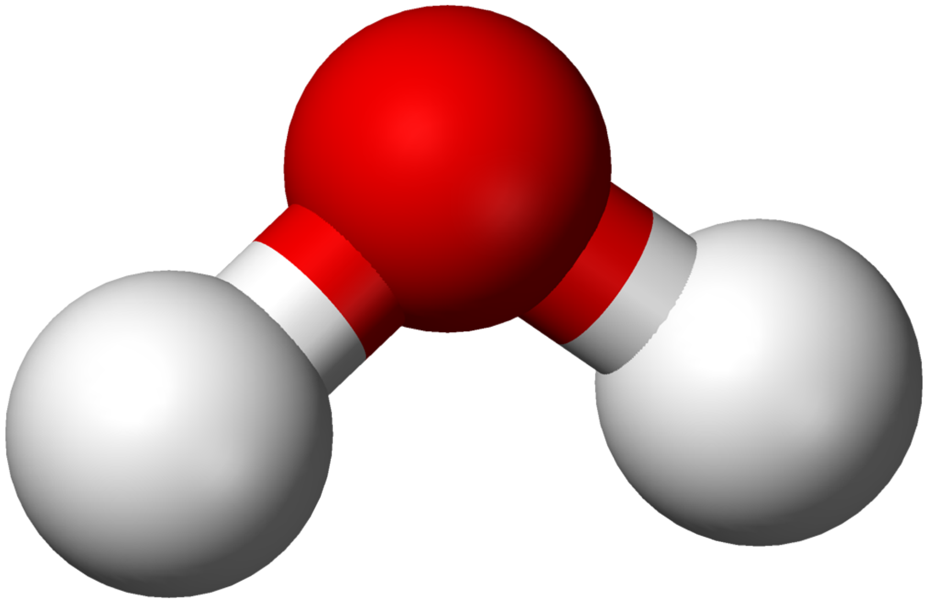

# Representation Theory Notes and Exercises

With 0 Figures

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**TODOs**

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**⟨⟨ Proper Chapter Header ⟩⟩**

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# Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. Occasionally, other sources may be used, such as the set of notes by Charles Rezk created for Math 427 [Rez20]. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

# Chapter 1

# Generalities on Linear Representations

Unless otherwise specified, $V$ will denote a vector space, usually over the field $\mathbb{C}$. We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

**Definition 1.1 (Linear Representation, Representation Space)**: Let $G$ be a group with identity $e$. A *linear representation* of $G$ in $V$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will frequently, and often interchangeably, write $\rho_s := \rho(s)$. Given $\rho$, we will say that $V$ is a *representation space* or *representation* of $G$.

**Definition 1.2 (Degree)**: Let $\rho : G \to V$ be a representation of $G$ in a vector space $V$. Then the *degree* of $\rho$ is $\dim(V)$.

Let $\rho : G \to V$ be a representation of $G$ in a vector space $V$ with $n := \dim(V)$. Fix a basis $(e_j)$ of $V$. Then since each $\rho_s$ is an invertible linear transformation of $V$, we may define an $n \times n$ matrix $R_s \equiv (r_{ij}(s))$ where each $r_{ij}(s)$ is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^{n} r_{ij}(s) e_i.$$

**Definition 1.3 (Matrix of a Representation)**: We call $R_s = (r_{ij}(s))$ above the *matrix of* $\rho_s$ with respect to the basis $(e_j)$.

Note that $R_s$ satisfies the following:

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^{n} r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two $n \times n$ matrices $A, A'$ are *similar* if there exists an invertible matrix $T$ such that $TA = A'T$. We may extend this notion to representations.

**Definition 1.4 (Similar/Isomorphic Representations)**: Let $\rho$ and $\rho'$ be two representations of the same group $G$ in vector spaces $V$ and $V'$ respectively. We say $\rho$ and $\rho'$ are *similar* or *isomorphic* if there exists an isomorphism $\tau : V \to V'$ such that for all $s \in G$, $\tau$ satisfies $\tau \circ \rho(s) = \rho'(s) \circ \tau$. If $R_s, R'_s$ are the corresponding matrices then this is equivalent to saying there exists an invertible matrix $T$ such that $TR_s = R'_s T$ for all $s \in G$.

Note that if $\rho$ and $\rho'$ are isomorphic, then they must have the same degree.

We now give some examples of these things.

**Example 1.5 (Unit/Trivial Representation)**: Let $G$ be a finite group. Representations of degree 1 must be of the form $\rho : G \to \mathbb{C}^\times$. Since elements $s$ of $G$ are of finite order, $\rho(s)$ must also be of finite order. Thus, for all $s \in G$, $\rho(s)$ is a root of unity. If we take $\rho(s) = 1$ for all $s \in G$, we obtain the *unit* or *trivial* representation of $G$. This also means that $R_s = 1$ for all $s$.

**Example 1.6 (Regular Representation)**: Let $g$ be the order of $G$, and let $V$ be a vector space of dimension $g$ with a basis $(e_t)_{t \in G}$. For each $s \in G$, define $\rho_s$ as the linear map $\rho_s : V \to V$ such that $\rho_s(e_t) = e_{st}$. This is a linear representation of $G$ called the *regular* representation of $G$. Since for each $s \in G$, $e_s = \rho_s(e_1)$ and thus the images of $e_1$ form a basis of $V$. Conversely, let $W$ be a representation of $G$ with a vector $w$ satisfying the collection of all $\rho_s(w)$, $s \in G$, forms a basis of $W$. Then $W$ is isomorphic to the regular representation of $G$ by the isomorphism $\tau(e_s) = \rho_s(w)$.

For example, let $G = \mathbb{Z}_3$ and $V = \mathbb{C}^3$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$. Then for example, $\rho_0, \rho_1, \rho_2 : \mathbb{C}^3 \to \mathbb{C}^3$ are the linear maps such that

$$
\begin{aligned}
\rho_0(e_0) &= e_{0+0} = e_0 & \rho_0(e_1) &= e_{0+1} = e_1 & \rho_0(e_2) &= e_{0+2} = e_2 \\
\rho_1(e_0) &= e_{1+0} = e_1 & \rho_1(e_1) &= e_{1+1} = e_2 & \rho_1(e_2) &= e_{1+2} = e_0 \\
\rho_2(e_0) &= e_{2+0} = e_2 & \rho_2(e_1) &= e_{2+1} = e_0 & \rho_2(e_2) &= e_{2+2} = e_1
\end{aligned}
$$

With this, the matrix representations of $\rho_0, \rho_1$ and $\rho_2$ is similarly straightforward:

$$
R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
$$

**Example 1.7 (Permutation Representation):** We may generalize the regular representation to any group action $G \curvearrowright X$, $X$ a finite set. Recall that for such an action, the map $x \mapsto sx$ for each $s \in G$ is a permutation $X \longleftrightarrow X$. Let $V$ be a vector space with dimension the size of $X$, and so a basis $(e_x)_{x \in X}$. Define a representation $\rho$ of $G$ by defining $\rho_s$ as the linear map sending $e_x \mapsto e_{sx}$. This representation is known as the *Permutation* representation of $G$ associated with $X$. If we consider $X = [n]$ and $G = S_n$, then take $V = \mathbb{C}^n$ as our vector space and $e_i$ as the standard basis vector. Then $\rho_\sigma(e_j) = e_{\sigma_j}$. Thus for each $\sigma \in S_n$, we have that $R_\sigma = (r_{ij}(\sigma))$ where entry $r_{ij}(\sigma) = 1$ if $i = \sigma_j$ and 0 otherwise.

**Definition 1.8 (Stable/Invariant Subspaces, Subrepresentation):** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation and $W \subseteq V$ a subspace of $V$. We say that $W$ is *stable* under the action of $G$ if $x \in W$ implies that $\rho_s(x) \in W$ for all $s \in G$., Thus, the restriction $\rho_s^W := \rho_s \mid_W$ is an isomorphism of $W$ onto itself. Restrictions satisfy the property that $\rho_s^W \circ \rho_t^W = \rho_{st}^W$. Thus, $\rho^W : G \to \mathrm{GL}(W)$ is a linear representation of $G$ in $W$ and we say that $W$ is a *subrepresentation* of $V$.

**Example 1.9 (Subrepresentations of the Regular Representation):** Let $G$ be a group. Recall the regular representation $V$ given in Example 1.6. Let $W$ be the 1 dimensional subspace of $V$ generated by the element $x = \sum_{s \in G} e_s$. Then note that $\rho_s(x) = x$ for all $s \in G$ and thus $W$ is a subrepresentation of $V$. Furthermore, this is isomorphic to the unit representation Example 1.5 with $\tau : C^\times \to W$ such that $\tau(1) = x$. For example, let $G = \mathbb{Z}_3$ and $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ the representation given in Example 1.6. Then $x = (1, 1, 1)$ and for example we have that

$$\rho_1(x) = \rho_1(e_0) + \rho_1(e_1) + \rho_1(e_2) = e_1 + e_2 + e_0 = x.$$

**Theorem 1.10:** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of $G$ in $V$ and let $W$ be a subspace of $V$ stable under $G$. Then there exists a complement $W^0$ of $W$ in $V$ which is stable under $G$.

**Proof:** Let $W'$ be an arbitrary complement of $W$ in $V$, and let $p : V \to W$ be the projection. Then we form the average $p^0$ of conjugates of $p$ by elements in $G$:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Since $p : V \to W$ and $\rho_t$ preserves $W$, we have that $p^0$ maps $V$ onto $W$. Furthermore, note that $\rho_t^{-1}$ also preserves $W$.

Thus we have that

$$(p\circ\rho_t^{-1})(x)=\rho_t^{-1}(x),\qquad (\rho_t\circ p\circ\rho_t^{-1})(x)=x,\qquad p^0(x)=x.$$

Thus, $p^0$ is a projection of $V$ onto $W$, corresponding to some complement $W^0$ of $W$. Moreover, we have that $\rho_s\circ p^0=p^0\circ\rho_s$ for all $s\in G$ because

$$\rho_s\circ p^0\circ\rho_s^{-1}=\frac{1}{|G|}\sum_{t\in G}\rho_s\circ\rho_t\circ p\circ\rho_t^{-1}\circ\rho_s^{-1}=\frac{1}{|G|}\sum_{t\in G}\rho_{st}\circ p\rho_{st}^{-1}=p^0.$$

Now suppose that $x\in W^0$ and $s\in G$, we have that $p^0(x)=0$ and hence $(p^0\circ\rho_s)(x)=(\rho_s\circ p^0)(x)=0$, meaning that $\rho_s(x)\in W^0$. This, $W^0$ is stable under $G$. □

Suppose that $V$ had an innerproduct $\langle x,y\rangle$, and furthermore suppose this inner product was invariant under $G$ meaning that for all $s\in G$, $\langle\rho_s(x),\rho_s(y)\rangle=\langle x,y\rangle$. We may also reduce to this case by replacing $\langle x,y\rangle$ with $\sum_{t\in G}\langle\rho_t(x),\rho_t(y)\rangle$. With this, the orthogonal complement $W^\perp$ of $W$ in $V$ is a complement of $W$ stable under $G$. Note that the invariance of $\langle x,y\rangle$ means that if $(e_i)$ is an orthonormal basis of $V$, then $R_s$ is a unitary matrix.

Using the notation of Theorem 1.10, let $x\in V$ and $w,w^0$ be the projections of $x$ on $W$ and $W^0$ respectively. Thus for all $s\in G$, $\rho_s(x)=\rho_s(w)+\rho_s(w^0)$. Since $W$ and $W^0$ are stable under $G$, we have that $\rho_s(w)\in W$ and $\rho_s(w^0)\in W^0$. This means that $\rho_s(w)$ and $\rho_s(w^0)$ are the projections of $\rho_s(x)$ and in turn the representations of $W$ and $W^0$ determine the representations of $V$.

**Definition 1.11 (Direct Sum of Representations):** Given the above, we write $V=W\oplus W^0$ as the *direct sum* of $W$ and $W^0$. We identify elements $v\in V$ as pairs $(w,w^0)$ given by their projections.

If the representations $W$ and $W^0$ are given in matrices $R_s$ and $R_s^0$, then the matrix form of the representation $V$ is given by

$$\begin{pmatrix}R_s & 0\\ 0 & R_s^0\end{pmatrix}.$$

Similar results hold for arbitrarily many, but finite, direct sums of representations.

The above is a discussion on how to compose two or more representations into one larger representation. The natural question is then about the opposite.

**Definition 1.12 (Irreducible/Simple Representations):** Let $\rho: G\to \mathrm{GL}(V)$ be a linear representation of $G$. Then this representation is *irreducible* or *simple* if $V$ has no subspaces stable under $G$ besides 0 and $V$ itself.

By Theorem 1.10, this is equivalent to saying that $V$ is not the direct sum of two representations besides $V=0\oplus V$. A representation of degree 1 is reducible. We may use irreducible representations to construct other ones via the direct sum.

**Theorem 1.13**: Every representation is a direct sum of irreducible representations.

**Proof**: Let $V$ be a linear representation of $G$. We induct on $\dim(V)$. If $\dim(V) = 0$, then $V = 0$ which is the direct sum of an empty family of irreducible representations. So suppose that
$dim(V) \geq 1$. If $V$ is irreducible, then we are done. Otherwise, there exists a subspace $W \subsetneq V$ stable under $G$ and by Theorem 1.10 a stable complement $W^0$ such that $V = W \oplus W^0$. By assumption, $W \neq 0 \neq W^0$ and so $\dim(W) < V$ and $\dim(W^0) < \dim(V)$. By induction, we have obtained a decomposition of $V$ into irreducibles. $\square$

**Example 1.14 (Decomposition of Representation of $\mathbb{Z}_3$ into Irreducibles)**: Recall from Example 1.6 the regular representation $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$ and

$$\begin{array}{lll} \rho_0(e_0) = e_0 & \rho_0(e_1) = e_1 & \rho_0(e_2) = e_2 \\ \rho_1(e_0) = e_1 & \rho_1(e_1) = e_2 & \rho_1(e_2) = e_0 \\ \rho_2(e_0) = e_2 & \rho_2(e_1) = e_0 & \rho_2(e_2) = e_1 \end{array}$$

Our goal will be to decompose $\rho$ into $\rho^1 \oplus \rho^2 \oplus \rho^3$. We aim to find the elements fixed by $\mathbb{Z}_3$. Note that if an element is fixed by 1, the generator of $\mathbb{Z}_3$, then it is fixed by all of $\mathbb{Z}_3$. We want to find 1-dimensional $\mathbb{Z}_3$-invarient subspaces of $\mathbb{C}^3$. This is equivalent to finding the eigenvalues of the matrix

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues their eigenvectors of $R_1$ are as follows, note the inclusion of the trivial subrepresentation from Example 1.9:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}, v_2 = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \qquad \lambda_3 = \frac{-1 + i\sqrt{3}}{2}, v_3 = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Thus $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$ where $V_i := \mathrm{span}(v_i)$. Note that there are only 3 morphisms $\mathbb{Z}_3 \to \mathbb{C}^\times$ mapping 1 to 1, $\omega$, or $\omega^2$ where $\omega$ is a cube root of unity. Thus $\rho^1, \rho^2$, and $\rho^3$ must correspond to these morphisms **⟨⟨ but which ones ⟩⟩** .

A natural question is if such a decomposition $V = W_1 \oplus \cdots \oplus W_k$ is unique. However, suppose that $\rho$ is the trivial representation (Example 1.5). Then each component of the decomposition is a line and we can decompose a vector space into the direct sum of lines in a number of ways. However, we will see that the number of $W_i$ that are isomorphic to a given irreducible representation does not depend on the choice of decomposition.

**Definition 1.15 (Tensor/Kronekcer Product of Representations):** Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two representations of a group $G$. We construct a representation $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ such that

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \circ \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

The existence and uniqueness of $\rho$ follow immediately from the existence and uniqueness of the tensor product. We write $\rho_s \equiv \rho_s^1 \otimes \rho_s^2$ as the *tensor product* of the given representations.

Recall that if $(e_{i_1})$ and $(e_{i_2})$ be bases of $V_1$ and $V_2$ respectively, then $(e_{i_1} \otimes e_{i_2})$ is a basis of $V_1 \otimes V_2$. If $(r_{i_1 j_1}(s))$ and $(r_{i_2 j_2}(s))$ are the matrices of $\rho_s^1$ and $\rho_s^2$ respectively satisfying

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1} e_{i_1} \qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2} e_{i_2}$$

then the matrix of $\rho_s$ is $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$ satisfying

$$\rho_s(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \otimes e_{i_2}.$$

**⟨⟨ TODO: example of tensor product ⟩⟩** Note that the tensor product of two irreducible representations is not in general irreducible **⟨⟨ TODO: example? ⟩⟩** .

We now consider the special case of $V \otimes V$. Let $(e_i)$ be a basis of $V$ and define an automorphism $\theta$ of $V \otimes V$ such that $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Then note that $\theta^2 \equiv \mathrm{id}_{V \otimes V}$. We may decompose $V \otimes V$ into the direct sum

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Here, $\mathrm{Sym}^2(V)$ is the set of $z \in V \otimes V$ such that $\theta(z) = z$ and $\mathrm{Alt}^2(V)$ is the set of $z \in V \otimes V$ where $\theta(z) = -z$. These have bases $(e_i \otimes e_j + e_j \otimes e_i)_{i \le j}$ and $(e_i \otimes e_j - e_j \otimes e_i)_{i < j}$ respectively. As such, $\dim(\mathrm{Sym}^2(V)) = \frac{n(n+1)}{2}$ and $\dim(\mathrm{Alt}^2(V)) = \frac{n(n-1)}{2}$ where $n \coloneqq \dim(V)$.

**Definition 1.16 (Symmetric Square, Alternating Square):** These subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ of $V \otimes V$ are respectively called the *symmetric square* and *alternative square* of the given representation.

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