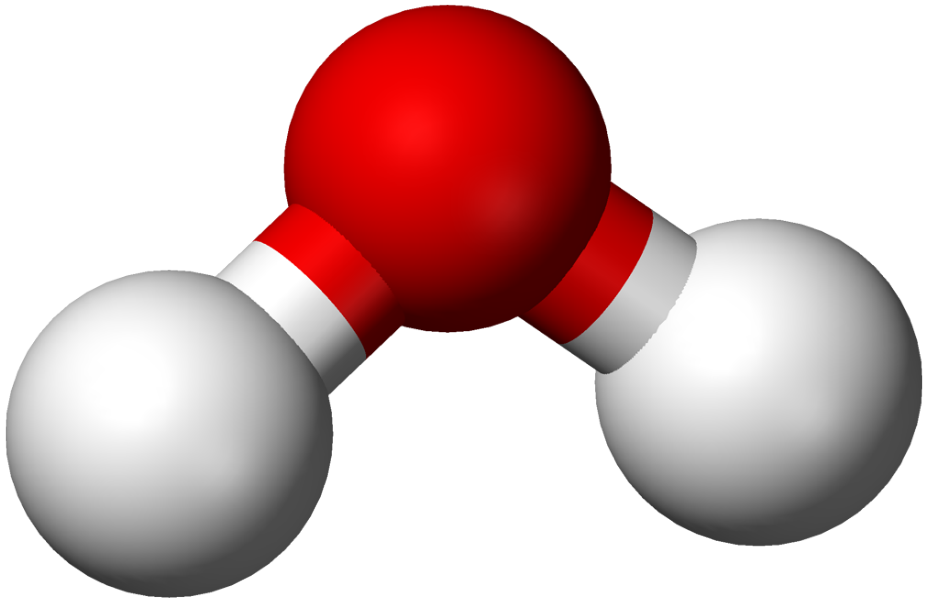

Representation Theory Notes and Exercises

With 0 Figures

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TODOs

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Preface

This is a set of notes on group representation theory mainly based on J.P. Serre's text *Linear Representations of Finite Groups* [Ser77]. Occasionally, other sources may be used, such as the set of notes by Charles Rezk created for Math 427 [Rez20]. The goal of these notes is to eventually work towards algebraic combinatorics such as Fulton's text *Young Tableaux* [Ful96], as much of algebraic combinatorics is motivated by questions stemming from representation theory. At the time of writing this, another goal is the applications of representation theory to computational complexity: see [Pan23] for a recent survey on this connection.

Chapter 1

Generalities on Linear Representations

Unless otherwise specified, V will denote a vector space, usually over the field $\mathbb{C}$. We will restrict ourselves to finite dimensional vector spaces. Similarly, we will restrict ourselves to finite groups.

Definition 1.1 (Linear Representation, Representation Space): Let G be a group with identity e. A *linear representation* of G in V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will frequently, and often interchangeably, write $\rho_s := \rho(s)$. Given ρ, we will say that V is a *representation space* or *representation* of G.

Definition 1.2 (Degree): Let $\rho : G \to V$ be a representation of G in a vector space V. Then the *degree* of ρ is $\dim(V)$.

Let $\rho : G \to V$ be a representation of G in a vector space V with $n := \dim(V)$. Fix a basis (e_j) of V. Then since each ρ_s is an invertible linear transformation of V, we may define an $n \times n$ matrix $R_s \equiv (r_{ij}(s))$ where each $r_{ij}(s)$ is defined by the identity

$$\rho_s(e_j) = \sum_{i=1}^{n} r_{ij}(s) e_i.$$

Definition 1.3 (Matrix of a Representation): We call $R_s = (r_{ij}(s))$ above the *matrix of* ρ_s with respect to the basis (e_j).

Note that R_s satisfies the following:

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t \equiv r_{ij}(st) = \sum_{k=1}^{n} r_{ik}(s) \cdot r_{kj}(s) \quad \forall s, t \in G.$$

Recall that two $n \times n$ matrices A, A' are *similar* if there exists an invertible matrix T such that $TA = A'T$. We may extend this notion to representations.

Definition 1.4 (Similar/Isomorphic Representations): Let ρ and ρ' be two representations of the same group G in vector spaces V and V' respectively. We say ρ and ρ' are *similar* or *isomorphic* if there exists an isomorphism $\tau: V \to V'$ such that for all $s \in G$, τ satisfies $\tau \circ \rho(s) = \rho'(s) \circ \tau$. If R_s, R_s' are the corresponding matrices then this is equivalent to saying there exists an invertible matrix T such that $TR_s = R_s'T$ for all $s \in G$.

Note that if ρ and ρ' are isomorphic, then they must have the same degree.

We now give some examples of these things.

Example 1.5 (Unit/Trivial Representation): Let G be a finite group. Representations of degree 1 must be of the form $\rho: G \to \mathbb{C}^\times$. Since elements s of G are of finite order, $\rho(s)$ must also be of finite order. Thus, for all $s \in G$, $\rho(s)$ is a root of unity. If we take $\rho(s) = 1$ for all $s \in G$, we obtain the *unit* or *trivial* representation of G. This also means that $R_s = 1$ for all s.

Example 1.6 (Regular Representation): Let g be the order of G, and let V be a vector space of dimension g with a basis $(e_t)_{t \in G}$. For each $s \in G$, define ρ_s as the linear map $\rho_s: V \to V$ such that $\rho_s(e_t) = e_{st}$. This is a linear representation of G called the *regular* representation of G. Since for each $s \in G$, $e_s = \rho_s(e_1)$ and thus the images of e_1 form a basis of V. Conversely, let W be a representation of G with a vector w satisfying the collection of all $\rho_s(w)$, $s \in G$, forms a basis of W. Then W is isomorphic to the regular representation of G by the isomorphism $\tau(e_s) = \rho_s(w)$.

For example, let $G = \mathbb{Z}_3$ and $V = \mathbb{C}^3$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$. Then for example, $\rho_0, \rho_1, \rho_2: \mathbb{C}^3 \to \mathbb{C}^3$ are the linear maps such that

$$\begin{array}{lll}
\rho_0(e_0) = e_{0+0} = e_0 & \rho_0(e_1) = e_{0+1} = e_1 & \rho_0(e_2) = e_{0+2} = e_2 \\
\rho_1(e_0) = e_{1+0} = e_1 & \rho_1(e_1) = e_{1+1} = e_2 & \rho_1(e_2) = e_{1+2} = e_0 \\
\rho_2(e_0) = e_{2+0} = e_2 & \rho_2(e_1) = e_{2+1} = e_0 & \rho_2(e_2) = e_{2+2} = e_1
\end{array}$$

With this, the matrix representations of ρ_0, ρ_1 and ρ_2 is similarly straightforward:

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 1.7 (Permutation Representation): We may generalize the regular representation to any group action $G \curvearrowright X$, X a finite set. Recall that for such an action, the map $x \mapsto sx$ for each $s \in G$ is a permutation $X \longleftrightarrow X$. Let V be a vector space with dimension the size of X, and so a basis $(e_x)_{x \in X}$. Define a representation ρ of G by defining ρ_s as the linear map sending $e_x \mapsto e_{sx}$. This representation is known as the *Permutation* representation of G associated with X. If we consider $X = [n]$ and $G = S_n$, then take $V = \mathbb{C}^n$ as our vector space and e_i as the standard basis vector. Then $\rho_\sigma(e_j) = e_{\sigma_j}$. Thus for each $\sigma \in S_n$, we have that $R_\sigma = (r_{ij}(\sigma))$ where entry $r_{ij}(\sigma) = 1$ if $i = \sigma(j)$ and 0 otherwise.

Definition 1.8 (Stable/Invariant Subspaces, Subrepresentation): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation and $W \subseteq V$ a subspace of V. We say that W is *stable* under the action of G if $x \in W$ implies that $\rho_s(x) \in W$ for all $s \in G$., Thus, the restriction $\rho_s^W := \rho_s |_W$ is an isomorphism of W onto itself. Restrictions satisfy the property that $\rho_s^W \circ \rho_t^W = \rho_{st}^W$. Thus, $\rho^W : G \to \mathrm{GL}(W)$ is a linear representation of G in W and we say that W is a *subrepresentation* of V.

Example 1.9 (Subrepresentations of the Regular Representation): Let G be a group. Recall the regular representation V given in Example 1.6. Let W be the 1 dimensional subspace of V generated by the element $x = \sum_{s \in G} e_s$. Then note that $\rho_s(x) = x$ for all $s \in G$ and thus W is a subrepresentation of V. Furthermore, this is isomorphic to the unit representation Example 1.5 with $\tau : C^\times \to W$ such that $\tau(1) = x$. For example, let $G = \mathbb{Z}_3$ and $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ the representation given in Example 1.6. Then $x = (1, 1, 1)$ and for example we have that

$$\rho_1(x) = \rho_1(e_0) + \rho_1(e_1) + \rho_1(e_2) = e_1 + e_2 + e_0 = x.$$

Theorem 1.10: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V and let W be a subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.

Proof: Let W' be an arbitrary complement of W in V, and let $p : V \to W$ be the projection. Then we form the average p^0 of conjugates of p by elements in G:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Since $p : V \to W$ and ρ_t preserves W, we have that p^0 maps V onto W. Furthermore, note that ρ_t^{-1} also preserves W.

Thus we have that

$$(p \circ \rho_t^{-1})(x) = \rho_t^{-1}(x), \qquad (\rho_t \circ p \circ \rho_t^{-1})(x) = x, \qquad p^0(x) = x.$$

Thus, p^0 is a projection of V onto W, corresponding to some complement W^0 of W. Moreover, we have that $\rho_s \circ p^0 = p^0 \circ \rho_s$ for all $s \in G$ because

$$\rho_s \circ p^0 \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \circ \rho_t \circ p \circ \rho_t^{-1} \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \circ p \rho_{st}^{-1} = p^0.$$

Now suppose that $x \in W^0$ and $s \in G$, we have that $p^0(x) = 0$ and hence $(p^0 \circ \rho_s)(x) = (\rho_s \circ p^0)(x) = 0$, meaning that $\rho_s(x) \in W^0$. This, W^0 is stable under G. □

Suppose that V had an inner product $\langle x, y \rangle$, and furthermore suppose this inner product was invariant under G meaning that for all $s \in G$, $\langle \rho_s(x), \rho_s(y) \rangle = \langle x, y \rangle$. We may also reduce to this case by replacing $\langle x, y \rangle$ with $\sum_{t \in G} \langle \rho_t(x), \rho_t(y) \rangle$. With this, the orthogonal complement $W^\perp$ of W in V is a complement of W stable under G. Note that the invariance of $\langle x, y \rangle$ means that if (e_i) is an orthonormal basis of V, then R_s is a unitary matrix.

Using the notation of Theorem 1.10, let $x \in V$ and w, w^0 be the projections of x on W and W^0 respectively. Thus for all $s \in G$, $\rho_s(x) = \rho_s(w) + \rho_s(w^0)$. Since W and W^0 are stable under G, we have that $\rho_s(w) \in W$ and $\rho_s(w^0) \in W^0$. This means that $\rho_s(w)$ and $\rho_s(w^0)$ are the projections of $\rho_s(x)$ and in turn the representations of W and W^0 determine the representations of V.

Definition 1.11 (Direct Sum of Representations): Given the above, we write $V = W \oplus W^0$ as the *direct sum* of W and W^0. We identify elements $v \in V$ as pairs (w, w^0) given by their projections.

If the representations W and W^0 are given in matrices R_s and R_s^0, then the matrix form of the representation V is given by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

Similar results hold for arbitrarily many, but finite, direct sums of representations.

The above is a discussion on how to compose two or more representations into one larger representation. The natural question is then about the opposite.

Definition 1.12 (Irreducible/Simple Representations): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Then this representation is *irreducible* or *simple* if V has no subspaces stable under G besides 0 and V itself.

By Theorem 1.10, this is equivalent to saying that V is not the direct sum of two representations besides $V = 0 \oplus V$. A representation of degree 1 is reducible. We may use irreducible representations to construct other ones via the direct sum.

Theorem 1.13: Every representation is a direct sum of irreducible representations.

Proof: Let V be a linear representation of G. We induct on $\dim(V)$. If $\dim(V) = 0$, then $V = 0$ which is the direct sum of an empty family of irreducible representations. So suppose that
$dim(V) \geq 1$. If V is irreducible, then we are done. Otherwise, there exists a subspace $W \subsetneq V$ stable under G and by Theorem 1.10 a stable complement W^0 such that $V = W \oplus W^0$. By assumption, $W \neq 0 \neq W^0$ and so $\dim(W) < V$ and $\dim(W^0) < \dim(V)$. By induction, we have obtained a decomposition of V into irreducibles. $\square$

Example 1.14 (Decomposition of Representation of $\mathbb{Z}_3$ into Irreducibles): Recall from Example 1.6 the regular representation $\rho : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ with $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$ and

$$\begin{array}{lll} \rho_0(e_0) = e_0 & \rho_0(e_1) = e_1 & \rho_0(e_2) = e_2 \\ \rho_1(e_0) = e_1 & \rho_1(e_1) = e_2 & \rho_1(e_2) = e_0 \\ \rho_2(e_0) = e_2 & \rho_2(e_1) = e_0 & \rho_2(e_2) = e_1 \end{array}$$

Our goal will be to decompose ρ into $\rho^1 \oplus \rho^2 \oplus \rho^3$. We aim to find the elements fixed by $\mathbb{Z}_3$. Note that if an element is fixed by 1, the generator of $\mathbb{Z}_3$, then it is fixed by all of $\mathbb{Z}_3$. We want to find 1-dimensional $\mathbb{Z}_3$-invarient subspaces of $\mathbb{C}^3$. This is equivalent to finding the eigenvalues of the matrix

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues their eigenvectors of R_1 are as follows, note the inclusion of the trivial subrepresentation from Example 1.9:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}, v_2 = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \qquad \lambda_3 = \frac{-1 + i\sqrt{3}}{2}, v_3 = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Thus $\mathbb{C}^3 = V_1 \oplus V_2 \oplus V_3$ where $V_i := \mathrm{span}(v_i)$. Note that there are only 3 morphisms $\mathbb{Z}_3 \to \mathbb{C}^\times$ mapping 1 to 1, ω, or ω^2 where ω is a cube root of unity. Thus ρ^1, ρ^2, and ρ^3 must correspond to these morphisms $\langle\langle$ **but which ones** $\rangle\rangle$.

A natural question is if such a decomposition $V = W_1 \oplus \cdots \oplus W_k$ is unique. However, suppose that ρ is the trivial representation (Example 1.5). Then each component of the decomposition is a line and we can decompose a vector space into the direct sum of lines in a number of ways. However, we will see that the number of W_i that are isomorphic to a given irreducible representation does not depend on the choice of decomposition.

Definition 1.15 (Tensor/Kronekcer Product of Representations): Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two representations of a group G. We construct a representation $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ such that

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \circ \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

The existence and uniqueness of ρ follow immediately from the existence and uniqueness of the tensor product. We write $\rho_s \equiv \rho_s^1 \otimes \rho_s^2$ as the *tensor product* of the given representations.

Recall that if (e_{i_1}) and (e_{i_2}) be bases of V_1 and V_2 respectively, then $(e_{i_1} \otimes e_{i_2})$ is a basis of $V_1 \otimes V_2$. If $(r_{i_1 j_1}(s))$ and $(r_{i_2 j_2}(s))$ are the matrices of ρ_s^1 and ρ_s^2 respectively satisfying

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1} e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2} e_{i_2}$$

then the matrix of ρ_s is $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$ satisfying

$$\rho_s(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \otimes e_{i_2}.$$

Note that the tensor product of two irreducible representations is not in general irreducible.

Example 1.16 (Tensor Product of Two Irreducible Representations that is not Irreducible): Consider $G = \mathbb{Z}/4\mathbb{Z}$. Consider the representation $\rho : G \to \mathrm{GL}(\mathbb{R}^2)$ such that

$$\rho_1 = M := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since M does not have 1 as an eigenvalue, there are no $\mathbb{Z}/4\mathbb{Z}$-invariant subspaces. Thus, ρ is irreducible.

Now consider $\rho' = \rho \otimes \rho$. Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$, and so $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ a basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$. We have that

$$\begin{aligned}
\rho_1'(e_1 \otimes e_1) &= Me_1 \otimes Me_1 = e_2 \otimes e_2, \\
\rho_1'(e_1 \otimes e_2) &= Me_1 \otimes Me_2 = e_2 \otimes -e_1, \\
\rho_1'(e_2 \otimes e_1) &= Me_2 \otimes Me_1 = -e_1 \otimes e_2, \\
\rho_1'(e_2 \otimes e_2) &= Me_2 \otimes Me_2 = e_2 \otimes e_2.
\end{aligned}$$

Thus the matrix of ρ_1' is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that the subspace generated by $(e_1 \otimes e_1, e_2 \otimes e_2)$ is a $\mathbb{Z}/4\mathbb{Z}$-invariant subspace.

We now consider the special case of $V \otimes V$. Let (e_i) be a basis of V and define an automorphism θ of $V \otimes V$ such that $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Then note that $\theta^2 \equiv \mathrm{id}_{V \otimes V}$. We may decompose $V \otimes V$ into the direct sum

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Here, $\mathrm{Sym}^2(V)$ is the set of $z \in V \otimes V$ such that $\theta(z) = z$ and $\mathrm{Alt}^2(V)$ is the set of $z \in V \otimes V$ where $\theta(z) = -z$. These have bases $(e_i \otimes e_j + e_j \otimes e_i)_{i \leq j}$ and $(e_i \otimes e_j - e_j \otimes e_i)_{i<j}$ respectively. As such, $\dim(\mathrm{Sym}^2(V)) = \frac{n(n+1)}{2}$ and $\dim(\mathrm{Alt}^2(V)) = \frac{n(n-1)}{2}$ where $n \coloneqq \dim(V)$.

Definition 1.17 (Symmetric Square, Alternating Square): These subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ of $V \otimes V$ are respectively called the *symmetric square* and *alternative square* of the given representation.

Chapter 2

Character Theory

Definition 2.1 (Character): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of a finite group G in V. Then the *character* χ_ρ of ρ is the function

$$\chi_\rho(s) := \mathrm{Tr}(R_s) \equiv \mathrm{Tr}(\rho_s).$$

for each $s \in G$.

Proposition 2.2: If χ is the character of a representation ρ of degree n then

1. $\chi(e) = 1$;
2. $\chi(s^{-1}) = \chi(s)^*$, the complex conjugate of $\chi(s)$,
3. $\chi(tst^{-1}) = \chi(s)$.

Proof: The first is immediate since ρ_1 is the identity matrix I and $\mathrm{Tr}(I) = n$. Then recall that we may choose our basis to be orthonormal, and as such ρ_s is a unitary matrix. Thus, each eigenvalue $\lambda_1, \dots, \lambda_n$ has an absolute value equal to 1. Thus

$$\chi(s)^* = \mathrm{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \mathrm{Tr}(\rho_s)^{-1} = \mathrm{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

This uses the fact that the conjugate of the sum is the sum of the conjugates as well as the fact that the eigenvalues of R_s^{-1} are the inverses of the eigenvalues of R_s. Finally, letting $u = ts$ and $v = t^{-1}$ allows us to write $\chi(tst^{-1}) = \chi(s)$ as $\chi(uv) = \chi(vu)$ which is immediate since for any complex matrices A, B we have that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. $\square$

Proposition 2.3: Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two linear representations with characters χ_1 and χ_2 respectively. Then

1. The character χ of the direct sum representation $V_1 \oplus V_2$ is $\chi_1 + \chi_2$.
2. The character ψ of the tensor product representation $V_1 \otimes V_2$ is $\chi_1 \cdot \chi_2$.

Proof: Let R_s^1, R_s^2 be the matrix forms of ρ_s^1 and ρ_s^2 respectively. Then the matrix form R_s of the representation of $V_1 \oplus V_2$ is given by

$$R_s = \begin{pmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{pmatrix}$$

and thus $\mathrm{Tr}(R_s) = \mathrm{Tr}\big(R_s^1\big) + \mathrm{Tr}\big(R_s^2\big)$. Let (e_{i_1}) and (e_{i_2}) be bases for V_1 and V_2. Then we have that

$$\psi(s) = \sum_{i_1 i_2} r_{i_1 i_1}(s) \cdot r_{i_2 i_2}(s) = \left(\sum_{i_1} r_{i_1 i_1}(s) \right) \cdot \left(\sum_{i_2} r_{i_2 i_2}(s) \right) = \chi_1(s) \cdot \chi_2(s).$$

$\square$

Proposition 2.4: Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G with character χ. Let χ_σ^2 be the character of $\mathrm{Sym}^2(V)$ and χ_α^2 be the character of $\mathrm{Alt}^2(V)$ from Definition 1.17. Then

$$\chi_\sigma^2(s) = \frac{1}{2}\big(\chi(s)^2 + \chi(s^2)\big)$$
$$\chi_\alpha^2(s) = \frac{1}{2}\big(\chi(s)^2 - \chi(s^2)\big)$$

which directly implies that $\chi_\sigma^2 + \chi_\alpha^2 = \chi$.

Proof: Let $s \in G$ and (e_i) a basis of V consisting solely of eigenvectors for ρ_s. Then $\rho_s(e_i) = \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$. Thus

$$\chi(s) = \sum \lambda_i \qquad\qquad \chi(s^2) = \sum \lambda_i^2.$$

We also have that

$$(\rho_s \otimes \rho_s)(e_i \otimes e_j + e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i)$$
$$(\rho_s \otimes \rho_s)(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

which yields that

$$\chi_\sigma^2(s) = \sum_{i \le j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\Big(\sum \lambda_i\Big)^2 + \frac{1}{2} \sum \lambda_i^2 \chi_\alpha^2(s) \quad = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\Big(\sum \lambda_i\Big)^2 - \frac{1}{2} \sum \lambda_i^2.$$

The proposition then directly follows. Note that the equality $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$ directly reflects the fact that $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$. □

Proposition 2.5 (Schur's Lemma): Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $f : V_1 \to V_2$ be a linear map such that $f \circ \rho_s^1 = \rho_s^2 \circ f$ for all $s \in G$. Then

1. If ρ^1 and ρ^2 are not isomorphic, then $f = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$ then f is a *homothety*, a scalar multiple of the identity.

Proof: The case of $f = 0$ is trivial, so suppose that $f \neq 0$. Let $W_1 = \ker(f)$ and $W_2 = \mathrm{im}(f)$. Then for $x \in W_1$ we have that $f(\rho_s^1(x)) = \rho_s^2(f(x)) = 0$ which means that $\rho_s^1(x) \in W_1$. Thus W_1 is stable under G and irreducibility of V_1 combined with the assumption that $f \neq 0$ implies that $W_1 = 0$. Similarly, we have that for $f(x) \in W_2$, we have that $\rho_s^2(f(x)) = f(\rho_s^1(x)) \in W_2$, so $\rho_s^2(f(x)) \in W_2$. Thus W_2 is also stable under G meaning that by a similar argument, $W_2 = V_2$. Since $\ker(f) = 0$ and $\mathrm{im}(f) = V_2$, we must have that f is an isomorphism $V_1 \to V_2$. This proves the first claim.

Now suppose that $V_1 = V_2$, $\rho^1 = \rho^2$, and that λ is some eigenvalue of f. Let $f' = f - \lambda$. Since λ is an eigenvalue, then $\ker(f') \neq 0$. However, we also have that $f' \circ \rho_s^1 = \rho_s^2 \circ f'$. The first part of this proof shows that this implies that $f' = 0$. Thus, $f = \lambda$ and f is a homothety. □

Corollary 2.6: Let $\rho^1 \colon G \to \mathrm{GL}(V_1)$ and $\rho^2 \colon G \to \mathrm{GL}(V_2)$ be two irreducible representations of G. Let $h \colon V_1 \to V_2$ and define h^0 such that

$$h^0 = \frac{1}{|G|} \sum_{t \in G} (\rho_t^2)^{-1} \circ h \circ \rho_t^1.$$

Then

1. If ρ^1 and ρ^2 are not isomorphic, then $h^0 = 0$
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$, then h^0 is a homothety of ratio $\frac{1}{n}\mathrm{Tr}(h)$, with $n = \dim(V_1)$.

Proof: First for $s \in G$ we have that

$$(\rho_s^2)^{-1} \circ h^0 \circ \rho_s^1 = \frac{1}{|G|} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} \circ h \circ \rho_t^1 \rho_s^1. \qquad = \frac{1}{|G|} \sum_{t \in G} (\rho_{ts}^2)^{-1} \circ h \circ \rho_{ts}^1 = h^0.$$

Thus, Proposition 2.5 applies to h^0 and in the first case $h^0 = 0$ and in the second h^0 is a homothety of scalar λ. Moreover we have that

$$n \cdot \lambda = \mathrm{Tr}(\lambda) = \frac{1}{|G|} \sum_{t \in G} \mathrm{Tr}\left((\rho_t^1)^{-1} \circ h \circ \rho_t^1\right) = \mathrm{Tr}(h).$$

Thus, $\lambda = \frac{1}{n}\mathrm{Tr}(h)$. □

Consider Corollary 2.6 in matrix form where $\rho_s^1 = (r_{i_1 j_1}(s))$ and $\rho_s^2 = (r_{i_2 j_2}(s))$. Then our linear map h is given by the matrix $(x_{i_2 i_1})$ and similarly h^0 is given by the matrix $(x^0_{i_2 i_1})$. Then by definition of h^0 we have that

$$x^0_{i_2 i_1} = \frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t).$$

In case (1) of Corollary 2.6, the right hand side must vanish and so all coefficients of 0:

Corollary 2.7: In case (1) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = 0$$

for all i_1, j_1, i_2, j_2.

We now consider case (2) of Corollary 2.6 in the matrix form. We have that $h^0 = \lambda$, with $\lambda = \frac{1}{n}\operatorname{Tr}(h)$, meaning that $x^0_{i_2 i_1} = \lambda \delta_{i_2 i_1}$. That is, $\lambda = \frac{1}{n} \sum \delta_{i_2 i_1} \cdot x_{i_2 i_1}$. This we have that

$$\frac{1}{|G|} \sum_{t \in G, j_1, j_2} r_{i_2 j_2}(t^{-1}) \cdot x_{j_2 j_1} \cdot r_{j_1 i_1}(t) = \frac{1}{n} \sum_{j_1, j_2} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1} \cdot x_{j_2 j_1}.$$

Equating coefficients of the $x_{j_2 j_1}$ yields the following corollary:

Corollary 2.8: In case (2) of Corollary 2.6 we have that

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) \cdot r_{j_1 i_1}(t) = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

We may generalize these notions to further functions. Let ϕ, ψ be complex valued functions on G. Define

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{t \in G} \phi(t^{-1}) \cdot \psi(t) = \frac{1}{|G|} \sum_{t \in G} \phi(t) \cdot \psi(t^{-1}).$$

Then $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$ and $\langle \phi, \psi \rangle$ is linear in ϕ and in ψ. Thus, Corollary 2.7 and Corollary 2.8 respectively become

$$\left\langle r_{i_2 j_2}, r_{j_1 i_1} \right\rangle = 0 \qquad\qquad \left\langle r_{i_2 j_2}, r_{j_1 i_1} \right\rangle = \frac{1}{n} \delta_{i_2 i_1} \cdot \delta_{j_2 j_1}.$$

If the matrices $(r_{ij}(t))$ are unitary, realized by a suitable choice of basis, then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and Corollary 2.7 and Corollary 2.8 are just *orthogonality relations*.

Definition 2.9 (Scalar Product): If ϕ, ψ are two complex valued functions on G, then let

$$(\phi \mid \psi) := \frac{1}{|G|} \sum_{t \in G} \phi(t)\psi(t)^*.$$

This is a *scalar product*. It is linear in ϕ, semilinear in ψ, and $(\phi \mid \phi) > 0$ for all $\phi \neq 0$.

Define $\check{\psi}(t) := \psi(t^{-1})^*$. Then $(\phi \mid \psi) = \langle \phi, \check{\psi} \rangle$. In particular, suppose χ is a character so that by Proposition 2.2 we have that $\chi = \check{\chi}$ then for all complex valued functions ϕ on G we have that $(\phi \mid \chi) = \langle \phi, \chi \rangle$. Thus, we may use the two interchangeably in the context of characters.

Theorem 2.10:

1. If χ is the character of an irreducible representation, we have that $(\chi \mid \chi) = 1$, i.e. χ has "norm 1."
2. If χ and χ' are characters of two non-isomorphic irreducible representations, then $(\chi \mid \chi') = 0$, i.e. χ and χ' are "orthogonal."

Proof: Suppose ρ is an irreducible representation with matrix form $\rho_t = (r_{ij}(t))$ and χ its character. Then $\chi(t) = \sum r_{ii}(t)$ and so

$$(\chi \mid \chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle = \frac{\delta_{ij}}{n}$$

where the last equality is by Corollary 2.8 and n is the degree of ρ. Thus

$$(\chi \mid \chi) = \sum_{i,j} \frac{\delta_{ij}}{n} = \frac{n}{n} = 1.$$

This proves the first claim. Applying Corollary 2.7 yields the second claim □

Theorem 2.11: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi \mid \chi) = \langle \phi, \chi \rangle$.

Proof: Let χ_i be the character of W_i. Then by Proposition 2.3 we have that $\phi = \chi_1 + \cdots + \chi_k$. By linearity of $(\cdot \mid \cdot)$ in the first argument we have that $(\phi \mid \chi) = (\chi_1 \mid \chi) + \cdots + (\chi_k \mid \chi)$. The result follows by Theorem 2.10. □

Corollary 2.12: Let V be a linear representation of G with character ϕ such that V decomposes into a direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$. Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W does not depend on the chosen decomposition.

Proof: Note that $(\phi \mid \chi)$ does not depend on choice of decomposition. □

Corollary 2.13: Two representations are isomorphic if and only if they have the same character.

Proof: The forward direction is obvious, and the reverse is true by the prior corollary. □

Thus, our study of representations is reduced to that of the study of characters. If $\chi_1, \dots, \chi_k$ are the distinct irreducible characters of G and if $W_1, \dots, W_k$ their corresponding representation, then each representation V of G is isomorphic to a direct sum. We will see later how we know that there are finitely many irreducible representations, and thus characters, of a finite group G.

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \qquad\qquad m_i \neq 0.$$

The character ϕ of V is equal to $m_1\chi_1 + \cdots + m_h\chi_h$ and we have that $m_i = (\phi \mid \chi_i)$. This is especially useful when considering the tensor product $W_i \otimes W_j$ of two irreducible representations. It shows that the product $\chi_i \cdot \chi_j$ decomposes into a sum $\chi_i\chi_j = \sum m_{ij}^k \chi_k$, each integer $m_{ij}^k \geq 0$. The orthogonality relations among the χ_i imply that

$$(\phi \mid \phi) = \sum_{i=1}^{h} m_i^2.$$

We now obtain a useful irreducibility criterion:

Theorem 2.14: If ϕ is the character of a representation V, $(\phi \mid \phi)$ is a positive integer and $(\phi \mid \phi) = 1$ if and only if V is irreducible.

Proof: We have that $\sum m_i^2 = 1$ if and only if one of the $m_i = 1$ and all the others are equal to 0. This means that V is isomorphic to one of the W_i. □

We now explore the decomposition of the regular representation $\rho : G \to \mathrm{GL}(R)$ of a group G (Example 1.6). Suppose $\chi_1, \dots, \chi_h$ are the irreducible characters of G with degrees $n_1, \dots, n_k$. Note that by Proposition 2.2, $n_i = \chi_i(e)$. Recall that R has basis $(e_t)_{t \in G}$ where $\rho_s(e_t) = e_{st}$. This means that for $s \neq e$, the diagonal terms of the matrix for ρ_s are all 0, so $\mathrm{Tr}(\rho_s) = 0$. On the otherhand, we have that

$$\mathrm{Tr}(\rho_e) = \dim(R) = |G|.$$

Proposition 2.15: The character r_G of the regular representation is given by

$$r_G(e) = |G| \qquad\qquad r_G(s) = 0 \text{ if } s \neq e.$$

Corollary 2.16: Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i.

Proof: By Theorem 2.11, the number of times W_i is contained in the regular representation is $\langle r_G, \chi_i \rangle$. We have that

$$\langle r_G, \chi_i \rangle = \frac{1}{|G|} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{|G|} \cdot |G| \chi_i(1) = \chi_i(1) = n_i.$$

□

Corollary 2.17:

1. The degrees satisfy $\sum_{i=1}^h n_i^2 = |G|$.
2. if $e \neq s \in G$, we have that $\sum_{i=1}^h n_i \chi_i(s) = 0$.

Proof: By Corollary 2.16, we have that $r_G(s) = \sum n_i \chi_i(s)$ for all $s \in G$. A priori we know that r_G is the sum of irreducibles χ_i, and Corollary 2.16 gives the multiplicities. Plugging in $s = e$ and $s \neq e$ yields the claim. □

The above result lets us determine the irreducible representations of a group G. Suppose we have constructed some mutually non-isomorphic irreducible representations of degrees $n_i, \ldots, n_h$. In order to check if we have found all such representations, it is necessary and sufficient to verify that $n_1^2 + \cdots + n_h^2 = |G|$. Also, we shall later see that each of the n_i divide the order of G.

Definition 2.18 (Class Function): A function f on a group G is a *class function* if for all $s, t \in G$, $f(tst^{-1}) = f(s)$.

Proposition 2.19: Let f be a class function on a group G and $\rho : G \to \mathrm{GL}(V)$ a linear representation of G with character χ. Define $\rho_f : V \to V$ by $\rho_f = \sum_{t \in G} f(t) \rho_t$. If V is irreducible of degree n, the ρ_f is a homothety of ratio λ where

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*).$$

Proof: We have that

$$\rho_s^{-1} \rho_f \rho_s = \sum_{t \in G} f(t) \rho_s^{-1} \rho_t \rho_s = \sum_{t \in G} f(t) \rho_{s^1 ts} = \sum_{t \in G} f(s^{-1}ts) \rho_{s^{-1}ts} = \rho_f.$$

Thus, by Proposition 2.5 we have that ρ_f us a homothety λ. The trace of λ is $n\lambda$. Thus, the trace of ρ_f is $\sum_{t \in G} f(t) \operatorname{Tr}(\rho_t) = \sum_{t \in G} f(t) \chi(t)$. Thus, $\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f \mid \chi^*)$. □

Let H be the space of class functions on G.

Theorem 2.20: The characters $\chi_1, \dots, \chi_h$ of G form an orthonormal basis of H.

Proof: Note that Theorem 2.10 says that the χ_i are all orthonormal to each other. To show that they generate H, it is enough to show that the only element of H orthogonal to χ_i^* is 0. Let f be such an element. For each representation ρ of G, let ρ_f be as in Proposition 2.19. Since f is orthogonal to the χ_i^*, Proposition 2.19 says that ρ_f is 0 as long as ρ is irreducible. From the decomposition of a representation into a direct sum of irreducible representation, with possible multiplicities, we conclude that ρ_f is always 0. Now consider the regular representation of G and compute the image of the basis vector e_e under ρ_f:

$$0 = \rho_f(e_e) = \sum_{t \in G} f(t)\rho_t(e_e) = \sum_{t \in G} f(t)\rho_t.$$

Thus, $f(t) = 0$ for each $t \in G$ and $f = 0$. □

Theorem 2.21: The number of irreducible representations of G, up to isomorphism, is the number of conjugacy classes of G.

Proof: Let $C_1, \dots, C_k$ be the distinct conjugacy classes of G. Then all class functions are constant on each class, their value determined by some λ_i for each C_i. These λ_i may be chosen arbitrarily. Thus, the dimension of the space H of class functions is equal to k. But we already know by Theorem 2.20 that the dimension of H is h, the number of irreducible representations of G. □

Proposition 2.22: Let $s \in G$ and $c(s)$ the number of elements in the conjugacy class of s.

1. We have $\sum_{i=1}^h \chi_i(s)^* \chi_i(s) = \frac{|G|}{c(s)}$.
2. For t not conjugate to s, we have $\sum_{i=1}^h \chi_i(s)^* \chi_i(t) = 0$.

Proof: Let f_s be the class function equal to to 1 on the class of s and 0 otherwise. By Theorem 2.21, we have that

$$f_s = \sum_{i=1}^h \lambda_i \chi_i \qquad \lambda_i = (f_s \mid \chi_i) = \frac{c(s)}{|G|} \chi_i(s)^*.$$

We have then, for each $t \in G$, that

$$f_s(t) = \frac{c(s)}{|G|} \sum_{i=1}^h \chi_i(s)^* \chi_i(t).$$

If $t = s$, we get claim 1 and for t not conjugate to s we get claim 2. □

Example 2.23 (Character Table of S_3): Consider the group S_3. There are three conjugacy classes: the identity (), the 3 transpositions, and the 2 cyclic permutations. Let t be one of the transpositions and c one of the cyclic permutations. Then $t^2 = 1 = c^3$ and $tc = c^2 t$. There are just two characters of degree 1: the unit character χ_1 and the character χ_2 giving the sign of the permutation. This is because $t^2 = 1$ means that $\chi(t) = 1$ or -1. Each choice then determines the character of c, which ends up corresponding to the unit character or the sign. By Theorem 2.21, there exists one more irreducible character θ. If n is the degree of θ, then we must have that $1 + 1 + n^2 = 6$, so $n = 2$. By Proposition 2.15, we have that $\chi_1 + \chi_2 + 2\theta$ is the character of the regular representation. Thus, we get the following *character table*:

⟨⟨ TODO: Center ⟩⟩

	1	t	c
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

We obtain an irreducible representation of G with character θ by having G permute the coordinates of elements of $\mathbb{C}^3$ satisfying $x + y + z = 0$.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation of G. Recall that the direct sum decomposition of V into irreducible representation is not necessarily unique. Thus, we shall now define a "coarser" decomposition which has the advantage of being unique.

Definition 2.24 (Canonical Decomposition of a Representation): Let $\chi_1, \ldots, \chi_h$ be the distinct characters of the irreducible representations of $W_1, \ldots, W_h$ of G with degrees $n_1, \ldots, n_h$. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into a direct sum of irreducible representations. For $i = 1, \ldots, h$, let V_i be the direct sum of the U_i which are isomorphic to W_i. Then $V = V_1 \oplus \cdots \oplus V_h$. We have decomposed V into a direct sum of irreducible representations and combined the ones which are isomorphic to each other.

This decomposition satisfies some nice properties:

Theorem 2.25: 1. The decomposition $V = V_1 \oplus V_h$ does not depend on the initially chosen decomposition of V into irreducibles.

2. The projection $p_i : V \to V_i$ is given by

$$p_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i^*(t) \rho_t.$$

Proof: We shall prove claim 2 since claim 1 follows as the p_i determine the V_i. Let $q_i = \frac{n_i}{|G|} \sum_{t \in G} \chi_i(t)^* \rho_t$. By Proposition 2.19, we have that the restriction of q_i to an irreducible representation W with character χ and degree n is a homothety of ratio $\frac{n_i}{n}(\chi_i | \chi)$. Thus, q_i is 0 if $\chi_i \neq \chi$ and 1 if $\chi = \chi_i$. This yields that q_i is the identity on an irreducible representation isomorphic to W_i, and 0 on the others. Thus, q_i is the identity on V_i and 0 on V_j for $j \neq i$. Decomposing $x \in V$ into $x_i \in V_i$ such that $x = x_1 + \cdots + x_h$ yields that

$$q_i(x) = q_i(x_1) + \cdots + q_i(x_h) = x_i.$$

Thus $q_i = p_i$. □

This allows us to decompose representations V in two stages. First, we determine $V_1 \oplus \cdots \oplus V_h$. This is done easily using the given formula for p_i in Theorem 2.25. Finally, for each V_i we may choose a decomposition of V_i into a direct sum of irreducible representations, each isomorphic to W_i, This last decomposition may be done in any number of ways.

Example 2.26 (Decomposition of C_2): Let $G = C_2 = \{e, s\}$ be the cyclic group of two elements generated by s. Let $\rho : G \to \mathrm{GL}(V)$ be any representation of C_2. Note that C_2 has two irreducible representations of degree 1, W^+ and W^- with respective characters $\rho^+ = 1$ and $\rho_s = -1$. The canonical decomposition of V is $V = V^+ \oplus V^-$, where V^+ consists of elements $x \in V$ which are symmetric and V^- consists of elements which are antisymmetric. In other words, V^+ consists of elements $x \in V$ where $\rho_s(x) = x$ and V^- consists of elements $x \in V$ where $\rho_s(x) = -x$. This, the projections are

$$p^+(x) = \frac{1}{2}(x + \rho_s(x)) \qquad\qquad p^-(x) = \frac{1}{2}(x - \rho_s(x)).$$

To decompose V^+ and V^- into irreducible components means to decompose these subspaces into a direct sum of lines, which can be in arbitrarily many ways.

We now have the tools to explicitly compute the components V_i of this canonical decomposition of $\rho : G \to \mathrm{GL}(V)$. Let $V = V_1 \oplus \cdots \oplus V_h$ be this decomposition. The projection given in Theorem 2.25 will allow us to do this. Let W_i have matrix form $(r_{\alpha\beta}(s))$ with respect to a basis $(e_1, \ldots, e_n)$. Then $\chi_i(s) = \sum_\alpha r_{\alpha\alpha}(s)$. For each $1 \leq \alpha, \beta \leq n$ define

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t.$$

Proposition 2.27:

1. The map $p_{\alpha\alpha}$ is a projection. It is 0 on V_j for $j \neq i$ and its image $V_{i,\alpha}$ is contained in V_i where V_i is the direct sum of the $V_{i,\alpha}$, $1 \leq \alpha \leq n$. We have that $p_i = \sum_\alpha p_{\alpha\alpha}$.

2. The linear map $p_{\alpha\beta}$ is 0 on V_j for $j \neq i$ as well as on $V_{i,\gamma}$ for $\gamma \neq \beta$. It defines an isomorphism $V_{i,\beta} \to V_{i,\alpha}$.

3. Let $x_1 \neq 0 \in V_{i,1}$ and $x_\alpha := p_{\alpha,1}(x_1) \in V_{i\alpha}$. Then the x_α are linearly independent and generate a subspace $W(x_1)$ stable under G and of dimension n. For each $s \in G$, we have that

$$\rho_s(x_\alpha) = \sum_\beta r_{\beta\alpha}(s) x_\beta.$$

   In particular, $W(x_1)$ is isomorphic to W_i.

4. If $(x_1^{(1)}, \ldots, x_1^{(m)})$ is a basis of $V_{i,1}$, then the representation V_i is the direct sum of the subrepresentations $W(x_1^{(1)}), \ldots, W(x_1^{(m)})$.

Proof: Observe that the definition of $p_{\alpha\beta}$ is defined in terms of arbitrary representations of G, and in particular in the irreducible representations W_j. For W_i, we have that

$$p_{\alpha\beta}(e_\gamma) = \frac{n}{|G|} \sum_{t \in G} r_{\beta\alpha}(t^{-1}) \rho_t(e_\gamma) = \frac{n}{|G|} \sum_\delta \sum_{t \in G} r_{\beta\alpha}(t^{-1}) r_{\delta\gamma}(t) e_\delta.$$

By Corollary 2.8 we have that

$$p_{\alpha\beta}(e_\gamma) = \begin{cases} e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

We get from this that $\sum_\alpha p_{\alpha\alpha} = \mathrm{id}_{W_i}$. We also get the formulas

$$p_{\alpha\beta} \circ p_{\gamma\delta} = \begin{cases} p_{\alpha\delta} & \text{if } \beta = \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_s \circ p_{\alpha\gamma} = \sum_\beta r_{\beta\alpha}(s) p_{\beta\gamma}.$$

For W_j, $j \neq i$, we use Corollary 2.7 and the same argument to show that all the $p_{\alpha\beta}$ are 0.

With this, we can now decompose V into subrepresentations each isomorphic to W_j and apply the above to these representations. The first two assertions follow. Moreover, these formulas are valid in V. Assuming the hypothesis of claim 3 holds, we have that

$$\rho_s(x_\alpha) = \rho_s \circ p_{\alpha 1}(x_1) = \sum_\beta r_{\beta\alpha}(s) p_{\beta 1}(x_1) = \sum_\beta r_{\beta\alpha}(s) x_\beta.$$

This proves claim 3. Finally, claim 4 follows from the first 3. $\square$

Exercises

Exercise 2.1 (Ser77 2.1): Let χ, χ' be the characters of two representations. Prove the following formulas:

$$(\chi + \chi')^2_\sigma = \chi^2_\sigma + \chi'^2_\sigma + \chi\chi'$$
$$(\chi + \chi')^2_\alpha = \chi^2_\alpha + \chi'^2_\alpha + \chi\chi'$$

Proof: Let $s \in G$. Then by Proposition 2.4 we have that

$$\begin{aligned}
(\chi + \chi')^2_\sigma(s) &= \frac{1}{2}\big((\chi + \chi')^2(s) + (\chi + \chi')(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi'(s)^2 + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)\big) \\
&= \frac{1}{2}\big(\chi(s)^2 + \chi(s^2)\big) + \frac{1}{2}\big(\chi'(s) + \chi'(s^2)\big) + \chi(s)\chi'(s) = \chi^2_\sigma(s) + \chi'^2_\sigma(s) + \chi(s)\chi'(s).
\end{aligned}$$

Since this holds for all $s \in G$, the formula holds in general. The proof of the other formula is similar. □

Exercise 2.2 (Ser77 2.2): Let X be a finite set on which G acts, and $\rho: G \to \mathrm{GL}(V)$ the corresponding permutation representation (Example 1.7), and χ_X the character of ρ. Then show that for $s \in G$, $\chi_X(s)$ is equal to the number of elements fixed by s.

Proof: Suppose $X = [n]$ and so $s \in S_n$, meaning $G \leq S_n$. We may assume this without loss of generality. Note that $R_s = (r_{ij}(s))$ where $r_{ij}(s) = 1$ if $s(j) = i$ and 0 otherwise. We want to count the number of elements in $[n]$ fixed by s, i.e. the number of i such that $\sigma(i) = i$. These correspond exactly to the entries in R_S where $r_{ii}(s) = 1$. Thus, the claim follows. □

Exercise 2.3 (Ser77 2.3): Let $\rho: G \to \mathrm{GL}(V)$ be a linear representation with character χ. Recall that V^* is the dual vector space of V. For $x \in V$, $x^* \in V^*$ let $\langle x, x^* \rangle = x^*(x)$. Then there exists a unique linear representation $\rho^*: G \to \mathrm{GL}(V^*)$ such that

$$\big\langle \rho_s(x), \rho^*_s(x^*) \big\rangle = \langle x, x^* \rangle$$

for $s \in G$, $x \in V$, and $x^* \in V^*$. Note that ρ^* has character χ^*, the conjugate of χ.

Proof: Let $\rho^*_s = \left(\rho^T_s\right)^{-1}$. Then

$$\big\langle \rho_s(x), \rho^*_s(x^*) \big\rangle = \Big\langle \rho_s(x), \left(\rho^T_s\right)^{-1}(x^*) \Big\rangle = \Big(x, \rho^T_s\Big(\left(\rho^T_s\right)^{-1}(x^*)\Big)\Big) = \langle x, x^* \rangle.$$

Now suppose that $\rho' \colon G \to \mathrm{GL}(V^*)$ was another representation satisfying the above property. Then we would have that

$$\left\langle \rho_s(x), (\rho^* - \rho')(x^*) \right\rangle = \left\langle \rho_s(x), \rho_s^*(x^*) \right\rangle - \left\langle \rho_s(x), \rho_s'(x^*) \right\rangle = 0.$$

Note that this holds for all $x \in V$ and $x^* \in V^*$. Thus, we must have that $(\rho^* - \rho')(x^*) = 0$, and thus $\rho^* = \rho'$. $\square$

Exercise 2.4 (Ser77 2.5): Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation with character χ. Then the number of times ρ contains the unit representation is equal to $(\chi \mid 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$.

Proof: The equality $(\chi \mid 1) = \frac{1}{|G|} \sum_{s \in G} \chi(s)$ is immediate by definition of the scalar product $(\cdot \mid \cdot)$ and the fact that $1^* = 1$. By Theorem 2.11, $(\chi \mid 1)$ counts the number of times an irreducible representation with character 1 appears in V. By Corollary 2.13, the only irreducible representation with character 1 is the unit representation. $\square$

Exercise 2.5 (Ser77 2.6): Let G act on a finite set X, ρ the corresponding permutation representation, and χ its character.

1. Let c be the number of distinct orbits. Show that c is equal to the number of times ρ contains the unit representation 1. Deduce that $(\chi \mid 1) = c$. In particular if G is transitive and thus $c = 1$, then $\rho = 1 \oplus \theta$ where θ does not contain the unit representation. If ψ is the character of θ, then $\chi = 1 + \psi$ and $(\psi \mid 1) = 0$.

2. Let G act on the product $X \times X$ in the natural way. Show that the character of the corresponding permutation representation is equal to χ^2.

3. Suppose that G is transitive on X and $|X| \geq 2$. We say G is *doubly transitive* if for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y$ there exists $s \in G$ such that $s(x, y) = (sx, sy) = (x', y')$. Prove that the following are equivalent:

   (a) G is doubly transitive.

   (b) The action of G on $X \times X$ has two orbits, the diagonal and the complement.

   (c) $(\chi^2 \mid 1) = 2$

   (d) The representation θ defined in the first part of this exercise is irreducible.

Proof: We know that the number of times the unit representation is contained in χ is equal to $(\chi \mid 1)$ by Theorem 2.11. By Exercise 2.4, we have that $(\chi \mid 1) = \frac{1}{|G|}\sum_{s\in G}\chi(s)$. We prove that $\frac{1}{|G|}\sum\chi(s) = c$ by double counting. Consider the set $\{(s,x) \in G \times X \mid s \cdot x = x\}$. Then we have that

$$\sum_{x\in X}|G_x| = \sum_{x\in X}|\{s \in G \mid s\cdot x = x\}| = |\{(s,x)\in G\times X \mid s\cdot x = x\}| = \sum_{s\in G}|\{x\in X \mid s\cdot x = x\}| = \sum_{s\in G}\chi(s).$$

Let $O_1, \dots, O_c$ be the distinct orbits. By the Orbit-Stabilizer theorem, each O_i is in bijection with G/G_x for all $x \in O_i$. Note that the orbits O_i partition X. Thus we have that

$$\sum_{s\in G}\chi(s) = \sum_{i=1}^{c}\sum_{x\in O_i}|G_x| = \sum_{i=1}^{c}\sum_{x\in O_i}\frac{|G|}{|O_i|} = c\cdot|G|$$

and $\frac{1}{|G|}\sum_{s\in G}\chi_s = c$. Following this, the rest of the claim is immediate.

Now suppose that ϕ is the character of the permutation representation of $G \curvearrowright X \times X$. Then by Exercise 2.2, $\phi(s)$ is equal to the number of elements fixed by s. An element $(x,y) \in X \times X$ is fixed by $s \in G$ if and only if both x and y are fixed. Thus if there are $\chi(s)$ elements of X fixed by s, then $\chi^2(s)$ elements of $X \times X$ are fixed by s and $\phi = \chi^2$.

To prove 3, we have that $(a) \iff (b)$ is immediate and $(b) \iff (c)$ follows from 1 and 2. Now suppose (c) holds and let ψ be the character of θ. Then $1 + \psi = \theta$. Since $(\chi \mid 1) = (1 \mid 1) = 1$ we must have that $(\psi \mid 1) = 0$. Since $\chi^2 = 1 + 2\psi + \psi^2$, we have that (c) is equivalent to saying $(\psi^2 \mid 1) = 1$. Thus

$$\frac{1}{|G|}\sum_{s\in G}\psi(s)^2 = 1.$$

However, note that $\psi(s)$ is real valued, not just complex valued. This is because χ is real valued, it counts fixed points, and clearly 1 is real valued. Thus $\psi^* = \psi(s)^*$ and so the above equality implies that $(\psi \mid \psi = 1)$. By Theorem 2.14, we have that this is true if and only if θ is irreducible, i.e. $(c) \iff (d)$ holds. $\square$

Exercise 2.6 (Ser77 Exercise 2.8): Let $\rho: G \to \mathrm{GL}(V)$ be any representation of a group G with $V = V_1 \oplus \cdots \oplus V_h$ the canonical decomposition, $W_1, \ldots, W_h$ all irreducible representations of G. Let H_i be the vector space of linear mappins $h: W_i \to V$ such that $\rho_s \circ h = h \circ \rho_s$ for all $s \in G$. Each $h \in H_i$ maps W_i into V_i.

1. Show that $\dim(H_i)$ is equal to $\dim(V_i)/\dim(W_i)$, the multiplicity of W_i in V_i.

2. Let G act on $H_i \otimes V_i$ through the tensor product of the trivial representation of G on H_i and the given representation on W_i. Show that the linear map

$$F: H_i \otimes W_i \to V_i$$
$$\sum h_\alpha \otimes w_\alpha \mapsto \sum h_\alpha(w_\alpha)$$

   is an isomorphism.

3. Let $(h_1, \ldots, h_k)$ be a basis of H_i and form the direct sum $W_i \oplus \cdots \oplus W_i$ of k copies of W_i. This basis defines an obvious mapping $h: W_i \oplus \cdots \oplus W_i \to V_i$. Show that h is an isomorphism of representations. In particular, to decompose V_i into a direct sum of representations isomorphic to W_i amounts to choosing a basis for H_i.

Proof:

1. Let $h \in H_i$. Then h maps W_i into say k_i copies of W_i. Each copy of W_i comes with a projection function $V_i \to W_i$. Composing h with this projection function shows that h is a linear combination of maps $W_i \to W_i$. Thus, it suffices to consider the case of $V = W_i$. But Schur's Lemma (Proposition 2.5) says that in this case h is a scalar multiple of the identity, and thus onto. Thus, $\dim(H_i) = 1 = \frac{\dim(V_i)}{\dim(W_i)}$.

2. By composing F with one of the k_i projection functions, we get that F is a linear combination of maps $H_i \otimes W_i \to W_i$. Thus, we may again reduce to the case that $V = W_i$. In this case, by the proof of 1 we get that F is surjective. Dimension counting yields that it is an isomorphism of vector spaces.

   To see that F is an isomorphism of representations, let $\rho': G \to \mathrm{GL}(H_i \otimes W_i)$ be the given tensor product representation. We have that

$$F(\rho'_s(h_{alpha} \otimes w_\alpha)) = F(h_\alpha \otimes \rho_s(w_\alpha)) = h_\alpha(\rho_s(w_\alpha)) = \rho_s(h_\alpha(w_\alpha)) = \rho_s(F(h_\alpha \otimes w_\alpha)).$$

   Thus $F \circ \rho'_s = \rho_s \circ F$ for all generators, and thus on all of $H_i \otimes W_i$. Thus F is an isomorphism of representations.

3. Define the map

$$\begin{aligned} h\colon W_i \oplus \cdots \oplus W_i &\to V_i \\ (w_1, \ldots, w_k) &\mapsto h_1(w_1) + \cdots + h_k(w_k). \end{aligned}$$

Clearly h is linear. From 2 we see that every element of V_i is of the form $\sum w_\alpha h_\alpha$ and the h_i form a basis. Thus h is surjective and dimension counting yields that h is a linear isomorphism. The proof that h is an isomorphism of representations is similar.

Now suppose we are given an isomorphism of representations $h\colon W_i \oplus \cdots \oplus W_i \to V_i$. Let $i_j\colon W_i \to W_i \oplus \cdots \oplus W_i$ be the inclusions of W_i into the j-th component of $W_i \oplus \cdots \oplus W_i$. Define $h_j\colon W_i \to V_i := h \circ i_j$. Since h is an isomorphism of representations, we have that h_j commutes with ρ and so $h_j \in H_i$. We claim that the h_j form a basis of H_i. Suppose the h_j we linearly dependent. Then this would contradict the face that h is an isomorphism of vector spaces since we would be able to show that $\ker(h) \neq 0$. Thus the h_j form a basis of H_i and every isomorphism of representations arises in the way described.

$\square$

Exercise 2.7 (Ser77 Exercise 2.9): Let W_i be a representation of G with matrix form $(r_{\alpha\beta}(s))$ with respect to a basis $(e_1, \ldots, e_n)$. Let H_i be the space of linear maps $h\colon W_i \to V$ such that $h \circ \rho_s = \rho_s \circ h$. Show that $h \mapsto h(e_\alpha)$ is an isomorphism of H_i onto $V_{i,\alpha}$.

Proof: Suppose that $h \mapsto 0$. So $h(e_\alpha) = 0$. By Exercise 2.6 (2) we have that $h \otimes e_\alpha = 0$. But $e_\alpha \neq 0$ so $h = 0$. Then note that $\dim(V_i) = \dim(V_{i\alpha}) \cdot n = \dim(V_{i\alpha}) \cdot \dim(W_i)$. By Exercise 2.6 (1) we have that $\dim(H_i) = \dim(V_{i\alpha})$ and so by a dimension argument, the map $h \mapsto h(e_\alpha)$ is an isomorphism. $\square$

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