

Algebraic Curves

With 0 Figures

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Last Edited on 8/30/24 at 02:37

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Preface

At the time of writing this, I am starting my PhD at The Ohio State University. Currently a large part of my interests in algebra are about algorithms as they relate to polynomials and algebraic geometry. As such, I've been primarily looking at *Ideals, Varieties, and Algorithms* [CLO15], I've been doing a bunch of problems from *Ideals, Varieties, and Algorithms* [CLO15], *Using Algebraic Geometry* [CLO05] and stealing glances at Sturmfel's *Algorithms in Invariant Theory* [Str08]. However, this book is such a classic and I'm of the opinion that you can never know too much about curves and varieties. To put my money where my mouth is, I've started to \TeX up some exercises from Fulton's *Algebraic Curves* [Ful08].

Chapter 1

Affine Algebraic Sets

1.1 Algebraic Preliminaries

Solution: [Ful08] Ex. 1.1: As normal, we denote a monomial $x_1^{i_1} \cdots x_n^{i_n}$ as \bar{x}^I where $|I| = i_1 + \cdots + i_n = d \in \mathbb{N}$ and $I \in \mathbb{N}^n$. Thus, if $f(x_1, \dots, x_n), g(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ are forms of degree r and s respectively, we have that

$$\begin{aligned} f(x_1, \dots, x_n) &= \sum_{I \in \mathbb{N}^n, |I|=r} f_I \bar{x}^I, \\ g(x_1, \dots, x_n) &= \sum_{J \in \mathbb{N}^n, |J|=s} g_J \bar{x}^J. \end{aligned}$$

Then, we have by distribution that

$$f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n) = \sum_{\substack{I \in \mathbb{N}^n, |I|=r \\ J \in \mathbb{N}^n, |J|=s}} f_I \bar{x}^I g_J \bar{x}^J.$$

For each I, J we have that if $f_I, g_J \neq 0$ then $f_I g_J \bar{x}^I \bar{x}^J = (f g)_{I,J} \bar{x}^{I+J}$ where $(f g)_{I,J} = (f g)_{I,J} = f_I g_J$. Notice that $|I + J| = r + s$ by coordinatewise addition and so each term is either 0 or a degree $r + s$ homogeneous form. Thus after combining like terms, we have that $f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)$ is either degree 0 or a degree $r + s$ homogeneous form. As 0 has any degree we choose, we have that $f \cdot g$ is a degree $r + s$ form.

If we wanted to show that the product of two nonzero polynomials is still nonzero, it can be shown that R a domain implies $R[x_1, \dots, x_n]$ is a domain. This can be seen by looking at leading terms and inducting on the number of variables. □

Solution: [Ful08] Ex. 1.2: The expression of $z \in K = \text{Frac}(R)$ as $z = \frac{a}{b}$ for some $a \in R, b \in R$ such that a, b have no common factors, is obvious as we only consider finite sums so we can always take common denominators. Suppose that $z = \frac{a}{b} = \frac{a'}{b'}$ two different representatives. Since R is a UFD, say that $a = u_a a_1 \cdots a_{n_a}$ and $a' = u_{a'} a'_1 \cdots a'_{n_{a'}}$ are factorizations of a and a' . We have similar factorizations for b and b' . Then we have that $ab' = a'b \in R$. However, as R is a unique factorization domain, we must have

$$u_a a_1 \cdots a_{n_a} u_{b'} b'_1 \cdots b'_{n_{b'}} = u_{a'} a'_1 \cdots a'_{n_{a'}} u_b b_1 \cdots b_{n_b}$$

are two equivalent factorizations up to unit. This implies that $\frac{a}{b}$ and $\frac{a'}{b'}$ differ only up to unit. \square

Solution: [Ful08] Ex. 1.3:

- (a) We know that $P = (p)$ for some nonzero, nonunit $p \in R$. Suppose that p was reducible, so $p = ab$ for some nonzero nonunit $a, b \in R$. Then as $p \nmid a$ and $p \nmid b$, since such divisibility would imply that a or b were equal to p up to unit, we have that $a, b \notin (p)$. But then we have elements $a, b \in R$ such that $a \notin (p)$, $b \notin (p)$ but $ab = p \in (p)$ which contradicts the primality of $P = (p)$. Then p is irreducible.
- (b) We show that $P = (p)$ is maximal. There exists a maximal ideal $M = (m)$ such that $(p) \subseteq (m)$. Thus, $p \mid m$. Notice that by (a) that both p and m are irreducible elements of R . As such, if $p \mid m$ we must have that $m \mid p$ as if not, m would be reducible. Thus, $p = um$ for some unit $u \in R^\times$. Thus, $(p) = (m)$ and $P = (p)$ is maximal.

\square

Solution: [Ful08] Ex. 1.4: We prove by induction on the number of variables n . If $n = 1$ then any nonzero polynomial of degree d has at most d distinct zeros. Thus if $F(x) \in k[x]$ is a nonzero polynomial, there exists a $z \in k$ such that $F(z) \neq 0$. Now let $n \geq 1$. Let $F(x_1, \dots, x_n, y) \in k[x_1, \dots, x_n, y]$ be a nonzero polynomial. Letting $d = \deg_y(F)$, we can find $F_0[x_1, \dots, x_n], \dots, F_d[x_1, \dots, x_n]$ such that

$$F(x_1, \dots, x_n, y) = \sum_{i=0}^d F_i(x_1, \dots, x_n) \cdot y^i.$$

Regard $F(x_1, \dots, x_n, y)$ as a nonzero polynomial $g(y)$ in $F(x_1, \dots, x_n)[y]$. As k is infinite and $g(y)$ is nonzero, then we must have there exists $z_y \in k$ such that $g(z_y) \neq 0$. Thus, we have that $F(x_1, \dots, x_n, z_y) \in k[x_1, \dots, x_n]$ is nonzero. By induction, there exists $(z_1, \dots, z_n) \in k^n$ such that $F(z_1, \dots, z_n, z_y) \neq 0$. \square

Solution: [Ful08] Ex. 1.5: This is Euclid's proof of there being infinitely many primes. Omitted. \square

Solution: [Ful08] Ex. 1.6: Let k be an algebraically closed field. Then the irreducible monic polynomials of $k[x]$ are of the form $(x - a)$ for each $a \in k$. Thus, they are in bijection with the elements of k . By [Ful08] Ex. 1.5, there are infinitely many monic irreducible polynomials over any field. Thus, k is an infinite field. \square

Solution: [Ful08] Ex. 1.7:

(a) Let $F(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. Then we may express $F(x_1, \dots, x_n)$ as

$$F(x_1, \dots, x_n) = \sum_{I=(i_1, \dots, i_n) \in \mathbb{N}^n} \lambda_I (x_1)^{i_1} \cdots (x_n)^{i_n} = \sum_{I=(i_1, \dots, i_n) \in \mathbb{N}^n} \lambda_I ((x_1 - a_1) + a_1)^{i_1} \cdots ((x_n - a_n) + a_n)^{i_n}.$$

Expanding and combining like terms yields the desired results.

(b) Suppose $F(a_1, \dots, a_n) = 0$. By (a), we may express $F(x_1, \dots, x_n)$ as

$$F(x_1, \dots, x_n) = \sum_{I=(i_1, \dots, i_n) \in \mathbb{N}^n} \lambda_I (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$$

The claim says that this expression has no constant term, i.e. $\lambda_{\vec{0}} = 0$. This is immediate as the constant term corresponds to $F(a_1, \dots, a_n) = 0$. \square

Bibliography

- [CLO05] D.A. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Springer-Verlag, 2005. ISBN: 0387207066. DOI: [10.1007/b138611](https://doi.org/10.1007/b138611). URL: <http://dx.doi.org/10.1007/b138611>.
- [CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. ISBN: 9783319167213. URL: <https://books.google.com/books?id=yL7yCAAQBAJ>.
- [Ful08] W. Fulton. *Algebraic Curves: An Introduction to Algebraic Geometry*. 3rd ed. 2008.
- [Str08] Bernd Strumfels. *Algorithms in Invariant Theory*. Springer Vienna, 2008. ISBN: 9783211774175. DOI: [10.1007/978-3-211-77417-5](https://doi.org/10.1007/978-3-211-77417-5). URL: <http://dx.doi.org/10.1007/978-3-211-77417-5>.