



# Representation Theory Notes and Exercises

With 0 Figures

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# Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. I make 0 claims that my writing is original, and anything that is well written most likely is a transcription from Serre. Occasionally, other sources may be used, such as the set of notes by Charles Rezk created for Math 427 [Rez20]. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

# Chapter 1

## Generalities on Linear Representations

Unless otherwise specified,  $V$  will denote a vector space, usually over the field  $\mathbb{C}$ . We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

**Definition 1.1 (Linear Representation, Representation Space):** Let  $G$  be a group with identity  $e$ . A *linear representation* of  $G$  in  $V$  is a homomorphism  $\rho : G \rightarrow \text{GL}(V)$ . We will frequently, and often interchangeably, write  $\rho_s := \rho(s)$ . Given  $\rho$ , we will say that  $V$  is a *representation space* or *representation* of  $G$ .

**Definition 1.2 (Degree):** Let  $\rho : G \rightarrow V$  be a representation of  $G$  in a vector space  $V$ . Then the *degree* of  $\rho$  is  $\dim(V)$ .

Let  $\rho : G \rightarrow V$  be a representation of  $G$  in a vector space  $V$  with  $n := \dim(V)$ . Fix a basis  $(e_j)$  of  $V$ . Then since each  $\rho_s$  is an invertible linear transformation of  $V$ , we may define an  $n \times n$  matrix  $R_s \equiv (r_{ij}(s))$  where each  $r_{ij}(s)$  is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^n r_{ij}(s) e_i.$$

**Definition 1.3 (Matrix of a Representation):** We call  $R_s = (r_{ij}(s))$  above the *matrix of  $\rho_s$*  with respect to the basis  $(e_j)$ .

Note that  $R_s$  satisfies the following:

$$\det(R_s) \neq 0, \quad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^n r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two  $n \times n$  matrices  $A, A'$  are *similar* if there exists an invertible matrix  $T$  such that  $TA = A'T$ . We may extend this notion to representations.

**Definition 1.4 (Similar/Isomorphic Representations):** Let  $\rho$  and  $\rho'$  be two representations of the same group  $G$  in vector spaces  $V$  and  $V'$  respectively. We say  $\rho$  and  $\rho'$  are *similar* or *isomorphic* if there exists an isomorphism  $\tau: V \rightarrow V'$  such that for all  $s \in G$ ,  $\tau$  satisfies  $\tau \circ \rho(s) = \rho'(s) \circ \tau$ . If  $R_s, R'_s$  are the corresponding matrices then this is equivalent to saying there exists an invertible matrix  $T$  such that  $TR_s = R'_s T$  for all  $s \in G$ .

Note that if  $\rho$  and  $\rho'$  are isomorphic, then they must have the same degree.

We now give some examples of these things.

**Example 1.5 (Unit/Trivial Representation):** Let  $G$  be a finite group. Representations of degree 1 must be of the form  $\rho: G \rightarrow \mathbb{C}^\times$ . Since elements  $s$  of  $G$  are of finite order,  $\rho(s)$  must also be of finite order. Thus, for all  $s \in G$ ,  $\rho(s)$  is a root of unity. If we take  $\rho(s) = 1$  for all  $s \in G$ , we obtain the *unit* or *trivial* representation of  $G$ . This also means that  $R_s = 1$  for all  $s$ .

**Example 1.6 (Regular Representation):** Let  $g$  be the order of  $G$ , and let  $V$  be a vector space of dimension  $g$  with a basis  $(e_t)_{t \in G}$ . For each  $s \in G$ , define  $\rho_s$  as the linear map  $\rho_s: V \rightarrow V$  such that  $\rho_s(e_t) = e_{st}$ . This is a linear representation of  $G$  called the *regular* representation of  $G$ . Since for each  $s \in G$ ,  $e_s = \rho_s(e_1)$  and thus the images of  $e_1$  form a basis of  $V$ . Conversely, let  $W$  be a representation of  $G$  with a vector  $w$  satisfying the collection of all  $\rho_s(w)$ ,  $s \in G$ , forms a basis of  $W$ . Then  $W$  is isomorphic to the regular representation of  $G$  by the isomorphism  $\tau(e_s) = \rho_s(w)$ .

For example, let  $G = \mathbb{Z}_3$  and  $V = \mathbb{C}^3$  with  $e_0 = (1, 0, 0)$ ,  $e_1 = (0, 1, 0)$ , and  $e_2 = (0, 0, 1)$ . Then for example,  $\rho_0, \rho_1, \rho_2: \mathbb{C}^3 \rightarrow \mathbb{C}^3$  are the linear maps such that

$$\begin{array}{lll} \rho_0(e_0) = e_{0+0} = e_0 & \rho_0(e_1) = e_{0+1} = e_1 & \rho_0(e_2) = e_{0+2} = e_2 \\ \rho_1(e_0) = e_{1+0} = e_1 & \rho_1(e_1) = e_{1+1} = e_2 & \rho_1(e_2) = e_{1+2} = e_0 \\ \rho_2(e_0) = e_{2+0} = e_2 & \rho_2(e_1) = e_{2+1} = e_0 & \rho_2(e_2) = e_{2+2} = e_1 \end{array}$$

With this, the matrix representations of  $\rho_0, \rho_1$  and  $\rho_2$  is similarly straightforward:

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



**Example 1.7 (Permutation Representation):** We may generalize the regular representation to any group action  $G \curvearrowright X$ ,  $X$  a finite set. Recall that for such an action, the map  $x \mapsto sx$  for each  $s \in G$  is a permutation  $X \leftrightarrow X$ . Let  $V$  be a vector space with dimension the size of  $X$ , and so a basis  $(e_x)_{x \in X}$ . Define a representation  $\rho$  of  $G$  by defining  $\rho_s$  as the linear map sending  $e_x \mapsto e_{sx}$ . This representation is known as the *Permutation* representation of  $G$  associated with  $X$ . If we consider  $X = [n]$  and  $G = S_n$ , then take  $V = \mathbb{C}^n$  as our vector space and  $e_i$  as the standard basis vector. Then  $\rho_\sigma(e_j) = e_{\sigma(j)}$ . Thus for each  $\sigma \in S_n$ , we have that  $R_\sigma = (r_{ij}(\sigma))$  where entry  $r_{ij}(\sigma) = 1$  if  $i = \sigma(j)$  and 0 otherwise.

**Definition 1.8 (Stable/Invariant Subspaces, Subrepresentation):** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation and  $W \subseteq V$  a subspace of  $V$ . We say that  $W$  is *stable* under the action of  $G$  if  $x \in W$  implies that  $\rho_s(x) \in W$  for all  $s \in G$ . Thus, the restriction  $\rho_s^W := \rho_s|_W$  is an isomorphism of  $W$  onto itself. Restrictions satisfy the property that  $\rho_s^W \circ \rho_t^W = \rho_{st}^W$ . Thus,  $\rho^W: G \rightarrow \text{GL}(W)$  is a linear representation of  $G$  in  $W$  and we say that  $W$  is a *subrepresentation* of  $V$ .

**Example 1.9 (Subrepresentations of the Regular Representation):** Let  $G$  be a group. Recall the regular representation  $V$  given in Example 1.6. Let  $W$  be the 1 dimensional subspace of  $V$  generated by the element  $x = \sum_{s \in G} e_s$ . Then note that  $\rho_s(x) = x$  for all  $s \in G$  and thus  $W$  is a subrepresentation of  $V$ . Furthermore, this is isomorphic to the unit representation Example 1.5 with  $\tau: C^\times \rightarrow W$  such that  $\tau(1) = x$ . For example, let  $G = \mathbb{Z}_3$  and  $\rho: \mathbb{Z}_3 \rightarrow \text{GL}(\mathbb{C}^3)$  the representation given in Example 1.6. Then  $x = (1, 1, 1)$  and for example we have that

$$\rho_1(x) = \rho_1(e_0) + \rho_1(e_1) + \rho_1(e_2) = e_1 + e_2 + e_0 = x.$$

**Theorem 1.10:** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$  in  $V$  and let  $W$  be a subspace of  $V$  stable under  $G$ . Then there exists a complement  $W^0$  of  $W$  in  $V$  which is stable under  $G$ .

**Proof:** Let  $W'$  be an arbitrary complement of  $W$  in  $V$ , and let  $p: V \rightarrow W$  be the projection. Then we form the average  $p^0$  of conjugates of  $p$  by elements in  $G$ :

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Since  $p: V \rightarrow W$  and  $\rho_t$  preserves  $W$ , we have that  $p^0$  maps  $V$  onto  $W$ . Furthermore, note that  $\rho_t^{-1}$  also preserves  $W$ .

Thus we have that

$$(p \circ \rho_t^{-1})(x) = \rho_t^{-1}(x), \quad (\rho_t \circ p \circ \rho_t^{-1})(x) = x, \quad p^0(x) = x.$$

Thus,  $p^0$  is a projection of  $V$  onto  $W$ , corresponding to some complement  $W^0$  of  $W$ . Moreover, we have that  $\rho_s \circ p^0 = p^0 \circ \rho_s$  for all  $s \in G$  because

$$\rho_s \circ p^0 \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \circ \rho_t \circ p \circ \rho_t^{-1} \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \circ p \rho_{st}^{-1} = p^0.$$

Now suppose that  $x \in W^0$  and  $s \in G$ , we have that  $p^0(x) = 0$  and hence  $(p^0 \circ \rho_s)(x) = (\rho_s \circ p^0)(x) = 0$ , meaning that  $\rho_s(x) \in W^0$ . This,  $W^0$  is stable under  $G$ .  $\square$

Suppose that  $V$  had an inner product  $\langle x, y \rangle$ , and furthermore suppose this inner product was invariant under  $G$  meaning that for all  $s \in G$ ,  $\langle \rho_s(x), \rho_s(y) \rangle = \langle x, y \rangle$ . We may also reduce to this case by replacing  $\langle x, y \rangle$  with  $\sum_{t \in G} \langle \rho_t(x), \rho_t(y) \rangle$ . With this, the orthogonal complement  $W^\perp$  of  $W$  in  $V$  is a complement of  $W$  stable under  $G$ . Note that the invariance of  $\langle x, y \rangle$  means that if  $(e_i)$  is an orthonormal basis of  $V$ , then  $R_s$  is a unitary matrix.

Using the notation of Theorem 1.10, let  $x \in V$  and  $w, w^0$  be the projections of  $x$  on  $W$  and  $W^0$  respectively. Thus for all  $s \in G$ ,  $\rho_s(x) = \rho_s(w) + \rho_s(w^0)$ . Since  $W$  and  $W^0$  are stable under  $G$ , we have that  $\rho_s(w) \in W$  and  $\rho_s(w^0) \in W^0$ . This means that  $\rho_s(w)$  and  $\rho_s(w^0)$  are the projections of  $\rho_s(x)$  and in turn the representations of  $W$  and  $W^0$  determine the representations of  $V$ .

**Definition 1.11 (Direct Sum of Representations):** Given the above, we write  $V = W \oplus W^0$  as the *direct sum* of  $W$  and  $W^0$ . We identify elements  $v \in V$  as pairs  $(w, w^0)$  given by their projections.

If the representations  $W$  and  $W^0$  are given in matrices  $R_s$  and  $R_s^0$ , then the matrix form of the representation  $V$  is given by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

Similar results hold for arbitrarily many, but finite, direct sums of representations.

The above is a discussion on how to compose two or more representations into one larger representation. The natural question is then about the opposite.

**Definition 1.12 (Irreducible/Simple Representations):** Let  $\rho : G \rightarrow \text{GL}(V)$  be a linear representation of  $G$ . Then this representation is *irreducible* or *simple* if  $V$  has no subspaces stable under  $G$  besides 0 and  $V$  itself.

By Theorem 1.10, this is equivalent to saying that  $V$  is not the direct sum of two representations besides  $V = 0 \oplus V$ . A representation of degree 1 is reducible. We may use irreducible representations to construct other ones via the direct sum.

**Theorem 1.13:** Every representation is a direct sum of irreducible representations.

**Proof:** Let  $V$  be a linear representation of  $G$ . We induct on  $\dim(V)$ . If  $\dim(V) = 0$ , then  $V = 0$  which is the direct sum of an empty family of irreducible representations. So suppose that

$\dim(V) \geq 1$ . If  $V$  is irreducible, then we are done. Otherwise, there exists a subspace  $W \subsetneq V$  stable under  $G$  and by Theorem 1.10 a stable complement  $W^0$  such that  $V = W \oplus W^0$ . By assumption,  $W \neq 0 \neq W^0$  and so  $\dim(W) < \dim(V)$  and  $\dim(W^0) < \dim(V)$ . By induction, we have obtained a decomposition of  $V$  into irreducibles.  $\square$

**Example 1.14 (Decomposition of Representation of  $\mathbb{Z}_3$  into Irreducibles):** Recall from Example 1.6 the regular representation  $\rho : \mathbb{Z}_3 \rightarrow \text{GL}(\mathbb{C}^3)$  with  $e_0 = (1, 0, 0)$ ,  $e_1 = (0, 1, 0)$ , and  $e_2 = (0, 0, 1)$  and

$$\begin{array}{lll} \rho_0(e_0) = e_0 & \rho_0(e_1) = e_1 & \rho_0(e_2) = e_2 \\ \rho_1(e_0) = e_1 & \rho_1(e_1) = e_2 & \rho_1(e_2) = e_0 \\ \rho_2(e_0) = e_2 & \rho_2(e_1) = e_0 & \rho_2(e_2) = e_1 \end{array}$$

Our goal will be to decompose  $\rho$  into  $\rho^1 \oplus \rho^2 \oplus \rho^3$ . We aim to find the elements fixed by  $\mathbb{Z}_3$ . Note that if an element is fixed by 1, the generator of  $\mathbb{Z}_3$ , then it is fixed by all of  $\mathbb{Z}_3$ . We want to find 1-dimensional  $\mathbb{Z}_3$ -invariant subspaces of  $\mathbb{C}^3$ . This is equivalent to finding the eigenvalues of the matrix

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues and their eigenvectors of  $R_1$  are as follows, note the inclusion of the trivial subrepresentation from Example 1.9:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}, v_2 = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \quad \lambda_3 = \frac{-1 + i\sqrt{3}}{2}, v_3 = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Thus  $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$  where  $V_i := \text{span}(v_i)$ . Note that there are only 3 morphisms  $\mathbb{Z}_3 \rightarrow \mathbb{C}^\times$  mapping 1 to 1,  $\omega$ , or  $\omega^2$  where  $\omega$  is a cube root of unity. Thus  $\rho^1, \rho^2$ , and  $\rho^3$  must correspond to these morphisms **<< but which ones >>**.

A natural question is if such a decomposition  $V = W_1 \oplus \cdots \oplus W_k$  is unique. However, suppose that  $\rho$  is the trivial representation (Example 1.5). Then each component of the decomposition is a line and we can decompose a vector space into the direct sum of lines in a number of ways. However, we will see that the number of  $W_i$  that are isomorphic to a given irreducible representation does not depend on the choice of decomposition.

**Definition 1.15 (Tensor/Kronecker Product of Representations):** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two representations of a group  $G$ . We construct a representation  $\rho: G \rightarrow \text{GL}(V_1 \otimes V_2)$  such that

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \circ \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

The existence and uniqueness of  $\rho$  follow immediately from the existence and uniqueness of the tensor product. We write  $\rho_s \equiv \rho_s^1 \otimes \rho_s^2$  as the *tensor product* of the given representations.

Recall that if  $(e_{i_1})$  and  $(e_{i_2})$  be bases of  $V_1$  and  $V_2$  respectively, then  $(e_{i_1} \otimes e_{i_2})$  is a basis of  $V_1 \otimes V_2$ . If  $(r_{i_1 j_1}(s))$  and  $(r_{i_2 j_2}(s))$  are the matrices of  $\rho_s^1$  and  $\rho_s^2$  respectively satisfying

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1}(s) e_{i_1} \qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2}(s) e_{i_2}$$

then the matrix of  $\rho_s$  is  $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$  satisfying

$$\rho_s(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \otimes e_{i_2}.$$

Note that the tensor product of two irreducible representations is not in general irreducible.

**Example 1.16 (Tensor Product of Two Irreducible Representations that is not Irreducible):** Consider  $G = \mathbb{Z}/4\mathbb{Z}$ . Consider the representation  $\rho: G \rightarrow \text{GL}(\mathbb{R}^2)$  such that

$$\rho_1 = M := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since  $M$  does not have 1 as an eigenvalue, there are no  $\mathbb{Z}/4\mathbb{Z}$ -invariant subspaces. Thus,  $\rho$  is irreducible.

Now consider  $\rho' = \rho \otimes \rho$ . Let  $(e_1, e_2)$  be the standard basis of  $\mathbb{R}^2$ , and so  $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$  a basis of  $\mathbb{R}^2 \otimes \mathbb{R}^2$ . We have that

$$\begin{aligned} \rho'_1(e_1 \otimes e_1) &= M e_1 \otimes M e_1 = e_2 \otimes e_2, \\ \rho'_1(e_1 \otimes e_2) &= M e_1 \otimes M e_2 = e_2 \otimes -e_1, \\ \rho'_1(e_2 \otimes e_1) &= M e_2 \otimes M e_1 = -e_1 \otimes e_2, \\ \rho'_1(e_2 \otimes e_2) &= M e_2 \otimes M e_2 = e_2 \otimes e_2. \end{aligned}$$

Thus the matrix of  $\rho'_1$  is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that the subspace generated by  $(e_1 \otimes e_1, e_2 \otimes e_2)$  is a  $\mathbb{Z}/4\mathbb{Z}$ -invariant subspace.

We now consider the special case of  $V \otimes V$ . Let  $(e_i)$  be a basis of  $V$  and define an automorphism  $\theta$  of  $V \otimes V$  such that  $\theta(e_i \otimes e_j) = e_j \otimes e_i$ . Then note that  $\theta^2 \equiv \text{id}_{V \otimes V}$ . We may decompose  $V \otimes V$  into the direct sum

$$V \otimes V = \text{Sym}^2(V) \oplus \text{Alt}^2(V).$$

Here,  $\text{Sym}^2(V)$  is the set of  $z \in V \otimes V$  such that  $\theta(z) = z$  and  $\text{Alt}^2(V)$  is the set of  $z \in V \otimes V$  where  $\theta(z) = -z$ . These have bases  $(e_i \otimes e_j + e_j \otimes e_i)_{i \leq j}$  and  $(e_i \otimes e_j - e_j \otimes e_i)_{i < j}$  respectively. As such,  $\dim(\text{Sym}^2(V)) = \frac{n(n+1)}{2}$  and  $\dim(\text{Alt}^2(V)) = \frac{n(n-1)}{2}$  where  $n := \dim(V)$ .

**Definition 1.17 (Symmetric Square, Alternating Square):** These subspaces  $\text{Sym}^2(V)$  and  $\text{Alt}^2(V)$  of  $V \otimes V$  are respectively called the *symmetric square* and *alternating square* of the given representation.

## Chapter 2

# Character Theory

**Definition 2.1 (Character):** Let  $\rho : G \rightarrow \text{GL}(V)$  be a linear representation of a finite group  $G$  in  $V$ . Then the character  $\chi_\rho$  of  $\rho$  is the function

$$\chi_\rho(s) := \text{Tr}(R_s) \equiv \text{Tr}(\rho_s).$$

for each  $s \in G$ .

**Proposition 2.2:** If  $\chi$  is the character of a representation  $\rho$  of degree  $n$  then

1.  $\chi(e) = 1$ ;
2.  $\chi(s^{-1}) = \chi(s)^*$ , the complex conjugate of  $\chi(s)$ ,
3.  $\chi(tst^{-1}) = \chi(s)$ .

**Proof:** The first is immediate since  $\rho_1$  is the identity matrix  $I$  and  $\text{Tr}(I) = n$ . Then recall that we may choose our basis to be orthonormal, and as such  $\rho_s$  is a unitary matrix. Thus, each eigenvalue  $\lambda_1, \dots, \lambda_n$  has an absolute value equal to 1. Thus

$$\chi(s)^* = \text{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \text{Tr}(\rho_s)^{-1} = \text{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

This uses the fact that the conjugate of the sum is the sum of the conjugates as well as the fact that the eigenvalues of  $R_s^{-1}$  are the inverses of the eigenvalues of  $R_s$ . Finally, letting  $u = ts$  and  $v = t^{-1}$  allows us to write  $\chi(tst^{-1}) = \chi(s)$  as  $\chi(uv) = \chi(vu)$  which is immediate since for any complex matrices  $A, B$  we have that  $\text{Tr}(AB) = \text{Tr}(BA)$ . □

**Proposition 2.3:** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two linear representations with characters  $\chi_1$  and  $\chi_2$  respectively. Then

1. The character  $\chi$  of the direct sum representation  $V_1 \oplus V_2$  is  $\chi_1 + \chi_2$ .
2. The character  $\psi$  of the tensor product representation  $V_1 \otimes V_2$  is  $\chi_1 \cdot \chi_2$ .

**Proof:** Let  $R_s^1, R_s^2$  be the matrix forms of  $\rho_s^1$  and  $\rho_s^2$  respectively. Then the matrix form  $R_s$  of the representation of  $V_1 \oplus V_2$  is given by

$$R_s = \begin{pmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{pmatrix}$$

and thus  $\text{Tr}(R_s) = \text{Tr}(R_s^1) + \text{Tr}(R_s^2)$ . Let  $(e_{i_1})$  and  $(e_{i_2})$  be bases for  $V_1$  and  $V_2$ . Then we have that

$$\psi(s) = \sum_{i_1 i_2} r_{i_1 i_1}(s) \cdot r_{i_2 i_2}(s) = \left( \sum_{i_1} r_{i_1 i_1}(s) \right) \cdot \left( \sum_{i_2} r_{i_2 i_2}(s) \right) = \chi_1(s) \cdot \chi_2(s).$$

□

**Proposition 2.4:** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$  with character  $\chi$ . Let  $\chi_\sigma^2$  be the character of  $\text{Sym}^2(V)$  and  $\chi_\alpha^2$  be the character of  $\text{Alt}^2(V)$  from Definition 1.17. Then

$$\begin{aligned} \chi_\sigma^2(s) &= \frac{1}{2}(\chi(s)^2 + \chi(s^2)) \\ \chi_\alpha^2(s) &= \frac{1}{2}(\chi(s)^2 - \chi(s^2)) \end{aligned}$$

which directly implies that  $\chi_\sigma^2 + \chi_\alpha^2 = \chi$ .

**Proof:** Let  $s \in G$  and  $(e_i)$  a basis of  $V$  consisting solely of eigenvectors for  $\rho_s$ . Then  $\rho_s(e_i) = \lambda_i e_i$  for some  $\lambda_i \in \mathbb{C}$ . Thus

$$\chi(s) = \sum \lambda_i \quad \chi(s^2) = \sum \lambda_i^2.$$

We also have that

$$\begin{aligned} (\rho_s \otimes \rho_s)(e_i \otimes e_j + e_j \otimes e_i) &= \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i) \\ (\rho_s \otimes \rho_s)(e_i \otimes e_j - e_j \otimes e_i) &= \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i) \end{aligned}$$

which yields that

$$\chi_\sigma^2(s) = \sum_{i \leq j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2} \left( \sum \lambda_i \right)^2 + \frac{1}{2} \sum \lambda_i^2 \chi_\alpha^2(s) = \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2} \left( \sum \lambda_i \right)^2 - \frac{1}{2} \sum \lambda_i^2.$$

The proposition then directly follows. Note that the equality  $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$  directly reflects the fact that  $V \otimes V = \text{Sym}^2(V) \oplus \text{Alt}^2(V)$ .  $\square$

**Proposition 2.5 (Schur's Lemma):** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two irreducible representations of  $G$ . Let  $f: V_1 \rightarrow V_2$  be a linear map such that  $f \circ \rho_s^1 = \rho_s^2 \circ f$  for all  $s \in G$ . Then

1. If  $\rho^1$  and  $\rho^2$  are not isomorphic, then  $f = 0$
2. If  $V_1 = V_2$  and  $\rho^1 = \rho^2$  then  $f$  is a *homothety*, a scalar multiple of the identity.

**Proof:** The case of  $f = 0$  is trivial, so suppose that  $f \neq 0$ . Let  $W_1 = \ker(f)$  and  $W_2 = \text{im}(f)$ . Then for  $x \in W_1$  we have that  $f(\rho_s^1(x)) = \rho_s^2(f(x)) = 0$  which means that  $\rho_s^1(x) \in W_1$ . Thus  $W_1$  is stable under  $G$  and irreducibility of  $V_1$  combined with the assumption that  $f \neq 0$  implies that  $W_1 = 0$ . Similarly, we have that for  $f(x) \in W_2$ , we have that  $\rho_s^2(f(x)) = f(\rho_s^1(x)) \in W_2$ , so  $\rho_s^2(f(x)) \in W_2$ . Thus  $W_2$  is also stable under  $G$  meaning that by a similar argument,  $W_2 = V_2$ . Since  $\ker(f) = 0$  and  $\text{im}(f) = V_2$ , we must have that  $f$  is an isomorphism  $V_1 \rightarrow V_2$ . This proves the first claim.

Now suppose that  $V_1 = V_2$ ,  $\rho^1 = \rho^2$ , and that  $\lambda$  is some eigenvalue of  $f$ . Let  $f' = f - \lambda$ . Since  $\lambda$  is an eigenvalue, then  $\ker(f') \neq 0$ . However, we also have that  $f' \circ \rho_s^1 = \rho_s^2 \circ f'$ . The first part of this proof shows that this implies that  $f' = 0$ . Thus,  $f = \lambda$  and  $f$  is a homothety.  $\square$

**Corollary 2.6:** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two irreducible representations of  $G$ . Let  $h: V_1 \rightarrow V_2$  and define  $h^0$  such that

$$h^0 = \frac{1}{|G|} \sum_{t \in G} (\rho_t^2)^{-1} \circ h \circ \rho_t^1.$$

Then

1. If  $\rho^1$  and  $\rho^2$  are not isomorphic, then  $h^0 = 0$
2. If  $V_1 = V_2$  and  $\rho^1 = \rho^2$ , then  $h^0$  is a homothety of ratio  $\frac{1}{n} \text{Tr}(h)$ , with  $n = \dim(V_1)$ .

**Proof:** First for  $s \in G$  we have that

$$(\rho_s^2)^{-1} \circ h^0 \circ \rho_s^1 = \frac{1}{|G|} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} \circ h \circ \rho_t^1 \rho_s^1 = \frac{1}{|G|} \sum_{t \in G} (\rho_{ts}^2)^{-1} \circ h \circ \rho_{ts}^1 = h^0.$$

Thus, Proposition 2.5 applies to  $h^0$  and in the first case  $h^0 = 0$  and in the second  $h^0$  is a homothety of scalar  $\lambda$ . Moreover we have that

$$n \cdot \lambda = \text{Tr}(\lambda) = \frac{1}{|G|} \sum_{t \in G} \text{Tr}((\rho_t^1)^{-1} \circ h \circ \rho_t^1) = \text{Tr}(h).$$

Thus,  $\lambda = \frac{1}{n} \text{Tr}(h)$ .  $\square$



Consider Corollary 2.6 in matrix form where  $\rho_s^1 = (r_{i_1 j_1}(s))$  and  $\rho_s^2 = (r_{i_2 j_2}(s))$ . Then our linear map  $h$  is given by the matrix  $(x_{i_2 i_1})$  and similarly  $h^0$  is given by the matrix  $(x_{i_2 i_1}^0)$ . Then by definition of  $h^0$  we have that

$$x_{i_2 i_1}^0 = \frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t).$$

In case (1) of Corollary 2.6, the right hand side must vanish and so all coefficients of 0:

**Corollary 2.7:** In case (1) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = 0$$

for all  $i_1, j_1, i_2, j_2$ .

We now consider case (2) of Corollary 2.6 in the matrix form. We have that  $h^0 = \lambda$ , with  $\lambda = \frac{1}{n} \text{Tr}(h)$ , meaning that  $x_{i_2 i_1}^0 = \lambda \delta_{i_2 i_1}$ . That is,  $\lambda = \frac{1}{n} \sum \delta_{i_2 i_1} \cdot x_{i_2 i_1}$ . This we have that

$$\frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t) = \frac{1}{n} \sum_{j_1, j_2} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1} \cdot x_{j_2 j_1}.$$

Equating coefficients of the  $x_{j_2 j_1}$  yields the following corollary:

**Corollary 2.8:** In case (2) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

We may generalize these notions to further functions. Let  $\phi, \psi$  be complex valued functions on  $G$ . Define

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{t \in G} \phi(t^{-1}) \cdot \psi(t) = \frac{1}{|G|} \sum_{t \in G} \phi(t) \cdot \psi(t^{-1}).$$

Then  $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$  and  $\langle \phi, \psi \rangle$  is linear in  $\phi$  and in  $\psi$ . Thus, Corollary 2.7 and Corollary 2.8 respectively become

$$\langle r_{i_2 j_2}, r_{j_1 i_1} \rangle = 0 \qquad \langle r_{i_2 j_2}, r_{j_1 i_1} \rangle = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

If the matrices  $(r_{ij}(t))$  are unitary, realized by a suitable choice of basis, then  $r_{ij}(t^{-1}) = r_{ji}(t)^*$  and Corollary 2.7 and Corollary 2.8 are just *orthogonality relations*.

**Definition 2.9 (Scalar Product):** If  $\phi, \psi$  are two complex valued functions on  $G$ , then let

$$(\phi | \psi) := \frac{1}{|G|} \sum_{t \in G} \phi(t) \psi(t)^*.$$

This is a *scalar product*. It is linear in  $\phi$ , semilinear in  $\psi$ , and  $(\phi | \phi) > 0$  for all  $\phi \neq 0$ .

Define  $\check{\psi}(t) := \psi(t^{-1})^*$ . Then  $(\phi | \psi) = \langle \phi, \check{\psi} \rangle$ . In particular, suppose  $\chi$  is a character so that by Proposition 2.2 we have that  $\chi = \check{\chi}$  then for all complex valued functions  $\phi$  on  $G$  we have that  $(\phi | \chi) = \langle \phi, \chi \rangle$ . Thus, we may use the two interchangeably in the context of characters.

**Theorem 2.10:**

1. If  $\chi$  is the character of an irreducible representation, we have that  $(\chi | \chi) = 1$ , i.e.  $\chi$  has “norm 1.”
2. If  $\chi$  and  $\chi'$  are characters of two non-isomorphic irreducible representations, then  $(\chi | \chi') = 0$ , i.e.  $\chi$  and  $\chi'$  are “orthogonal.”

**Proof:** Suppose  $\rho$  is an irreducible representation with matrix form  $\rho_t = (r_{ij}(t))$  and  $\chi$  its character. Then  $\chi(t) = \sum r_{ii}(t)$  and so

$$(\chi | \chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle = \frac{\delta_{ij}}{n}$$

where the last equality is by Corollary 2.8 and  $n$  is the degree of  $\rho$ . Thus

$$(\chi | \chi) = \sum_{i,j} \frac{\delta_{ij}}{n} = \frac{n}{n} = 1.$$

This proves the first claim. Applying Corollary 2.7 yields the second claim □

**Theorem 2.11:** Let  $V$  be a linear representation of  $G$  with character  $\phi$  such that  $V$  decomposes into a direct sum of irreducible representations  $V = W_1 \oplus \cdots \oplus W_k$ . Then if  $W$  is an irreducible representation with character  $\chi$ , the number of  $W_i$  isomorphic to  $W$  is equal to the scalar product  $(\phi | \chi) = \langle \phi, \chi \rangle$ .

**Proof:** Let  $\chi_i$  be the character of  $W_i$ . Then by Proposition 2.3 we have that  $\phi = \chi_1 + \cdots + \chi_k$ . By linearity of  $(\cdot | \cdot)$  in the first argument we have that  $(\phi | \chi) = (\chi_1 | \chi) + \cdots + (\chi_k | \chi)$ . The result follows by Theorem 2.10. □

**Corollary 2.12:** Let  $V$  be a linear representation of  $G$  with character  $\phi$  such that  $V$  decomposes into a direct sum of irreducible representations  $V = W_1 \oplus \cdots \oplus W_k$ . Then if  $W$  is an irreducible representation with character  $\chi$ , the number of  $W_i$  isomorphic to  $W$  does not depend on the chosen decomposition.

**Proof:** Note that  $(\phi | \chi)$  does not depend on choice of decomposition. □

**Corollary 2.13:** Two representations are isomorphic if and only if they have the same character.

**Proof:** The forward direction is obvious, and the reverse is true by the prior corollary.  $\square$

Thus, our study of representations is reduced to that of the study of characters. If  $\chi_1, \dots, \chi_k$  are the distinct irreducible characters of  $G$  and if  $W_1, \dots, W_k$  their corresponding representation, then each representation  $V$  of  $G$  is isomorphic to a direct sum. We will see later how we know that there are finitely many irreducible representations, and thus characters, of a finite group  $G$ .

$$V = m_1 W_1 \oplus \dots \oplus m_h W_h \quad m_i \neq 0.$$

The character  $\phi$  of  $V$  is equal to  $m_1 \chi_1 + \dots + m_h \chi_h$  and we have that  $m_i = (\phi | \chi_i)$ . This is especially useful when considering the tensor product  $W_i \otimes W_j$  of two irreducible representations. It shows that the product  $\chi_i \cdot \chi_j$  decomposes into a sum  $\chi_i \chi_j = \sum m_{ij}^k \chi_k$ , each integer  $m_{ij}^k \geq 0$ . The orthogonality relations among the  $\chi_i$  imply that

$$(\phi | \phi) = \sum_{i=1}^h m_i^2.$$

We now obtain a useful irreducibility criterion:

**Theorem 2.14:** If  $\phi$  is the character of a representation  $V$ ,  $(\phi | \phi)$  is a positive integer and  $(\phi | \phi) = 1$  if and only if  $V$  is irreducible.

**Proof:** We have that  $\sum m_i^2 = 1$  if and only if one of the  $m_i = 1$  and all the others are equal to 0. This means that  $V$  is isomorphic to one of the  $W_i$ .  $\square$

We now explore the decomposition of the regular representation  $\rho: G \rightarrow \text{GL}(R)$  of a group  $G$  (Example 1.6). Suppose  $\chi_1, \dots, \chi_h$  are the irreducible characters of  $G$  with degrees  $n_1, \dots, n_k$ . Note that by Proposition 2.2,  $n_i = \chi_i(e)$ . Recall that  $R$  has basis  $(e_t)_{t \in G}$  where  $\rho_s(e_t) = e_{st}$ . This means that for  $s \neq e$ , the diagonal terms of the matrix for  $\rho_s$  are all 0, so  $\text{Tr}(\rho_s) = 0$ . On the otherhand, we have that

$$\text{Tr}(\rho_e) = \dim(R) = |G|.$$

**Proposition 2.15:** The character  $r_G$  of the regular representation is given by

$$r_G(e) = |G| \quad r_G(s) = 0 \text{ if } s \neq e.$$

**Corollary 2.16:** Every irreducible representation  $W_i$  is contained in the regular representation with multiplicity equal to its degree  $n_i$ .

**Proof:** By Theorem 2.11, the number of times  $W_i$  is contained in the regular representation is  $\langle r_G, \chi_i \rangle$ . We have that

$$\langle r_G, \chi_i \rangle = \frac{1}{|G|} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{|G|} \cdot |G| \chi_i(1) = \chi_i(1) = n_i.$$

□

**Corollary 2.17:**

1. The degrees satisfy  $\sum_{i=1}^h n_i^2 = |G|$ .
2. if  $e \neq s \in G$ , we have that  $\sum_{i=1}^h n_i \chi_i(s) = 0$ .

**Proof:** By Corollary 2.16, we have that  $r_G(s) = \sum n_i \chi_i(s)$  for all  $s \in G$ . A priori we know that  $r_G$  is the sum of irreducibles  $\chi_i$ , and Corollary 2.16 gives the multiplicities. Plugging in  $s = e$  and  $s \neq e$  yields the claim. □

The above result lets us determine the irreducible representations of a group  $G$ . Suppose we have constructed some mutually non-isomorphic irreducible representations of degrees  $n_1, \dots, n_h$ . In order to check if we have found all such representations, it is necessary and sufficient to verify that  $n_1^2 + \dots + n_h^2 = |G|$ . Also, we shall later see that each of the  $n_i$  divide the order of  $G$ .

**Definition 2.18 (Class Function):** A function  $f$  on a group  $G$  is a *class function* if for all  $s, t \in G$ ,  $f(tst^{-1}) = f(s)$ .

**Proposition 2.19:** Let  $f$  be a class function on a group  $G$  and  $\rho: G \rightarrow \text{GL}(V)$  a linear representation of  $G$  with character  $\chi$ . Define  $\rho_f: V \rightarrow V$  by  $\rho_f = \sum_{t \in G} f(t) \rho_t$ . If  $V$  is irreducible of degree  $n$ , the  $\rho_f$  is a homothety of ratio  $\lambda$  where

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*).$$

**Proof:** We have that

$$\rho_s^{-1} \rho_f \rho_s = \sum_{t \in G} f(t) \rho_s^{-1} \rho_t \rho_s = \sum_{t \in G} f(t) \rho_{s^{-1}ts} = \sum_{t \in G} f(s^{-1}ts) \rho_{s^{-1}ts} = \rho_f.$$

Thus, by Proposition 2.5 we have that  $\rho_f$  is a homothety  $\lambda$ . The trace of  $\lambda$  is  $n\lambda$ . Thus, the trace of  $\rho_f$  is  $\sum_{t \in G} f(t) \text{Tr}(\rho_t) = \sum_{t \in G} f(t) \chi(t)$ . Thus,  $\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*)$ . □

Let  $H$  be the space of class functions on  $G$ .

**Theorem 2.20:** The characters  $\chi_1, \dots, \chi_h$  of  $G$  form an orthonormal basis of  $H$ .

**Proof:** Note that Theorem 2.10 says that the  $\chi_i$  are all orthonormal to each other. To show that they generate  $H$ , it is enough to show that the only element of  $H$  orthogonal to  $\chi_i^*$  is 0. Let  $f$  be such an element. For each representation  $\rho$  of  $G$ , let  $\rho_f$  be as in Proposition 2.19. Since  $f$  is orthogonal to the  $\chi_i^*$ , Proposition 2.19 says that  $\rho_f$  is 0 as long as  $\rho$  is irreducible. From the decomposition of a representation into a direct sum of irreducible representation, with possible multiplicities, we conclude that  $\rho_f$  is always 0. Now consider the regular representation of  $G$  and compute the image of the basis vector  $e_e$  under  $\rho_f$ :

$$0 = \rho_f(e_e) = \sum_{t \in G} f(t) \rho_t(e_e) = \sum_{t \in G} f(t) \rho_t.$$

Thus,  $f(t) = 0$  for each  $t \in G$  and  $f = 0$ . □

**Theorem 2.21:** The number of irreducible representations of  $G$ , up to isomorphism, is the number of conjugacy classes of  $G$ .

**Proof:** Let  $C_1, \dots, C_k$  be the distinct conjugacy classes of  $G$ . Then all class functions are constant on each class, their value determined by some  $\lambda_i$  for each  $C_i$ . These  $\lambda_i$  may be chosen arbitrarily. Thus, the dimension of the space  $H$  of class functions is equal to  $k$ . But we already know by Theorem 2.20 that the dimension of  $H$  is  $h$ , the number of irreducible representations of  $G$ . □

**Proposition 2.22:** Let  $s \in G$  and  $c(s)$  the number of elements in the conjugacy class of  $s$ .

1. We have  $\sum_{i=1}^h \chi_i(s)^* \chi_i(s) = \frac{|G|}{c(s)}$ .
2. For  $t$  not conjugate to  $s$ , we have  $\sum_{i=1}^h \chi_i(s)^* \chi_i(t) = 0$ .

**Proof:** Let  $f_s$  be the class function equal to 1 on the class of  $s$  and 0 otherwise. By Theorem 2.21, we have that

$$f_s = \sum_{i=1}^h \lambda_i \chi_i \qquad \lambda_i = (f_s | \chi_i) = \frac{c(s)}{|G|} \chi_i(s)^*.$$

We have then, for each  $t \in G$ , that

$$f_s(t) = \frac{c(s)}{|G|} \sum_{i=1}^h \chi_i(s)^* \chi_i(t).$$

If  $t = s$ , we get claim 1 and for  $t$  not conjugate to  $s$  we get claim 2. □

**Example 2.23 (Character Table of  $S_3$ ):** Consider the group  $S_3$ . There are three conjugacy classes: the identity  $()$ , the 3 transpositions, and the 2 cyclic permutations. Let  $t$  be one of the transpositions and  $c$  one of the cyclic permutations. Then  $t^2 = 1 = c^3$  and  $tc = c^2t$ . There are just two characters of degree 1: the unit character  $\chi_1$  and the character  $\chi_2$  giving the sign of the permutation. This is because  $t^2 = 1$  means that  $\chi(t) = 1$  or  $-1$ . Each choice then determines the character of  $c$ , which ends up corresponding to the unit character or the sign. By Theorem 2.21, there exists one more irreducible character  $\theta$ . If  $n$  is the degree of  $\theta$ , then we must have that  $1 + 1 + n^2 = 6$ , so  $n = 2$ . By Proposition 2.15, we have that  $\chi_1 + \chi_2 + 2\theta$  is the character of the regular representation. Thus, we get the following *character table*:

	1	$t$	$c$
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\theta$	2	0	-1

We obtain an irreducible representation of  $G$  with character  $\theta$  by having  $G$  permute the coordinates of elements of  $\mathbb{C}^3$  satisfying  $x + y + z = 0$ .

Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$ . Recall that the direct sum decomposition of  $V$  into irreducible representation is not necessarily unique. Thus, we shall now define a “coarser” decomposition which has the advantage of being unique.

**Definition 2.24 (Canonical Decomposition of a Representation):** Let  $\chi_1, \dots, \chi_h$  be the distinct characters of the irreducible representations of  $W_1, \dots, W_h$  of  $G$  with degrees  $n_1, \dots, n_h$ . Let  $V = U_1 \oplus \dots \oplus U_m$  be a decomposition of  $V$  into a direct sum of irreducible representations. For  $i = 1, \dots, h$ , let  $V_i$  be the direct sum of the  $U_i$  which are isomorphic to  $W_i$ . Then  $V = V_1 \oplus \dots \oplus V_h$ . We have decomposed  $V$  into a direct sum of irreducible representations and combined the ones which are isomorphic to each other.

This decomposition satisfies some nice properties:

**Theorem 2.25:** 1. The decomposition  $V = V_1 \oplus V_h$  does not depend on the initially chosen decomposition of  $V$  into irreducibles.

2. The projection  $p_i: V \rightarrow V_i$  is given by

$$p_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i^*(t) \rho_t.$$

**Proof:** We shall prove claim 2 since claim 1 follows as the  $p_i$  determine the  $V_i$ . Let  $q_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i(t)^* \rho_t$ . By Proposition 2.19, we have that the restriction of  $q_i$  to an irreducible representation  $W$  with character  $\chi$  and degree  $n$  is a homothety of ratio  $\frac{n_i}{n}(\chi_i | \chi)$ . Thus,  $q_i$  is 0 if  $\chi_i \neq \chi$  and 1 if  $\chi = \chi_i$ . This yields that  $q_i$  is the identity on an irreducible representation isomorphic to  $W_i$ , and 0 on the others. Thus,  $q_i$  is the identity on  $V_i$  and 0 on  $V_j$  for  $j \neq i$ . Decomposing  $x \in V$  into  $x_i \in V_i$  such that  $x = x_1 + \cdots + x_h$  yields that

$$q_i(x) = q_i(x_1) + \cdots + q_i(x_h) = x_i.$$

Thus  $q_i = p_i$ . □

This allows us to decompose representations  $V$  in two stages. First, we determine  $V_1 \oplus \cdots \oplus V_h$ . This is done easily using the given formula for  $p_i$  in Theorem 2.25. Finally, for each  $V_i$  we may choose a decomposition of  $V_i$  into a direct sum of irreducible representations, each isomorphic to  $W_i$ . This last decomposition may be done in any number of ways.

**Example 2.26 (Decomposition of  $C_2$ ):** Let  $G = C_2 = \{e, s\}$  be the cyclic group of two elements generated by  $s$ . Let  $\rho: G \rightarrow \text{GL}(V)$  be any representation of  $C_2$ . Note that  $C_2$  has two irreducible representations of degree 1,  $W^+$  and  $W^-$  with respective characters  $\rho^+ = 1$  and  $\rho_s = -1$ . The canonical decomposition of  $V$  is  $V = V^+ \oplus V^-$ , where  $V^+$  consists of elements  $x \in V$  which are symmetric and  $V^-$  consists of elements which are antisymmetric. In other words,  $V^+$  consists of elements  $x \in V$  where  $\rho_s(x) = x$  and  $V^-$  consists of elements  $x \in V$  where  $\rho_s(x) = -x$ . This, the projections are

$$p^+(x) = \frac{1}{2}(x + \rho_s(x)) \quad p^-(x) = \frac{1}{2}(x - \rho_s(x)).$$

To decompose  $V^+$  and  $V^-$  into irreducible components means to decompose these subspaces into a direct sum of lines, which can be in arbitrarily many ways.

We now have the tools to explicitly compute the components  $V_i$  of this canonical decomposition of  $\rho: G \rightarrow \text{GL}(V)$ . Let  $V = V_1 \oplus \cdots \oplus V_h$  be this decomposition. The projection given in Theorem 2.25 will allow us to do this. Let  $W_i$  have matrix form  $(r_{\alpha\beta}(s))$  with respect to a basis  $(e_1, \dots, e_n)$ . Then  $\chi_i(s) = \sum_{\alpha} r_{\alpha\alpha}(s)$ . For each  $1 \leq \alpha, \beta \leq n$  define

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t.$$

**Proposition 2.27:**

1. The map  $p_{\alpha\alpha}$  is a projection. It is 0 on  $V_j$  for  $j \neq i$  and its image  $V_{i,\alpha}$  is contained in  $V_i$  where  $V_i$  is the direct sum of the  $V_{i,\alpha}$ ,  $1 \leq \alpha \leq n$ . We have that  $p_i = \sum_{\alpha} p_{\alpha\alpha}$ .
2. The linear map  $p_{\alpha\beta}$  is 0 on  $V_j$  for  $j \neq i$  as well as on  $V_{i,\gamma}$  for  $\gamma \neq \beta$ . It defines an isomorphism  $V_{i,\beta} \rightarrow V_{i,\alpha}$ .
3. Let  $x_1 \neq 0 \in V_{i,1}$  and  $x_{\alpha} := p_{\alpha,1}(x_1) \in V_{i,\alpha}$ . Then the  $x_{\alpha}$  are linearly independent and generate a subspace  $W(x_1)$  stable under  $G$  and of dimension  $n$ . For each  $s \in G$ , we have that

$$\rho_s(x_{\alpha}) = \sum_{\beta} r_{\beta\alpha}(s)x_{\beta}.$$

In particular,  $W(x_1)$  is isomorphic to  $W_i$ .

4. If  $(x_1^{(1)}, \dots, x_1^{(m)})$  is a basis of  $V_{i,1}$ , then the representation  $V_i$  is the direct sum of the subrepresentations  $W(x_1^{(1)}), \dots, W(x_1^{(m)})$ .

**Proof:** Observe that the definition of  $p_{\alpha\beta}$  is defined in terms of arbitrary representations of  $G$ , and in particular in the irreducible representations  $W_j$ . For  $W_i$ , we have that

$$p_{\alpha\beta}(e_{\gamma}) = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t(e_{\gamma}) = \frac{n}{|G|} \sum_{\delta} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) r_{\delta\gamma}(t) e_{\delta}.$$

By Corollary 2.8 we have that

$$p_{\alpha\beta}(e_{\gamma}) = \begin{cases} e_{\alpha} & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

We get from this that  $\sum_{\alpha} p_{\alpha\alpha} = \text{id}_{W_i}$ . We also get the formulas

$$p_{\alpha\beta} \circ p_{\gamma\delta} = \begin{cases} p_{\alpha\delta} & \text{if } \beta = \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_s \circ p_{\alpha\gamma} = \sum_{\beta} r_{\beta\alpha}(s) p_{\beta\gamma}.$$

For  $W_j$ ,  $j \neq i$ , we use Corollary 2.7 and the same argument to show that all the  $p_{\alpha\beta}$  are 0.



With this, we can now decompose  $V$  into subrepresentations each isomorphic to  $W_j$  and apply the above to these representations. The first two assertions follow. Moreover, these formulas are valid in  $V$ . Assuming the hypothesis of claim 3 holds, we have that

$$\rho_s(x_\alpha) = \rho_s \circ p_{\alpha 1}(x_1) = \sum_{\beta} r_{\beta \alpha}(s) p_{\beta 1}(x_1) = \sum_{\beta} r_{\beta \alpha}(s) x_{\beta}.$$

This proves claim 3. Finally, claim 4 follows from the first 3. □

## Exercises

**Exercise 2.1 (Ser77 2.1):** Let  $\chi, \chi'$  be the characters of two representations. Prove the following formulas:

$$(\chi + \chi')_{\sigma}^2 = \chi_{\sigma}^2 + \chi'_{\sigma}{}^2 + \chi \chi'$$

$$(\chi + \chi')_{\alpha}^2 = \chi_{\alpha}^2 + \chi'_{\alpha}{}^2 + \chi \chi'$$

**Proof:** Let  $s \in G$ . Then by Proposition 2.4 we have that

$$\begin{aligned} (\chi + \chi')_{\sigma}^2(s) &= \frac{1}{2}((\chi + \chi')^2(s) + (\chi + \chi')(s^2)) \\ &= \frac{1}{2}(\chi(s)^2 + \chi'(s)^2 + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)) \\ &= \frac{1}{2}(\chi(s)^2 + \chi(s^2)) + \frac{1}{2}(\chi'(s) + \chi'(s^2)) + \chi(s)\chi'(s) = \chi_{\sigma}^2(s) + \chi'_{\sigma}{}^2(s) + \chi(s)\chi'(s). \end{aligned}$$

Since this holds for all  $s \in G$ , the formula holds in general. The proof of the other formula is similar.  $\square$

**Exercise 2.2 (Ser77 2.2):** Let  $X$  be a finite set on which  $G$  acts, and  $\rho: G \rightarrow \text{GL}(V)$  the corresponding permutation representation (Example 1.7), and  $\chi_X$  the character of  $\rho$ . Then show that for  $s \in G$ ,  $\chi_X(s)$  is equal to the number of elements fixed by  $s$ .

**Proof:** Suppose  $X = [n]$  and so  $s \in S_n$ , meaning  $G \leq S_n$ . We may assume this without loss of generality. Note that  $R_s = (r_{ij}(s))$  where  $r_{ij}(s) = 1$  if  $s(j) = i$  and 0 otherwise. We want to count the number of elements in  $[n]$  fixed by  $s$ , i.e. the number of  $i$  such that  $\sigma(i) = i$ . These correspond exactly to the entries in  $R_s$  where  $r_{ii}(s) = 1$ . Thus, the claim follows.  $\square$

**Exercise 2.3 (Ser77 2.3):** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation with character  $\chi$ . Recall that  $V^*$  is the dual vector space of  $V$ . For  $x \in V$ ,  $x^* \in V^*$  let  $\langle x, x^* \rangle = x^*(x)$ . Then there exists a unique linear representation  $\rho^*: G \rightarrow \text{GL}(V^*)$  such that

$$\langle \rho_s(x), \rho_s^*(x^*) \rangle = \langle x, x^* \rangle$$

for  $s \in G$ ,  $x \in V$ , and  $x^* \in V^*$ . Note that  $\rho^*$  has character  $\chi^*$ , the conjugate of  $\chi$ .

**Proof:** Let  $\rho_s^* = (\rho_s^T)^{-1}$ . Then

$$\langle \rho_s(x), \rho_s^*(x^*) \rangle = \langle \rho_s(x), (\rho_s^T)^{-1}(x^*) \rangle = \langle x, \rho_s^T((\rho_s^T)^{-1}(x^*)) \rangle = \langle x, x^* \rangle.$$

Now suppose that  $\rho' : G \rightarrow \text{GL}(V^*)$  was another representation satisfying the above property. Then we would have that

$$\langle \rho_s(x), (\rho^* - \rho')(x^*) \rangle = \langle \rho_s(x), \rho_s^*(x^*) \rangle - \langle \rho_s(x), \rho'_s(x^*) \rangle = 0.$$

Note that this holds for all  $x \in V$  and  $x^* \in V^*$ . Thus, we must have that  $(\rho^* - \rho')(x^*) = 0$ , and thus  $\rho^* = \rho'$ .  $\square$

**Exercise 2.4 (Ser77 2.5):** Let  $\rho : G \rightarrow \text{GL}(V)$  be a linear representation with character  $\chi$ . Then the number of times  $\rho$  contains the unit representation is equal to  $(\chi | 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$ .

**Proof:** The equality  $(\chi | 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$  is immediate by definition of the scalar product  $(\cdot | \cdot)$  and the fact that  $1^* = 1$ . By Theorem 2.11,  $(\chi | 1)$  counts the number of times an irreducible representation with character 1 appears in  $V$ . By Corollary 2.13, the only irreducible representation with character 1 is the unit representation.  $\square$

**Exercise 2.5 (Ser77 2.6):** Let  $G$  act on a finite set  $X$ ,  $\rho$  the corresponding permutation representation, and  $\chi$  its character.

1. Let  $c$  be the number of distinct orbits. Show that  $c$  is equal to the number of times  $\rho$  contains the unit representation 1. Deduce that  $(\chi | 1) = c$ . In particular if  $G$  is transitive and thus  $c = 1$ , then  $\rho = 1 \oplus \theta$  where  $\theta$  does not contain the unit representation. If  $\psi$  is the character of  $\theta$ , then  $\chi = 1 + \psi$  and  $(\psi | 1) = 0$ .
2. Let  $G$  act on the product  $X \times X$  in the natural way. Show that the character of the corresponding permutation representation is equal to  $\chi^2$ .
3. Suppose that  $G$  is transitive on  $X$  and  $|X| \geq 2$ . We say  $G$  is *doubly transitive* if for all  $x, y, x', y' \in X$  with  $x \neq y$  and  $x' \neq y'$  there exists  $s \in G$  such that  $s(x, y) = (sx, sy) = (x', y')$ . Prove that the following are equivalent:
  - (a)  $G$  is doubly transitive.
  - (b) The action of  $G$  on  $X \times X$  has two orbits, the diagonal and the complement.
  - (c)  $(\chi^2 | 1) = 2$
  - (d) The representation  $\theta$  defined in the first part of this exercise is irreducible.

**Proof:** We know that the number of times the unit representation is contained in  $\chi$  is equal to  $(\chi | 1)$  by Theorem 2.11. By Exercise 2.4, we have that  $(\chi | 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$ . We prove that  $\frac{1}{|G|} \sum \chi(s) = c$  by double counting. Consider the set  $\{(s, x) \in G \times X \mid s \cdot x = x\}$ . Then we have that

$$\sum_{x \in X} |G_x| = \sum_{x \in X} |\{s \in G \mid s \cdot x = x\}| = |\{(s, x) \in G \times X \mid s \cdot x = x\}| = \sum_{s \in G} |\{x \in X \mid s \cdot x = x\}| = \sum_{s \in G} \chi(s).$$

Let  $O_1, \dots, O_c$  be the distinct orbits. By the Orbit-Stabilizer theorem, each  $O_i$  is in bijection with  $G/G_x$  for all  $x \in O_i$ . Note that the orbits  $O_i$  partition  $X$ . Thus we have that

$$\sum_{s \in G} \chi(s) = \sum_{i=1}^c \sum_{x \in O_i} |G_x| = \sum_{i=1}^c \sum_{x \in O_i} \frac{|G|}{|O_i|} = c \cdot |G|$$

and  $\frac{1}{|G|} \sum_{s \in G} \chi(s) = c$ . Following this, the rest of the claim is immediate.

Now suppose that  $\phi$  is the character of the permutation representation of  $G \curvearrowright X \times X$ . Then by Exercise 2.2,  $\phi(s)$  is equal to the number of elements fixed by  $s$ . An element  $(x, y) \in X \times X$  is fixed by  $s \in G$  if and only if both  $x$  and  $y$  are fixed. Thus if there are  $\chi(s)$  elements of  $X$  fixed by  $s$ , then  $\chi^2(s)$  elements of  $X \times X$  are fixed by  $s$  and  $\phi = \chi^2$ .

To prove 3, we have that  $(a) \iff (b)$  is immediate and  $(b) \iff (c)$  follows from 1 and 2. Now suppose  $(c)$  holds and let  $\psi$  be the character of  $\theta$ . Then  $1 + \psi = \theta$ . Since  $(\chi | 1) = (1 | 1) = 1$  we must have that  $(\psi | 1) = 0$ . Since  $\chi^2 = 1 + 2\psi + \psi^2$ , we have that  $(c)$  is equivalent to saying  $(\psi^2 | 1) = 1$ . Thus

$$\frac{1}{|G|} \sum_{s \in G} \psi(s)^2 = 1.$$

However, note that  $\psi(s)$  is real valued, not just complex valued. This is because  $\chi$  is real valued, it counts fixed points, and clearly 1 is real valued. Thus  $\psi^* = \psi(s)^*$  and so the above equality implies that  $(\psi | \psi = 1)$ . By Theorem 2.14, we have that this is true if and only if  $\theta$  is irreducible, i.e.  $(c) \iff (d)$  holds.  $\square$

**Exercise 2.6 (Ser77 Exercise 2.8):** Let  $\rho : G \rightarrow \text{GL}(V)$  be any representation of a group  $G$  with  $V = V_1 \oplus \cdots \oplus V_h$  the canonical decomposition,  $W_1, \dots, W_h$  all irreducible representations of  $G$ . Let  $H_i$  be the vector space of linear mappings  $h : W_i \rightarrow V$  such that  $\rho_s \circ h = h \circ \rho_s$  for all  $s \in G$ . Each  $h \in H_i$  maps  $W_i$  into  $V_i$ .

1. Show that  $\dim(H_i)$  is equal to  $\dim(V_i)/\dim(W_i)$ , the multiplicity of  $W_i$  in  $V_i$ .
2. Let  $G$  act on  $H_i \otimes W_i$  through the tensor product of the trivial representation of  $G$  on  $H_i$  and the given representation on  $W_i$ . Show that the linear map

$$F : H_i \otimes W_i \rightarrow V_i$$

$$\sum h_\alpha \otimes w_\alpha \mapsto \sum h_\alpha(w_\alpha)$$

is an isomorphism.

3. Let  $(h_1, \dots, h_k)$  be a basis of  $H_i$  and form the direct sum  $W_i \oplus \cdots \oplus W_i$  of  $k$  copies of  $W_i$ . This basis defines an obvious mapping  $h : W_i \oplus \cdots \oplus W_i \rightarrow V_i$ . Show that  $h$  is an isomorphism of representations. In particular, to decompose  $V_i$  into a direct sum of representations isomorphic to  $W_i$  amounts to choosing a basis for  $H_i$ .

**Proof:** 1. Let  $h \in H_i$ . Then  $h$  maps  $W_i$  into say  $k_i$  copies of  $W_i$ . Each copy of  $W_i$  comes with a projection function  $V_i \rightarrow W_i$ . Composing  $h$  with this projection function shows that  $h$  is a linear combination of maps  $W_i \rightarrow W_i$ . Thus, it suffices to consider the case of  $V = W_i$ . But Schur's Lemma (Proposition 2.5) says that in this case  $h$  is a scalar multiple of the identity, and thus onto. Thus,  $\dim(H_i) = 1 = \frac{\dim(V_i)}{\dim(W_i)}$ .

2. By composing  $F$  with one of the  $k_i$  projection functions, we get that  $F$  is a linear combination of maps  $H_i \otimes W_i \rightarrow W_i$ . Thus, we may again reduce to the case that  $V = W_i$ . In this case, by the proof of 1 we get that  $F$  is surjective. Dimension counting yields that it is an isomorphism of vector spaces.

To see that  $F$  is an isomorphism of representations, let  $\rho' : G \rightarrow \text{GL}(H_i \otimes W_i)$  be the given tensor product representation. We have that

$$F(\rho'_s(h_\alpha \otimes w_\alpha)) = F(h_\alpha \otimes \rho_s(w_\alpha)) = h_\alpha(\rho_s(w_\alpha)) = \rho_s(h_\alpha(w_\alpha)) = \rho_s(F(h_\alpha \otimes w_\alpha)).$$

Thus  $F \circ \rho'_s = \rho_s \circ F$  for all generators, and thus on all of  $H_i \otimes W_i$ . Thus  $F$  is an isomorphism of representations.

3. Define the map

$$h: W_i \oplus \cdots \oplus W_i \rightarrow V_i$$

$$(w_1, \dots, w_k) \mapsto h_1(w_1) + \cdots + h_k(w_k).$$

Clearly  $h$  is linear. From 2 we see that every element of  $V_i$  is of the form  $\sum w_\alpha h_\alpha$  and the  $h_i$  form a basis. Thus  $h$  is surjective and dimension counting yields that  $h$  is a linear isomorphism. The proof that  $h$  is an isomorphism of representations is similar.

Now suppose we are given an isomorphism of representations  $h: W_i \oplus \cdots \oplus W_i \rightarrow V_i$ . Let  $i_j: W_i \rightarrow W_i \oplus \cdots \oplus W_i$  be the inclusions of  $W_i$  into the  $j$ -th component of  $W_i \oplus \cdots \oplus W_i$ . Define  $h_j: W_i \rightarrow V_i := h \circ i_j$ . Since  $h$  is an isomorphism of representations, we have that  $h_j$  commutes with  $\rho$  and so  $h_j \in H_i$ . We claim that the  $h_j$  form a basis of  $H_i$ . Suppose the  $h_j$  are linearly dependent. Then this would contradict the fact that  $h$  is an isomorphism of vector spaces since we would be able to show that  $\ker(h) \neq 0$ . Thus the  $h_j$  form a basis of  $H_i$  and every isomorphism of representations arises in the way described.

□

**Exercise 2.7 (Ser77 Exercise 2.9):** Let  $W_i$  be a representation of  $G$  with matrix form  $(r_{\alpha\beta}(s))$  with respect to a basis  $(e_1, \dots, e_n)$ . Let  $H_i$  be the space of linear maps  $h: W_i \rightarrow V$  such that  $h \circ \rho_s = \rho_s \circ h$ . Show that  $h \mapsto h(e_\alpha)$  is an isomorphism of  $H_i$  onto  $V_{i,\alpha}$ .

**Proof:** Suppose that  $h \mapsto 0$ . So  $h(e_\alpha) = 0$ . By Exercise 2.6 (2) we have that  $h \otimes e_\alpha = 0$ . But  $e_\alpha \neq 0$  so  $h = 0$ . Then note that  $\dim(V_i) = \dim(V_{i,\alpha}) \cdot n = \dim(V_{i,\alpha}) \cdot \dim(W_i)$ . By Exercise 2.6 (1) we have that  $\dim(H_i) = \dim(V_{i,\alpha})$  and so by a dimension argument, the map  $h \mapsto h(e_\alpha)$  is an isomorphism.

□

## Chapter 3

# Subgroups, Products, and Induced Representations

As always, all groups are assumed to be finite.

**Theorem 3.1:** Let  $G$  be a group. The following are equivalent.

1.  $G$  is abelian.
2. All the irreducible representations of  $G$  have degree 1.

**Proof:** If  $(n_1, \dots, n_h)$  are the degrees of the distinct irreducible representations of  $G$ , we know that  $h$  is the number of conjugacy classes of  $G$ . We know that  $G$  is abelian if and only if  $h = |G|$ . By Corollary 2.17 we have  $n_1^2 + \dots + n_h^2 = |G|$ . Thus  $G$  is abelian if and only if all of the  $n_i = 1$ .  $\square$

**Corollary 3.2:** Let  $A$  be an abelian subgroup of  $G$ . Then every irreducible representation has degree  $\leq \frac{|G|}{|A|}$ .

**Proof:** Let  $\rho : G \rightarrow \text{GL}(V)$  be an irreducible representation of  $G$ . Then  $\rho|_A$  is a representation of  $A$ . Let  $W \subseteq V$  be an irreducible subrepresentation of  $\rho|_A$ , which by Theorem 3.1 has dimension 1. Let  $V'$  be the vector subspace generated by the images  $\rho_s(W)$  as  $s$  ranges over  $G$ . Then  $V'$  is stable under  $G$  and since  $\rho$  is irreducible we must have that  $V' = V$ . But for  $s \in G$  and  $t \in A$  we have

$$\rho_{st}(W) = \rho_s(\rho_t(W)) = \rho_s(W).$$

Thus, the maximum number of distinct images  $\rho_s(W)$  is  $\frac{|G|}{|A|}$ . This yields that  $\dim(V) \leq \frac{|G|}{|A|}$  since  $V$  is the sum of the  $\rho_s(W)$ .  $\square$

**Example 3.3 (Degrees of Irreducible Representations of a Dihedral Group):** Consider any dihedral group  $G$ . This group has a cyclic subgroup  $A$  of index  $\frac{|G|}{|A|} = 2$ . Thus, every irreducible representation of  $G$  has degree 1 or 2.

**Definition 3.4 (Representation of a Direct Product, Tensor Product of Representations):** Suppose that  $\rho^1: G_1 \rightarrow \text{GL}(V_1)$  and  $\rho^2: G_2 \rightarrow \text{GL}(V_2)$  are two representations. Then we may define a representation for the direct product as follows:

$$\begin{aligned}\rho^1 \otimes \rho^2: G_1 \times G_2 &\rightarrow \text{GL}(V_1 \otimes V_2) \\ (s_1, s_2) &\mapsto \rho_{s_1}^1 \otimes \rho_{s_2}^2.\end{aligned}$$

We will also call  $\rho^1 \otimes \rho^2$  the *tensor product* of the representations  $\rho^1$  and  $\rho^2$ . Similarly, if  $\chi_1$  and  $\chi_2$  are the characters of the representations  $\rho^1$  and  $\rho^2$ , then  $\chi(s_1, s_2) := \chi_1(s_1) \cdot \chi_2(s_2)$  is the character of  $\rho^1 \otimes \rho^2$ .

There may be complaints between the similarities in naming for the tensor products of representations in Definition 1.15 and Definition 3.4. The earlier definition took two representations  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  and created a new representation  $\rho^1 \otimes \rho^2: G \rightarrow \text{GL}(V_1 \otimes V_2)$ . However, notice that if you take  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  and apply Definition 3.4, and then restrict this tensor product representation to the diagonal  $\Delta_G$ , you recover Definition 1.15 as a special case of Definition 3.4.

**Theorem 3.5:**

1. If  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  are irreducible, then  $\rho^1 \otimes \rho^2$  is an irreducible representation of  $G_1 \times G_2$ .
2. Every irreducible representation of  $G_1 \times G_2$  takes the form  $\rho^1 \otimes \rho^2$  for some irreducible representations  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$ .

**Proof:** We use the notation from Definition 3.4. By Theorem 2.14, we have that

$$\frac{1}{|G_1|} \sum_{s_1 \in G_1} |\chi_1(s_1)|^2 = 1, \quad \frac{1}{|G_2|} \sum_{s_2 \in G_2} |\chi_2(s_2)|^2 = 1.$$

Thus, we have that

$$\frac{1}{|G_1 \times G_2|} \sum_{(s_1, s_2) \in G_1 \times G_2} |\chi(s_1, s_2)|^2 = 1.$$

Thus  $\rho^1 \otimes \rho^2$  is irreducible.

In order to prove the second claim, it suffices to show that each class function  $f$  on  $G_1 \times G_2$ , which must be orthogonal to the characters of the form  $\chi_1(s_1)\chi_2(s_2)$ , is 0. So suppose that

$$\sum_{s_1, s_2} f(s_1, s_2) \chi_1(s_1)^* \chi_2(s_2)^* = 0.$$



Fix  $\chi_2$  and define  $g(s_1) = \sum_{s_2} f(s_1, s_2) \chi_2(s_2)^*$ . Thus, for all  $\chi_1$  we have that

$$\sum_{s_1} g(s_1) \chi_1(s_1)^* = 0.$$

This implies that  $g = 0$ . Swapping the roles of  $\chi_1$  and  $\chi_2$  also yields that  $g = 0$ . Thus, we must have that  $f = 0$ .  $\square$

Thus, we have reduced the study of representations of  $G_1 \times G_2$  to representations of  $G_1$  and  $G_2$ .

Now let  $\rho : G \rightarrow \text{GL}(V)$  be a linear representation of  $G$  and  $H \leq G$  a subgroup. Let  $W \leq V$  be a subspace stable under  $H$ , i.e. a subrepresentation of  $\rho|_H$ . Then let  $\theta : H \rightarrow \text{GL}(W)$  be the resulting representation. Consider  $s \in G$  and the associated coset  $sH$ . Then the vector space  $\rho_s(W)$  only depends on  $sH$ . If we consider  $st$  for  $t \in H$ , then since  $W$  is invariant under  $H$  we have that

$$\rho_{st}(W) = \rho_s(\rho_t W) = \rho_s(W).$$

Suppose  $\sigma$  is any left coset of  $H$ . Then we may define a subspace  $W_\sigma$  of  $V$  to be  $\rho_s W$  for any  $s \in \sigma$  by what we just saw. Then we have that  $\rho_s$  permutes the  $W_\sigma$  amongst themselves. The sum  $\sum_{\sigma \in G/H} W_\sigma$  is a subrepresentation of  $V$ .

**Definition 3.6 (Induced Representation):** Using the notation above, we say a representation  $\rho : G \rightarrow \text{GL}(V)$  is *induced* by the representation  $\theta : H \rightarrow \text{GL}(W)$  if  $V$  is equal to the direct sum of the  $W_\sigma$ , i.e. if  $V = \bigoplus_{\sigma \in G/H} W_\sigma$ . This may be equivalently stated in two ways:

1. Each  $x \in V$  can be written uniquely as  $\sum_{\sigma \in G/H} x_\sigma$  with  $x_\sigma \in W_\sigma$ .
2. For a choice of representatives  $R$  for  $G/H$ , the vector space  $V$  is the direct sum of  $\rho_r(W)$  for  $r \in R$ .

We have that  $\dim(V) = \sum_{r \in R} \dim(\rho_r(W)) = [G : H] \cdot \dim(W)$ .

**Example 3.7 (Regular Representation is Induced by Any Subspace):** Let  $V$  be the regular representation of a group  $G$ . Then the space  $V$  has a basis  $(e_t)_{t \in G}$  indexed by  $G$  such that  $\rho_s(e_t) = e_{st}$  for all  $s, t \in G$ . Let  $H$  be any subgroup of  $G$  and  $W$  the subspace generated by  $(e_t)_{t \in H}$ . Then  $W$  is the regular representation of  $H$  and it is clear that  $V$  is induced by  $W$ . This follows immediately since the cosets in  $G/H$  partition the full basis  $(e_t)_{t \in G}$ .

**Example 3.8 (Permutation Representation is Induced by Unit Representation):** Let  $V$  be a vector space with a basis  $(e_\sigma)$  indexed by  $\sigma \in G/H$ . Define a representation  $\rho : G \rightarrow \text{GL}(V)$  such that for  $s \in G$  we have  $\rho_s(e_\sigma) = e_{s\sigma}$ . This is the permutation representation of  $G$  associated with  $G/H$ . The vector  $e_H$  corresponding to  $H$  is invariant under  $H$ . The representation of  $H$  in the subspace  $\mathbb{C}e_H$  is the unit representation of  $H$ . It is clear that this unit representation induces  $\rho$ .

**Example 3.9 (Direct Sum of Induced Representations):** Suppose that  $\rho^1$  and  $\rho^2$  are representations induced by  $\theta^1$  and  $\theta^2$  respectively. Then  $\rho^1 \oplus \rho^2$  is induced by  $\theta^1 \oplus \theta^2$ . This is immediate since induced representations are characterized by direct sums.

**Example 3.10 (Stable Subspaces of Induced Representations):** Suppose  $\rho : G \rightarrow V$  is induced by  $\theta : H \rightarrow W$ . Let  $W_1$  be a stable subspace of  $W$ . Then the subspace  $V_1 = \sum_{r \in R} \rho_r(W_1)$ , where  $R$  is a system of representatives for  $G/H$ , is a stable subspace of  $V$ . We have that  $V_1$  is induced by  $W_1$ .

**Example 3.11 (Tensor Product of Induced Representations):** If  $\rho$  is induced by  $\theta$  and  $\rho'$  is a representation of  $G$ , then  $\rho \otimes \rho'$  is induced by  $\theta \otimes \rho'|_H$ .

We now aim to prove that induced representations exist and are unique.

**Lemma 3.12:** Suppose that  $\rho : G \rightarrow \text{GL}(V)$  is induced by  $\theta : H \rightarrow W$ . Let  $\rho' : G \rightarrow \text{GL}(V')$  be a representation of  $G$  and  $f : W \rightarrow V'$  a linear map such that  $f(\theta_t(w)) = \rho'_t(f(w))$  for all  $t \in H$ ,  $w \in W$ . Then there exists a unique linear map  $F : V \rightarrow V'$  which extends  $f$  and satisfies  $F \circ \rho_s = \rho'_s \circ F$  for all  $s \in G$ .

**Proof:** Let  $\sigma \in G/H$  and  $s \in \sigma$ . Define  $F$  to be  $F := \rho'_s \circ f \circ \rho_s^{-1}$ . Note that this does not depend on the choice of  $s \in \sigma$ . If we take  $st$  with  $t \in H$  then on  $W_\sigma$

$$\rho'_{st} \circ f \circ \rho_{st}^{-1} = \rho'_s \circ \rho'_t \circ f \circ \rho_t^{-1} \circ \rho_s^{-1} = \rho'_s \circ \rho'_t \circ f \circ \theta_t^{-1} \circ \rho_s^{-1} = \rho'_s \circ f \circ \theta_t \circ \theta_t^{-1} \circ \rho_s^{-1} = \rho'_s \circ f \circ \rho_s^{-1}.$$

Since  $V$  is a direct sum of the  $W_\sigma$ , there exists a linear map  $F : V \rightarrow V'$  extending the partial mappings on each of the  $W_\sigma$ . It is easy to check that  $\rho_s \circ F = F \circ \rho_s$ .

For uniqueness, we have for any  $F$  satisfying the above conditions that for  $x \in W$

$$F(x) = (F \circ \rho_s \circ \rho_s^{-1})(x) = (\rho'_s \circ F \circ \rho_s)(x) = (\rho'_s \circ f \circ \rho_s)(x).$$

This holds because if  $x \in \rho_s(W)$  then  $\rho_s^{-1}(x) \in W$ . This computation determines  $F$  on  $\rho_s(W)$ , and thus on  $V$  since  $V$  is the direct sum of these  $\rho_s(W)$ .  $\square$

**Theorem 3.13:** Let  $G$  be a group and  $H$  a subgroup of  $G$ . Let  $\theta : H \rightarrow \text{GL}(W)$  be a linear representation. Then there exists a representation  $\rho : G \rightarrow \text{GL}(V)$  of  $G$  induced by  $\theta$  which is unique up to isomorphism.

**Proof:** By Example 3.9, we may take without loss of generality that  $\theta$  is irreducible. Recall that Corollary 2.16 says that the regular representation contains every irreducible representation. Then  $\theta$  is isomorphic to a subrepresentation of the regular representation of  $H$ . This may be induced to the regular representation of  $G$ . Applying Example 3.10 yields that  $\theta$  itself may be induced.

Now suppose that  $\rho : G \rightarrow \text{GL}(V)$  and  $\rho' : G \rightarrow \text{GL}(V')$  are two representations of  $G$  induced by  $\theta$ . Let  $f : W \hookrightarrow V'$  be the injection and apply Lemma 3.12. Then there exists a unique linear map  $F : V \rightarrow V'$  which is

the identity on  $W$  and satisfies  $F \circ \rho_s = \rho'_s \circ F$  for all  $s \in G$ . Thus, the image of  $F$  contains all  $\rho'_s(W)$  and thus the image of  $F$  is  $V'$ . Since  $V$  and  $V'$  are the same dimension,  $F$  is an isomorphism of representations.  $\square$

**Theorem 3.14:** Suppose that  $\rho: G \rightarrow \text{GL}(V)$  is induced by  $\theta: H \rightarrow \text{GL}(W)$  with corresponding characters  $\chi_\rho$  and  $\chi_\theta$ . Let  $R$  be a system of representatives for  $G/H$ . Then for each  $u \in G$  we have

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}us \in H}} \chi_\theta(s^{-1}us).$$

**Proof:** We have that  $V$  is the direct sum of  $\rho_r(W)$  for  $r \in R$ . Moreover,  $\rho_u$  permutes the  $\rho_r(W)$  in that if we write  $ur$  in the form  $r_u t$  for some  $r_u \in R$  and  $t \in H$  then  $\rho_u(\rho_r(W)) = \rho_{r_u}(W)$ . Give  $V$  a basis by taking the union of the bases of the  $\rho_r(W)$ . Whenever  $r_u \neq r$ , the diagonal terms of  $\rho_u$  are 0. Thus we have that

$$\chi_\rho(u) = \sum_{r \in R_u} \text{Tr}_{\rho_r(W)}(\rho_u|_{\rho_r(W)})$$

where  $R_u$  is the set of  $r \in R$  such that  $r_u = r$ . Note that  $r \in R_u$  if and only if  $ur = rt$  with  $t \in H$ , meaning that  $t = r^{-1}ur$ .

Now note that  $\rho_r$  is an isomorphism from  $W$  onto  $\rho_r(W)$ . We also have that

$$\rho_r \circ \theta_t = \rho_u|_{\rho_r(W)} \circ \rho_r$$

where  $t = r^{-1}ur \in H$ . Thus the trace of  $\rho_u|_{\rho_r(W)}$  is equal to the trace of  $\theta_t$ . **<< Why? >>** Thus

$$\chi_\rho(u) = \sum_{r \in R_u} \text{Tr}_{\rho_r(W)}(\rho_u|_{\rho_r(W)}) = \sum_{r \in R_u} \chi_\theta(r^{-1}ur).$$

The second formula is immediate via a counting argument since  $\rho_s(W)$  only depends on  $sH$ , not just  $s$ .  $\square$

## Exercises

**Exercise 3.1 (Ser77 Exercise 3.1):** Using Schur's Lemma (Proposition 2.5), show that each irreducible representation of an abelian group, finite or not, has degree 1.

**Proof:** Let  $\rho : G \rightarrow \text{GL}(V)$  be an irreducible representation of an abelian group  $G$ . Then note that since  $G$  is abelian, we have that  $\rho_s \circ \rho_t = \rho_t \circ \rho_s$  for all  $s, t \in G$ . Thus, by Proposition 2.5 we have that  $\rho_s$  is a homothety. This means that  $V$  must be of dimension 1.  $\square$

**Definition 3.15 (Dual of an Abelian Group):** Let  $G$  be an abelian group. Let  $\widehat{G}$  be the *dual* of the group  $G$ , the set of irreducible characters of  $G$ .

We now analyze this group. Note that by Theorem 3.1, we have that any representation is of degree 1. Then since  $G$  is finite, we have that the image of any element  $s \in G$  must also be of finite order. Thus, the image of any element of  $s \in G$  is a root of unity. If  $\chi_1, \chi_2 \in \widehat{G}$ , then  $\chi_1 \cdot \chi_2 \in \widehat{G}$ . This is immediate since the product of two roots of unity is another root of unity.

**Exercise 3.2 (Ser77 Exercise 3.3):** Show that  $\widehat{G}$  is an abelian group isomorphic to  $G$  where for  $s \in G$ , the map  $\chi \mapsto \chi(s)$  is the element in  $\widehat{G}$  dual to  $s$ .

**Proof:** It is clear that  $\widehat{G}$  is an abelian group since the product of any two roots of unity is another root of unity and  $\mathbb{C}^\times$  is an abelian group. Also we know by the proof of Theorem 3.1 that the number of irreducible characters of an abelian group  $G$  is  $|G|$ . Thus  $\widehat{G}$  has order  $|G|$ . Now consider the mapping

$$\begin{aligned} \phi : G &\rightarrow \widehat{G} \\ s &\mapsto (\chi \mapsto \chi(s)). \end{aligned}$$

If  $s \in G$ , then there is some irreducible representation  $\chi$  where  $\chi(s)$  is an  $n$ -th root of unity. Thus, if  $s \neq e$ , then  $\phi(s)$  is not the identity map. Thus,  $\phi$  is injective, and  $\phi$  is clearly a homomorphism. Since  $|G| = |\widehat{G}|$ , we have that  $G \simeq \widehat{G}$ .  $\square$

**Exercise 3.3 (Ser77 Exercise 3.4):** Let  $H$  be a subgroup of a group  $G$ . Show that each irreducible representation of  $G$  is contained in a representation induced by an irreducible representation of  $H$ .

**Proof:** The trivial representation on  $H$  is irreducible. We know by Example 3.7 that any subgroup, and thus  $H$ , induced the regular representation of  $G$ . Since every irreducible representation is contained in  $G$ , we are done.  $\square$

We may use this to give an alternative proof of Corollary 3.2.

**Proof:** Let  $\rho: G \rightarrow \text{GL}(V)$  be an irreducible representation. Let  $A \leq G$  be an abelian subgroup of  $G$  with irreducible representation  $\theta: A \rightarrow \text{GL}(W)$ . Then by Exercise 3.3, we have that  $\rho$  is contained in some representation, say  $\phi: G \rightarrow \text{GL}(W')$ , induced by  $\theta$ . Thus  $\dim(W') = [G:A]\dim(W)$ . Since  $A$  is abelian and  $\theta$  is irreducible,  $\dim(W) = 1$  by Theorem 3.1. Then since  $\rho$  is contained in  $\phi$ , we have that  $\dim(V) \leq \dim(W')$ . Thus  $\dim(V) \leq \frac{|G|}{|A|}$ .  $\square$

**Exercise 3.4 (Ser77 Exercise 3.6):** Suppose  $G = H \times K$ . Let  $\rho: G \rightarrow \text{GL}(V)$  be a representation of  $G$  induced by a representation  $\theta: H \rightarrow \text{GL}(W)$  of  $H$ . Show that  $\rho$  is isomorphic to  $\theta \otimes r_K$  where  $r_K$  is the regular representation of  $K$ .

**Proof:** Note that  $H$  is normal in  $G$  and so the representatives of our cosets of  $G/H$  are given by  $(e, k)$  where  $e$  is the identity of  $H$  and  $k$  ranges over  $K$ . Let  $\chi_\rho, \chi_\theta, \chi_r$ , and  $\chi$  be the characters of  $\rho, \theta, r_K$  and  $\theta \otimes r_K$  respectively. By Proposition 2.15 we have that

$$\chi(h, k) = \chi_\theta(h) \cdot \chi_r(k) = \begin{cases} \chi_\theta(h)|K| & k = e, \\ 0 & k \neq e. \end{cases}$$

Then by Theorem 3.14, Proposition 2.2, and normality of  $H$  we have that

$$\chi_\rho(h, e) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^1(h, e)s \in H}} \chi_\theta(s^{-1}(h, e)s) = \frac{1}{|H|} \sum_{s \in G} \chi_\theta(h) = \frac{|G|}{|H|} \chi_\theta(h) = \chi_\theta(h)|K|.$$

Then note that for  $\chi_\rho(h, k)$  where  $k \neq e$ , the sum in Theorem 3.14 is empty and thus 0. Thus,  $\chi_\rho = \chi$  meaning the representations are isomorphic by Corollary 2.13.  $\square$

## Chapter 4

# Compact Groups



I am skipping this for now since I (naively?) think I don't need these generalizations to compact groups.

# Chapter 5

## Examples

**Example 5.1 (Character Table of  $C_n$ ):** Consider the group  $C_n$  generated by an element  $r$  such that  $r^n = 1$ . Since  $C_n$  is abelian, its representation are of degree 1 by Theorem 3.1. Since  $1 = \chi(r^n) = \chi(r)^n$ , we have that  $\chi(r)$  is  $e^{2\pi i h/n}$  where  $h = 0, \dots, n-1$ , a root of unity. This gives  $n$  distinct irreducible representations of degree 1 with character  $\chi_0, \dots, \chi_{n-1}$  such that

$$\chi_h(r^k) = e^{2\pi i h k/n}.$$

We also have that  $\chi_h \cdot \chi_{h'} = \chi_{h+h'}$  with addition taking modulo  $n$ . If we take  $n = 3$  we get the following character table: where  $w = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ .

	1	$r$	$r^2$
$\chi_0$	1	1	1
$\chi_1$	1	$w$	$w^2$
$\chi_2$	1	$w^2$	$w$

**Example 5.2 (Character Table of  $D_n$ ):** This is the group of rotations and reflections of a polygon with  $n$  vertices. It is generated by two elements  $r$  and  $s$  satisfying  $r^n = 1$ ,  $s^2 = 1$ , and  $srs = r^{-1}$ . Since  $C_n \leq D_n$ , we have by Corollary 3.2 that every irreducible representation of  $D_n$  is of degree 1 or 2. We split our analysis into the case of  $n \geq 2$  even and  $n$  odd.

Suppose  $n \geq 2$  is even. Then we have 4 representations of degree 1, obtained by letting  $\pm 1$  correspond to  $r$  and  $s$  in all possible ways. Their characters are  $\psi_1, \psi_2, \psi_3$ , and  $\psi_4$  given by the following table:

	$r^k$	$s r^k$
$\psi_1$	1	1
$\psi_2$	1	-1
$\psi_3$	$(-1)^k$	$(-1)^k$
$\psi_4$	$(-1)^k$	$(-1)^{k+1}$

The representations of degree 2 are more interesting. Let  $w = e^{2\pi i/n}$  and  $h$  an arbitrary integer. Then define a representation  $\rho^h: D_n \text{ GL}(\mathbb{C}^2)$  by setting

$$\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \quad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{-hk} \\ w^{hk} & 0 \end{pmatrix}.$$

This is induced by the representation of  $C_n$  with character  $\chi_h$  from Example 5.1. It only depends on the residue class of  $n$  modulo  $h$ . Moreover, we have that  $\rho^h$  and  $\rho^{n-h}$  are isomorphic. Thus, we may assume  $0 \leq h \leq \frac{n}{2}$ . The cases of  $h = 0$  and  $h = \frac{n}{2}$  are not interesting since these representations are reducible with characters  $\psi_1 + \psi_2$  and  $\psi_3 + \psi_4$  respectively. The other cases are irreducible. Since  $w^h \neq w^{-h}$  for  $0 < h < \frac{n}{2}$ , the only lines stable under  $\rho^h(r)$  are the coordinate axes, which are not stable under  $\rho^h(s)$ . This also shows that these representations are all pairwise non-isomorphic. The character  $\chi_h$  of  $\rho^h$  are given by

$$\begin{aligned} \chi_h(r^k) &= w^{hk} + w^{-hk} = 2 \cos \frac{2\pi hk}{n}, \\ \chi_h(sr^k) &= 0. \end{aligned}$$

Since the sums of the squares of their degrees is  $4 \cdot 1^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 = 2n$ , we have found all the irreducible representations.

Now suppose  $n$  is odd. The only representations of degree 1 with characters  $\chi_1$  and  $\chi_2$  are given by The

	$r^k$	$sr^k$
$\psi_1$	1	1
$\psi_2$	1	-1

representations  $\rho^h$  of degree 2 are the same as in the even case, and are also irreducible and pairwise non-isomorphic. Similarly we have that  $2 \cdot 1^2 + \frac{n-1}{2} \cdot 2^2 = 2n$ , and thus we have found all the irreducible representations.

**Example 5.3 (Character Table of  $D_{nh}$ ):** We now consider the group  $D_{nh} := D_n \times C_2$  where  $C_2$  is the group of order two generated by an element  $\iota$ . By Theorem 3.5, the irreducible representations of  $D_{nh}$  are just the tensor products of the irreducible representations of  $D_n$  and  $C_2$ . By Example 5.1, the character table of  $C_2$  with characters  $g, u$  is given by the table:

	1	$\iota$
$g$	1	1
$u$	1	-1

From this, for each character  $\chi$  of  $D_n$  we get two irreducible characters  $\chi_g$  and  $\chi_u$  given by the table

	$x$	$\iota x$
$\chi_g$	$\chi(x)$	$\chi(x)$
$\chi_u$	$\chi(x)$	$-\chi(x)$



**Example 5.4 (Character Table of  $A_4$ ):** Recall that the alternating group  $A_4$  has 12 elements:

- The identity element  $()$ ,
- Three elements of order 2:  $x = (12)(34)$ ,  $y = (13)(24)$ ,  $z = (14)(23)$ ,
- Eight elements of order 3. Let  $t$  be the element  $(123)$  of order 3.

Let  $K = \{1, t, t^2\}$  and  $H = \{1, x, y, z\}$ . Then we have

$$txt^{-1} = z,$$

$$tzt^{-1} = y,$$

$$tyt^{-1} = x.$$

Note that  $H$  and  $K$  are subgroups of  $A_4$ ,  $H \trianglelefteq A_4$  and  $H \cap K = e$ . Thus  $A_4$  is the semidirect product of  $H$  and  $K$  and every element of  $A_4$  is uniquely expressed as a product of elements  $hk$  with  $h \in H$  and  $k \in K$ .

There are 4 conjugacy classes of  $A_4$ , and thus by Theorem 2.21, there are 4 irreducible characters. There these are  $\{e\}$ ,  $\{x, y, z\}$ ,  $\{t, tx, ty, tz\}$ , and  $\{t^2, t^2x, t^2y, t^2z\}$ . Three of these are degree 1, corresponding to the characters  $\chi_0, \chi_1$ , and  $\chi_2$  of Example 5.1 and extending them  $A_4$  by  $\chi_i(hk) = \chi_i(k)$ . The last character  $\psi$  we may determine using Corollary 2.17. It is the character of the **⟨ natural representation ⟩** of  $A_4$  in  $\mathbb{R}^3$  extended to  $\mathbb{C}^3$  by linearity. Thus, we have the following character table for  $A_4$ : where  $w = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ .

	1	$x$	$t$	$t^2$
$\chi_0$	1	1	1	1
$\chi_1$	1	1	$w$	$w^2$
$\chi_2$	1	1	$w^2$	$w$
$\psi$	3	-1	0	0

**Example 5.5 (Character Table of  $S_4$ ):** Recall that  $S_4$  has 24 elements. Let  $x, y, z$  be as in Example 5.4. Let  $H = \{e, x, y, z\}$ , where  $H \trianglelefteq S_4$ , and  $L$  the group of permutations leaving 4 fixed. Then  $S_4$  is the semidirect product of  $H$  by  $L$ . Similarly, each element of  $S_4$  is a unique product of elements  $hl$  with  $h \in H$  and  $l \in L$ . Each representation  $\rho$  of  $L$  is extended to  $S_4$  by setting  $\rho(hl) = \rho(l)$  for  $h \in H$  and  $l \in L$ . This gives three irreducible representations of  $S_4$  of degrees 1, 1, and 2. We know the irreducible representations of  $L \simeq S_3$  by Example 2.23. On the other hand, the **⟨ natural representation ⟩** of  $S_4$  in  $\mathbb{C}^3$  is irreducible since its restriction to  $A_4$  is. This is also true of the tensor product of this representation with the nontrivial degree 1 representation of  $S_4$ . Thus the character table of  $S^4$  is

	1	(12)	(12)(34)	(123)	(1234)
$\chi_0$	1	1	1	1	1
$\varepsilon$	1	-1	1	1	-1
$\theta$	2	0	2	-1	0
$\psi$	3	1	-1	0	-1
$\varepsilon\psi$	3	-1	-1	0	1

# Bibliography

- [Ful96] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996. DOI: [10.1017/CB09780511626241](https://doi.org/10.1017/CB09780511626241).
- [Pan23] Greta Panova. *Computational Complexity in Algebraic Combinatorics*. 2023. arXiv: [2306.17511](https://arxiv.org/abs/2306.17511) [math.CO].
- [Rez20] Charles Rezk. *A short course on finite group representations*. 2020. URL: <https://rezk.web.illinois.edu/Finite%5C%20Group%5C%20Reps/short-course-finite-group-representations.html>.
- [Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*. Springer New York, 1977. ISBN: 9781468494587. DOI: [10.1007/978-1-4684-9458-7](https://doi.org/10.1007/978-1-4684-9458-7).