

# Young Tableau, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci

With 0 Figures

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Last Edited on 1/5/25 at 14:47

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# Preface

These are notes for a reading course under Professor [Dave Anderson](#). The primary focus is Manivel's *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci* [\[Man01\]](#) which one could see as a quasi-sequel to Fulton's *Young Tableaux*<sup>1</sup> [\[Ful97\]](#). Primarily, the solutions will be to exercises from [\[Man01\]](#). However, as needed there will be solutions to material from [\[Ful97\]](#), or perhaps even other texts such as [\[Mac98\]](#).

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<sup>1</sup>which throughout these notes will be spelled as “tableaux” or “tableau” with no real consistency.

# Chapter 1

## [Ful97] Geometry

**Solution:** [Ful97] §9.1 Ex. 1: Choose a basis  $\{e_1, \dots, e_m\}$  so that  $E$  can be identified with  $\mathbb{C}^m$ . Let  $i_1 < \dots < i_{d-1}$  and  $j_1 < \dots < j_{d+1}$  be sequences in  $[m]$ . Apply §9.1 Equation (1) with  $k = 1$  to the sequences  $j_2 < \dots < j_{d+1}$  and  $i_1 < \dots < i_{d-1}, j_1$  by fixing  $j_1$  to be the vector swapped successively with the  $j_2 < \dots < j_{d+1}$ . Reordering the indices and applying the appropriate sign change yields the desired alternating summation.  $\square$

**Solution:** [Ful97] §9.1 Ex. 2: We have that  $V \subseteq E = \mathbb{C}^4$  is given as the kernel of multiplication of a matrix  $A = (a_{i,j})_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq 2}}$ . To find this matrix, the given conditions of the  $x_{i,j}$  describe the following determinantal conditions on the entries of  $A$ :

$$x_{1,2} = 1 \iff \Delta_{1,2}(A) = 1,$$

$$x_{1,3} = 2 \iff \Delta_{1,3}(A) = 2,$$

$$x_{1,4} = 1 \iff \Delta_{1,4}(A) = 1,$$

$$x_{2,3} = 1 \iff \Delta_{2,3}(A) = 1,$$

$$x_{2,4} = 2 \iff \Delta_{2,4}(A) = 2,$$

$$x_{3,4} = 3 \iff \Delta_{3,4}(A) = 3.$$

From here, we must make an assumption based on which affine portion of  $\mathbb{P}^5$  our matrix lives in. This amounts to picking some  $i_1, i_2$  so that the minor given by those columns is the identity matrix. For the given conditions, we could pick  $(i_1, i_2) = (1, 2), (1, 4),$  or  $(2, 3)$ . We give  $A$  for each of these choices respectively:

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 & 0 & -3 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

It is clear that these are the same matrix up to row operations.  $\square$

**Solution:** [Ful97] §9.1 Ex. 4: Following [Ful97, Corollary of Theorem 1, §8.1], we have that  $S^\bullet(m; d_1, \dots, d_s)$  is canonically isomorphic to the subalgebra of  $\mathbb{C}[Z]$  generated by all  $D_T$ , where  $T$  varies over all tableaux on Young diagrams whose columns have lengths in  $\{d_1, \dots, d_s\}$  and entries in  $[m]$  where

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j), T(2,j), \dots, T_{\mu_j, j}}$$

where  $\mu_j$  is the length of the  $j^{\text{th}}$  column of  $\lambda$  the shape of  $T$  and  $\ell = \lambda_1$ .

- (a) We mimic the proof of [Ful97, Proposition 2, §9.1]. **« I think this proof needs to be rewritten, perhaps with a highest weight argument? »** Let  $G = G(d_1, \dots, d_s) \leq \text{GL}(V)$ . The dimension of the vector space of polynomials of homogeneous polynomials of degree  $a$  in the span of all the  $D_{i_1, \dots, i_p}$  for  $p \in \{d_1, \dots, d_s\}$  is  $\sum d_\lambda(m)$  where the sum ranges over all partitions of  $a$  of shape  $\lambda$  with columns whose lengths lie in  $\{d_1, \dots, d_s\}$ . Viewing  $V^{\oplus m}$  by identifying  $Z_{i,j}$  with the  $i^{\text{th}}$  basis vector of the  $j^{\text{th}}$  copy of  $V$ , we have by [Ful97, Corollary 3(a), §8.3] that  $\mathbb{C}[Z]_a = \text{Sym}^a(V^{\oplus m}) \simeq \bigoplus_{\lambda \vdash a} (V^\lambda)^{d_\lambda(m)}$  where  $\lambda \vdash a$  has at most  $n$  rows. Thus, we would like to show that  $(V^\lambda)^G$  has dimension 1 when the lengths of the columns of  $\lambda$  lie in  $\{d_1, \dots, d_s\}$  and 0 otherwise.

We recall the construction of  $V^\lambda$  in §8.1 of [Ful97]. Elements of  $V^{\times \lambda}$  are specified by specifying an element of  $V$  for each box in  $\lambda$ . Fillings by basis vectors  $\{e_1, \dots, e_n\}$  corresponding to semistandard Young Tableaux  $T$  of shape  $\lambda$  with entries in  $[n]$ . The images of such elements in  $V^{\times \lambda}$  in  $V^\lambda$  form a basis  $\{e_T\}$  of  $V^\lambda$ . Consider the basis element corresponding to the tableaux  $U(\lambda)$  given by filling every box on row  $i$  with the number  $i$ . For maps in  $G$ , the first  $d_i$  basis vectors must map to linear combinations of the first  $i$  basis vectors and the restrictions of such maps to the  $V_i$  have determinant 1. As such, we can only consider  $\lambda$  whose columns have lengths lying in  $\{d_1, \dots, d_s\}$ . To see that  $e_{U(\lambda)}$  is the only such fixed basis vector,

(b)

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# Bibliography

- [Ful97] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997. ISBN: 0521567246. DOI: [10.1017/cbo9780511626241](https://doi.org/10.1017/cbo9780511626241).
- [Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.
- [Man01] L. Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*. Collection SMF. American Mathematical Society, 2001. ISBN: 9780821821541. URL: <https://books.google.com/books?id=yz7gyKYgIuwC>.