Using Algebraic Geometry

With 0 Figures

Anakin Dey

Last Edited on 7/26/24 at 10:50

Contents

Preface

At the time of writing this, I am starting my PhD at The Ohio State University. Currently a large part of my interests in algebra are about algorithms as they relate to polynomials and algebraic geometry. I've been doing a bunch of problems from *Ideals, Varieties, and Algorithms* [CLO15]. However, it seems that *Using Algebraic Geometry* [CLO05] moves through the material faster as it assumes you know more algebra. So I've moved onto working through this book as well as trying to comprehend Sturmfel's *Algorithms in Invariant Theory* [Str08].

Chapter 1

Introduction

1.1 Polynomials and Ideals

Exercise 1.1 (CLO05 1.1.1):

(a) Show that $x^2 \in \langle x - y^2, xy \rangle$ in $k[x, y]$.

(b) Show that $\langle x - y^2, xy, y^2 \rangle = \langle x, y^2 \rangle$.

(c) Is $\langle x - y^2, xy \rangle = \langle x^2, xy \rangle$? Why or why not?

Proof:

(a) We have that $x(x - y^2) + y(xy) = x^2 - xy^2 + xy^2 = x^2$.

(b) It suffices to check for generators. We have that $x + (-1)(y^2) = x - y^2$, $y(x) = xy$, and $y^2 = y^2$ showing that $\langle x - y^2, xy, y^2 \rangle \subseteq \langle x, y^2 \rangle$. Then $x - y^2 + y^2 = x$ and $y^2 = y^2$ shows the reverse containment and overall the ideals are equal.

(c) We already know from 1. that x^2 lives in $\langle x - y^2, xy \rangle$. Since $xy = xy$, we overall have that $\langle x^2, xy \rangle \subseteq \langle x - y^2, xy \rangle$. It remains to check if $x - y^2 \in \langle x^2, xy \rangle$. However, notice that every element of $\langle x^2, xy \rangle$ is divisible by x while $x - y^2$ is clearly not divisible by x. Thus $x - y^2 \notin \langle x^2, xy \rangle$ and the two ideals are not equal.

$\square$

Exercise 1.2 (CLO05 1.1.2):

Show that $\langle f_1, \ldots, f_s \rangle$ is closed under sums in $k[x_1, \ldots, x_n]$. Also show that if $f \in \langle f_1, \ldots, f_s \rangle$ and $p \in k[x_1, \ldots, x_n]$ then $p \cdot f \in \langle f_1, \ldots, f_s \rangle$.

Proof:

Let $f, g \in \langle f_1, \ldots, f_s \rangle$. Then $\exists p_1, \ldots, p_s, q_1, \ldots, q_s$ such that $f = \sum_{i=1}^{s} p_i \cdot f_i$ and $g = \sum_{i=1}^{s} q_i \cdot f_i$. Thus $f + g = \sum_{i=1}^{s} (p_i + q_i) \cdot f_i$ which shows that $f + g \in \langle f_1, \ldots, f_s \rangle$. Then let $p \in k[x_1, \ldots, x_n]$. We have that $p \cdot f = p \sum_{i=1}^{s} p_i f_i = \sum_{i=1}^{s} (p \cdot p_i) \cdot f_i$ which shows that $\langle f_1, \ldots, f_s \rangle$ is an ideal. $\qquad \square$

Exercise 1.3 (CLO05 1.1.3):

Show that $\langle f_1, \ldots, f_s \rangle$ is the smallest ideal containing $\{ f_1, \ldots, f_s \}$.

Proof:

We already know that $\langle f_1, \ldots, f_s \rangle$ is an ideal by Exercise 1.2. Now suppose that J is an ideal containing $\{ f_1, \ldots, f_s \}$. Then, since ideals are closed under addition and scaling, we have that for all $p_1, \ldots, p_s \in k[x_1, \ldots, x_n]$ that $\sum_{i=1}^{s} p_i \cdot f_i \in J$. Thus, $\langle f_1, \ldots, f_s \rangle \subseteq J$. $\qquad \square$

Exercise 1.4 (CLO05 1.1.4):

Using Exercise 1.3, formulate and prove a general criterion for the equality of $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$.

Proof:

We claim that $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ if and only if $\{ g_1, \ldots, g_t \} \subseteq I$ and $\{ f_1, \ldots, f_s \} \subseteq J$. The forward implication is immediate. Then by Exercise 1.3, if $\{ g_1, \ldots, g_t \} \subseteq I$ then $J \subseteq I$. Similarly, $\{ f_1, \ldots, f_s \} \subseteq J \implies I \subseteq J$ and overall $I = J$. This fact was used in Exercise 1.1 (b). $\qquad \square$

Exercise 1.5 (CLO05 1.1.5):

Show that $\left\langle y - x^2, z - x^3 \right\rangle = \left\langle y - x^2, z - xy \right\rangle$ in $\mathbb{Q}[x, y, z]$.

Proof:

It suffices to show that $z - x^3 \in \left\langle y - x^2, z - xy \right\rangle$ and and $z - xy \in \left\langle x - y^2, z - x^3 \right\rangle$. Indeed we have that $(z - xy) + x(y - x^2) = z - x^3$ which also yields that $z - xy = z - x^3 - x(y - x^2)$. $\qquad \square$

Bibliography

[CLO05] D.A. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Springer-Verlag, 2005. ISBN: 0387207066. DOI: 10.1007/b138611. URL: http://dx.doi.org/10.1007/b138611.

[CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. ISBN: 9783319167213. URL: https://books.google.com/books?id=yL7yCAAAQBAJ.

[Str08] Bernd Strumfels. *Algorithms in Invariant Theory*. Springer Vienna, 2008. ISBN: 9783211774175. DOI: 10.1007/978-3-211-77417-5. URL: http://dx.doi.org/10.1007/978-3-211-77417-5.