Math Things we may have to memorize! :) 24's edition

(*) Single valued function:

A function $w = f(z)$ is called a single valued function in domain S if only one value of w corresponds to each value of z in S

(*) Multiple valued functions:

A function $w = f(z)$ is called a multiple valued function in a domain S, if more than one value of w corresponds to each value of z in S..

(*) Derivative or diff. coefficient in a region of domain of the Argand plane:

Let $f(z)$ be defined as a single valued function in a region or domain R, then the derivative of $f(z)$ is defined as $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

(*) A single valued function is said to be differentiable at $z = z_0$ of the Argand plane

Analytic function:

· A single valued function $f(z)$ is said to be analytic in a region R, if the derivative $f'(z)$ exists at all points z in the Region R.

Necessary condition:

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R, it is necessary that the four partial derivatives $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$ and $v_y = \frac{\partial v}{\partial y}$ should exist and satisfy the cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$

Sufficient condition for $f(z)$ to be analytic:

The continuous single-valued function $f(z) = u(x,y) + iv(x,y)$ is a analytic in a region R if the four partial derivatives exist u_x, u_y, v_x and v_y exist and satisfy the cauchy riemann equations.

(*) Cachy Riemann equations in polar form:

$$\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}, \quad \frac{\delta v}{\delta r} = -\frac{1}{r}\frac{\delta u}{\delta \theta}$$

(*) Laplace's equations in two dimensions:

In two dimensions a second order partial differential equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$ is called laplace's equation.

The vector operator ∇^2 is called laplacian where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(*) Harmonic function:

A real valued function $\phi(x,y)$ is called a harmonic function or simply harmonic in a region R if all its second order partial derivatives are continuous in R and at each point of the region it satisfies the laplacian equation

$$\frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2} = 0$$

Conjugate Harmonic Functions:

If $f(z) = u + iv$ is an analytic function, the u and v are called conjugate harmonic function or conjugate har... or simply conjugate functions.

Laplace equations in polar form:

$$\frac{\delta^2 \psi}{\delta r^2} + \frac{1}{r}\frac{\delta \psi}{\delta r} + \frac{1}{r^2} \cdot \frac{\delta \psi}{\delta \theta^2} = 0$$

Milne Thomson method / simple Milne method:

If $f(z) = u(x,y) + iv(x,y)$ is analytic and if $\frac{\delta u}{\delta x} = \phi_1(x,y)$, $\frac{\delta u}{\delta y} = \phi_2(x,y)$ and $\frac{\delta v}{\delta x} = \psi_1$, $\frac{\delta v}{\delta y} = \psi_2$ then,

(i) $f'(z) = \phi_1(z,0) - \phi_2(z,0)$

(ii) $f(z) = \int [\phi_1(z,0) - \phi_2(z,0)]\, dz + c$

....

(*) Singular point / critical point / singularity

A point at which a function fails or ceases to be analytic is called a singular point.

→ Isolated singular point / singularity

A singular point $z = z_0$ is called an isolated singular point of a function $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no other singularity of the function other than z_0. If there's no such ~~point~~ δ, in this case z_0 is called a non-isolated singularity of $f(z)$.

→ Ordinary point

A point z_0 is called an ordinary point of a function $f(z)$ if z_0 is not a singular point and we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point.

(4) **Pole**: A singular point $z = z_0$ is called a pole of order n of the function $f(z)$ if $\lim_{z \to z_0} \lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, where n is a positive integer.

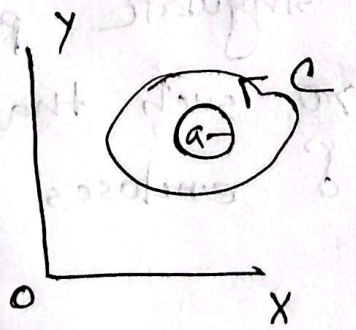
Ch 5

Cauchys integral formula:

If $f(z)$ is an analytic inside and on a simple closed curve C, If a is any point inside C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} \cdot dz$$

where C is traversed in the positive sense. e.g. in the counter-clockwise direction.



Theorem

If $f(z)$ is analytic inside and on the boundary C of a simply connected closed region R, then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)\pm(z-b)}\cdot dz$$

$$= \begin{cases} f(a)\pm f(b) & \text{if the points } a, b \text{ inside } C \\ f(a) & \text{———} \\ f(b) & \text{———} \\ 0 & \text{———} \quad \text{not inside } C \end{cases}$$

For n'th derivative,

$$f^{n}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\cdot dz$$

ch 6

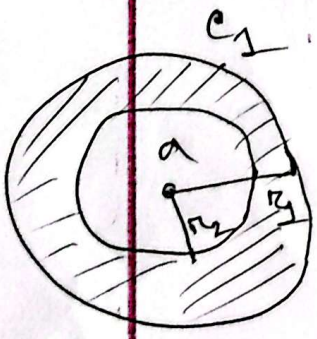
Taylor's theorem :

If $f(z)$ is analytic inside for all values of z inside a circle c with center a, then

$$f(z) = f(z) + \frac{(z-a)}{1!} f'(z) + \frac{(z-a)^2}{2!} f''(z) + \frac{(z-a)^3}{3!} f'''(z) + \dots$$

Laurent's theorem :

If $f(z)$ is analytic inside and on the boundary of a ring shaped region R bounded by two concentric circles c_1 and c_2 of center a and radi r_1, r_2 $(r_2 < r_1)$ respectively then for all z in R



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where,

$$\left.\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_c \frac{f(w)}{(w-a)^{n+1}} \\ a_{-n} &= \frac{1}{2\pi i} \oint_c \frac{f(w)}{(w-a)^{-n+1}} \end{aligned}\right\} \text{where } n = 1, 2, 3, \dots$$

Ch 7

Residue

If the function $f(z)$ has an isolated singularity at the point $z=a$, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the laurent's expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

$$+ \frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \dots$$

Theorem:

If $f(z)$ is analytic inside and on a closed curve C except at the point $z=a$ inside C, then

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \cdot dz$$

or, $a_{-1} = \lim_{z \to a} (z-a) f(z)$, where a_{-1} is the residue of $f(z)$ at the point $z=a$

Residue at multiple point

If $f(z)$ is analytic inside and on a simple closed curve C except at pole $z=a$ of order m inside C the the residue of $f(z)$ at $z=a$ is,

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

Cauchys residue theorem

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, \ldots, a_n$ inside C then

$$\oint_C f(z) \cdot dz = 2\pi i \left[\text{Res}(a_1) + \text{Res}(a_2) + \ldots + \text{Res}(a_n) \right]$$

Contour Integration

Form 1

$\int_0^{2\pi} f(\cos\theta, \sin\theta) \cdot d\theta$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Let, $I = \int_0^{2\pi} f(\cos\theta, \sin\theta) \cdot d\theta$ ..... ①

where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$. Let $z = e^{i\theta}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2} = \frac{z^2 - 1}{2z}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2 + 1}{2iz}$$

Again, $z = e^{i\theta}$

or, $dz = \cancel{ie\theta}\ i \cdot e^{i\theta} \cdot d\theta = z \cdot i \cdot d\theta$

or, $d\theta = \frac{dz}{zi}$

Now using these

$$I = \oint_c g(z) \cdot dz$$

where c is the circle $|z| = 1$, whose center is at $(0,0)$ and radius ~~z~~ equal to 1

Form 2

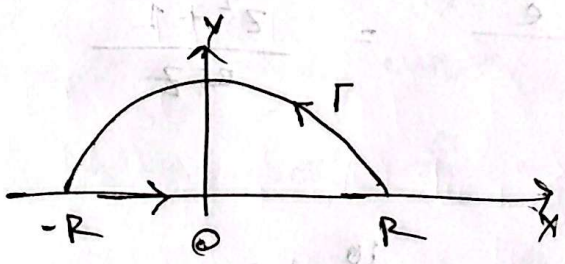
Integrals of the form $\int_{-\infty}^{\infty} f(x).dx$

Consider $\oint_c f(z).dz$, where c is the contour consisting of:

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Note Then $R \to \infty$ and if $f(x)$ is an even function this can be evaluated to the integral $\int_{0}^{\infty} f(x).dx$



(*) $\cos mz$

(*) $e^{imz} = \cos(mz) + i\sin(mz)$

$e^{i\theta} + e^{-i\theta}$

$\cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

Form 3

$$\int_{-\infty}^{\infty} f(x) \cdot \cos mx \cdot dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cdot \sin mx \cdot dx$$

where $f(x)$ is a rational function of x

Consider $\oint_C e^{imz} \cdot f(z) \cdot dz$

← prev. page

(*) Jordan's lemma (Theorem 221)

$\lim$ If $|f(z)| < \frac{M}{R^k}$ for $z = Re^{i\theta}$

where $k > 0$

and M is a constant

$$\lim_{R \to \infty} \int_{\Gamma} e^{imz} \cdot f(z) \cdot dz = 0$$

where Γ is the upper semi-circle of ~~radi~~ radius R and m is a positive constant.

(*) Finite Fourier sine transform :

$$F_s\{F(x)\} = f_s(n) = \int_0^l F(x) \cdot \sin \frac{n\pi x}{l} \cdot dx$$

$n \rightarrow$ অনুক্রমিক সংখ্যা

(*) Inverse FFST

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \cdot \sin \frac{n\pi x}{l}$$

(*) Finite fourier cosine transform :

$$F_{cs}\{F(x)\} = f_c(n) = \int_0^l F(x) \cdot \cos \frac{n\pi x}{l}$$

(*) Inverse FFCT

$$F(x) = \frac{1}{l} f(0) + \frac{2}{l} \sum_{n=1}^{\infty} f(n) \cos \frac{n\pi x}{l}$$

(*) Fourier transform

$$F\{F(x)\} = f(n) = \int_{-\infty}^{\infty} e^{inx} \cdot F(x) \cdot dx$$

(*) Inverse fourier transform

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-inx} \cdot f(n) \cdot dx$$

Applications of ...

* Fourier sine transform

$$f_s(n) = \int_0^\infty F(x) \cdot \sin nx \cdot dx$$

* Inverse FST

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \cdot \sin nx \cdot dn$$

* Fourier cosine transform

$$f_c(n) = \int_0^\infty F(n) \cdot \cos nx \cdot dx$$

* Inverse FCT

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cdot \cos nx \cdot dn$$

Periodic function:

If $f(x+T) = f(x)$, then $f(x)$ is called periodic function and T is called it's period

At the end points $x = \pm\pi$. If $f(x)$ is discontinuous then

$$f(\pi) = \lim_{\delta \to 0} \frac{1}{2}\left[f(-\pi+\delta) + f(\pi-\delta)\right]$$

$$f(\pi) = \frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right]$$

Fourier series:

Let $f(x)$ be defined in the interval $(-\pi, \pi)$ with period 2π which can be expanded in a trigonometric series

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Here,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx$$

Some results

(I) $\int_{-\pi}^{\pi} \sin mx \cdot dx = \int_{-\pi}^{\pi} \cos mx \cdot dx = 0$

(II) $\int_{-\pi}^{\pi} \sin mx \cdot \cos nx \cdot dx = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \cdot dx = 0$

(III) $\int_{-\pi}^{\pi} \cos^2 mx \cdot dx = \int_{-\pi}^{\pi} \sin^2 nx \cdot dx = \pi$

Even function

If $f(-x) = f(x)$, then $f(x)$ is called even function

Odd function

If $f(-x) = -f(x)$, then $f(x)$ is called odd function

EXTRAS

Application of

- Laplace transform
- Fourier transform
- Cauchys residue theorem

Dirichlet's conditions

Let $f(x)$ function satisfy the following conditions

① $f(x)$ is defined in the interval $-l<x<l$

② $f(x)$ and $f'(x)$ are sectionally continuous in the interval $-l<x<l$

③ $f(x+2l)=f(x)$, i.e. $f(x)$ is periodic and it's period is $2l$.

Laplace Transform

Let $F(t)$ be a function of t specified for $t > 0$. Then the laplace transform of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$ or $f(s)$,

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} \cdot F(t) \cdot dt$$

If $F(t) = 1$ then, $\mathcal{L}\{1\} = f(s) = \frac{1}{s}$

If $F(t) = t$ then, $\mathcal{L}\{t\} = f(s) = \frac{1}{s^2}$

If $F(t) = t^n$ then, $\mathcal{L}\{t^n\} = f(s) = \frac{n!}{s^{n+1}}$

If $F(t) = e^{at}$ then $\mathcal{L}\{e^{at}\} = f(s) = \frac{1}{s-a}$

If $F(t) = e^{-at}$ then $\mathcal{L}\{e^{-at}\} = f(s) = \frac{1}{s+a}$

If $F(t) = \sin at$ then $\mathcal{L}\{\sin at\} = f(s) = \frac{a}{s^2+a^2}$

If $F(t) = \cos at$ then $\mathcal{L}\{\cos at\} = f(s) = \frac{s}{s^2+a^2}$

If $F(t) = \sinh at$ then $\mathcal{L}\{\sinh at\} = f(s) = \frac{a}{s^2-a^2}$

If $F(t) = \cosh at$ then $\mathcal{L}\{\cosh at\} = f(s) = \frac{s}{s^2-a^2}$

First translation/ shifting

If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at} F(s)\} = f(s-a)$$

Scale property

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Laplace transform of derivatives

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$$

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - s F(0) - F'(0)$$

$$\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$$

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \dots$$

$$- s F^{n-2}(0) - F^{n-1}(0)$$

Multiplication by t^n

$$\mathcal{L}\{t^n F(t)\} = (-1)^n f^n(s)$$

Laplace transform of a function division by t

If $\mathcal{L}\{F(t)\} = f(s)$,

then

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(u).du$$

Laplace transform of integrals

If $\mathcal{L}\{F(t)\} = f(s)$,

① $\mathcal{L}\left\{\int_0^t F(u).du\right\} = \frac{f(s)}{s}$

② $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}.du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$

(17) Inverse laplace transform

If $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse laplace transform of $f(s)$ and it is denoted by

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$