Bisection Method:

Binary chopping interval having
Bolzano's method
Incremental search method

If $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$, $f(x_u)$ have opposite signs, that is $\boxed{f(x_l)\, f(x_u) < 0}$, then there is at least one real root between x_l and x_u.

Algorithm:

Step-1: Choose lower x_l and upper x_u guesses for the root such that, the function changes sign over the interval. This can be checked by ensuring that, $f(x_l) f(x_u) < 0$

Step-2: An estimate of the root x_r is determined by, $x_r = \dfrac{x_l + x_u}{2}$

Step-3: Make the following evaluations to determine in which sub-interval the root is:

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(a) If $\boxed{f(x_l)\, f(x_r) < 0,}$ the root lies in the lower sub-interval. Therefore, set $\boxed{x_u = x_r}$ and return to step 2.

(b) If $\boxed{f(x_l)\, f(x_r) > 0}$, the root lies in the upper subinterval. Therefore set $\boxed{x_l = x_r}$ and return to step 2.

(c) If $\boxed{f(x_l)\, f(x_r) = 0}$, the root equals to x_r, terminate the computation.

Example : 5.3 :

Use Bisection method to solve the following problem up to approximate percent relative error $\epsilon_a \leq 0.422$.

Solution:

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$

Let, $a = 12$

$b = 16$

$\therefore f(a) = 6.0669$

$f(b) = -2.2687$

$\therefore f(a) * f(b) < 0$

so, Root interval is present on limit $\boxed{12, 16}$

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positive হলে lower change
negative হলে upper change হবে।

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a	b	f(a)	f(b)	$c = \frac{a+b}{2}$	f(c)	$\varepsilon_a = \frac{c^{new} - c^{old}}{c^{new}} \times 100$
12	16	6.0669	-2.2687	14	1.5687	100%
14	16	1.5687	-2.2687	15	-0.4248	6.67%
14	15	1.5687	-0.4248	14.5	0.5523	3.4487%
14.5	15	0.5523	-0.4248	14.75	0.0589	1.69%
14.75	15	0.0589	-0.4248	14.875	-0.1841	0.84%
14.75	14.875	0.0589	-0.1841	14.8125	-0.0628	0.422%
14.75	14.8125	0.0589	-0.0628	14.78125	-0.00203	0.21%

Thus after 6 iterations ε_a falls equal to 0.422%.

False position Method:

Linear interpolations method
Regular Falsi

False position method joins $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line and x axis represents an improved estimates of the root.

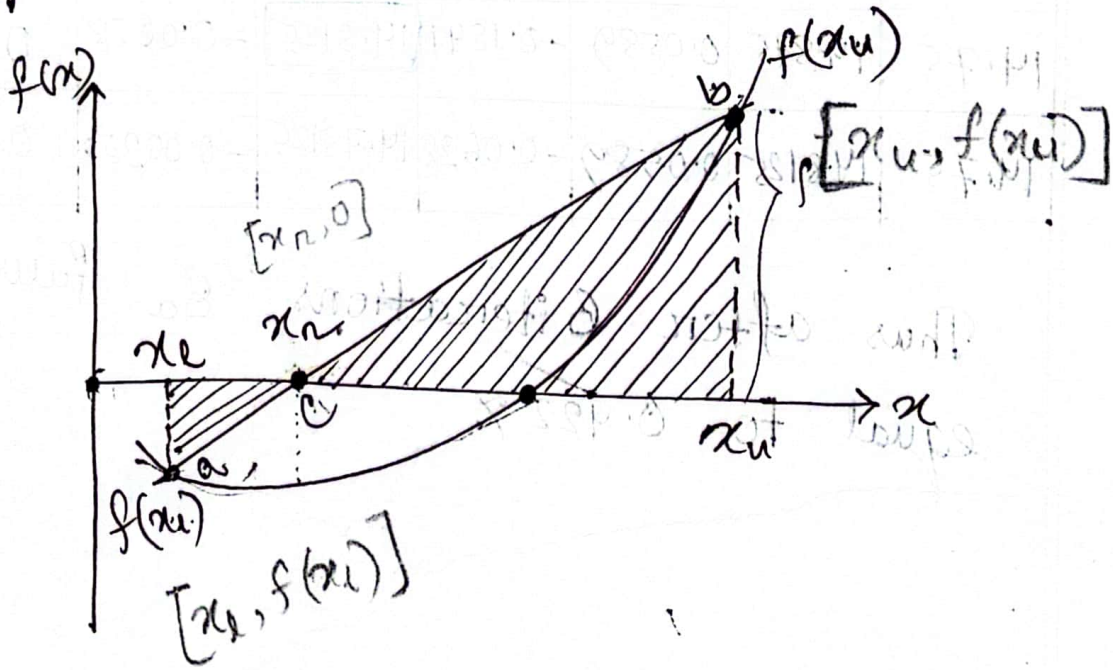

Fig: Graphical depiction of the method of false position

slop of ac = slop of bc

$$\frac{f(x_l)-0}{x_l-x_r} = \frac{f(x_u)-0}{x_u-x_r}$$

$$\Rightarrow \frac{f(x_l)}{x_r-x_l} = \frac{f(x_u)}{x_r-x_u}$$

$$\Rightarrow f(x_l)(x_r-x_u) = f(x_u)(x_r-x_l)$$

$$\Rightarrow x_r f(x_l) - x_u f(x_l) = x_r f(x_u) - x_l f(x_u)$$

$$\Rightarrow x_r\left[f(x_l)-f(x_u)\right] = x_u f(x_l) - x_l f(x_u)$$

$$\Rightarrow x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = \frac{x_u f(x_l)}{f(x_l)-f(x_u)} - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = x_u + \frac{x_u f(x_l)}{f(x_l)-f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = x_u + \frac{\cancel{x_u f(x_l)} - \cancel{x_u f(x_l)} + x_u f(x_u)}{f(x_l)-f(x_u)} - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\boxed{\therefore x_r = x_u - \frac{f(x_u)(x_l-x_u)}{f(x_l)-f(x_u)}}$$

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Pitfall of False position:

A case where bisection is preferable than ~~to~~ false position method.

Although False-position method would seem to be the bracketing method of preference, there are certain cases where it performs poorly.

Example 5.6:

using Bisection, the

$f(x) = x^{10} - 1$

between $x = 0$ and 1.3

Solution:

using Bisection, the result can be summarized as,

Iteration	x_l	x_u	x_r	ε_a (%)
1	0	1.3	0.65	100%
2	0.65	1.3	0.975	33.3%
3	0.975	1.3	1.1375	14.28%
4	0.975	1.1375	1.05625	7.69%
5	0.975	1.05625	1.015625	4%

using False position:

Iteration	x_l	x_u	x_r	ε_a (%)
1	0	1.3	0.0943	
2	0.0943	1.3	0.18176	48.1%
3	0.18176	1.3	0.26287	30.9%
4	0.26287	1.3	0.33811	22.3%
5	0.33811	1.3	0.40788	17.1%

After 5 iterations, appropriate error from bisection is 4.0% and from false position is 17.1. This proves that, there are cases where Bisection is preferable.

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Simple Fixed point Iteration

Direct substitution method

Convergence Divergence with a suitable example.

Example:

$$\boxed{2x^2 - 4x + 1 = 0}$$

$$\Rightarrow 4x = 2x^2 + 1$$

$$\Rightarrow x = \frac{x^2}{2} + \frac{1}{4}$$

$$\therefore x = \frac{1}{2}(x^2) + \frac{1}{4}$$

n	x_n
1	1
2	0.75
3	0.53125
4	0.391113
5	0.326484
6	0.303295
7	0.295993

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8	0.293806
9	0.293161
10	0.292972
11	0.292916
12	0.292899
13	0.292894
14	0.292893
15	0.292893

Convergence

1st step − 2nd step > 2nd step − 3rd step

n	x_n
1	2
2	2.25
3	2.78125
4	4.117676
5	8.727627
6	38.335736
⋮	⋮

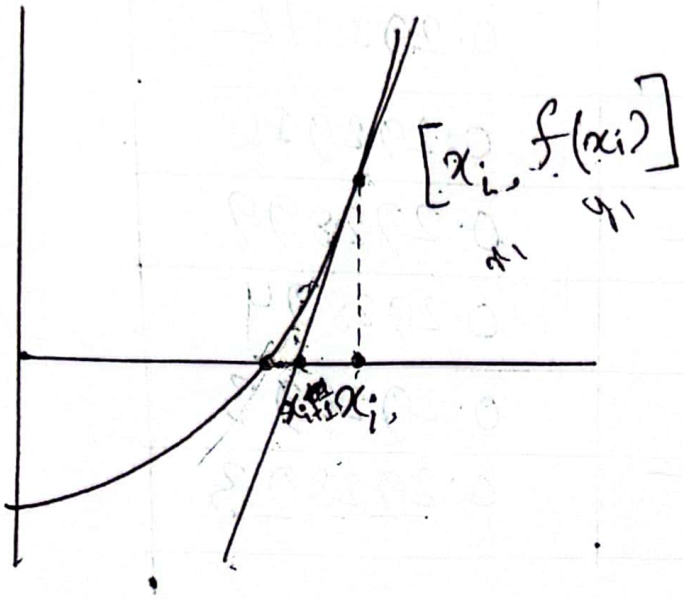
Divergence:

3rd step − 2nd step < 2nd step − 1st step

Newton Repson Method

NR Method

→ দ্রুত কাজ করে (Derivative লাগে)



$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - f(x_i) = f'(x_i) \cdot (x - x_i)$$

For $[(x_{i+1}), 0]$ point
x_1 y_1

$$0 - f(x_i) = f'(x_i) \cdot (x_{i+1} - x_i)$$

$$\therefore f'(x_i) = \frac{f(x_i)}{(x_i - x_{i+1})}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

This is called the Newton-Repson formula

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If the initial guess at root is x_i, a tangent can be extended from the point $[x_i, f(x_i)]$. The point where this tangent crosses the x axis usually represents an improved estimate of the root.

Example 6.3:

Use the ~~Netwo~~ Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$. True value $= 0.56714329$ until $\varepsilon_t(\%) < 10^{-8}$

Solution:

The first derivative of function can be evaluated as

$$f'(x) = -e^{-x} - 1$$

Which can be substitute along with the original function into the equation to give

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

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Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute.

i	x_i	ε_t (%)
0	0	100
1	0.50000	11.8
2	0.566311003	0.147
3	0.567193165	0.0000220
4	0.567143290	$< 10^{-8}$

$$\varepsilon_t = \frac{x_{real} - x_{new}}{x_{real}} \times 100\%$$

Thus this approach converges rapidly on the true root. The true percent relative error at each iteration decreases much faster than it does in simple fixed point iteration.

Pitfall of the Newtwon Raphson Method:

performs poorly for multiple roots, sometimes for simple roots.

Example 6.5:

Determine the positive root of ~~of~~ $f(x) = x^{10} - 1$ using the Newton Raphson method and an initial guess of $x = 0.5$

Solution:

The newton Raphson formula for this case is

$$x_{i+1} = x_i - \frac{x_i^{10} - 1}{10 x_i^9}$$

which can be used to compute

iteration	x
0	0.5
1	51.65
2	46.485
3	41.8365
4	37.6585
5	33.887565
⋮	⋮
∞	1.000000

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Thus after the first poor prediction, the technique is converging on the true of 1. but at a very slow rate.

Four cases where the Newton Raphson method exhibit convergence:

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[derivative লাগে না]

Secant Method:

N-R method is the evaluation of the derivative. Although this is not inconvenient for polynomials and many other functions,

For these cases, the derivative can be approximated by a backward finite divided difference.

$$f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

From N-R,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\boxed{= x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}}$$

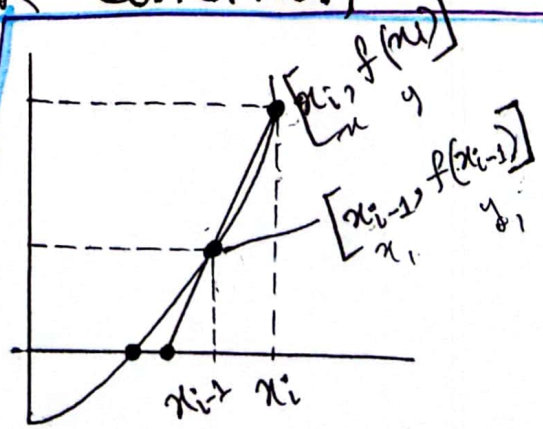
এটা bracketing method হিসেবে ধরা হয় না।

কারণ, "$f(x_1) f(x_2) < 0$" এই condition satisfy করা লাগে না।

$y - y_1 = \frac{dy}{dx}(x - x_1)$

$$f(x_i) - f(x_{i-1}) = f'(x_i)(x_i - x_{i-1})$$

$$\therefore f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$



$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

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Example - 6.6

Use the secant method to estimate the root of $f(x) = e^{-x} - x$. Start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$.

True root is 0.56714329

Solution:

$x_{-1} = 0 \quad f(x_{-1}) = 1.0000$

$x_0 = 1 \quad f(x_0) = -0.63212$

$i=0$ 1st iteration:

$$x_1 = 1 - \frac{-0.63212\,(0-1)}{1-(-0.63212)}$$

$= 0.61270$

$\varepsilon_t = 8.0\%$

2nd iteration:

$x_0 = 1 \quad f(x_0) = -0.63212$

$x_1 = 0.61270 \quad f(x_1) = -0.07081$

} Both are same side on the root

$$x_2 = 0.61270 - \frac{-0.07081\,(1-0.61270)}{-0.63212-(-0.07081)}$$

$= 0.56384$

$\varepsilon_t = 0.58\%$

Third iteration:

$x_1 = 0.61270$ $\quad f(x_1) = -0.07081$

$x_2 = 0.56384$ $\quad f(x_2) = 0.00518$

$$x_3 = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - (-0.00518)}$$

$$= 0.56717$$

$$\varepsilon_t = 0.0048\%$$

(Ans)

Difference between Secant and False position Method:

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Cramer's Rule:

Best suited to small number of equations.

Example-9.3:

Use Cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Solution:

The determinant D can be written as

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

$$= 0.3(0.5 - 1.9\times0.3) - 0.52(0.5\times0.5 - 1.9\times0.1)$$
$$+ 1(0.5\times0.3 - 1\times0.1)$$

$$= -0.0022$$

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$$D_{x_1} = \begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}$$

$$= -0.01\left(0.5 - 1.9 \times 0.3\right) - 0.52\left(0.67 \times 0.5 + 1.9 \times 0.44\right)$$

$$+ 1\left(0.67 \times 0.3 + 0.44\right)$$

$$= 0.03278$$

$$D_{x_2} = \begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}$$

$$= 0.3\left(0.67 \times 0.5 + 1.9 \times 0.44\right) + 0.01\left(0.5 \times 0.5 - 1.9 \times 0.1\right)$$

$$+ 1\left(-0.5 \times 0.44 - 0.67 \times 0.1\right)$$

$$= 0.0649$$

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$$D_{x_3} = \begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}$$

$$= 0.3(-0.44 - 0.67 \times 0.3) - 0.52(-0.5 \times 0.44 - 0.67 \times 0.1)$$

$$- 0.01(0.5 \times 0.3 - 0.1)$$

$$= -0.04356$$

$$x_1 = \frac{D_{x_1}}{D} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{D_{x_2}}{D} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{D_{x_3}}{D} = \frac{-0.04356}{-0.0022} = 19.8$$

Ans

Gauss Elimination :

Two steps: 1) Forward elimination
2) Back Substitution

Example 9.5:

Use gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{—— (1)}$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{—— (2)}$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{—— (3)}$$

Solution :

The first part of the procedure is forward elimination.

Now to remove x_1 from (2), (3)

$$(2) - (1) \times \frac{0.1}{3} \text{ and } (3) - (1) \times \frac{0.3}{3} \Rightarrow$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{—— (4)}$$
$$7.00333x_2 - 0.293333x_3 = -19.5617 \quad \text{—— (5)}$$
$$-0.190000x_2 + 10.0200x_3 = 70.6150 \quad \text{—— (6)}$$

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To complete the forward elimination x_2 must be removed from (6).

To remove x_2, $(6) - (5) \times \frac{(-0.19000)}{7.00333}$.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{--- (7)}$$

$$7.00333x_2 - 0.29333x_3 = -19.5617 \quad \text{--- (8)}$$

$$10.0120x_3 = 70.0843 \quad \text{--- (9)}$$

We can now solve these equations by back substitution. From equation (9) we get,

$$x_3 = \frac{70.0843}{10.0120}$$

$$= 7.0000 \quad \text{--- (10)}$$

This result can be back substituted into equation (8)

$$7.00333\,x_2 - (0.29333 \times 7.0000) = -19.5617$$

$$\Rightarrow x_2 = \frac{-19.5617 + 0.29333 \times 7.0000}{7.00333}$$

$$= -2.5000 \quad \text{——— (11)}$$

Finally equation (10) & (11) can be substituted in equation (4)

$$3x_1 - 0.1 \times (-2.5) - 0.2\,(7.000) = 7.85$$

$$\Rightarrow x_1 = \frac{7.85 - 0.25 + 1.4}{3}$$

$$= 3.000.$$

The results are identical to the exact solution of

$x_1 = 3$, $x_2 = -2.5$ and $x_3 = 7$

(Ans.)

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Gauss - Jordan :

The Gauss - Jordan method is a variation of gauss elimination.

Example 9.12 :

use gauss jordan technique to solve.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution :

First, we have to express the coefficients and the right hand side as an argumented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

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Now,

$$\begin{bmatrix} 1 & -0.03333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix} \quad r_1' = r_1/3$$

Then,

$$\begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0 & 7.00833 & -0.029333 & -19.5617 \\ 0 & -0.19000 & 10.020001 & 70.615 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \times 0. \\ r_3' = r_3 - r_1 \times 0. \end{matrix}$$

$$= \begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.19001 & 10.0200 & 70.6150 \end{bmatrix} \quad r_2' = r_2/7.00$$

$$= \begin{bmatrix} 1 & 0 & -0.06806 & 2.52357 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.012 & 70.0843 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 + 0.0333 r_2 \\ r_3' = r_3 + 0.19001 r_2 \end{matrix}$$

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$$= \begin{bmatrix} 1 & 0 & -0.06806 & 2.52357 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.000 \end{bmatrix} \quad r_3' = r_3/10.012$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3.000 \\ 0 & 1 & 0 & -2.5 \\ 0 & 0 & 1 & 7.000 \end{bmatrix} \quad \begin{array}{l} r_1' = r_1 + 0.06806\, r_3' \\ r_2' = r_2 + 0.0418848\, r_3' \end{array}$$

$\therefore x_1 = 3$

$x_2 = -2.5$

$x_3 = 7$

Ans.

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Factorization Method:

Triangularization Method

Cholesky's process

LU Decomposition

Matrix A in the equation $AX=B$ can always be resolved into factors L.U. where L and U are the lower and upper triangular matrix respectively.

Hence, the equation $AX=B$ can be written as

$$LUX=B$$

Let, $UX=Y$ so that, $LY=B$

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Example:

Apply the cholesky's process to locate the root of the following system -

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

$$B = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} ; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Comparing the values,

$u_{11} = 1$
$u_{12} = 1$
$u_{13} = -1$

$l_{21}u_{11} = 2$
$\therefore l_{21} = 2$
$l_{31}u_{11} = 3$
$\therefore l_{31} = 3$

$l_{31}u_{12} + l_{32}u_{22} = 2$
or, $3 + l_{32} = 2$
$\therefore l_{32} = -1$
$l_{21}u_{12} + u_{22} = 3$
$\therefore u_{22} = 3 - 2$

$u_{23} = 2 + 5$
$= 7$
$u_{33} = -3 + 3 + 7$
$= 7$

$$\begin{cases} AX = B \\ LUX = B \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} X = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$\begin{cases} \text{Let, } UX = Y \\ LY = B \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$y_1 = 2$$

$$2y_1 + y_2 = -3$$

$$\therefore y_2 = -3 - 2 \times 2$$
$$= -7$$

$$3y_1 - y_2 + y_3 = 6$$

$$\therefore y_3 = 6 - 3 \times 2 - 7$$
$$= -7$$

Now,

$$UX = Y$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}$$

$$7x_3 = -7$$

$$\therefore x_3 = -1$$

$$x_2 + 7x_3 = -7$$

or, $x_2 = -7 - 7(-1)$

$$\therefore x_2 = 0$$

$$x_1 + x_2 - x_3 = 2$$

~~or, $x_1 = 2 - (-1) + 0$~~

or, $x_1 = 2 - 0 + (-1)$

$$\therefore x_1 = 1$$

$\therefore$

$$x_1 = 1$$
$$x_2 = 0$$
$$x_3 = -1$$

(Ans.)

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Lagrange Interpolation:

It is a way to find a polynomial, called lagrange polynomial, that takes on certain values at arbitary points.

Advantage of Lagrange Interpolation:

- This formula is used to find the value of the function even when the arguments are not equally spaced.
- This formula is used to find the value of independent variable x corresponding to a given value of a function.

** If the values of x are at equidistant or not at equidistant, we use Lagrange's interpolation formula.

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Let, $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ Corresponding to $x = x_0, x_1, x_2, \ldots x_n$ that is

$$y_i = f(x_i); \quad i = 0, 1, 2, \ldots n$$

Now, there are $(n+1)$ paired values (x_i, y_i) $i = 0, 1, 2 \ldots n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x:

Then the Lagrange formula is:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots +$$

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n.$$

Example:

Using Lagrange's interpolation formula find $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

Solution:

Here the intervals are not equal. By Lagrange's interpolation formula, we have

$$x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$$

$$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

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$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

putting $x=10$ we get,

$$y(10) = f(10) = \frac{(10-6)(10-9)(10-11)}{(-1)(-4)(-6)} \times 12 + \frac{(10-5)(10-9)(10-11)}{1 \times (-3) \times (-5)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{4 \times 3 \times (-2)} \times 14 + \frac{(10-5)(10-6)(10-9)}{6 \times 5 \times 2} \times 16$$

$$= \frac{12}{6} - \frac{13}{3} + \frac{5 \times 14}{3 \times 2} + \frac{4 \times 16}{12}$$

$$= 14.6663$$

Ans

Trapezoidal Rule:

The trapezoidal rule is the first of the Newton-cotes closed integration formulas. It corresponds to the case where the polynomial in the equation is first order;

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

A straight line can be represented as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

The area under this straight line is an estimate of the integral of $f(x)$ between the limits a and b:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right] dx$$

$$= \left[f(a)x + \frac{f(b) - f(a)}{b-a} \cdot \frac{(x-a)^2}{2} \right]_a^b$$

$$= f(a)b - f(a)a + \frac{f(b) - f(a)}{b-a} \cdot \frac{(b-a)^2}{2}$$

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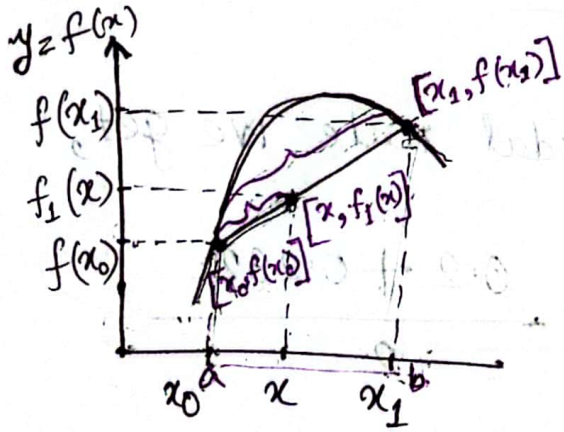
$$= f(a)(b-a) + \frac{[f(b)-f(a)]\cdot(b-a)}{2}$$

$$= \frac{2f(a)(b-a) + f(b)(b-a) - f(a)(b-a)}{2}$$

$$= \frac{f(a)(b-a) + f(b)(b-a)}{2}$$

$$\therefore I = \underbrace{(b-a)}_{\text{width}} \times \underbrace{\cdot \frac{f(a)+f(b)}{2}}_{\text{average height}}$$

This is the formula for the trapezoidal rule.



$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$f(x_0) = a \qquad x_1 = b$$

$x_0 = a$

$x_1 = b$

Example 21.1:

use Trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$;

exact value $= 1.640533$

Solution:

The function values

$f(0) = 0.2$

$f(0.8) = 0.232$

using trapezoidal rule we get,

$$I \cong (0.8-0)\frac{0.2+0.232}{2}$$

$$= 0.1728$$

Which represents an error of

$$\varepsilon_t = \frac{1.640533 - 0.1728}{1.640533} \times 100$$

$$= 89.5\%$$

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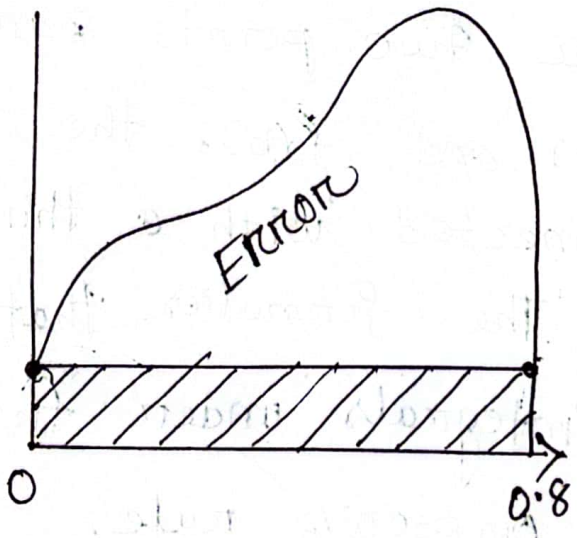
Which corresponds to percent relative error, $\varepsilon_t = 89.5\%$

The reason for this large error is evident from the figure as we can see that the area under the straight line neglects a significant portion of the integral lying above the line.



How to remove the error of Trapezoidal rule?

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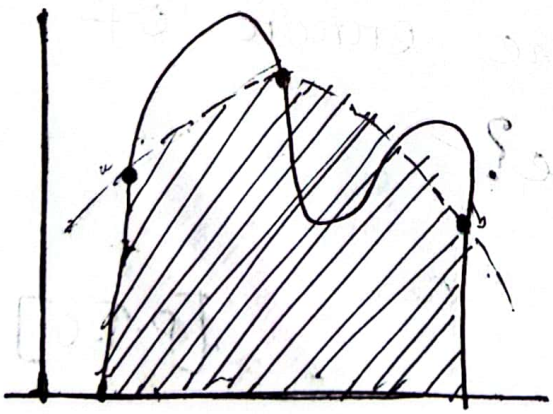
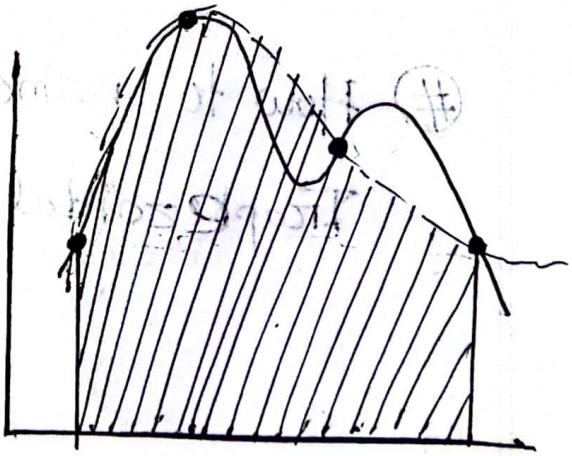
Simpson Rules:

Another way to obtain a more accurate estimate of an integral is to use higher order polynomials to connect the points. For example, if there is an extra point midway between $f(a)$ and $f(b)$, the three points can be connected with a parabola.

If there are two points equally spaced between $f(a)$ and $f(b)$, the four points can be connected with a third order polynomial. The formulas that result from taking the integrals under these polynomials are called Simpson's rule.



[1/3 Rule]



[3/8 Rule]

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Simpson's 1/3 Rule:

Simpson's 1/3 rule results when a second order interpolating polynomial is substitute into

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are ~~designated~~ designated as x_0 and x_2 and $f_2(x)$ is represented by a 2nd order Lagrange polynomial, the integral becomes,

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integration and algebric manipulation, the following formula result:

$$\boxed{I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}$$

Where, $h = \frac{b-a}{2}$ for this case

$$\boxed{I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{average height}}}$$

This is called simpson's 1/3 Rule as h is divided by 3.

Example 21.4:

use simpson's 1/3 rule to integrate

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

from $a=0$ to $b=0.8$;

exact integral is 1.640533

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Solution :

The function values

$f(0) = 0.2$

$f(0.4) = 2.456$

$f(0.8) = 0.232$

Using Simpson's 1/3 Rule:

$$I \cong 0.8 \frac{0.2 + 4(2.456) + 0.232}{6}$$

$$= 1.367467$$

Which represents an exact error of

$$\varepsilon_t = \frac{1.640533 - 1.367467}{1.640533} \times 100\%$$

$$= 16.6\%$$

Which is 5 times more accurate than a single application of the trapezoidal rule.

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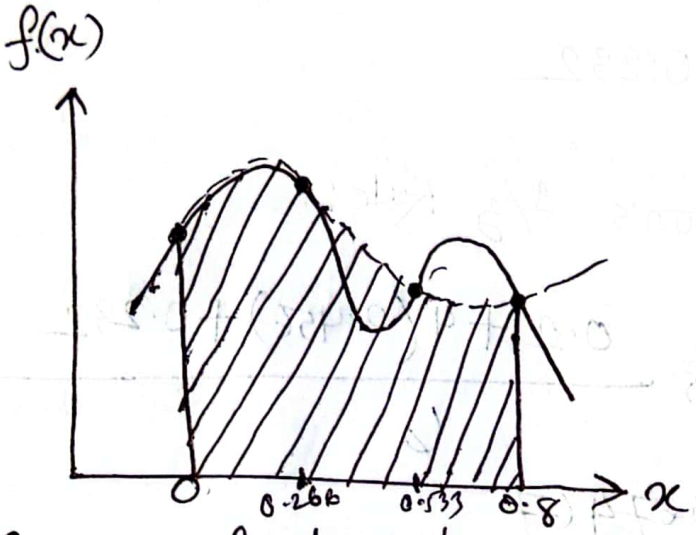
Simpson's 3/8 Rule:

If there are two points equally spaced between f(a) and f(b) the four points can be connected with a third order polynomial.



Graphical depiction of Simpson's 3/8 Rule

A third order Lagrange polynomial can be fit to four points and integrated.

$$I \cong \int_a^b f_3(x)\,dx$$

$$I \cong \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

Where, $h = \dfrac{b-a}{3}$

This equation is called Simpson's 3/8 rule because h is manipulated by 3/8. It is the third Newton cotes closed integration formula.

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The 3/8 rule can also be expressed as,

$$I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}}_{\text{Average height}}$$

Example-21.6:

a) use Simpson's 3/8 rule to integrate

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

from $a=0$ to $b=0.8$

Solution: A single application of Simpson's 3/8 rule requires four equally spaced points:

$= \frac{a+b}{3}$

$= \frac{0+0.8}{3}$

$= 0.2667$

$f(0) = 0.2$ $\quad f(0.2667) = 1.432724$

$f(0.5333) = 3.487177$ $\quad f(0.8) = 0.232$

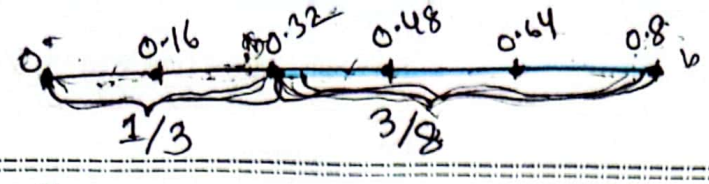
using Simpson's 3/8 rule,

$$I \cong 0.8 \cdot \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8}$$

$$= 1.519170$$

$$\varepsilon_t = \frac{1.640533 - 1.519170}{1.640533}$$

$$= 7.4\%$$



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b) use it in conjunction with Simpson's 3/8 Rule to integrate the same function for five segments.

Solution: The data needed for a five segment application

0.8/5 = 0.16

$f(0) = 0.2$ $\quad f(0.16) = 1.296919$

$f(0.32) = 1.743393$ $\quad f(0.48) = 3.186015$

$f(0.64) = 3.181929$ $\quad f(0.80) = 0.232$

The Integral for the first to segments is obtained using simpson's 1/3 rule:

$$I \cong 0.32 \frac{0.2 + 4(1.296919) + 1.743393}{6}$$

$$= 0.3803237$$

For the last three segments, the 3/8 rule can be used to obtain:

$$I \cong (0.8 - 0.32) \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8}$$

$$= 1.264754$$

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The total integral is computed by summing the two results.

$$I = 0.3803237 + 1.264754$$

$$= 1.645077$$

$$\varepsilon_t = \frac{1.640533 - 1.645077}{1.640533} \times 100\%$$

$$= -0.28\%$$

(Ans.)

$$\frac{12-3}{2}, \quad \frac{12+3}{3}, \quad \frac{12-3}{3}$$

[P.T.O]

Milne's Method:

Milne's predictor corrector method

x	y
0	0
0.2	0.02
0.4	0.0795
0.6	0.1762
↓	
0.8	0.3049
	0.3046

Example:

Given,

$$\frac{dy}{dx} = \frac{1}{2}(x+y)$$

$y(0) = 2$

$y(0.5) = 2.636$

$y(1.0) = 3.595$

$y(1.5) = 4.968$

Now find $y(2)$ by milne's method.

Solution:

Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \quad \text{—— (i)}$$

putting $n=3$ we get,

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \quad \text{—— (ii)}$$

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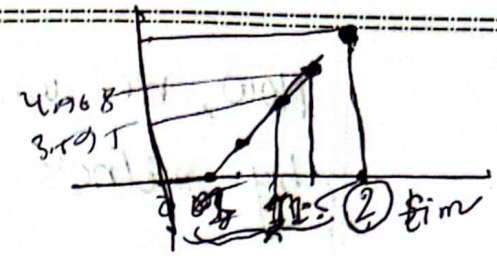
Given,



$x_0 = 0$
$y_0 = 2$
$= 0.5$

$x_0 = 0 \qquad y_0 = 2$

$x_1 = 0.5 \qquad y_1 = 2.636$

$x_2 = 1.0 \qquad y_2 = 3.595$

$x_3 = 1.5 \qquad y_3 = 4.968$

given differential equation is $y' = \frac{1}{2}(x+y)$

From above equation we will calculate

y_1', y_2', y_3'

$$y_1' = \frac{1}{2}(x_1+y_1) = \frac{1}{2}(0.5+2.636) = 1.568$$

$$y_2' = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1+3.595) = 2.2975$$

$$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+4.968) = 3.324$$

(11) $\Rightarrow y_{4p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$

$$= 2 + \frac{4\times 0.5}{3}\left[2\times 1.568 - 2.2975 + 2\times 3.324\right]$$

$$= 6.871$$

So, our predictore value is 6.871

$$y_4' = \frac{1}{2}(x_4+y_4) = \frac{1}{2}(2+6.871) = 4.4355$$

y(2) বের করতে বলা হয়েছে।

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Now, we will correct it to get actual value by milne's Corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right] \quad \text{--- (iii)}$$

By putting $n=3$, we get

$$y_{4c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$= 3.505 + \frac{0.5}{3}\left[2.2975 + 4\times 3.324 + 4.4355\right]$$

$$= 6.8731$$

[P.T.O]

***

If four values of y are not given you have to use Taylor's method to find remaining values of y.

Milne's Method + Taylor Series method

Example:

Find $y(0.4)$ by milne's method for $y' = y - x^3$ $y(0) = 1$. Obtain starting values by Taylor Series method. Step size 0.1.

Solution:

We need total four values of y for predictor corrector method to work. Only one value of y is given, remaining values are to be calculated by Taylor series method. We need to calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ by Taylor method.

Given, $y' = y - x^3$ —— (1)

and $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Taylor Algorithm is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad (2)$$

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Differentiating (i) w.r.t x, we get

$$y' = y - x^3$$

$$y_0' = y_0 - x_0^3 = 1 - 0^3 = 1$$

$$y_0'' = y_0' - 3x_0^2 = 1 - 3\times 0^2 = 1$$

$$y_0''' = y_0'' - 6x_0 = 1 - 6\times 0 = 1$$

putting these values in (2) we get,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= 1 + \left(\frac{0.1}{1}\times 1\right) + \left(\frac{0.1^2}{2!}\right)\times 1 + \left(\frac{0.1^3}{3!}\right)\times 1$$

$$= 1.0005$$

$$x_1 = x_0 + h = 0.1$$

$$y_1 = 1.0005$$

Now,

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad (3)$$

$$y_1' = y_1 - x_1^3 = 1.0005 - (0.1)^3 = 1.0000$$

$$y_1'' = y_1' - 3x_1^2 = 0.97$$

$$y_1''' = y_1'' - 6x_1 = 0.37$$

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③ $\Rightarrow y_2 = 1.005 + \frac{0.1}{1} \times 1.0 + \frac{(0.1)^2}{2} \times 0.97 + \frac{(0.1)^3}{6} \times 0.37$

$= 1.105$

$x_2 = x_1 + h$

$= 0.1 + 0.1$

$= 0.2$

$y_2 = 1.105$

Again,

$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2'''$ —— ④

$y_2' = y_2 - x_2^3 = 1.105 - (0.2)^3 = 1.097$

$y_2'' = y_2' - 3x_2^2 = 1.097 - 3(0.2)^2 = 0.977$

$y_2''' = y_2'' - 6x_2 = 0.977 - 6(0.2) = -0.223$

④ $\Rightarrow y_3 = 1.105 + \frac{0.1}{1} \times 1.097 + \frac{(0.1)^2}{2} \times 0.977 + \frac{(0.1)^3}{6} \times (-0.223)$

$= 1.497$

$x_3 = x_2 + h$

$= 0.2 + 0.1$

$= 0.3$

$y_3 = 1.497$

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$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1.0005$

$x_2 = 0.2 \qquad y_2 = 1.105$

$x_3 = 0.3 \qquad y_3 = 1.497$

Milne's predictor formula is

$$y_{4p} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right) \quad \text{——— (5)}$$

$$y_1' = y_1 - x_1^3 = 1.0005 - (0.1)^3 = 0.9995$$

$$y_2' = y_2 - x_2^3 = 1.105 - (0.2)^3 = 1.097$$

$$y_3' = y_3 - x_3^3 = 1.2197 - (0.3)^3 = 1.1927$$

$$(5) \Rightarrow y_{4p} = 1 + \frac{4 \times 0.1}{3}\left(2 \times 0.995 - 1.097 + 2 \times 1.1927\right)$$

$$= 1.43712$$

So the predictor value is 1.437

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Now we will correct it using the corrector formula to get the

Milne's corrector formula is

$$y_{4c} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right) \quad \text{--- (6)}$$

$$y_4' = y_4 - x_4^3$$

$$= 1.4372 - (0.4)^3$$

$$= 1.373$$

$$(6) \Rightarrow y_{4c} = 1.105 + \frac{0.1}{3}\left(1.097 + 4 \times 1.1927 + 1.373\right)$$

$$= 1.34636$$

Linear Regression:

The simplest example of a least squares approximation is fitting a straight line to a set of paired observations:

$$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$$

The mathematical expression for the straight line is:

$$y = \underbrace{a_0}_{\text{intercept}} + \underbrace{a_1}_{\text{slop}} x + \underbrace{e}_{\text{error}}$$

Where a_0 and a_1 are coefficients representing the intercept and the slope respectively.

e is the error between the model and observation.

$$e = y - a_0 - a_1 x \quad \text{——— (1)}$$

Criteria for best fit:

For fitting a best line through the data would be to minimize the sum of the residual errors for all the available data.

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$\sum a_0 = n a_0$

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i) \quad \text{--- (1)}$$

Where,
n = total no. of points.

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i|$$

Let,

$$\underbrace{S_r = \sum_{i=1}^{n} e_i^2}_{} = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \quad \text{--- (2)}$$

$$\frac{\delta S_r}{\delta a_0} = -2 \sum (y_i - a_0 - a_1 x_i) \quad \text{--- (3)}$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum (y_i - a_0 - a_1 x_i) x_i \quad \text{--- (4)}$$

derivaties equal to zero will result in a minimum S_r.

$$(3) \Rightarrow 0 = \sum y_i - \sum a_0 - \sum a_1 x_i \quad \text{--- (5)}$$

$$(4) \Rightarrow 0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2 \quad \text{--- (6)}$$

From equation (5),

$$n a_0 + (\sum x_i) a_1 = \sum y_i \quad \text{--- (7)}$$

From equation (6),

$$(\sum x_i) a_0 + (\sum x_i^2) a_1 = \sum x_i y_i \quad \text{--- (8)}$$

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From ~~6 &~~ 7 & 8

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$\frac{\sum \text{sum}}{n} = \overline{\text{sum}}$$

From equation 7

$$n a_0 + (\sum x_i) a_1 = \sum y_i$$

or, $n a_0 = \sum y_i - (\sum x_i) a_1$

or, $a_0 = \frac{\sum y_i}{n} - \frac{(\sum x_i) a_1}{n}$

$$\therefore a_0 = \bar{y} - a_1 \bar{x}$$

▶ Find out the value of a_0 and a_1 in the case of least square regression.

Example 17.1:

Fit a straight line to the x and y value in the first two columns of the table. Computations for an error analysis of the linear fit.

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Given value

x_i	y_i	$(y_i-\bar{y})^2$	$(y_i-a_0-a_1x_i)^2$
1	0.5	8.5765	0.1687
2	2.5	0.8622	0.5625
3	2.0	2.0408	0.3473
4	4.0	0.3265	0.3265
5	3.5	0.0051	0.5896
6	6.0	6.6122	0.7972
7	5.5	4.2908	0.1993
Σ 28	Σ 24.0	22.7143	2.9911

Solution:

The following quantities can be computed

$n = 7$ $\quad \Sigma x_i y_i = 119.5$ $\quad \Sigma x_i^2 = 140$

$\Sigma x_i = 28$ $\quad \bar{x} = \frac{28}{7} = 4$

$\Sigma y_i = 24$ $\quad \bar{y} = \frac{24}{7} = 3.428571$

Now, $a_1 = \dfrac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$

$= \dfrac{7(119.5) - 28(24)}{7(140) - (28)^2} = 0.8392857$

$a_0 = \bar{y} - a_1\bar{x}$

$= 3.428571 - 0.8392857 \times 4$

$= 0.07142857$

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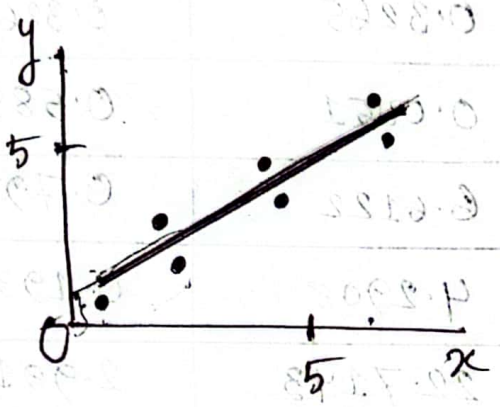
Therefore, the least square fit is :

$$y = a_0 + a_1 x$$

$$\Rightarrow y = 0.07142857 + 0.8392857\, x$$



This line along with the data shown in this figure.

Estimate of ERRORS for the linear least square fit:

Standard deviation:

$$S_y = \sqrt{\frac{22.7143}{7-1}} = 1.9457$$

$$\boxed{S_y = \sqrt{\frac{(y_i - \bar{y})^2}{n-1}}}$$

Standard error:

$$S_{y/x} = \sqrt{\frac{2.9911}{7-2}} = 0.7735$$

$$\boxed{S_{y/x} = \sqrt{\frac{S_r}{n-2}}}$$

$$r^2 = \frac{22.7143 - 2.9911}{22.7143} = 0.868$$

$$\boxed{r^2 = \frac{S_t - S_r}{S_t}}$$

$$r = \sqrt{0.868} = 0.93$$

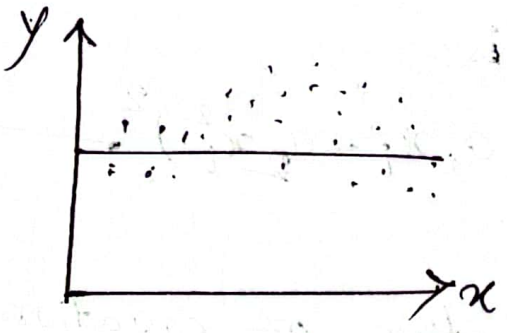
$r^2 \rightarrow$ Coefficient of determination

$r \rightarrow$ correlation coefficient

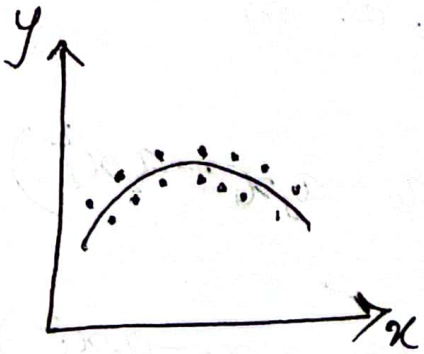
These result indicate that 86.8 percent of the original uncertainity has been explained by the linear model.

polinomial Regression:

The process of going back to an earlier or less advanced from an state.



ill-suited data for linear regression



parabola is preferable

The least squares procedure can be readily extended to fit the data to a higher order polinomial.

For example, Lets suppose that we fit a second order polynomial or quadratic.

$$y = a_0 + a_1 x + a_2 x^2 + e$$

For this case the sum of the squares of the residuals is:

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 \quad \text{--- (1)}$$

Taking the derivative of equation (1) w.r.t. each of the unknown coefficients of the polynomial, as in

$$\frac{\delta S_r}{\delta a_0} = -2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum x_i (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{\delta S_r}{\delta a_2} = -2 \sum x_i^2 (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

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These equations can be set equal to zero and rearranged to developed the following set of normal equations:

$$n a_0 + (\Sigma x_i) a_1 + (\Sigma x_i^2) a_2 = \Sigma y_i$$

$$(\Sigma x_i) a_0 + (\Sigma x_i^2) a_1 + (\Sigma x_i^3) a_2 = \Sigma x_i y_i$$

$$(\Sigma x_i^2) a_0 + (\Sigma x_i^3) a_1 + (\Sigma x_i^4) a_2 = \Sigma x_i^2 y_i$$

standard error is formulated as:

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

$$\begin{bmatrix} n & \Sigma x & \Sigma x^2 \\ \Sigma x & \Sigma x^2 & \Sigma x^3 \\ \Sigma x^2 & \Sigma x^3 & \Sigma x^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma xy \\ \Sigma x^2 y \end{bmatrix}$$

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Example-17.5:

Fit a second order polinomial to the data in the first two columns of the data.

x_i	y_i	$(y_i-\bar{y})^2$	$(y_i-a_0-a_1x_i-a_2x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08158
3	27.2	3.12	0.80491
4	40.9	239.22	0.61951
5	61.1	1272.11	0.09439
$\Sigma 15$	$\Sigma 152.6$	2513.39	3.74657

Solution:

From the given data,

order, $m=2$

$n=6$

$\bar{x}=2.5$

$\bar{y}=25.433$

$\Sigma x_i=15$

$\Sigma y_i=152.6$

$\Sigma x_i^2=55$

$\Sigma x_i^3=255$

$\Sigma x_i^4=979$

$\Sigma x_iy_i=585.6$

$\Sigma x_i^2y_i=2488.8$

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Therefore simultaneous linear equation are -

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix}$$

using gauss elimination we get,

$a_0 = 2.47857$

$a_1 = 2.35929$

$a_2 = 1.86071$

The least square quadratic equation for this case is,

$$y = a_0 + a_1 x + a_2 x^2$$

$$\Rightarrow y = 2.47857 + 2.35929\,x + 1.86071\,x^2$$

The standard error of the estimate based on the regression polinomial is

$$S_{y/x} = \sqrt{\frac{3.74657}{6-3}}$$

$$= 1.12$$

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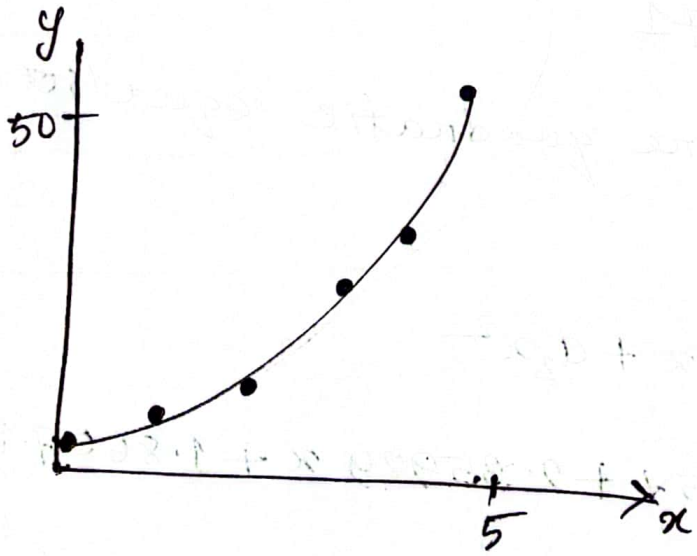
The coefficient of determination

$$r^2 = \frac{2513.39 - 3.74657}{2513.39}$$

$$= 0.99851$$

and the correlation coefficient is, $r = 0.99925$

These result indicate that 99.851 percent of the original uncertainty has been explained by the model.



Fit of a second order polinomial

Runge-Kutta Method

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Euler's Method:

New value = old value + slopx step size

Euler-cauchy method

point slop method

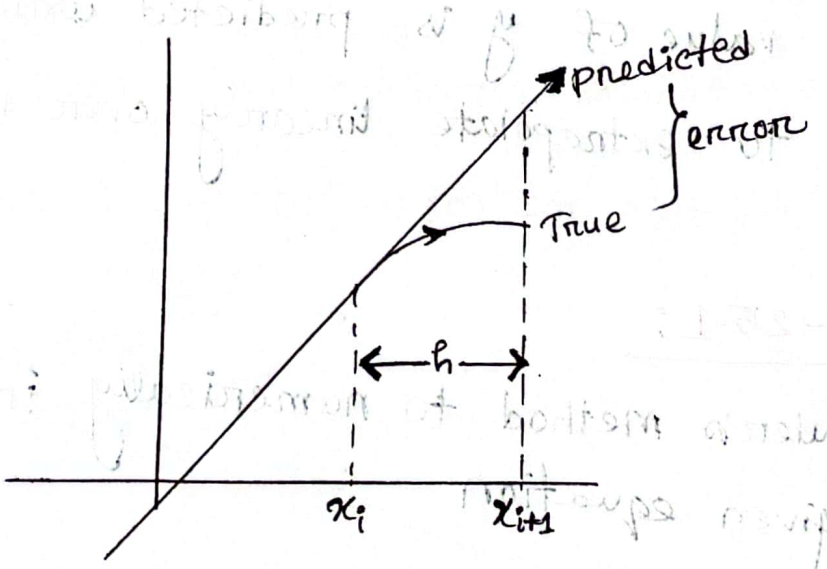
$$y_{i+1} = y_i + \phi h$$

$$\boxed{y_{i+1} = y_i + f(x,y)h}$$

slop = $\frac{dy}{dx} = f(x,y)$

- This formula referred to as Euler's method.

Graphical Picard



Euler's method

The first derivative provides a direct estimate of the slope at x_i

$$\phi = f(x_i, y_i)$$

where, $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i.

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This estimate can be substitute into the equation,

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)\,h}$$

This formula is referred to as Euler's method.

A new value of y is predicted using the slope to extrapolate linearly over the step size h.

Example - 25.1 :

Use Euler's method to numerically integrate the given equation

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x=0$ to $x=4$ with a step size of 0.5. The initial condition at $x=0$ is $y=1$.

Recall that the exact equation is given by

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

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$x=0 \rightarrow y=1$

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Solution:

$y_{i+1} = y_i + f(x_i, y_i)h$ can be use used to implement Euler's method.

$$y(0.5) = y(0) + f(0,1)\,0.5$$

Where,

$y(0)=1$ and the slope estimate at $x=0$ is

$$f(0,1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5$$
$$= 8.5$$

Therefore,

$$y(0.5) = 1 + 8.5 \times 0.5$$
$$= 5.25$$

The true solution at $x=0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1$$
$$= 3.21875$$

Thus the error is,

$$E_t = \text{true} - \text{approximate}$$
$$= 3.21875 - 5.25$$
$$= -2.03125$$

~~or, expected~~

or, expressed as percent relative error, $\varepsilon_t = -63.1\%$

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$$y(1) = y(0.5) + f(0.5, 5.25)\,0.5$$

$$= 5.25 + \left[-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5\right]0.5$$

$$= 5.875$$

The true solution at $x=1$ is 3.0

The percent relative error is −95.8%.

The computation is repeated.

x	y_{true}	y_{Euler}	percent Relative ERROR
0	1	1	
0.5	3.21875	5.25	−63.1
1	3	5.875	−95.8
1.5	2.21875	5.125	−131
2	2	4.5	−125
2.5	2.71875	4.75	−74.7
3	4	5.875	−46.9
3.5	4.71875	7.125	−51.0
4	3	7	−133.3

Picards Method:

Consider the initial value problem

$$\frac{dy}{dx} = f(x,y) \; ; \quad y(x_0) = y_0$$

Integrating the above differential equation between x_0 and x. we can write

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\, dx$$

or, $$[y]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\, dx$$

or, $$y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

or, $$\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\, dx}$$

As a first approximation y in right hand side is replace by y_0, the value of at $x = x_0$ and the resulting value of y in left hand side is denoted by y_1.

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So that, we have

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

A second approximation y_2 is similarly obtained by replacing y_0 by y_1 in the above result, we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$\vdots \qquad \vdots$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx$$

Example

Given $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$,
Calculate $y(0.1), y(0.2)$ using picard's method.

Solution:

Given, $\frac{dy}{dx} = 1 + xy$

Let, $f(x, y) = 1 + xy$

$y_0 = 1$
$x_0 = 0$

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From Picard's ordinary equation

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx$$

or, $$y_{n+1} = y_0 + \int_{x_0}^{x} (1 + x y_n)\, dx$$

put, $n=0$

$$y_1 = y_0 + \int_{x_0}^{x} (1 + x y_0)\, dx$$

$$= 1 + \int_0^x (1+x)\, dx$$

$$= 1 + \int_0^x dx + \int_0^x x\, dx$$

$$= 1 + [x]_0^x + \left[\frac{x^2}{2}\right]_0^x$$

$$\therefore y_1 = 1 + x + \frac{x^2}{2}$$

put, $n=1$

$$y_2 = y_0 + \int_{x_0}^{x} (1 + x y_1)\, dx$$

$$= 1 + \int_0^x dx + \int_0^x x\left(1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

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put $n=2$, we get,

$$y_3 = y_0 + \int_{x_0}^{x} (1 + x y_2)\, dx$$

$$= 1 + \int_0^x dx + \int_0^x x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$\frac{(0.1)^2}{2}$$

$$\therefore y(0.1) = 1 + 0.1 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

$$= \cancel{0.10034}\ 1.10534$$

$$\therefore y(0.2) = \cancel{0.20288}\ 1.22286$$

Ans.

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Heun's Method

predictor:

$$y^0_{i+1} = y_i + f(x_i, y_i)h$$

Corrector:

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} \times h$$

Example:

$f(x, y) = x + 2y$

$x_0 = 0$, $y_0 = 0$, $h = 0.1$

$y(0.2) = ?$

$x_1 + 2y_1$

$0.1 + 0$

Solution: 1st step:

predictor:

$$y^0_1 = y_0 + f(x_0, y_0) \times h$$

$$= 0 + 0 \times 0.1$$

$$= 0$$

Corrector:

$$y_1 = 0 + \frac{0 + 0.1 + 2 \times 0}{2} \times 0.1$$

$$= 0.005$$

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$x_1 + 2y_1$

2nd step:

Predictor:

$$y_2^0 = y_1 + f(x_1, y_1)h$$

1st step correction formula এর মতো

$$= 0.005 + (0.1 + 2(0.005)) \times 0.1$$

$$= 0.005 + 0.11 \times 0.1$$

$$= 0.016$$

x_0 0
$x_1 - 0.1$
$x_2 \to 0.2$

$x_2 + 2y_2$

(02)

Corrector:

$$y_2 = y_1 + h\left(\frac{f(x_1, y_1) + f(x_2, y_2)}{2}\right)$$

$$= 0.005 + 0.1\left(\frac{0.11 + 0.2 + 2(0.016)}{2}\right)$$

$$= 0.0221$$

Ans

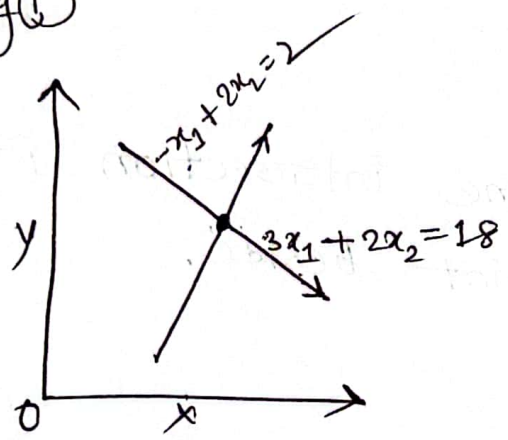
$$\therefore y(0.2) = 0.0221$$

Graphical Method

(#) Sketch and state graphical depiction of Singular and ill condition system.

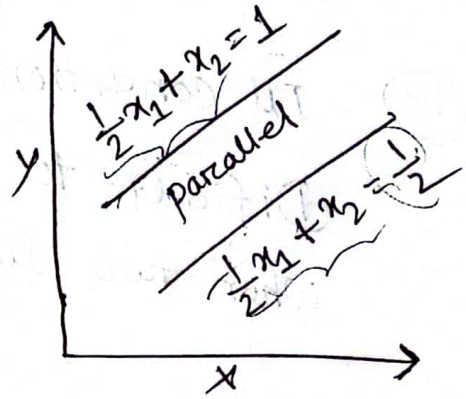
Solution:

Fig (i)



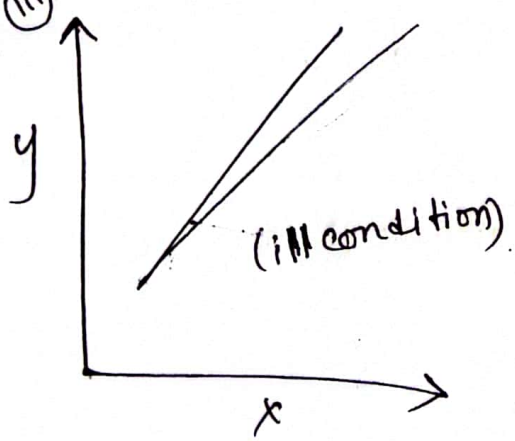
There exist only one Solution / Root

Fig (ii)



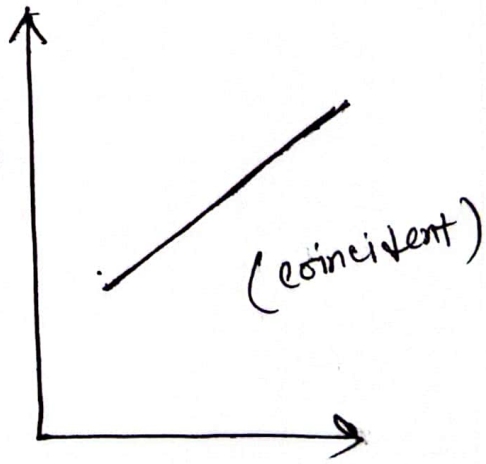
As, the lines are parallel so, there no solution.

Fig (iii)



Slop is very close, point of intersection is difficult to detect.

Fig (iv)



infinite solution

(III) & (IV) are called Singular.

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Decision:

(i) Beyond three equation graphical method breakdown

(ii) Between two parallel straight line there is no solution

(iii) Ill condition:

Dificult to Determine intersection point when two line convint beside.

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Weddle's Rule:

The Newton cotes formula is:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

putting $n=6$ in the equation of Newton-cotes formula. so,

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_6(x)\,dx$$

And neglecting the differences of order higher than six, we obtain weddle's rule

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

the error in which is given by, $\varepsilon = \frac{h}{140} y_6$

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1(A):

$$\begin{bmatrix} 0.40 & -0.1 & -0.2 \\ 0.10 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{bmatrix}$$

The determinant D can be written as:

$$D = \begin{vmatrix} 0.4 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{vmatrix}$$

$$= 0.4\left(7\times10 - 0.3\times0.2\right) + 0.1\left(0.1\times10 + 0.3\times0.3\right)$$

$$- 0.2\left(0.1\times0.2 - 7\times0.3\right)$$

$$= 28.509$$

$$D_{x_1} = \begin{vmatrix} 7.85 & -0.1 & -0.2 \\ -19.3 & 7 & -0.3 \\ 71.4 & -0.2 & 10 \end{vmatrix}$$

$$= 7.85\left(7\times10 - 0.3\times0.2\right) + 0.1\left(-19.3\times10 + 0.3\times71.4\right)$$

$$- 0.2\left(19.3\times0.2 - 7\times71.4\right)$$

$$= 631.059$$

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$$Dx_2 = \begin{vmatrix} 0.4 & 7.85 & -0.2 \\ 0.1 & -19.3 & -0.3 \\ 0.3 & 71.4 & 10 \end{vmatrix}$$

$$= 0.4(-19.3\times10 + 0.3\times71.4) - 7.85(0.1\times10 + 0.3\times0.3)$$
$$- 0.2(0.1\times71.4 + 19.3\times0.3)$$

$$= -79.7745$$

$$Dx_3 = \begin{vmatrix} 0.4 & -0.1 & 7.85 \\ 0.1 & 7 & -19.3 \\ 0.3 & -0.2 & 71.4 \end{vmatrix}$$

$$= 0.4(7\times71.4 - 19.3\times0.2) + 0.1(0.1\times71.4 + 19.3\times0.3)$$
$$+ 7.85(-0.1\times0.2 - 7\times0.3)$$

$$= 183.027$$

$$x_1 = \frac{Dx_1}{D} = \frac{631.059}{28.509} = 22.1354$$

$$x_2 = \frac{Dx_2}{D} = \frac{-79.7745}{28.509} = -2.7982$$

$$x_3 = \frac{Dx_3}{D} = \frac{183.027}{28.509} = 6.4199$$

28.027
-2.97
4.23

1(B)

First part of the procedure is Forward elimination.

$$0.3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (I)}$$

$$0.10x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{——— (II)}$$

$$x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{——— (III)}$$

To remove x_1 ~~and~~ From equation (II) and (III)

$$(II) - (I) \times \frac{0.1}{0.3} \quad \text{and} \quad (III) - (I) \times \frac{1}{0.3}$$

$$0.3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (IV)}$$

$$7.0333x_2 - 0.2333x_3 = -21.91667 \quad \text{——— (V)}$$

$$0.1333x_2 + 10.667x_3 = 45.2333 \quad \text{——— (VI)}$$

$$(VI) - (V) \times \frac{0.1333}{7.0333}$$

$$0.3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{——— (VII)}$$

$$7.0333x_2 - 0.2333x_3 = -21.91667 \quad \text{——— (VIII)}$$

$$10.671x_3 = 45.64867$$

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We can solve this by back substitution,

$$x_3 = \frac{45.64867}{10.671}$$

$$\therefore x_3 = 4.2778$$

The result can be back substituted into equation (iii)

$$7.0333x_2 - 0.2333 \times 4.2778 = -21.91667$$

$$\text{or, } x_2 = \frac{-21.91667 + 0.2333 \times 4.2778}{7.0333}$$

$$\therefore x_2 = -2.974$$

Now,

$$0.3x_1 - 0.1 \times (-2.974) - 0.2 \times 4.2778 = 7.85$$

$$\text{or, } x_1 = \frac{71.4 - 0.1 \times 2.974 - 10 \times 4.2778}{0.3}$$

$$\therefore x_1 = 28.0272$$

The results are:

$x_1 = 28.0272$

$x_2 = -2.974$

$x_3 = 4.2778$

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2(A)

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

Comparing the values,

$u_{11} = 1$

$u_{12} = 1$

$u_{13} = -1$

$l_{21} = 2$

$2\times(1) + u_{22} = 3$

$\therefore u_{22} = 1$

$2\times(-1) + u_{23} = 5$

$\therefore u_{23} = 7$

$l_{31} = 3$

$3\times 1 + l_{32}\times 1 = 2$

$\therefore l_{32} = -1$

$3\times(-1) + (-1)\times 7 + u_{33} = -3$

$\therefore u_{33} = -3 + 3 + 7$

$= 7$

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$AX = B$

$LUX = B$

Let, $UX = Y$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$y_1 = 2$

$2y_1 + y_2 = -3$

$\therefore y_2 = -3 - 2 \times 2$

$= -7$

$3y_1 - y_2 + y_3 = 6$

$\therefore y_3 = 6 - 3 \times 2 - -7$

$= -7$

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Now,

$$UX = Y$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}$$

$$\therefore x_3 = \frac{-7}{7} = -1$$

$$x_2 + 7x_3 = -7$$

$$\therefore x_2 = -7 - 7 \times (-1)$$
$$= 0$$

$$x_1 + x_2 - x_3 = 2$$

$$\therefore x_1 = 2 + (-1)$$
$$= 1$$

The result are:

$x_1 = 1$

$x_2 = 0$

$x_3 = -1$

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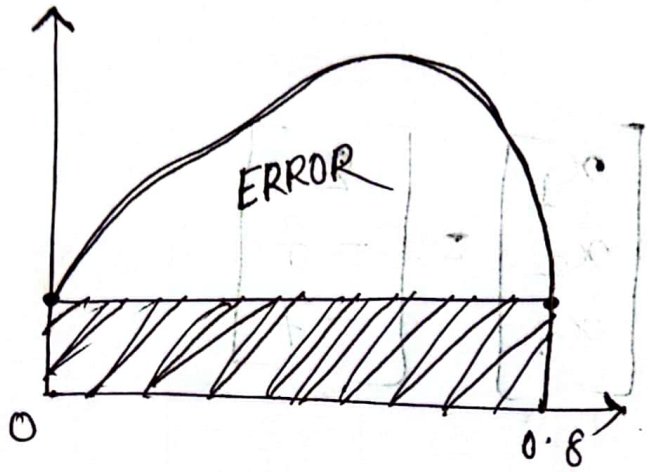
2(B)

i



The reason for this large error is seen from the figure. that, the area under the straight line neglects a significant portion of the integral lying above the line.

By using Simpson Rule we can eliminate the error in the Trapezoidal rule. To obtain a more accurate estimate of an integral is to use higher order polinomials to connect the points.

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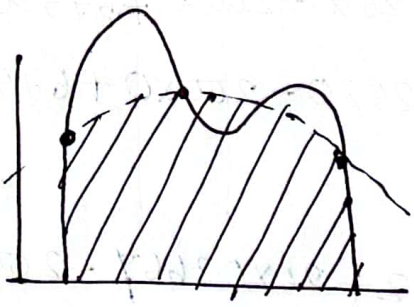
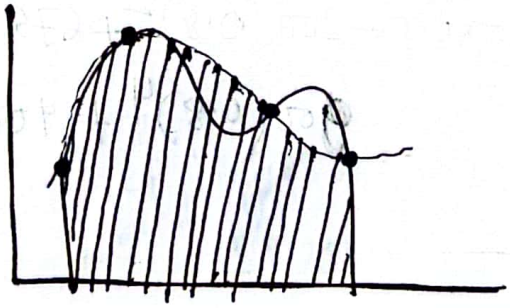
By using Simpson 1/3 Rule,
we add extra point midway between $f(a)$ and $f(b)$, the three point connected with a parabola.



To get this area we can use this formula,

$$I = (b-a) \cdot \frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

By using Simpson 3/8 Rule,
There will be two points equally spaced between $f(a)$ and $f(b)$, the four points can be connected with a third order polinomial.



To get this area we can use this formula,

$$I = (b-a) \frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

By this process we can eliminate the error in Trapezoidal Rule by applying the Simpson Rule.

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(ii) Simpson's 3/8 rule required 4 equally spaced points.

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$f(0) = 0.2 + 25\times0 - 200\times0 + 675\times0 - 900\times0 + 400\times0$$
$$= 0.2$$

$$f(0.2667) = 0.2 + 25\times0.2667 - 200(2.667)^2 + 675(2.667)^3 - 900(2.667)^4 + 400(2.667)^5$$
$$= 1.4329$$

$$f(0.5333) = 0.2 + 25\times0.5333 - 200(0.5333)^2 + 675(0.5333)^3 - 900(0.5333)^4 + 400(0.5333)^5$$
$$= 3.487$$

$$f(0.8) = 0.2 + 25\times0.8 - 200(0.8)^2 + 675(0.8)^3 + 900(0.8)^4 + 400(0.8)^5$$
$$= 0.232$$

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using simpson 3/8 Rule,

$$I \simeq 0.8 \frac{0.2 + 3(1.4329 + 3.487) + 0.232}{8}$$

$$= \cancel{1.8989} = 1.5102$$

3(A) Augmented matrix:

$$\begin{bmatrix} 0.7 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 1 & -0.2 & 10 & 71.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.1428 & -0.2857 & 11.214 \\ 0.1 & 7 & -0.3 & -19.3 \\ 1 & -0.2 & 10 & 71.4 \end{bmatrix} \quad r_1' = r_1/0.7$$

$$= \begin{bmatrix} 1 & -0.1428 & -0.2857 & 11.214 \\ 0 & 7.01428 & -0.27143 & -20.4214 \\ 0 & -0.0572 & 10.2857 & 60.186 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - 0.1 r_1 \\ r_3' = r_3 - r_1 \end{matrix}$$

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$$= \begin{bmatrix} 1 & -0.1428 & -0.2857 & 11.214 \\ 0 & 1 & -0.0387 & -2.9114 \\ 0 & -0.0572 & 10.2857 & 60.186 \end{bmatrix} \quad r_2' = r_2 / 7.01428$$

$$= \begin{bmatrix} 1 & 0 & -0.2912 & 10.7982 \\ 0 & 1 & -0.0387 & -2.9114 \\ 0 & 0 & 10.2835 & 60.0194 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 + 0.1428 r_2 \\ r_3' = r_3 + 0.0572 r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -0.2912 & 10.7982 \\ 0 & 1 & -0.0387 & -2.9114 \\ 0 & 0 & 1 & 5.8364 \end{bmatrix} \quad r_3' = r_3 / 10.2835$$

$$= \begin{bmatrix} 1 & 0 & 0 & 12.5 \\ 0 & 1 & 0 & -2.68 \\ 0 & 0 & 1 & 5.836 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 + 0.2912 r_3 \\ r_2' = r_2 + 0.0387 r_3 \end{matrix}$$

The solution is

$x_1 = 12.5$

$x_2 = -2.68$

$x_3 = 5.836$

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3(B)

(i) In the case of iteration process, there is two scenario

(I) Convergence

(II) Divergence

Let's take an example too see this too scenarios.

$$2x^2 - 4x + 1 = 0$$

or, $4x = 2x^2 + 1$

$$\therefore x = \frac{x^2}{2} + \frac{1}{4}$$

n	x_n
1	1
2	0.75
3	0.53125
4	0.3911
5	0.32648
6	0.303296
7	0.29599
8	0.2938

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9	0.29316
10	0.29297
11	0.292916
12	0.29289
13	0.292895
14	0.292893
15	0.292893

1st step - 2nd step

$= 1 - 0.75$

$= 0.25$

2nd step - 3rd step

$= 0.75 - 0.53125$

$= 0.21875$

1st step - 2nd step > 2nd step - 3rd step

This condition is called convergence.

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n	x_n
1	2
2	2.25
3	2.78125
4	4.1176
5	8.7276
6	38.335
⋮	⋮

3rd step − 2nd step

$= 2.78125 - 2.25$

$= 0.53125$

2nd step − 1st step

$= 2.25 - 2$

$= 0.25$

2nd step − 1st step < 3rd step − 2nd step

This condition is called divergence.

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n	x_n
1	2
2	2.25
3	2.78125
4	4.1176
5	8.7276
6	38.335
⋮	⋮

3rd step − 2nd step

$= 2.78125 - 2.25$

$= 0.53125$

2nd step − 1st step

$= 2.25 - 2$

$= 0.25$

2nd step − 1st step < 3rd step − 2nd step

This condition is called divergence.

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(ii) Algorithm of Bisection method:

Step 1: Choose lower x_l and upper x_u for the root such that the function changes sign over the interval. This can be checked by $f(x_l) \cdot f(x_u) < 0$.

Step 2: An estimate of the root is determined by,

$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Make the following evaluations.

(a) If $f(x_l) \cdot f(x_r) < 0$, the root is lies in the lower subinterval. set $x_u = x_r$ and return to step 2.

(b) If $f(x_l) \cdot f(x_r) > 0$, the root is lies in the upper subinterval. set $x_l = x_r$ and return to step 2.

(c) If $f(x_l) f(x_r) = 0$, the root present in x_r and terminate the computation.

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4(A)

x_i	y_i		$(y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$
0	2.1		0
1	7.7		0
2	13.6		0
$\Sigma x_i = 3$	$\Sigma y_i = 23.4$		$\Sigma = 0$

$\Sigma x_i = 3$ $\quad n = 3$ $\quad \bar{y} = 7.8$

$m = 2$

$\bar{x} = 1$

$\Sigma x^2 = 5$ $\quad \Sigma y = 23.4$

$\Sigma x^3 = 9$ $\quad \Sigma xy = 34.9$

$\Sigma x^4 = 17$ $\quad \Sigma x^2 y = 62.1$

Therefore Simultaneous linear equation is:

$$\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 23.4 \\ 34.9 \\ 62.1 \end{bmatrix}$$

using gauss elimination we get,

$a_0 = 2.1$

$a_1 = 5.45$

$a_2 = 0.15$

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The least square quadratic equation for this case is:

$$y = a_0 + a_1 x + a_2 x^2$$

$$= 2.1 + 5.45x + 0.15x^2$$

Standard error $S_{y/x}$ is

$$= \sqrt{\frac{0}{3-3}}$$

$$= \sqrt{\frac{0}{0}}$$

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4(B)

The mathematical expression for the straight line is:

$$y = a_0 + a_1 x + e$$

Where a_0 and a_1 coefficient representing respectively intercept and slop. And e is ~~mod~~ error between model and observation.

$$e = y - a_0 - a_1 x$$

For Fitting best line through the data would be minimize the sum of the error of the available data,

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} y_i - a_0 - a_1 x_i$$; where n is the number of point

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i|$$

Let,

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

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$$\frac{\delta S_r}{\delta a_0} = -2\sum(y_i - a_0 - a_1 x_i) \quad \text{——} \quad (1)$$

$$\frac{\delta S_r}{\delta a_1} = -2\sum(y_i - a_0 - a_1 x_i)x_i \quad \text{——} \quad (2)$$

derivative equal to zero will result minimum S_r.

$$(1) \Rightarrow 0 = -2\sum(y_i - a_0 - a_1 x_i) \quad \text{——} \quad (3)$$

$$(2) \Rightarrow 0 = -2\sum(y_i - a_0 - a_1 x_i)x_i \quad \text{——} \quad (4)$$

From equation (3)

$$\sum y_i = \sum a_0 + \sum a_1 x_i$$

$$n a_0 + (\sum x_i) a_1 = \sum y_i \quad \text{——} \quad (5)$$

From equation (4)

$$(\sum x_i) a_0 + (\sum x_i^2) a_1 = \sum x_i y_i \quad \text{——} \quad (6)$$

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From (5) and (6)

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

From equation (5),

$$n a_0 = \sum y_i - (\sum x_i) a_1$$

or, $$a_0 = \frac{\sum y_i}{n} - \frac{(\sum x_i) a_1}{n}$$

$$\therefore a_0 = \bar{y} - a_1 \bar{x}$$

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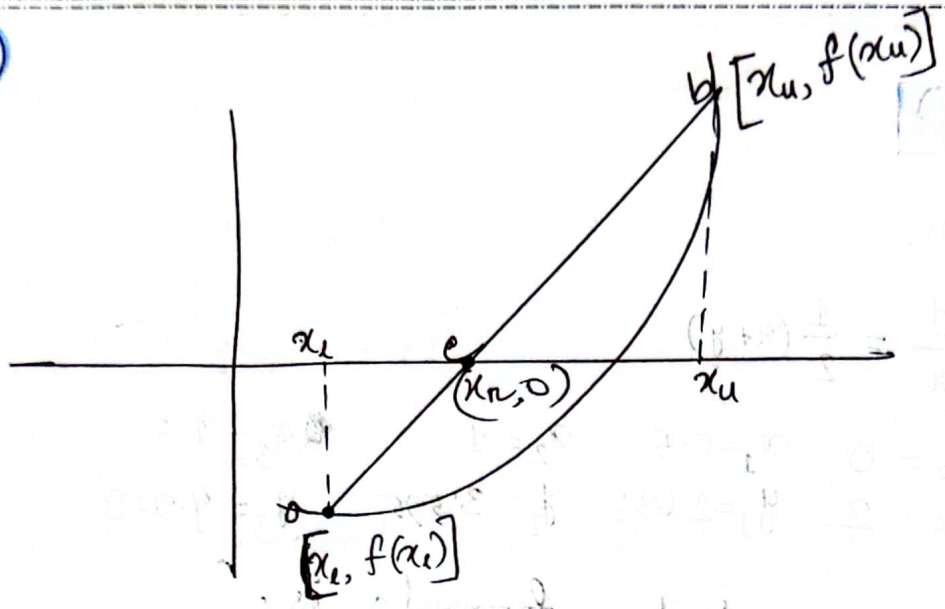
(ii)



Slop of ac = slop of bc

$$\frac{f(x_l)-0}{x_l-x_r} = \frac{f(x_u)-0}{x_u-x_r}$$

or, $f(x_l)(x_u-x_r) = f(x_u)(x_l-x_r)$

or, $x_u f(x_l) - x_r f(x_l) = x_l f(x_u) - x_r f(x_u)$

or, $x_r f(x_u) - x_r f(x_l) = x_l f(x_u) - x_u f(x_l)$

or, $x_r f(x_l) - x_r f(x_u) = x_u f(x_l) - x_l f(x_u)$

or, $x_r = \dfrac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$

or, $x_r = \dfrac{x_u f(x_l)}{f(x_l)-f(x_u)} - \dfrac{x_l f(x_u)}{f(x_l)-f(x_u)}$

$$x_r = x_u + \frac{x_u f(x_l)}{f(x_l)-f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$x_r = x_u + \frac{\cancel{x_u f(x_l)} - \cancel{x_u f(x_l)} + x_u f(x_u)}{f(x_l)-f(x_u)} - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\therefore x_r = x_u - \frac{f(x_u)(x_l-x_u)}{f(x_l)-f(x_u)}$$

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5 (B)

Given,

$$\frac{dy}{dx} = \frac{1}{2}(x+y)$$

$x_0 = 0 \quad x_1 = 0.5 \quad x_2 = 1 \quad x_3 = 1.5$

$y_0 = 2 \quad y_1 = 2.636 \quad y_2 = 3.595 \quad y_3 = 4.968$

Milne's predictor formula is:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

putting $n=3$ we get,

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$y' = \frac{1}{2}(x+y$$

$$y'_1 = \frac{1}{2}(x_1+y_1) = \frac{1}{2}(0.5+2.636) = 1.568$$

$$y'_2 = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1+3.595) = 2.2975$$

$$y'_3 = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+4.968) = 3.234.$$

$$y_{4p} = 2 + \frac{4\times 0.5}{3}\left(2\times 1.568 - 2.2975 + 2\times 3.234\right)$$

$$= 6.871$$

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So, our predictor value is 6.871.

$$y_4' = \frac{1}{2}(x_4 + y_4)$$

$$= \frac{1}{2}(2 + 6.871)$$

$$= 4.4355$$

Now, we correct ~~formula~~ by milne's Corrector formula:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

By putting $n=3$, we get

$$y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 3.595 + \frac{0.5}{3}\left[2.2975 + 4 \times 3.234 + 4.4355\right]$$

$$= 6.8731$$

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6(B)

$y_{i+1} = y_i + f(x_i, y_i) \times h$

this equation can be used to implement Euler's method.

$y(0.5) = y(0) + f(0,1) \times 0.5$

Given, $y(0) = 1$

$f(0,1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5$

$= 8.5$

$\therefore y(0.5) = 1 + 8.5 \times 0.5$

$= 5.25$

$y(1) = y(0.5) + f(0.5, 5.25) \times 0.5$

$= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5] \times 0.5$

$= 5.875$

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The computation is repeated:

X	y_{true}	y_{Euler}	Relative ERROR
0		1	
0.5		5.25	
1		5.875	
1.5		5.125	
2		4.5	
2.5		4.75	
3		5.875	
3.5		7.125	
4		7	

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