Young Tableau, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci

With 0 Figures

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Preface

These are notes for a reading course under Professor Dave Anderson. They begin with a review of some material from Fulton's *Young Tableaux*[1] [Ful97]. However, the primary focus is Manivel's *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci* [Man01] which one could see as a quasi-sequel.

[1]which throughout these notes will be spelled as "tableaux" or "tableau" with no real consistency.

Chapter 1

[Ful97] Geometry

Solution: [Ful97] §9.1 Ex. 1: Choose a basis $\{e_1, \ldots, e_m\}$ so that E can be identified with $\mathbb{C}^m$. Let $i_1 < \cdots < i_{d-1}$ and $j_1 < \cdots j_{d+1}$ be sequences in $[m]$. Apply §9.1 Equation (1) with $k = 1$ to the sequences $j_2 < \cdots < j_{d+1}$ and $i_1 < \cdots < i_{d-1}, j_1$ by fixing j_1 to be the vector swapped successively with the $j_2 < \cdots < j_{d+1}$. Reordering the indices and applying the appropriate sign change yields the desired alternating summation. □

Solution: [Ful97] §9.1 Ex. 2: We have that $V \subseteq E = \mathbb{C}^4$ is given as the kernel of multiplication of a matrix $A = (a_{i,j})_{\substack{1 \le i \le 4 \\ 1 \le j \le 2}}$. To find this matrix, the given conditions of the $x_{i,j}$ describe the following determinantal conditions on the entries of A:

$$\begin{aligned}
x_{1,2} = 1 &\iff \Delta_{1,2}(A) = 1,\\
x_{1,3} = 2 &\iff \Delta_{1,3}(A) = 2,\\
x_{1,4} = 1 &\iff \Delta_{1,4}(A) = 1,\\
x_{2,3} = 1 &\iff \Delta_{2,3}(A) = 1,\\
x_{2,4} = 2 &\iff \Delta_{2,4}(A) = 2,\\
x_{3,4} = 3 &\iff \Delta_{3,4}(A) = 3.
\end{aligned}$$

From here, we must make an assumption based on which affine portion of $\mathbb{P}^5$ our matrix lives in. This amounts to picking some i_1, i_2 so that the minor given by those columns is the identity matrix. For the given conditions, we could pick $(i_1, i_2) = (1,2)$, $(1,4)$, or $(2,3)$. We give A for each of these choices respectively:

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 0 & -3 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

It is clear that these are the same matrix up to row operations. □

Bibliography

[Ful97] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997. ISBN: 0521567246. DOI: 10.1017/cbo9780511626241.

[Man01] L. Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*. Collection SMF. American Mathematical Society, 2001. ISBN: 9780821821541. URL: https://books.google.com/books?id=yz7gyKYgIuwC.