Energy in space

Dot product

Vector multiplication

$\vec{a} \times \vec{b} = \vec{c}$ cross product

$|\vec{c}| = |\vec{a}||\vec{b}| \sin(\theta)$

$\vec{a} \cdot \vec{b} = c$ dot product

$c = |\vec{a}||\vec{b}| \cos(\theta)$

A cross product, represented as "x", multiplies two vectors and gives a new, perpendicular, vector as a result.

A dot product, represented as "*" or " · ", multiplies two vectors, but gives a scalar as a result.

Dot product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

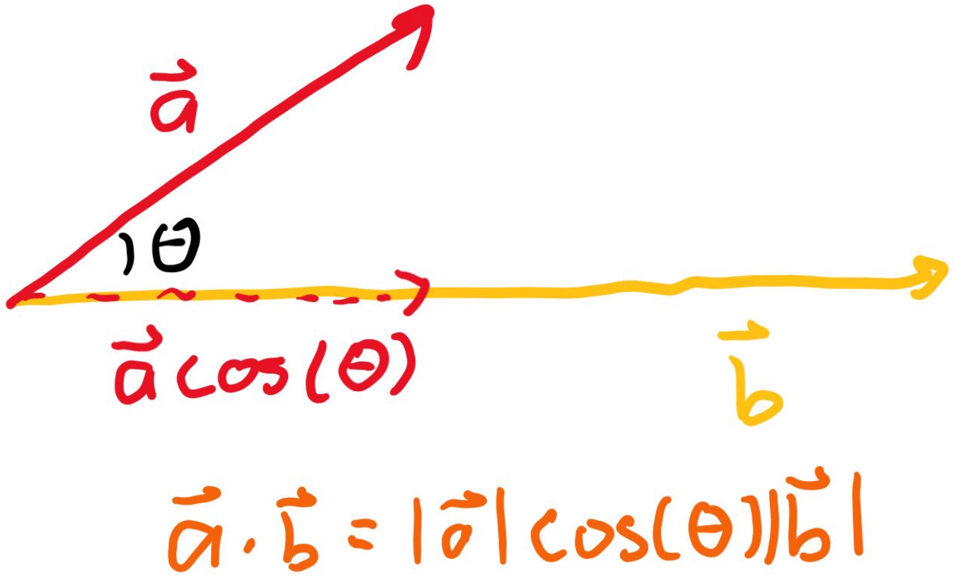
$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Unlike a cross product which only works for 2, 3, 7 dimensions, dot products apply to every dimension.

Vectors <a, b, c> and <x, y, z> have a dot product of ax+by+cz, where you sum the product of each matching term in the vectors.

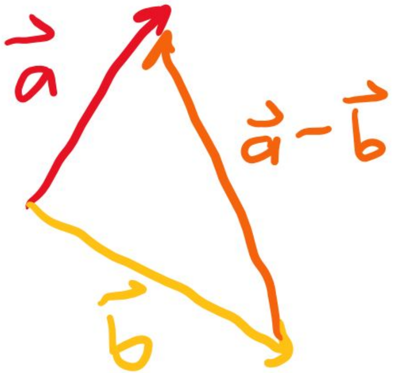
Magnitude approach



Another way to take the dot product of **a** * **b** for vectors **a**, **b** is to do |**a**||**b**|cos(θ).

This is equivalent to multiplying the magnitude of vector components parallel to each other.

Both approaches give the same thing



law of cosines:

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\vec{x}\cdot\vec{x} = |\vec{x}|^2\cos(\theta) = |\vec{x}|^2$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

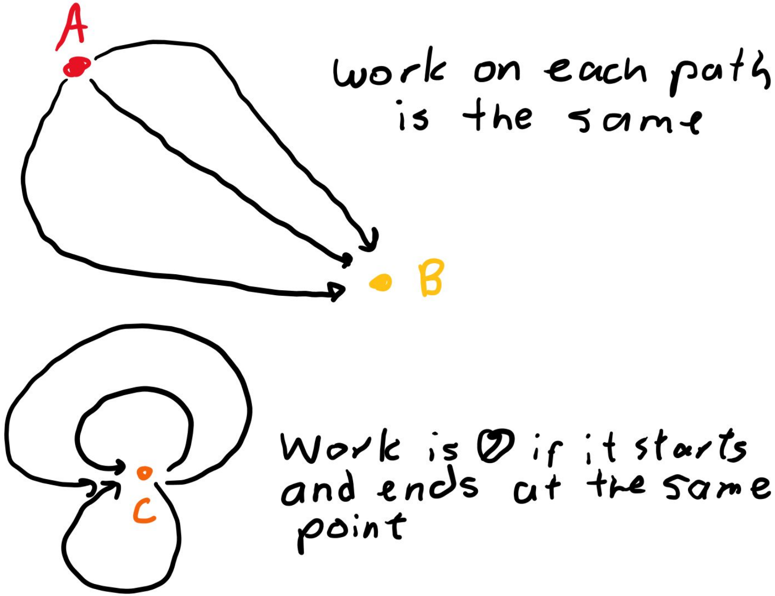
$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}-\vec{b}|^2$$

$$|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

$$|\vec{a}||\vec{b}|\cos(\theta) = \vec{a}\cdot\vec{b}$$

Conservative forces

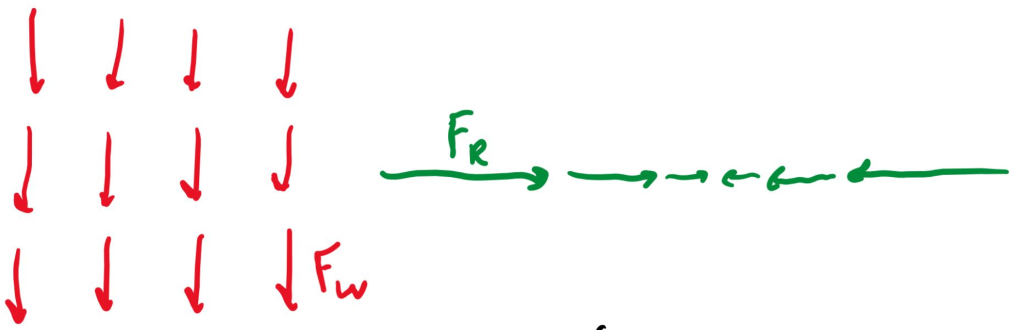
What are they



These are forces that decrease potential energy of a system.

Since potential energy is about the position and structure of a system, the work done by a conservative force from point a to point b is the same no matter what path is taken.

Examples

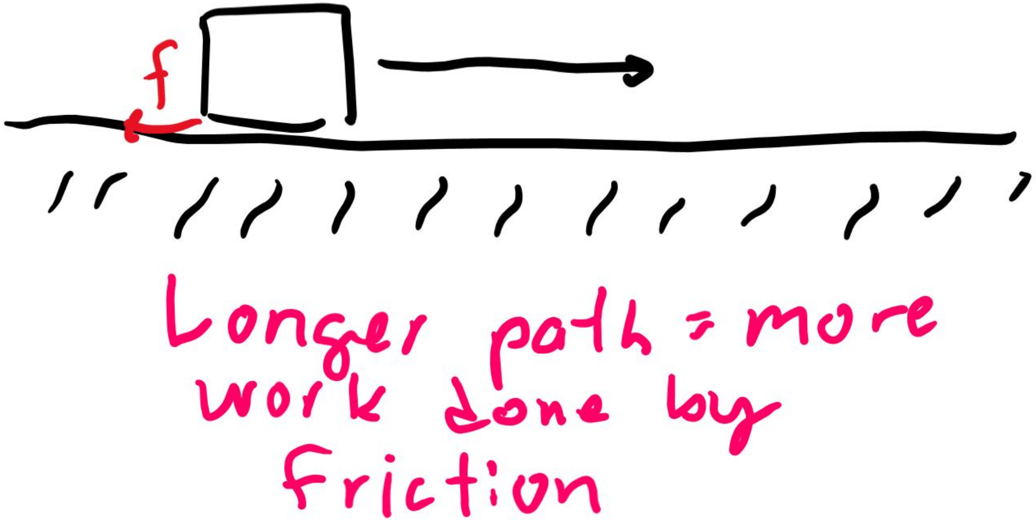


Conservative forces

The restoring force of a spring reduces the spring potential energy and depends only on how far you stretch it out.

Gravitational force reduces gravitational potential energy and depends on how far you are from a mass.

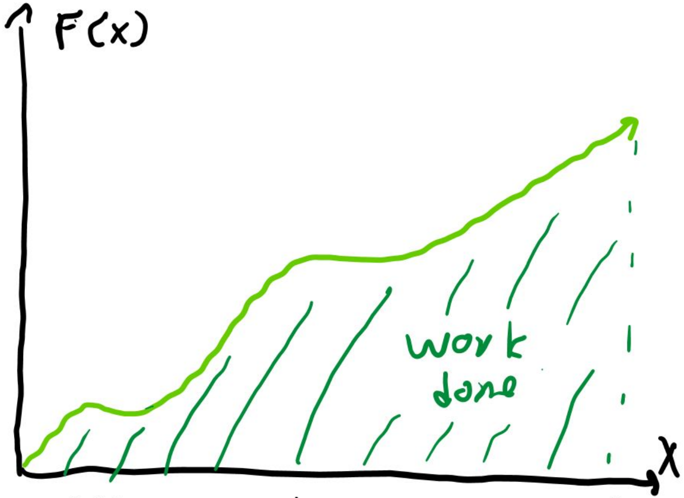
Non-examples



Friction does more work the more you travel: if you travel from a to b to a to b, the work done is different than just going from a to b despite the overall change being from point a to point b for both.

Non-conservative forces like friction generally increase the internal energy of a system (produce heat).

Energy relationship



All work done converts U to other forms of energy (like KE)

As conservative forces decrease the potential energy of a system, $\int F(x)dx = -U(x)$ for some conservative force $F(x)$ and some potential energy $U(x)$.

Spring potential example

$$U_s = \frac{1}{2} k x^2$$

$$-\frac{dU_s}{dx} = -\frac{d\left(\frac{1}{2} k x^2\right)}{dx}$$

$$F_s = -kx \rightarrow \text{Hooke's law}$$

Taking the negative of the derivative of spring potential energy gives the formula for the spring restoring force, Hooke's law.

Gravitational potential

Derivation of gravitational potential

$$F = -\frac{GMm}{r^2}$$

Because "down" is negative

$$-\int F\,dr = \int GMmr^{-2}\,dr$$

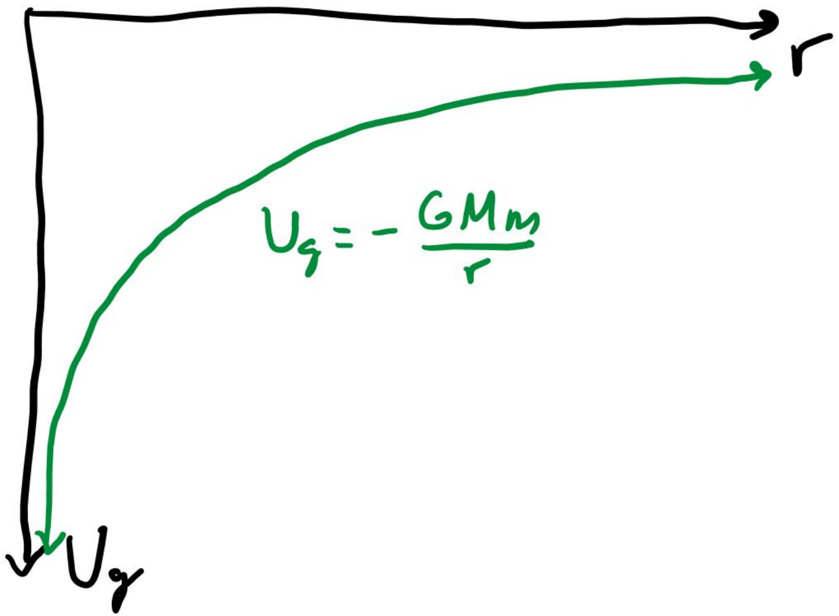
$$U = -GMmr^{-1} + C$$

$$= -\frac{GMm}{r} \qquad C = 0$$

You take newton’s law of gravitation: F = -GmM/r^2 and integrate it by r, since the integral of force with respect to distance is work.

This gives U = -GmM/r, where U represents gravitational potential energy.

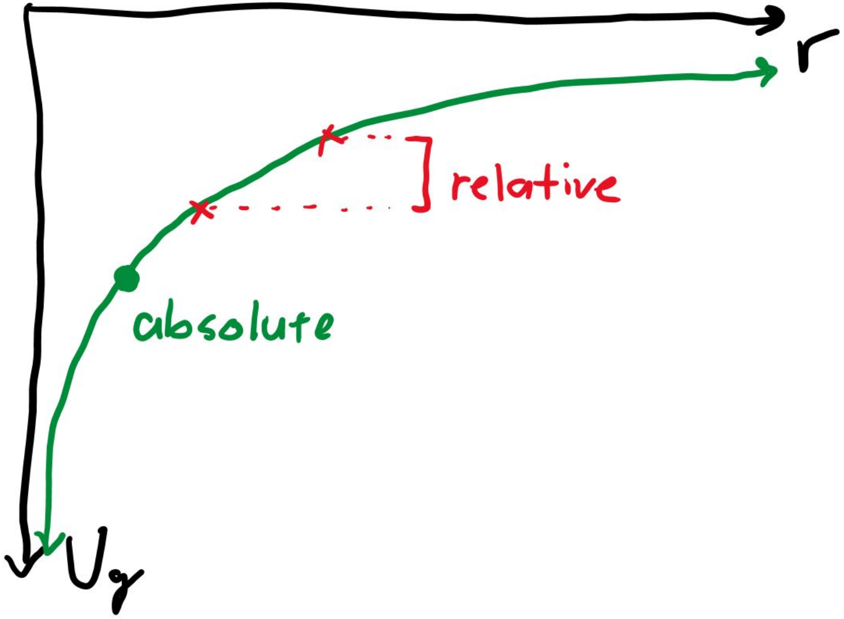
Bounds



In U = -GmM/r, note that G, m, M, and r are all positive, so U is never positive.

An object infinite distance away from a mass has 0 gravitational potential energy while an object 0 distance away from the center of a mass has negative infinity gravitational potential energy.

Relative vs. absolute potential



Absolute gravitational potential energy is given by the formula U=-GmM/r.

However, I can also ask for the potential energy difference relative to, say, the floor, which would be non-absolute.

Just saying “potential energy” could refer to either, so use context to find which one.

Relative potential equation

Let's say we're on a cliff h meters higher from the surface of the earth with radius r.

$U_{surface} = -GmM/r$ and $U_{cliff} = -GmM/(r+h)$, so $\Delta U = GmMh/(r^2+rh)$.

Since h is small compared to r, rh is approximately zero.

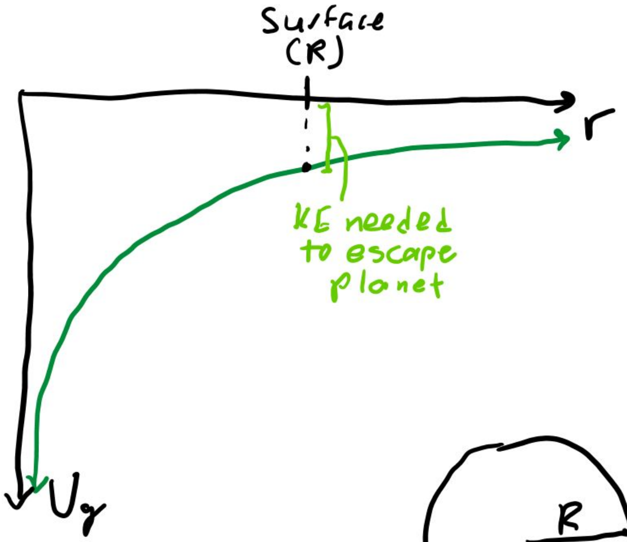
Thus, we can say $\Delta U = GmMh/r^2$.

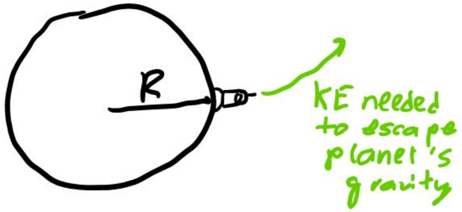
We know $g = GM/r^2$, and $mgh = GmMh/r^2 = \Delta U$.

This is why we can say relative potential energy has the equation mgh.

Escape velocity

Overcoming potential





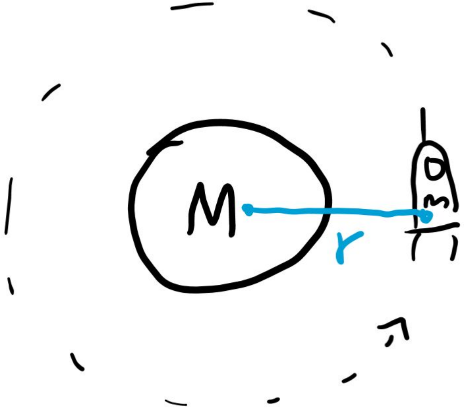
We know U = -GmM/r, so to overcome this potential energy, we need KE = +GmM/r = ½ mv^2.

Thus, v = sqrt(2GM/r) is enough velocity for a mass to overcome gravitational potential energy.

This is called the escape velocity.

Orbital kinetic energy

Velocity in orbit



$$F_g = \frac{GMm}{r^2}$$

$$F_c = \frac{mv^2}{r}$$

$$F_g = F_c$$

$$mv^2 = \frac{GMm}{r}$$

$$KE = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Unlike escape velocity, this KE comes from an object having velocity in orbit.

Since $F_{centripetal} = GmM/r^2 = mv^2/r$, $v = \text{sqrt}(GM/r)$.

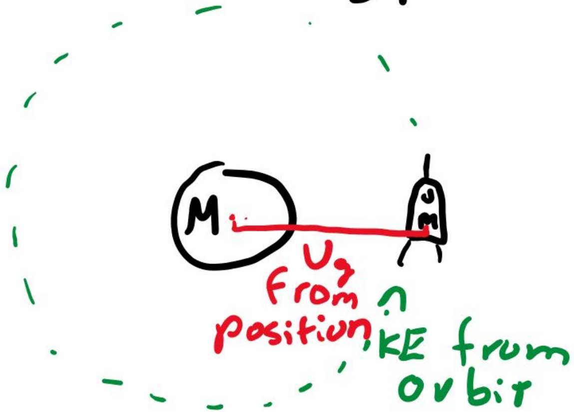
$KE = ½\, mv^2 = GmM/(2r)$, which is orbital kinetic energy.

Total orbital energy

Gravitational mechanical energy

$$U_g = -\frac{GMm}{r} \qquad KE = \frac{GMm}{2r}$$

$$TME = -\frac{GMm}{2r}$$



We know TME = U + KE.

As U = -GmM/r and KE = +GmM/(2r), the sum of the two is -GmM/(2r) = TME in orbit.

Notice that TME = ½ U = -KE.

Rotational kinetic energy

Formula

$$KE = \frac{1}{2} m v^2$$

$$RKE = \frac{1}{2} I \omega^2$$

depends on the object

$$= \frac{1}{2} c\, m r^2 \omega^2$$

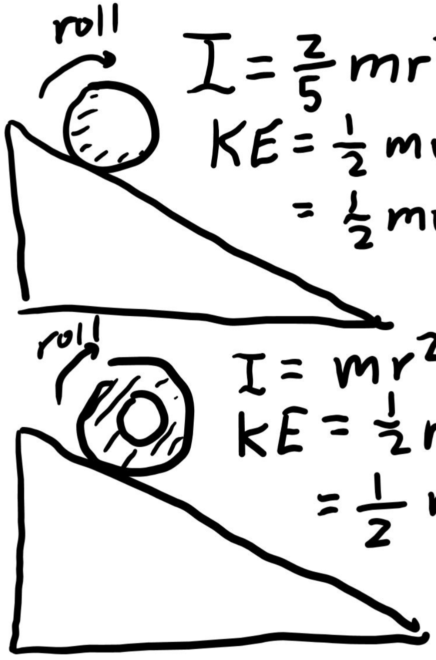
$$= \frac{c}{2} m v^2$$

KE = ½ mv^2 for linear motion, so
KE = ½ $I\omega^2$ for rotational motion.

Since I is usually cmr^2 for some constant c and $\omega = v/r$, rotational KE = ½ cmv^2.

Total KE is the sum of the two, which is usually ½ mv^2 + ½ cmv^2 = kmv^2 for some constant k.

Total energy of rolling objects



$$I = \frac{2}{5} m r^2$$

$$KE = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{5} m v^2$$

$$I = m r^2$$

$$KE = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m v^2 + \underbrace{\frac{1}{2} m v^2}_{\text{needs more energy to roll}}$$

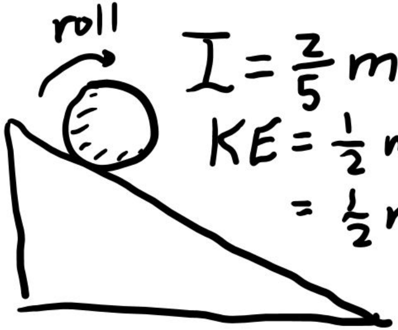
Objects rolling down a ramp will convert U to KE.

Objects with larger I have larger rotational KE, meaning they have less linear KE.

Thus, objects with larger I take longer to get down a ramp (move slower linearly).

Rolling

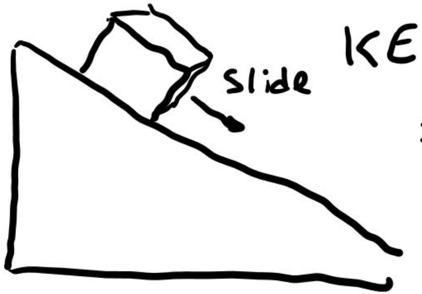
Friction as torque



$$I = \frac{2}{5} m r^2$$

$$KE = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{5} m v^2 = \frac{7}{10} m v^2$$



$$KE = \frac{1}{2} m v^2 + 0\, RKE$$

$$= \frac{1}{2} m v^2$$

takes less energy to slide

Friction acts perpendicular to the axis of rotation of an object, causing them to roll.

Rolling without slipping means that v/r = ω, but an object that is slipping will have less ω than v/r.

Note objects that slip will have less rotational KE, so will have more linear KE and go down ramps faster.