Young Tableau, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci

With 0 Figures

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Preface

These are notes for a reading course under Professor Dave Anderson. The primary focus is Manivel's *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci* [Man01] which one could see as a quasi-sequel to Fulton's *Young Tableaux*[1] [Ful97]. Primarily, the solutions will be to exercises from [Man01]. However, as needed there will be solutions to material from [Ful97], or perhaps even other texts such as [Mac98] or [Sta01].

[1]which throughout these notes will be spelled as "tableaux" or "tableau" with no real consistency.

Chapter 1

[Ful97] Geometry

Solution: [Ful97] §9.1 Ex. 1: Choose a basis $\{e_1, \ldots, e_m\}$ so that E can be identified with $\mathbb{C}^m$. Let $i_1 < \cdots < i_{d-1}$ and $j_1 < \cdots j_{d+1}$ be sequences in $[m]$. Apply §9.1 Equation (1) with $k = 1$ to the sequences $j_2 < \cdots < j_{d+1}$ and $i_1 < \cdots < i_{d-1}, j_1$ by fixing j_1 to be the vector swapped successively with the $j_2 < \cdots < j_{d+1}$. Reordering the indices and applying the appropriate sign change yields the desired alternating summation. □

Solution: [Ful97] §9.1 Ex. 2: We have that $V \subseteq E = \mathbb{C}^4$ is given as the kernel of multiplication of a matrix $A = (a_{i,j})_{\substack{1 \le i \le 4 \\ 1 \le j \le 2}}$. To find this matrix, the given conditions of the $x_{i,j}$ describe the following determinantal conditions on the entries of A:

$$\begin{aligned}
x_{1,2} = 1 &\iff \Delta_{1,2}(A) = 1, \\
x_{1,3} = 2 &\iff \Delta_{1,3}(A) = 2, \\
x_{1,4} = 1 &\iff \Delta_{1,4}(A) = 1, \\
x_{2,3} = 1 &\iff \Delta_{2,3}(A) = 1, \\
x_{2,4} = 2 &\iff \Delta_{2,4}(A) = 2, \\
x_{3,4} = 3 &\iff \Delta_{3,4}(A) = 3.
\end{aligned}$$

From here, we must make an assumption based on which affine portion of $\mathbb{P}^5$ our matrix lives in. This amounts to picking some i_1, i_2 so that the minor given by those columns is the identity matrix. For the given conditions, we could pick $(i_1, i_2) = (1,2)$, $(1,4)$, or $(2,3)$. We give A for each of these choices respectively:

$$A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 0 & -3 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

It is clear that these are the same matrix up to row operations. □

Solution: [Ful97] §9.1 Ex. 4: Following [Ful97, Corollary of Theorem 1, §8.1], we have that $S^\bullet(m; d_1, \ldots, d_s)$ is canonically isomorphic to the subalgebra of $\mathbb{C}[Z]$ generated by all D_T, where T varies over all tableaux on Young diagrams whose columns have lengths in $\{ d_1, \ldots, d_s \}$ and entries in $[m]$ where

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j),T(2,j),\ldots,T_{\mu_j,j}}$$

where μ_j is the length of the j^{th} column of λ the shape of T and $\ell = \lambda_1$.

(a) We mimic the proof of [Ful97, Proposition 2, §9.1]. **⟨⟨ I think this proof needs to be rewritten, perhaps with a highest weight argument? ⟩⟩** Let $G = G(d_1, \ldots, d_s) \leq \mathrm{GL}(V)$. The dimension of the vector space of polynomials of homogeneous polynomials of degree a in the span of all the $D_{i_1,\ldots,i_p}$ for $p \in \{ d_1, \ldots, d_s \}$ is $\sum d_\lambda(m)$ where the sum ranges over all partitions of a of shape λ with columns whose lengths lie in $\{ d_1, \ldots, d_s \}$. Viewing $V^{\oplus m}$ by identifying $Z_{i,j}$ with the i^{th} basis vector of the j^{th} copy of V, we have by [Ful97, Corollary 3(a), §8.3] that $\mathbb{C}[Z]_a = \mathrm{Sym}^a(V^{\oplus m}) \simeq \bigoplus_{\lambda \vdash a} (V^\lambda)^{d_\lambda(m)}$ where $\lambda \vdash a$ has at most n rows. Thus, we would like to show that $(V^\lambda)^G$ has dimension 1 when the lengths of the columns of λ lie in $\{ d_1, \ldots, d_s \}$ and 0 otherwise.

We recall the construction of V^λ in §8.1 of [Ful97]. Elements of $V^{\times\lambda}$ are specified by specifying an element of V for each box in λ. Fillings by basis vectors $\{ e_1, \ldots, e_n \}$ corresponding to semistandard Young Tableaux T of shape λ with entries in $[n]$. The images of such elements in $V^{\times\lambda}$ in V^λ form a basis $\{ e_T \}$ of V^λ. Consider the basis element corresponding to the tableaux $U(\lambda)$ given by filling every box on row i with the number i. For maps in G, the first d_i basis vectors must map to linear combinations of the first i basis vectors and the restrictions of such maps to the V_i have determinant 1. As such, we can only consider λ whose columns have lengths lying in $\{ d_1, \ldots, d_s \}$. To see that $e_{U(\lambda)}$ is the only such fixed basis vector,

(b)

$\square$

Chapter 2

[Man01] The Ring of Symmetric Functions

2.1 Ordinary Functions

Solution: [Man01] Ex. 1.1.2: We will denote the dominance ordering by $\lambda \leq \mu$ and the ordering given by inclusion of Ferrers diagrams by $\lambda \subseteq \mu$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ and $\lambda' = (\lambda'_1 \geq \cdots \geq \lambda'_l \geq 0)$ be two partitions.

We first consider the ordering $\subseteq$. Note that $\lambda \subseteq \lambda'$ if and only if $k \leq l$ and for all $1 \leq i \leq k$ we have that $\lambda_i \leq \lambda'_i$. Let $m = \min\{k, l\}$. Then define a partition $\mu = (\min\{\lambda_1, \lambda'_1\} \geq \cdots \min\{\lambda_m, \lambda'_m\} \geq 0)$. Then we have that $\mu \subseteq \lambda$ and $\mu \subseteq \lambda'$. Now suppose that $\nu \subseteq \lambda$ and $\nu \subseteq \lambda'$ where $\nu = (\nu_1 \geq \cdots \geq \nu_n \geq 0)$. Then we must have that $n \leq \min\{k, l\} = m$ and that for all $1 \leq i \leq n$ that $\nu_i \leq \min\{\lambda_i, \lambda'_i\} = \mu_i$. Thus, $\nu \subseteq \mu$ and so $\mu = \lambda \wedge \lambda'$ with respect to $\subseteq$. The existence and uniqueness of $\lambda \vee \lambda'$ is similar.

We now consider the ordering $\leq$, now assuming that $|\lambda| = \left|\lambda'\right|$. Before we define $\lambda \vee \lambda'$ for $\leq$, we prove that $\lambda \leq \lambda'$ if and only if $\lambda'^* \leq \lambda^*$. This follows a proof given by [Ros]. Note that $\lambda \leq \lambda'$ if and only if λ can be obtained from λ' by moving boxes successively down from higher rows to lower rows such that every intermediate diagram is still a Ferrers diagram. This immediately implies the duality.

First, for a partition λ let $\hat{\lambda} := (0, \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_k)$. We remark that $\lambda \leq \lambda'$ if and only $\hat{\lambda} \leq_\ell \hat{\lambda}'$ where $\leq_\ell$ is *lexicographic ordering*. One can easily recover λ from $\hat{\lambda}$. By taking componentwise minimums as above for $\hat{\lambda}$ and $\hat{\lambda}'$, one recovers a tuple $\hat{\mu}$ which yields a partition μ. By the remark, we have that $\mu = \lambda \wedge \lambda'$ with respect to $\leq$. Then to define $\lambda \vee \lambda'$, we have that $\lambda \vee \lambda' := (\lambda^* \wedge \lambda'^*)^*$. That this is our desired maximum follows from the above paragraph on conjugation being a lattice antiautomorphism on partitions of the same weight. Uniqueness is also immediate. □

Solution: [Man01] Ex. 1.1.7: These ideas come from [Sta01, Proposition 7.4.1]. Let $X = (x_{ij})$ be the matrix of variables where $x_{ij} = x_j$, so the first column of X is all x_1, the second column is all x_2, etc. We can obtain a term from of e_λ from X by choosing λ_1 elements from the first row, λ_2 elements from the second row, corresponding to picking a term from e_{λ_1}, then a term from e_{λ_2}, etc. After choosing all elements, let the result be $\overline{x}^\alpha$. Replace all the chosen elements with 1's and all the other elements with 0's. This gives a matrix with row sums given by λ and all column sums given by α. Note that α is not necessarily a partition, but rather just a tuple (also called a *weak composition*), which is in line with the fact that monomial symmetric functions are sums over arbitrary orderings of tuples with a fixed set of entries. Conversely, any such 0-11matrix with the prescribed row and column sums describes a term of e_λ. Thus, we have that $e_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$.

Similarly, with X as before, we can obtain a term of h_λ as follows. Choose λ_1 elements from the first row, but we allow each term to be chosen more than once. Next, choose λ_2 elements from the second row, again allowing for each term to be chosen more than once. Continuing on in this manner yields a term $\overline{x}^\alpha$. This again give a matrix, however this time with entries in $\mathbb{N}$ given by numbering the elements with however many times they were chosen. This matrix has the prescribed row and column sums. Conversely, any such matrix with entries in $\mathbb{N}$ with the given row and column sums gives a term of h_λ and so $h_\lambda = \sum_\mu b_{\lambda\mu} m_\mu$.

Now suppose that $a_{\lambda\mu} > 0$. Then we want to show that $\mu \leq \lambda^*$, i.e. that $|\lambda| = |\mu|$ and that for all i we have that $\mu_1 + \cdots + \mu_i \leq \lambda_1^* + \cdots + \lambda_i^*$. If $|\lambda| \neq |\mu|$, then we must have that $a_{\lambda\mu=0}$ and so we know that $|\lambda| = |\mu|$. So by the above argument, there exist a 0-1-matrix M with row sums given by λ and column sums given by μ. ⟨⟨ **Stuck, look at terms in polynomials but what is the correspondence?** ⟩⟩ □

2.2 Pieri's Formulas

Solution: [Man01] Ex. 1.2.4: We have that $a_{\delta+\delta} = \det\left(x_i^{\delta_j+n-j}\right) = \det\left(x_i^{2n-2j}\right)$. This is the Vandermonde determinant again, but now every term is squared. Thus, $a_{\delta+\delta} = \prod_{1\le i<j\le n}(x_i^2 - x_j^2)$. Thus, we have that

$$s_\delta = \frac{a_{\delta+\delta}}{a_\delta} = \frac{\prod_{1\le i<j\le n}(x_i^2 - x_j^2)}{\prod_{1\le i<j\le n}(x_i - x_j)} = \prod_{1\le i<j\le n}(x_i + x_j).$$

□

Solution: [Man01] Ex. 1.2.7: By Pieri's formulas, we have that

$$\left(\sum_{\mu \text{ even}} s_\mu\right)\cdot\left(\sum_{n=0}^{k} e_k\right) = \sum_{\mu \text{ even}}\sum_{k=0}^{n} s_\mu e_k = \sum_{\mu \text{ even}}\sum_{k=0}^{n}\sum_{\lambda\in\mu\otimes 1^k} s_\lambda. \tag{2.1}$$

Clearly, every s_λ term, *not monomial terms*, in the last summation of Equation (2.1) is a term in $\sum_\lambda s_\lambda$, except possibly with a coefficient > 1 We claim that all the coefficients are indeed 1 and that every term in $\sum_\lambda s_\lambda$ appears in the in the last summation of Equation (2.1). This follows from the fact that for any λ, we can decompose λ into an even μ by removing at most one box from each row of λ in each row which is odd and that this removal is unique. □

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