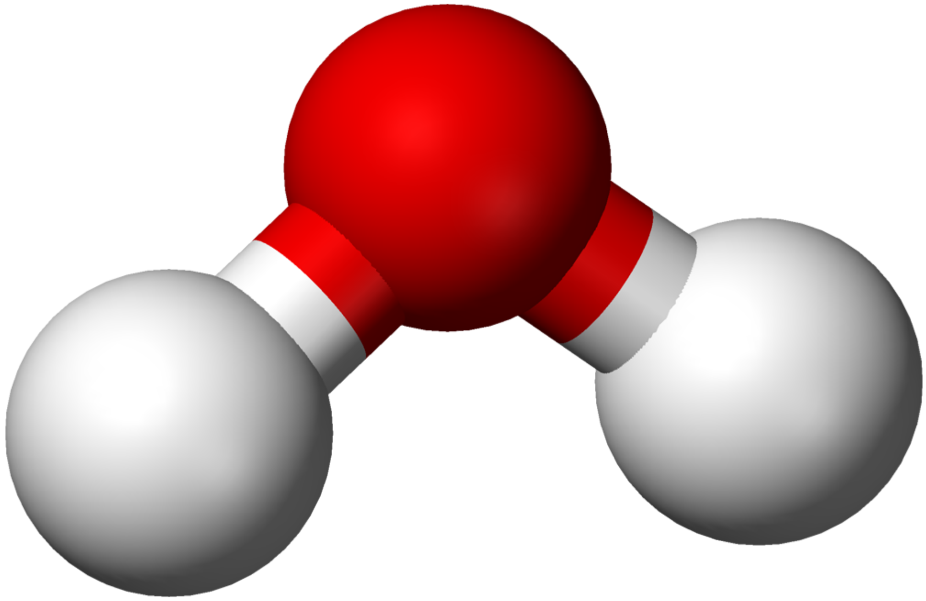

Representation Theory Notes and Exercises

With 0 Figures

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TODOs

⟨⟨ Change the style of enumerates from “1.” to “(1)” ⟩⟩

⟨⟨ Proper Exercise Header ⟩⟩

⟨⟨ Proper Chapter Header ⟩⟩

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Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. Occasionally, other sources may be used. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

Chapter 1

Generalities on Linear Representations

Unless otherwise specified, V will denote a vector space, usually over the field $\mathbb{C}$. We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

Definition 1.1 (Linear Representation, Representation Space): Let G be a group with identity e. A *linear representation* of G in V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will frequently, and often interchangably, write $\rho_s := \rho(s)$. Given ρ, we will say that V is a *representation space* or *representation* of G.

Definition 1.2 (Degree): Let $\rho : G \to V$ be a representation of G in a vector space V. Then the *degree* of ρ is $\dim(V)$.

Let $\rho : G \to V$ be a representation of G in a vector space V with $n := \dim(V)$. Fix a basis (e_j) of V. Then since each ρ_s is an invertible linear transformation of V, we may define an $n \times n$ matrix $R_s \equiv (r_{ij}(s))$ where each $r_{ij}(s)$ is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^{n} r_{ij}(s) e_i.$$

Definition 1.3 (Matrix of a Representation): We call $R_s = (r_{ij}(s))$ above the *matrix of* ρ_s with respect to the basis (e_j).

Note that R_s satisfies the following:

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^{n} r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two $n \times n$ matrices A, A' are *similar* if there exists an invertible matrix T such that $TA = A'T$. We may extend this notion to representations.

Definition 1.4 (Similar/Isomorphic Representations): Let ρ and ρ' be two representations of the same group G in vector spaces V and V' respectively. We say ρ and ρ' are *similar* or *isomorphic* if there exists an isomorphism $\tau : V \to V'$ such that for all $s \in G$, τ satisfies $\tau \circ \rho(s) = \rho'(s) \circ \tau$. If R_s, R'_s are the corresponding matrices then this is equivalent to saying there exists an invertible matrix T such that $TR_s = R'_s T$ for all $s \in G$.

Note that if ρ and ρ' are isomorphic, then they must have the same degree.

We now give some examples of these things.

Example 1.5 (Unit/Trivial Representation): Let G be a finite group. Representations of degree 1 must be of the form $\rho : G \to \mathbb{C}^\times$. Since elements s of G are of finite order, $\rho(s)$ must also be of finite order. Thus, for all $s \in G$, $\rho(s)$ is a root of unity. If we take $\rho(s) = 1$ for all $s \in G$, we obtain the *unit* or *trivial* representation of G.

Example 1.6 (Regular Representation): Let g be the order of G, and let V be a vector space of dimension g with a basis $(e_t)_{t \in G}$. For each $s \in G$, define ρ_s as the linear map $\rho_s : V \to V$ sending $e_t \mapsto e_{st}$. This is a linear representation of G called the *regular* representation of G. Since for each $s \in G$, $e_s = \rho_s(e_1)$ and thus the images of e_1 form a basis of V. Conversely, let W be a representation of G with a vector w satisfying the collection of all $\rho_s(w)$, $s \in G$, forms a basis of W. Then W is isomorphic to the regular representation of G by the isomorphism $\tau(e_s) = \rho_s(w)$.

Example 1.7 (Permutation Representation): We may generalize the regular representation to any group action $G \curvearrowright X$, X a finite set. Recall that for such an action, the map $x \mapsto sx$ for each $s \in G$ is a permutation $X \longleftrightarrow X$. Let V be a vector space with dimension the size of X, and so a basis $(e_x)_{x \in X}$. Define a representation ρ of G by defining ρ_s as the linear map sending $e_x \mapsto e_{sx}$. This representation is known as the *Permutation* representation of G associated with X.

Bibliography

[Ful96] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996. DOI: 10.1017/CBO9780511626241.

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