Using Algebraic Geometry

With 0 Figures

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Preface

At the time of writing this, I am starting my PhD at The Ohio State University. Currently a large part of my interests in algebra are about algorithms as they relate to polynomials and algebraic geometry. I've been doing a bunch of problems from *Ideals, Varieties, and Algorithms* [CLO15]. However, it seems that *Using Algebraic Geometry* [CLO05] moves through the material faster as it assumes you know more algebra. So I've moved onto working through this book as well as trying to comprehend Sturmfel's *Algorithms in Invariant Theory* [Str08].

Chapter 1

Introduction

1.1 Polynomials and Ideals

Exercise 1.1 (CLO05 1.1.1):

(a) Show that $x^2 \in \langle x - y^2, xy \rangle$ in $k[x, y]$.

(b) Show that $\langle x - y^2, xy, y^2 \rangle = \langle x, y^2 \rangle$.

(c) Is $\langle x - y^2, xy \rangle = \langle x^2, xy \rangle$? Why or why not?

Proof:

(a) We have that $x(x - y^2) + y(xy) = x^2 - xy^2 + xy^2 = x^2$.

(b) It suffices to check for generators. We have that $x + (-1)(y^2) = x - y^2$, $y(x) = xy$, and $y^2 = y^2$ showing that $\langle x - y^2, xy, y^2 \rangle \subseteq \langle x, y^2 \rangle$. Then $x - y^2 + y^2 = x$ and $y^2 = y^2$ shows the reverse containment and overall the ideals are equal.

(c) We already know from 1. that x^2 lives in $\langle x - y^2, xy \rangle$. Since $xy = xy$, we overall have that $\langle x^2, xy \rangle \subseteq \langle x - y^2, xy \rangle$. It remains to check if $x - y^2 \in \langle x^2, xy \rangle$. However, notice that every element of $\langle x^2, xy \rangle$ is divisible by x while $x - y^2$ is clearly not divisible by x. Thus $x - y^2 \notin \langle x^2, xy \rangle$ and the two ideals are not equal.

$\square$

Exercise 1.2 (CLO05 1.1.2):

Show that $\langle f_1,\dots,f_s\rangle$ is closed under sums in $k[x_1,\dots,x_n]$. Also show that if $f\in\langle f_1,\dots,f_s\rangle$ and $p\in k[x_1,\dots,x_n]$ then $p\cdot f\in\langle f_1,\dots,f_s\rangle$.

Proof:

Let $f,g\in\langle f_1,\dots,f_s\rangle$. Then $\exists p_1,\dots,p_s,q_1,\dots,q_s$ such that $f=\sum_{i=1}^s p_i\cdot f_i$ and $g=\sum_{i=1}^s q_i\cdot f_i$. Thus $f+g=\sum_{i=1}^s(p_i+q_i)\cdot f_i$ which shows that $f+g\in\langle f_1,\dots,f_s\rangle$. Then let $p\in k[x_1,\dots,x_n]$. We have that $p\cdot f=p\sum_{i=1}^s p_if_i=\sum_{i=1}^s(p\cdot p_i)\cdot f_i$ which shows that $\langle f_1,\dots,f_s\rangle$ is an ideal. $\square$

Exercise 1.3 (CLO05 1.1.3):

Show that $\langle f_1,\dots,f_s\rangle$ is the smallest ideal containing $\{f_1,\dots,f_s\}$.

Proof:

We already know that $\langle f_1,\dots,f_s\rangle$ is an ideal by Exercise 1.2. Now suppose that J is an ideal containing $\{f_1,\dots,f_s\}$. Then, since ideals are closed under addition and scaling, we have that for all $p_1,\dots,p_s\in k[x_1,\dots,x_n]$ that $\sum_{i=1}^s p_i\cdot f_i\in J$. Thus, $\langle f_1,\dots,f_s\rangle\subseteq J$. $\square$

Exercise 1.4 (CLO05 1.1.4):

Using Exercise 1.3, formulate and prove a general criterion for the equality of $I=\langle f_1,\dots,f_s\rangle$ and $J=\langle g_1,\dots,g_t\rangle$.

Proof:

We claim that $\langle f_1,\dots,f_s\rangle=\langle g_1,\dots,g_t\rangle$ if and only if $\{g_1,\dots,g_t\}\subseteq I$ and $\{f_1,\dots,f_s\}\subseteq J$. The forward implication is immediate. Then by Exercise 1.3, if $\{g_1,\dots,g_t\}\subseteq I$ then $J\subseteq I$. Similarly, $\{f_1,\dots,f_s\}\subseteq J\implies I\subseteq J$ and overall $I=J$. This fact was used in Exercise 1.1 (b). $\square$

Exercise 1.5 (CLO05 1.1.5):

Show that $\left\langle y-x^2,z-x^3\right\rangle=\left\langle y-x^2,z-xy\right\rangle$ in $\mathbb{Q}[x,y,z]$.

Proof:

It suffices to show that $z-x^3\in\left\langle y-x^2,z-xy\right\rangle$ and and $z-xy\in\left\langle x-y^2,z-x^3\right\rangle$. Indeed we have that $(z-xy)+x(y-x^2)=z-x^3$ which also yields that $z-xy=z-x^3-x(y-x^2)$. $\square$

Exercise 1.6 (CLO05 1.1.6):

Show that every ideal $I \subseteq k[x]$ is generated by a single polynomial.

Proof:

If $I = \{0\}$ then $I = \langle 0 \rangle$. So suppose $I \neq 0$. Let $d \in I$ be of minimal degree. **$\langle\langle$ d = gcd(I) but I need infinite Bezout. $\rangle\rangle$** Then we claim that $\langle d \rangle = I$. Since $d \in I$, we have that $\langle d \rangle \subseteq I$. Now let $f \in I$. By Euclidean division, there exists $q, r \in k[x]$ such that $f = qd + r$ where either $r = 0$ or $0 \leq \deg(r) \leq \deg(d) - 1$. If $r = 0$ then $f \in \langle d \rangle$ and we are done. So suppose $r \neq 0$. Then $f, qd \in I \implies r = f - qd \in I$. Thus, $r \in I$ is of degree strictly less than d, contradicting the minimality of the degree of d. So we must have that $r = 0$ and overall $\langle d \rangle = I$. $\square$

Bibliography

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