

# Young Tableau, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci

With 0 Figures

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# Preface

These are notes for a reading course under Professor [Dave Anderson](#). The primary focus is Manivel's *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci* [[Man01](#)] which one could see as a quasi-sequel to Fulton's *Young Tableaux*<sup>1</sup> [[Ful97](#)]. Primarily, the solutions will be to exercises from [[Man01](#)]. However, as needed there will be solutions to material from [[Ful97](#)], or perhaps even other texts such as [[Mac98](#)] or [[Sta01](#)].

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<sup>1</sup>which throughout these notes will be spelled as “tableaux” or “tableau” with no real consistency.

# Chapter 1

## [Ful97] Geometry

**Solution:** [Ful97] §9.1 Ex. 1: Choose a basis  $\{e_1, \dots, e_m\}$  so that  $E$  can be identified with  $\mathbb{C}^m$ . Let  $i_1 < \dots < i_{d-1}$  and  $j_1 < \dots < j_{d+1}$  be sequences in  $[m]$ . Apply §9.1 Equation (1) with  $k = 1$  to the sequences  $j_2 < \dots < j_{d+1}$  and  $i_1 < \dots < i_{d-1}, j_1$  by fixing  $j_1$  to be the vector swapped successively with the  $j_2 < \dots < j_{d+1}$ . Reordering the indices and applying the appropriate sign change yields the desired alternating summation.  $\square$

**Solution:** [Ful97] §9.1 Ex. 2: We have that  $V \subseteq E = \mathbb{C}^4$  is given as the kernel of multiplication of a matrix  $A = (a_{i,j})_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq 2}}$ . To find this matrix, the given conditions of the  $x_{i,j}$  describe the following determinantal conditions on the entries of  $A$ :

$$x_{1,2} = 1 \iff \Delta_{1,2}(A) = 1,$$

$$x_{1,3} = 2 \iff \Delta_{1,3}(A) = 2,$$

$$x_{1,4} = 1 \iff \Delta_{1,4}(A) = 1,$$

$$x_{2,3} = 1 \iff \Delta_{2,3}(A) = 1,$$

$$x_{2,4} = 2 \iff \Delta_{2,4}(A) = 2,$$

$$x_{3,4} = 3 \iff \Delta_{3,4}(A) = 3.$$

From here, we must make an assumption based on which affine portion of  $\mathbb{P}^5$  our matrix lives in. This amounts to picking some  $i_1, i_2$  so that the minor given by those columns is the identity matrix. For the given conditions, we could pick  $(i_1, i_2) = (1, 2), (1, 4),$  or  $(2, 3)$ . We give  $A$  for each of these choices respectively:

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 & 0 & -3 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

It is clear that these are the same matrix up to row operations.  $\square$

**Solution:** [Ful97] §9.1 Ex. 4: Following [Ful97, Corollary of Theorem 1, §8.1], we have that  $S^\bullet(m; d_1, \dots, d_s)$  is canonically isomorphic to the subalgebra of  $\mathbb{C}[Z]$  generated by all  $D_T$ , where  $T$  varies over all tableaux on Young diagrams whose columns have lengths in  $\{d_1, \dots, d_s\}$  and entries in  $[m]$  where

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j), T(2,j), \dots, T_{\mu_j, j}}$$

where  $\mu_j$  is the length of the  $j^{\text{th}}$  column of  $\lambda$  the shape of  $T$  and  $\ell = \lambda_1$ .

- (a) We mimic the proof of [Ful97, Proposition 2, §9.1]. **⟨ I think this proof needs to be rewritten, perhaps with a highest weight argument? ⟩** Let  $G = G(d_1, \dots, d_s) \leq \text{GL}(V)$ . The dimension of the vector space of polynomials of homogeneous polynomials of degree  $a$  in the span of all the  $D_{i_1, \dots, i_p}$  for  $p \in \{d_1, \dots, d_s\}$  is  $\sum d_\lambda(m)$  where the sum ranges over all partitions of  $a$  of shape  $\lambda$  with columns whose lengths lie in  $\{d_1, \dots, d_s\}$ . Viewing  $V^{\oplus m}$  by identifying  $Z_{i,j}$  with the  $i^{\text{th}}$  basis vector of the  $j^{\text{th}}$  copy of  $V$ , we have by [Ful97, Corollary 3(a), §8.3] that  $\mathbb{C}[Z]_a = \text{Sym}^a(V^{\oplus m}) \simeq \bigoplus_{\lambda \vdash a} (V^\lambda)^{d_\lambda(m)}$  where  $\lambda \vdash a$  has at most  $n$  rows. Thus, we would like to show that  $(V^\lambda)^G$  has dimension 1 when the lengths of the columns of  $\lambda$  lie in  $\{d_1, \dots, d_s\}$  and 0 otherwise.

We recall the construction of  $V^\lambda$  in §8.1 of [Ful97]. Elements of  $V^{\times \lambda}$  are specified by specifying an element of  $V$  for each box in  $\lambda$ . Fillings by basis vectors  $\{e_1, \dots, e_n\}$  corresponding to semistandard Young Tableaux  $T$  of shape  $\lambda$  with entries in  $[n]$ . The images of such elements in  $V^{\times \lambda}$  in  $V^\lambda$  form a basis  $\{e_T\}$  of  $V^\lambda$ . Consider the basis element corresponding to the tableaux  $U(\lambda)$  given by filling every box on row  $i$  with the number  $i$ . For maps in  $G$ , the first  $d_i$  basis vectors must map to linear combinations of the first  $i$  basis vectors and the restrictions of such maps to the  $V_i$  have determinant 1. As such, we can only consider  $\lambda$  whose columns have lengths lying in  $\{d_1, \dots, d_s\}$ . To see that  $e_{U(\lambda)}$  is the only such fixed basis vector,

(b)

□

## Chapter 2

# [Man01] The Ring of Symmetric Functions

### 2.1 Ordinary Functions

**Solution:** [Man01] Ex. 1.1.2: We will denote the dominance ordering by  $\lambda \leq \mu$  and the ordering given by inclusion of Ferrers diagrams by  $\lambda \subseteq \mu$ . Let  $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$  and  $\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_l \geq 0)$  be two partitions.

We first consider the ordering  $\subseteq$ . Note that  $\lambda \subseteq \lambda'$  if and only if  $k \leq l$  and for all  $1 \leq i \leq k$  we have that  $\lambda_i \leq \lambda'_i$ . Let  $m = \min\{k, l\}$ . Then define a partition  $\mu = (\min\{\lambda_1, \lambda'_1\} \geq \dots \geq \min\{\lambda_m, \lambda'_m\} \geq 0)$ . Then we have that  $\mu \subseteq \lambda$  and  $\mu \subseteq \lambda'$ . Now suppose that  $\nu \subseteq \lambda$  and  $\nu \subseteq \lambda'$  where  $\nu = (\nu_1 \geq \dots \geq \nu_n \geq 0)$ . Then we must have that  $n \leq \min\{k, l\} = m$  and that for all  $1 \leq i \leq n$  that  $\nu_i \leq \min\{\lambda_i, \lambda'_i\} = \mu_i$ . Thus,  $\nu \subseteq \mu$  and so  $\mu = \lambda \wedge \lambda'$  with respect to  $\subseteq$ . The existence and uniqueness of  $\lambda \vee \lambda'$  is similar.

We now consider the ordering  $\leq$ , now assuming that  $|\lambda| = |\lambda'|$ . Before we define  $\lambda \vee \lambda'$  for  $\leq$ , we prove that  $\lambda \leq \lambda'$  if and only if  $\lambda'^* \leq \lambda^*$ . This follows a proof given by [Ros]. Note that  $\lambda \leq \lambda'$  if and only if  $\lambda$  can be obtained from  $\lambda'$  by moving boxes successively down from higher rows to lower rows such that every intermediate diagram is still a Ferrers diagram. This immediately implies the duality.

First, for a partition  $\lambda$  let  $\hat{\lambda} := (0, \lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \dots + \lambda_k)$ . We remark that  $\lambda \leq \lambda'$  if and only if  $\hat{\lambda} \leq_\ell \hat{\lambda}'$  where  $\leq_\ell$  is *lexicographic ordering*. One can easily recover  $\lambda$  from  $\hat{\lambda}$ . By taking componentwise minimums as above for  $\hat{\lambda}$  and  $\hat{\lambda}'$ , one recovers a tuple  $\hat{\mu}$  which yields a partition  $\mu$ . By the remark, we have that  $\mu = \lambda \wedge \lambda'$  with respect to  $\leq$ . Then to define  $\lambda \vee \lambda'$ , we have that  $\lambda \vee \lambda' := (\lambda^* \wedge \lambda'^*)^*$ . That this is our desired maximum follows from the above paragraph on conjugation being a lattice antiautomorphism on partitions of the same weight. Uniqueness is also immediate.  $\square$

**Solution:** [Man01] Ex. 1.1.7: These ideas come from [Sta01, Proposition 7.4.1]. Let  $X = (x_{ij})$  be the matrix of variables where  $x_{ij} = x_j$ , so the first column of  $X$  is all  $x_1$ , the second column is all  $x_2$ , etc. We can obtain a term from  $e_\lambda$  from  $X$  by choosing  $\lambda_1$  elements from the first row,  $\lambda_2$  elements from the second row, corresponding to picking a term from  $e_{\lambda_1}$ , then a term from  $e_{\lambda_2}$ , etc. After choosing all elements, let the result be  $\bar{x}^\alpha$ . Replace all the chosen elements with 1's and all the other elements with 0's. This gives a matrix with row sums given by  $\lambda$  and all column sums given by  $\alpha$ . Note that  $\alpha$  is not necessarily a partition, but rather just a tuple (also called a *weak composition*), which is in line with the fact that monomial symmetric functions are sums over arbitrary orderings of tuples with a fixed set of entries. Conversely, any such 0-1 matrix with the prescribed row and column sums describes a term of  $e_\lambda$ . Thus, we have that  $e_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$ .

Similarly, with  $X$  as before, we can obtain a term of  $h_\lambda$  as follows. Choose  $\lambda_1$  elements from the first row, but we allow each term to be chosen more than once. Next, choose  $\lambda_2$  elements from the second row, again allowing for each term to be chosen more than once. Continuing on in this manner yields a term  $\bar{x}^\alpha$ . This again give a matrix, however this time with entries in  $\mathbb{N}$  given by numbering the elements with however many times they were chosen. This matrix has the prescribed row and column sums. Conversely, any such matrix with entries in  $\mathbb{N}$  with the given row and column sums gives a term of  $h_\lambda$  and so  $h_\lambda = \sum_\mu b_{\lambda\mu} m_\mu$ .

Now suppose that  $a_{\lambda\mu} > 0$ . Then we want to show that  $\mu \leq \lambda^*$ , i.e. that  $|\lambda| = |\mu|$  and that for all  $i$  we have that  $\mu_1 + \dots + \mu_i \leq \lambda_1^* + \dots + \lambda_i^*$ . If  $|\lambda| \neq |\mu|$ , then we must have that  $a_{\lambda\mu} = 0$  as both  $|\lambda|$  and  $|\mu|$  are equal to the total number of ones and so we must have that  $|\lambda| = |\mu|$ . So by the above argument, there exist a 0-1-matrix  $M$  with row sums given by  $\lambda$  and column sums given by  $\mu$ . Suppose there exists  $i$  such that  $\mu_1 + \dots + \mu_i > \lambda_1^* + \dots + \lambda_i^*$ .

⟨⟨ **Morally** ⟩⟩ I would like to say the  $\lambda_i^*$  correspond to column sums as well in some manner but I am not sure how to phrase that. □

## 2.2 Schur Functions

**Solution:** [Man01] Ex. 1.2.4: We have that  $a_{\delta+\delta} = \det(x_i^{\delta_j+n-j}) = \det(x_i^{2n-2j})$ . This is the Vandermonde determinant again, but now every term is squared. Thus,  $a_{\delta+\delta} = \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)$ . Thus, we have that

$$s_{\delta} = \frac{a_{\delta+\delta}}{a_{\delta}} = \frac{\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} = \prod_{1 \leq i < j \leq n} (x_i + x_j).$$

□

**Solution:** [Man01] Ex. 1.2.7: By Pieri's formulas, we have that

$$\left( \sum_{\mu \text{ even}} s_{\mu} \right) \cdot \left( \sum_{n=0}^k e_n \right) = \sum_{\mu \text{ even}} \sum_{k=0}^n s_{\mu} e_k = \sum_{\mu \text{ even}} \sum_{k=0}^n \sum_{\lambda \in \mu \otimes 1^k} s_{\lambda}. \quad (2.1)$$

Clearly, every  $s_{\lambda}$  term, *not monomial terms*, in the last summation of Equation (2.1) is a term in  $\sum_{\lambda} s_{\lambda}$ , except possibly with a coefficient  $> 1$ . We claim that all the coefficients are indeed 1 and that every term in  $\sum_{\lambda} s_{\lambda}$  appears in the in the last summation of Equation (2.1). This follows from the fact that for any  $\lambda$ , we can decompose  $\lambda$  into an even  $\mu$  by removing at most one box from each row of  $\lambda$  in each row which is odd and that this removal is unique. □

**Solution:** [Man01] Ex. 1.2.12: The first identity comes from noticing that if you take any standard Young tableaux with  $n$  boxes and remove the box labelled  $n$ , then you obtain a standard Young tableaux with  $n - 1$  boxes. Furthermore, if you add a box labelled  $n$  to any valid position of a Young tableaux with  $n - 1$  boxes, valid meaning the resulting shape is still a partition, then you obtain a standard Young tableaux with  $n$  boxes. This gives a combinatorial bijection between the two sets described by each side of first identity.

For the second identity, suppose that  $|\lambda| = (1)$ . Then  $\lambda$  is just a single box and thus we must have that  $K_{\lambda} = K_{(1)} = 1$  and so  $(1 + |(1)|)K_{(1)} = 2$ . Then  $(1) \otimes 1 = \{(1, 1), (2)\}$  which each have exactly one standard filling and so we have that  $K_{(1,1)} = K_{(2)} = 1$  and thus  $\sum_{\mu \in (1) \otimes 1} K_{\mu} = 2$ . Now suppose that  $|\lambda| = n > 1$ . We have that

$$\begin{aligned} (1 + |\lambda|)K_{\lambda} &= (1 + |\lambda|) \sum_{\lambda \in \mu \otimes 1} K_{\mu} \\ &= \sum_{\lambda \in \mu \otimes 1} K_{\mu} + \sum_{\lambda \in \mu \otimes 1} (1 + |\mu|)K_{\mu} \\ &= \sum_{\lambda \in \mu \otimes 1} K_{\mu} + \sum_{\lambda \in \mu \otimes 1} \sum_{\nu \in \mu \otimes 1} K_{\nu} \end{aligned}$$

⟨⟨ Not sure ⟩⟩ how to work with this double summation.



For the third identity, as  $K_{(1)} = 1$  we immediately have that  $\sum_{|\lambda|=1} K_\lambda^2 = K_{(1)}^2 = 1 = 1!$ . Now suppose that  $|\lambda| = \ell > 0$ . Then we have that

$$\begin{aligned} \ell! &= \ell \cdot (\ell - 1)! \\ &= \ell \sum_{|\lambda|=\ell-1} K_\lambda^2 \\ &= \sum_{|\lambda|=\ell-1} K_\lambda \cdot (\ell K_\lambda) \\ &= \sum_{|\lambda|=\ell-1} K_\lambda \cdot \sum_{\mu \in \lambda \otimes 1} K_\mu \end{aligned}$$

**<< Not sure >>** how to work with this double summation. □

**Solution:** [Man01] Ex. 1.2.15: Recall that  $h_j = s_{(j)}$  and  $e_k = s_{1^k}$ . Using the Pieri formulas, we can express  $h_j e_k$  as

$$\sum_{\mu \in 1^k \otimes j} s_\mu = s_{1^k} h_j = h_j e_k = s_{(j)} e_k = \sum_{\mu \in (j) \otimes 1^k} s_\mu.$$

**<< Expanding either side >>** gives  $h_j s_k = s_{(j-1|k)} + s_{(j|k-1)}$  which is already stated. Not sure what a second way would be, nor how to introduce the variable  $q$  in a generating-function sort of way. □

## 2.3 The Knuth Correspondence

**Solution:** [Man01] Ex. 1.3.1: Already saw this as the *Row Bumping Lemma* in [Ful97] which gives a slightly stronger characterization. □

## 2.4 Some Applications to Symmetric Functions

**Solution:** [Man01] Ex. 1.4.4: ⟨ Why ⟩ are these bases?

Let  $M_{\lambda\nu}$  and  $N_{\lambda\nu}$  be such that  $s_\lambda = \sum_\mu M_{\lambda\mu} = \sum_\nu N_{\lambda\nu} b_\nu$ . Then we have that

$$\begin{aligned} \sum_\lambda a_\lambda(\bar{x}) b_\lambda(\bar{y}) &= \prod_{i,j} (1 - x_i y_j)^{-1} \\ &= \sum_\lambda s_\lambda(\bar{x}) s_\lambda(\bar{y}) \\ &= \sum_\lambda \left( \sum_\rho M_{\lambda\rho} a_\rho(\bar{x}) \right) \left( \sum_\nu N_{\lambda\nu} b_\nu(\bar{y}) \right) = \sum_{\rho,\nu} \left( \sum_\lambda M_{\lambda\rho} N_{\lambda\nu} \right) a_\rho(\bar{x}) b_\nu(\bar{y}). \end{aligned}$$

Thus by the fact that the  $a_\rho$  and  $b_\nu$  form bases in their respective variables, we have that  $\sum_\lambda M_{\lambda\rho} N_{\lambda\nu} = \langle a_\rho, b_\nu \rangle$ . We want to show that  $\langle a_\rho, b_\nu \rangle = \delta_{\rho\nu}$ . Indeed, this follows from that

$$\sum_\lambda s_\lambda(\bar{x}) s_\lambda(\bar{y}) = \sum_\lambda a_\lambda(\bar{x}) b_\lambda(\bar{y}) \implies \sum_\lambda M_{\lambda\rho} N_{\lambda\nu} = \delta_{\rho\nu}.$$

□

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