Calculus III
UNIT 14 –MULTIPLE INTEGRATION
Notes 14.1
Iterated Integrals and Area in the Plane

Iterated Integrals

In Chapter 13, you saw that it is meaningful to differentiate functions of several variables with respect to one variable while holding the other variables constant. You can *integrate* functions of several variables by a similar procedure. For example, if you are given the partial derivative

$$f_x(x, y) = 2xy$$

then, by considering y constant, you can integrate with respect to x to obtain

The "constant" of integration, $C(y)$, is a function of y. In other words, by integrating with respect to x, you are able to recover $f(x, y)$ only partially. The total recovery of a function of x and y from its partial derivatives is a topic you will study in Chapter 15. For now, we are more concerned with extending definite integrals to functions of several variables. For instance, by considering y constant, you can apply the Fundamental Theorem of Calculus to evaluate

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\, dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \qquad \text{With respect to } x$$

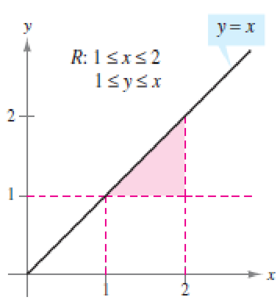
$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\, dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \qquad \text{With respect to } y$$

EXAMPLE 1 Integrating with Respect to y

Evaluate $\int_1^x (2x^2y^{-2} + 2y)\, dy$.

EXAMPLE 2 The Integral of an Integral

Evaluate $\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\, dy \right] dx$.



The region of integration for

$\int_1^2 \int_1^x f(x, y)\, dy\, dx$

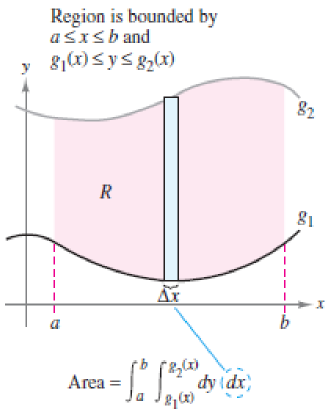
Figure 14.1

The integral in Example 2 is an **iterated integral.** The brackets used in Example 2 are normally not written. Instead, iterated integrals are usually written simply as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

The **inside limits of integration** can be variable with respect to the outer variable of integration. However, the **outside limits of integration** *must be* constant with respect to both variables of integration. After performing the inside integration, you obtain a "standard" definite integral, and the second integration produces a real number. The limits of integration for an iterated integral identify two sets of boundary intervals for the variables. For instance, in Example 2, the outside limits indicate that x lies in the interval $1 \leq x \leq 2$ and the inside limits indicate that y lies in the interval $1 \leq y \leq x$. Together, these two intervals determine the **region of integration R** of the iterated integral, as shown in Figure 14.1.

Because an iterated integral is just a special type of definite integral—one in which the integrand is also an integral—you can use the properties of definite integrals to evaluate iterated integrals.



Vertically simple region
Figure 14.2

Area of a Plane Region

In the remainder of this section, you will take a new look at an old problem—that of finding the area of a plane region. Consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, as shown in Figure 14.2. The area of R is given by the definite integral

$$\int_a^b [g_2(x) - g_1(x)]\, dx. \qquad \text{Area of } R$$

Using the Fundamental Theorem of Calculus, you can rewrite the integrand $g_2(x) - g_1(x)$ as a definite integral. Specifically, if you consider x to be fixed and let y vary from $g_1(x)$ to $g_2(x)$, you can write

$$\int_{g_1(x)}^{g_2(x)} dy = y\Big]_{g_1(x)}^{g_2(x)} = g_2(x) - g_1(x).$$

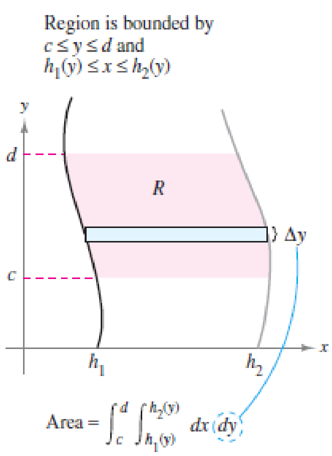
Combining these two integrals, you can write the area of the region R as an iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b y\Big]_{g_1(x)}^{g_2(x)} dx \qquad \text{Area of } R$$

$$= \int_a^b [g_2(x) - g_1(x)]\, dx.$$

Vertical rectangle – the region is **vertically simple.**

Horizontal rectangle -- the region is **horizontally simple.**



Horizontally simple region
Figure 14.3

Area of a Region in the Plane

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \qquad \text{Figure 14.2 (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \qquad \text{Figure 14.3 (horizontally simple)}$$

NOTE Be sure you see that the order of integration of these two integrals is different—the order $dy\, dx$ corresponds to a vertically simple region, and the order $dx\, dy$ corresponds to a horizontally simple region.

EXAMPLE 3 The Area of a Rectangular Region

Use an iterated integral to represent the area of the rectangle shown in Figure 14.4.

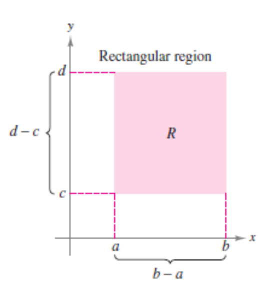


Figure 14.4

EXAMPLE 4 Finding Area by an Iterated Integral

Use an iterated integral to find the area of the region bounded by the graphs of

$f(x) = \sin x$ Sine curve forms upper boundary.

$g(x) = \cos x$ Cosine curve forms lower boundary.

between $x = \pi/4$ and $x = 5\pi/4$.

EXAMPLE 5 Comparing Different Orders of Integration

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx\, dy.$$

Then find another iterated integral using the order *dy dx* to represent the same area and show that both integrals yield the same value.

EXAMPLE 6 An Area Represented by Two Iterated Integrals

Find the area of the region R that lies below the parabola

$$y = 4x - x^2 \qquad \text{Parabola forms upper boundary.}$$

above the x-axis, and above the line

$$y = -3x + 6. \qquad \text{Line and } x\text{-axis form lower boundary.}$$

Calculus III
Notes 14.2
Double Integrals & Volume

Double Integrals and Volume of a Solid Region

You already know that a definite integral over an *interval* uses a limit process to assign measure to quantities such as area, volume, arc length, and mass. In this section, you will use a similar process to define the **double integral** of a function of two variables over a *region in the plane*.

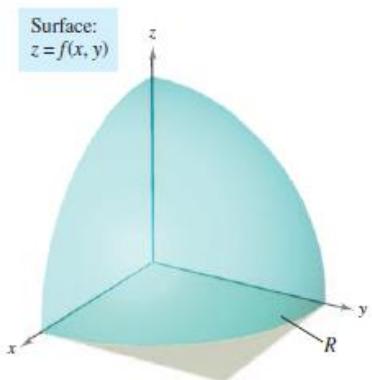


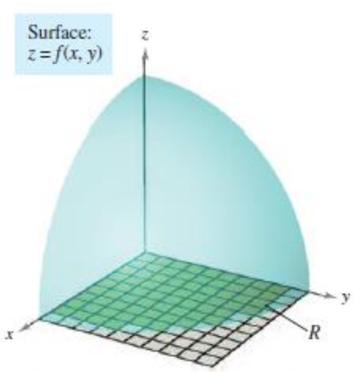
Figure 14.8

Consider a continuous function *f* such that $f(x, y) \geq 0$ for all (x, y) in a region *R* in the *xy*-plane. The goal is to find the volume of the sold region lying between the surface given in Figure 14.8.

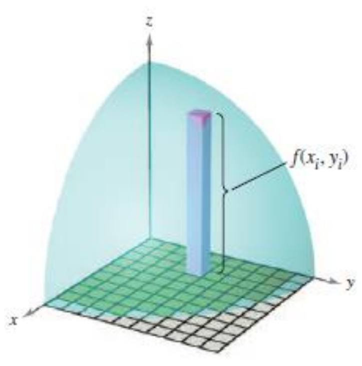
We begin by superimposing a rectangular grid over the region. The rectangles lying entirely within *R* form an **inner partition.** Next choose a point in each rectangle and form the rectangular prism whose height is $f(x_i, y_i)$.

You can approximate the volume of the solid region by the Riemann sum of the volumes all all *n* prisms.

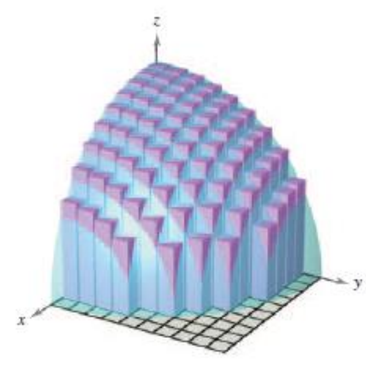
This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles.



The rectangles lying within *R* form an inner partition of *R*.
Figure 14.9



Rectangular prism whose base has an area of ΔA_i and whose height is $f(x_i, y_i)$
Figure 14.10



Volume approximated by rectangular prisms
Figure 14.11

EXAMPLE 1 Approximating the Volume of a Solid

Approximate the volume of the solid lying between the paraboloid

$$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

and the square region R given by $0 \leq x \leq 1$, $0 \leq y \leq 1$. Use a partition made up of squares whose sides have a length of $\frac{1}{4}$.

In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

$$\text{Volume} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i.$$

Using the limit of a Riemann sum to define volume is a special case of using the limit to define a **double integral.** The general case, however, does not require that the function be positive or continuous.

Definition of Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is given by

$$\int_R\int f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

A double integral can be used to find the volume of a solid region that lies between the xy-plane and the surface given by $z = f(x, y)$.

Volume of a Solid Region

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is defined as

$$V = \int_R\int f(x, y)\, dA.$$

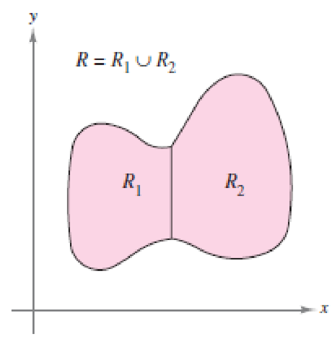
Properties of Double Integrals

Double integrals share many properties of single integrals.

THEOREM 14.1 Properties of Double Integrals

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle \int_R\int cf(x, y)\, dA = c\int_R\int f(x, y)\, dA$
2. $\displaystyle \int_R\int [f(x, y) \pm g(x, y)]\, dA = \int_R\int f(x, y)\, dA \pm \int_R\int g(x, y)\, dA$
3. $\displaystyle \int_R\int f(x, y)\, dA \geq 0, \qquad \text{if } f(x, y) \geq 0$
4. $\displaystyle \int_R\int f(x, y)\, dA \geq \int_R\int g(x, y)\, dA, \qquad \text{if } f(x, y) \geq g(x, y)$
5. $\displaystyle \int_R\int f(x, y)\, dA = \int_{R_1}\int f(x, y)\, dA + \int_{R_2}\int f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.



Two regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to R_1 and R_2 is 0.

Figure 14.14

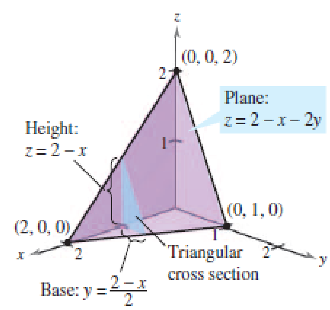
Evaluation of Double Integrals

Normally, the first step in evaluating a double integral is to rewrite it as an iterated integral. To show how this is done, a geometric model of a double integral is used as the volume of a solid.

Consider the solid region bounded by the plane $z = f(x, y) = 2 - x - 2y$ and the three coordinate planes, as shown in Figure 14.15. Each vertical cross section taken parallel to the yz-plane is a triangular region whose base has a length of $y = (2 - x)/2$ and whose height is $z = 2 - x$. This implies that for a fixed value of x, the area of the triangular cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\left(\frac{2-x}{2}\right)(2 - x) = \frac{(2-x)^2}{4}.$$

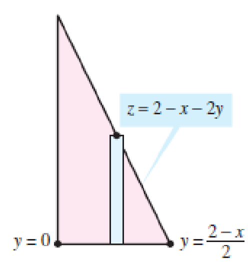
By the formula for the volume of a solid with known cross sections (Section 7.2), the volume of the solid is



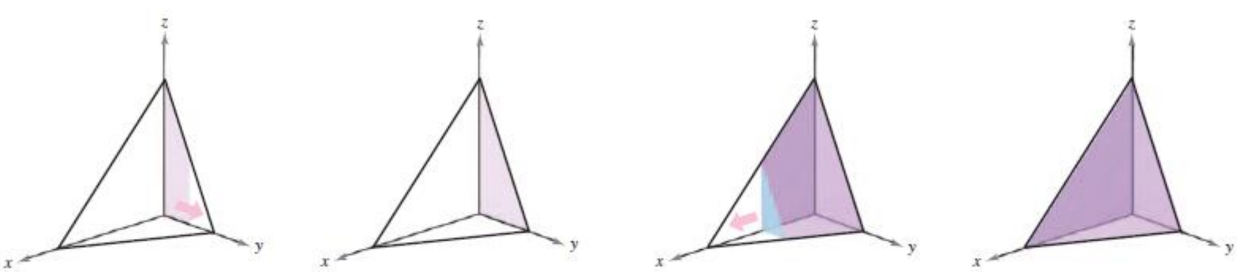
Volume: $\int_0^2 A(x)\,dx$

Figure 14.15

Rotatable Graph



Triangular cross section



Integrate with respect to y to obtain the area of the cross section.

Figure 14.17

Integrate with respect to x to obtain the volume of the solid.

The following theorem was proved by the Italian mathematician Guido Fubini (1879–1943). The theorem states that if R is a vertically or horizontally simple region and f is continuous on R, the double integral of f on R is equal to an iterated integral.

THEOREM 14.2 Fubini's Theorem

Let f be continuous on a plane region R.

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 2 Evaluating a Double Integral as an Iterated Integral

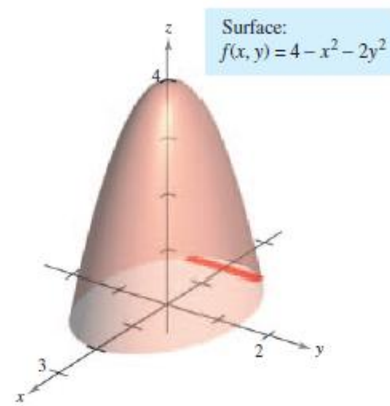
Evaluate

$$\iint_R \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dA$$

where R is the region given by $0 \le x \le 1$, $0 \le y \le 1$.

EXAMPLE 3 Finding Volume by a Double Integral

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.

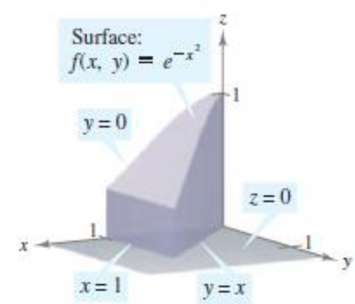


EXAMPLE 4 Comparing Different Orders of Integration

Find the volume of the solid region R bounded by the surface

$$f(x, y) = e^{-x^2} \qquad \text{Surface}$$

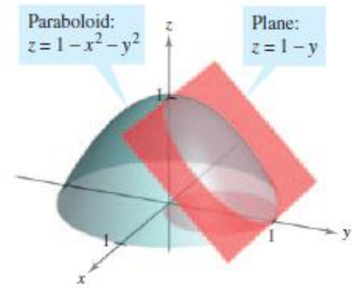
and the planes $z = 0$, $y = 0$, $y = x$, and $x = 1$, as shown in Figure 14.20.



Base is bounded by $y = 0$, $y = x$, and $x = 1$.
Figure 14.20

EXAMPLE 5 Volume of a Region Bounded by Two Surfaces

Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$, as shown in Figure 14.22.



R: $0 \le y \le 1$
$-\sqrt{y - y^2} \le x \le \sqrt{y - y^2}$

Calculus III
Notes 14.3
Change of Variables: Polar Coordinates

Double Integrals in Polar Coordinates

Some double integrals are *much* easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles, cardioids, and rose curves, and for integrands that involve $x^2 + y^2$.

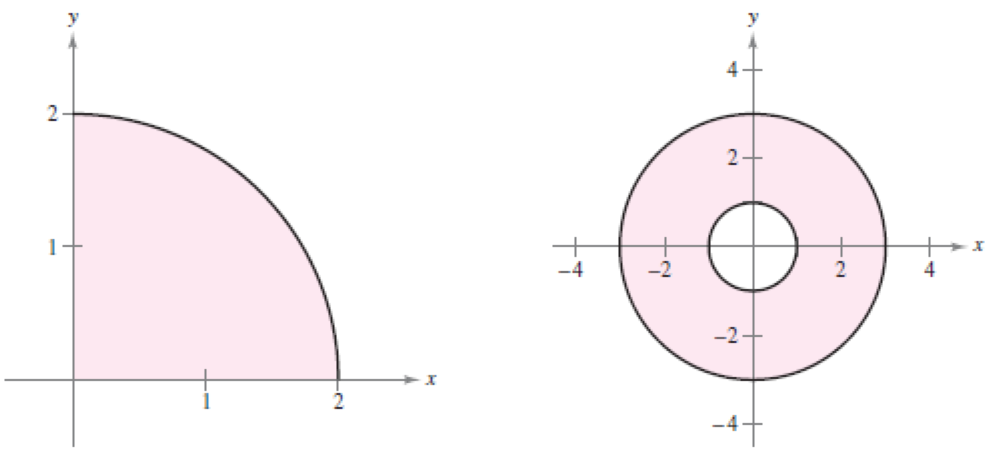
In Section 10.4, you learned that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

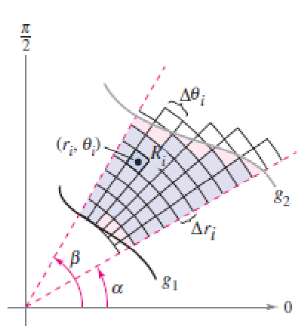
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

EXAMPLE 1 Using Polar Coordinates to Describe a Region

Use polar coordinates to describe each region shown in Figure 14.23.



(a) **(b)**
Figure 14.23



Polar grid is superimposed over region R.
Figure 14.25

To define a double integral of a continuous function $z = f(r,\theta)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and by the lines $\theta = \alpha$ and $\theta = \beta$. On R superimpose a polar grid made of rays and circular arcs. The polar sectors lying entirely within R form an **inner polar partition** whose **norm** is the length of the longest diagonal of the n polar sectors.

This allows us to define a double integral:

Theorem: Polar Form of Fubini's Theorem

Let f be continuous on a plane region R.

1. If R is defined by $\theta_1 \le \theta \le \theta_2$ and $g_1(\theta) \le r \le g_2(\theta)$, where g_1 and g_2 are continuous on $[\theta_1, \theta_2]$, then $\iint\limits_R f(r,\theta)dA = \int\limits_{\theta_1}^{\theta_2}\int\limits_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)r\,dr\,d\theta$.
2. If R is defined by $r_1 \le r \le r_2$ and $h_1(r) \le \theta \le h_2(r)$, where h_1 and h_2 are continuous on $[r_1, r_2]$, then $\iint\limits_R f(r,\theta)dA = \int\limits_{r_1}^{r_2}\int\limits_{h_1(r)}^{h_2(r)} f(r,\theta)r\,d\theta\,dr$.

Ex. 1: Evaluate $\iint\limits_R \sin\theta\, dA$ where R is the first-quadrant region lying inside the circle given by $r = 4\cos\theta$ and outside the circle given by $r = 2$.

Ex. 2: Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

THEOREM 14.3 Change of Variables to Polar Form

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\int_R\int f(x, y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

EXAMPLE 3 Change of Variables to Polar Coordinates

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2} \qquad \text{Hemisphere forms upper surface.}$$

and below by the circular region R given by

$$x^2 + y^2 \le 4 \qquad \text{Circular region forms lower surface.}$$

as shown in Figure 14.30.

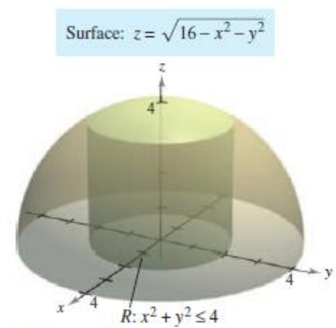
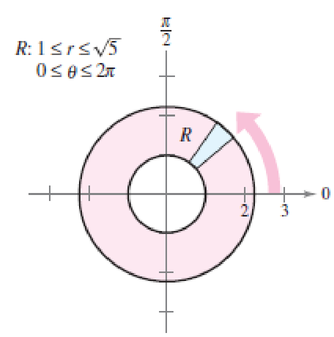


Figure 14.30

EXAMPLE 2 Evaluating a Double Polar Integral

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\int_R\int (x^2 + y)\,dA$.



r-Simple region
Figure 14.29

Calculus III
Notes 14.5
Surface Area

In this section, you will learn how to find the upper **surface area** of the solid. Later, you will learn how to find the centroid of the solid (Section 14.6) and the lateral surface area (Section 15.2).

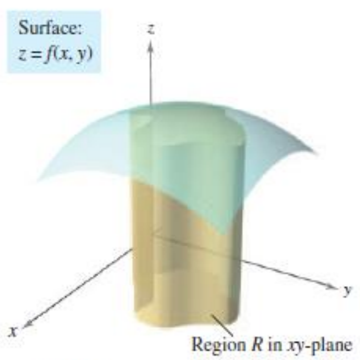


Figure 14.42

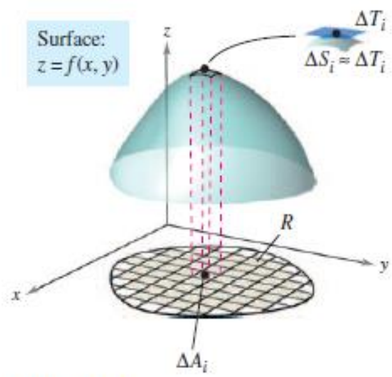


Figure 14.43

Definition of Surface Area

If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface** S given by $z = f(x, y)$ over R is given by

$$\text{Surface area} = \iint_R dS$$
$$= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

Length on x-axis: $\displaystyle\int_a^b dx$

Arc length in xy-plane: $\displaystyle\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Area in xy-plane: $\displaystyle\iint_R dA$

Surface area in space: $\displaystyle\iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$

EXAMPLE 1 The Surface Area of a Plane Region

Find the surface area of the portion of the plane

$$z = 2 - x - y$$

that lies above the circle $x^2 + y^2 \leq 1$ in the first quadrant, as shown in Figure 14.44.

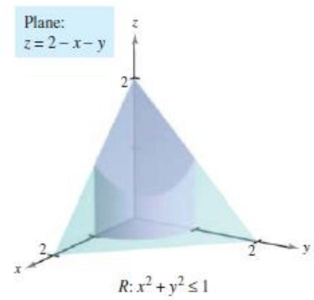


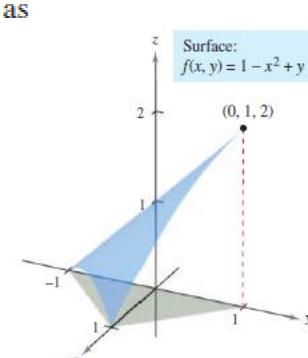
Figure 14.44

EXAMPLE 2 Finding Surface Area

Find the area of the portion of the surface

$$f(x, y) = 1 - x^2 + y$$

that lies above the triangular region with vertices (1, 0, 0), (0, −1, 0), and (0, 1, 0), as shown in Figure 14.45(a).



(a)

EXAMPLE 3 Change of Variables to Polar Coordinates

Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle, as shown in Figure 14.46.

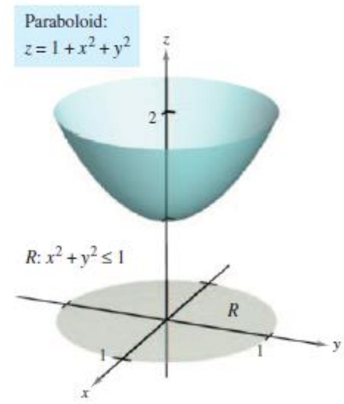


Figure 14.46

EXAMPLE 4 Finding Surface Area

Find the surface area S of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2} \qquad \text{Hemisphere}$$

that lies above the region R bounded by the circle $x^2 + y^2 \leq 9$, as shown in Figure 14.47.

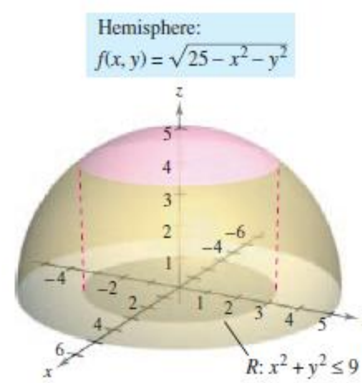


Figure 14.47

EXAMPLE 5 Approximating Surface Area by Simpson's Rule

Find the area of the surface of the paraboloid

$$f(x, y) = 2 - x^2 - y^2 \qquad \text{Paraboloid}$$

that lies above the square region bounded by $-1 \le x \le 1$ and $-1 \le y \le 1$, as shown in Figure 14.49.

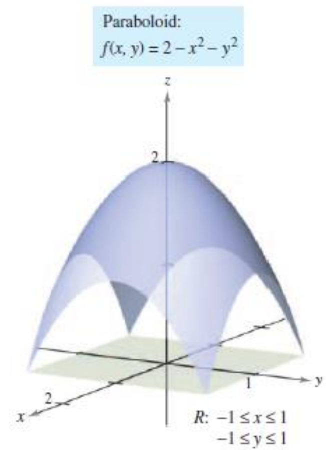


Figure 14.49

Calculus III
Notes 14.6
Triple Integrals

The procedure used to define a **triple integral** follows that used for double integrals.

Definition of Triple Integral

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

Some of the properties of double integrals in Theorem 14.1 can be restated in terms of triple integrals.

1. $\displaystyle\iiint_Q cf(x, y, z)\, dV = c\iiint_Q f(x, y, z)\, dV$

2. $\displaystyle\iiint_Q [f(x, y, z) \pm g(x, y, z)]\, dV = \iiint_Q f(x, y, z)\, dV \pm \iiint_Q g(x, y, z)\, dV$

3. $\displaystyle\iiint_Q f(x, y, z)\, dV = \iiint_{Q_1} f(x, y, z)\, dV + \iiint_{Q_2} f(x, y, z)\, dV$

In the properties above, Q is the union of two nonoverlapping solid subregions Q_1 and Q_2. If the solid region Q is simple, the triple integral $\iiint f(x, y, z)\, dV$ can be evaluated with an iterated integral using one of the six possible orders of integration:

$$dx\,dy\,dz \quad dy\,dx\,dz \quad dz\,dx\,dy \quad dx\,dz\,dy \quad dy\,dz\,dx \quad dz\,dy\,dx.$$

THEOREM 14.4 Evaluation by Iterated Integrals

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x, y) \le z \le g_2(x, y)$$

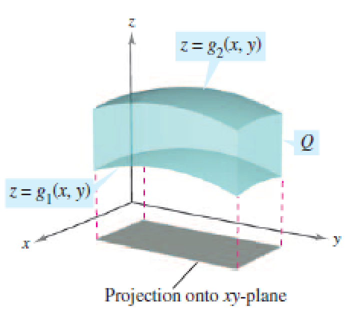
where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

EXAMPLE 1 Evaluating a Triple Iterated Integral

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\, dz\, dy\, dx.$$



Solid region Q lies between two surfaces.
Figure 14.52

Rotatable Graph

To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables. Then, by projecting the solid Q onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals. For instance, to evaluate

$$\iiint_Q f(x, y, z)\, dz\, dy\, dx$$

first determine the limits for z, and then the integral has the form

$$\iint \left[\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\, dz \right] dy\, dx.$$

By projecting the solid Q onto the xy-plane, you can determine the limits for x and y as you did for double integrals, as shown in Figure 14.52.

EXAMPLE 2 Using a Triple Integral to Find Volume

Find the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

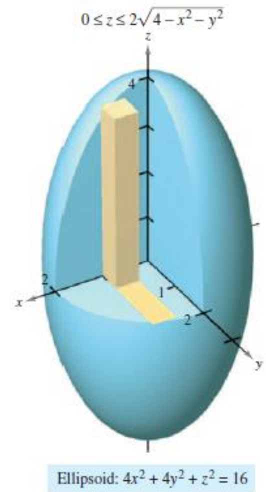
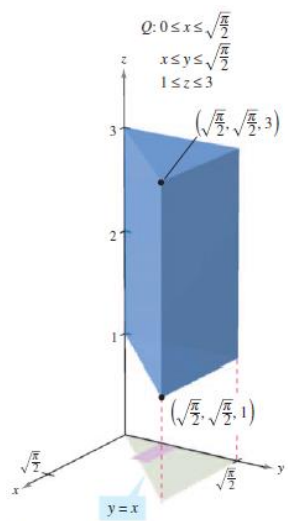


Figure 14.53

EXAMPLE 3 Changing the Order of Integration

Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\, dz\, dy\, dx.$



The volume of the solid region Q is 1.
Figure 14.55

EXAMPLE 4 Determining the Limits of Integration

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes $x + y = 1$ and $x + y = 3$

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$

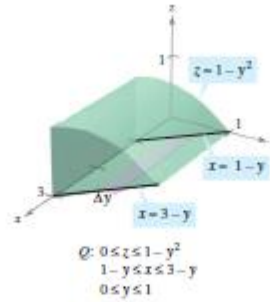


Figure 14.56

Rotatable Graph

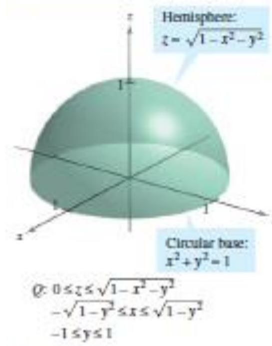


Figure 14.57

Rotatable Graph

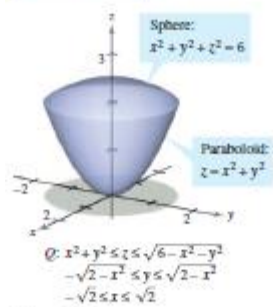
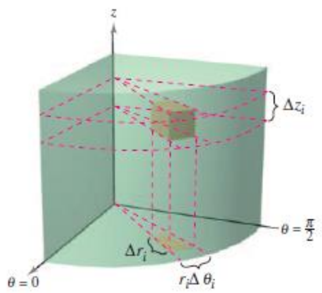


Figure 14.58

Calculus III
Notes 14.7
Triple Integrals in Cylindrical and Spherical Coordinates

Triple Integrals in Cylindrical Coordinates

Many common solid regions such as spheres, ellipsoids, cones, and paraboloids can yield difficult triple integrals in rectangular coordinates. In fact, it is precisely this difficulty that led to the introduction of nonrectangular coordinate systems. In this section, you will learn how to use *cylindrical* and *spherical* coordinates to evaluate triple integrals.

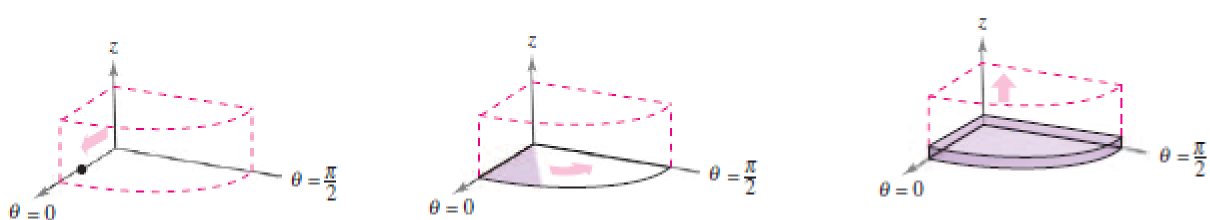


Volume of cylindrical block:
$\Delta V_i = r_i \Delta r_i \Delta \theta_i \Delta z_i$
Figure 14.62

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

NOTE This is only one of six possible orders of integration. The other five are $dz\, d\theta\, dr$, $dr\, dz\, d\theta$, $dr\, d\theta\, dz$, $d\theta\, dz\, dr$, and $d\theta\, dr\, dz$.

To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, in the order $dr\, d\theta\, dz$, the first integration occurs in the r-direction as a point sweeps out a ray. Then, as θ increases, the line sweeps out a sector. Finally, as z increases, the sector sweeps out a solid wedge, as shown in Figure 14.63.



Integrate with respect to r.
Integrate with respect to θ.
Integrate with respect to z.
Figure 14.63

EXAMPLE 1 Finding Volume by Cylindrical Coordinates

Find the volume of the solid region Q cut from the sphere

$$x^2 + y^2 + z^2 = 4 \qquad \text{Sphere}$$

by the cylinder $r = 2 \sin \theta$, as shown in Figure 14.64.

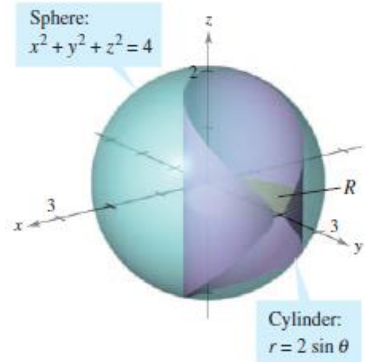
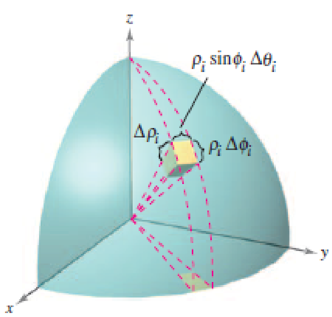


Figure 14.64

Triple Integrals in Spherical Coordinates

Triple integrals involving spheres or cones are often easier to evaluate by converting to spherical coordinates. Recall from Section 11.7 that the rectangular conversion equations for spherical coordinates are

$$x = \rho \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta$$
$$z = \rho \cos \phi.$$



Spherical block:
$\Delta V_i \approx \rho_i^2 \sin \phi_i \, \Delta\rho_i \, \Delta\theta_i \, \Delta\phi_i$
Figure 14.67

Rotatable Graph

In this coordinate system, the simplest region is a spherical block determined by

$$\{(\rho, \theta, \phi): \rho_1 \le \rho \le \rho_2, \quad \theta_1 \le \theta \le \theta_2, \quad \phi_1 \le \phi \le \phi_2\}$$

where $\rho_1 \ge 0$, $\theta_2 - \theta_1 \le 2\pi$, and $0 \le \phi_1 \le \phi_2 \le \pi$, as shown in Figure 14.67. If (ρ, θ, ϕ) is a point in the interior of such a block, then the volume of the block can be approximated by $\Delta V \approx \rho^2 \sin \phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$ (see Exercise 17 in the Problem Solving exercises for this chapter).

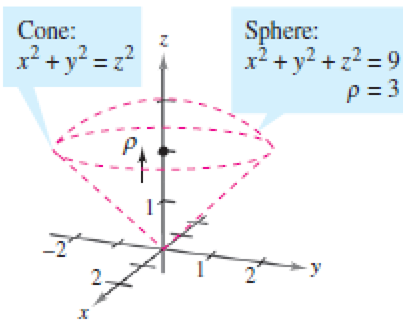
Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function f defined on the solid region Q.

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

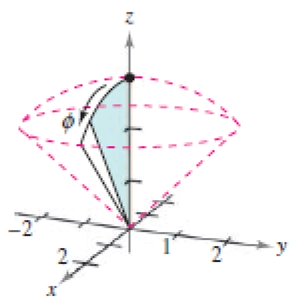
Like triple integrals in cylindrical coordinates, triple integrals in spherical coordinates are evaluated with iterated integrals. As with cylindrical coordinates, you can visualize a particular order of integration by viewing the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, the iterated integral

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

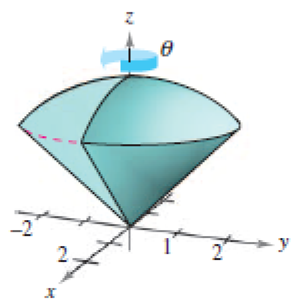
(which is used in Example 4) is illustrated in Figure 14.68.



ρ varies from 0 to 3 with ϕ and θ held constant.



ϕ varies from 0 to $\pi/4$ with θ held constant.



θ varies from 0 to 2π.

Figure 14.68

EXAMPLE 4 Finding Volume in Spherical Coordinates

Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$, as shown in Figure 14.69.

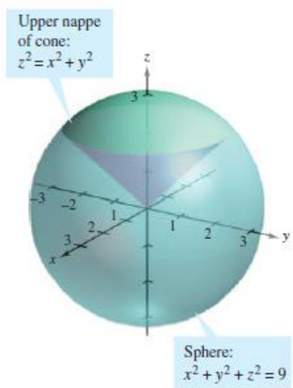


Figure 14.69

Calculus III
Notes 14.8
Change of Variables: Jacobians

Jacobians

For the single integral

$$\int_a^b f(x)\,dx$$

you can change variables by letting $x = g(u)$, so that $dx = g'(u)\,du$, and obtain

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$. Note that the change-of-variables process introduces an additional factor $g'(u)$ into the integrand. This also occurs in the case of double integrals

$$\int_R\int f(x, y)\,dA = \int_S\int f(g(u, v), h(u, v))\underbrace{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|}_{\text{Jacobian}}du\,dv$$

where the change of variables $x = g(u, v)$ and $y = h(u, v)$ introduces a factor called the **Jacobian** of x and y with respect to u and v. In defining the Jacobian, it is convenient to use the following determinant notation.

Definition of the Jacobian

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

EXAMPLE 1 The Jacobian for Rectangular-to-Polar Conversion

Find the Jacobian for the change of variables defined by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Example 1 points out that the change of variables from rectangular to polar coordinates for a double integral can be written as

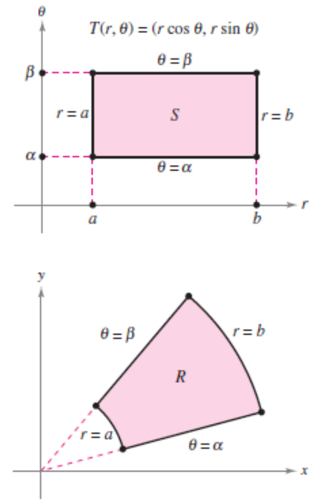
$$\int_R\int f(x, y)\,dA = \int_S\int f(r\cos\theta, r\sin\theta)r\,dr\,d\theta, \quad r > 0$$
$$= \int_S\int f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane, as shown in Figure 14.70. This formula is similar to that found on page 1003.

In general, a change of variables is given by a one-to-one **transformation** T from a region S in the uv-plane to a region R in the xy-plane, to be given by

$$T(u, v) = (x, y) = (g(u, v), h(u, v))$$

where g and h have continuous first partial derivatives in the region S. Note that the point (u, v) lies in S and the point (x, y) lies in R. In most cases, you are hunting for a transformation in which the region S is simpler than the region R.



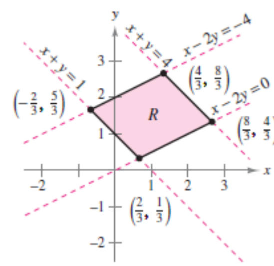
S is the region in the $r\theta$-plane that corresponds to R in the xy-plane.
Figure 14.70

EXAMPLE 2 Finding a Change of Variables to Simplify a Region

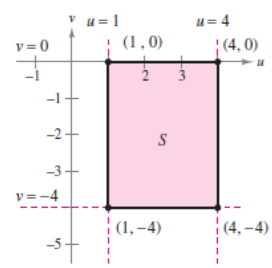
Let R be the region bounded by the lines

$$x - 2y = 0, \qquad x - 2y = -4, \qquad x + y = 4, \qquad \text{and} \qquad x + y = 1$$

as shown in Figure 14.71. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- or v-axis).



Region R in the xy-plane
Figure 14.71



Region S in the uv-plane
Figure 14.72

Change of Variables for Double Integrals

> **THEOREM 14.5 Change of Variables for Double Integrals**
>
> Let R and S be regions in the xy- and uv-planes that are related by the equations $x = g(u, v)$ and $y = h(u, v)$ such that each point in R is the image of a unique point in S. If f is continuous on R, g and h have continuous partial derivatives on S, and $\partial(x, y)/\partial(u, v)$ is nonzero on S, then
>
> $$\int_R\int f(x, y)\,dx\,dy = \int_S\int f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$

EXAMPLE 3 Using a Change of Variables to Simplify a Region

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 14.75. Evaluate the double integral

$$\int_R\int 3xy\,dA.$$

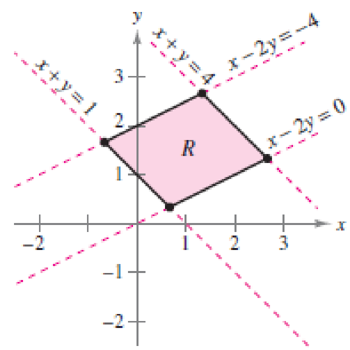
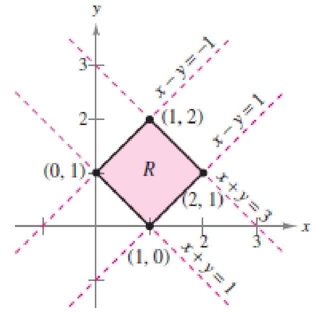


Figure 14.75

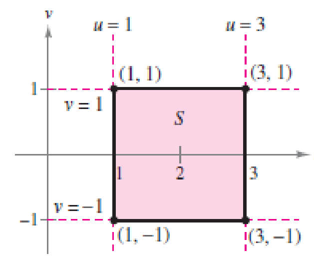
EXAMPLE 4 Using a Change of Variables to Simplify an Integrand

Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral

$$\int_R\int (x + y)^2 \sin^2(x - y)\, dA.$$



Region R in the xy-plane
Figure 14.76



Region S in the uv-plane
Figure 14.77

In each of the change-of-variables examples in this section, the region S has been a rectangle with sides parallel to the u- or v-axis. Occasionally, a change of variables can be used for other types of regions. For instance, letting $T(u, v) = \left(x, \frac{1}{2}y\right)$ changes the circular region $u^2 + v^2 = 1$ to the elliptical region $x^2 + (y^2/4) = 1$.