A Taste of Differentiation

Slopes and Derivatives

Limits

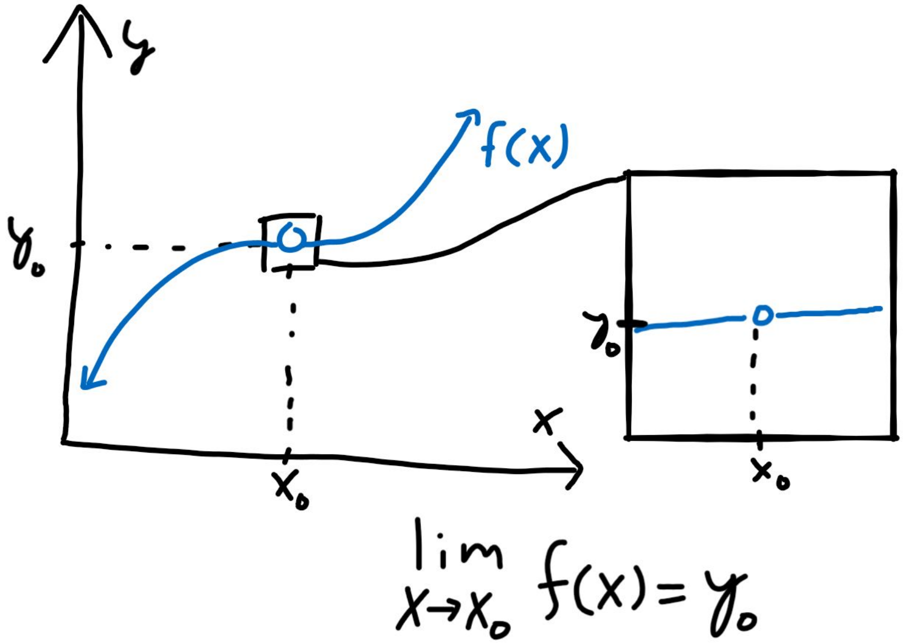
Let's take a graph and remove a point from it.

How do we know which point we removed?

We can zoom in and see what the values of nearby points are and deduce that the removed point should be "very close" to the nearby points.

More concretely: we approach the point from both sides of the graph and see the trend very near, but not at, a particular point.

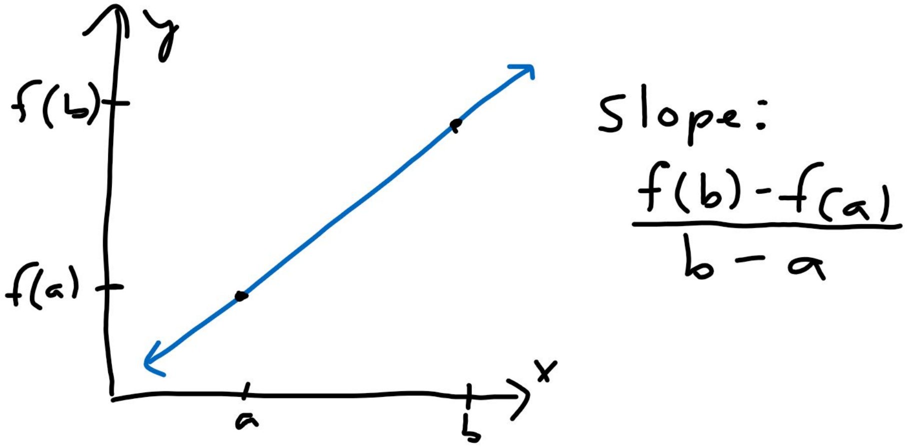
This is the idea behind taking a "limit" of a function.



Slopes

We can calculate the “slope” of a line by taking the rise (change in y coordinate) and dividing it by the run (change in x coordinate).

For a line of the form y = f(x), the slope from a point x = a to x = b can be written as the fraction [f(b) - f(a)]/[b - a].



Derivatives

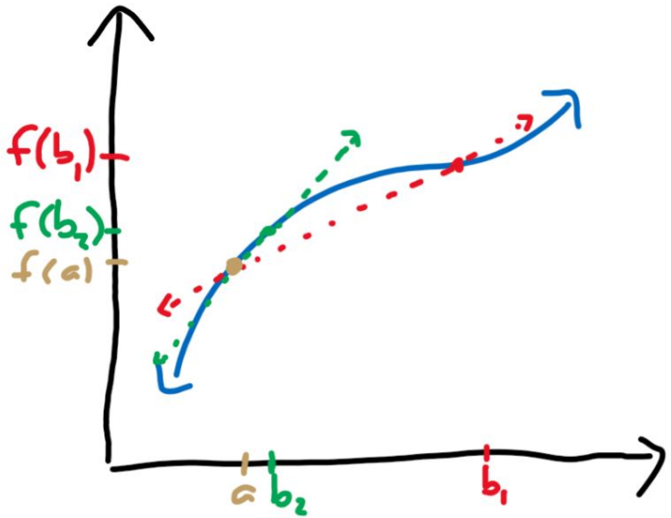
What about the slope for a non-linear curve?

We can approximate the slope at a certain point “a” as the slope of the line that passes through “a” and a nearby point “b.”

As b gets closer to a, our approximation gets better.

If we use our slope formula and take the limit as b approaches a, we get the slope of the curve at a point a.

We call the slope at a the value of the “derivative” of the curve at the point a.

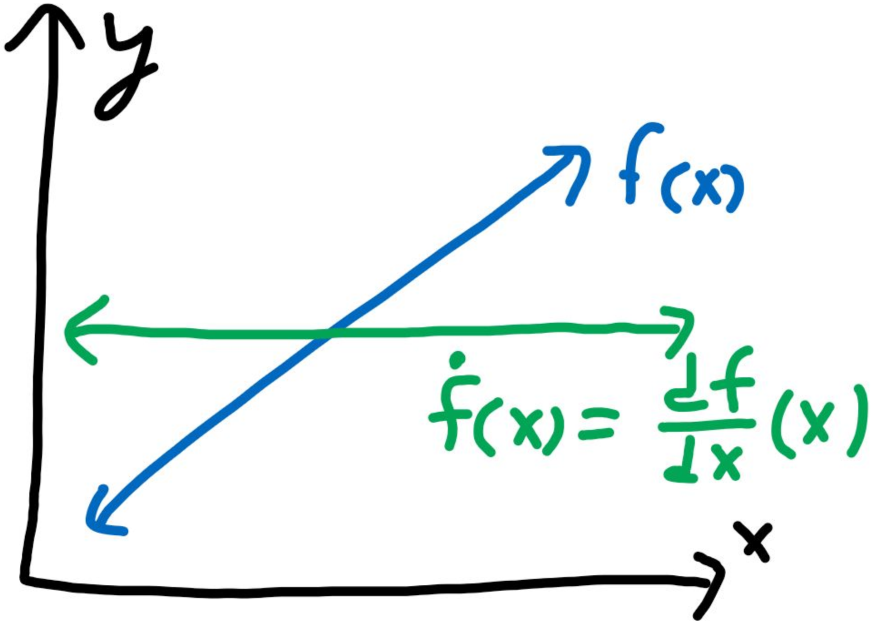


$$\text{slope} = f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Derivatives as a function

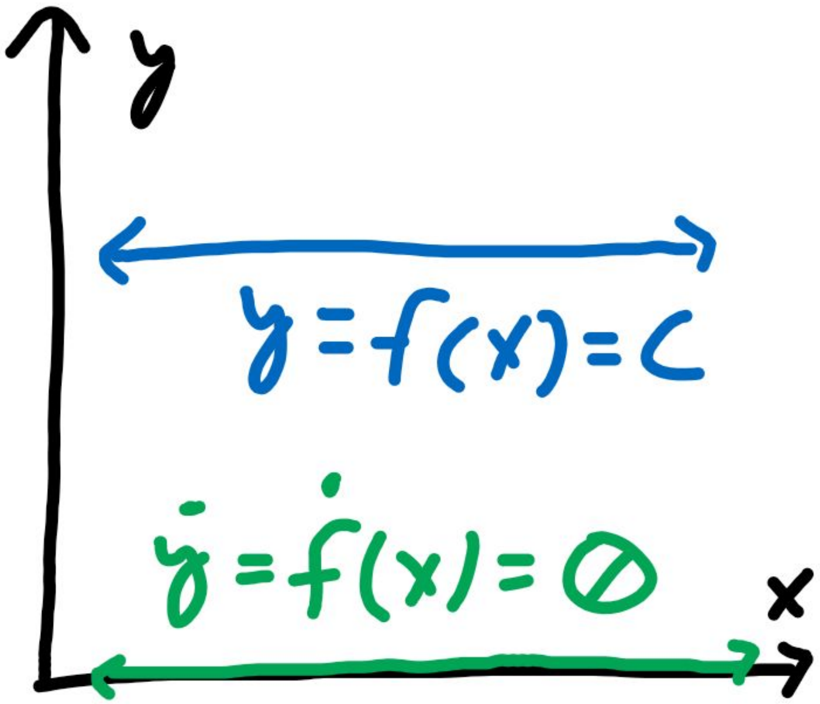
We can plot the value of the derivative at every point to get the derivative as a function, which we denote as dy/dx where d represents a small change in.

Thus, dy/dx represents a small change in y divided by a small change in x, which is rise over run at a very small scale.



Derivative of a constant

The constant function y = c has a slope of 0 everywhere, so the derivative dy/dx of the function y = c is 0.



Taylor Expansions

Polynomials

A polynomial $p(x)$ is the sum of terms $c_n x^n$ where c_n is some constant and n is a non-negative integer.

In general, we can write a polynomial as the expression $p(x) = c_0 + c_1 x + c_2 x^2 + . . . + c_n x^n$.

Power rule

What about for a polynomial?

We can do some manipulation of the derivative formula for a polynomial and find that, in general, the derivative of the function $y = x^n$ is $dy/dx = nx^{n-1}$ where n is a non-zero integer.

For $f(x) = x^2$

$$\dot{f}(x) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

$$= \lim_{b \to a} \frac{b^2 - a^2}{b - a}$$

$$= \lim_{b \to a} \frac{(b-a)(b+a)}{b - a}$$

$$= \lim_{b \to a} (b + a)$$

$$= 2a$$

Coefficients of a polynomial

Notice that $p(0) = c_0$.

Let $p' = dp/dx$. Then, $p'(x) = c_1 + 2c_2x + 3c_3x^2 \ldots$, so $p'(0) = c_1$.

We can continue taking derivatives to get $p'' = dp'/dx = 2c_2 + 6c_3x + \ldots$ and $p''' = dp''/dx = 3c_3$.

Thus, $p''(0) = 2c_2$ and $p'''(0) = 6c_3 = (3!)c_3$.

Let us define $p^{(a)}$ to be the "a"th derivative of the polynomial $p(x)$. In other words, the result when we take the derivative of $p(x)$ "a" times.

Then $p^{(a)}(0)$ represents the value of the function $p^{(a)}(x)$ at $x = 0$.

In general to find c_a, where a is an integer between 0 and n inclusive, we can use the equation $c_a = p^{(a)}(0)/(a!)$.

Taylor approximation (at x = 0)

Let us try to approximate a function f(x) as a polynomial p(x). Thus, we treat f as a polynomial.

The polynomial is formed from many “monomial” terms $c_a x^a$.

We know $c_a = f^{(a)}(0)/(a!)$, so $c_a x^a = [f^{(a)}(0)/(a!)]x^a$.

We can write out each and every term to get $f(x) \approx f(0) + f'(0)x + [1/2]f''(0)x^2 + [1/(3!)]f'''(0)x^3 \ldots$ with more terms.

This is called the Taylor expansion of f(x) at the value x = 0, also known as the Maclaurin series of a function.

$$f(x) \approx \lim_{n \to \infty} \sum_{a=0}^{n} \frac{1}{a!} f^{(a)}(0) x^a$$

$$= f(0) + f'(0)x + \frac{1}{2} f'(0) x^2 + \ldots$$