Using Algebraic Geometry

With 0 Figures

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Contents

Preface

At the time of writing this, I am starting my PhD at The Ohio State University. Currently a large part of my interests in algebra are about algorithms as they relate to polynomials and algebraic geometry. I've been doing a bunch of problems from *Ideals, Varieties, and Algorithms* [CLO15]. However, it seems that *Using Algebraic Geometry* [CLO05] moves through the material faster as it assumes you know more algebra. So I've moved onto working through this book as well as trying to comprehend Sturmfel's *Algorithms in Invariant Theory* [Str08].

Chapter 1

Introduction

1.1 Polynomials and Ideals

Exercise 1 (CLO05 1.1.1):

(a) Show that $x^2 \in \langle x - y^2, xy \rangle$ in $k[x, y]$.

(b) Show that $\langle x - y^2, xy, y^2 \rangle = \langle x, y^2 \rangle$.

(c) Is $\langle x - y^2, xy \rangle = \langle x^2, xy \rangle$? Why or why not?

Proof:

(a) We have that $x(x - y^2) + y(xy) = x^2 - xy^2 + xy^2 = x^2$.

(b) It suffices to check for generators. We have that $x + (-1)(y^2) = x - y^2$, $y(x) = xy$, and $y^2 = y^2$ showing that $\langle x - y^2, xy, y^2 \rangle \subseteq \langle x, y^2 \rangle$. Then $x - y^2 + y^2 = x$ and $y^2 = y^2$ shows the reverse containment and overall the ideals are equal.

(c) We already know from 1. that x^2 lives in $\langle x - y^2, xy \rangle$. Since $xy = xy$, we overall have that $\langle x^2, xy \rangle \subseteq \langle x - y^2, xy \rangle$. It remains to check if $x - y^2 \in \langle x^2, xy \rangle$. However, notice that every element of $\langle x^2, xy \rangle$ is divisible by x while $x - y^2$ is clearly not divisible by x. Thus $x - y^2 \notin \langle x^2, xy \rangle$ and the two ideals are not equal.

□

Exercise 2 (CLO05 1.1.2):

Show that $\langle f_1,\dots,f_s\rangle$ is closed under sums in $k[x_1,\dots,x_n]$. Also show that if $f\in\langle f_1,\dots,f_s\rangle$ and $p\in k[x_1,\dots,x_n]$ then $p\cdot f\in\langle f_1,\dots,f_s\rangle$.

Proof:

Let $f,g\in\langle f_1,\dots,f_s\rangle$. Then $\exists p_1,\dots,p_s,q_1,\dots,q_s$ such that $f=\sum_{i=1}^s p_i\cdot f_i$ and $g=\sum_{i=1}^s q_i\cdot f_i$. Thus $f+g=\sum_{i=1}^s(p_i+q_i)\cdot f_i$ which shows that $f+g\in\langle f_1,\dots,f_s\rangle$. Then let $p\in k[x_1,\dots,x_n]$. We have that $p\cdot f=p\sum_{i=1}^s p_i f_i=\sum_{i=1}^s(p\cdot p_i)\cdot f_i$ which shows that $\langle f_1,\dots,f_s\rangle$ is an ideal. □

Exercise 3 (CLO05 1.1.3):

Show that $\langle f_1,\dots,f_s\rangle$ is the smallest ideal containing $\{f_1,\dots,f_s\}$.

Proof:

We already know that $\langle f_1,\dots,f_s\rangle$ is an ideal by Exercise 2. Now suppose that J is an ideal containing $\{f_1,\dots,f_s\}$. Then, since ideals are closed under addition and scaling, we have that for all $p_1,\dots,p_s\in k[x_1,\dots,x_n]$ that $\sum_{i=1}^s p_i\cdot f_i\in J$. Thus, $\langle f_1,\dots,f_s\rangle\subseteq J$. □

Exercise 4 (CLO05 1.1.4):

Using Exercise 3, formulate and prove a general criterion for the equality of $I=\langle f_1,\dots,f_s\rangle$ and $J=\langle g_1,\dots,g_t\rangle$.

Proof:

We claim that $\langle f_1,\dots,f_s\rangle=\langle g_1,\dots,g_t\rangle$ if and only if $\{g_1,\dots,g_t\}\subseteq I$ and $\{f_1,\dots,f_s\}\subseteq J$. The forward implication is immediate. Then by Exercise 3, if $\{g_1,\dots,g_t\}\subseteq I$ then $J\subseteq I$. Similarly, $\{f_1,\dots,f_s\}\subseteq J\implies I\subseteq J$ and overall $I=J$. This fact was used in Exercise 1 (b). □

Exercise 5 (CLO05 1.1.5):

Show that $\left\langle y-x^2,z-x^3\right\rangle=\left\langle y-x^2,z-xy\right\rangle$ in $\mathbb{Q}[x,y,z]$.

Proof:

It suffices to show that $z-x^3\in\left\langle y-x^2,z-xy\right\rangle$ and and $z-xy\in\left\langle x-y^2,z-x^3\right\rangle$. Indeed we have that $(z-xy)+x(y-x^2)=z-x^3$ which also yields that $z-xy=z-x^3-x(y-x^2)$. □

Exercise 6 (CLO05 1.1.6):

Show that every ideal $I \subseteq k[x]$ is generated by a single polynomial.

Proof:

If $I = \{0\}$ then $I = \langle 0 \rangle$. So suppose $I \neq 0$. Let $d \in I$ be of minimal degree. **⟨⟨ $d = \gcd(I)$ but I need infinite Bezout. ⟩⟩** Then we claim that $\langle d \rangle = I$. Since $d \in I$, we have that $\langle d \rangle \subseteq I$. Now let $f \in I$. By Euclidean division, there exists $q, r \in k[x]$ such that $f = qd + r$ where either $r = 0$ or $0 \leq \deg(r) \leq \deg(d) - 1$. If $r = 0$ then $f \in \langle d \rangle$ and we are done. So suppose $r \neq 0$. Then $f, qd \in I \implies r = f - qd \in I$. Thus, $r \in I$ is of degree strictly less than d, contradicting the minimality of the degree of d. So we must have that $r = 0$ and overall $\langle d \rangle = I$. □

Exercise 7 (CLO05 1.1.7):

(a) Show that $\sqrt{\langle x^n \rangle} = \langle x \rangle$ in $k[x]$.

(b) If $p(x) = (x - a_1)^{e_1} \cdots (x - a_m)^{e_m}$, find $\sqrt{\langle p(x) \rangle}$.

(c) Let $k = \mathbb{C}$. What are the radical ideals in $\sqrt{\mathbb{C}[x]}$?

Proof:

(a) Suppose $f(x) \in \langle x \rangle$. Then $f(x)^m \in \langle x^n \rangle$ so $f(x) \in \sqrt{\langle x^n \rangle}$ Now suppose that $f(x) \in \sqrt{\langle x^n \rangle}$. Then $\exists k$ such that $f(x)^k \in \langle x^n \rangle$. Thus $f(x)^k$ is a multiple of x^n. This implies that $f(x)^k$ is a multiple of x. Then notice that the unique factorization of $f(x)^k$ into irreducibles is the kth power of the factorization of $f(x)$ into irreducibles. Thus x must be a factor of $f(x)$ and so $f(x) \in \langle x \rangle$. Note, this heavily uses the fact that $k[x]$ is a unique factorization domain for all fields k.

(b) We claim that $\sqrt{\langle p(x) \rangle} = \langle (x - a_1) \cdots (x - a_m) \rangle = I$. Suppose $f(x) \in I$. Let $k = \max e_1, \ldots, e_n$. Then $p(x) \mid f(x)^k$ so $f(x) \in \sqrt{\langle p(x) \rangle}$. Now suppose that $f(x) \in \sqrt{\langle p(x) \rangle}$. Then $\exists k$ such that $f(x)^k \in \langle p(x) \rangle$. Thus $f(x)^k$ is a multiple of each $(x - a_i)$. Then notice that the unique factorization of $f(x)^k$ into irreducibles is the kth power of the factorization of $f(x)$ into irreducibles. Thus $f(x)$ is a multiple of each $(x - a_i)$ and so $f(x) \in I$.

(c) Radical ideals are the ideals I such that $\sqrt{I} = I$. Notice that $\mathbb{C}[x]$ is a principal ideal domain and so any such I must be generated by a single polynomial. Since every polynomial in $\mathbb{C}[x]$ splits into linear factors, (b) immediately implies that the only radical ideals of $\mathbb{C}[x]$ are the ones which are of the form $\langle (x - a_1) \cdots (x - a_m) \rangle$ for $a_1, \ldots, a_m \in \mathbb{C}[x]$.

□

Exercise 8 (CLO05 1.1.8):

(a) Show that a prime ideal is radical.

(b) What are the prime ideals in $\mathbb{C}[x]$? What about the prime ideals in $\mathbb{R}[x]$ or $\mathbb{Q}[x]$?

Proof:

(a) Let $\mathfrak{p}$ be a prime ideal in $k[\overline{x}]$. Clearly $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$ always. Let $f(\overline{x}) \in \sqrt{\mathfrak{p}}$. Then $f(\overline{x})^m \in \mathfrak{p}$ for some $m \in \mathbb{Z}_{\geq 1}$. We prove the reverse inclusion by induction on m. If $m = 1$ then $f(\overline{x}) = f(\overline{x})^1 \in \mathfrak{p}$. Now let $m > 1$ and suppose the claim holds for all $k \leq m$. Then suppose $f(\overline{x})^{m+1} \in \mathfrak{p}$. Then $f(\overline{x}) \cdot f(\overline{x})^m \in \mathfrak{p}$/ Either $f(\overline{x}) \in \mathfrak{p}$ or $f(\overline{x})^m \in \mathfrak{p}$ which by induction implies that $f(\overline{x}) \in \mathfrak{p}$. Thus, $f(\overline{x})^m \in \mathfrak{p} \implies f(\overline{x}) \in \mathfrak{p}$ for all $m \in \mathbb{Z}_{\geq 1}$ and so $\sqrt{\mathfrak{p}} \subseteq \mathfrak{p}$. Thus, all prime ideals are radical.

(b) Notice that for all fields k that $k[x]$ is a principal ideal domain. Thus, all the prime ideals are the ones generated by a single irreducible polynomial. Also, in $k[x]$ we have that (0) is a prime ideal as well as $k[x]$ is an integral domain. In $\mathbb{C}[x]$, these are the ideals generated by $x - z$ for some $z \in \mathbb{C}$. In $\mathbb{R}[x]$, the primes are the ideals generated by $x - r$ for some $r \in \mathbb{R}$ or $x^2 + r$ for some positive $r \in R$. **⟨⟨ What would be a general condition for $\mathbb{Q}[x]$? ⟩⟩**

□

Exercise 9 (CLO05 1.1.9):

(a) Show that $\langle x_1, \ldots, x_n \rangle$ is maximal in $k[x_1, \ldots, x_n]$.

(b) Show that for any point $(a_1, \ldots, a_n) \in k^n$ that $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ is maximal in $k[x_1, \ldots, x_n]$.

(c) Show that $\left\langle x^2 + 1 \right\rangle$ is maximal in $\mathbb{R}[x]$. Is $\left\langle x^2 + 1 \right\rangle$ maximal in $\mathbb{C}[x]$?

Proof:

(a) First, observe that $\langle x_1, \ldots, x_n \rangle$ is the ideal consisting exactly of polynomials which have no constant term. Let I be an ideal in $k[x_1, \ldots, x_n]$ such that $\langle x_1, \ldots, x_n \rangle \subsetneq I$. Thus there exists $f(x_1, \ldots, x_n) \in I \setminus \langle x_1, \ldots, x_n \rangle$. We have by our observation that f has a nonzero constant term z. Then note that the non-constant terms of f form a polynomial $g(x_1, \ldots, x_n)$ in $\langle x_1, \ldots, x_n \rangle$. Thus, we have that $z = f(x) - g(x) \in I$. Since I contains a nonzero constant term, we must have that $I = k[x_1, \ldots, x_n]$.

(b) Recall that an ideal I is maximal if and only if R/I is a field. Let $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Consider the evaluation map $\mathrm{ev}_{\overline{a}} \colon k[x_1, \ldots, x_n] \to k$ sending $f(x_1, \ldots, x_n) \mapsto f(a_1, \ldots, a_n)$. Clearly this map is surjective. Then since for all i we have that $x_i \equiv a_i \pmod{I}$, we have that $f(x_1, \ldots, x_n) \equiv f(a_1, \ldots, a_n) \pmod{I}$ for all $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$. Thus, $\mathrm{ev}_{\overline{a}}(f) = f(a_1, \ldots, a_n) = 0$ if and only if $f(x_1, \ldots, x_n) \in I$. Thus, $\ker(\mathrm{ev}_{\overline{a}}) = I$ and $k[x_1, \ldots, x_n]/I$ is a field, meaning $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ is maximal.

(c) Since $\mathbb{R}[x]$ is a principal ideal domain, any ideal I strictly containing $\left\langle x^2 + 1 \right\rangle$ is of the form $\langle g(x) \rangle$ for some $g(x) \mid x^2 + 1$. However, since $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, we have that $g(x)$ is either $z(x^2 + 1)$ for some nonzero $z \in \mathbb{C}$ or $g(x) = z$ for some nonzero $z \in \mathbb{C}$, meaning $\langle g(x) \rangle = \left\langle x^2 + 1 \right\rangle$ or or $\langle g(x) \rangle = \mathbb{R}[x]$. Thus, $\left\langle x^2 + 1 \right\rangle$ is maximal. However, in $\mathbb{C}[x]$, we have that $x^2 + 1 = (x + i)(x - i)$ and so $\left\langle x^2 + 1 \right\rangle \subsetneq \langle x - i \rangle \subsetneq \mathbb{C}[x]$.

$\square$

Exercise 10 (CLO05 1.1.10):

(a) Let $I = \left\langle x^2+y^2, x^2-z^3 \right\rangle \subseteq k[x,y,z]$. Show that y^2+z^3 is in the first elimination ideal with respect to the ordering $x > y > z$.

(b) Show that if I is an ideal in $k[x_1,\dots,x_n]$ then for all $\ell \geq 1$, I_ℓ is an ideal in $k[x_{\ell+1},\dots,x_n]$.

Proof:

(a) Since $x^2+y^2-(x^2-z^3) = y^2+z^3$ is an element of I which does not depend on x, y^2+z^3 is in I_1.

(b) For all $\ell \geq 1$, we have that $0 \in I_\ell$. Then, if $f(x_{\ell+1},\dots,x_n), g(x_{\ell+1},\dots,x_n)$ are two polynomials in I who do not depend on the first ℓ variables, then so is $f+g$. Finally, let $r(x_\ell+1,\dots,x_n) \in k[x_{\ell+1},\dots,x_n]$. Then $r \cdot f \in I_\ell$ since $r \cdot f \in I$ and still does not depend on any of the first ℓ variables.

$\square$

Exercise 11 (CLO05 1.1.11):

Let I, J be ideals in $k[\overline{x}]$.

(a) Show that $I+J$ is an ideal.

(b) Show that $I+J$ is the smallest ideal containing $I \cup J$.

(c) If $I = \langle f_1,\dots,f_s \rangle$ and $J = \langle g_1,\dots,g_t \rangle$, what is a finite generating set of $I+J$?

Proof:

(a) **⟨⟨ meh ⟩⟩**

(b) **⟨⟨ meh ⟩⟩**

(c) We claim that $I+J = \langle f_1,\dots,f_s,g_1,\dots,g_t \rangle$. Clearly $I, J \subseteq \langle f_1,\dots,f_s,g_1,\dots,g_t \rangle$ and thus so is $I \cup J$. By (b), this shows that $I+J \subseteq \langle f_1,\dots,f_s,g_1,\dots,g_t \rangle$. Then, since $f_i = f_i + 0$ and $g_j = 0 + g_j$ for all i, j, we have the reverse inclusion and thus the two ideals are equal.

$\square$

Exercise 12 (CLO05 1.1.12):

Let I, J be ideals in $k[\overline{x}]$.

(a) Show that $I \cap J$ is an ideal.

(b) Show that $IJ \subseteq I \cap J$. Give an example where $IJ \subsetneq I \cap J$.

Proof:

(a) **⟨⟨ meh ⟩⟩**

(b) Suppose that $h(\overline{x}) \in IJ$. Note that IJ is generated by the products $f(\overline{x}) \cdot g(\overline{x})$ for $f(\overline{x}) \in I$, and $g(\overline{x}) \in J$. Then $h(\overline{x})$ consists of sums of terms of the form $r(\overline{x}) \cdot f(\overline{x}) \cdot g(\overline{x})$ for $r(\overline{x}) \in k[\overline{x}]$, $f(\overline{x}) \in I$, and $g(\overline{x}) \in J$. Thus, each term is in both I and J and overall so is $h(\overline{x})$.

To see an example where $IJ \subsetneq I \cap J$, consider $I = \left\langle x^2 y \right\rangle$ and $J = \left\langle x y^2 \right\rangle$ in $k[x, y]$. Then $I \cap J = \left\langle x^2 y^2 \right\rangle$ and $IJ = \left\langle x^3 y^3 \right\rangle$. Thus $IJ \subsetneq I \cap J$ as $I \cap J$ contains $x^2 y^2$ and IJ does not contain $x^2 y^2$.

□

Chapter 2

Solving Polynomial Equations

2.1 Solving Polynomial Systems by Elimination

Exercise 1 (⟨⟨ CLO05 2.1.1 ⟩⟩):

Exercise 2 (⟨⟨ CLO05 2.1.2 ⟩⟩):

Exercise 3 (CLO05 2.1.3):

Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a monic polynomial in $\mathbb{C}[z]$. Then all roots $\overline{z}$ of $p(z)$ satisfy

$$\overline{z} \leq B := \max\{1, |a_{n-1}| + \cdots + |a_0|\}.$$

Proof:

We may freely rewrite the polynomial as $p(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_0$ We have that $0 = \overline{z}^n - a_{n-1}\overline{z}^{n-1} - \cdots - a_0$ and so $\overline{z}^n = a_{n-1}\overline{z}^{n-1} + \cdots + a_0$. Suppose now that $|\overline{z}| \geq 1$. Then

$$|\overline{z}|^n = \left|a_{n-1}\overline{z}^{n-1} + \cdots + a_0\right| \leq |a_{n-1}||z|^{n-1} + \cdots + a_0 \leq |a_{n-1}|\overline{z}^{n-1} + \cdots |a_0|\overline{z}^{n-1}.$$

Thus, $|\overline{z}| \leq |a_{n-1}| + \cdots + |a_0|$. However, we assumed that $|\overline{z}| \geq 1$. This may not be the case. Thus, $|\overline{z}| \leq B := \max\{1, |a_{n-1}| + \cdots + |a_0|\}$. □

Exercise 4 (⟨⟨ CLO05 2.1.4 ⟩⟩):

Numerically find all roots of $2z^6 + 2z^5 - z^4 - z^3 - 2z^2 - 2z - 2$.

Exercise 5 (CLO05 2.1.5):

Verify that if $x > y$ then $G = \left[x^2 + 2x + 3 + y^5 - y, y^6 - y^2 + 2y\right]$ is a lex Gröbner basis for the ideal that G generates in $\mathbb{R}[x, y]$

Proof:

We apply Buchberger's Criterion. Let $f(x, y) = x^2 + 2x + 3 + y^5 - y$ and $g(x, y) = y^6 - y^2 + 2y$. Then we have that

$$S(f,g) = \frac{x^2y^6}{x^2} \cdot (x^2+2x+3+y^5-y) - \frac{x^2y^6}{y^6} \cdot (y^6-y^2+2y) = y^6 \cdot (x^2+2x+3+y^5-y) - x^2 \cdot (y^6-y^2+2y).$$

This shows that $\overline{S(f,g)}^G = 0$ which yields that G is a Gröbner basis. □

Exercise 6 (⟨⟨ CLO05 2.1.6 ⟩⟩):

Exercise 7 (⟨⟨ CLO05 2.1.7 ⟩⟩):

Exercise 8 (CLO05 2.1.8):

Newton's method for an equation $p(z) = 0$ is the sequence of points $\{z_k\}_{k \geq 0}$ starting from a chosen z_0 and defining $z_{k+1} = N_p(z_k)$ for $N_p(z) = z - \frac{p(z)}{p'(z)}$.

(a) Prove that a simple root of a polynomial $p(z)$ is a fixed point of $N_p(z)$.

(b) Show that multiple roots of $p(z)$ are removable singularities of $N_p(z)$. That is, show that $\left|N_p(z)\right|$ is bounded in a neighborhood of each multiple root. How should $N_p(z)$ be defined at a multiple root of $p(z)$ to make $N_p(z)$ continuous.

(c) Show that $N_p'(\overline{z}) = 0$ if $\overline{z}$ is a simple root, meaning that $p(\overline{z}) = 0$ and $p'(\overline{z}) \neq 0$.

(d) Show that if $\overline{z}$ is a root of multiplicity k of $p(z)$, meaning $p(overline z) = p'(\overline{z}) = \cdots = p^{(k-1)}(\overline{z}) = 0$ and $p^{(k)}(\overline{z}) \neq 0$, then

$$\lim_{z \to \overline{z}} N_p'(z) = 1 - \frac{1}{k}.$$

(e) Show that by replacing $p(z)$ with

$$p_{red}(z) = \frac{p(z)}{\gcd p(z), p'(z)}$$

that the difficulty in (d) is eliminated as all roots of $p_{\text{red}}(z)$ are simple.

Proof:

(a) Let $\bar{z}$ be a simple root of $p(z)$, so $p(z) = 0$ but $p'(z) \neq 0$. Then $N_p(\bar{z}) = \bar{z} - \frac{p(\bar{z})}{p'(\bar{z})} = \bar{z}$ meaning $\bar{z}$ is a fixed point of $N_p(z)$.

(b) Suppose that $\bar{z}$ is a multiple root of $p(z)$ with multiplicity $m \geq 2$. Then we may express $p(z) = \tilde{(p)}(z)(z-\bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Thus, we have that

$$\begin{aligned} N_p(z) &:= z - \frac{p(z)}{p'(z)} \\ &= z - \frac{\tilde{p}(z)(z-\bar{z})^m}{\tilde{p}'(z)(z-\bar{z})^m + m\tilde{p}(z)(z-\bar{z})^{m-1}} = z - \frac{\tilde{p}(z)(z-\bar{z})}{\tilde{p}'(z)(z-\bar{z}) + m\tilde{p}(z)} \end{aligned}$$

Note that $m\tilde{p}(\bar{z}) \neq 0$. Thus, we have that

$$\left|N_p(\bar{z})\right| = \left|\bar{z} - \frac{\tilde{p}(\bar{z})(\bar{z}-\bar{z})}{\tilde{p}'(\bar{z})(\bar{z}-\bar{z}) + m\tilde{p}(\bar{z})}\right| = |\bar{z}| \leq \textsc{lc}(p) \cdot B$$

where B is the value from Exercise 3 and $\textsc{lc}(p)$ is the leading coefficient of $p(z)$.

(c) Suppose now that $\bar{z}$ is a simple root of $p(\bar{z})$. Then we may express $p(z) = \tilde{p}(z)(z-\bar{z})$ such that $\tilde{p}(\bar{z}) \neq 0$. We have that

$$p'(z) = \tilde{p}'(z)(z-\bar{z}) + \tilde{p}(z)$$

and evaluation of $p'(z)$ at $\bar{z}$ is nonzero.

(d) Let $\bar{z}$ be a root of multiplicity m. Following (b), we write $p(z) = \tilde{p}(z)(z-\bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Then we have, by differentiating the expression for $N_p(z)$ from (b), that

$$N_p'(z) = 1 - \frac{(\tilde{p}'(z)(z-\bar{z}) + \tilde{p}(z))(\tilde{p}'(z)(z-\bar{z}) + m\tilde{p}(z)) - (\tilde{p}(z)(z-\tilde{z}))(\tilde{p}''(z)(z-\bar{z}) + \tilde{p}'(z) + m\tilde{p}'(z))}{(\tilde{p}'(z)(z-\bar{z}) + m\tilde{p}(z))^2}.$$

Evaluation at $z = \bar{z}$ yields that $\lim_{z \to \bar{z}} N_p'(z) = 1 - \frac{1}{m}$.

(e) Let $\bar{z}$ be a root of multiplicity m. Following (b), we write $p(z) = \tilde{p}(z)(z-\bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Then

$$p'(z) = \tilde{p}'(z-\bar{z})^m + m\tilde{p}(z)(z-\bar{z})^{m-1} = (z-\bar{z})^{m-1}(\tilde{p}'(z)(z-\bar{z}) + m\tilde{p}(z)).$$

Notice that $\tilde{p}'(\bar{z})(\bar{z}-\bar{z}) + m\tilde{p}(\bar{z}) = m\tilde{p}(\bar{z}) \neq 0$. Thus, a root of multiplicity $m \geq 1$ of $p(z)$ is a root of multiplicity $m-1$ of $p'(z)$. This implies that if we have roots $\bar{z}_1, \ldots, \bar{z}_k$ with multiplicities $m_1, \ldots, m_k \geq 1$, then $\gcd(p(z), p'(z)) = (z-\bar{z}_1)^{m_1} \cdots (z-\bar{z}_k)^{m_k}$. Thus, the polynomial $p_{\text{red}}(z) = \frac{p(z)}{\gcd(p(z),p'(z))}$ has the same roots of $p(z)$ but all with multiplicity 1 which is the best case for Newton's method.

$\square$

Exercise 9 (CLO05 2.1.9):

(a) What happens if you do Newton's method to solve $z^2+1=0$ starting from a real z_0 versus a complex z_0?

(b) Let $p(z)=z^4-z^2-\frac{11}{36}$. Let $N_p(z)=z-\frac{p(z)}{p'(z)}$. Show that $N_p\left(\pm\frac{1}{\sqrt{6}}\right)=\mp\frac{1}{\sqrt{6}}$ and $N_p'\left(\frac{1}{\sqrt{6}}\right)=0$.

Proof:

(a) Let $p(z)=z^2+1$. We have that

$$N_p(z)=z-\frac{z^2+1}{2z}=\frac{2z^2-z^2+1}{2z}=\frac{z^2+1}{2z}=\frac{x^2+2ixy-y^2+1}{2x+2iy}.$$

If z is real then $y=0$ and so $N_p(x)=\frac{x^2+1}{2x}$ which is always real. Thus, Newton's method will never reach the imaginary roots of z^2+1. However, if we begin with a guess with nonzero imaginary part, then the guess does converge as expected.

(b) **⟨⟨ Just basic arithmetic not worth doing. ⟩⟩**

□

Exercise 10 (CLO05 2.1.10):

Let $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_0$ be a monic polynomial in $\mathbb{C}[z]$. Then all roots $\overline{z}$ of $p(z)$ satisfy

$$\overline{z}\leq B:=1+\max\{|a_{n-1}|,\ldots,|a_0|\}.$$

Proof:

Let $\overline{z}$ be a root of $p(z)$. Then $-\overline{z}^n=a_{n-1}\overline{z}^{n-1}+\cdots+a_0$ and so

$$\begin{aligned}
|\overline{z}|^n &= \left|a_{n-1}\overline{z}^{n-1}+\cdots+a_0\right| \\
&\leq \max_i\{|a_i|\}\cdot\left|\overline{z}^{n-1}+\cdots+1\right| \\
&\leq \max_i\{|a_i|\}\cdot\left(|\overline{z}|^{n-1}+\cdots+1\right) \\
&= \max_i\{|a_i|\}\cdot\frac{|\overline{z}|^n+1}{|\overline{z}|-1}\leq\max_i\{|a_i|\}\cdot\frac{|\overline{z}|^n}{|\overline{z}|-1}.
\end{aligned}$$

Thus, $|\overline{z}|^n\leq\max_i\{|a_i|\}\cdot\frac{|z|^n}{|z|-1}$ which implies that $|z|-1\leq\max_i\{|a_i|\}$. Thus, $|z|\leq 1+\max_i\{|a_i|\}$. □

2.2 Finite Dimensional Algebras

Exercise 1 (⟨⟨ CLO05 2.2.1 ⟩⟩):

Exercise 2 (CLO05 2.2.2):

Let $p_i(x_i)$ as in [CLO05, 2.2 (2.6)]. Then show that $p_i(x_i)$ generates $I \cap k[x_i]$.

Proof:

It is clear that $\langle p_i(x_i)\rangle \subseteq I \cap k[x_i]$. Now suppose that $f(x_i) \in I \cap k[x_i]$. Then $\deg(f(x_i))$ must be $\geq m_i$. If not, then by the minimality of m_i we would arrive at a contradiction. Now by the division algorithm, write $f(x_i) = q(x_i)p_i(x_i) + r(x_i)$ where $\deg(r_{x_i}) < m_i$. Then $r(x_i) = f(x_i) - q(x_i)p(x_i) \in I$ and so $r(x_i)$ must be 0 since if not, we would arrive at a contradiction of the minimality of m_i.

This gives us an algorithm to compute $p_i(x_i)$. Let I be a zero dimensional ideal and G a Gröbner basis for I. Then we know there exists m_i such that $\{1, [x_i], \dots, [x^{m_i}]\}$ is linearly dependent in $k[\overline{x}]/I$. In fact, we may use the Finiteness Theorem to set m_i to the smallest integer such that $x_i^{m_i} = \mathrm{LT}(g)$ for some $g \in G$. Since $k[x_1, \dots, x_n]/I$ is a vector space, we can check linear independence in the usual way. See **⟨⟨ Sage code? ⟩⟩** for a SageMath implementation of this. □

Exercise 3 (CLO05 2.2.3):

Let $p(x) \in k[x]$ be a nonzero polynomial. Then $\sqrt{\langle p(x)\rangle} = \langle p_{\text{red}}(x)\rangle$.

Proof:

Let $0 \neq f(x) \in \sqrt{\langle p(x)\rangle}$. Then there exists $m \geq 1$ such that $f^m \in \langle p(x)\rangle$ and so $p(x) \mid f(x)^m$. In particular, each linear factor $(x - \overline{z})$ of $p(x)$ divides $f(x)^n$ and so divides $f(x)$ as $(x - \overline{z})$ is irreducible. Thus, $p_{\text{red}}(x) \mid f(x)$ and so $f(x) \in \langle p_{\text{red}}(x)\rangle$. Conversely, suppose $f(x) \in \langle p_{\text{red}}(x)\rangle$ so that $\langle p_{\text{red}}\rangle \mid f(x)$. Label the roots of $p(x)$ as $\overline{z}_1, \dots, \overline{z}_r$, each $\overline{z}_i \in \overline{k}$. Then for each i, $(x - \overline{z}_i) \mid f(x)$ Let m_i be the multiplicity of z_i in $p(x)$ and $m = \max\{m_1, \dots, m_r\}$. Then $p(x) \mid f(x)^m$ and so $f(x) \in \sqrt{\langle p(x)\rangle}$ □

Exercise 4 (⟨⟨ CLO05 2.2.4 ⟩⟩):

Exercise 5 (⟨⟨ **CLO05 2.2.5** ⟩⟩):

Let I as in Exercise 4. Compare the dimensions of $\mathbb{C}[x,y]/I$ and $\mathbb{C}[x,y]/\sqrt{I}$. How many points are in $V(I)$?

Proof:

Then $\sqrt{I} = I + \langle x(x-1), y(y-2)\rangle$. Since $I \subseteq \sqrt{I}$, we see that $\dim \mathbb{C}[x,y]/I \geq \dim \mathbb{C}[x,y]/\sqrt{I}$. A quick ⟨⟨ **SageMath** ⟩⟩ computation confirms this: $\dim \mathbb{C}[x,y]/I = 9$ and $\dim \mathbb{C}[x,y]/\sqrt{I} = 2$. Then, since $I \subseteq \sqrt{I}$ we have that $V(\sqrt{I}) \subseteq V(I)$. Notice that

$$\begin{aligned} y^4x + 3x^3 - y^4 - 3x^2 &= y^4(x-1) + 3x^2(x-1) = (y^4 + 3x^2)(x-1) \\ x^2y - 2x^2 &= x^2(y-2) \\ 2y^4x - x^3 - 2y^4 + x^2 &= 2y^4(x-1) - x^2(x-1) = (2y^4 - x^2)(x-1). \end{aligned}$$

Thus, $(1,2)$ and $(0,0)$ are the only two points in $V(I)$. Since it is evident that $V(\sqrt{I})$ contains these two points, we see in this case that $V(\sqrt{I}) = V(I)$. □

Bibliography

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