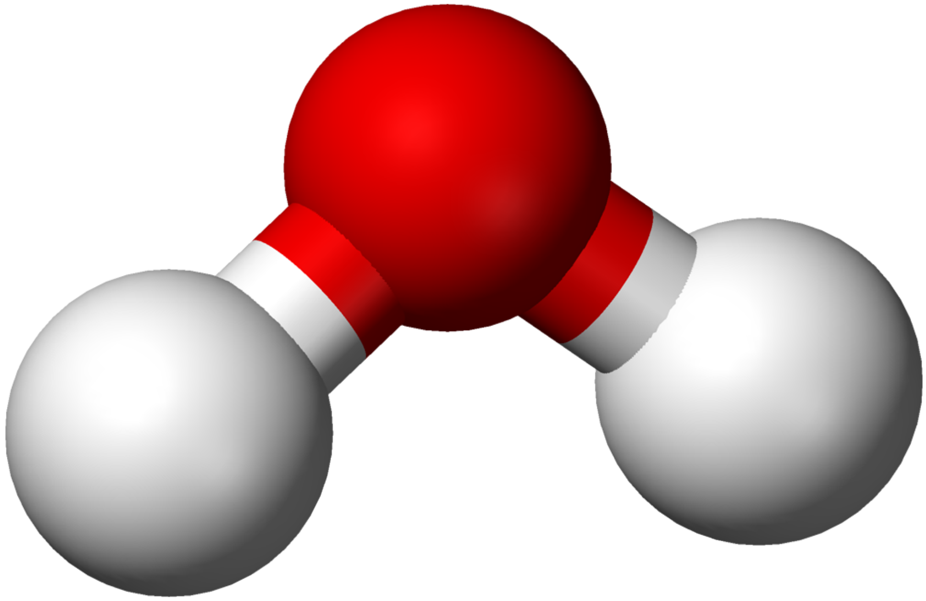

Representation Theory Notes and Exercises

With 0 Figures

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TODOs

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Contents

List of Definitions

List of Examples and Counterexamples

Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. Occasionally, other sources may be used, such as the set of notes by Charles Rezk created for Math 427 [Rez20]. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

Chapter 1

Generalities on Linear Representations

Unless otherwise specified, V will denote a vector space, usually over the field $\mathbb{C}$. We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

Definition 1.1 (Linear Representation, Representation Space): Let G be a group with identity e. A *linear representation* of G in V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will frequently, and often interchangeably, write $\rho_s := \rho(s)$. Given ρ, we will say that V is a *representation space* or *representation* of G.

Definition 1.2 (Degree): Let $\rho : G \to V$ be a representation of G in a vector space V. Then the *degree* of ρ is $\dim(V)$.

Let $\rho : G \to V$ be a representation of G in a vector space V with $n := \dim(V)$. Fix a basis (e_j) of V. Then since each ρ_s is an invertible linear transformation of V, we may define an $n \times n$ matrix $R_s \equiv (r_{ij}(s))$ where each $r_{ij}(s)$ is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^{n} r_{ij}(s) e_i.$$

Definition 1.3 (Matrix of a Representation): We call $R_s = (r_{ij}(s))$ above the *matrix of ρ_s* with respect to the basis (e_j).

Note that R_s satisfies the following:

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^{n} r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two $n \times n$ matrices A, A' are *similar* if there exists an invertible matrix T such that $TA = A'T$. We may extend this notion to representations.

Definition 1.4 (Similar/Isomorphic Representations): Let ρ and ρ' be two representations of the same group G in vector spaces V and V' respectively. We say ρ and ρ' are *similar* or *isomorphic* if there exists an isomorphism $\tau\colon V \to V'$ such that for all $s \in G$, τ satisfies $\tau \circ \rho(s) = \rho'(s) \circ \tau$. If R_s, R'_s are the corresponding matrices then this is equivalent to saying there exists an invertible matrix T such that $TR_s = R'_s T$ for all $s \in G$.

Note that if ρ and ρ' are isomorphic, then they must have the same degree.

We now give some examples of these things.

Example 1.5 (Unit/Trivial Representation): Let G be a finite group. Representations of degree 1 must be of the form $\rho\colon G \to \mathbb{C}^\times$. Since elements s of G are of finite order, $\rho(s)$ must also be of finite order. Thus, for all $s \in G$, $\rho(s)$ is a root of unity. If we take $\rho(s) = 1$ for all $s \in G$, we obtain the *unit* or *trivial* representation of G. This also means that $R_s = 1$ for all s.

Example 1.6 (Regular Representation): Let g be the order of G, and let V be a vector space of dimension g with a basis $(e_t)_{t\in G}$. For each $s \in G$, define ρ_s as the linear map $\rho_s\colon V \to V$ such that $\rho_s(e_t) = e_{st}$. This is a linear representation of G called the *regular* representation of G. Since for each $s \in G$, $e_s = \rho_s(e_1)$ and thus the images of e_1 form a basis of V. Conversely, let W be a representation of G with a vector w satisfying the collection of all $\rho_s(w)$, $s \in G$, forms a basis of W. Then W is isomorphic to the regular representation of G by the isomorphism $\tau(e_s) = \rho_s(w)$.

For example, let $G = \mathbb{Z}_3$ and $V = \mathbb{C}^3$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$. Then for example, $\rho_0, \rho_1, \rho_2\colon \mathbb{C}^3 \to \mathbb{C}^3$ are the linear maps such that

$$\begin{aligned}
&\rho_0(e_0) = e_{0+0} = e_0 &\qquad &\rho_0(e_1) = e_{0+1} = e_1 &\qquad &\rho_0(e_2) = e_{0+2} = e_2\\
&\rho_1(e_0) = e_{1+0} = e_1 &&\rho_1(e_1) = e_{1+1} = e_2 &&\rho_1(e_2) = e_{1+2} = e_0\\
&\rho_2(e_0) = e_{2+0} = e_2 &&\rho_2(e_1) = e_{2+1} = e_0 &&\rho_2(e_2) = e_{2+2} = e_1
\end{aligned}$$

With this, the matrix representations of ρ_0, ρ_1 and ρ_2 is similarly straightforward:

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 1.7 (Permutation Representation): We may generalize the regular representation to any group action $G \curvearrowright X$, X a finite set. Recall that for such an action, the map $x \mapsto sx$ for each $s \in G$ is a permutation $X \longleftrightarrow X$. Let V be a vector space with dimension the size of X, and so a basis $(e_x)_{x \in X}$. Define a representation ρ of G by defining ρ_s as the linear map sending $e_x \mapsto e_{sx}$. This representation is known as the *Permutation* representation of G associated with X. If we consider $X = [n]$ and $G = S_n$, then take $V = \mathbb{C}^n$ as our vector space and e_i as the standard basis vector. Then $\rho_\sigma(e_j) = e_{\sigma_j}$. Thus for each $\sigma \in S_n$, we have that $R_\sigma = (r_{ij}(\sigma))$ where entry $r_{ij}(\sigma) = 1$ if $i = \sigma(j)$ and 0 otherwise.

Definition 1.8 (Stable/Invariant Subspaces, Subrepresentation): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation and $W \subseteq V$ a subspace of V. We say that W is *stable* under the action of G if $x \in W$ implies that $\rho_s(x) \in W$ for all $s \in G$., Thus, the restriction $\rho_s^W := \rho_s \mid_W$ is an isomorphism of W onto itself. Restrictions satisfy the property that $\rho_s^W \circ \rho_t^W = \rho_{st}^W$. Thus, $\rho^W : G \to \mathrm{GL}(W)$ is a linear representation of G in W and we say that W is a *subrepresentation* of V.

Example 1.9 (Subrepresentations of the Regular Representation): Let G be a group. Recall the regular representation V given in Example 1.6. Let W be the 1 dimensional subspace of V generated by the element $x = \sum_{s \in G} e_s$. Then note that $\rho_s(x) = x$ for all $s \in G$ and thus W is a subrepresentation of V. Furthermore, this is isomorphic to the unit representation Example 1.5 with $\tau : C^\times \to W$ such that $\tau(1) = x$. For example, let $G = \mathbb{Z}_3$ and $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ the representation given in Example 1.6. Then $x = (1, 1, 1)$ and for example we have that

$$\rho_1(x) = \rho_1(e_0) + \rho_1(e_1) + \rho_1(e_2) = e_1 + e_2 + e_0 = x.$$

Theorem 1.10: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V and let W be a subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.

Proof: Let W' be an arbitrary complement of W in V, and let $p : V \to W$ be the projection. Then we form the average p^0 of conjugates of p by elements in G:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Since $p : V \to W$ and ρ_t preserves W, we have that p^0 maps V onto W. Furthermore, note that ρ_t^{-1} also preserves W.

Thus we have that

$$(p \circ \rho_t^{-1})(x) = \rho_t^{-1}(x), \qquad (\rho_t \circ p \circ \rho_t^{-1})(x) = x, \qquad p^0(x) = x.$$

Thus, p^0 is a projection of V onto W, corresponding to some complement W^0 of W. Moreover, we have that $\rho_s \circ p^0 = p^0 \circ \rho_s$ for all $s \in G$ because

$$\rho_s \circ p^0 \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \circ \rho_t \circ p \circ \rho_t^{-1} \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \circ p \rho_{st}^{-1} = p^0.$$

Now suppose that $x \in W^0$ and $s \in G$, we have that $p^0(x) = 0$ and hence $(p^0 \circ \rho_s)(x) = (\rho_s \circ p^0)(x) = 0$, meaning that $\rho_s(x) \in W^0$. This, W^0 is stable under G. □

Suppose that V had an inner product $\langle x, y \rangle$, and furthermore suppose this inner product was invariant under G meaning that for all $s \in G$, $\langle \rho_s(x), \rho_s(y) \rangle = \langle x, y \rangle$. We may also reduce to this case by replacing $\langle x, y \rangle$ with $\sum_{t \in G} \langle \rho_t(x), \rho_t(y) \rangle$. With this, the orthogonal complement $W^\perp$ of W in V is a complement of W stable under G. Note that the invariance of $\langle x, y \rangle$ means that if (e_i) is an orthonormal basis of V, then R_s is a unitary matrix.

Using the notation of Theorem 1.10, let $x \in V$ and w, w^0 be the projections of x on W and W^0 respectively. Thus for all $s \in G$, $\rho_s(x) = \rho_s(w) + \rho_s(w^0)$. Since W and W^0 are stable under G, we have that $\rho_s(w) \in W$ and $\rho_s(w^0) \in W^0$. This means that $\rho_s(w)$ and $\rho_s(w^0)$ are the projections of $\rho_s(x)$ and in turn the representations of W and W^0 determine the representations of V.

Definition 1.11 (Direct Sum of Representations): Given the above, we write $V = W \oplus W^0$ as the *direct sum* of W and W^0. We identify elements $v \in V$ as pairs (w, w^0) given by their projections.

If the representations W and W^0 are given in matrices R_s and R_s^0, then the matrix form of the representation V is given by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

Similar results hold for arbitrarily many, but finite, direct sums of representations.

The above is a discussion on how to compose two or more representations into one larger representation. The natural question is then about the opposite.

Definition 1.12 (Irreducible/Simple Representations): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Then this representation is *irreducible* or *simple* if V has no subspaces stable under G besides 0 and V itself.

By Theorem 1.10, this is equivalent to saying that V is not the direct sum of two representations besides $V = 0 \oplus V$. A representation of degree 1 is reducible. We may use irreducible representations to construct other ones via the direct sum.

Theorem 1.13: Every representation is a direct sum of irreducible representations.

Proof: Let V be a linear representation of G. We induct on $\dim(V)$. If $\dim(V) = 0$, then $V = 0$ which is the direct sum of an empty family of irreducible representations. So suppose that $dim(V) \geq 1$. If V is irreducible, then we are done. Otherwise, there exists a subspace $W \subsetneq V$ stable under G and by Theorem 1.10 a stable complement W^0 such that $V = W \oplus W^0$. By assumption, $W \neq 0 \neq W^0$ and so $\dim(W) < V$ and $\dim(W^0) < \dim(V)$. By induction, we have obtained a decomposition of V into irreducibles. □

Example 1.14 (Decomposition of Representation of $\mathbb{Z}_3$ into Irreducibles): Recall from Example 1.6 the regular representation $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$ and

$$\begin{aligned}
\rho_0(e_0) &= e_0 & \rho_0(e_1) &= e_1 & \rho_0(e_2) &= e_2\\
\rho_1(e_0) &= e_1 & \rho_1(e_1) &= e_2 & \rho_1(e_2) &= e_0\\
\rho_2(e_0) &= e_2 & \rho_2(e_1) &= e_0 & \rho_2(e_2) &= e_1
\end{aligned}$$

Our goal will be to decompose ρ into $\rho^1 \oplus \rho^2 \oplus \rho^3$. We aim to find the elements fixed by $\mathbb{Z}_3$. Note that if an element is fixed by 1, the generator of $\mathbb{Z}_3$, then it is fixed by all of $\mathbb{Z}_3$. We want to find 1-dimensional $\mathbb{Z}_3$-invarient subspaces of $\mathbb{C}^3$. This is equivalent to finding the eigenvalues of the matrix

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues their eigenvectors of R_1 are as follows, note the inclusion of the trivial subrepresentation from Example 1.9:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}, v_2 = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \qquad \lambda_3 = \frac{-1 + i\sqrt{3}}{2}, v_3 = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Thus $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$ where $V_i := \mathrm{span}(v_i)$. Note that there are only 3 morphisms $\mathbb{Z}_3 \to \mathbb{C}^\times$ mapping 1 to 1, ω, or ω^2 where ω is a cube root of unity. Thus ρ^1, ρ^2, and ρ^3 must correspond to these morphisms **⟨⟨ but which ones ⟩⟩** .

A natural question is if such a decomposition $V = W_1 \oplus \cdots \oplus W_k$ is unique. However, suppose that ρ is the trivial representation (Example 1.5). Then each component of the decomposition is a line and we can decompose a vector space into the direct sum of lines in a number of ways. However, we will see that the number of W_i that are isomorphic to a given irreducible representation does not depend on the choice of decomposition.

Definition 1.15 (Tensor/Kronekcer Product of Representations): Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two representations of a group G. We construct a representation $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ such that

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \circ \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

The existence and uniqueness of ρ follow immediately from the existence and uniqueness of the tensor product. We write $\rho_s \equiv \rho_s^1 \otimes \rho_s^2$ as the *tensor product* of the given representations.

Recall that if (e_{i_1}) and (e_{i_2}) be bases of V_1 and V_2 respectively, then $(e_{i_1} \otimes e_{i_2})$ is a basis of $V_1 \otimes V_2$. If $(r_{i_1 j_1}(s))$ and $(r_{i_2 j_2}(s))$ are the matrices of ρ_s^1 and ρ_s^2 respectively satisfying

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1} e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2} e_{i_2}$$

then the matrix of ρ_s is $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$ satisfying

$$\rho_s(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \otimes e_{i_2}.$$

⟨⟨ TODO: example of tensor product ⟩⟩ Note that the tensor product of two irreducible representations is not in general irreducible **⟨⟨ TODO: example? ⟩⟩** .

We now consider the special case of $V \otimes V$. Let (e_i) be a basis of V and define an automorphism θ of $V \otimes V$ such that $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Then note that $\theta^2 \equiv \mathrm{id}_{V \otimes V}$. We may decompose $V \otimes V$ into the direct sum

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Here, $\mathrm{Sym}^2(V)$ is the set of $z \in V \otimes V$ such that $\theta(z) = z$ and $\mathrm{Alt}^2(V)$ is the set of $z \in V \otimes V$ where $\theta(z) = -z$. These have bases $(e_i \otimes e_j + e_j \otimes e_i)_{i \le j}$ and $(e_i \otimes e_j - e_j \otimes e_i)_{i < j}$ respectively. As such, $\dim(\mathrm{Sym}^2(V)) = \frac{n(n+1)}{2}$ and $\dim(\mathrm{Alt}^2(V)) = \frac{n(n-1)}{2}$ where $n := \dim(V)$.

Definition 1.16 (Symmetric Square, Alternating Square): These subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ of $V \otimes V$ are respectively called the *symmetric square* and *alternative square* of the given representation.

Chapter 2

Character Theory

Definition 2.1 (Character): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of a finite group G in V. Then the *character* χ_ρ of ρ is the function

$$\chi_\rho(s) := \mathrm{Tr}(R_s) \equiv \mathrm{Tr}(\rho_s).$$

for each $s \in G$.

Proposition 2.2: If χ is the character of a representation ρ of degree n then

1. $\chi(e) = 1$;
2. $\chi(s^{-1}) = \chi(s)^*$, the complex conjugate of $\chi(s)$,
3. $\chi(tst^{-1}) = \chi(s)$.

Proof: The first is immediate since ρ_1 is the identity matrix I and $\mathrm{Tr}(I) = n$. Then recall that we may choose our basis to be orthonormal, and as such ρ_s is a unitary matrix. Thus, each eigenvalue $\lambda_1, \ldots, \lambda_n$ has an absolute value equal to 1. Thus

$$\chi(s)^* = \mathrm{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \mathrm{Tr}(\rho_s)^{-1} = \mathrm{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

This uses the fact that the conjugate of the sum is the sum of the conjugates as well as the fact that the eigenvalues of R_s^{-1} are the inverses of the eigenvalues of R_s. Finally, letting $u = ts$ and $v = t^{-1}$ allows us to write $\chi(tst^{-1}) = \chi(s)$ as $\chi(uv) = \chi(vu)$ which is immediate since for any complex matrices A, B we have that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. $\square$

Proposition 2.3: Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two linear representations with characters χ_1 and χ_2 respectively. Then

1. The character χ of the direct sum representation $V_1 \oplus V_2$ is $\chi_1 + \chi_2$.
2. The character ψ of the tensor product representation $V_1 \otimes V_2$ is $\chi_1 \cdot \chi_2$.

Proof: Let R_s^1, R_s^2 be the matrix forms of ρ_s^1 and ρ_s^2 respectively. Then the matrix form R_s of the representation of $V_1 \oplus V_2$ is given by

$$R_s = \begin{pmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{pmatrix}$$

and thus $\mathrm{Tr}(R_s) = \mathrm{Tr}\big(R_s^1\big) + \mathrm{Tr}\big(R_s^2\big)$. Let (e_{i_1}) and (e_{i_2}) be bases for V_1 and V_2. Then we have that

$$\psi(s) = \sum_{i_1 i_2} r_{i_1 i_1}(s) \cdot r_{i_2 i_2}(s) = \left(\sum_{i_1} r_{i_1 i_1}(s)\right) \cdot \left(\sum_{i_2} r_{i_2 i_2}(s)\right) = \chi_1(s) \cdot \chi_2(s).$$

□

Proposition 2.4: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G with character χ. Let χ_σ^2 be the character of $\mathrm{Sym}^2(V)$ and χ_α^2 be the character of $\mathrm{Alt}^2(V)$ from Definition 1.16. Then

$$\chi_\sigma^2(s) = \frac{1}{2}\big(\chi(s)^2 + \chi(s^2)\big)$$
$$\chi_\alpha^2(s) = \frac{1}{2}\big(\chi(s)^2 - \chi(s^2)\big)$$

which directly implies that $\chi_\sigma^2 + \chi_\alpha^2 = \chi$.

Proof: Let $s \in G$ and (e_i) a basis of V consisting solely of eigenvectors for ρ_s. Then $\rho_s(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$. Thus

$$\chi(s) = \sum \lambda_i \qquad\qquad \chi(s^2) = \sum \lambda_i^2.$$

We also have that

$$(\rho_s \otimes \rho_s)(e_i \otimes e_j + e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i)$$
$$(\rho_s \otimes \rho_s)(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

which yields that

$$\chi_\sigma^2(s) = \sum_{i \le j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\Big(\sum \lambda_i\Big)^2 + \frac{1}{2}\sum \lambda_i^2 \chi_\alpha^2(s) \quad = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\Big(\sum \lambda_i\Big)^2 - \frac{1}{2}\sum \lambda_i^2.$$

The proposition then directly follows. Note that the equality $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$ directly reflects the fact that $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$. $\square$

Proposition 2.5 (Schur's Lemma): Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $f : V_1 \to V_2$ be a linear map such that $f \circ \rho_s^1 = \rho_s^2 \circ f$ for all $s \in G$. Then

1. If ρ^1 and ρ^2 are not isomorphic, then $f = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$ then f is a *homothety*, a scalar multiple of the identity.

Proof: The case of $f = 0$ is trivial, so suppose that $f \neq 0$. Let $W_1 = \ker(f)$ and $W_2 = \mathrm{im}(f)$. Then for $x \in W_1$ we have that $f(\rho_s^1(x)) = \rho_s^2(f(x)) = 0$ which means that $\rho_s^1(x) \in W_1$. Thus W_1 is stable under G and irreducibility of V_1 combined with the assumption that $f \neq 0$ implies that $W_1 = 0$. Similarly, we have that for $f(x) \in W_2$, we have that $\rho_s^2(f(x)) = f(\rho_s^1(x)) \in W_2$, so $\rho_s^2(f(x)) \in W_2$. Thus W_2 is also stable under G meaning that by a similar argument, $W_2 = V_2$. Since $\ker(f) = 0$ and $\mathrm{im}(f) = V_2$, we must have that f is an isomorphism $V_1 \to V_2$. This proves the first claim.

Now suppose that $V_1 = V_2$, $\rho^1 = \rho^2$, and that λ is some eigenvalue of f. Let $f' = f - \lambda$. Since λ is an eigenvalue, then $\ker(f') \neq 0$. However, we also have that $f' \circ \rho_s^1 = \rho_s^2 \circ f'$. The first part of this proof shows that this implies that $f' = 0$. Thus, $f = \lambda$ and f is a homothety. $\square$

Corollary 2.6: Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $h \colon V_1 \to V_2$ and define h^0 such that

$$h^0 = \frac{1}{|G|} \sum_{t \in G} (\rho_t^2)^{-1} \circ h \circ \rho_t^1.$$

Then

1. If ρ^1 and ρ^2 are not isomorphic, then $h^0 = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$, then h^0 is a homothety of ratio $\frac{1}{n}\mathrm{Tr}(h)$, with $n = \dim(V_1)$.

Proof: First for $s \in G$ we have that

$$(\rho_s^2)^{-1} \circ h^0 \circ \rho_s^1 = \frac{1}{|G|} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} \circ h \circ \rho_t^1 \rho_s^1. \qquad = \frac{1}{|G|} \sum_{t \in G} (\rho_{ts}^2)^{-1} \circ h \circ \rho_{ts}^1 = h^0.$$

Thus, Proposition 2.5 applies to h^0 and in the first case $h^0 = 0$ and in the second h^0 is a homothety of scalar λ. Moreover we have that

$$n \cdot \lambda = \mathrm{Tr}(\lambda) = \frac{1}{|G|} \sum_{t \in G} \mathrm{Tr}\left((\rho_t^1)^{-1} \circ h \circ \rho_t^1\right) = \mathrm{Tr}(h).$$

Thus, $\lambda = \frac{1}{n}\mathrm{Tr}(h)$. $\square$

Consider Corollary 2.6 in matrix form where $\rho_s^1 = (r_{i_1j_1}(s))$ and $\rho_s^2 = (r_{i_2j_2}(s))$. Then our linear map h is given by the matrix $(x_{i_2i_1})$ and similarly h^0 is given by the matrix $(x^0_{i_2i_1})$. Then by definition of h^0 we have that

$$x^0_{i_2i_1} = \frac{1}{|G|}\sum_{t\in G,j_1,j_2} r_{i_2j_2}(t^{-1})\cdot x_{j_2j_1}\cdot r_{j_1i_1}(t).$$

In case (1) of Corollary 2.6, the right hand side must vanish and so all coefficients of 0:

Corollary 2.7: In case (1) of Corollary 2.6 we have that

$$\frac{1}{|G|}\sum_{t\in G} r_{i_2j_2}(t^{-1})\cdot r_{j_1i_1}(t) = 0$$

for all i_1, j_1, i_2, j_2.

We now consider case (2) of Corollary 2.6 in the matrix form. We have that $h^0 = \lambda$, with $\lambda = \frac{1}{n}\operatorname{Tr}(h)$, meaning that $x^0_{i_2i_1} = \lambda\delta_{i_2i_1}$. That is, $\lambda = \frac{1}{n}\sum\delta_{i_2i_1}\cdot x_{i_2i_1}$. This we have that

$$\frac{1}{|G|}\sum_{t\in G,j_1,j_2} r_{i_2j_2}(t^{-1})\cdot x_{j_2j_1}\cdot r_{j_1i_1}(t) = \frac{1}{n}\sum_{j_1,j_2}\delta_{i_2i_1}\cdot\delta_{j_2j_1}\cdot x_{j_2j_1}.$$

Equating coefficients of the $x_{j_2j_1}$ yields the following corollary:

Corollary 2.8: In case (2) of Corollary 2.6 we have that

$$\frac{1}{|G|}\sum_{t\in G} r_{i_2j_2}(t^{-1})\cdot r_{j_1i_1}(t) = \frac{1}{n}\delta_{i_2i_1}\cdot\delta_{j_2j_1}.$$

We may generalize these notions to further functions. Let ϕ, ψ be **⟨⟨ complex valued? ⟩⟩** functions on G. Define

$$\langle\phi,\psi\rangle := \frac{1}{|G|}\sum_{t\in G}\phi(t^{-1})\cdot\psi(t) = \frac{1}{|G|}\sum_{t\in G}\phi(t)\cdot\psi(t^{-1}).$$

Then $\langle\phi,\psi\rangle = \langle\psi,\phi\rangle$ and $\langle\phi,\psi\rangle$ is linear in ϕ and in ψ. Thus, Corollary 2.7 and Corollary 2.8 respectively become

$$\left\langle r_{i_2j_2}, r_{j_1i_1}\right\rangle = 0 \qquad\qquad \left\langle r_{i_2j_2}, r_{j_1i_1}\right\rangle = \frac{1}{n}\delta_{i_2i_1}\cdot\delta_{j_2j_1}.$$

If the matrices $(r_{ij}(t))$ are unitary, realized by a suitable choice of basis, then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and Corollary 2.7 and Corollary 2.8 are just *orthogonality relations*.

Definition 2.9 (Scalar Product): If ϕ, ψ are two complex valued functions on G, then let

$$(\phi \mid \psi) := \frac{1}{|G|} \sum_{t \in G} \phi(t)\psi(t)^*.$$

This is a *scalar product*. It is linear in ϕ, semilinear in ψ, and $(\phi \mid \phi) > 0$ for all $\phi \neq 0$.

Define $\check{\psi}(t) := \psi(t^{-1})^*$. Then $(\phi \mid \psi) = \langle \phi, \check{\psi} \rangle$. In particular, suppose χ is a character so that by Proposition 2.2 we have that $\chi = \check{\chi}$ then for all complex valued functions ϕ on G we have that $(\phi \mid \chi) = \langle \phi, \chi \rangle$. Thus, we may use the two interchangeably in the context of characters.

Theorem 2.10:

1. If χ is the character of an irreducible representation, we have that $(\chi \mid \chi) = 1$, i.e. χ has "norm 1."
2. If χ and χ' are characters of two non-isomorphic irreducible representations, then $(\chi \mid \chi') = 0$, i.e. χ and χ' are "orthogonal."

Proof: Suppose ρ is an irreducible representation with matrix form $\rho_t = (r_{ij}(t))$ and χ its character. Then $\chi(t) = \sum r_{ii}(t)$ and so

$$(\chi \mid \chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle = \frac{\delta_{ij}}{n}$$

where the last equality is by Corollary 2.8 and n is the degree of ρ. Thus

$$(\chi \mid \chi) = \sum_{i,j} \frac{\delta_{ij}}{n} = \frac{n}{n} = 1.$$

This proves the first claim. Applying Corollary 2.7 yields the second claim □

Theorem 2.11: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi \mid \chi) = \langle \phi, \chi \rangle$.

Proof: Let χ_i be the character of W_i. Then by Proposition 2.3 we have that $\phi = \chi_1 + \cdots + \chi_k$. By linearity of $(\cdot \mid \cdot)$ in the first argument we have that $(\phi \mid \chi) = (\chi_1 \mid \chi) + \cdots + (\chi_k \mid \chi)$. The result follows by Theorem 2.10. □

Corollary 2.12: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W does not depend on the chosen decomposition.

Proof: Note that $(\phi \mid \chi)$ does not depend on choice of decomposition. □

Corollary 2.13: Two representations are isomorphic if and only if they have the same character.

Proof: The forward direction is obvious, and the reverse is true by the prior corollary. □

Thus, our study of representations is reduced to that of the study of characters. If $\chi_1, \dots, \chi_k$ are the distinct irreducible characters of G ⟨⟨ **How do we know there are finitely many?** ⟩⟩ and if $W_1, \dots, W_k$ their corresponding representation, then each representation V of G is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \qquad\qquad m_i \neq 0.$$

The character ϕ of V is equal to $m_1\chi_1 + \cdots + m_h\chi_h$ and we have that $m_i = (\phi \mid \chi_i)$. This is especially useful when considering the tensor product $W_i \otimes W_j$ of two irreducible representations. It shows that the product $\chi_i \cdot \chi_j$ decomposes into a sum $\chi_i\chi_j = \sum m_{ij}^k \chi_k$, each integer $m_{ij}^k \geq 0$. The orthogonality relations among the χ_i imply that

$$(\phi \mid \phi) = \sum_{i=1}^{h} m_i^2.$$

We now obtain a useful irreducibility criterion:

Theorem 2.14: If ϕ is the character of a representation V, $(\phi \mid \phi)$ is a positive integer and $(\phi \mid \phi) = 1$ if and only if V is irreducible.

Proof: We have that $\sum m_i^2 = 1$ if and only if one of the $m_i = 1$ and all the others are equal to 0. This means that V is isomorphic to one of the W_i. □

We now explore the decomposition of the regular representation $\rho : G \to \mathrm{GL}(R)$ of a group G (Example 1.6). Suppose $\chi_1, \dots, \chi_h$ are the irreducible characters of G with degrees $n_1, \dots, n_k$. Note that by Proposition 2.2, $n_i = \chi_i(e)$. Recall that R has basis $(e_t)_{t\in G}$ where $\rho_s(e_t) = e_{st}$. This means that for $s \neq e$, the diagonal terms of the matrix for ρ_s are all 0, so $\mathrm{Tr}(\rho_s) = 0$. On the otherhand, we have that

$$\mathrm{Tr}(\rho_e) = \dim(R) = |G|.$$

Proposition 2.15: The character r_G of the regular representation is given by

$$r_G(e) = |G| \qquad\qquad r_G(s) = 0 \text{ if } s \neq e.$$

Corollary 2.16: Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i.

Proof: By Theorem 2.11, the number of times W_i is contained in the regular representation is $\langle r_G, \chi_i \rangle$. We have that

$$\langle r_G, \chi_i \rangle = \frac{1}{|G|} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{|G|} \cdot |G| \chi_i(1) = \chi_i(1) = n_i.$$

□

Corollary 2.17: 1. The degrees satisfy $\sum_{i=1}^{h} n_i^2 = |G|$.

2. if $e \neq s \in G$, we have that $\sum_{i=1}^{h} n_i \chi_i(s) = 0$.

Proof: By Corollary 2.16, we have that $r_G(s) = \sum n_i \chi_i(s)$ for all $s \in G$ ⟨⟨ **how?** ⟩⟩ . The claim is thus immediate. □

The above result lets us determine the irreducible representations of a group G. Suppose we have constructed some mutually non-isomorphic irreducible representations of degrees $n_i, \ldots, n_h$. In order to check if we have found all such representations, it is necessary and sufficient to verify that $n_1^2 + \cdots + n_h^2 = |G|$. Also, we shall later see that each of the n_i divide the order of g.

Exercises

Exercise 2.1 (Ser77 2.1): Let χ, χ' be the characters of two representations. Prove the following formulas:

$$(\chi + \chi')^2_\sigma = \chi^2_\sigma + \chi'^2_\sigma + \chi\chi'$$
$$(\chi + \chi')^2_\alpha = \chi^2_\alpha + \chi'^2_\alpha + \chi\chi'$$

Proof: Let $s \in G$. Then by Proposition 2.4 we have that

$$\begin{aligned}
(\chi + \chi')^2_\sigma(s) &= \frac{1}{2}\big((\chi + \chi')^2(s) + (\chi + \chi')(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi'(s)^2 + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi(s^2)\big) + \frac{1}{2}\big(\chi'(s) + \chi'(s^2)\big) + \chi(s)\chi'(s) = \chi^2_\sigma(s) + \chi'^2_\sigma(s) + \chi(s)\chi'(s).
\end{aligned}$$

Since this holds for all $s \in G$, the formula holds in general. The proof of the other formula is similar. □

Exercise 2.2 (Ser77 2.2): Let X be a finite set on which G acts, and $\rho : G \to \mathrm{GL}(V)$ the corresponding permutation representation (Example 1.7), and χ_X the character of ρ. Then show that for $s \in G$, $\chi_X(s)$ is equal to the number of elements fixed by s.

Proof: Suppose $X = [n]$ and so $s \in S_n$, meaning $G \leq S_n$. We may assume this without loss of generality. Note that $R_s = (r_{ij}(s))$ where $r_{ij}(s) = 1$ if $s(j) = i$ and 0 otherwise. We want to count the number of elements in $[n]$ fixed by s, i.e. the number of i such that $\sigma(i) = i$. These correspond exactly to the entries in R_S where $r_{ii}(s) = 1$. Thus, the claim follows. □

Exercise 2.3 (Ser77 2.6): Let G act on a finite set X, ρ the corresponding permutation representation, and χ its character.

1. Let c be the number of distinct orbits. Show that c is equal to the number of times ρ contains the unit representation 1. Deduce that $(\chi \mid 1) = c$. In particular if G is transitive and thus $c = 1$, then $\rho = 1 \oplus \theta$ where θ does not contain the unit representation. If ψ is the character of θ, then $\chi = 1 + \psi$ and $(\psi \mid 1) = 0$.

2. Let G act on the product $X \times X$ in the natural way. Show that the character of the corresponding permutation representation is equal to χ^2.

3. Suppose that G is transitive on X and $|X| \geq 2$. We say G is *doubly transitive* if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y$ there exists $s \in G$ such that $s(x, y) = (sx, sy) = (x', y')$. Prove that the following are equivalent:

(a) G is doubly transitive.

(b) The action of G on $X \times X$ has two orbits, the diagonal and the complement.

(c) $(\chi^2 \mid 1) = 2$

(d) The representation θ defined in the first part of this exercise is irreducible.

Proof: First suppose that $G \curvearrowright X$ is transitive. Then by Exercise 2.2, $s \in G$, $\chi(s) = 1$ and so by **⟨⟨ Exercise 2.5 ⟩⟩** we have that $(\chi \mid 1) = \frac{1}{|G|}\sum \chi(s) = 1$. Note that the unit representation is irreducible, and so by Theorem 2.11 we have that the number of times ρ contains the unit representation is equal to $(\chi \mid 1) = 1$ as desired. Now let $G \curvearrowright X$ be a general group action. Let V be the underlying vector space with basis $(e_s)_{s\in G}$. Then each orbit of $G \curvearrowright X$ induces a transitive subaction of G on that orbit. Each orbit corresponds uniquely to some set of basis vectors as the set of orbits partitions X, and thus partitions $(e_s)_{s\in G}$. Let ψ_i, $1 \le i \le c$ be the character associated to the permutation representation of the induced subaction of G on the i-th orbit. Then using the prior argument and the linearity of the scalar product we get that

$$(\chi \mid 1) = (\psi_1 \mid 1) + \cdots + (\psi_c \mid 1) = c.$$

Following this, the rest of the claim is immediate.

Now suppose that ϕ is the character of the permutation representation of $G \curvearrowright X \times X$. Then by Exercise 2.2, $\phi(s)$ is equal to the number of elements fixed by s. An element $(x, y) \in X \times X$ is fixed by $s \in G$ if and only if both x and y are fixed. Thus if there are $\chi(s)$ elements of X fixed by s, then $\chi^2(s)$ elements of $X \times X$ are fixed by s and $\phi = \chi^2$.

To prove 3, we have that $(a) \iff (b)$ is immediate and $(b) \iff (c)$ follows from 1 and 2. Now suppose (c) holds and let ψ be the character of θ. Then $1 + \psi = \theta$. Since $(\chi \mid 1) = (1 \mid 1) = 1$ we must have that $(\psi \mid 1) = 0$. Since $\chi^2 = 1 + 2\psi + \psi^2$, we have that (c) is equivalent to saying $(\psi^2 \mid 1) = 1$. Thus

$$\frac{1}{|G|}\sum_{s\in G} \psi(s)^2 = 1.$$

However, note that $\psi(s)$ is real valued, not just complex valued. This is because χ is real valued, it counts fixed points, and clearly 1 is real valued. Thus $\psi^* = \psi(s)^*$ and so the above equality implies that $(\psi \mid \psi = 1)$. By Theorem 2.14, we have that this is true if and only if θ is irreducible, i.e. $(c) \iff (d)$ holds. □

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