid \phi) > 0$ for all $\phi \neq 0$.

Define $\check{\psi}(t) := \psi(t^{-1})^*$. Then $(\phi \mid \psi) = \langle \phi, \check{\psi} \rangle$. In particular, suppose χ is a character so that by Proposition 2.2 we have that $\chi = \check{\chi}$ then for all complex valued functions ϕ on G we have that $(\phi \mid \chi) = \langle \phi, \chi \rangle$. Thus, we may use the two interchangeably in the context of characters.

Theorem 2.10:

1. If χ is the character of an irreducible representation, we have that $(\chi \mid \chi) = 1$, i.e. χ has "norm 1."
2. If χ and χ' are characters of two non-isomorphic irreducible representations, then $(\chi \mid \chi') = 0$, i.e. χ and χ' are "orthogonal."

Proof: Suppose ρ is an irreducible representation with matrix form $\rho_t = (r_{ij}(t))$ and χ its character. Then $\chi(t) = \sum r_{ii}(t)$ and so

$$(\chi \mid \chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle = \frac{\delta_{ij}}{n}$$

where the last equality is by Corollary 2.8 and n is the degree of ρ. Thus

$$(\chi \mid \chi) = \sum_{i,j} \frac{\delta_{ij}}{n} = \frac{n}{n} = 1.$$

This proves the first claim. Applying Corollary 2.7 yields the second claim □

Theorem 2.11: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi \mid \chi) = \langle \phi, \chi \rangle$.

Proof: Let χ_i be the character of W_i. Then by Proposition 2.3 we have that $\phi = \chi_1 + \cdots + \chi_k$. By linearity of $(\cdot \mid \cdot)$ in the first argument we have that $(\phi \mid \chi) = (\chi_1 \mid \chi) + \cdots + (\chi_k \mid \chi)$. The result follows by Theorem 2.10. □

Corollary 2.12: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W does not depend on the chosen decomposition.

Proof: Note that $(\phi \mid \chi)$ does not depend on choice of decomposition. □

Corollary 2.13: Two representations are isomorphic if and only if they have the same character.

Proof: The forward direction is obvious, and the reverse is true by the prior corollary. □

Thus, our study of representations is reduced to that of the study of characters. If $\chi_1, \ldots, \chi_k$ are the distinct irreducible characters of G and if $W_1, \ldots, W_k$ their corresponding representation, then each representation V of G is isomorphic to a direct sum. We will see later how we know that there are finitely many irreducible representations, and thus characters, of a finite group G.

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \qquad\qquad m_i \neq 0.$$

The character ϕ of V is equal to $m_1\chi_1 + \cdots + m_h\chi_h$ and we have that $m_i = (\phi \mid \chi_i)$. This is especially useful when considering the tensor product $W_i \otimes W_j$ of two irreducible representations. It shows that the product $\chi_i \cdot \chi_j$ decomposes into a sum $\chi_i\chi_j = \sum m_{ij}^k \chi_k$, each integer $m_{ij}^k \geq 0$. The orthogonality relations among the χ_i imply that

$$(\phi \mid \phi) = \sum_{i=1}^{h} m_i^2.$$

We now obtain a useful irreducibility criterion:

Theorem 2.14: If ϕ is the character of a representation V, $(\phi \mid \phi)$ is a positive integer and $(\phi \mid \phi) = 1$ if and only if V is irreducible.

Proof: We have that $\sum m_i^2 = 1$ if and only if one of the $m_i = 1$ and all the others are equal to 0. This means that V is isomorphic to one of the W_i. □

We now explore the decomposition of the regular representation $\rho : G \to \mathrm{GL}(R)$ of a group G (Example 1.6). Suppose $\chi_1, \ldots, \chi_h$ are the irreducible characters of G with degrees $n_1, \ldots, n_k$. Note that by Proposition 2.2, $n_i = \chi_i(e)$. Recall that R has basis $(e_t)_{t \in G}$ where $\rho_s(e_t) = e_{st}$. This means that for $s \neq e$, the diagonal terms of the matrix for ρ_s are all 0, so $\mathrm{Tr}(\rho_s) = 0$. On the otherhand, we have that

$$\mathrm{Tr}(\rho_e) = \dim(R) = |G|.$$

Proposition 2.15: The character r_G of the regular representation is given by

$$r_G(e) = |G| \qquad\qquad r_G(s) = 0 \text{ if } s \neq e.$$

Corollary 2.16: Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i.

Proof: By Theorem 2.11, the number of times W_i is contained in the regular representation is $\langle r_G, \chi_i \rangle$. We have that

$$\langle r_G, \chi_i \rangle = \frac{1}{|G|} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{|G|} \cdot |G| \chi_i(1) = \chi_i(1) = n_i.$$

□

Corollary 2.17:

1. The degrees satisfy $\sum_{i=1}^{h} n_i^2 = |G|$.
2. if $e \neq s \in G$, we have that $\sum_{i=1}^{h} n_i \chi_i(s) = 0$.

Proof: By Corollary 2.16, we have that $r_G(s) = \sum n_i \chi_i(s)$ for all $s \in G$. A priori we know that r_G is the sum of irreducibles χ_i, and Corollary 2.16 gives the multiplicities. Plugging in $s = e$ and $s \neq e$ yields the claim. □

The above result lets us determine the irreducible representations of a group G. Suppose we have constructed some mutually non-isomorphic irreducible representations of degrees $n_i, \ldots, n_h$. In order to check if we have found all such representations, it is necessary and sufficient to verify that $n_1^2 + \cdots + n_h^2 = |G|$. Also, we shall later see that each of the n_i divide the order of G.

Definition 2.18 (Class Function): A function f on a group G is a *class function* if for all $s, t \in G$, $f(tst^{-1}) = f(s)$.

Proposition 2.19: Let f be a class function on a group G and $\rho : G \to \mathrm{GL}(V)$ a linear representation of G with character χ. Define $\rho_f : V \to V$ by $\rho_f = \sum_{t \in G} f(t) \rho_t$. If V is irreducible of degree n, the ρ_f is a homothety of ratio λ where

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*).$$

Proof: We have that

$$\rho_s^{-1} \rho_f \rho_s = \sum_{t \in G} f(t) \rho_s^{-1} \rho_t \rho_s = \sum_{t \in G} f(t) \rho_{s^1 ts} = \sum_{t \in G} f(s^{-1}ts) \rho_{s^{-1}ts} = \rho_f.$$

Thus, by Proposition 2.5 we have that ρ_f us a homothety λ. The trace of λ is $n\lambda$. Thus, the trace of ρ_f is $\sum_{t \in G} f(t) \mathrm{Tr}(\rho_t) = \sum_{t \in G} f(t) \chi(t)$. Thus, $\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*)$. □

Let H be the space of class functions on G.

Theorem 2.20: The characters $\chi_1, \ldots, \chi_h$ of G form an orthonormal basis of H.

Proof: Note that Theorem 2.10 says that the χ_i are all orthonormal to each other. To show that they generate H, it is enough to show that the only element of H orthogonal to χ_i^* is 0. Let f be such an element. For each representation ρ of G, let ρ_f be as in Proposition 2.19. Since f is orthogonal to the χ_i^*, Proposition 2.19 says that ρ_f is 0 as long as ρ is irreducible. From the decomposition of a representation into a direct sum of irreducible representation, with possible multiplicities, we conclude that ρ_f is always 0. Now consider the regular representation of G and compute the image of the basis vector e_e under ρ_f:

$$0 = \rho_f(e_e) = \sum_{t \in G} f(t)\rho_t(e_e) = \sum_{t \in G} f(t)\rho_t.$$

Thus, $f(t) = 0$ for each $t \in G$ and $f = 0$. □

Theorem 2.21: The number of irreducible representations of G, up to isomorphism, is the number of conjugacy classes of G.

Proof: Let $C_1, \ldots, C_k$ be the distinct conjugacy classes of G. Then all class functions are constant on each class, their value determined by some λ_i for each C_i. These λ_i may be chosen arbitrarily. Thus, the dimension of the space H of class functions is equal to k. But we already know by Theorem 2.20 that the dimension of H is h, the number of irreducible representations of G. □

Proposition 2.22: Let $s \in G$ and $c(s)$ the number of elements in the conjugacy class of s.

1. We have $\sum_{i=1}^h \chi_i(s)^* \chi_i(s) = \frac{|G|}{c(s)}$.
2. For t not conjugate to s, we have $\sum_{i=1}^h \chi_i(s)^* \chi_i(t) = 0$.

Proof: Let f_s be the class function equal to to 1 on the class of s and 0 otherwise. By Theorem 2.21, we have that

$$f_s = \sum_{i=1}^h \lambda_i \chi_i \qquad\qquad \lambda_i = (f_s \mid \chi_i) = \frac{c(s)}{|G|}\chi_i(s)^*.$$

We have then, for each $t \in G$, that

$$f_s(t) = \frac{c(s)}{|G|} \sum_{i=1}^h \chi_i(s)^* \chi_i(t).$$

If $t = s$, we get claim 1 and for t not conjugate to s we get claim 2. □

Example 2.23 (Character Table of S_3): Consider the group S_3. There are three conjugacy classes: the identity (), the 3 transpositions, and the 2 cyclic permutations. Let t be one of the transpositions and c one of the cyclic permutations. Then $t^2 = 1 = c^3$ and $tc = c^2 t$. There are just two characters of degree 1: the unit character χ_1 and the character χ_2 giving the sign of the permutation. This is because $t^2 = 1$ means that $\chi(t) = 1$ or -1. Each choice then determines the character of c, which ends up corresponding to the unit character or the sign. By Theorem 2.21, there exists one more irreducible character θ. If n is the degree of θ, then we must have that $1 + 1 + n^2 = 6$, so $n = 2$. By Proposition 2.15, we have that $\chi_1 + \chi_2 + 2\theta$ is the character of the regular representation. Thus, we get the following *character table*:

⟨⟨ TODO: Center ⟩⟩

	1	t	c
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

We obtain an irreducible representation of G with character θ by having G permute the coordinates of elements of $\mathbb{C}^3$ satisfying $x + y + z = 0$.

Let $\rho \colon G \to \mathrm{GL}(V)$ be a linear representation of G. Recall that the direct sum decomposition of V into irreducible representation is not necessarily unique. Thus, we shall now define a "coarser" decomposition which has the advantage of being unique.

Definition 2.24 (Canonical Decomposition of a Representation): Let $\chi_1, \ldots, \chi_h$ be the distinct characters of the irreducible representations of $W_1, \ldots, W_h$ of G with degrees $n_1, \ldots, n_h$. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into a direct sum of irreducible representations. For $i = 1, \ldots, h$, let V_i be the direct sum of the U_i which are isomorphic to W_i. Then $V = V_1 \oplus \cdots \oplus V_h$. We have decomposed V into a direct sum of irreducible representations and combined the ones which are isomorphic to each other.

This decomposition satisfies some nice properties:

Theorem 2.25: 1. The decomposition $V = V_1 \oplus V_h$ does not depend on the initially chosen decomposition of V into irreducibles.

2. The projection $p_i : V \to V_i$ is given by

$$p_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i^*(t) \rho_t.$$

Proof: We shall prove claim 2 since claim 1 follows as the p_i determine the V_i. Let $q_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i(t)^* \rho_t$. By Proposition 2.19, we have that the restriction of q_i to an irreducible representation W with character χ and degree n is a homothety of ratio $\frac{n_i}{n}(\chi_i|\chi)$. Thus, q_i is 0 if $\chi_i \neq \chi$ and 1 if $\chi = \chi_i$. This yields that q_i is the identity on an irreducible representation isomorphic to W_i, and 0 on the others. Thus, q_i is the identity on V_i and 0 on V_j for $j \neq i$. Decomposing $x \in V$ into $x_i \in V_i$ such that $x = x_1 + \cdots + x_h$ yields that

$$q_i(x) = q_i(x_1) + \cdots + q_i(x_h) = x_i.$$

Thus $q_i = p_i$. □

This allows us to decompose representations V in two stages. First, we determine $V_1 \oplus \cdots \oplus V_h$. This is done easily using the given formula for p_i in Theorem 2.25. Finally, for each V_i we may choose a decomposition of V_i into a direct sum of irreducible representations, each isomorphic to W_i, This last decomposition may be done in any number of ways.

Example 2.26 (Decomposition of C_2): Let $G = C_2 = \{e, s\}$ be the cyclic group of two elements generated by s. Let $\rho : G \to \mathrm{GL}(V)$ be any representation of C_2. Note that C_2 has two irreducible representations of degree 1, W^+ and W^- with respective characters $\rho^+ = 1$ and $\rho_s = -1$. The canonical decomposition of V is $V = V^+ \oplus V^-$, where V^+ consists of elements $x \in V$ which are symmetric and V^- consists of elements which are antisymmetric. In other words, V^+ consists of elements $x \in V$ where $\rho_s(x) = x$ and V^- consists of elements $x \in V$ where $\rho_s(x) = -x$. This, the projections are

$$p^+(x) = \frac{1}{2}(x + \rho_s(x)) \qquad p^-(x) = \frac{1}{2}(x - \rho_s(x)).$$

To decompose V^+ and V^- into irreducible components means to decompose these subspaces into a direct sum of lines, which can be in arbitrarily many ways.

We now have the tools to explicitly compute the components V_i of this canonical decomposition of $\rho : G \to \mathrm{GL}(V)$. Let $V = V_1 \oplus \cdots \oplus V_h$ be this decomposition. The projection given in Theorem 2.25 will allow us to do this. Let W_i have matrix form $(r_{\alpha\beta}(s))$ with respect to a basis $(e_1, \ldots, e_n)$. Then $\chi_i(s) = \sum_\alpha r_{\alpha\alpha}(s)$. For each $1 \leq \alpha, \beta \leq n$ define

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t.$$

Proposition 2.27:

1. The map $p_{\alpha\alpha}$ is a projection. It is 0 on V_j for $j \neq i$ and its image $V_{i,\alpha}$ is contained in V_i where V_i is the direct sum of the $V_{i,\alpha}$, $1 \leq \alpha \leq n$. We have that $p_i = \sum_\alpha p_{\alpha\alpha}$.
2. The linear map $p_{\alpha\beta}$ is 0 on V_j for $j \neq i$ as well as on $V_{i,\gamma}$ for $\gamma \neq \beta$. It defines an isomorphism $V_{i,\beta} \to V_{i,\alpha}$.
3. Let $x_1 \neq 0 \in V_{i,1}$ and $x_\alpha := p_{\alpha,1}(x_1) \in V_{i\alpha}$. Then the x_α are linearly independent and generate a subspace $W(x_1)$ stable under G and of dimension n. For each $s \in G$, we have that
$$\rho_s(x_\alpha) = \sum_\beta r_{\beta\alpha}(s) x_\beta.$$
In particular, $W(x_1)$ is isomorphic to W_i.
4. If $(x_1^{(1)}, \ldots, x_1^{(m)})$ is a basis of $V_{i,1}$, then the representation V_i is the direct sum of the subrepresentations $W(x_1^{(1)}), \ldots, W(x_1^{(m)})$.

Proof: Observe that the definition of $p_{\alpha\beta}$ is defined in terms of arbitrary representations of G, and in particular in the irreducible representations W_j. For W_i, we have that

$$p_{\alpha\beta}(e_\gamma) = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t(e_\gamma) = \frac{n}{|G|} \sum_\delta \sum_{t \in G} r_{\beta\alpha}(t^{-1}) r_{\delta\gamma}(t) e_\delta.$$

By Corollary 2.8 we have that

$$p_{\alpha\beta}(e_\gamma) = \begin{cases} e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

We get from this that $\sum_\alpha p_{\alpha\alpha} = \mathrm{id}_{W_i}$. We also get the formulas

$$p_{\alpha\beta} \circ p_{\gamma\delta} = \begin{cases} p_{\alpha\delta} & \text{if } \beta = \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_s \circ p_{\alpha\gamma} = \sum_\beta r_{\beta\alpha}(s) p_{\beta\gamma}.$$

For W_j, $j \neq i$, we use Corollary 2.7 and the same argument to show that all the $p_{\alpha\beta}$ are 0.

With this, we can now decompose V into subrepresentations each isomorphic to W_j and apply the above to these representations. The first two assertions follow. Moreover, these formulas are valid in V. Assuming the hypothesis of claim 3 holds, we have that

$$\rho_s(x_\alpha) = \rho_s \circ p_{\alpha 1}(x_1) = \sum_\beta r_{\beta\alpha}(s) p_{\beta 1}(x_1) = \sum_\beta r_{\beta\alpha}(s) x_\beta.$$

This proves claim 3. Finally, claim 4 follows from the first 3. $\square$

Exercises

Exercise 2.1 (Ser77 2.1): Let χ, χ' be the characters of two representations. Prove the following formulas:

$$(\chi + \chi')^2_\sigma = \chi^2_\sigma + \chi'^2_\sigma + \chi\chi'$$
$$(\chi + \chi')^2_\alpha = \chi^2_\alpha + \chi'^2_\alpha + \chi\chi'$$

Proof: Let $s \in G$. Then by Proposition 2.4 we have that

$$\begin{aligned}
(\chi + \chi')^2_\sigma(s) &= \frac{1}{2}\big((\chi + \chi')^2(s) + (\chi + \chi')(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi'(s)^2 + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi(s^2)\big) + \frac{1}{2}\big(\chi'(s) + \chi'(s^2)\big) + \chi(s)\chi'(s) = \chi^2_\sigma(s) + \chi'^2_\sigma(s) + \chi(s)\chi'(s).
\end{aligned}$$

Since this holds for all $s \in G$, the formula holds in general. The proof of the other formula is similar. □

Exercise 2.2 (Ser77 2.2): Let X be a finite set on which G acts, and $\rho : G \to \mathrm{GL}(V)$ the corresponding permutation representation (Example 1.7), and χ_X the character of ρ. Then show that for $s \in G$, $\chi_X(s)$ is equal to the number of elements fixed by s.

Proof: Suppose $X = [n]$ and so $s \in S_n$, meaning $G \leq S_n$. We may assume this without loss of generality. Note that $R_s = (r_{ij}(s))$ where $r_{ij}(s) = 1$ if $s(j) = i$ and 0 otherwise. We want to count the number of elements in $[n]$ fixed by s, i.e. the number of i such that $\sigma(i) = i$. These correspond exactly to the entries in R_S where $r_{ii}(s) = 1$. Thus, the claim follows. □

Exercise 2.3 (Ser77 2.3): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation with character χ. Recall that V^* is the dual vector space of V. For $x \in V$, $x^* \in V^*$ let $\langle x, x^* \rangle = x^*(x)$. Then there exists a unique linear representation $\rho^* : G \to \mathrm{GL}(V^*)$ such that

$$\left\langle \rho_s(x), \rho^*_s(x^*) \right\rangle = \langle x, x^* \rangle$$

for $s \in G$, $x \in V$, and $x^* \in V^*$. Note that ρ^* has character χ^*, the conjugate of χ.

Proof: Let $\rho^*_s = \left(\rho^T_s\right)^{-1}$. Then

$$\left\langle \rho_s(x), \rho^*_s(x^*) \right\rangle = \left\langle \rho_s(x), \left(\rho^T_s\right)^{-1}(x^*) \right\rangle = \left(x, \rho^T_s\left(\left(\rho^T_s\right)^{-1}(x^*)\right)\right) = \langle x, x^* \rangle.$$

Now suppose that $\rho' \colon G \to \mathrm{GL}(V^*)$ was another representation satisfying the above property. Then we would have that

$$\left\langle \rho_s(x), (\rho^* - \rho')(x^*) \right\rangle = \left\langle \rho_s(x), \rho_s^*(x^*) \right\rangle - \left\langle \rho_s(x), \rho_s'(x^*) \right\rangle = 0.$$

Note that this holds for all $x \in V$ and $x^* \in V^*$. Thus, we must have that $(\rho^* - \rho')(x^*) = 0$, and thus $\rho^* = \rho'$. □

Exercise 2.4 (Ser77 2.5): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation with character χ. Then the number of times ρ contains the unit representation is equal to $(\chi \mid 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$.

Proof: The equality $(\chi \mid 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$ is immediate by definition of the scalar product $(\cdot \mid \cdot)$ and the fact that $1^* = 1$. By Theorem 2.11, $(\chi \mid 1)$ counts the number of times an irreducible representation with character 1 appears in V. By Corollary 2.13, the only irreducible representation with character 1 is the unit representation. □

Exercise 2.5 (Ser77 2.6): Let G act on a finite set X, ρ the corresponding permutation representation, and χ its character.

1. Let c be the number of distinct orbits. Show that c is equal to the number of times ρ contains the unit representation 1. Deduce that $(\chi \mid 1) = c$. In particular if G is transitive and thus $c = 1$, then $\rho = 1 \oplus \theta$ where θ does not contain the unit representation. If ψ is the character of θ, then $\chi = 1 + \psi$ and $(\psi \mid 1) = 0$.

2. Let G act on the product $X \times X$ in the natural way. Show that the character of the corresponding permutation representation is equal to χ^2.

3. Suppose that G is transitive on X and $|X| \geq 2$. We say G is *doubly transitive* if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y$ there exists $s \in G$ such that $s(x, y) = (sx, sy) = (x', y')$. Prove that the following are equivalent:

   (a) G is doubly transitive.

   (b) The action of G on $X \times X$ has two orbits, the diagonal and the complement.

   (c) $(\chi^2 \mid 1) = 2$

   (d) The representation θ defined in the first part of this exercise is irreducible.

Proof: We know that the number of times the unit representation is contained in χ is equal to $(\chi \mid 1)$ by Theorem 2.11. By Exercise 2.4, we have that $(\chi \mid 1) = \frac{1}{|G|}\sum_{s\in G}\chi(s)$. We prove that $\frac{1}{|G|}\sum\chi(s) = c$ by double counting. Consider the set $\{(s,x)\in G\times X \mid s\cdot x = x\}$. Then we have that

$$\sum_{x\in X}|G_x| = \sum_{x\in X}|\{s\in G \mid s\cdot x = x\}| = |\{(s,x)\in G\times X \mid s\cdot x = x\}| = \sum_{s\in G}|\{x\in X \mid s\cdot x = x\}| = \sum_{s\in G}\chi(s).$$

Let $O_1,\ldots,O_c$ be the distinct orbits. By the Orbit-Stabilizer theorem, each O_i is in bijection with G/G_x for all $x\in O_i$. Note that the orbits O_i partition X. Thus we have that

$$\sum_{s\in G}\chi(s) = \sum_{i=1}^{c}\sum_{x\in O_i}|G_x| = \sum_{i=1}^{c}\sum_{x\in O_i}\frac{|G|}{|O_i|} = c\cdot|G|$$

and $\frac{1}{|G|}\sum_{s\in G}\chi_s = c$. Following this, the rest of the claim is immediate.

Now suppose that ϕ is the character of the permutation representation of $G \curvearrowright X\times X$. Then by Exercise 2.2, $\phi(s)$ is equal to the number of elements fixed by s. An element $(x,y)\in X\times X$ is fixed by $s\in G$ if and only if both x and y are fixed. Thus if there are $\chi(s)$ elements of X fixed by s, then $\chi^2(s)$ elements of $X\times X$ are fixed by s and $\phi = \chi^2$.

To prove 3, we have that $(a) \iff (b)$ is immediate and $(b) \iff (c)$ follows from 1 and 2. Now suppose (c) holds and let ψ be the character of θ. Then $1+\psi = \theta$. Since $(\chi \mid 1) = (1 \mid 1) = 1$ we must have that $(\psi \mid 1) = 0$. Since $\chi^2 = 1 + 2\psi + \psi^2$, we have that (c) is equivalent to saying $(\psi^2 \mid 1) = 1$. Thus

$$\frac{1}{|G|}\sum_{s\in G}\psi(s)^2 = 1.$$

However, note that $\psi(s)$ is real valued, not just complex valued. This is because χ is real valued, it counts fixed points, and clearly 1 is real valued. Thus $\psi^* = \psi(s)^*$ and so the above equality implies that $(\psi \mid \psi = 1)$. By Theorem 2.14, we have that this is true if and only if θ is irreducible, i.e. $(c) \iff (d)$ holds. □

Exercise 2.6 (Ser77 Exercise 2.8): Let $\rho : G \to \mathrm{GL}(V)$ be any representation of a group G with $V = V_1 \oplus \cdots \oplus V_h$ the canonical decomposition, $W_1, \ldots, W_h$ all irreducible representations of G. Let H_i be the vector space of linear mappins $h \colon W_i \to V$ such that $\rho_s \circ h = h \circ \rho_s$ for all $s \in G$. Each $h \in H_i$ maps W_i into V_i.

1. Show that $\dim(H_i)$ is equal to $\dim(V_i)/\dim(W_i)$, the multiplicity of W_i in V_i.

2. Let G act on $H_i \otimes V_i$ through the tensor product of the trivial representation of G on H_i and the given representation on W_i. Show that the linear map

$$\begin{aligned} F : H_i \otimes W_i &\to V_i \\ \sum h_\alpha \otimes w_\alpha &\mapsto \sum h_\alpha(w_\alpha) \end{aligned}$$

   is an isomorphism.

3. Let $(h_1, \ldots, h_k)$ be a basis of H_i and form the direct sum $W_i \oplus \cdots \oplus W_i$ of k copies of W_i. This basis defines an obvious mapping $h \colon W_i \oplus \cdots \oplus W_i \to V_i$. Show that h is an isomorphism of representations. In particular, to decompose V_i into a direct sum of representations isomorphic to W_i amounts to choosing a basis for H_i.

Proof:

1. Let $h \in H_i$. Then h maps W_i into say k_i copies of W_i. Each copy of W_i comes with a projection function $V_i \to W_i$. Composing h with this projection function shows that h is a linear combination of maps $W_i \to W_i$. Thus, it suffices to consider the case of $V = W_i$. But Schur's Lemma (Proposition 2.5) says that in this case h is a scalar multiple of the identity, and thus onto. Thus, $\dim(H_i) = 1 = \frac{\dim(V_i)}{\dim(W_i)}$.

2. By composing F with one of the k_i projection functions, we get that F is a linear combination of maps $H_i \otimes W_i \to W_i$. Thus, we may again reduce to the case that $V = W_i$. In this case, by the proof of 1 we get that F is surjective. Dimension counting yields that it is an isomorphism of vector spaces.

   To see that F is an isomorphism of representations, let $\rho' \colon G \to \mathrm{GL}(H_i \otimes W_i)$ be the given tensor product representation. We have that

$$F(\rho'_s(h_{alpha} \otimes w_\alpha)) = F(h_\alpha \otimes \rho_s(w_\alpha)) = h_\alpha(\rho_s(w_\alpha)) = \rho_s(h_\alpha(w_\alpha)) = \rho_s(F(h_\alpha \otimes w_\alpha)).$$

   Thus $F \circ \rho'_s = \rho_s \circ F$ for all generators, and thus on all of $H_i \otimes W_i$. Thus F is an isomorphism of representations.

3. Define the map

$$\begin{aligned} h \colon W_i \oplus \cdots \oplus W_i &\to V_i \\ (w_1, \ldots, w_k) &\mapsto h_1(w_1) + \cdots + h_k(w_k). \end{aligned}$$

Clearly h is linear. From 2 we see that every element of V_i is of the form $\sum w_\alpha h_\alpha$ and the h_i form a basis. Thus h is surjective and dimension counting yields that h is a linear isomorphism. The proof that h is an isomorphism of representations is similar.

Now suppose we are given an isomorphism of representations $h \colon W_i \oplus \cdots \oplus W_i \to V_i$. Let $i_j \colon W_i \to W_i \oplus \cdots \oplus W_i$ be the inclusions of W_i into the j-th component of $W_i \oplus \cdots \oplus W_i$. Define $h_j \colon W_i \to V_i := h \circ i_j$. Since h is an isomorphism of representations, we have that h_j commutes with ρ and so $h_j \in H_i$. We claim that the h_j form a basis of H_i. Suppose the h_j we linearly dependent. Then this would contradict the face that h is an isomorphism of vector spaces since we would be able to show that $\ker(h) \neq 0$. Thus the h_j form a basis of H_i and every isomorphism of representations arises in the way described.

□

Exercise 2.7 (Ser77 Exercise 2.9): Let W_i be a representation of G with matrix form $(r_{\alpha\beta}(s))$ with respect to a basis $(e_1, \ldots, e_n)$. Let H_i be the space of linear maps $h \colon W_i \to V$ such that $h \circ \rho_s = \rho_s \circ h$. Show that $h \mapsto h(e_\alpha)$ is an isomorphism of H_i onto $V_{i,\alpha}$.

Proof: Suppose that $h \mapsto 0$. So $h(e_\alpha) = 0$. By Exercise 2.6 (2) we have that $h \otimes e_\alpha = 0$. But $e_\alpha \neq 0$ so $h = 0$. Then note that $\dim(V_i) = \dim(V_{i\alpha}) \cdot n = \dim(V_{i\alpha}) \cdot \dim(W_i)$. By Exercise 2.6 (1) we have that $\dim(H_i) = \dim(V_{i\alpha})$ and so by a dimension argument, the map $h \mapsto h(e_\alpha)$ is an isomorphism. □

Chapter 3

Subgroups, Products, and Induced Representations

As always, all groups are assumed to be finite.

Theorem 3.1: Let G be a group. The following are equivalent.

1. G is abelian.

2. All the irreducible representations of G have degree 1.

Proof: If $(n_1, \dots, n_h)$ are the degrees of the distinct irreducible representations of G, we know that h is the number of conjugacy classes of G. We know that G is abelian if and only if $h = |G|$. By Corollary 2.17 we have $n_1^2 + \cdots + n_h^2 = |G|$. Thus G is abelian if and only if all of the $n_i = 1$. □

Corollary 3.2: Let A be an abelian subgroup of G. Then every irreducible representation has degree $\leq \frac{|G|}{|A|}$.

Proof: Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Then $\rho\mid_A$ is a representation of A. Let $W \subseteq V$ be an irreducible subrepresentation of ρ_A, which by Theorem 3.1 has dimension 1. Let V' be the vector subspace generated by the images $\rho_s(W)$ as s ranges over G. Then V' is stable under G and since ρ is irreducible we must have that $V' = V$. But for $s \in G$ and $t \in A$ we have

$$\rho_{st}(W) = \rho_s(\rho_t(W)) = \rho_s(W).$$

Thus, the maximum number of distinct images $\rho_s(W)$ is $\frac{|G|}{|A|}$. This yields that $\dim(V) \leq \frac{|G|}{|A|}$ since V is the sum of the $\rho_s(W)$. □

Example 3.3 (Degrees of Irreducible Representations of a Dihedral Group): Consider any dihedral group G. This group has a cyclic subgroup A of index $\frac{|G|}{|A|} = 2$. Thus, every irreducible representation of G has degree 1 or 2.

Exercises

Exercise 3.1: Using Schur's Lemma (Proposition 2.5), show that each irreducible representation of an abelian group, finite or not, has degree 1.

Proof: Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of an abelian group G. Then note that since G is abelian, we have that $\rho_s \circ \rho_t = \rho_t \circ \rho_s$ for all $s, t \in G$. Thus, by Proposition 2.5 we have that ρ_s is a homothety. This means that V must be of dimension 1. $\square$

Definition 3.4 (Dual of an Abelian Group): Let G be an abelian group. Let $\widehat{G}$ be the *dual* of the group G, the set of irreducible characters of G.

We now analyze this group. Note that by Theorem 3.1, we have that any representation is of degree 1. Then since G is finite, we have that the image of any element $s \in G$ must also be of finite order. Thus, the image of any element of $s \in G$ is a root of unity. If $\chi_1, \chi_2 \in \widehat{G}$, then $\chi_1 \cdot \chi_2 \in \widehat{G}$. This is immediate since the product of two roots of unity is another root of unity.

Exercise 3.2: Show that $\widehat{G}$ is an abelian group isomorphic to G where for $s \in G$, the map $\chi \mapsto \chi(s)$ is the element in $\widehat{G}$ dual to s.

Proof: It is clear that $\widehat{G}$ is an abelian group since the product of any two roots of unity is another root of unity and $\mathbb{C}^\times$ is an abelian group. Also we know by the proof of Theorem 3.1 that the number of irreducible characters of an abelian group G is $|G|$. Thus $\widehat{G}$ has order $|G|$. Now consider the mapping

$$\begin{aligned} \phi : G &\to \widehat{G} \\ s &\mapsto (\chi \mapsto \chi(s)). \end{aligned}$$

If $s \in G$, then there is some irreducible representation χ where $\chi(s)$ is an n-th root of unity. Thus if $s \neq e$, then $phi(s)$ is not the identity map. Thus ϕ is injective, and ϕ is clearly a homomorphism. Since $|G| = \left|\widehat{G}\right|$, we have that $G \simeq \widehat{G}$. $\square$

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