# A Walk Through Physics

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# Taylor Expansions

# Limits

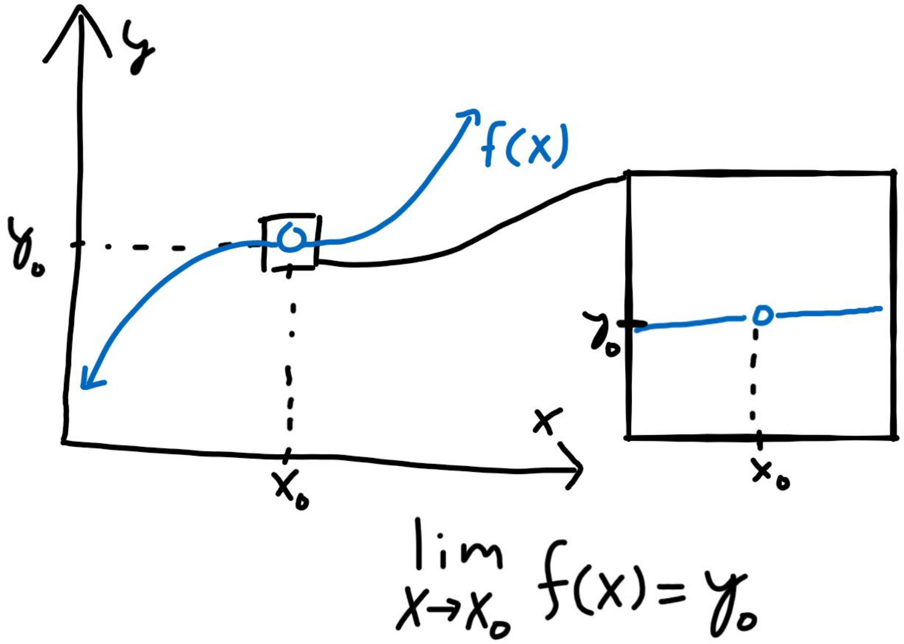
Let's take a graph and remove a point from it.

How do we know which point we removed?

We can zoom in and see what the values of nearby points are and deduce that the removed point should be "very close" to the nearby points.

More concretely: we approach the point from both sides of the graph and see the trend very near, but not at, a particular point.

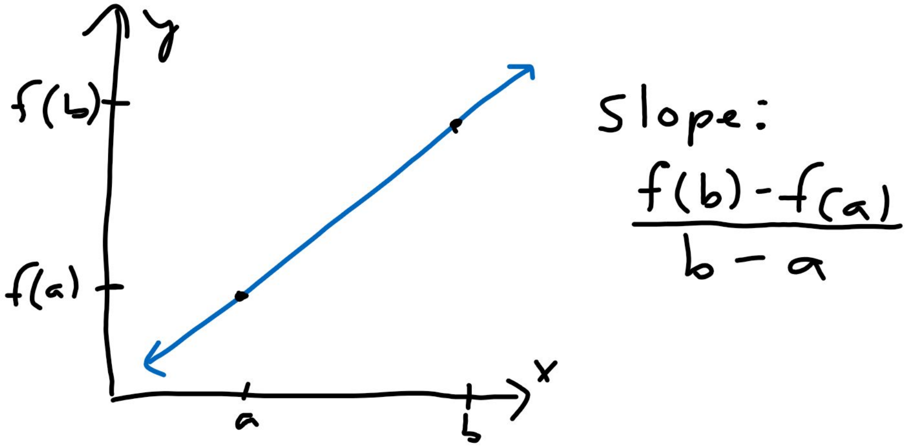
This is the idea behind taking a "limit" of a function.



# Slopes

We can calculate the “slope” of a line by taking the rise (change in y coordinate) and dividing it by the run (change in x coordinate).

For a line of the form y = f(x), the slope from a point x = a to x = b can be written as the fraction [f(b) - f(a)]/[b - a].



# Derivatives

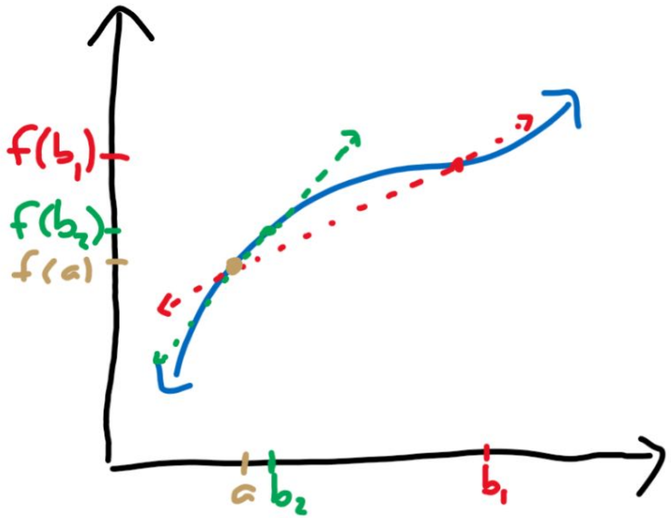
What about the slope for a non-linear curve?

We can approximate the slope at a certain point “a” as the slope of the line that passes through “a” and a nearby point “b.”

As b gets closer to a, our approximation gets better.

If we use our slope formula and take the limit as b approaches a, we get the slope of the curve at a point a.

We call the slope at a the value of the “derivative” of the curve at the point a.

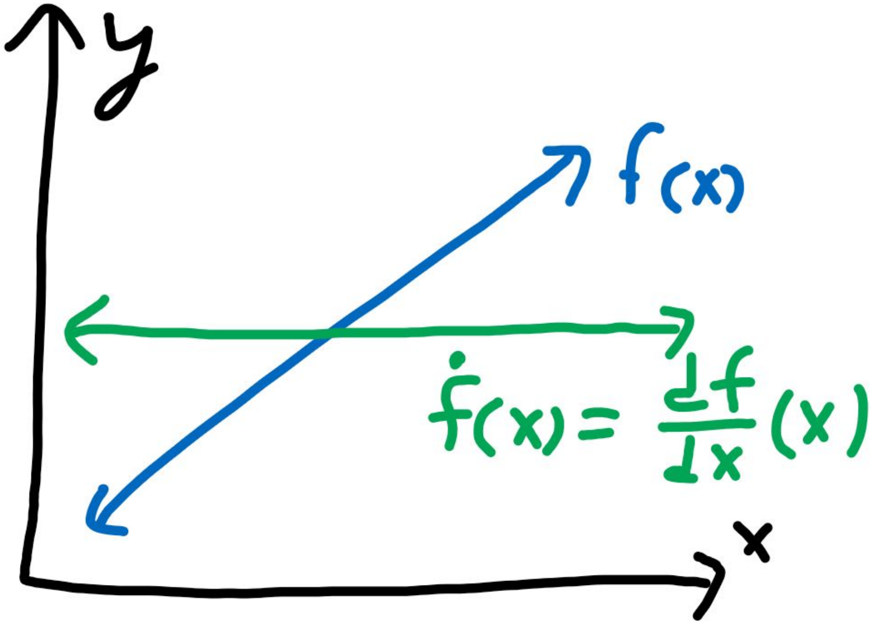



$$\text{slope} = f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

# Derivatives as a function

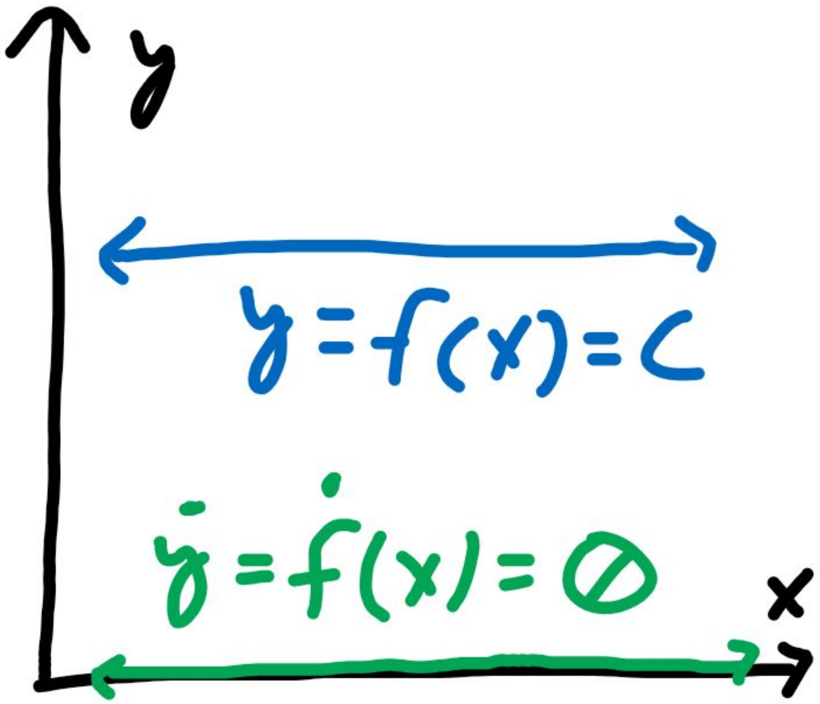
We can plot the value of the derivative at every point to get the derivative as a function, which we denote as dy/dx where d represents a small change in.

Thus, dy/dx represents a small change in y divided by a small change in x, which is rise over run at a very small scale.



# Derivative of a constant

The constant function y = c has a slope of 0 everywhere, so the derivative dy/dx of the function y = c is 0.



# Polynomials

A polynomial $p(x)$ is the sum of terms $c_n x^n$ where $c_n$ is some constant and $n$ is a non-negative integer.

In general, we can write a polynomial as the expression $p(x) = c_0 + c_1 x + c_2 x^2 + . . . + c_n x^n$.

# Power rule

What about for a polynomial?

We can do some manipulation of the derivative formula for a polynomial and find that, in general, the derivative of the function $y = x^n$ is $dy/dx = nx^{n-1}$ where n is a non-zero integer.

For $f(x) = x^2$

$$\dot{f}(x) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

$$= \lim_{b \to a} \frac{b^2 - a^2}{b - a}$$

$$= \lim_{b \to a} \frac{(b-a)(b+a)}{b - a}$$

$$= \lim_{b \to a} (b + a)$$

$$= 2a$$

# Coefficients of a polynomial

Notice that $p(0) = c_0$.

Let $p' = dp/dx$. Then, $p'(x) = c_1 + 2c_2x + 3c_3x^2 \ldots$, so $p'(0) = c_1$.

We can continue taking derivatives to get $p'' = dp'/dx = 2c_2 + 6c_3x + \ldots$ and $p''' = dp''/dx = 3c_3$.

Thus, $p''(0) = 2c_2$ and $p'''(0) = 6c_3 = (3!)c_3$.

Let us define $p^{(a)}$ to be the "a"th derivative of the polynomial $p(x)$. In other words, the result when we take the derivative of $p(x)$ "a" times.

Then $p^{(a)}(0)$ represents the value of the function $p^{(a)}(x)$ at $x = 0$.

In general to find $c_a$, where a is an integer between 0 and n inclusive, we can use the equation $c_a = p^{(a)}(0)/(a!)$.

# Taylor approximation (at x = 0)

Let us try to approximate a function f(x) as a polynomial p(x). Thus, we treat f as a polynomial.

The polynomial is formed from many “monomial” terms $c_a x^a$.

We know $c_a = f^{(a)}(0)/(a!)$, so $c_a x^a = [f^{(a)}(0)/(a!)]x^a$.

We can write out each and every term to get $f(x) \approx f(0) + f'(0)x + [1/2]f''(0)x^2 + [1/(3!)]f'''(0)x^3 \ldots$ with more terms.

This is called the Taylor expansion of f(x) at the value x = 0, also known as the Maclaurin series of a function.

$$f(x) \approx \lim_{n \to \infty} \sum_{a=0}^{n} \frac{1}{a!} f^{(a)}(0) x^a$$

$$= f(0) + f'(0)x + \frac{1}{2}f'(0)x^2 + \ldots$$

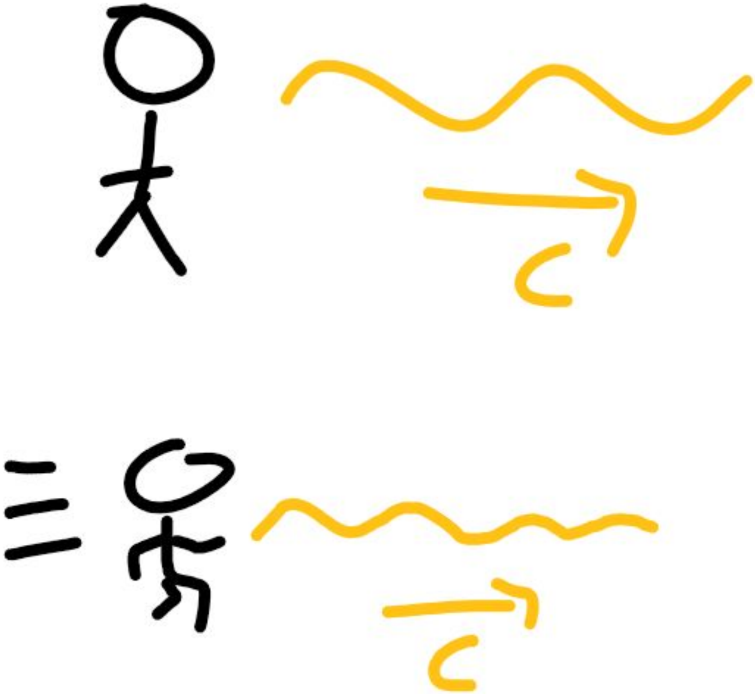
# Relativity

# The one constant

The speed of light (in a vacuum) is constant in any reference frame.

That means that a stationary observer and an observer moving at a velocity v will both observe the sleep of light to be constant.

We call the constant “c.”

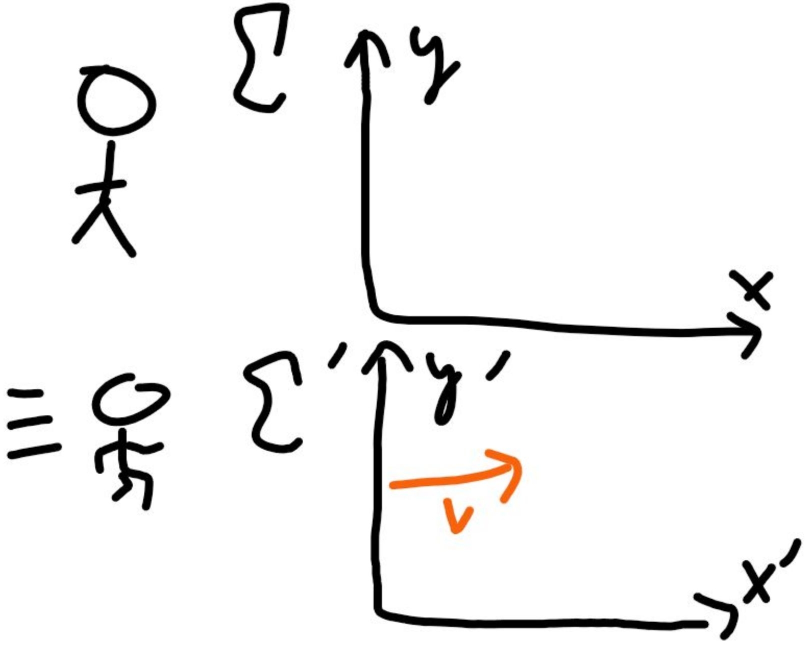


# Reference frames

We denote reference frames with the symbol sigma: Σ.

Generally, a moving reference frame is denoted as Σ' and a stationary one is Σ.

For us, we will define Σ' to be moving along the positive x'-axis at a constant velocity v compared to Σ where the x and x' axes are parallel and the y and y' axes are parallel.

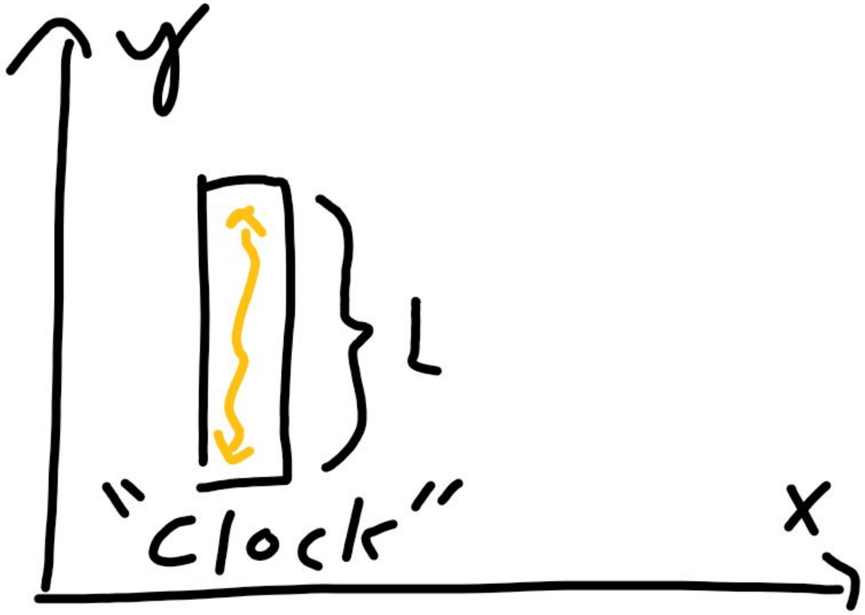


# Light clock

Let us define a clock of a constant length L and a light beam bouncing up and down along the y-axis.

Since the speed of light is constant, we can change the length of the clock so it takes light one second to go from the bottom to the top.

But what if we put this clock on a spaceship moving at a constant velocity v?



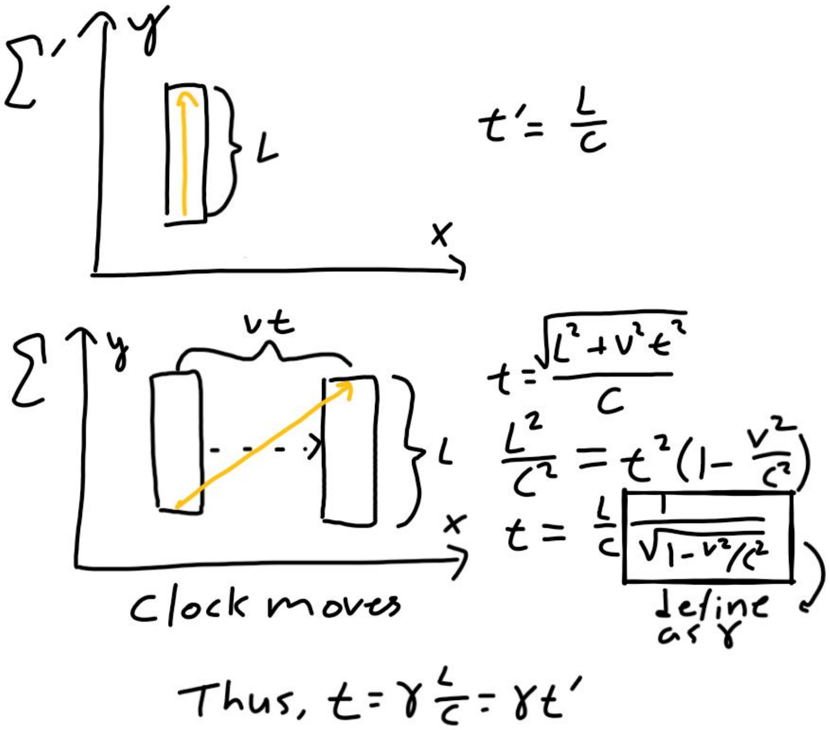
# Time dilation

In the Σ′ (spaceship) system, the light beam still seems to go straight from the bottom of the clock to the top.

However, in the Σ (stationary) system, the light beam travels a further distance.

Thus, the stationary observer thinks more time passed compared to the spaceship observer.

The relationship can be written as t = γt′ where t′ is time passed through the spaceship system and t is time passed in the stationary system. Sometimes t and t′ are defined oppositely.



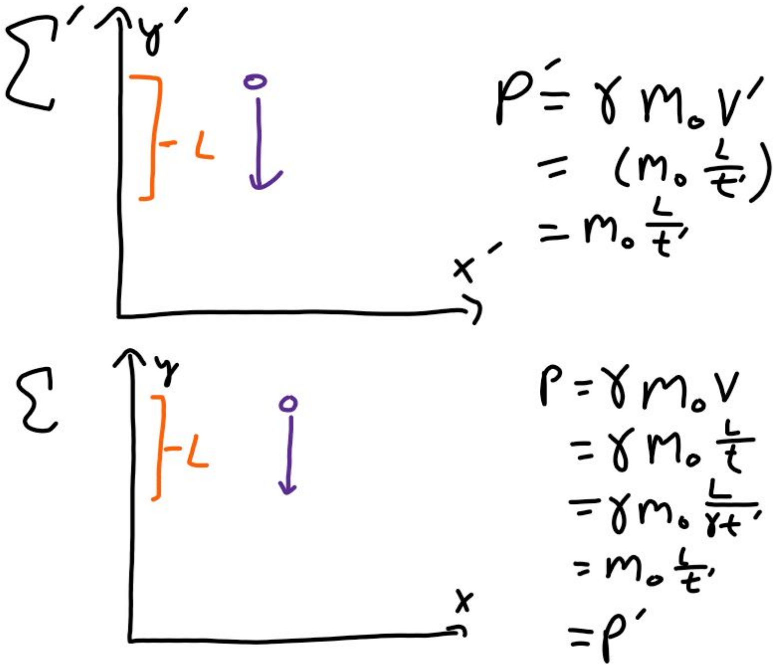
# Relativistic momentum

Let us introduce a quantity called momentum, represented as p.

Momentum is generally calculated as mass times velocity, but in the relativistic sense, this is not true.

An object in different reference frames will have different momentums due to time dilation if we use p = mv, so we need a new momentum in relativity.

Relativistic momentum is defined as $\gamma m_0 v$.



# Relativistic kinetic energy

We can use some integration techniques to go from relativistic momentum to relativistic kinetic energy.

This gives us the formula KE = $\gamma m_0 c^2 - m_0 c^2$.

$$KE = \int_0^{x} \frac{d}{dt} p \, dx$$

$$= \int_0^{t} \frac{d}{dt} \gamma m_0 v \frac{dx}{dt} dt$$

$$= m_0 \int_0^{\gamma} \frac{d}{dt} \gamma v^2 dt$$

$$= m_0 \left( \gamma v^2 - \int_0^{v} \gamma v \, dv \right)$$

$$= m_0 \left( \gamma v^2 + c^2 \left( \frac{1}{\gamma} - 1 \right) \right)$$

$$= \gamma m_0 c^2 - m_0 c^2$$

# Taylor approximation of kinetic energy

Of course, we can try to Taylor approximate this kinetic energy to approximate the term $\gamma m_0 c^2 \approx m_0 c^2[1 + (1/2)(v^2/c^2) + (3/8)(v^4/c^4) + \ldots]$.

For objects moving very slowly ($v \lll c$), then $v^4/c^4$ and subsequent terms disappear, giving us $\gamma m_0 c^2 \approx m_0 c^2[1 + (1/2)(v^2/c^2)]$.

$$KE \approx \underbrace{\frac{1}{2} m_0 v^2}_{\text{good approximation}} + \underbrace{\frac{3}{8} m_0 \frac{v^4}{c^2} + \frac{5}{16} m_0 \frac{v^6}{c^4} + \ldots}_{\text{very small}}$$

Thus, $KE = \gamma m_0 c^2 - m_0 c^2 \approx m_0 c^2 (1/2)(v^2/c^2)$, which simplifies to $KE \approx (1/2) m_0 v^2$.

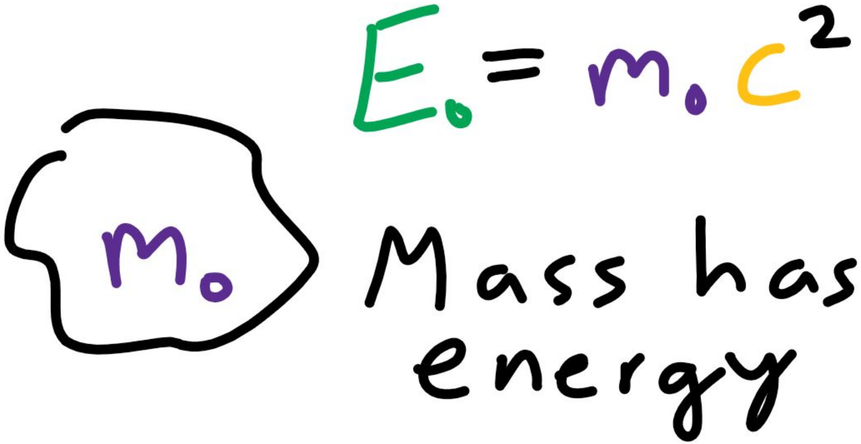
# Einstein’s equation

$E_0 = m_0c^2$, but what does this really mean?

$E_0$ denotes the energy of an object at rest (the 0 represents that the object is moving at a velocity v = 0).

The mass of an object is $m_0$, and c is the speed of light. Thus, an object at rest intrinsically has energy due to its mass.

This is called mass-energy equivalence.



# Relativistic energy

We can sum kinetic energy and the energy due to the mass of an object to get the total mass of an object.

Let relativistic momentum be p and relativistic energy be E. We can do some manipulation to get $p^2 - (E/c)^2 = -m_0^2c^2$.

If we rearrange this, we get $E^2 = p^2c^2 + m_0^2c^4$. This is the generalized form of Einstein's equivalence principle for objects with velocity.

$$p^2 - \left(\frac{E}{c}\right)^2 = \gamma^2 m_0^2 v^4 - \frac{\gamma^2 m_0^2 c^4}{c^2}$$

$$= \gamma^2 m_0^2 (v^2 - c^2)$$

$$= -\gamma^2 m_0^2 c^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$= -\gamma^2 m_0^2 c^2 \frac{1}{\gamma^2}$$

$$= -m_0^2 c^2$$

$$p^2 - \left(\frac{E}{c}\right)^2 = -m_0^2 c^2$$

$$p^2c^2 - E^2 = -m_0^2 c^4$$

$$E^2 = p^2c^2 + m_0^2c^4$$

# A “simple” question

# The “simple” question

Let a block of mass M on a circular table be connected by a string by a block also of mass M hanging under the center of the table.

At first, let the block on the table be spinning at a constant angular velocity $\omega_0$ such that the spinning of the block counteracts the force of gravity ($F_a$ = Mg) applied by the hanging block.

Now let’s give the spinning block a little push into the center of the table. What is the subsequent motion of the block?