

Algorithms in Invariant Theory

With 0 Figures

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Contents

1	Introduction	1
1.1	Symmetric Polynomials	1
1.2	Gröbner Bases	3
1.3	What is Invariant Theory?	5

Preface

A core interest of mine (at the time of writing this) is algorithms in the context of commutative algebra and algebraic geometry. As such, Bernd Sturmfel's text *Algorithms in Invariant Theory* [\[Str08\]](#) is a good resource for first learning this stuff. Perhaps in the future I'll include notes and some source code

Chapter 1

Introduction

1.1 Symmetric Polynomials

Solution: [Str08] 1.1.5: From [Page 23, 2.11'] [Mac98], we have the following identities

$$1 \cdot \sigma_1 = p_1,$$

$$2 \cdot \sigma_2 = p_1 \sigma_1 - p_2,$$

$$3 \cdot \sigma_3 = p_1 \sigma_2 - p_2 \sigma_1 + p_3,$$

$$\vdots$$

$$k \cdot \sigma_k = \sum_{r=1}^k (-1)^{r-1} p_r \sigma_{k-r}.$$

Treating the σ_i as indeterminants, we can re-express the above system of equations:

$$p_1 = 1 \cdot \sigma_1,$$

$$p_2 = p_1 \sigma_1 - 2 \cdot \sigma_2,$$

$$p_3 = p_2 \sigma_1 - p_1 \sigma_2 + 3 \cdot \sigma_3,$$

$$\vdots$$

$$p_k = \left(\sum_{r=1}^{k-1} (-1)^{r-1} p_{k-r} \sigma_r \right) + (-1)^{k-1} \cdot k \sigma_k.$$

Consider $\sigma_1, -\sigma_2, \sigma_3, \dots, (-1)^n \sigma_{n-1}, \sigma_n$ as indeterminants. Thus, we obtain the following matrix equation:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k \end{pmatrix} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \\ \vdots \\ (-1)^k \sigma_{k-1} \\ \sigma_k \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \\ p_k \end{pmatrix}.$$

Then, Cramer's rule yields that

$$\begin{aligned} \sigma_k &= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & p_1 \\ p_1 & 2 & 0 & \cdots & p_2 \\ p_2 & p_1 & 3 & \cdots & p_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & p_k \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & (-1)^k \cdot k \end{pmatrix}^{-1} \\ &= \frac{(-1)^k}{k!} \det \begin{pmatrix} 1 & 0 & 0 & \cdots & p_1 \\ p_1 & 2 & 0 & \cdots & p_2 \\ p_2 & p_1 & 3 & \cdots & p_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & \cdots & p_k \end{pmatrix} \\ &= \frac{(-1)^k}{k!} \cdot (-1)^k \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix} = \frac{1}{k!} \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{k-1} & p_{k-2} & \cdots & p_1 & k-1 \\ p_k & p_{k-1} & \cdots & \cdots & p_1 \end{pmatrix}. \end{aligned}$$

□

1.2 Gröbner Bases

Lemma 1.2.1:

Let $R = \mathbb{C}[x_1, \dots, x_n]$. Then with the usual grading, let $H(R, z) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(R_d) z^d$. We have that

$$H(R, z) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(R_d) z^d = \sum_{d=0}^{\infty} \binom{d+n-1}{n-1} z^d = \frac{1}{(1-z)^n}.$$

Proof: To see that $H(R, z) = \sum_{d=0}^{\infty} \binom{d+n-1}{n-1} z^d$, just count the number of monomials of degree d in n variables. The value $\binom{d+n-1}{n-1}$ is the number of non-negative integer solutions to $a_1 + \dots + a_n = d$. Each solution corresponds to a monomial $x_1^{a_1} \dots x_n^{a_n}$. Then to see that $H(R, z) = \frac{1}{(1-z)^n}$, consider the product of infinite sums $(1 + z + z^2 + \dots) \dots (1 + z + z^2 + \dots)$ a total of n -times. Then the coefficient of z^d again corresponds to the number of such non-negative integer solutions. Since $\frac{1}{1-z} = 1 + z + z^2 + \dots$, we obtain the desired equality. \square

Lemma 1.2.2:

For $1 \leq k \leq n$, we have that

$$h_k(x_k, \dots, x_n) + \sum_{i=1}^k (-1)^i h_{k-i}(x_k, \dots, x_n) \sigma_i(x_1, \dots, x_n) = 0.$$

Proof: Using the generating functions for the h_i and σ_i , we have that the above expression is the coefficient of t^k in the product

$$\prod_{i=k}^n (1 - x_i t)^{-1} \cdot \prod_{i=1}^n (1 - x_i t) = \prod_{i=1}^{k-1} (1 - x_i t).$$

However, the right-hand side of this has degree $k-1$ in t . Thus, the coefficient of t^k is indeed 0. \square

Solution: [Str08] 1.2.1: Let M be a set of monomial generators for $\text{init}(I)$ and let m be minimally nonstandard. Since m is a monomial and in $\text{init}(I)$, we have that $m' \mid m$ for some monomial $m' \in M$. However, note that $m' \in \text{init}(I)$ and furthermore by the fact that m is minimally nonstandard, we cannot have that m' strictly divides m . Thus, $m' = m$ and $m \in M$. Then by Dickson's Lemma, we have that M is a finite set and thus there are finitely many minimally nonstandard monomials. \square

Solution: [Str08] 1.2.2: This is [CLO15, Chapter 2, §7, Theorem 5]. \square

Solution: [Str08] 1.2.3: This is [CLO15, Chapter 3, §1, Theorem 2]. \square

Solution : [Str08] 1.2.4: This is following [Rob85] and [GP07]. Let \geq be a monomial ordering on $\mathbb{C}[x_1, \dots, x_n]$. This is equivalent to a total semigroup ordering \geq on \mathbb{Z}^n . Such a semigroup ordering gives a unique total ordering on \mathbb{Q}^n . To see this, for $\bar{q} = (q_1, \dots, q_n) \in \mathbb{Q}^n$, let $m \in \mathbb{Z}$ such that $m \cdot q_i \in \mathbb{Z}$ for all i . Then say that $\bar{q} \geq 0$ if and only if $m \cdot \bar{q} \geq 0$ where the latter ordering is in \mathbb{Z}^n .

Let $V \subseteq \mathbb{Q}^n$ be a \mathbb{Q} -vector space with $\dim_{\mathbb{Q}}(V) = r$. Then let

$$V_0 := \{z \in \mathbb{R}^n \mid \forall \varepsilon > 0, \exists z_+(\varepsilon), z_-(\varepsilon) \in V \cap B_\varepsilon(z) \text{ such that } z_+(\varepsilon) > 0, z_-(\varepsilon) < 0\}.$$

Then V_0 is clearly a \mathbb{R} -subspace of \mathbb{R}^n . With the ordering \geq on \mathbb{Q}^n , we can define V_+ and V_- depending on if $\bar{q} \geq 0$ or $\bar{q} < 0$. We define a map $\pi: V \rightarrow \{-1, 1\}$, where V has the Euclidean topology and $\{-1, 1\}$ has the discrete topology. Let $\pi(q) = 1$ if there exists an open ball $U_\varepsilon(q)$ such that $U_\varepsilon(q) \cap V \subseteq V_+$ and $\pi(q) = -1$ if there exists an open ball $U_\varepsilon(q)$ such that $U_\varepsilon(q) \cap V \subseteq V_-$. Then π is continuous. Recall that topological vector spaces over \mathbb{R} are connected. Thus, we cannot have that $\dim_{\mathbb{R}} V_0 < r - 1$ as if it were, then $V_{\mathbb{R}} \setminus V_0$ would be connected **⟨ why is this bad? ⟩**. Then suppose that $\dim_{\mathbb{R}} V_0 = r$. Then we have an ordered basis e_1, \dots, e_r such that $e_i > 0$ for all i . But then the linear combinations of the e_i with positive coefficients are a subspace of V_+ which is a contradiction **⟨ why? ⟩**.

To construct the first row of the matrix, start with $V = \mathbb{Q}^n$ and consider the obtained V_0 . Then the dimension 1 subspace orthogonal to V_0 in \mathbb{R}^n defines the first row of A . We can continue this construction inductively to obtain the full matrix A . □

Solution: [Str08] 1.2.6: First, we recall some definitions. The S -polynomial of f and g is

$$S(f, g) := \frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{\text{LT}(f)} f - \frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{\text{LT}(g)} g.$$

For a set of polynomials $\mathcal{F} = \{f_1, \dots, f_t\} \subseteq k[x_1, \dots, x_n]$, we write $f \rightarrow_{\mathcal{F}} 0$ if there exists $a_1, \dots, a_t \in k[x_1, \dots, x_n]$ such that $a_1 f_1 + \dots + a_t f_t = 0$. Then [CLO15, Chapter 2, §9, Theorem 3] says that a basis $\mathcal{F} = \{f_1, \dots, f_t\}$ is a Gröbner basis for G if and only if $S(f_i, f_j) \rightarrow_{\mathcal{F}} 0$ for all $i \neq j$. But [CLO15, Chapter 2, §9, Proposition 4] says that for $f, g \in \mathcal{F}$ with relatively prime initial monomials, we have that $S(f, g) \rightarrow_{\mathcal{F}} 0$. This proves the claim. □

1.3 What is Invariant Theory?

Solution: [Str08] 1.3.1: Let Γ be a finite group. Consider $f(x) = \prod_{g \in \Gamma} g \cdot x$. Then f is well defined as Γ is finite, invariant under the action of Γ , and of degree $|\Gamma| > 0$.

Now suppose Γ is the subgroup of matrices λI_n for $\lambda \in \mathbb{C}^\times$. Then for any polynomial $f(\bar{x}) = \sum_I \bar{a}^I \bar{x}^I \in \mathbb{C}[\bar{x}]^\Gamma$ and for any such $\lambda I_n \in \Gamma$, we have that

$$\sum_I \bar{a}^I \bar{x}^I = f(\bar{x}) = \lambda I_n \cdot f(\bar{x}) = \sum_I \bar{a}^I \lambda^{|I|} \bar{x}^I.$$

Then comparing coefficients, we deduce that $f(\bar{x})$ is fixed if and only if $f(\bar{x}) \in \mathbb{C}$. Thus, $\mathbb{C}[\bar{x}]^\Gamma = \mathbb{C}$. \square

Solution: [Str08] 1.3.3: Fix $a_1, \dots, a_n \in \mathbb{Z}$ and let $\Gamma = \{ \text{diag}(t^{a_1}, \dots, t^{a_n}) \mid t \in \mathbb{C}^\times \}$. For $d \in \Gamma$ and a monomial $x_1^{v_1} \cdots x_n^{v_n}$, we have that $d \cdot x_1^{v_1} \cdots x_n^{v_n} = t^{a_1 v_1 + \dots + a_n v_n} x_1^{v_1} \cdots x_n^{v_n}$. Thus, we want to determine the set of fixed exponent vectors

$$\mathcal{M} = \{ (v_1, \dots, v_n) \in \mathbb{Z}^n \mid v_1, \dots, v_n \geq 0, a_1 v_1 + \dots + a_n v_n = 0 \}.$$

This is exactly the object of student in §1.4, and in particular is solved by [Str08, Algorithm 1.4.5]. **⟨⟨ Is there a more direct way to see this? ⟩⟩** \square

Solution: [Str08] 1.3.4: Recall that $\text{GL}_n(\mathbb{C})$ is an affine algebraic subvariety of $\mathbb{A}_{\mathbb{C}}^{n^2+1}$. Consider the subspace of matrices in $\text{GL}_n(\mathbb{C})$ which have distinct eigenvalues. Note that this is a Zariski open, and thus dense, subspace of $\text{GL}_n(\mathbb{C})$. Indeed, let $A \in \text{GL}_n(\mathbb{C})$ have eigenvalues $\lambda_1, \dots, \lambda_n$. Then the discriminant of the characteristic polynomial p_A of A is $D(p_A) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)$. Recall that the discriminant of a degree d polynomial $f(x)$ is $\frac{(-1)^{\binom{d}{2}}}{\text{lc}(f)} \text{Res}_x(f(x), f'(x))$. For the characteristic polynomials, this is all expressible in terms of the entries of the matrix. Thus, the subspace of matrices in $\text{GL}_n(\mathbb{C})$ is a dense open subset of $\text{GL}_n(\mathbb{C})$. As this is an infinite set of matrices, any polynomial invariant on this dense subset must be invariant everywhere.

Let $f(\bar{X}) \in \mathbb{C}[\bar{X}]^{\text{GL}_n(\mathbb{C})}$, where \bar{X} is a matrix of indeterminates. Let A have distinct eigenvalues, and so A is diagonalizable so that there exists a matrix $M \in \text{GL}_n(\mathbb{C})$ such $A = MDM^{-1}$ for some diagonal matrix D . In particular, the entries of D are the eigenvalues of A . Thus, $f(A) = f(MDM^{-1}) = f(D)$. Furthermore, we may conjugate A by permutation matrices to reorder the eigenvalues. Thus, f must be a *symmetric* polynomial in the eigenvalues of A , denote these by $e_i := e_i(\lambda_1, \dots, \lambda_n)$. Recall that via the characteristic polynomial, we can express these e_i in terms of the entries of A in general so that each $e_i \in \mathbb{C}[\bar{X}]$ and it makes sense to write that $\mathbb{C}[e_1, \dots, e_n] \subseteq \mathbb{C}[\bar{X}]$. Thus, $\mathbb{C}[\bar{X}]^{\text{GL}_n(\mathbb{C})} \subseteq \mathbb{C}[e_1, \dots, e_n]$. Denote by $f_i(\bar{X})$ the coefficient of t^i in $\det(tI_n - \bar{X})$. Then by noting that the characteristic polynomial is fixed under conjugation, and by comparing coefficients, we see that each f_i is also fixed under conjugation. Thus, we overall have that each e_i is fixed under conjugation and overall, $\mathbb{C}[\bar{X}]^{\text{GL}_n(\mathbb{C})} = \mathbb{C}[e_1, \dots, e_n]$. \square

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