Topology Notes and Exercises

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1 Metric Spaces

The most familiar example of a metric space is $\mathbb{R}^n$ with $\operatorname{dist}(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$. We also have a notion of continuity. A function f is *continuous at a point* x if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{dist}(x,y) < \delta \implies \operatorname{dist}(f(x), f(y)) < \varepsilon$. We wish to generalize this notion.

Definition 1.1 (Metric Spaces): A *metric space* is a set X equipped with a distance function $\operatorname{dist}\colon X \times X \to \mathbb{R}$ called a *metric* satisfying

1. $\operatorname{dist}(x,y) \geq 0$ with equality if and only if $x = y$
2. $\operatorname{dist}(x.y) = \operatorname{dist}(y,x)$
3. $\operatorname{dist}(x,z) \leq \operatorname{dist}(x,y) + \operatorname{dist}(y,z)$.

Definition 1.2 (ε-ball): For a point x in some metric space X, the *ε-ball about x* is $B_\varepsilon(x) := \{\, y \in X \mid \operatorname{dist}(x,y) < \varepsilon \,\}$.

It is not hard to rephrase the definition of continuity in terms of ε-balls.

Definition 1.3 (Open and Closed Sets in $\mathbb{R}$): A subset U of some metric space X is *open* if for all $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. A subset C is *closed* if its complement is open.

We can actually even rephrase continuity in terms of open sets.

Proposition 1.4 (Continuity in terms of Open Sets): A function $f : X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open set $U \subseteq Y$.

Proof: Suppose $f : X \to Y$ is continuous. Let $U \subset Y$ be some open set and let $f(x) \in U$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. By continuity of f, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_e(f(x))$. Thus $B_\delta(x) \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is open.

Conversely, suppose that $f^{-1}(U)$ is open for any open $U \subseteq Y$. Let $\varepsilon > 0$. Then $B_\varepsilon(f(x))$ is open and $f^{-1}(B_\varepsilon(f(x)))$ is open and contains x. Thus there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$. Thus $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ and f is continuous. $\square$

This can also all be rephrased in terms of closed sets, rather than open sets.

Example 1.5 (Other Metrics on $\mathbb{R}$): We can also equip other metrics on $\mathbb{R}^n$.

$$\operatorname{dist}_2(x,y) = \sum_{i=1}^n |x_i - y_i|$$
$$\operatorname{dist}_3(x,y) = \max_{i=1}^n (|x_i - y_i|)$$

However it turns out that for $\mathbb{R}^n$, choice between these 3 metrics is irrelevant.

Proposition 1.6 (Equivalence of Open Sets): Suppose dist_1 and dist_2 are two metrics on the same set X such that for any $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{dist}_1(x, y) < \delta \implies \mathrm{dist}_2(x, y) < \varepsilon$$

and

$$\mathrm{dist}_2(x, y) < \delta \implies \mathrm{dist}_1(x, y) < \varepsilon.$$

Then these metrics define the same open sets in X.

Proof: **⟨⟨ TODO: PROOF ⟩⟩** □

Corollary 1.7 (Equivalence of Open Sets in $\mathbb{R}^n$): The following metrics define the same open sets in $\mathbb{R}^n$:

$$\mathrm{dist}(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$$
$$\mathrm{dist}_2(x, y) = \sum_{i=1}^{n}|x_i - y_i|$$
$$\mathrm{dist}_3(x, y) = \max_{i=1}^{n}(|x_i - y_i|)$$

Exercises

Exercise 1.1 (Bre93 1.1.1): Consider the set X of all continuous real valued functions on $[0,1]$. Show that

$$\operatorname{dist}(f,g) = \int_0^1 |f(x) - g(x)|\, \mathrm{d}x$$

defines a metric on X. Is this still the case if continuity is weakened to integrability?

Proof: Positivity follows from the positivity of absolute value. If $f = g$ then clearly for all $x \in [0,1]$ we have that $|f(x) - g(x)| = 0$. Then if $f \neq g$ then there exists $x \in [0,1]$ such that $|f(x) - g(x)| \neq 0$. Since these functions are continuous, this yields that $\int_0^1 |f(x) - g(x)|\, \mathrm{d}x > 0$.

Symmetry follows from $|f(x) - g(x)| = |g(x) - f(x)|$.

The triangle inequality follows from the triangle inequality for absolute value.

Note that we require continuity. Let $f(x) := 0$ and let $g(x) = 0$ for all $x > 0$ and $g(0) = 1$. Then f and g are integrable and $\operatorname{dist}(f,g) = 0$ but $f \neq g$. □

Exercise 1.2 (Bre93 1.1.2): If X is a metric space and x_0 is a given point in X, show that the function $f : X \to \mathbb{R}$ given by $f(x) = \operatorname{dist}(x, x_0)$ is continuous.

Proof: Let $\varepsilon > 0$. Suppose that $\operatorname{dist}(x, y) < \varepsilon$. Then we have that

$$|f(x), f(y)| = |\operatorname{dist}(x, x_0) - \operatorname{dist}(y, x_0)| \leq |\operatorname{dist}(x, y)| = \operatorname{dist}(x, y) < \varepsilon.$$

□

Exercise 1.3 (Bre93 1.1.3): If A is a subset of a metric space X then define a real valued function d on X by $d(x) = \operatorname{dist}(x, A) := \int \{\operatorname{dist}(x, y) \mid y \in A\}$. Show that d is continuous.

Proof: Note that for any $x, y \in X$ and $z \in A$ we have that

$$\operatorname{dist}(x, z) \leq \operatorname{dist}(x, y) + \operatorname{dist}(y, z).$$

Taking infimum yields that

$$\operatorname{dist}(x, A) \leq \operatorname{dist}(x, y) + \operatorname{dist}(y, A).$$

Thus $\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq \operatorname{dist}(x, y)$ and similarly we have $\operatorname{dist}(y, A) - \operatorname{dist}(x, A) \leq \operatorname{dist}(x, y)$. This implies that $|\operatorname{dist}(y, A) - \operatorname{dist}(x, A)| \leq \operatorname{dist}(x, y)$.

Now let $\varepsilon > 0$. Suppose that $\operatorname{dist}(x, y) < \varepsilon$. Then we have that $|d(x), d(y)| \leq \operatorname{dist}(x, y) < \varepsilon$. □

2 Topological Spaces

We usually only care about continuity, not the actual metrics. Continuity can be formulated in terms of open sets. It can be shown that a function is continuous if and only if $f^{-1}(U)$ is open for all open U in the codomain.

Definition 2.1 (Topological Spaces): A *topological* space is a set X with a collection of subsets of X called "open sets" such that

1. the intersection of two open sets is open;
2. the union of any collection of open sets is open; and
3. $X, \emptyset$ are open.

A subset $X \subseteq X$ is *closed* if $X \setminus C$ is open.

Definition 2.2 (Continuous Functions): A function of topological spaces $f : X \to Y$ is *continuous* if $f^{-1}(U)$ is open for all open $U \subseteq Y$. A *map* is a continuous function.

It isn't hard to see that a function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

Definition 2.3 (Neighborhood): If X is a topological space, a set $N \subseteq X$ is a *neighborhood* of $x \in X$ if it contains an open set $U \subseteq N$ such that $x \in U$.

It is immediate that arbitrary unions and finite intersections of neighborhoods of a point $x \in X$ are still neighborhoods of x.

Definition 2.4 (Neighborhood Basis): Let X be a topological space and $x \in X$. A collection $\mathbf{B}_x$ of subsets of X containing x is called a *neighborhood basis* at x in X if each neighborhood of x contains some element of $\mathbf{B}_x$ and each element of $\mathbf{B}_x$ is a neighborhood of x.

Neighborhood bases let us define continuity at a single point.

Definition 2.5 (Continuity at a Point): A function $f : X \to Y$ between topological spaces is said to be *continuous at* x, $x \in X$, if, given any neighborhood N of $f(x)$ in Y, there is a neighborhood M of x in X such that $f(M) \subseteq N$.

This is the same as saying $f^{-1}(N)$ is a neighborhood of x.

Proposition 2.6: A function $f : X \to Y$ between topological spaces is continuous if and only if it is continuous at each point $x \in X$.

Proof: Suppose f is continuous and let N be a neighborhood of $f(x)$ in Y with open $U \subseteq N$ such that $f(x) \in U$. Then $x \in f^{-1}(U) \subseteq f^{-1}(N)$ where $f^{-1}(U)$ is open in X. Thus $f^{-1}(N)$ is a neighborhood of x in X and $f\left(f^{-1}(N)\right) = N \subseteq N$ and f is continuous at $x \in X$.

Conversely now suppose that f is continuous at each point in X and let $U \subseteq Y$ be open. Then for any $x \in f^{-1}(U)$, we have that $f^{-1}(U)$ is a neighborhood of x. Thus there exists open $V_x \subseteq f^{-1}(U)$ with $x \in V_x$. Thus $f^{-1}(U)$ is the union of open sets V_x ranging over $x \in f^{-1}(U)$ and $f^{-1}(U)$ is open which yields that f is continuous. $\square$

Definition 2.7 (Homeomorphic Functions): A function $f : X \to Y$ between topological spaces is called a *homeomorphism* if $f^{-1} : Y \to X$ exists and both f and f^{-1} are continuous. We notate that $X \approx Y$ meaning that X is *homeomorphic* to Y.

Topological spaces are homeomorphic then if there is a bijection between them as sets but also there is a correspondence between the open sets. These sets can then essentially be regarded as the same sets.

Describing topological spaces can be described in a more simple manner than listing all the open sets using the concept of a basis.

Definition 2.8 (Basis): Let X be a topological space and $\mathbf{B}$ a collection of subsets of X. We say that $\mathbf{B}$ is a *basis* for the topology of X if the open sets of X are precisely the unions of members of $\mathbf{B}$. A collection $\mathbf{S}$ of subsets of X is called a subbasis for the topology of X if the set $\mathbf{B}$ of finite intersections of members of $\mathbf{S}$ forms a basis of X.

Any collection $\mathbf{S}$ of subsets of any set X is a subbasis for some topology on X, namely the topology where the open sets are arbitrary unions of finite intersections of members of $\mathbf{S}$. The empty set is the union of an empty collection and X is the intersection of an empty collection. Thus to specify a topology, a subbasis suffices. In a metric space, the collection of all ε-balls, for all $\varepsilon > 0$, forms a basis. So is the collection of ε-balls for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$.

Example 2.9 (Examples of Topologies): We now give examples of topological spaces:

1. (Trivial Topology) Any set X where the only open sets are X and $\emptyset$.
2. (Discrete Topology) Any set X where every subset of X is open.
3. Any sets X where the closed sets are finite sets and X itself.
4. $X = \mathbb{N} \cup \{\mathbb{N}\}$ with the open sets being all subsets of $\mathbb{N}$ together with complements of finite sets.
5. Let X be any poset. For $\alpha \in X$ consider the one-sided intervals $\{\beta \in X \mid \alpha < \beta\}$ and $\{\beta \in X \mid \alpha > \beta\}$. The "order topology" on X is the topology generated by these intervals. The "strong order topology" is the topology generated by these intervals together with the complements of finite sets.
6. Let $X = I^2$ where $I = [0, 1]$ the unit intervals. Give this the "dictionary ordering" where $(x, y) < (s, t)$ if and only if either $x < s$ or ($x = s$ and $y < t$). Let X have the order topology for this ordering.
7. Let X be the real line bu with the topology generated by the "half open intervals" $[x, y)$. This is called the "half open interval topology" or the "lower limit topology."
8. Let $X = \Omega \cup \{\Omega\}$ be the set of ordinal numbers up to and including the least uncountable ordinal Ω. Give this the order topology.

Definition 2.10 (Countability): A topological space is *first countable* if each point has a countable neighborhood basis. A topological space is *second countable* if its topology has a countable basis.

Example 2.11 (Metric Spaces are First Countable but some are not Second Countable): Let $x \in X$ a metric space. Then the set of open balls around x with rational radius forms a countable neighborhood basis.

Consider the space if any uncountable set with metric $\text{dist}(x, y) = 1$ if $x \neq y$ and 0 otherwise (which yields the discrete topology). Euclidian spaces are second countable since the ε-balls, with ε rational, about points with rational coordinates, is a countable basis.

Proposition 2.12: A topological space X is second countable if and only if every basis for X has a countable subbasis.

Proof: Suppose X is second countable. Let $B = \{B_\alpha\}_{\alpha\in A}$ be a basis for X and $C = \{C_i\}_{i\in\mathbb{N}}$ be a countable basis for X. It suffices to show that each C_i can be expressed as a countable union of some subset of the B_α's. Let C_k be a member of the countable basis. B is a basis so $C_k = \bigcup_{\alpha\in I\subseteq A}$ for some I. For each $x \in C_k$, let B_{i_x} be a member of the basis such that $i_x \in I$ and $x \in B_{i_x}$. But then C is also a basis, so we can find C_{i_x} such that $x \in C_{i_x} \subseteq B_{i_x}$. We can see that $\{B_{i_x} \mid x \in C_k\}$ is our desired countable subset. Clearly for $x \in C_k$ we have that there is some i_x such that $x \in B_{i_x}$ so $C_k \subseteq \bigcup_{x\in C_k} B_{i_x}$. We also see that we only need countably many i_x since each of the $x \in C_i$ are in some C_{i_x} and there are only countably many C_{i_x}. □

Definition 2.13 (Uniform Convergence of Functions): A sequence $f_1, f_2, \ldots$ of functions from a topological space X to a metric space Y is said to *converge uniformly* to a function $f : X \to Y$ if, for all $\varepsilon > 0$, there is a number n such that for all $i > n$, $\operatorname{dist}(f_i(x), f(x)) < \varepsilon$ for all $x \in X$.

Theorem 2.14: If a sequence $f_1, f_2, \ldots$ of continuous functions from a topological space X to a metric space Y converges uniformly to a function $f : X \to Y$, then f is continuous.

Proof: Given $\varepsilon > 0$, let n_0 be such that for all $x \in X$, $n \geq n_0$ implies that $\operatorname{dist}(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Given n_0, continuity of f_{n_0} implies that there is a neighborhood N of x_0 such that $x \in N$ implies that $\operatorname{dist}(f_{n_0}(x), f_{n_0}(x_0)) < \frac{\varepsilon}{3}$. Thus for any $x \in N$ we have that

$$
\begin{aligned}
\operatorname{dist}(f(x), f(x_0)) &\leq \operatorname{dist}(f(x), f_{n_0}(x)) + \operatorname{dist}(f_{n_0}(x), f_{n_0}(x_0)) + \operatorname{dist}(f_{n_0}(x), f(x_0)) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus f is continuous. □

Definition 2.15 (Open and Closed Functions): A function $f : X \to Y$ between topological spaces is said to be *open* if $f(U)$ is open in Y for all open $U \subseteq X$. It is said to be *closed* if $f(C)$ is closed in Y for all closed $C \subseteq X$.

Definition 2.16 (Smallest and Largest Topologies): If X is a set and some condition is given on subsets of X, which may or may not hold for any particular subset, then if there is a topology T whose open sets satisfy the condition, and such that, for any topology T' whose open sets satisfy the condition, then the T-open sets are also T'-open (i.e. $T \subseteq T'$), then T is called the *smallest* (or *weakest* or *coarsest*) topology satisfying the condition. If, instead, for any topology T' whose open sets satisfy the condition, any T'-open sets are also T-open (i.e. $T' \subseteq T$), then T is called the *largest* (or *strongest* or *finest*) topology satisfying the condition.

Example 2.17 (Largest and Smallest Topology for a Condition): If $f : X \to Y$ is a function and X is a topological space, then there is a largest topology on Y making f continuous by having open sets $\{V \subseteq Y \mid f^{-1}(V) \text{ is open in } X\}$. The smallest such topology is the trivial topology.

If a topology is the largest one satisfying some condition, then there is always some other condition where the given topology is the smallest one satisfying the new condition. For example, the topology described in the prior example is the smallest topology satisfying the condition "for all spaces Z and functions $g : Y \to Z$, $g \circ f$ being continuous implies g is continuous." Thus there is no way to argue that a topology is "large" or "small" without knowing the defining condition.

2.1 Exercises

Exercise 2.1 (Bre93 1.2.1): Show that in a topological space X:

a. the union of two closed sets is closed;

b. the intersection of any collection of closed sets is closed; and

c. the empty set and the whole space X are closed.

Proof: a. Suppose that C_1, C_2 are closed in X. Then we have that $X \setminus (C_1 \cup \mathbb{C}_2) = (X \setminus C_1) \cup (X \setminus C_2)$ which is the union of two open sets. Thus $C_1 \cup C_2$ is closed.

b. **⟨⟨ Sim. ⟩⟩**

c. We have that $X \setminus \emptyset = X$ which is open so $\emptyset$ is closed. Similarly we have that $X \setminus X = \emptyset$ which is open so X is closed.

□

Exercise 2.2 (Bre93 1.2.2): Consider the topology on $\mathbb{R}$ generated by half open intervals $[x, y)$ together with those of the form $(x, y]$. Show that this coincides with the discrete topology.

Proof: Note that $(x-1, x] \cup [x, x+1) = \{x\}$. Thus every singleton, and therefore every subset of $\mathbb{R}$, is open and we recover the discrete topology. □

Exercise 2.3 (Bre93 1.2.4): If $f : X \to Y$ is a function between topological space, and $f^{-1}(U)$ is open for each open U in some subbasis for the topology of Y, show that f is continuous.

Proof: Let $\mathbf{S}$ be a subbasis for Y and let U be open in Y. We have that $\mathbf{S}$ generates some basis $\mathbf{B}$ for Y and for some indexing set I we have $U = \bigcup_{i \in I} B_i$ where $B_i \in \mathbf{B}$. But then each $B_i = S_{i_1} \cap S_{i_n}$ for $S_{i_j} \in \mathbf{S}$. We have that

$$f^{-1}(B_i) = \bigcap_{j=1}^{n} f^{-1}(S_{i_j}) \text{ where each } f^{-1}(S_{i_j}) \text{ is open so } f^{-1}(B_i) \text{ is open ; and}$$
$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B_i) \text{ where each } f^{-1}(B_i) \text{ is open so } f^{-1}(U) \text{ is open .}$$

Thus f is continuous. □

Exercise 2.4 (Bre93 1.2.5): Suppose that S is a set and we are given, for each $x \in S$, a collection $\mathbf{N}(x)$ of subsets of S satisfying:

1. $N \in \mathbf{N}(x) \implies x \in N$;
2. $N, M \in \mathbf{N}(x) \implies \exists P \in \mathbf{N}(x)$ such that $P \subseteq N \cap M$; and
3. $x \in S \implies \mathbf{N}(x) \neq \emptyset$.

Then show that there is a unique topology on S such that $\mathbf{N}(x)$ is a neighborhood basis at x, for each $x \in S$. Thus a topology can be defined by giving such a collection of neighborhoods at each point.

Proof: Let S be the topology with open sets $\{U \subseteq S \mid \forall x \in U, \exists N \in \mathbf{N}(x) \text{ such that } N \subseteq U\}$.

- $\emptyset$ is open in S vacuously. Now take $x \in S$. $\mathbf{N}(x) \neq \emptyset$ and so there exists $N \subseteq S$ such that $x \in N$. Thus S is also open.
- Let U, V be open. Then let $x \in U \cap V$. $x \in U$ implies there exists $N \in \mathbf{N}(x)$ such that $N \subseteq U$. Similarly there exists $M \in \mathbf{N}(x)$ such that $M \subseteq V$. We have then there exists $P \in \mathbf{N}(x)$ such that $P \subseteq N \cap M$. Thus $U \cap V$ is open.
- **⟨⟨ TODO: Union ⟩⟩**

Suppose that Y is some other topology such that for each $x \in S$. $\mathbf{N}(x)$ is a neighborhood basis of x. Let U be open in Y and $x \in U$. Then there must be $N \in \mathbf{N}(x)$ with $N \subseteq U$ because U is an open set containing x and $\mathbf{N}(x)$ is a neighborhood basis. Thus U is also open in S.

Now suppose that U is open in S. Then for $x \in U$ there exists $N \in \mathbf{N}(x)$ such that $N \subseteq U$. N is a neighborhood of x i Y also. Thus there exists open $V_x \subseteq N$ such that $x \in V_x$. Thus since $V_x \subseteq N \subseteq U$ we have that $U = \bigcup_{x \in U} V_x$. Thus U is the union of open sets in Y and U is open in Y.

Overall $S = Y$ and we have that S is the unique topology we want. $\square$

3 Subspaces

Definition 3.1 (Subspace): If X is a topological space and $A \subseteq X$, then the *relative topology* or *subspace topology* on A is the collection of intersections of A with open sets of X. With this topology, A is called a *subspace* of X.

We have a series of basic consequences of this definition of subspace.

Proposition 3.2: If Y is a subspace of A, then $A \subseteq Y$ is closed if and only if $A = Y \cap B$ for some closed $B \subseteq X$

Proof: Suppose A is closed in Y. Then $Y \setminus A$ is open in Y. Then $Y \setminus A = Y \cap U$ for some open $U \subseteq X$. $X \setminus U$ is closed and $(Y \setminus A) \sqcup A = Y = Y \cap U \sqcup (Y \cap (X \setminus U))$. This implies $A = Y \cap (X \setminus U)$.

Now suppose that $A = Y \cap B$ for some closed $B \subseteq X$. Then $X \setminus B$ is open in X. We have that $Y \setminus A = Y \setminus (Y \cap B) = Y \cap (X \setminus B)$ and so $Y \cap A$ is open meaning that A is closed. □

Proposition 3.3: If X is a topological space and $A \subseteq X$, then there exists a largest open set U with $U \subseteq A$.

Proof: Let $\{U_i\}_{i \in I}$ be an indexed set of all open sets $\subseteq A$. Then $U := \bigcup_{i \in I} U_i$ is the largest open set in A. This is because the union of open sets is open and if $O \subseteq A$ is open then $O = U_j$ for some j and thus $O \subseteq U$. □

Definition 3.4 (Interior): let X be a topological space and $A \subseteq X$. The largest open set contained in A is called the *interior* of A in X and is denoted by $\mathrm{int}(A)$ or A°.

Proposition 3.5: If X is a toplogical space and $A \subseteq X$, then there exists a smallest closed set F such that $A \subseteq F \subseteq X$.

Proof: Let $\{F_i\}_{i \in I}$ be the indexed set of all closed sets $A \subseteq F_i \subseteq X$. Then $F := \bigcup_{i \in I} F_i$ is the closure of A. F is closed since the intersection of closed sets is closed and $A \subseteq F$ since $A \subseteq F_i$ for all $i \in I$. If $A \subseteq C \subseteq X$ is closed then $C = F_j$ for some j and thus $F \subseteq C$. □

Definition 3.6 (Closure): let X be a topological space and $A \subseteq X$. The smallest closed set containing A is called the *closure* of A in X and is denoted by $\mathrm{cl}(A)$ or $\overline{A}$.

We give more precise definitions of the interior and closure of a set.

Proposition 3.7 (Bre93 1.3.1.a): Let X be a topological space and $A \subseteq X$. Then we have that

$$\mathrm{int}(A) = \{ a \in A \mid \exists U \text{ open such that } a \in U \subseteq X \}$$
$$\overline{A} = \{ x \in X \mid \forall U \text{ open such that } x \in U,\ U \cap A \neq \emptyset \}$$

Proof: **⟨⟨ Immediate from defn. ⟩⟩** □

Proposition 3.8 (Bre93 1.3.1.b): Let X be a topological space and $A \subseteq X$. Then A is open if and only if $\mathrm{int}(A) = A$ and A is closed if and only if $\overline{A} = A$.

Proof: **⟨⟨ Immediate from Proposition 3.7 ⟩⟩** . □

Proposition 3.9 (Bre93 1.3.1.c): Let X be a topological space and $A \subseteq X$. Then $X \setminus \operatorname{int}(A) = \overline{X \setminus A}$ and $X \setminus \overline{A} = \operatorname{int}(X \setminus A)$.

Proof: Let $x \in X \setminus \operatorname{int}(A)$. Then for any open U such that $x \in U$, we have that $U \nsubseteq A$ and so $U \cap (X \setminus A) \neq \emptyset$. Thus by Proposition 3.7, $x \in \overline{X \setminus A}$. Conversely let $x \in \overline{X \setminus A}$. Then for any open set U with $x \in U$, we have $U \cap (X \setminus A) \neq \emptyset$. Thus $U \nsubseteq A$ and $x \in X \setminus \operatorname{int}(A)$.

Now let $x \in X \setminus \overline{A}$. Then by Proposition 3.7 there exists an open set U such that $x \in U$ and $U \cap A = \emptyset$. Thus $U \subseteq X \setminus A$ and $x \in \operatorname{int}(X \setminus A)$. **⟨⟨ TODO: Reverse Inclusion ⟩⟩** □

Proposition 3.10 (Bre93 1.3.1.d): Let X be a topological space and $A, B \subseteq X$. Then $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof: Suppose that $x \in \operatorname{int}(A \cap B)$. Then there exists open U such that $x \in U \subseteq A \cap B$. Thus $U \subseteq A$ and $U \subseteq B$ and $x \operatorname{int}(A) \cap \operatorname{int}(B)$. Now suppose $x \in \operatorname{int}(A) \cap \operatorname{int}(B)$. Then there exists open sets U_1 and U_2 such that $x \in U_1 \subseteq A$ and $x \in U_B \subseteq B$. Thus $x \in U_1 \cap U_2 \subseteq A \cap B$ which implies that $x \in \operatorname{int}(A \cap B)$.

Now suppose that $x \in \overline{A \cup B}$. Then for all open U with $x \in U$, $U \cap (A \cup B) \neq \emptyset$. Thus for all open U with $x \in U$, $U \cap A$ or $U \cap B$ is nonempty and $x \in \overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. Then for all open U with $x \in U$, $U \cap A$ or $U \cap B$ is nonempty. Thus $U \cap (A \cup B) \neq \emptyset$ and $x \in \overline{A \cup B}$. □

Proposition 3.11 (Bre93 1.3.1.e): Let X be a topological space and $\{A_\alpha\}$ an indexed collection of subsets of X.

1. $\bigcap_\alpha \operatorname{int}(A_\alpha) \supseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right) = \operatorname{int}\left(\bigcap_\alpha \operatorname{int}(A_\alpha)\right)$.
2. $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha} = \overline{\bigcap_\alpha \overline{A_\alpha}}$.
3. $\bigcup_\alpha \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcup_\alpha A_\alpha\right)$.
4. $\bigcap_\alpha \overline{A_\alpha} \supseteq \overline{\bigcap_\alpha A_\alpha}$.

Proof:

1. In the finite case, by induction, we have that $\bigcap_\alpha \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right)$. **⟨⟨ The proof does not invoke finiteness / infiniteness and so this holds in the general case. ⟩⟩** We actually have equality in the case of finite intersection.

   Note that $\operatorname{int}\left(\bigcap_\alpha A_\alpha\right)$ is open and $\operatorname{int}\left(\bigcap_\alpha A_\alpha\right) \subseteq \bigcap_\alpha \operatorname{int}(A_\alpha)$ and so $\operatorname{int}\left(\bigcap_\alpha A_\alpha\right) \subseteq \operatorname{int}\left(\bigcap_\alpha \operatorname{int}(A_\alpha)\right)$. Now let $x \in \operatorname{int}\left(\bigcap_\alpha \operatorname{int}(A_\alpha)\right)$. Then there is some open U with $x \in U$ such that $U \subseteq \bigcap_\alpha \operatorname{int}(A_\alpha)$. Thus $U \subseteq \operatorname{int}(A_\alpha) \subseteq A_\alpha$ for all α and so $U \subseteq \bigcap_\alpha A_\alpha$ and $x \in \operatorname{int}\left(\bigcap_\alpha A_\alpha\right)$.

2. Let $x \in \bigcup_\alpha \overline{A_\alpha}$. Then for some α, $x \in \overline{A_\alpha}$. So for all open U with $x \in U$, $U \cap A_\alpha \neq \emptyset$. Thus for all open U with $x \in U$, $U \cap \left(\bigcup_\alpha A_\alpha\right) \neq \emptyset$ and $x \in \overline{\bigcup_\alpha A_\alpha}$.

   $A_\alpha \subseteq \overline{A_\alpha}$ for all α. So $\overline{\bigcup_\alpha A_\alpha} \subseteq \overline{\bigcup_\alpha \overline{A_\alpha}}$. Now let $x \in \overline{\bigcup_\alpha \overline{A_\alpha}}$. Then for all open U with $x \in U$, $U \cap \bigcup_\alpha A_\alpha \neq \emptyset$. Since $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$, we have that $U \cap \overline{\bigcup_\alpha A_\alpha} \neq \emptyset$. Thus $x \in \overline{\bigcup_\alpha}$.

3. Suppose that $x \in \bigcup_\alpha \operatorname{int}(A_\alpha)$. Then there exists α such that $x \in \operatorname{int}(A_\alpha)$. Thus there exists open U with $x \in U$ such that $U \subseteq A_\alpha$. So $U \subseteq \bigcup_\alpha A_\alpha$ and overall $x \in \operatorname{int}\left(\bigcup_\alpha A_\alpha\right)$.

4. Let $x \in \overline{\bigcap_\alpha A_\alpha}$. Then for all open U with $x \in U$, $U \cap \bigcap_\alpha A_\alpha \neq \emptyset$. Thus for all α, $U \cap A_\alpha \neq \emptyset$ and $x \in \overline{A_\alpha}$ for all α. Thus $x \in \bigcap_\alpha \overline{A_\alpha}$.

$\square$

Example 3.12 (Interior of Intersection is not Equal to Intersection of Interiors): We have that $\bigcap_\alpha \operatorname{int}(A_\alpha) \supseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right)$. We also know that we have equality in the case of finite intersections. However in general we have that $\bigcap_\alpha \operatorname{int}(A_\alpha) \nsubseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right)$. Let $A_n := \left(1-\frac{1}{n}, 1+\frac{1}{n}\right) \subseteq \mathbb{R}$ for all $n \geq 1$. Then

$$\bigcap_{n\geq 1} \operatorname{int}(A_n) = \bigcap_{n\geq 1}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right) = \{1\}; \text{ and}$$
$$\operatorname{int}\left(\bigcap_{n\geq 1} A_n\right) = \operatorname{int}\{1\} = \emptyset.$$

Example 3.13 (Union of Closures is not Equal to Closure of Union): We have that $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$. However in general we have that $\overline{\bigcup_\alpha A_\alpha} \nsubseteq \bigcup_\alpha \overline{A_\alpha}$. Let $A_n = \left(\frac{1}{n}, 1\right]$ for $n \geq 1$. Then

$$\bigcup_{n\geq 1} \overline{\left(\frac{1}{n}, 1\right]} = \bigcup_{n\geq 1}\left[\frac{1}{n}, 1\right] = (0,1]$$
$$\overline{\bigcup_{n\geq 1}\left(\frac{1}{n}, 1\right]} = \overline{(0,1]} = [0,1]$$

Example 3.14 (Union of Interiors is not Equal to Interior of Union): We have that $\bigcup_\alpha \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcup_\alpha A_\alpha\right)$. However in general we have that $\operatorname{int}\left(\bigcup_\alpha A_\alpha\right) \nsubseteq \bigcup_\alpha \operatorname{int}(A_\alpha)$. Let $A_1 = \left[0, \frac{1}{2}\right]$ and let $A_2 = \left[\frac{1}{2}, 1\right]$. Then $\operatorname{int}(A_1 \cup A_2) = \operatorname{int}([0,1]) = (0,1)$. However $\operatorname{int}(A_1) \cup \operatorname{int}(A_2) = \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$.

Example 3.15 (Intersection of Closures is not Equal to Closure of Intersection): We have that $\overline{\bigcap_\alpha A_\alpha} \subseteq \bigcap_\alpha \overline{A_\alpha}$. However in general we have that $\bigcap_\alpha \overline{A_\alpha} \nsubseteq \overline{\bigcap_\alpha A_\alpha}$. Let $A_1 = \left(0, \frac{1}{2}\right)$ and $A_2 = \left(\frac{1}{2}, 1\right)$. Then $\overline{A_1 \cap A_2} = \overline{\emptyset} = \emptyset$. However $\overline{A_1} \cap \overline{A_2} = \left[0, \frac{1}{2}\right] \cap \left[\frac{1}{2}, 1\right] = \{\frac{1}{2}\}$.

Proposition 3.16 (Bre93 1.3.1.f): Let X be a topological space and A, B subsets of X. If $A \subset B$, then we have that $\overline{A} \subseteq \overline{B}$ and $\operatorname{int}(A) \subseteq \operatorname{int}(B)$.

Proof: Suppose $A \subseteq B$ and $x \in \overline{A}$. Then for all open sets U with $x \in U$ we have $U \cap A \neq \emptyset$. Thus $U \cap B \neq \emptyset$ and $x \in \overline{B}$.

Now let $x \in \operatorname{int}(A)$. Then there exists open U such that $x \in U \subseteq A \subseteq B$. Thus $x \in \operatorname{int}(B)$. $\square$

We may specify the space in which a closure is taken with the notation $\overline{A}^X$. However, we do not need this notation very often.

Proposition 3.17: If $A \subseteq Y \subseteq X$, then $\overline{A}^Y = \overline{A}^X \cap Y$. Thus if Y is closed, then $\overline{A}^Y = \overline{A}^X$.

Proof: Suppose $a \in \overline{A}^Y$. Then $a \in X$ since $\overline{A} \subseteq Y \subseteq X$. Thus $A \subseteq \overline{A} \subseteq X$ and since $a \in Y$ also, we have that $a \in \overline{A}^X \cap Y$. The reverse inclusion is also a similar argument and thus $\overline{A}^Y = \overline{A}^X \cap Y$. If Y is closed, then $\overline{A} \subseteq Y$ and so $\overline{A}^X \cap Y = \overline{A}^X$. $\square$

Proposition 3.18: If $Y \subseteq X$ then the set of intersection of Y with a basis of X is a basis of the relative topology of X.

Proof: **⟨⟨ High lvl: Take open $\subseteq Y$. This is equal to $Y \cap$ open $\subseteq X$. Use basis of X, distribute intersection ⟩⟩** . $\square$

Proposition 3.19: If X, Y, Z are topological spaces and Z is a subspace of Y, and Y is a subspace of X, then Z is a subspace of X.

Proof: Let $U \subseteq Z$ be open. Then $U = Z \cap U'$ for some open $Z' \subseteq Y$. But $U' = Y \cap U''$ for some open $U'' \subseteq X$. Note that $Z \subseteq Y$ so $Z \cap Y = Z$. Thus $U = Z \cap U' = Z \cap Y \cap U'' = Z \cap U''$ and overall Z is a subspace of X. $\square$

Proposition 3.20: If X is a metric space and $A \subseteq X$, then $\overline{A}$ coincides with the set of limits in X of sequences of points in A.

Proof: If x is the limit point of a sequence of points in A, then any open sets about x contains a point of A. Thus $x \notin \text{int}(X \setminus A)$. We have that $X \setminus \text{int}(X \setminus A) = \overline{A}$ and so $x \in \overline{A}$.

Now suppose $x \in \overline{A}$. $B_{1/n}(x)$ must contain a point in A. If it didn't then $x \in \int(X \setminus A)$ which is disjoint from $\overline{A}$. Let x_n be a point in A which is also in $B_{a/n}(x)$. Thus x is a limit of a sequence of points in A. $\square$

Proposition 3.21: Let X be a metric space d. Recall that for $C \subseteq X$, $x \in X$, that $d(x, C) := \inf\{\, d(x, c) \mid x \in C \,\}$. Let C be closed. Then $d(x, C) = 0$ if and only if $x \in C$

Proof: If $x \in C$ then the claim is clear. Now suppose $x \in X$ such that $d(x, C) = 0$. Then for all $r > 0$ we have that $B_r(x) \cap C \neq \emptyset$. Thus $x \in \overline{C} = C$. $\square$

Definition 3.22 (Boundary): If X is a topological space and $A \subseteq X$, then the *boundary* or *frontier* of A is defined to be $\partial(A)$ or $\text{bdry}(A) = \overline{A} \cap \overline{X \setminus A}$.

Definition 3.23 (Dense and Nowhere Dense Sets): A subset A of a topological space X is called *dense* in X if $\overline{A} = X$. A subset A is said to be *nowhere dense* in X if $\text{int}(\overline{A}) = \emptyset$.

Exercises

Exercise 3.1 (Bre93 1.3.2): For $A \subseteq X$, we have that $X = \operatorname{int}(A) \sqcup \partial(A) \sqcup X \setminus \overline{A}$.

Proof: Recall that $X \setminus \overline{A} = \operatorname{int}(X \setminus A)$ and $\partial A := \overline{A} \cap \overline{X \setminus A}$. Let $x \in \operatorname{int}(A)$. Then there exists open $N \subseteq A$ such that $x \in N$. Thus $N \cap (X \setminus A) = \emptyset$ and $x \notin \overline{X \setminus A}$. So $x \notin \partial A$. Now suppose that $x \in \partial A = \overline{A} \cap \overline{X \setminus A}$. Then for every open set U such that $x \in U$, we have $U \cap (X \setminus A) \neq \emptyset$ and so in particular $U \nsubseteq A$. Thus overall $\operatorname{int}(A)$ and ∂A are disjoint. We automatically have that ∂A and $X \setminus \overline{A}$ are disjoint since $x \in \partial A \implies x \in \overline{A}$.

Now let $x \in X$ such that $x \notin \operatorname{int}(A) \sqcup X \setminus \overline{A}$. $x \notin \operatorname{int}(A)$ so for all open U such that $x \in U$ we have $U \nsubseteq A$. So $U \cap (X \setminus A) \neq \emptyset$ for all open U containing x. Thus $x \in \overline{X \setminus A}$. Also $x \notin X \setminus \overline{A}$ so $x \in \overline{A}$. Thus $x \in \partial A$ and overall $X = \operatorname{int}(A) \sqcup \partial(A) \sqcup X \setminus \overline{A}$. □

Exercise 3.2 (Bre93 1.3.3): Show that a metric space X is second countable if and only if it has a countable dense set.

Proof: Suppose that X is second countable with a countable basis $\{B_i\}_{i \in \mathbb{N}}$. Without loss of generality, suppose all the B_i are non-empty. Let $x_i \in B_i$ and $D = \{x_i \mid i \in \mathbb{N}\}$. We claim that D is dense in X. Clearly $\overline{D} \subseteq X$ so let $x \in X$. Then for all open U with $x \in U$, we have $U \cap D \neq \emptyset$. This is because U is a union of members of $\{B_i\}$ and so contains some x_i.

Now let D be a countable dense set of X. Let $B = \{B_q(x) \mid x \in D,\ q \in \mathbb{Q}^\times\}$. We claim that B is a countable basis for X. Let U be an open set in X. Then for all $x \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. D is dense in X and so there exists $d_x \in D$ such that $d_x \in B_{\varepsilon/2}(x)$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q_x \in \mathbb{Q}$ such that $\operatorname{dist}(x, d_x) < q_x < \frac{\varepsilon}{2}$. Thus $B_q(d_x)$ contains x and $B_{d_x}(x) \subseteq B_e(x) \subseteq U$ which means that

$$U = \bigcup_{x \in U} B_{q_x}(d_x)$$

and B is a countable basis for X. □

Definition 3.24 (Separable Space): A topological space is *separable* if it has a countable dense set.

Exercise 3.3 (Bre93 1.3.4): The union of two nowhere dense sets is nowhere dense.

Proof: Let A and B be two nowhere dense subsets of a topological space X. Suppose we have some non-empty $U \subseteq \overline{A \cup B} = \overline{A} \cup \overline{B}$. Let $V = U \cap (X \setminus \overline{A})$. Note that V is open. Suppose that V was empty, then $U \subseteq \overline{A}$ which is impossible. Thus V is nonempty. $U \subseteq \overline{A} \cup \overline{B}$ and has non-empty intersection with $X \setminus \overline{A}$ and thus $V \subseteq \overline{B}$. This is also impossible meaning that U must be empty. □

Definition 3.25 (Irreducible Space): A topological space X is said to be *irreducible* if whenever $X = F \cup G$ with F, G closed we have $X = F$ or $X = G$. A subspace is irreducible if it is irreducible in the subspace topology.

Definition 3.26 (Zariski Space): A *Zariski* space is a topological space such that every descending chain $F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \cdots$ of closed sets is eventually constant.

Exercise 3.4 (Bre93 1.3.8): Let $X = A \cup B$ where A, B are closed. Let $f : X \to Y$ be a function. If $f\mid_A$ and $f\mid_B$ are both continuous, then f is continuous.

Proof: Let $U \subseteq Y$ be open. Then $f\mid_A^{-1}(U)$ and $f\mid_B^{-1}(U)$ are both open. Thus their union is open. We have

$$\begin{aligned} f\mid_A^{-1}(U) \cup f\mid_B^{-1}(U) &= \{x \in A \mid f(x) \in U\} \cup \{x \in B \mid f(x) \in U\} \\ &= \{x \in A \cup B \mid f(x) \in U\} \\ &= f^{-1}(U) \end{aligned}$$

and so f is continuous. $\square$

4 Connectivity and Components

Intuitively, a connected space is a space where you can move from one space to another with no jumps. Another intuition is that the space doesn't have two or more separated pieces.

Definition 4.1 (Connected Sets and Separation): A topological space X is *connected* if it is not the disjoint union of two nonempty open subsets. If A, B are two disjoint nonempty open subsets of X such that $A \sqcup B = X$ then we say that A and B form a *separation* of X.

Definition 4.2 (Clopen Sets): A subset A of a topological space X is *clopen* if it is both open and closed in X.

Proposition 4.3: A topological space X is connected if and only if the open clopen sets are X and $\emptyset$.

Proposition 4.4: Suppose X is connected and suppose there exists clopen $\emptyset \subsetneq U \subsetneq X$. Then $X \setminus U$ is also clopen and $X = (X \setminus U) \sqcup U$ forms a separation of X, contradiction.

Definition 4.5 (Discrete Valued Maps (DVM)): A *discrete valued map (DVM)* is a map from a topological space X to a discrete space D.

Proposition 4.6: A topological space X is connected if and only if every discrete valued map on X is constant.

Proof: If X is connected and $d : X \to D$ is a DVM and y is in the range of d, then $d^{-1}(y)$ is clopen in and non-empty. Thus $d^{-1}(y) = X$ and d is constant.

Now suppose that X is not connected and $U \sqcup V$ form a separation of X. Then $d : X \to \{0, 1\}$ where $d(x) = 0$ if and only if $x \in U$ forms a nonconstant DVM. □

Proposition 4.7: If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof: Let $d : f(X) \to D$ be a DVM. Then $d \circ f$ is a DVM on X and thus constant. This implies that d is constant and thus $f(X)$ is connected. □

Proposition 4.8: If $\{Y_i\}_{i \in I}$ is a collection of connected sets in a topological space X such that no two Y_i are disjoint, then $\bigcup_{i \in I} Y_i$ is connected.

Proof: Let $d : \bigcup_{i \in I} Y_i \to D$ be a DVM. Let $p, q \in \bigcup_{i \in I} Y_i$ such that $p \in Y_i$, $q \in Y_j$ and $v \in Y_i \cap Y_j$. Then $d(p) = d(v) = d(q)$ and d is constant which yields that $\bigcup_{i \in I} Y_i$ is connected. □

Corollary 4.9: The relation "$p \sim q$ if and only if p and q belong to a connected subset of X" is an equivalence relation.

Proof: Immediate. □

Definition 4.10 (Components): The equivalence classes of the relation stated in Corollary 4.9 are called the *components* of X. These are the "maximal" connected subsets of X.

Lemma 4.11: Let X be a connected set. Then $\overline{X}$ is connected.

Proof: Let $d : \overline{X} \to \{0, 1\}$ be a DVM. X is connected so $f(X) =$, without loss of generality, $\{0\}$. Then $\{0\}$ is closed so $d^{-1}(0)$ is closed and contains X. Thus $\overline{X} \subseteq d^{-1}(0) \subseteq \overline{X}$ which means that f is constant. □

Proposition 4.12: Components of a topological space X are connected and closed. Each connected subset of X is contained in a component. Components are equal or disjoint and their union is X.

Proof: The last sentence is immediate based off of Corollary 4.9. By definition, the component of X containing a point p is the union of all connected sets containing p, which itself is connected. This implies that connected sets lie in components. We have by Lemma 4.11 that since a component C is connected $\overline{C}$ is connected and since $C \subseteq \overline{C}$ we overall have that $C = \overline{C}$ which means that C is closed. $\square$

Proposition 4.13: The relation "$p \sim q$ if and only if $d(p) = d(q)$ for every discrete valued map d on X" is an equivalence relation on X.

Proof: Immediate. $\square$

Definition 4.14 (Quasi-components): The equivalence classes of the relation stated in Proposition 4.13 are called the *quasi-components* of X.

Proposition 4.15: Quasi-components of a space X are closed. Each connected set is contained in a quasi-component and in particular each component is contained in a quasi-component. Quasi-component are either equal or disjoint and their union is X.

Proof: The last statement is immediate based off of Proposition 4.13. If $p \in X$, then the quasi-component is $\{ q \in X \mid d(q) = d(p)$ for all DV But this is $\bigcap \left\{ d^{-1}(d(p)) \,\middle|\, d \text{DVM on } X \right\}$. We have that $d^{-1}(d(p))$ is closed since d is continuous. The intersection of closed sets is closed, so quasi-components are closed. Components are constant on every DVM, so they are contained in some quasi-component. $\square$

There is another closely related notion of connectedness which can be easier to deal with.

Definition 4.16 (Arcwise / Pathwise Connected Space): A topological space X is *arcwise connected* or *pathwise connected* if for any two points $p, q \in X$ there exists a map $\lambda \colon [0,1] \to X$ with $\lambda(0) = p$ and $\lambda(1) = q$. λ is called a *path*.

Definition 4.17 (Locally Arcwise Connected Space): A topological space X is *locally arcwise connected* if every neighborhood of any point contains an arcwise connected neighborhood.

We now prove some immediate properties of arcwise connectivity relating it to the other definition of connectivity we have seen.

Proposition 4.18 (Bre93 1.4.5.a): An arcwise connected space is connected.

Proof: Suppose X is arcwise connected but not connected with separation $X = U \sqcup V$. Let $p \in U$ and $q \in V$ with λ the path connecting the two points. λ is continuous and so $\lambda^{-1}(U), \lambda^{-1}(V)$ are open and disjoint and thus form a separation of $[0,1]$. This implies $[0,1]$ is not connected which is a contradiction of the fact that $[0,1]$ is connected (Exercise 4.3). $\square$

Proposition 4.19 (Bre93 1.4.5.b): The relation "$p \sim q$ if and only if there exists a map $\lambda \colon [0,1] \to X$ with $\lambda(0) = p$ and $\lambda(1) = q$" over a space X is an equivalence relation.

Proof: Straightforward. $\square$

Definition 4.20 (Arc Component): The equivalence classes of Proposition 4.19 are called *arc components* and they are the maximally arcwise connected subsets of a space X.

Proposition 4.21 (Bre93 1.4.5.c): An arc component of a space is contained in some component.

Proof: Arc components are connected and so an arc component must be contained in some component. $\square$

Proposition 4.22 (Bre93 1.4.5.d): Arc components of locally arcwise connected space X are clopen and coincide with the components.

Proof: We already know that arc components are connected which means they are closed. Now let A be an arc component in a locally arcwise connected space X and $x \in A$. X is a neighborhood of x and so there exists an arcwise connected neighborhood of x, call it V. V is arcwise connected and contains x so $V \subseteq A$. V is a neighborhood of x and so it contains an open subset U such that $x \in U \subseteq V \subseteq A$ and thus A is also open.

Now let C be the component that A is contained in and suppose $A \subsetneq C$. let Q be the union of all path connected components $\neq A$ that intersect C. $Q \subseteq C$ and since arc components are disjoint we have $C = A \sqcup Q$. A is open and Q is open meaning that they form a separation of C which is a contradiction to the fact that C is connected. Thus $A = C$. $\square$

Example 4.23 (An Arcwise Connected Space with Two Points): Let the two points be p and q and suppose we have open sets $\emptyset, \{p\}$ and $\{p, q\}$. Clearly p and q are arcwise connected to themselves. p is arcwise connected to q with $\lambda : [0,1] \to \{p, q\}$ such that $\lambda(1) = q$ and $\lambda(x) = p$ otherwise. Note that $[0,1)$ is open in $[0,1]$ (but not in $\mathbb{R}$). Thus we have that $\lambda^{-1}(p)$ is open and $\lambda^{-1}(q)$ is closed as desired and λ is our desired path. Thus this space is arcwise connected.

Example 4.24 (Topologist's Sine Curve, A Space that is Connected but not Arcwise Connected): The following is due to [Con]. The *Topologist's Sine Curve* is the subset of the real plane

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$

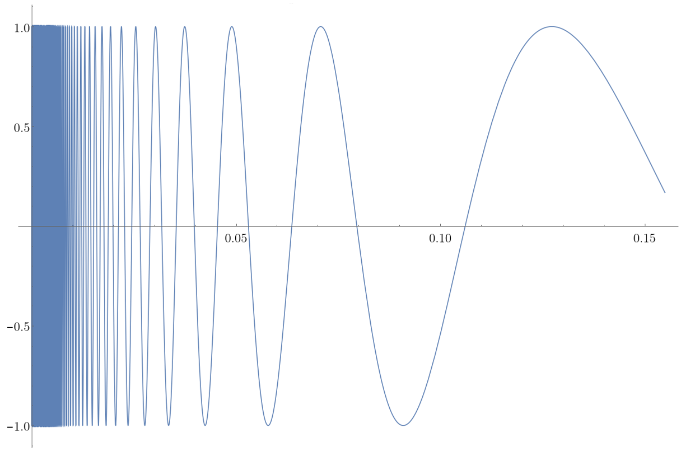

Figure 1: The Topologist's Sine Curve

Clearly S is arcwise connected and thus connected. This is because $\sin\left(\frac{1}{x}\right)$ is continuous and so we can just parameterize along the path.

Consider the set

$$T = (\{0\} \times [-1,1]) \sqcup \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$

We will show this path is connected but not arcwise connected. The intuition here is that we cannot "get to" a portion of the y-axis in finite time starting from a point on the sine curve due to the rapid oscillation.

We claim that $\overline{S} = T$. Note that the function $\sin\left(\frac{1}{x}\right)$ is continuous for all $x > 0$ and so every point on it is a limit point. Now consider a point $(0, y)$ where $y \in [-1, 1]$. Then due to the rapid oscillation of $\sin\left(\frac{1}{x}\right)$ as $x \to 0$ we have that every open ball around $(0, y)$ has non-trivial intersection with the curve infinitely often and so $(0, y)$ is a limit point of some sequence of points in S. Thus every point in T is a limit point of some sequence of points in S and $T \subseteq \overline{S}$.

To show that $\overline{S} \subseteq T$ it suffices to show that T is closed since $S \subseteq T$. Closed sets contain their limit points so let $\{(x_n, y_n)\}$ be a sequence of points in T with limit (x, y). We know that $x = \lim x_n$ and $y = \lim y_n$ so $x \geq 0$ and $y \in [-1, 1]$. If $x = 0$ then we are done since $(0, y) \in T$ for $y \in [-1, 1]$. So now suppose $x > 0$ and we can assume that after dropping some number of starting terms $x_n > 0$ for all n. Thus (x_n, y_n) lies on the curve $\sin\left(\frac{1}{x}\right)$. Since this function is continuous we get that (x, y) lies on the curve and thus T is closed. Thus $\overline{S} \subseteq T$ and overall we have equality.

Note that since S is connected, since it is arcwise connected, we have that T is also connected by Lemma 4.11. We now show that T is not path-connected. Suppose that we have some path λ connecting a point $\lambda(0) = p$ on the sine curve to $\lambda(1) = (0, 1)$. Let $\varepsilon = \frac{1}{2}$. By continuity of λ there is some $\delta > 0$ such that $|\lambda(t) - (0, 1)| < \frac{1}{2}$ for $t \in [1 - \delta, 1]$. The sine curve keeps escaping the disc of radius $\frac{1}{2}$ around $(0, 1)$. In particular this means that we cannot have continuity since the curve will eventually be of distance $> \frac{1}{2}$ from $(0, 1)$. Thus T is not path-connected.

Exercises

Exercise 4.1 (Bre93 1.4.1): If A is a connected subset of a topological space X and $A \subseteq B \subseteq \overline{A}$ then B is connected.

Proof: Suppose U, V are a separation of B. Then note that $A = (U \cap A) \sqcup (V \cap A)$. But $U \cap A$ and $V \cap A$ are open in A and since A is connected we have that one of these, say $U \cap A$, is empty. Thus $A \subseteq X \setminus U$ which is closed. This means that $\overline{A} \subseteq X \setminus U$. So overall now we have $U \subseteq B \subseteq \overline{A} \subseteq X \setminus U$. This can hold if and only if U is empty. Thus B is connected. $\square$

Definition 4.25 (Locally Connected Space): A topological space X is *locally connected* if for each $x \in X$ and each neighborhood N of x, there is a connected neighborhood V of x such that $V \subseteq N$.

Exercise 4.2 (Bre93 1.4.2): If X is locally connected, then its components are open and equal to its quasi-components.

Proof: Let C be a component of X and $x \in C$. Then X itself is a neighborhood of x and so there is a connected neighborhood V of x such that $V \subseteq X$. V is connected so $V \subseteq C$. Since V is a neighborhood of x, there exists an open $U \subseteq V$ such that $x \in U \subseteq C$. Thus C is open.

We already know that components are contained in quasi-components. Let C be a component and Q the quasi-component containing C and suppose $C \subsetneq Q$. C is open and we know that components are also closed and so C is clopen in X. Let $x \in C$ and $y \in Q \setminus C$. Then $X = C \sqcup (Q \setminus C)$ is a separation of X. Thus there exists a DVM d such that $d(x) \neq d(y)$ meaning that x and y are in different quasi-components. But $x \in C \subseteq Q$ and $y \in Q \setminus C \subseteq Q$ which is a contradiction and so $C = Q$. $\square$

Exercise 4.3 (Bre93 1.4.3): The unit interval $[0,1]$ is connected.

Proof: Suppose that U, V formed a separation of x such that $1 \in V$. Let $x = \sup(U) \in [0,1]$ since the unit interval is closed. Suppose $x \in U$, then since U is open there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$ meaning that $x + \frac{\varepsilon}{2} \in U$ which is a contradiction to our choice of x. Thus $x \in V$ which is also open meaning there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq V$. Thus $x - \frac{\varepsilon}{2} \in V$. Suppose that $x - \frac{\varepsilon}{2}$ is not an upper bound for U. Then there is some $p \in U$ such that $p > x - \frac{\varepsilon}{2}$. Thus $p \in \left(\frac{x-\varepsilon}{2}, x\right) \subseteq B_\varepsilon(x) \subseteq V$ and so $p \in V$ contradicting the disjointness of U and V. Thus $x - \frac{\varepsilon}{2}$ is a lower upper bound of U than $x = \sup(U)$ which is a contradiction. Thus we have found $x \in [0,1]$ such that $x \notin U$ and $x \notin V$ which is a contradiction as well and $[0,1]$ is connected. $\square$

5 Separation Axioms

Definition 5.1 (Separation Axioms, Hausdorff, Regular, Normal Spaces): The *Separation Axioms*:

(T_0) A topological space X is called a *T_0-space* if for any two points $x \neq y$ there is an open set containing one of them but not the other. This says that point can be distinguished by the open sets where they lie.

(T_1) A topological space X is called a *T_1-space* if for any two points $x \neq y$ there is an open set containing x but not y and another open set containing y but not x. This says that singletons, and thus finite sets, are closed. Let $x \in X$. For each point $y \neq x$ let U_y be an open set containing y but not x. Then $X \setminus \{x\} = \bigcup_y U_y$ which is open and thus $\{x\}$ is closed. Conversely if $\{x\}$ is closed, then $X \setminus \{x\}$ is open and contains the other point.

(T_2) A topological space X is called a *T_2-space* or *Hausdorff* if for any two points $x \neq y$ there are disjoint open sets U, V with $x \in U$ and $y \in V$. This is the most useful type of space. It essential means that "limits" are unique.

(T_3) A T_1-space X is called a *T_4-space* of *regular* if for any point x and closed set F not containing x there are disjoint open sets U, V with $x \in U$ and $F \subseteq V$.

(T_4) A T_1-space X is called a *T_5-space* of *normal* if for any two disjoint closed sets F, G there are disjoint open sets U, V with $F \subseteq U$ and $G \subseteq V$.

Example 5.2 (A Space that is not T_0): The topology on $\{x, y\}$ where the only open sets are $\emptyset$ and $\{x, y\}$ is not T_0. There is no open set containing x and not y nor vice versa.

Example 5.3 (A Space that is T_0 but not T_1): The topology on $\{x, y\}$ with open sets $\emptyset, \{x\}$, and $\{x, y\}$ is T_0. This is because $\{x\}$ is an open set containing x but not y. However, there is no open set containing y but not x so the space is not T_1.

Proposition 5.4: A Hausdorff space X is regular if and only if the closed neighborhoods of any point form a neighborhood basis of the point.

Proof: Suppose X is regular. Let $x \in V$ for V open and let $C := X \setminus V$. Then by regularity there exist open sets U and W with $x \in U$, $C \subseteq W$, and $U \cap W = \emptyset$. Then $X \setminus W$ is closed and $X \setminus W \subseteq X \setminus C = V$ and so any neighborhood of V of x contains a closed neighborhood of x.

Now suppose that every point has a closed neighborhood basis. Let $x \in C$ with C closed and $V = X \setminus C$. Then there exists open U with $\overline{U} \subseteq V = X \setminus C$ and $x \in U$. Then $C \subseteq X \setminus \overline{U}$ and $U \cap (X \setminus \overline{U}) = \emptyset$. Thus X is regular. □

Corollary 5.5: A subspace of a regular space X is regular.

Proof: If $A \subseteq X$ is a subspace, just intersect a closed neighborhood basis in X of some $a \in A$ with A to obtain a closed neighborhood basis of a in A. □

Exercises

Exercise 5.1 (Bre93 1.5.2): A finite T_1-space is discrete.

Proof: Let X be a finite T_1 space and $x \in X$. For all $y \neq x$ we can find an open set U_y such that $y \notin U_y$ but $x \in U_y$. Then $\{x\} = \bigcap_{y \neq x} U_y$ is open since it is a finite intersection. Thus since singletons are open, their unions are open. This means arbitrary subsets of X are open and we have the discrete topology. $\square$

Exercise 5.2 (Bre93 1.5.5): Subspaces of Hausdorff Spaces are Hausdorff.

Proof: Let X be Hausdorff and Y a subspace of X. Let $x \neq y$ be elements in Y. Then there exists disjoint U, V open in X such that $x \in U$ and $y \in V$. But $U \cap Y$ and $U \cap V$ are open in Y and contain x and y respectively and are still disjoint. Thus Y is Hausdorff. $\square$

Exercise 5.3 (Bre93 1.5.6): A Hausdorff space X is normal if and only if for all open sets U and closed sets $C \subseteq U$ there is an open set V with $C \subseteq V \subseteq \overline{V} \subseteq U$.

Proof: Suppose that X is normal and let U open and C closed in X such that $C \subseteq U \subseteq X$. U is open so $X \setminus U$ is closed and disjoint from C. Thus there exists disjoint open V, W such that $C \subseteq V$ and $(X \setminus U) \subseteq W$. But $V \subseteq X \setminus W$ and since $X \setminus W$ is closed, it contains $\overline{V}$. Then we also have that $X \setminus W \subseteq U$ since $X \setminus U \subseteq W$. Thus overall we have $C \subseteq V \subseteq \overline{V} \subseteq X \setminus W \subseteq U$.

Now suppose that for any U open, C closed such that $C \subseteq U$ we have V open such that $C \subseteq V \subseteq \overline{V} \subseteq U$. Let C_1, C_2 be disjoint closed subsets of X. Since C_1 is closed, $X \setminus C_1$ is open and contains C_2. Thus there exists open V such that $C_2 \subseteq V \subseteq \overline{V} \subseteq X \setminus C_1$. $C_2 \subseteq \overline{V}$ and $\overline{V}$ closed means that $C_1 \subseteq X \setminus \overline{V}$ which is open. Thus we have open sets $V, X \setminus \overline{V}$ that are disjoint and contain C_1 and C_2 respectively meaning that X is normal. $\square$

Exercise 5.4 (Bre93 1.5.9): Metric spaces are normal.

Proof: Let X be a metric space with metric d. Recall from Proposition 3.21 that for closed $C \subseteq X$, $x \in X$ that $d(x, C) = 0$ if and only if $x \in C$. Let C_1, C_2 be disjoint closed sets in X. Define open sets U_1, U_2 such that

$$U_1 := \bigcup_{x \in C_1} D_{\frac{d(x,C_2)}{3}}(x), \; U_2 := \bigcup_{x \in C_2} D_{\frac{d(x,C_1)}{3}}(x).$$

We have that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$ and U_1, U_2 are open and disjoint. Thus X is normal. $\square$

6 Nets (Moore-Smith Convergence)

Many results in metric spaces are stated in terms of sequences. We discuss a generalization of sequences called *nets*.

Definition 6.1 (Directed Sets): A *directed set* D is a poset such that for all $\alpha, \beta \in D$, there exists $\tau \in D$ such that $\tau \geq \alpha$ and $\tau \geq \beta$.

Definition 6.2 (Net): A *net* in a topological space X is a directed set D along with a function $\phi : D \to X$.

Example 6.3 (Sequences are Nets): Note that $\mathbb{N}$ with the usual ordering is a directed set. Sequences are nets with $\mathbb{N}$ as the directed set.

Definition 6.4 (Frequently, Eventually): If $\phi : D \to X$ is a net in a topological space X and $A \subseteq X$, we say that ϕ is *frequently* in A if for all $\alpha \in D$ there exists $\beta \geq \alpha$ such that $\phi(\beta) \in A$. We say that ϕ is *eventually* in A if there exists $\alpha \in D$ such that for all $\beta \geq \alpha$ we have that $\phi(\beta) \in A$.

Definition 6.5 (Convergence of a Net): A net $\phi : D \to X$ in a topological space X is said to *converge to* $x \in X$ if for any neighborhood U of x, ϕ is eventually in U.

Proposition 6.6: A topological space X is Hausdorff if and only if any two limits of a convergent net are equal. Thus it makes sense to speak of the limit of a convergent net.

Proof: Suppose that X is Hausdorff. If a net ϕ is eventually in two sets U and V, then it is eventually in $U \cap V$. Also this means that $U \cap V \neq \emptyset$. Thus the forward direction is immediate.

Now suppose that X is not Hausdorff and that $x \neq y \in X$ are two points which cannot be separated by open sets. Consider a directed whose elements are pairs of open sets (U, V) with $x \in U$, $y \in V$. We give this directed set the ordering $(U, V) \geq (A, B)$ if and only if $(U \subseteq A)$ and $(V \subseteq B)$ (so smaller sets are greater). Let ϕ be a net on this directed set such that $\phi(U, V) =$ some point in $U \cap V$.

We claim that this net converges to both x and y. Let W be any neighborhood of x. Take any open set V containint y and an open set U with $x \in U \subseteq W$. For any $(A, B) \geq (W, V)$ we have that $\phi(A, B) \in A \cap B \subseteq U \subseteq W$. Thus ϕ is eventually in W and ϕ converges to x. A similar argument yields that ϕ also converges to y. $\square$

Proposition 6.7: $f : X \to Y$ between topological spaces is continuous if and only if for any net ϕ converging to $x \in X$, the net $f \circ \phi$ in Y converges to $f(x)$.

Proof: Suppose that f is continuous. Let ϕ be a net in X converging to x. Let $V \subseteq Y$ be an open set containing $f(x)$. We have that $U = f^{-1}(V)$ is a neighborhood of x. ϕ is eventually in U so $f \circ \phi$ is eventually in V and thus converges to $f(x)$.

Now suppose that f is not continuous. Then there is some open $V \subseteq Y$ such that $K := f^{-1}(V)$ is not open. Let $x \in K \setminus \operatorname{int}(K)$. Consider the directed set of open neighborhoods of x with the ordering $A \geq B$ if and only if $A \subseteq B$. Choose any neighborhood A of x. Note that $A \nsubseteq K$ so let $\phi(A) = w_A \in A \setminus K$ be a net. If N is a neighborhood of x and $B \geq N$, so $B \subseteq N$, then $\phi(B) = w_B \in B \setminus K \subseteq N$ and so ϕ is eventually in N. Thus ϕ converges to x. However $(f \circ \phi)(A) \notin V$ for any A and so $f \circ \phi$ is not eventually in V and thus does not converge to $f(x)$. $\square$

Given a net $\phi : D \to X$, let $x_\alpha := \phi(\alpha)$. It's common to notate this net as $\{x_\alpha\}_{\alpha \in D}$. So the condition in Proposition 6.7 can be stated as

$$f : X \to Y \text{ continuous} \iff f(\lim x_\alpha) = \lim f(x_\alpha).$$

Proposition 6.8: if $A \subseteq X$ then $\overline{A}$ is the set of limits of nets in A which converge to X.

Proof: If $x \in \overline{A}$ then any open neighborhood of x intersects nontrivially. We can make a net of this set of neighborhoods ordered by inclusion and have $x_U \in U \cap A$. This clearly converges to x.

Now suppose that we have a net $\{x_\alpha\}$ of points in A which converges to a point $x \in X$. Then this net is eventually in any neighborhood of x. Thus any neighborhood of x has nontrivial intersection with a point in A and $x \in \overline{A}$. $\square$

Definition 6.9 (Final Functions): If D and D' are directed sets, a function $h : D' \to D$ is *final* if for all $\delta \in D$, there exists $\delta' \in D'$ such that $\alpha' \geq \delta'$ implies $h(\alpha') \geq \delta$.

Definition 6.10 (Subnets): A *subnet* of a net $\phi : D \to X$ is the composition of a final map $h : D' \to D$ to a net $\phi \circ h$.

Proposition 6.11: A net $\{x_\alpha\}$ is frequently in each neighborhood of a given point $x \in X$ if and only if it has a subnet which converges to x.

Proof: Consider the set D' be ordered pairs (α, U) where $\alpha \in D$, U is a neighborhood of x, and $x_\alpha \in U$. Give D' the ordering on D and by inclusion. If (α, U) and (β, V) are in D', then since $\{x_\alpha\}$ is frequently in U and V, it is frequently in $U \cap V$. Thus there is some $\tau \geq \alpha, \beta$ with $x_\tau \in U \cap V$. Thus $(\tau, U \cap V) \in D'$ and $(\tau, U \cap V) \geq (\alpha, U), (\beta, V)$. So D' is directed.

Map $D' \to D$ by $(\alpha, U) to \alpha$. For any $\delta \in D$, $(\delta, X) \in D'$, and $(\alpha, X) \geq (\delta, X)$ implies that $\alpha \geq \delta$. So this map is final and $\{x_{\alpha,U}\}$ is a subset of $\{x_\alpha\}$.

Let N be a neighborhood of x. Then by assumption there exists some $x_\beta \in N$. If $(\alpha, U) \geq (\beta, N)$, then $x_{\alpha,U} = x_\alpha \in U \subseteq N$. So $\{x_{\alpha,N}\}$ is eventually in N.

The converse is immediate. $\square$

Definition 6.12 (Universal Nets): A net in a set X is *universal* if for any $A \subseteq X$, the net is either eventually in A or $X \setminus A$.

Proposition 6.13: The composition of a universal net in X with a function $f : X \to Y$ is a universal net in Y.

Proof: If $A \subseteq Y$, then the net is eventually either in $f^{-1}(A)$ or $X \setminus f^{-1}(A)$. But $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$ so the composition is either in A or $Y \setminus A$. $\square$

Theorem 6.14: Every net has a universal subset

Proof: Let $\{x_\alpha \mid \alpha \in P\}$ be a net in X. Consider all collections $\mathbf{C}$ of subsets of X such that

1. $A \in \mathbf{C} \implies \{x_\alpha\}$ is frequently in A; and
2. $A, B \in \mathbf{C} \implies A \cap B \in \mathbf{C}$.

Note that $\mathbf{C} = \{X\}$ is such a collection. Other the family of all such collections by inclusion. The union of any **⟨⟨ simply ordered set ⟩⟩** of collections satisfying conditions is another such collection. Thus by the **⟨⟨ maximality principle ⟩⟩** there exists a maximal such collection $\mathbf{C}_0$.

Let $P_0 = \{(A, \alpha) \in \mathbf{C}_0 \times P \mid x_\alpha \in A\}$ and order it by

$$(B, \beta) \geq (A, \alpha) \iff B \subseteq A \text{ and } \beta \geq \alpha.$$

This makes P_0 a directed set. Map $(A, \alpha) \to \alpha$ which is clearly final and thus defines a subset $\{x_{A,\alpha}\}$.

We claim this subnet is universal. Suppose S is any subset of X such that $\{x_{A,\alpha}\}$ is frequently in S. Then for any $(A, \alpha) \in P_0$, there exists $(B, \beta) \geq (A, \alpha)$ in P_0 with $x_\beta = x_{B,\beta} \in S$. Then $B \subseteq A$, $\beta \geq \alpha$, and $x_\beta \in B$. Thus $x_\beta \in S \cap B \subseteq S \cap A$. This means that $\{x_\alpha\}$ is frequently in $S \cap A$ for any $A \in \mathbf{C}_0$. But S and $S \cap A$, for $A \in \mathbf{C}_0$, can be added to $\mathbf{C}_0$ and these conditions still hold. So by maximality, $S \in \mathbf{C}_0$.

If $\{x_{A,\alpha}\}$ was also frequently in $X \setminus S$, then $X \setminus S \in \mathbf{C}_0$ be a similar argument. Thus $S \cap (X \setminus S) = \emptyset \in \mathbf{C}_0$. This contradicts the first condition. Thus $\{X_{A,\alpha}\}$ is not frequently in $X \setminus S$ and so it is indeed eventually in S.

Overall we have that if $\{x_{(A,\alpha)}\}$ is frequently in S, it is eventually in S. Thus $\{x_{A,\alpha}\}$ is a universal subset. □

Note that here we have used the Axiom of Choice in the form of the **⟨⟨ maximality principle ⟩⟩**. This past theorem is in fact equal to the Axiom of Choice.

Proposition 6.15: Subnets of universal nets are universal.

Proof: Let $\{x_\alpha\}$ be a universal net $\phi : D \to X$ and $\{x_{h(\alpha)}\}$ a subnet with final $h : D' \to D$. Let $A \subseteq X$ and without loss of generality suppose that $\{x_\alpha\}$ is eventually in A. Then there exists $\alpha \in D$ such that for all $\beta \geq \alpha$, $x_\beta \in A$. We have that h is final and so there exists α' such that $\beta' \geq \alpha'$ implies that $h(\beta') \geq \beta$. Thus the subnet is eventually in A. □

Exercises

Exercise 6.1 (Bre93 1.6.1): A sequence is universal if and only if it is eventually constant.

Proof: Let $x = \{x_i\}_{i \in \mathbb{N}}$ be a sequence in a metric space with metric d. Suppose the sequence was not eventually constant. Then for any $i \in \mathbb{N}$ we can find $\varepsilon > 0$ and $j > i$ such that $x_j \notin B_\varepsilon(x_i)$. Thus the sequence is not universal.

The converse is clear. $\square$

7 Compactness

Definition 7.1 (Covering, Open Covering, Subcover): A *covering* of a topological space X is a collection of sets whose union is X. An *open covering* is a covering where each set is open. A *subcover* is a subset of a cover which is still a cover.

Definition 7.2 (Compact, Heine-Borel Property): A topological space X is said to be *compact* or have the *Heine-Borel property* if every open covering of X has a finite subcover.

Definition 7.3 (Finite Intersection Property): A collection of sets has the *finite intersection property* if the intersection of any finite subcollection is empty.

The following theorem is just a translation of compactness in terms of open sets to an equivalent statement of about the closed complements of those sets.

Theorem 7.4: A topological space X is compact if and only if for every collection of closed subsets of X which has the finite intersection property, the intersection of the whole collection is nonempty.

Proof: **⟨⟨ Trivial Proof ⟩⟩** □

Theorem 7.5: If X is a Hausdorff space, then any compact subset of X is closed.

Proof: Let $A \subseteq X$ be compact and suppose that $x \in X \setminus A$. For $a \in A$ let $a \in U_a$ and $x \in V_a$ be disjoint open sets. Now $A = \bigcup_{a \in A}(U_a \cap A)$ and so we have a cover. Thus by compactness of A, we have $a_1, \ldots, a_n \in A$ such that $A \subseteq u_{a_1} \cup \cdots \cup U_{a_n} = U$. But then $x \in V_{a_1} \cap \cdots \cap V_{a_n} = V$ which is open, and $U \cap V = \emptyset$. Thus $x \in V \subseteq X \setminus U \subseteq X \setminus A$ and V is open. Since this holds for any $x \in X \setminus A$, we have that $X \setminus A$ is open and thus A is closed. □

Theorem 7.6: If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

Theorem 7.7: We may as well replace Y by $f(X)$ and so assume that f is onto. For any open cover of $f(X)$, look at inverse images of the sets and apply compactness.

Theorem 7.8: If X is compact and $A \subseteq X$ is closed, then A is compact.

Proof: Cover A with open sets in X, add the open set $X \setminus A$, and then apply compactness of X. □

Theorem 7.9: Suppose that X is compact, Y is Hausdorff, $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof: We need to show that f^{-1} is continuous. This is equivalent to showing that f maps closed sets to closed sets. But if $A \subseteq X$ is closed, then A is compact by Theorem 7.8. Thus $f(A)$ is compact and $f(A)$ must be closed since A is Hausdorff. □

Example 7.10 (Closed Intervals are compact in $\mathbb{R}$): We have that $[0,1]$ is compact in $\mathbb{R}$.

Proof: Let $\mathbf{U}$ be an open covering of $[0,1]$. Let $S = \{s \in [0,1] \mid [0,s]$ is covered by a finite subcollection of $\mathbf{U}\}$. Then Let $b = \sup(S)$. Note that S takes the form $[0,b]$ or $[0,b)$. Suppose that S takes for form $[0,b)$. Consider a set $U \in \mathbf{U}$ containing b. U must contain the interval $[a,b]$ for some $a < b$. But we can throw this in with the interval $[0,a]$ and obtain $[0,b]$. So S must take the form $[0,b]$. Now suppose that $b < 1$. Then **⟨⟨ a similar argument ⟩⟩** yields that there exists $c > b$ such that there is a finite cover of $[0,c]$ which contradicts the definition of b. Thus $b = 1$ and we have a finite cover of $[0,1]$. □

Overall we have that closed intervals $[a,b]$ and their closed subsets are compact. Also note that these closed subsets must be bounded. So in $\mathbb{R}$ we have that a subset is compact if and only if it is closed and bounded. This does not hold in arbitrary metric spaces.

Theorem 7.11: A real-valued map on a compact space takes a maximum.

Proof: If $f : X \to \mathbb{R}$ is a real-valued map on a compact space, then $f(X)$ is compact. Since $f(X)$ is compact, it is closed and bounded. This means that $\sup(f(X))$ exists, is finite, and belongs to $f(X)$. □

Theorem 7.12: Compact Hausdorff spaces are normal

Proof: Suppose that X is a compact Hausdorff space. First we show that X is regular. Let C be a closed subset of X and let $x \notin C$. X is Hausdorff so for any $y \in C$ there exists open disjoint U_y, V_y such that $x \in U_y$ and $y \in V_y$. C is closed and so it is compact by Theorem 7.8. V_y is an open cover of C and so there exist $y_1, \ldots, y_n$ such that $C \subseteq V := V_{y_1} \cup \cdots \cup V_{y_n}$. Let $U := U_{y_1} \cap \cdots \cap U_{y_n}$. Then we have that $x \in U$, $C \subseteq V$, and U, V disjoint and open. Thus X is regular.

Now repeat this same proof letting a closed set F play the role of x and the other closed set G playing the role of C. Thus X is normal. □

Definition 7.13 (Proper Maps): A map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact for each compact $C \subseteq Y$.

Theorem 7.14: If $f : X \to Y$ is closed and $f^{-1}(y)$ is compact for each $y \in Y$, then f is proper.

Proof: Let $C \subseteq Y$ be compact and let $\{U_\alpha \mid \alpha \in A\}$ be an indexed collection of open sets whose union contains $f^{-1}(C)$. For any $y \in C$, there exists a finite subset of $A_y \subseteq A$ such that $f^{-1}(y) = \bigcup_{\alpha \in A_y} U_\alpha$. Now let

$$\begin{aligned} W_y &= \bigcup_{\alpha \in A_y} U_\alpha, \\ V_y &= Y \setminus f(X \setminus W_y). \end{aligned}$$

These are both open sets. $f^{-1}(V_y) \subseteq W_y$ and $y \in V_y$. Since C is compact and covered by V_y, there exists $y_1, \ldots, y_n$ such that $C \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$. Thus

$$\begin{aligned} f^{-1}(C) &\subseteq f^{-1}(V_{y_1}) \cup \cdots \cup f^{-1}(V_{y_n}) \\ &\subseteq W_{y_1} \cup \cdots \cup W_{y_n} \\ &= \bigcup_{\alpha \in A_{y_i}, 1 \le i \le n} U_\alpha \end{aligned}$$

which is a finite open subcover of $f^{-1}(C)$. □

Theorem 7.15: For a topological space X, the following are equivalent:

1. X is compact.
2. Every collection of closed subsets of X with the finite intersection property has non-empty intersection.
3. Every universal net in X converges.
4. Every net in X has a convergent subset.

Proof: We have already seen the equivalence of 1. and 2. and now we show the rest.

1. $\iff$ 2. See Theorem 7.4.

1. $\implies$ 3. Suppose that $\{x_\alpha\}$ is a universal net that does not converge. Given $x \in X$, there is an open neighborhood of U_x such that $\{x_\alpha\}$ is not eventually in U_x and so the net is eventually in $X \setminus U_x$. So there exists β_x such that for all $\alpha \ge \beta_x$ we have that $x_\alpha \notin U_x$. Now cover X by some finite cover $U_{x_1} \cup \cdots U_{x_n}$. Let $\alpha \ge \beta_{x_i}$ for all i, then $x_\alpha \notin U_{x_i}$ for any i. Thus $x_\alpha \notin X$ which is a contradiction.

3. $\implies$ 4. This is clear by Theorem 6.14.

4. $\implies$ 2. Let $\mathbf{F} = \{C\}$ be a collection of closed sets with the finite intersection property. Without loss of generality, we can assume that $\mathbf{F}$ is closed under finite intersection. Order $\mathbf{F}$ by $C \ge C'$ if and only if $C \subseteq C'$, making $\mathbf{F}$ a directed set. Let $\{x_C\}_{C \in \mathbf{F}}$ be a net. By assumption. there is a convergent subnet given by a final map $f : D \to \mathbf{F}$. Thus for $\alpha \in D$, $f(\alpha) \in \mathbf{F}$ and $x_{f(\alpha)} \in f(\alpha)$. Suppose that $x_{f(\alpha)}$ converges to x. Let $C \in \mathbf{F}$. Then there is $\beta \in D$ such that for all $\alpha \ge \beta$ we have $f(\alpha) \subseteq C$. Thus $x_{f(\alpha)} \in f(\alpha) \subseteq C$. Since C is closed, it contains its limit points and $x \in C$ meaning that $x \in \bigcap_{C \in \mathbf{F}} C$ and the total intersection is nonempty.

□

Exercises

Exercise 7.1 (Bre93 1.7.1): Show that if X is compact, then every net in X has a convergent subnet without using universal nets.

Proof: Recall that for any collection of closed subsets of X with the finite intersection property, the intersection of the whole collection has nonempty intersection. Let $\{x_\alpha\}_{\alpha\in A}$ be a net in X and for each α let $E_\alpha := \{x_\beta \mid \beta \geq \alpha\}$. Then note that the collection $\{\overline{E_\alpha}\}$ has the finite intersection property. Consider a finite intersection $\bigcap_{\alpha\in I}\overline{E_\alpha}$. Let $\alpha^* \geq \alpha$ for all $\alpha \in I$, which exists since A is a directed set. Then $\alpha^* \in \overline{E_\alpha}$ for all $\alpha \in I$ and so in particular the finite intersection is non-empty. Thus we have that the intersection the whole collection $\bigcap_{\alpha\in A}\overline{E_\alpha}$ is non-empty containing some element $x \in X$.

Consider the set $B = \{\alpha, U_\alpha \mid \alpha \in A,\ U_\alpha \text{ is a neighborhood of } x_\alpha\}$. This can be made directed by $(\alpha, U_\alpha) \geq (\beta, U_\beta)$ if and only if $\alpha \geq \beta$ and $U_\alpha \subseteq U_\beta$. Consider the map $h\colon B \to A$ such that $h(\alpha, U_\alpha) \to \alpha$. This is clearly final.

This subnet $\{x_{(\alpha,U_\alpha)}\}$ converges to x. Let U be a neighborhood of x. By construction, there exists some α such that $x_\alpha \in U$ since each of the $\overline{E_\alpha}$ are closed sets. So consider $(\alpha, U) \in B$. Then for all $(\beta, U_\beta) \geq (\alpha, U)$ we have that $h(\beta, U_\beta) \geq \alpha$ and thus $x_\beta \in U_\beta \subseteq U$, a convergent subnet. □

Exercise 7.2 (Bre93 1.7.2): Let X be a compact space and $\{C_\alpha \mid \alpha \in A\}$ a collection of closed sets which is also closed with respect to finite intersection. Let $C := \bigcap C_\alpha$ and suppose that $C \subseteq U$ for some U open. Then for some α, we have that $C_\alpha \subseteq U$.

Proof: Note that $V := X \setminus U$ and $V \cap C_\alpha$ are both closed and thus compact. Also note that

$$\bigcap_{\alpha\in A} V \cap C_\alpha = (X \setminus U) \cap \bigcap_{\alpha\in A} C_\alpha = (X \setminus U) \cap C = \emptyset.$$

Let $F_\alpha := X \setminus (V \cap C_\alpha)$ which is open. Fix some $\beta \in A$. We have that $\bigcup_{\alpha\in A} F_\alpha$ is an open cover for $V \cap C_\beta$. This is because if $x \in V \cap C_\beta$, then since $\bigcap_{\alpha\in A} V \cap C_\alpha = \emptyset$ there is some α such that $x \notin V \cap C_\alpha$ meaning that $x \in F_\alpha$. But $V \cap C_\beta$ is compact, so there is some finite subcover $A' \subseteq A$ such that

$$\bigcup_{\alpha\in A'} F_\alpha = \bigcup_{\alpha\in A'} X \setminus (V \cap C_\alpha) \supseteq V \cap C_\beta.$$

This implies that

$$(V \cap C_\beta) \cap \left(\bigcap_{\alpha\in A'} V \cap C_\alpha\right) = V \cap \left(C_{\beta \cap \bigcap_{\alpha\in A'} C_\alpha}\right) = \emptyset.$$

Thus we have found a finite intersection of closed sets from the collection, which itself is a member of the collection, disjoint from $X \setminus U$ meaning that is contained in U as desired. □

Exercise 7.3 (Bre93 1.7.3): Show that the hypothesis in Theorem 7.14 that f is closed is necessary.

Proof: Recall that subsets of $\mathbb{R}$ are compact if and only if they are closed and bounded. Note that $(0,1]$ is bounded but not closed. This is because $\mathbb{R} \setminus (0,1]$ is open, and in particular there is no open ball around 0 completely contained in $R \setminus (0,1]$.

Now consider the inclusion $f : (0,1] \hookrightarrow [0,1]$. This satisfies the hypothesis that $f^{-1}(y)$ is compact for all $y \in [0,1]$. Clearly $f^{-1}(0) = \emptyset$ is compact. We also have for $y > 0$ that because $[y,1]$ is compact and $\{y\}$ is closed, $f^{-1}(y) = \{y\}$ is compact. However f is not proper since $[0,1]$ is compact but $f^{-1}([0,1]) = (0,1]$ is not compact. □

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