

Using Algebraic Geometry

With 0 Figures

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Preface

At the time of writing this, I am starting my PhD at The Ohio State University. Currently a large part of my interests in algebra are about algorithms as they relate to polynomials and algebraic geometry. I've been doing a bunch of problems from *Ideals, Varieties, and Algorithms* [CLO15]. However, it seems that *Using Algebraic Geometry* [CLO05] Ex. moves through the material faster as it assumes you know more algebra. So I've moved onto working through this book as well as trying to comprehend Sturmfel's *Algorithms in Invariant Theory* [Str08].

Chapter 1

Introduction

1.1 Polynomials and Ideals

Solution: [CLO05] Ex. 1.1.1:

- (a) We have that $x(x - y^2) + y(xy) = x^2 - xy^2 + xy^2 = x^2$.
- (b) It suffices to check for generators. We have that $x + (-1)(y^2) = x - y^2$, $y(x) = xy$, and $y^2 = y^2$ showing that $\langle x - y^2, xy, y^2 \rangle \subseteq \langle x, y^2 \rangle$. Then $x - y^2 + y^2 = x$ and $y^2 = y^2$ shows the reverse containment and overall the ideals are equal.
- (c) We already know from 1. that x^2 lives in $\langle x - y^2, xy \rangle$. Since $xy = xy$, we overall have that $\langle x^2, xy \rangle \subseteq \langle x - y^2, xy \rangle$. It remains to check if $x - y^2 \in \langle x^2, xy \rangle$. However, notice that every element of $\langle x^2, xy \rangle$ is divisible by x while $x - y^2$ is clearly not divisible by x . Thus $x - y^2 \notin \langle x^2, xy \rangle$ and the two ideals are not equal.

□

Solution: [CLO05] Ex. 1.1.2: Let $f, g \in \langle f_1, \dots, f_s \rangle$. Then $\exists p_1, \dots, p_s, q_1, \dots, q_s$ such that $f = \sum_{i=1}^s p_i \cdot f_i$ and $g = \sum_{i=1}^s q_i \cdot f_i$. Thus $f + g = \sum_{i=1}^s (p_i + q_i) \cdot f_i$ which shows that $f + g \in \langle f_1, \dots, f_s \rangle$. Then let $p \in k[x_1, \dots, x_n]$. We have that $p \cdot f = p \cdot \sum_{i=1}^s p_i f_i = \sum_{i=1}^s (p \cdot p_i) \cdot f_i$ which shows that $\langle f_1, \dots, f_s \rangle$ is an ideal. □

Solution: [CLO05] Ex. 1.1.3: We already know that $\langle f_1, \dots, f_s \rangle$ is an ideal by [CLO05] Ex. 1.1.2. Now suppose that J is an ideal containing $\{f_1, \dots, f_s\}$. Then, since ideals are closed under addition and scaling, we have that for all $p_1, \dots, p_s \in k[x_1, \dots, x_n]$ that $\sum_{i=1}^s p_i \cdot f_i \in J$. Thus, $\langle f_1, \dots, f_s \rangle \subseteq J$. □

Solution: [CLO05] Ex. 1.1.4: We claim that $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$ if and only if $\{g_1, \dots, g_t\} \subseteq I$ and $\{f_1, \dots, f_s\} \subseteq J$. The forward implication is immediate. Then by [CLO05] Ex. 1.1.3, if $\{g_1, \dots, g_t\} \subseteq I$ then $J \subseteq I$. Similarly, $\{f_1, \dots, f_s\} \subseteq J \implies I \subseteq J$ and overall $I = J$. This fact was used in [CLO05] Ex. 1.1.1 (b). □

Solution: [CLO05] Ex. 1.1.5: It suffices to show that $z - x^3 \in \langle y - x^2, z - xy \rangle$ and $z - xy \in \langle x - y^2, z - x^3 \rangle$. Indeed we have that $(z - xy) + x(y - x^2) = z - x^3$ which also yields that $z - xy = z - x^3 - x(y - x^2)$. \square

Solution: [CLO05] Ex. 1.1.6: If $I = \{0\}$ then $I = \langle 0 \rangle$. So suppose $I \neq 0$. Let $d \in I$ be of minimal degree. **$\langle d = \gcd(I)$ but I need infinite Bezout. \rangle** Then we claim that $\langle d \rangle = I$. Since $d \in I$, we have that $\langle d \rangle \subseteq I$. Now let $f \in I$. By Euclidean division, there exists $q, r \in k[x]$ such that $f = qd + r$ where either $r = 0$ or $0 \leq \deg(r) < \deg(d)$. If $r = 0$ then $f \in \langle d \rangle$ and we are done. So suppose $r \neq 0$. Then $f, qd \in I \implies r = f - qd \in I$. Thus, $r \in I$ is of degree strictly less than d , contradicting the minimality of the degree of d . So we must have that $r = 0$ and overall $\langle d \rangle = I$. \square

Solution: [CLO05] Ex. 1.1.7:

- (a) Suppose $f(x) \in \langle x \rangle$. Then $f(x)^m \in \langle x^n \rangle$ so $f(x) \in \sqrt{\langle x^n \rangle}$. Now suppose that $f(x) \in \sqrt{\langle x^n \rangle}$. Then $\exists k$ such that $f(x)^k \in \langle x^n \rangle$. Thus $f(x)^k$ is a multiple of x^n . This implies that $f(x)^k$ is a multiple of x . Then notice that the unique factorization of $f(x)^k$ into irreducibles is the k th power of the factorization of $f(x)$ into irreducibles. Thus x must be a factor of $f(x)$ and so $f(x) \in \langle x \rangle$. Note, this heavily uses the fact that $k[x]$ is a unique factorization domain for all fields k .
- (b) We claim that $\sqrt{\langle p(x) \rangle} = \langle (x - a_1) \cdots (x - a_m) \rangle = I$. Suppose $f(x) \in I$. Let $k = \max e_1, \dots, e_n$. Then $p(x) \mid f(x)^k$ so $f(x) \in \sqrt{\langle p(x) \rangle}$. Now suppose that $f(x) \in \sqrt{\langle p(x) \rangle}$. Then $\exists k$ such that $f(x)^k \in \langle p(x) \rangle$. Thus $f(x)^k$ is a multiple of each $(x - a_i)$. Then notice that the unique factorization of $f(x)^k$ into irreducibles is the k th power of the factorization of $f(x)$ into irreducibles. Thus $f(x)$ is a multiple of each $(x - a_i)$ and so $f(x) \in I$.
- (c) Radical ideals are the ideals I such that $\sqrt{I} = I$. Notice that $\mathbb{C}[x]$ is a principal ideal domain and so any such I must be generated by a single polynomial. Since every polynomial in $\mathbb{C}[x]$ splits into linear factors, (b) immediately implies that the only radical ideals of $\mathbb{C}[x]$ are the ones which are of the form $\langle (x - a_1) \cdots (x - a_m) \rangle$ for $a_1, \dots, a_m \in \mathbb{C}$. \square

Solution: [CLO05] Ex. 1.1.8:

- (a) Let \mathfrak{p} be a prime ideal in $k[\bar{x}]$. Clearly $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$ always. Let $f(\bar{x}) \in \sqrt{\mathfrak{p}}$. Then $f(\bar{x})^m \in \mathfrak{p}$ for some $m \in \mathbb{Z}_{\geq 1}$. We prove the reverse inclusion by induction on m . If $m = 1$ then $f(\bar{x}) = f(\bar{x})^1 \in \mathfrak{p}$. Now let $m > 1$ and suppose the claim holds for all $k \leq m$. Then suppose $f(\bar{x})^{m+1} \in \mathfrak{p}$. Then $f(\bar{x}) \cdot f(\bar{x})^m \in \mathfrak{p}$. Either $f(\bar{x}) \in \mathfrak{p}$ or $f(\bar{x})^m \in \mathfrak{p}$ which by induction implies that $f(\bar{x}) \in \mathfrak{p}$. Thus, $f(\bar{x})^m \in \mathfrak{p} \implies f(\bar{x}) \in \mathfrak{p}$ for all $m \in \mathbb{Z}_{\geq 1}$ and so $\sqrt{\mathfrak{p}} \subseteq \mathfrak{p}$. Thus, all prime ideals are radical.
- (b) Notice that for all fields k that $k[x]$ is a principal ideal domain. Thus, all the prime ideals are the ones generated by a single irreducible polynomial. Also, in $k[x]$ we have that (0) is a prime ideal as well as $k[x]$ is an integral domain. In $\mathbb{C}[x]$, these are the ideals generated by $x - z$ for some $z \in \mathbb{C}$. In $\mathbb{R}[x]$, the primes are the ideals generated by $x - r$ for some $r \in \mathbb{R}$ or $x^2 + r$ for some positive $r \in \mathbb{R}$. **⟨ What would be a general condition for $\mathbb{Q}[x]$? ⟩**

□

Solution: [CLO05] Ex. 1.1.9:

- (a) First, observe that $\langle x_1, \dots, x_n \rangle$ is the ideal consisting exactly of polynomials which have no constant term. Let I be an ideal in $k[x_1, \dots, x_n]$ such that $\langle x_1, \dots, x_n \rangle \subsetneq I$. Thus there exists $f(x_1, \dots, x_n) \in I \setminus \langle x_1, \dots, x_n \rangle$. We have by our observation that f has a nonzero constant term z . Then note that the non-constant terms of f form a polynomial $g(x_1, \dots, x_n)$ in $\langle x_1, \dots, x_n \rangle$. Thus, we have that $z = f(x) - g(x) \in I$. Since I contains a nonzero constant term, we must have that $I = k[x_1, \dots, x_n]$.
- (b) Recall that an ideal I is maximal if and only if R/I is a field. Let $I = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. Consider the evaluation map $\text{ev}_{\bar{a}}: k[x_1, \dots, x_n] \rightarrow k$ sending $f(x_1, \dots, x_n) \mapsto f(a_1, \dots, a_n)$. Clearly this map is surjective. Then since for all i we have that $x_i \equiv a_i \pmod{I}$, we have that $f(x_1, \dots, x_n) \equiv f(a_1, \dots, a_n) \pmod{I}$ for all $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. Thus, $\text{ev}_{\bar{a}}(f) = f(a_1, \dots, a_n) = 0$ if and only if $f(x_1, \dots, x_n) \in I$. Thus, $\ker(\text{ev}_{\bar{a}}) = I$ and $k[x_1, \dots, x_n]/I$ is a field, meaning $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal.
- (c) Since $\mathbb{R}[x]$ is a principal ideal domain, any ideal I strictly containing $\langle x^2 + 1 \rangle$ is of the form $\langle g(x) \rangle$ for some $g(x) \mid x^2 + 1$. However, since $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, we have that $g(x)$ is either $z(x^2 + 1)$ for some nonzero $z \in \mathbb{C}$ or $g(x) = z$ for some nonzero $z \in \mathbb{C}$, meaning $\langle g(x) \rangle = \langle x^2 + 1 \rangle$ or $\langle g(x) \rangle = \mathbb{R}[x]$. Thus, $\langle x^2 + 1 \rangle$ is maximal. However, in $\mathbb{C}[x]$, we have that $x^2 + 1 = (x + i)(x - i)$ and so $\langle x^2 + 1 \rangle \subsetneq \langle x - i \rangle \subsetneq \mathbb{C}[x]$.

□

Solution: [CLO05] Ex. 1.1.10:

- (a) Since $x^2 + y^2 - (x^2 - z^3) = y^2 + z^3$ is an element of I which does not depend on x , $y^2 + z^3$ is in I_1 .
- (b) For all $\ell \geq 1$, we have that $0 \in I_\ell$. Then, if $f(x_{\ell+1}, \dots, x_n), g(x_{\ell+1}, \dots, x_n)$ are two polynomials in I who do not depend on the first ℓ variables, then so is $f + g$. Finally, let $r(x_{\ell+1}, \dots, x_n) \in k[x_{\ell+1}, \dots, x_n]$. Then $r \cdot f \in I_\ell$ since $r \cdot f \in I$ and still does not depend on any of the first ℓ variables.

□

Solution: [CLO05] Ex. 1.1.11:

- (a) **<< meh >>**
- (b) **<< meh >>**
- (c) We claim that $I + J = \langle f_1, \dots, f_s, g_1, \dots, g_t \rangle$. Clearly $I, J \subseteq \langle f_1, \dots, f_s, g_1, \dots, g_t \rangle$ and thus so is $I \cup J$. By (b), this shows that $I + J \subseteq \langle f_1, \dots, f_s, g_1, \dots, g_t \rangle$. Then, since $f_i = f_i + 0$ and $g_j = 0 + g_j$ for all i, j , we have the reverse inclusion and thus the two ideals are equal.

□

Solution: [CLO05] Ex. 1.1.12:

- (a) **<< meh >>**
- (b) Suppose that $h(\bar{x}) \in IJ$. Note that IJ is generated by the products $f(\bar{x}) \cdot g(\bar{x})$ for $f(\bar{x}) \in I$, and $g(\bar{x}) \in J$. Then $h(\bar{x})$ consists of sums of terms of the form $r(\bar{x}) \cdot f(\bar{x}) \cdot g(\bar{x})$ for $r(\bar{x}) \in k[\bar{x}]$, $f(\bar{x}) \in I$, and $g(\bar{x}) \in J$. Thus, each term is in both I and J and overall so is $h(\bar{x})$.

To see an example where $IJ \subsetneq I \cap J$, consider $I = \langle x^2y \rangle$ and $J = \langle xy^2 \rangle$ in $k[x, y]$. Then $I \cap J = \langle x^2y^2 \rangle$ and $IJ = \langle x^3y^3 \rangle$. Thus $IJ \subsetneq I \cap J$ as $I \cap J$ contains x^2y^2 and IJ does not contain x^2y^2 .

□

1.2 Gröbner Bases

Solution: $\langle \text{[CLO05] Ex. 1.3.11} \rangle :$



1.3 Affine Varieties

Solution: $\langle \text{[CLO05 Ex. 1.4.9]} \rangle :$



Chapter 2

Solving Polynomial Equations

2.1 Solving Polynomial Systems by Elimination

Solution: $\langle\langle$ [CLO05] Ex. 2.1.1 $\rangle\rangle$:

□

Solution: $\langle\langle$ [CLO05] Ex. 2.1.2 $\rangle\rangle$:

□

Solution: [CLO05] Ex. 2.1.3: We may freely rewrite the polynomial as $p(z) = z^n - a_{n-1}z^{n-1} - \dots - a_0$. We have that $0 = \bar{z}^n - a_{n-1}\bar{z}^{n-1} - \dots - a_0$ and so $\bar{z}^n = a_{n-1}\bar{z}^{n-1} + \dots + a_0$. Suppose now that $|\bar{z}| \geq 1$. Then

$$|\bar{z}|^n = |a_{n-1}\bar{z}^{n-1} + \dots + a_0| \leq |a_{n-1}||\bar{z}|^{n-1} + \dots + |a_0| \leq |a_{n-1}||\bar{z}|^{n-1} + \dots + |a_0||\bar{z}|^{n-1}.$$

Thus, $|\bar{z}| \leq |a_{n-1}| + \dots + |a_0|$. However, we assumed that $|\bar{z}| \geq 1$. This may not be the case. Thus, $|\bar{z}| \leq B := \max\{1, |a_{n-1}| + \dots + |a_0|\}$. □

Solution: $\langle\langle$ [CLO05] Ex. 2.1.4 $\rangle\rangle$: Numerically find all roots of $2z^6 + 2z^5 - z^4 - z^3 - 2z^2 - 2z - 2$. □

Solution: [CLO05] Ex. 2.1.5: We apply Buchberger's Criterion. Let $f(x, y) = x^2 + 2x + 3 + y^5 - y$ and $g(x, y) = y^6 - y^2 + 2y$. Then we have that

$$S(f, g) = \frac{x^2 y^6}{x^2} \cdot (x^2 + 2x + 3 + y^5 - y) - \frac{x^2 y^6}{y^6} \cdot (y^6 - y^2 + 2y) = y^6 \cdot (x^2 + 2x + 3 + y^5 - y) - x^2 \cdot (y^6 - y^2 + 2y).$$

This shows that $\overline{S(f, g)}^G = 0$ which yields that G is a Gröbner basis. □

Solution: << [CLO05] Ex. 2.1.6 >> : □

Solution: << [CLO05] Ex. 2.1.7 >> : □

Solution: [CLO05] Ex. 2.1.8:

- (a) Let \bar{z} be a simple root of $p(z)$, so $p(\bar{z}) = 0$ but $p'(\bar{z}) \neq 0$. Then $N_p(\bar{z}) = \bar{z} - \frac{p(\bar{z})}{p'(\bar{z})} = \bar{z}$ meaning \bar{z} is a fixed point of $N_p(z)$.
- (b) Suppose that \bar{z} is a multiple root of $p(z)$ with multiplicity $m \geq 2$. Then we may express $p(z) = \tilde{p}(z)(z - \bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Thus, we have that

$$\begin{aligned} N_p(z) &:= z - \frac{p(z)}{p'(z)} \\ &= z - \frac{\tilde{p}(z)(z - \bar{z})^m}{\tilde{p}'(z)(z - \bar{z})^m + m\tilde{p}(z)(z - \bar{z})^{m-1}} = z - \frac{\tilde{p}(z)(z - \bar{z})}{\tilde{p}'(z)(z - \bar{z}) + m\tilde{p}(z)} \end{aligned}$$

Note that $m\tilde{p}(\bar{z}) \neq 0$. Thus, we have that

$$|N_p(\bar{z})| = \left| \bar{z} - \frac{\tilde{p}(\bar{z})(\bar{z} - \bar{z})}{\tilde{p}'(\bar{z})(\bar{z} - \bar{z}) + m\tilde{p}(\bar{z})} \right| = |\bar{z}| \leq \text{LC}(p) \cdot B$$

where B is the value from [CLO05] Ex. 2.1.3 and $\text{LC}(p)$ is the leading coefficient of $p(z)$.

- (c) Suppose now that \bar{z} is a simple root of $p(\bar{z})$. Then we may express $p(z) = \tilde{p}(z)(z - \bar{z})$ such that $\tilde{p}(\bar{z}) \neq 0$. We have that

$$p'(z) = \tilde{p}'(z)(z - \bar{z}) + \tilde{p}(z)$$

and evaluation of $p'(z)$ at \bar{z} is nonzero.

- (d) Let \bar{z} be a root of multiplicity m . Following (b), we write $p(z) = \tilde{p}(z)(z - \bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Then we have, by differentiating the expression for $N_p(z)$ from (b), that

$$N'_p(z) = 1 - \frac{(\tilde{p}'(z)(z - \bar{z}) + \tilde{p}(z))(\tilde{p}'(z)(z - \bar{z}) + m\tilde{p}(z)) - (\tilde{p}(z)(z - \bar{z}))(\tilde{p}''(z)(z - \bar{z}) + \tilde{p}'(z) + m\tilde{p}'(z))}{(\tilde{p}'(z)(z - \bar{z}) + m\tilde{p}(z))^2}.$$

Evaluation at $z = \bar{z}$ yields that $\lim_{z \rightarrow \bar{z}} N'_p(z) = 1 - \frac{1}{m}$.

- (e) Let \bar{z} be a root of multiplicity m . Following (b), we write $p(z) = \tilde{p}(z)(z - \bar{z})^m$ such that $\tilde{p}(\bar{z}) \neq 0$. Then

$$p'(z) = \tilde{p}'(z)(z - \bar{z})^m + m\tilde{p}(z)(z - \bar{z})^{m-1} = (z - \bar{z})^{m-1}(\tilde{p}'(z)(z - \bar{z}) + m\tilde{p}(z)).$$

Notice that $\tilde{p}'(\bar{z})(\bar{z} - \bar{z}) + m\tilde{p}(\bar{z}) = m\tilde{p}(\bar{z}) \neq 0$. Thus, a root of multiplicity $m \geq 1$ of $p(z)$ is a root of multiplicity $m - 1$ of $p'(z)$. This implies that if we have roots $\bar{z}_1, \dots, \bar{z}_k$ with multiplicities $m_1, \dots, m_k \geq 1$, then $\gcd(p(z), p'(z)) = (z - \bar{z}_1)^{m_1} \dots (z - \bar{z}_k)^{m_k}$. Thus, the polynomial $p_{\text{red}}(z) = \frac{p(z)}{\gcd(p(z), p'(z))}$ has the same roots of $p(z)$ but all with multiplicity 1 which is the best case for Newton's method.

□

Solution: [CLO05] Ex. 2.1.9:

(a) Let $p(z) = z^2 + 1$. We have that

$$N_p(z) = z - \frac{z^2 + 1}{2z} = \frac{2z^2 - z^2 + 1}{2z} = \frac{z^2 + 1}{2z} = \frac{x^2 + 2ixy - y^2 + 1}{2x + 2iy}.$$

If z is real then $y = 0$ and so $N_p(x) = \frac{x^2+1}{2x}$ which is always real. Thus, Newton's method will never reach the imaginary roots of $z^2 + 1$. However, if we begin with a guess with nonzero imaginary part, then the guess does converge as expected.

(b) **<< Just basic arithmetic not worth doing. >>**

□

Solution: [CLO05] Ex. 2.1.10: Let \bar{z} be a root of $p(z)$. Then $-\bar{z}^n = a_{n-1}\bar{z}^{n-1} + \cdots + a_0$ and so

$$\begin{aligned} |\bar{z}|^n &= |a_{n-1}\bar{z}^{n-1} + \cdots + a_0| \\ &\leq \max_i \{|a_i|\} \cdot |\bar{z}^{n-1} + \cdots + 1| \\ &\leq \max_i \{|a_i|\} \cdot (|\bar{z}|^{n-1} + \cdots + 1) \\ &= \max_i \{|a_i|\} \cdot \frac{|\bar{z}|^n + 1}{|\bar{z}| - 1} \leq \max_i \{|a_i|\} \cdot \frac{|\bar{z}|^n}{|\bar{z}| - 1}. \end{aligned}$$

Thus, $|\bar{z}|^n \leq \max_i \{|a_i|\} \cdot \frac{|z|^n}{|z|-1}$ which implies that $|z| - 1 \leq \max_i \{|a_i|\}$. Thus, $|z| \leq 1 + \max_i \{|a_i|\}$.

□

2.2 Finite Dimensional Algebras

Solution: [⟨ \[CLO05\] Ex. 2.2.1 ⟩](#) :

□

Solution: [\[CLO05\] Ex. 2.2.2](#): It is clear that $\langle p_i(x_i) \rangle \subseteq I \cap k[x_i]$. Now suppose that $f(x_i) \in I \cap k[x_i]$. Then $\deg(f(x_i))$ must be $\geq m_i$. If not, then by the minimality of m_i we would arrive at a contradiction. Now by the division algorithm, write $f(x_i) = q(x_i)p_i(x_i) + r(x_i)$ where $\deg(r_{x_i}) < m_i$. Then $r(x_i) = f(x_i) - q(x_i)p_i(x_i) \in I$ and so $r(x_i)$ must be 0 since if not, we would arrive at a contradiction of the minimality of m_i .

This gives us an algorithm to compute $p_i(x_i)$. Let I be a zero dimensional ideal and G a Gröbner basis for I . Then we know there exists m_i such that $\{1, [x_i], \dots, [x_i^{m_i}]\}$ is linearly dependent in $k[\bar{x}]/I$. In fact, we may use the Finiteness Theorem to set m_i to the smallest integer such that $x_i^{m_i} = \text{LT}(g)$ for some $g \in G$. Since $k[x_1, \dots, x_n]/I$ is a vector space, we can check linear independence in the usual way. See `code/ch2/2_2_2.sage` for a SageMath implementation of this. □

Solution: [\[CLO05\] Ex. 2.2.3](#): Let $0 \neq f(x) \in \sqrt{\langle p(x) \rangle}$. Then there exists $m \geq 1$ such that $f^m \in \langle p(x) \rangle$ and so $p(x) \mid f(x)^m$. In particular, each linear factor $(x - \bar{z})$ of $p(x)$ divides $f(x)^m$ and so divides $f(x)$ as $(x - \bar{z})$ is irreducible. Thus, $p_{\text{red}}(x) \mid f(x)$ and so $f(x) \in \langle p_{\text{red}}(x) \rangle$. Conversely, suppose $f(x) \in \langle p_{\text{red}}(x) \rangle$ so that $\langle p_{\text{red}} \rangle \mid f(x)$. Label the roots of $p(x)$ as $\bar{z}_1, \dots, \bar{z}_r$, each $\bar{z}_i \in \bar{k}$. Then for each i , $(x - \bar{z}_i) \mid f(x)$. Let m_i be the multiplicity of z_i in $p(x)$ and $m = \max\{m_1, \dots, m_r\}$. Then $p(x) \mid f(x)^m$ and so $f(x) \in \sqrt{\langle p(x) \rangle}$ □

Solution: [\[CLO05\] Ex. 2.2.4](#): We use the algorithm from [\[CLO05\] Ex. 2.2.2](#) implemented in `code/ch2/2_2_2sage`. See `code/ch2/2_2_2sage` for the code in action. □

Solution : $\langle \langle \text{[CLO05] Ex. 2.2.5} \rangle \rangle$: Then $\sqrt{I} = I + \langle x(x-1), y(y-2) \rangle$. Since $I \subseteq \sqrt{I}$, we see that $\dim \mathbb{C}[x, y]/I \geq \dim \mathbb{C}[x, y]/\sqrt{I}$. A quick SageMath computation confirms this: $\dim \mathbb{C}[x, y]/I = 9$ and $\dim \mathbb{C}[x, y]/\sqrt{I} = 2$. See code/ch2/2_2_5.sage for the code in action. Then, since $I \subseteq \sqrt{I}$ we have that $V(\sqrt{I}) \subseteq V(I)$. Notice that

$$y^4x + 3x^3 - y^4 - 3x^2 = y^4(x-1) + 3x^2(x-1) = (y^4 + 3x^2)(x-1)$$

$$x^2y - 2x^2 = x^2(y-2)$$

$$2y^4x - x^3 - 2y^4 + x^2 = 2y^4(x-1) - x^2(x-1) = (2y^4 - x^2)(x-1).$$

Thus, $(1, 2)$ and $(0, 0)$ are the only two points in $V(I)$. Since it is evident that $V(\sqrt{I})$ contains these two points, we see in this case that $V(\sqrt{I}) = V(I)$. \square

Solution: [CLO05] Ex. 2.2.6: A grevlex Gröbner basis for I is $\{y^4 - 16y^2, x^3 - x^2, -2x^2\}$. Thus, by the Finiteness Theorem, we know that for monomials $x^a y^b$ in $\mathbb{C}[x, y]/I$ we must have that $0 \leq a \leq 1$ and $0 \leq b \leq 3$. See code/ch2/2_2_6.sage for the code in action to compute the table. \square

Solution: [CLO05] Ex. 2.2.7: We implement the algorithm described in $\langle \langle \text{[CLO05] Ex. 1.3.11} \rangle \rangle$. See /code/ch2/2_2_7.sage for the code in action. \square

Solution: [CLO05] Ex. 2.2.8:

(a) See code/ch2/2_2_8.sage for the code in action.

(b) Since each of the I_j are maximal ideals and $I_j \subseteq \sqrt{I_j}$, we must have that $I = \sqrt{I_j}$. Thus $I(V(I_j)) = I_j$ and we must have that $I_j = \sqrt{I_j}$. Since each I_j is radical and $I = \bigcap_{j=1}^5 I_j$, we have by [CLO05] Ex. 2.2.7 that I is radical. \square

Solution: [CLO05] Ex. 2.2.9:

- (a) Let $f(\bar{x}) \in I + \langle p \rangle$ and let $1 \leq j \leq d$. Then $f(\bar{x}) = g(\bar{x}) + h(\bar{x})p(x_1)$ for some $g(\bar{x}) \in I$ and $h(\bar{x}) \in k[\bar{x}]$. We have that $(x_1 - a_j) \mid p(x_1)$ and so $h(\bar{x})p(x_1) \in \langle x_1 - a_j \rangle$. Thus, $f(\bar{x}) = g(\bar{x}) + h(\bar{x})p(x_1) \in I + \langle x_1 - a_j \rangle$. As j was arbitrary, we have that $f(\bar{x}) \in \bigcap_j (I + \langle x_1 - a_j \rangle)$.
- (b) Let $f(\bar{x}) \in p_j \cdot (I + \langle x_1 - a_j \rangle)$. Then $f(\bar{x}) = p_j(x_1) \cdot (g(\bar{x}) + h(\bar{x})(x_1 - a_j))$ for some $g(\bar{x}) \in I$ and $h(\bar{x}) \in k[\bar{x}]$. We have that $p_j(x_1)g(\bar{x}) \in I$ and $p_j(x_1)h(\bar{x})(x_1 - a_j) = h(\bar{x})p(x_1) \in \langle p \rangle$. Thus, $f(\bar{x}) = p_j(x_1)g(\bar{x}) + h(\bar{x})p(x_1) \in I + \langle p \rangle$.
- (c) Let $d = \gcd(p_1, \dots, p_d)$. Then as $d \mid p_1$ and $d \mid p_2$, we have that $d \mid \prod_{j \neq 1, 2} (x_1 - a_j)$. Continuing on inductively, we have that for all $c \leq d$ that $d \mid \prod_{j \notin [c]} (x_1 - a_j)$. In particular, this means that $d \mid \prod_{j \notin [d]} (x_1 - a_j) = 1$. Thus, d itself is a unit in $k[\bar{x}]$ and p_i and p_j are coprime. By Bezout's Lemma, there exists polynomials $h_1, \dots, h_d \in k[\bar{x}]$ such that $1 = \sum_{j=1}^d h_j(\bar{x})p_j(x_1)$.
- (d) Now let $h(\bar{x}) \in \bigcap_{j=1}^d (I + \langle x_1 - a_j \rangle)$. As all the p_j are coprime, we have that there exist polynomials $h_1, \dots, h_d \in k[\bar{x}]$ such that $1 = \sum_{j=1}^d h_j(\bar{x})p_j(x_1)$. Thus, $h = \sum_{j=1}^d h_j(\bar{x})p_j(x_1)h(\bar{x})$. Then for all $1 \leq j \leq d$, we have that as $p_j(x_1)h(\bar{x}) \in p_j \cdot (I + \langle x_1 - a_j \rangle) \subseteq I + \langle p \rangle$. Thus, each summand of $\sum_{j=1}^d h_j(\bar{x})p_j(x_1)h(\bar{x})$ is in $I + \langle p \rangle$ and so overall $h \in I + \langle p \rangle$.

□

Solution: [CLO05] Ex. 2.2.10:

- (a) Let $\bar{f}^G = \sum_{j=1}^d c_j(f)x^{\alpha(j)}$ and $\bar{g}^G = \sum_{j=1}^d c_j(g)x^{\alpha(j)}$. Then by combining like terms, we have that $\bar{f}^G + \bar{g}^G = \sum_{j=1}^d (c_j(f) + c_j(g))x^{\alpha(j)}$. On the other hand, we have that $\overline{f+g}^G = \sum_{j=1}^d c_j(f+g)x^{\alpha(j)}$. Since $\bar{f}^G + \bar{g}^G = \overline{f+g}^G$ and each of the $x^{\alpha(j)}$ are linearly independent, we may equate coefficients and conclude that $c_j(f) + c_j(g) = c_j(f+g)$. For $\lambda \in k$, $\overline{\lambda f}^G = \sum_{j=1}^d c_j(\lambda f)x^{\alpha(j)}$. Now notice that $\overline{\lambda f}^G = \lambda \bar{f}^G$ as we are working over a field. Thus, we have by equating coefficients that $c_j(\lambda f) = \lambda c_j(f)$. Thus, c_j is a linear function $A \rightarrow k$.
- (b) Let $\alpha_j \in A^*$ be the linear map $\alpha_j(f) = c_j(f)$. Notice that for all $1 \leq i, j \leq d$ we have that $\alpha_j(x^{\alpha(i)}) = c_j(x^{\alpha(i)}) = \delta_{i,j}$. Suppose there exists $\lambda_1, \dots, \lambda_d \in k$ such that $\lambda_1 \alpha_1 + \dots + \lambda_d \alpha_d = 0$. Then for all $1 \leq i \leq d$ we have that

$$0 = \left(\sum_{j=1}^d \lambda_j \alpha_j \right) (x^{\alpha(i)}) = \sum_{j=1}^d \lambda_j \alpha_j(x^{\alpha(i)}) = \lambda_i$$

and so for all $1 \leq i \leq d$, $\lambda_i = 0$ meaning that $\{\alpha_1, \dots, \alpha_d\}$ is linearly independent. Since we know that $d = \dim A = \dim A^*$, we have that $\{\alpha_1, \dots, \alpha_d\}$ is a basis for A^* .

- (c) This was proven in (b).

□

Solution: [CLO05] Ex. 2.2.11:

- (a) We want a linear polynomial $\ell(\bar{x}) = \ell_1 x_1 + \cdots + \ell_n x_n$ takes distinct values at each of the $p_i \in \mathbb{C}^n$. Consider the space of all such (ℓ_1, \dots, ℓ_n) . This itself is a \mathbb{C} vector space, call it L . Let $L_{i,j}$ be the subspace of L corresponding to polynomials $\ell(\bar{x})$ such that $\ell(p_i) = \ell(p_j)$. There are finitely many such $L_{i,j}$. We know that vector spaces over an infinite field cannot be expressed as the finite union of proper subspaces. Thus, $L \neq \bigcup_{1 \leq i \neq j \leq m} L_{i,j}$. This means there exists $(\ell_1, \dots, \ell_n) \in L \setminus \bigcup_{1 \leq i \neq j \leq m} L_{i,j}$ such that $\ell(\bar{x}) = \ell_1 x_1 + \cdots + \ell_n x_n$ takes distinct values at each of the p_i .

<< Can we do this constructively? >>

- (b) Let $\ell(\bar{x})$ be our constructed polynomial from (a). For $1 \leq i \leq m$, we define $g_i \in \mathbb{C}[x_1, \dots, x_n]$ as

$$g_i(\bar{x}) = \frac{\prod_{1 \leq i \neq j \leq m} \ell(\bar{x}) - \ell(\bar{p}_j)}{\prod_{1 \leq i \neq j \leq m} \ell(\bar{p}_i) - \ell(\bar{p}_j)}.$$

Then clearly $g_i(p_j) = \delta_{ij}$ as desired.

□

Solution: [CLO05] Ex. 2.2.12:

- (a) Clearly the map is linear. We now show it is well defined. Let $[f] = [g] \in \mathbb{C}[\bar{x}]/I$ meaning that $f - g \in I$. Thus, we have that

$$\varphi([f]) - \varphi([g]) = \varphi([f - g]) = 0 \implies \varphi([f]) = \varphi([g])$$

as desired.

- (b) Let $[f], [g] \in \mathbb{C}[\bar{x}]/I$. Then we have

$$\begin{aligned} \varphi([f] \cdot [g]) &= \varphi([f \cdot g]) \\ &= ((f \cdot g)(p_1), \dots, (f \cdot g)(p_m)) \\ &= (f(p_1) \cdot g(p_1), \dots, f(p_m) \cdot g(p_m)) \\ &= (f(p_1), \dots, f(p_m)) \cdot (g(p_1), \dots, g(p_m)) = \varphi([f]) \cdot \varphi([g]). \end{aligned}$$

Thus, φ is a homomorphism of rings. In fact, it is a homomorphism of \mathbb{C} -algebras as $(\lambda \cdot f)(x) := \lambda \cdot f(\bar{x})$ for all $\lambda \in \mathbb{C}$ and $f \in \mathbb{C}[\bar{x}]$ and this descends to $\mathbb{C}[\bar{x}]/I$.

- (c) We have that φ is surjective and that $I \subseteq \ker(\varphi)$ as in the proof of [CLO05, Theorem 2.10]. So we want to show that $\ker(\varphi) \subseteq I$ if and only if $I = \sqrt{I}$ which holds exactly as in the proof of [CLO05, Theorem 2.10].

□

Bibliography

- [CLO05] D.A. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Springer-Verlag, 2005. ISBN: 0387207066. DOI: [10.1007/b138611](https://doi.org/10.1007/b138611). URL: <http://dx.doi.org/10.1007/b138611>.
- [CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. ISBN: 9783319167213. URL: <https://books.google.com/books?id=yL7yCAAQBAJ>.
- [Str08] Bernd Strumfels. *Algorithms in Invariant Theory*. Springer Vienna, 2008. ISBN: 9783211774175. DOI: [10.1007/978-3-211-77417-5](https://doi.org/10.1007/978-3-211-77417-5). URL: <http://dx.doi.org/10.1007/978-3-211-77417-5>.