Algorithms in Invariant Theory Exercises

With 0 Figures

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Preface

A core interest of mine (at the time of writing this) is algorithms in the context of commutative algebra and algebraic geometry. As such, Bernd Sturmfel's text *Algorithms in Invariant Theory* [Str08] is a good resource for first learning this stuff. This solely just contains exercises for now. Perhaps in the future I'll include notes and some source code

Chapter 1

Introduction

Symmetric Polynomials

Exercise 1.1 (Str08 1.1.5): Prove the following explicit formula for elementary symmetric polynomials in terms of the power sums [Mac98, Page 29].

$$\sigma_k = \frac{1}{k!} \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix}$$

Proof: From [Page 23, 2.11'][Mac98], we have the following identities

$$\begin{aligned} 1 \cdot \sigma_1 &= p_1, \\ 2 \cdot \sigma_2 &= p_1\sigma_1 - p_2, \\ 3 \cdot \sigma_3 &= p_1\sigma_2 - p_2\sigma_1 + p_3, \\ &\vdots \\ k \cdot \sigma_k &= \sum_{r=1}^{k} (-1)^{r-1} p_r \sigma_{k-r}. \end{aligned}$$

Treating the σ_i as indeterminants, we can re-express the above system of equations:

$$
\begin{aligned}
p_1 &= 1 \cdot \sigma_1, \\
p_2 &= p_1\sigma_1 - 2 \cdot \sigma_2, \\
p_3 &= p_2\sigma_1 - p_1\sigma_2 + 3 \cdot \sigma_3, \\
&\vdots \\
p_k &= \left(\sum_{r=1}^{k-1} (-1)^{r-1} p_{k-r}\sigma_r \right) + (-1)^{k-1} \cdot k\sigma_k.
\end{aligned}
$$

Consider $\sigma_1, -\sigma_2, \sigma_3, \ldots, (-1)^n\sigma_{n-1}, \sigma_n$ as indeterminants. Thus, we obtain the following matrix equation:

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
p_1 & 2 & 0 & \cdots & 0 \\
p_2 & p_1 & 3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\ -\sigma_2 \\ \vdots \\ (-1)^k\sigma_{k-1} \\ \sigma_k
\end{pmatrix}
=
\begin{pmatrix}
p_1 \\ p_2 \\ \vdots \\ p_{k-1} \\ p_k
\end{pmatrix}.
$$

Then, Cramer's rule yields that

$$
\begin{aligned}
\sigma_k &= \det \begin{pmatrix}
1 & 0 & 0 & \cdots & p_1 \\
p_1 & 2 & 0 & \cdots & p_2 \\
p_2 & p_1 & 3 & \cdots & p_3 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & \cdots & p_k
\end{pmatrix}
\det \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
p_1 & 2 & 0 & \cdots & 0 \\
p_2 & p_1 & 3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k
\end{pmatrix}^{-1} \\
&= \frac{(-1)^k}{k!} \det \begin{pmatrix}
1 & 0 & 0 & \cdots & p_1 \\
p_1 & 2 & 0 & \cdots & p_2 \\
p_2 & p_1 & 3 & \cdots & p_3 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & \cdots & p_k
\end{pmatrix} \\
&= \frac{(-1)^k}{k!} \cdot (-1)^k \det \begin{pmatrix}
p_1 & 1 & 0 & \cdots & 0 \\
p_2 & p_1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\
p_k & p_{k-1} & \cdots & \cdots & p_1
\end{pmatrix}
= \frac{1}{k!} \det \begin{pmatrix}
p_1 & 1 & 0 & \cdots & 0 \\
p_2 & p_1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\
p_k & p_{k-1} & \cdots & \cdots & p_1
\end{pmatrix}.
\end{aligned}
$$

$\square$

Gröbner Bases

Exercise 1.2 (Str08 1.2.1): Let $\prec$ be an monomial order and let I be any ideal in $\mathbb{C}[x_1, \ldots, x_n]$. A monomial m is called *minimally nonstandard* if M is nonstandard and all proper divisors of m are standard. Show that the set of minimally nonstandard monomials is finite.

Proof: **⟨⟨ Have ugly inductive proof ⟩⟩** □

Exercise 1.3 (Str08 1.2.2): Prove that the reduced Gröbner basis $\mathcal{G}_{\text{red}}$ of I with respect to $\prec$ is unique (up to multiplicative constants from $\mathbb{C}$). Given an algorithm which transforms an arbitrary Gröbner basis into $\mathcal{G}_{\text{red}}$.

Proof: This is [CLO15, Chapter 2, §7, Theorem 5]. □

Exercise 1.4 (Str08 1.2.3): Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal, given by a finite set of generators. Using Gröbner bases, describe an algorithm for computing the *elimination ideals* $I \cap \mathbb{C}[x_1, \ldots, x_i]$ for $i = 1, \ldots, n-1$, and prove its correctness.

Proof: This is [CLO15, Chapter 3, §1, Theorem 2]. □

Exercise 1.5 (Str08 1.2.4): Find a characterization of all monomial orders on the polynomial ring $\mathbb{C}[x_1, x_2]$. Hint: each variable receives a certain "weight" which behaves additively under multiplication of variables. Generalize your result to n variables.

Bibliography

[CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. ISBN: 9783319167213. URL: https://books.google.com/books?id=yL7yCAAAQBAJ.

[Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

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