Algorithms in Invariant Theory Exercises

With 0 Figures

Anakin Dey

Last Edited on 3/30/24 at 13:43

Contents

Preface

A core interest of mine (at the time of writing this) is algorithms in the context of commutative algebra and algebraic geometry. As such, Bernd Sturmfel's text *Algorithms in Invariant Theory* [Str08] is a good resource for first learning this stuff. This solely just contains exercises for now. Perhaps in the future I'll include notes and some source code

Chapter 1

Introduction

Symmetric Polynomials

Exercise 1.1 (Str08 1.1.5): Prove the following explicit formula for elementary symmetric polynomials in terms of the power sums [Mac98, Page 29].

$$\sigma_k = \frac{1}{k!} \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix}$$

Proof: From [Page 23, 2.11'][Mac98], we have the following identities

$$1 \cdot \sigma_1 = p_1,$$
$$2 \cdot \sigma_2 = p_1 \sigma_1 - p_2,$$
$$3 \cdot \sigma_3 = p_1 \sigma_2 - p_2 \sigma_1 + p_3,$$
$$\vdots$$
$$k \cdot \sigma_k = \sum_{r=1}^{k} (-1)^{r-1} p_r \sigma_{k-r}.$$

Treating the σ_i as indeterminants, we can re-express the above system of equations:

$$p_1 = 1 \cdot \sigma_1,$$

$$p_2 = p_1\sigma_1 - 2 \cdot \sigma_2,$$

$$p_3 = p_2\sigma_1 - p_1\sigma_2 + 3 \cdot \sigma_3,$$

$$\vdots$$

$$p_k = \left(\sum_{r=1}^{k-1}(-1)^{r-1}p_{k-r}\sigma_r\right) + (-1)^{k-1} \cdot k\sigma_k.$$

Consider $\sigma_1, -\sigma_2, \sigma_3, \ldots, (-1)^n\sigma_{n-1}, \sigma_n$ as indeterminants. Thus, we obtain the following matrix equation:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k \end{pmatrix} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \\ \vdots \\ (-1)^k\sigma_{k-1} \\ \sigma_k \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \\ p_k \end{pmatrix}.$$

Then, Cramer's rule yields that

$$\sigma_k = \det\begin{pmatrix} 1 & 0 & 0 & \cdots & p_1 \\ p_1 & 2 & 0 & \cdots & p_2 \\ p_2 & p_1 & 3 & \cdots & p_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & p_k \end{pmatrix} \det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k \end{pmatrix}^{-1}$$

$$= \frac{(-1)^k}{k!} \det\begin{pmatrix} 1 & 0 & 0 & \cdots & p_1 \\ p_1 & 2 & 0 & \cdots & p_2 \\ p_2 & p_1 & 3 & \cdots & p_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & p_k \end{pmatrix}$$

$$= \frac{(-1)^k}{k!} \cdot (-1)^k \det\begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix} = \frac{1}{k!} \det\begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix}.$$

$\square$

Bibliography

[Mac98]   Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[Str08]   Bernd Strumfels. *Algorithms in Invariant Theory*. Springer Vienna, 2008. ISBN: 9783211774175. DOI: 10.1007/978-3-211-77417-5. URL: http://dx.doi.org/10.1007/978-3-211-77417-5.