Young Tableau, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci

With 0 Figures

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Preface

These are notes for a reading course under Professor Dave Anderson. The primary focus is Manivel's *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci* [Man01] which one could see as a quasi-sequel to Fulton's *Young Tableaux*[1] [Ful97]. Primarily, the solutions will be to exercises from [Man01]. However, as needed there will be solutions to material from [Ful97], or perhaps even other texts such as [Mac98] or [Sta24].

[1]which throughout these notes will be spelled as "tableaux" or "tableau" with no real consistency.

Chapter 1

[Ful97] Geometry

Solution: [Ful97] §9.1 Ex. 1: Choose a basis $\{e_1, \ldots, e_m\}$ so that E can be identified with $\mathbb{C}^m$. Let $i_1 < \cdots < i_{d-1}$ and $j_1 < \cdots j_{d+1}$ be sequences in $[m]$. Apply §9.1 Equation (1) with $k = 1$ to the sequences $j_2 < \cdots < j_{d+1}$ and $i_1 < \cdots < i_{d-1}, j_1$ by fixing j_1 to be the vector swapped successively with the $j_2 < \cdots < j_{d+1}$. Reordering the indices and applying the appropriate sign change yields the desired alternating summation. □

Solution: [Ful97] §9.1 Ex. 2: We have that $V \subseteq E = \mathbb{C}^4$ is given as the kernel of multiplication of a matrix $A = (a_{i,j})_{\substack{1 \le i \le 4 \\ 1 \le j \le 2}}$. To find this matrix, the given conditions of the $x_{i,j}$ describe the following determinantal conditions on the entries of A:

$$\begin{aligned}
x_{1,2} = 1 &\iff \Delta_{1,2}(A) = 1, \\
x_{1,3} = 2 &\iff \Delta_{1,3}(A) = 2, \\
x_{1,4} = 1 &\iff \Delta_{1,4}(A) = 1, \\
x_{2,3} = 1 &\iff \Delta_{2,3}(A) = 1, \\
x_{2,4} = 2 &\iff \Delta_{2,4}(A) = 2, \\
x_{3,4} = 3 &\iff \Delta_{3,4}(A) = 3.
\end{aligned}$$

From here, we must make an assumption based on which affine portion of $\mathbb{P}^5$ our matrix lives in. This amounts to picking some i_1, i_2 so that the minor given by those columns is the identity matrix. For the given conditions, we could pick $(i_1, i_2) = (1,2)$, $(1,4)$, or $(2,3)$. We give A for each of these choices respectively:

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 0 & -3 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

It is clear that these are the same matrix up to row operations. □

Solution: [Ful97] §9.1 Ex. 4: Following [Ful97, Corollary of Theorem 1, §8.1], we have that $S^\bullet(m; d_1, \ldots, d_s)$ is canonically isomorphic to the subalgebra of $\mathbb{C}[Z]$ generated by all D_T, where T varies over all tableaux on Young diagrams whose columns have lengths in $\{d_1, \ldots, d_s\}$ and entries in $[m]$ where

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j),T(2,j),\ldots,T_{\mu_j,j}}$$

where μ_j is the length of the j^{th} column of λ the shape of T and $\ell = \lambda_1$.

(a) We mimic the proof of [Ful97, Proposition 2, §9.1]. **⟨⟨ I think this proof needs to be rewritten, perhaps with a highest weight argument? ⟩⟩** Let $G = G(d_1, \ldots, d_s) \leq \mathrm{GL}(V)$. The dimension of the vector space of polynomials of homogeneous polynomials of degree a in the span of all the $D_{i_1,\ldots,i_p}$ for $p \in \{d_1, \ldots, d_s\}$ is $\sum d_\lambda(m)$ where the sum ranges over all partitions of a of shape λ with columns whose lengths lie in $\{d_1, \ldots, d_s\}$. Viewing $V^{\oplus m}$ by identifying $Z_{i,j}$ with the i^{th} basis vector of the j^{th} copy of V, we have by [Ful97, Corollary 3(a), §8.3] that $\mathbb{C}[Z]_a = \mathrm{Sym}^a(V^{\oplus m}) \simeq \bigoplus_{\lambda \vdash a} (V^\lambda)^{d_\lambda(m)}$ where $\lambda \vdash a$ has at most n rows. Thus, we would like to show that $(V^\lambda)^G$ has dimension 1 when the lengths of the columns of λ lie in $\{d_1, \ldots, d_s\}$ and 0 otherwise.

We recall the construction of V^λ in §8.1 of [Ful97]. Elements of $V^{\times\lambda}$ are specified by specifying an element of V for each box in λ. Fillings by basis vectors $\{e_1, \ldots, e_n\}$ corresponding to semistandard Young Tableaux T of shape λ with entries in $[n]$. The images of such elements in $V^{\times\lambda}$ in V^λ form a basis $\{e_T\}$ of V^λ. Consider the basis element corresponding to the tableaux $U(\lambda)$ given by filling every box on row i with the number i. For maps in G, the first d_i basis vectors must map to linear combinations of the first i basis vectors and the restrictions of such maps to the V_i have determinant 1. As such, we can only consider λ whose columns have lengths lying in $\{d_1, \ldots, d_s\}$. To see that $e_{U(\lambda)}$ is the only such fixed basis vector,

(b)

□

Chapter 2

[Man01] The Ring of Symmetric Functions

2.1 Ordinary Functions

Solution: [Man01] Ex. 1.1.2: We will denote the dominance ordering by $\lambda \leq \mu$ and the ordering given by inclusion of Ferrers diagrams by $\lambda \subseteq \mu$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ and $\lambda' = (\lambda'_1 \geq \cdots \geq \lambda'_l \geq 0)$ be two partitions.

We first consider the ordering $\subseteq$. Note that $\lambda \subseteq \lambda'$ if and only if $k \leq l$ and for all $1 \leq i \leq k$ we have that $\lambda_i \leq \lambda'_i$. Let $m = \min\{k, l\}$. Then define a partition $\mu = (\min\{\lambda_1, \lambda'_1\} \geq \cdots \min\{\lambda_m, \lambda'_m\} \geq 0)$. Then we have that $\mu \subseteq \lambda$ and $\mu \subseteq \lambda'$. Now suppose that $\nu \subseteq \lambda$ and $\nu \subseteq \lambda'$ where $\nu = (\nu_1 \geq \cdots \geq \nu_n \geq 0)$. Then we must have that $n \leq \min\{k, l\} = m$ and that for all $1 \leq i \leq n$ that $\nu_i \leq \min\{\lambda_i, \lambda'_i\} = \mu_i$. Thus, $\nu \subseteq \mu$ and so $\mu = \lambda \wedge \lambda'$ with respect to $\subseteq$. The existence and uniqueness of $\lambda \vee \lambda'$ is similar.

We now consider the ordering $\leq$, now assuming that $|\lambda| = \left|\lambda'\right|$. Before we define $\lambda \vee \lambda'$ for $\leq$, we prove that $\lambda \leq \lambda'$ if and only if $\lambda'^* \leq \lambda^*$. This follows a proof given by [Ros]. Note that $\lambda \leq \lambda'$ if and only if λ can be obtained from λ' by moving boxes successively down from higher rows to lower rows such that every intermediate diagram is still a Ferrers diagram. This immediately implies the duality.

First, for a partition λ let $\hat{\lambda} := (0, \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_k)$. We remark that $\lambda \leq \lambda'$ if and only $\hat{\lambda} \leq_\ell \hat{\lambda}'$ where $\leq_\ell$ is *lexicographic ordering*. One can easily recover λ from $\hat{\lambda}$. By taking componentwise minimums as above for $\hat{\lambda}$ and $\hat{\lambda}'$, one recovers a tuple $\hat{\mu}$ which yields a partition μ. By the remark, we have that $\mu = \lambda \wedge \lambda'$ with respect to $\leq$. Then to define $\lambda \vee \lambda'$, we have that $\lambda \vee \lambda' := (\lambda^* \wedge \lambda'^*)^*$. That this is our desired maximum follows from the above paragraph on conjugation being a lattice antiautomorphism on partitions of the same weight. Uniqueness is also immediate. $\square$

Solution: [Man01] Ex. 1.1.7: These ideas come from [Sta24, Proposition 7.4.1]. Let $X = (x_{ij})$ be the matrix of variables where $x_{ij} = x_j$, so the first column of X is all x_1, the second column is all x_2, etc. We can obtain a term from of e_λ from X by choosing λ_1 elements from the first row, λ_2 elements from the second row, corresponding to picking a term from e_{λ_1}, then a term from e_{λ_2}, etc. After choosing all elements, let the result be $\overline{x}^\alpha$. Replace all the chosen elements with 1's and all the other elements with 0's. This gives a matrix with row sums given by λ and all column sums given by α. Note that α is not necessarily a partition, but rather just a tuple (also called a *weak composition*), which is in line with the fact that monomial symmetric functions are sums over arbitrary orderings of tuples with a fixed set of entries. Conversely, any such 0-11matrix with the prescribed row and column sums describes a term of e_λ. Thus, we have that $e_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$.

Similarly, with X as before, we can obtain a term of h_λ as follows. Choose λ_1 elements from the first row, but we allow each term to be chosen more than once. Next, choose λ_2 elements from the second row, again allowing for each term to be chosen more than once. Continuing on in this manner yields a term $\overline{x}^\alpha$. This again give a matrix, however this time with entries in $\mathbb{N}$ given by numbering the elements with however many times they were chosen. This matrix has the prescribed row and column sums. Conversely, any such matrix with entries in $\mathbb{N}$ with the given row and column sums gives a term of h_λ and so $h_\lambda = \sum_\mu b_{\lambda\mu} m_\mu$.

Now suppose that $a_{\lambda\mu} > 0$. Then we want to show that $\mu \leq \lambda^*$, i.e. that $|\lambda| = |\mu|$ and that for all i we have that $\mu_1 + \cdots + \mu_i \leq \lambda_1^* + \cdots + \lambda_i^*$. If $|\lambda| \neq |\mu|$, then we must have that $a_{\lambda\mu} = 0$ as both $|\lambda|$ and $|\mu|$ are equal to the total number of ones and so we must have that $|\lambda| = |\mu|$. So by the above argument, there exist a 0-1-matrix M with row sums given by λ and column sums given by μ. Suppose there exists i such that $\mu_1 + \cdots + \mu_i > \lambda_1^* + \cdots + \lambda_i^*$.

⟨⟨ **Morally** ⟩⟩ I would like to say the λ_i^* correspond to column sums as well in some manner but I am not sure how to phrase that. □

2.2 Schur Functions

Solution: [Man01] Ex. 1.2.4: We have that $a_{\delta+\delta} = \det\left(x_i^{\delta_j+n-j}\right) = \det\left(x_i^{2n-2j}\right)$. This is the Vandermonde determinant again, but now every term is squared. Thus, $a_{\delta+\delta} = \prod_{1\le i<j\le n}(x_i^2 - x_j^2)$. Thus, we have that

$$s_\delta = \frac{a_{\delta+\delta}}{a_\delta} = \frac{\prod_{1\le i<j\le n}(x_i^2-x_j^2)}{\prod_{1\le i<j\le n}(x_i-x_j)} = \prod_{1\le i<j\le n}(x_i+x_j).$$

$\square$

Solution: [Man01] Ex. 1.2.7: By Pieri's formulas, we have that

$$\left(\sum_{\mu \text{ even}} s_\mu\right)\cdot\left(\sum_{n=0}^{k} e_k\right) = \sum_{\mu \text{ even}}\sum_{k=0}^{n} s_\mu e_k = \sum_{\mu \text{ even}}\sum_{k=0}^{n}\sum_{\lambda\in\mu\otimes 1^k} s_\lambda. \tag{2.1}$$

Clearly, every s_λ term, *not monomial terms*, in the last summation of Equation (2.1) is a term in $\sum_\lambda s_\lambda$, except possibly with a coefficient > 1. We claim that all the coefficients are indeed 1 and that every term in $\sum_\lambda s_\lambda$ appears in the in the last summation of Equation (2.1). This follows from the fact that for any λ, we can decompose λ into an even μ by removing at most one box from each row of λ in each row which is odd and that this removal is unique. $\square$

Solution: [Man01] Ex. 1.2.12: The first identity comes from noticing that if you take any standard Young tableaux with n boxes and remove the box labelled n, then you obtain a standard Young tableaux with $n-1$ boxes. Furthermore, if you add a box labelled n to any valid position of a Young tableaux with $n-1$ boxes, valid meaning the resulting shape is still a partition, then you obtain a standard Young tableaux with n boxes. This gives a combinatorial bijection between the two sets described by each side of first identity.

For the second identity, suppose that $|\lambda| = (1)$. Then λ is just a single box and thus we must have that $K_\lambda = K_{(1)} = 1$ and so $(1+|(1)|)K_{(1)} = 2$. Then $(1)\otimes 1 = \{(1,1),(2)\}$ which each have exactly one standard filling and so we have that $K_{(1,1)} = K_{(2)} = 1$ and thus $\sum_{\mu\in(1)\otimes 1} K_\mu = 2$. Now suppose that $|\lambda| = n > 1$. We have that

$$\begin{aligned}(1+|\lambda|)K_\lambda &= (1+|\lambda|)\sum_{\lambda\in\mu\otimes 1} K_\mu \\ &= \sum_{\lambda\in\mu\otimes 1} K_\mu + \sum_{\lambda\in\mu\otimes 1}(1+|\mu|)K_\mu \qquad = \sum_{\lambda\in\mu\otimes 1} K_\mu + \sum_{\lambda\in\mu\otimes 1}\sum_{\nu\in\mu\otimes 1} K_\nu\end{aligned}$$

⟨⟨ **Not sure** ⟩⟩ how to work with this double summation.

For the third identity, as $K_{(1)} = 1$ we immediately have that $\sum_{\lambda=1} K_\lambda^2 = K_{(1)}^2 = 1 = 1!$. Now suppose that $|\lambda| = \ell > 0$. Then we have that

$$\begin{aligned}
\ell! &= \ell \cdot (\ell - 1)! \\
&= \ell \sum_{|\lambda|=\ell-1} K_\lambda^2 \\
&= \sum_{|\lambda|=\ell-1} K_\lambda \cdot (\ell K_\lambda) \\
&= \sum_{|\lambda|=\ell-1} K_\lambda \cdot \sum_{\mu \in \lambda \otimes 1} K_\mu
\end{aligned}$$

⟨⟨ **Not sure** ⟩⟩ how to work with this double summation. □

Solution: [Man01] Ex. 1.2.15: Recall that $h_j = s_{(j)}$ and $e_k = s_{1^k}$. Using the Pieri formulas, we can express $h_j e_k$ as

$$\sum_{\mu \in 1^k \otimes j} s_\mu = s_{1^k} h_j = h_j e_k = s_{(j)} e_k = \sum_{\mu \in (j) \otimes 1^k} s_\mu.$$

⟨⟨ **Expanding either side** ⟩⟩ gives $h_j s_k = s_{(j-1|k)} + s_{(j|k-1)}$ which is already stated. Not sure what a second way would be, nor how to introduce the variable q in a generating-function sort of way. □

2.3 The Knuth Correspondence

Solution: [Man01] Ex. 1.3.1: Already saw this as the *Row Bumping Lemma* in [Ful97] which gives a slightly stronger characterization. □

2.4 Some Applications to Symmetric Functions

Solution: [Man01] Ex. 1.4.4: ⟨⟨ **Why** ⟩⟩ are these bases?

Let $M_{\lambda\nu}$ and $N_{\lambda\nu}$ be such that $s_\lambda = \sum_\mu M_{\lambda\mu} = \sum_\nu N_{\lambda\nu} b_\nu$. Then we have that

$$\begin{aligned}
\sum_\lambda a_\lambda(\overline{x}) b_\lambda(\overline{y}) &= \prod_{i,j} (1 - x_i y_j)^{-1} \\
&= \sum_\lambda s_\lambda(\overline{x}) s_\lambda(\overline{y}) \\
&= \sum_\lambda \left(\sum_\rho M_{\lambda\rho} a_\rho(\overline{x}) \right) \left(\sum_\nu N_{\lambda\nu} b_\nu(\overline{y}) \right) = \sum_{\rho,\nu} \left(\sum_\lambda M_{\lambda\rho} N_{\lambda\nu} \right) a_\rho(\overline{x}) b_\nu(\overline{y}).
\end{aligned}$$

Thus by the fact that the a_ρ and b_ν form bases in their respective variables, we have that $\sum_\lambda M_{\lambda\rho} N_{\lambda\nu} = \left\langle a_\rho, b_\nu \right\rangle$. We want to show that $\left\langle a_\rho, b_\nu \right\rangle = \delta_{\rho\nu}$. Indeed, this follows from that

$$\sum_\lambda s_\lambda(\overline{x}) s_\lambda(\overline{y}) = \sum_\lambda a_\lambda(\overline{x}) b_\lambda(\overline{y}) \implies \sum_\lambda M_{\lambda\rho} N_{\lambda\nu} = \delta_{\rho\nu}.$$

□

2.5 The Littlewood-Richardson Rule

Solution: [Man01] Ex. 1.5.4: We consider the coefficient of $\overline{x}^{\alpha}$ on both sides. We have that

$$\prod_i (1-x_i)^{-1} \cdot \prod_{i<j} (1-x_i x_j)^{-1} = \left(\prod_i \sum_{n\geq 0} x_i^n\right) \cdot \left(\prod_{i<j} \sum_{n\geq 0} x_i^n x_j^n\right).$$

Notices that the coefficient of $\overline{x}^{\alpha}$ is equal to the number of symmetric matrices A such that the vector of row-sums of A is equal to α. Then, by the combinatorial definition of Schur polynomials, the coefficient of $\overline{x}^{\alpha}$ in $\sum_\lambda s_\lambda(\overline{x})$ is equal to the number of semistandard Young tableaux with weight vector α. Then by [Man01, Knuth Correspondence 1.3.4] and in particular [Man01, Corollary 1.5.3], we know these two quantities must be equivalent, and thus the identity holds.

Next, recall from **??** that $\left(\sum_{\mu \text{ even}} s_\mu(\overline{x})\right) \cdot \left(\sum_{k=0}^n e_k\right) = \sum_\lambda s_\lambda$. To see that

$$\sum_{\mu \text{ even}} s_\mu(\overline{x}) = \prod_i (1-x_i)^{-2} \cdot \prod_{i<j} (1-x_i x_j)^{-1} = \prod_{i\leq j} (1-x_i x_j)^{-1}$$

simply apply the above identity and the fact that $\sum_k e_k = \prod_i (1+x_i)$ and divide. To prove the other identity, apply the involution ω using the fact that the sum over all λ is just a reordering of the sum over all λ^*. The same arguments above generalize to the generating function with $t^{o(\lambda)}$ by multiplying/dividing appropriately by $1+tx_i$ corresponding to odd parts of λ. □

Solution: [Man01] Ex. 1.5.6: By [Man01, Corollary 1.5.3], every standard tableau with n boxes corresponds to an involution $\sigma : [n] \longleftrightarrow [n]$. Thus, we can establish the recurrence for involution. Every involution $\sigma : [n+1] \longleftrightarrow [n+1]$ takes one of two forms. The first is that $\sigma(n+1) = n+1$ and the rest is an involution $\sigma|_{[n]} : [n] \longleftrightarrow [n]$. Otherwise, σ swaps $n+1$ and some $i \in [n]$ and the rest of σ is an involution on the other $n-1$ elements which up to relabelling is an involution from $n-1$ to $n-1$ elements. As these cases are disjoint, this yields the recursive formula.

To prove the identity, we work with exponential generating functions. This follows ideas from [Sta24, Example 7.8.5]. □

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