Chapter-5

mid Bisection Method

*Bisection Method: It is also called binary chopping interval halving or Bolzano's method. It is the incremental search method in which the interval is always divided in half.

Final 2018-19

* Algorithm for bisection method:

Step 1: choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)f(x_u) < 0$

Step 2: An estimate of the root x_r is determined by

$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Make the following evaluations to determine in which subinterval the

root lies!

a) If $f(x_l)f(x_r) < 0$, the root lies in the ~~upper~~ lower subinterval. Therefore, set $x_u = x_r$ and return to step 2

b) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval Therefore, set $x_l = x_r$, and return to step 2

c) If $f(x_l)f(x_r) = 0$, the root equals x, terminate the computation.

Example: 5.3 $f(c) = \frac{9.81(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right) - 40$

Solution:

Step-1: The first step in bisection is to give two values of the unknown that gives values for $f(c)$ with different signs. We can see that the function changes sign between values 12 and 16. Therefore, initial estimate of the root x_r lies at the midpoint of the interval.

Step-2: $x_r = \frac{12+16}{2} = 14$ (iteration 1)

Step-3: Now we compute the product of the function value at the lower bound and midpoint

$$f(12)\,f(14) = \left\{\frac{9.81(68.1)}{12}\left(1 - e^{-(12/68.1)10}\right) - 40\right\}\left\{\frac{9.81(68.1)}{14}\left(1 - e^{-(14/68.1)10}\right) - 40\right\}$$

$$= 6.114(1.611) = 9.850$$

which is greater than zero and hence no sign change between lower and midpoint. So, the root must be located between 14 and 16.

Let, $x_l = 12$, $x_u = 16$

Step 1: $f(12)f(16) = 6.0669 \times (-2.2687)$

$= -13.76$

$\therefore f(x_l)\, f(x_u) < 0$; $x_u = x_r$

x_l এবং x_u এমনভাবে নিতে হবে যেন একটির +ve মান আরেকটির −ve মান আসবে

$f(x_l)\, f(x_u) < 0$ হবে always

$$x_r = \frac{14+16}{2} = 15 \text{ (iteration 2)}$$

This process can be repeated to obtain refined estimates. For example,

$$f(14)\, f(15) = 1.611(-0.384) = -0.619$$

Therefore, the root is between 14 and 15.

$$x_r = \frac{14+15}{2} = 14.5 \text{ (iteration 3)}$$

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100\%$$

Continuing this process until the appropriate error $\varepsilon_a <= 0.422$

Iteration	x_l	x_u	x_r	$\varepsilon_a(\%)$
1	12	16	14	
2	14	16	15	6.677
3	14	15	14.5	3.448
4	14.5	15	14.75	1.695
5	14.75	15	14.875	0.840
6	14.75	14.875	14.8125	0.422

repeat হচ্ছে তাই এটা বাদ

Thus after 6 iterations, ε_a falls below 0.422%

* What is numerical method used in engineering?

Ans: Numerical methods are techniques and algorithms used to solve mathematical problems ~~and simulate p~~ step by step using computers.

* Why do engineers need to know numerical method?

Ans: Engineers need to know numerical methods for several important reasons:

⇒ Complex problem solving: Many engineering problems are highly complex and can't be solved analytically using traditional mathematic techniques. Numerical methods provide solutio

⇒ Real-world application: Engineering is all about real-world problem and designing practical solutions. Numerical methods allow physical ~~solutions~~ phenomena to model and simulate.

⇒ Validation and verification: Engineers use numerical simulations to ~~so~~ validate and verify their designs.

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⇒ Handling data

⇒ Optimization

⇒ Dynamic system

⇒ Uncertainty and Probability.

⇒ Time and cost savings.

⇒ Innovation and Research.

⇒ Global Challenges.

*What are the real life applications of N.M ?

Ans: Here are some example of numerical method in real life applications:

1. Aerospace Engineering.
2. civil Engineering.
3. Mechanical Engineering.
4. Electrical Engineering.
5. computer science and data science.
6. Finance and Economics.
7. Environmental science.

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8. Biomedical Engineering.
9. Material science.
10. Geophysics and earth science.

*What is numerical Methods ~~in~~ used in engineering

Ans: Numerical methods are techniques and algorithms used to solve mathematical problems and simulate step wise using computer programming ~~tog~~ language. It is the fast solution for mathematical problems. such problems be in any field in engineering.

1. Solving Linear / Non-linear equations and finding the real roots
2. Fit some points to curve and simple solution

It is actually used to appropriate the solution of complicated problems. They are suitable for the use with computers. It can ~~be~~ solve a problem in an arithmetic sense. They can be converted into computer program. Sometimes you need to solve equations or a set of equations which are solvable by hand, in

such cases methods, such as gauss elimination methods are used. So, we can say that, a numerical method is a complete and definite set of procedures for the solution problem, together with computable error estimates.

*Difference between Determinants and Matrix:

Ans:

Determinant	Matrix
1. Determinant is a set of numbers covered by two bars.	1. Matrix is a set o numbers covered by two brackets.
2. It is necessary that number of rows are equal to number of column in determinants.	2. It is necessary that number of rows will be equal to number of colums in matrix.
3. Determinants can be used for calculating the value of x, y, z with Cramer's rule and to for calculating inverse matrix	3. Matrix can be used for adding, subtracting and multiplying the coefficients

Determinant	Matrix
4. Determinant is a unique number related to that matrix	4. Matrix is a group of numbers.
5. It has fixed value	5. It has no value.

* Mid Cramer's Rule: Best suited to small number of equation.

Example-9.3) Use cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01 \ (b_1)$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67 \ (b_2)$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44 \ (b_3)$$

Solution:

The Determinant D can be written as

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} \rightarrow \left[D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \right]$$

$$\left[D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right]$$

The minors are,

$$A_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 0.5 - 0.57 = -0.07$$

$$A_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.25 - 0.09 = 0.16$$

$$A_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.15 - 0.1 = 0.05$$

$$\left[D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{2} \\ a_{31} & a_{3} \end{vmatrix} \right.$$

These minors can be used to evaluate the determinant.

$$D = 0.3(-0.07) - 0.52 \times 0.06 + 1(0.05)$$

$$= -0.0022$$

Now applying the equation of the cramer's rule we get,

$$\left[x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \right]$$

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{D} = \frac{0.3278}{-0.0022} = -149$$

Using calculator:

[Mode → Matrix → MatA → select order → Give value → vale → = → Press AC for save → shift 4 → select "det" → Again shift 4 → select MatA → close first bracket and press "=".]

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}$$

$$= \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

$$= \frac{-0.04356}{-0.0022} = 19.8 \text{ (Ans)}$$

* When cramer's Rule becomes impractical?

Ans: For more than three equations, cramer's rule becomes impractical because, as the number of equations increases, the determinants are time consuming to evaluate by hand.

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* Gauss Elimination Method :

Exm + Lab এ থাকা 2D Array দিয়ে develop করতে দিবে।

1. Have to make triangular matrix (upper triangular)
2. Have back substitution.

Steps :

1. Forward Elimination → equation এ সংগুলো 0 বানাতে হবে x_1 থেকে শু
2. Back substitution → x_3 থেকে x_1 এর মান বের করা।

Example 9.5 Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{—— (I)}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{—— (II)}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{—— (III)}$$

Solution: The first part of the procedure is forward elimination. Now,

to remove x_1 from (II) and (III),

(II) $- (I) \times \frac{0.1}{3}$ and (III) $- (I) \times \frac{0.3}{3}$ ⇒

After this operation the set of the equation is,

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$ —— (IV)

$$\left(0.1 - 3\times\frac{0.1}{3}\right)x_1 + \left(7 + 0.1\times\frac{0.1}{3}\right)x_2 - \left(0.3 - 0.2\times\frac{0.1}{3}\right)x_3 = -19.3 - 7.85\times\frac{0.1}{3}$$

$$\Rightarrow 0 + 7.00333x_2 - 0.293333x_3 = -19.5617 \quad \text{—— (V)}$$

$$\cancel{\left(0.1 - 0.3\times\frac{0.1}{3}\right)x_1 + \left(7 + 0.2\times\frac{0.1}{3}\right)x_2 - \left(0.3 - 10\times\frac{0.1}{3}\right)}$$

$$\left(0.3 - 3\times\frac{0.3}{3}\right)x_1 - \left(0.2 + 0.1\times\frac{0.3}{3}\right)x_2 + \left(10 + 0.2\times\frac{0.3}{3}\right)x_3 = 7.14 - 7.85\times\frac{0.3}{3}$$

$$\Rightarrow -0.190000x_2 + 10.0200x_3 = 70.6150 \quad \text{—— (VI)}$$

To complete forward elimination, x_2 must be removed from (VI). To remove x_2

$$(VI) - (V)\left(\frac{-0.190000}{7.00333}\right) \Rightarrow$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{—— (VII)}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \quad \text{—— (VIII)}$$

$$\left(-0.190000 - \left(7.00333 \times \frac{-0.190000}{7.00333}\right)\right) x_2$$

$$+ \left\{10.0200 + \left(0.293333 \times \frac{-0.190000}{7.00333}\right)\right\}$$

$$= 70.6150 + 19.5617 \times \frac{-0.3}{7.0}$$

$$\Rightarrow 10.0120\, x_3 = 70.0843 \quad \text{——— (IX)}$$

We can now solve these equations by back substitution. From eqn (IX) we get,

$$x_3 = \frac{70.0843}{10.0120} = 7.0000 \quad \text{——— (X)}$$

This reseut can be back substituted into eqn (VIII)

$$7.00333 x_2 - (0.293333 \times 7.0000) = -19.5617$$

$$\Rightarrow x_2 = \frac{-19.5617 + 0.293333 \times 7.0000}{7.00333}$$

$$= -2.50000 \quad \text{——— (XI)}$$

Finally eqn (x) and (xi) can be substituted in eqn (iv),

$$3x_1 - 0.1\times(-2.50000) - (0.2\times 7.0000) = 7.8$$

$$\Rightarrow x_1 = \frac{7.85 - 0.25 + 1.4}{3}$$

$$= 3.00000$$

The results are identical to the exact solution of $x_1 = 3$, $x_2 = -2.5$, $x_3 = 7$ (Ans)

Mid

* Gauss-Jordan: The Gauss-Jordan method is a variation of Gauss Elimination.

* Gauss Elimination using Matrix:

Example: 9.5

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Matrix form:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} & c_2 & c_3 & \end{array} \\ \begin{bmatrix} 1 & -0.0333 & -0.06667 & 2.6167 \\ 0 & 6.9 & -0.4 & -19.4 \\ 0 & -0.5 & 9.7 & 71.1 \end{bmatrix} \end{array} \quad \begin{bmatrix} r_1' = \frac{r_1}{3} \\ r_2' = r_2 - 0.1 \\ r_3' = r_3 - 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.0333 & -6.873\times10^{-3} & 2.6167 \\ 0 & 6.9 & -0.041 & -19.4 \\ 0 & -0.07246 & 1 & 71.1 \end{bmatrix} \quad \begin{bmatrix} c_3' = \frac{c_3}{9.7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4.826\times10^{-3} & -6.873\times10^{-3} & 2.6167 \\ 0 & 1 & -0.041 & -19.4 \\ 0 & 0 & 1.07246 & 71.17246 \end{bmatrix} \quad \begin{bmatrix} r_3'' = r_3 + 0.07246 \\ r_3' = r_3 + 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.0333 & -0.0667 & 2.6167 \\ 0 & 6.9 & -0.4 & -19.4 \\ 0 & 0 & 10.2 & 71.1 \end{bmatrix} \quad \begin{bmatrix} r_3' = r_3 + 0.5 \end{bmatrix}$$

Date: ...../...../.....

$\cancel{x_3 =}$

$1.07246x_3 = 71.17246$

$\therefore x_3 = 66.36$

$x_2 - 0.041x_3 = -19.4$

$x_2 = -19.4 + 0.041 \times 3$

$=$

$10.2x_3 = 71.6$

$\therefore x_3 = 7.0196$

$6.9x_2 - 0.4x_3 = -19.4$

$6.9x_2 = -19.4 + 0.4x_3$

$x_2 = \dfrac{-19.4 + 0.4 \times 7.0196}{6.9}$

$= -2.5$

$x_1 = 3$ Ans

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arthrofen®
misoprostol-diclofenac sodium

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* Example 9.12 Use Gaus-Jordan technique to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution:

First we have to express the coefficients and the right hand side as an argumented matrix:

$$\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array}$$

$$\left[\begin{array}{cccc} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] \begin{array}{l} \rightarrow r_1 \\ \rightarrow r_2 \\ \rightarrow r_3 \end{array}$$

Then normalize the first row by dividing it by 3,

$$\left[\begin{array}{cccc} 1 & -0.03333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right]$$

$$r_1' = \frac{r_1}{3}$$

Then,

$$\begin{array}{c} r_1 \rightarrow \\ r_2 \rightarrow \\ r_3 \rightarrow \end{array} \begin{bmatrix} 1 & -0.03333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.29333 & -19.3617 \\ 0 & -0.19000 & 10.0200 & 70.6150 \end{bmatrix}$$

$$\begin{bmatrix} r_2' = r_2 - r_1 \times 0.1 \\ r_3' = r_3 - r_1 \times 0.3 \end{bmatrix}$$

Next,

$$\begin{array}{c} r_1 \rightarrow \\ r_2 \rightarrow \\ r_3 \rightarrow \end{array} \begin{bmatrix} 1 & -0.03333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.19000 & 10.0200 & 70.6150 \end{bmatrix} \quad r_2' = \frac{r_2}{7.0033}$$

Reduction of the x_2 terms from the first and third equation,

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.5236 \\ 0 & 1 & -0.0418848 & -2.793 \\ 0 & 0 & 10.01200 & 70.084 \end{bmatrix} \quad \begin{array}{l} r_1' = r_1 + 0.0333 r \\ r_3' = r_3 + 0.1900 r \end{array}$$

The third row is then normalized by dividing it by 10.01200.

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix} \quad r_3' = \frac{r_3}{10.01200}$$

Finally, ~~reedu~~ x_3 terms can be reduced from the first and second equations to give

$$\begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 + 0.0680 r_3 \\ r_2' = r_2 + 0.0418 r_3 \end{matrix}$$

(Ans)

[Step by Step: (3x3)

i) কর্ণের প্রথম উপাদানকে 1 বানাও (1st row 1st column)

ii) ১ম column এর বাকি উপাদানকে 0 বানাও।

iii) কর্ণের ২য় উপাদানকে 1 বানাও (2nd row 2nd column)

iv) ২য় column এর বাকি উপাদানগুলোকে 0 বানাও

v) কর্ণের ৩য় উপাদানকে 1 বানাও (3rd row 3rd column)

vi) ৩য় column এর বাকী উপাদানগুলোকে 0 বানাও.

* Difference between Gauss elimination and Gauss Jordan: (16-17) (12-13)

Gauss elimination	Gauss Jordan
1. In this method just the subsequent ones are eliminated.	1. In this method when an unknown is eliminated, ~~all~~ it ~~other~~ is eliminated ~~from~~ all other equations.
2. In Gauss elimination the elimination steps results in a triangular matrix.	2. In Gauss Jordan method the elimination steps result in an identity matrix.
3. Back substitution is necessary for the solution	3. It is not necessary to employ back substitution to obtain the solution.
4. It doesn't seem easier but require about 50% fewer operation than Gauss Jordan	4. It seems easier but requires 50% more operations than Gauss elimination
5. For large system, Gauss elimination is not preferred	5. For large system, Gauss Jordan is preferred to Gauss Elimination method.

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Triangularisation Method / LU Decomposition / Factorization Process / Cholesky's Process:

$$AX = B \Rightarrow \underbrace{LU}_{\text{lower, upper triangular}}X = \underbrace{B}_{\text{constant}} \Rightarrow LY = B \text{ then } UX = Y$$

A → coefficient, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

[একটা matrix কে তার lower & upper triangulate matrix multiplication হিসেবে দেখানো যায়.

Example: (11-12)

Apply the Cholesky's process to locate the root of the following system:

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + 5x_3 &= -3 \\ 3x_1 + 2x_2 - 3x_3 &= 6 \end{aligned}\right\} \text{constant}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} \cdot u_{11} & l_{31} u_{12} + l_{32} u_{22} & l_{31} u_{13} + l_{32} u_{23} + u_{33} \end{bmatrix}$$

Comparing the values,

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = -1$, $l_{21} u_{11} = 2$, so that $l_{21} = 2$

$l_{21} u_{12} + u_{22} = 3$, $u_{22} = 1$, $l_{21} u_{13} + u_{23} = 5$, $u_{23} = 7$

$\because u_{11} = 1$ $\quad l_{31} u_{11} = 3$, $l_{31} = 3$

Similarly we can find out another unknown.

$$AX = B \Rightarrow LUX = B$$

i.e $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} X = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Let, $UX = Y$, $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$y_1 = 2$

$2y_1 + y_2 = -3 \Rightarrow -4 - 3 = y_2 \Rightarrow y_2 = -7$

$3y_1 - y_2 + y_3 = 6 \Rightarrow 3 \times 2 + 7 + y_3 = 6 \Rightarrow y_3 = -7$

$UX = Y$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 2$ — (I)

$x_2 + 7x_3 = -7$ — (II)

$7x_3 = -7$

$x_3 = -1$

Putting values of x_3 in (II),

$x_2 + 7 \times (-1) = -7$

$\Rightarrow x_2 - 7 = -7 \Rightarrow x_2 = 0$

$x_1 + 0 + 1 = 2$

$\Rightarrow x_1 = 1$

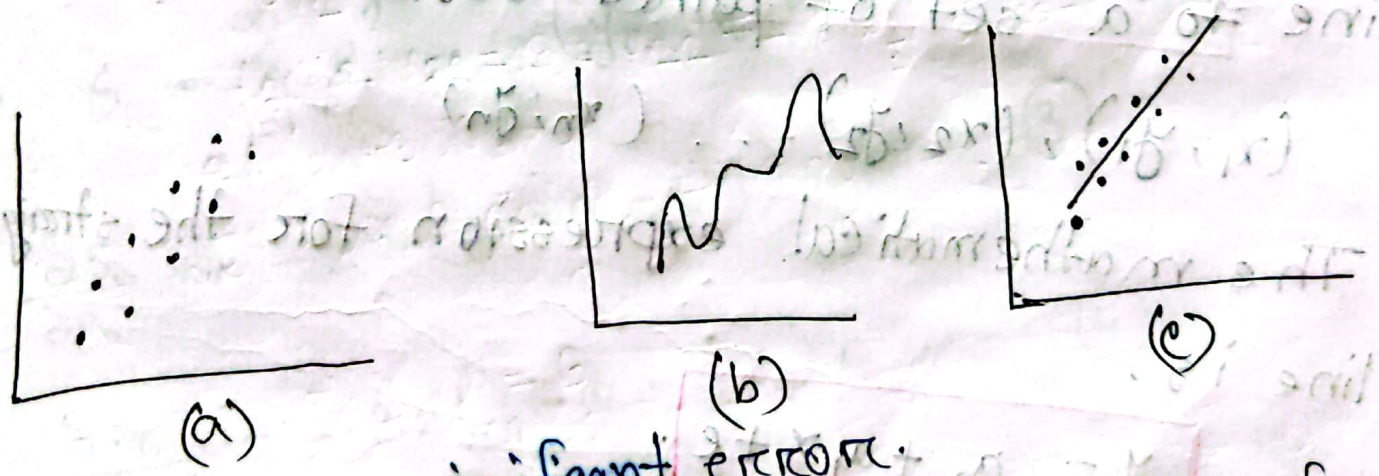
$x_1 = 1,\ x_2 = 0,\ x_3 = -1$ (Ans)

[Gauss Elimination & Gauss Jordan - code করে নিতে হবে।]

Least-Square Regression

* 2 parallel lines - no solution
* 1 straight line on another - infinite solution
* straight lines very close (কাছাকাছি) - ill condition

Linear Regression



a) Data Exhibiting significant error.

b) Polynomial fit oscillating beyond the range of the data.

c) More satisfactory result using the least-square fit

* Linear Regression:

Linear regression finds a linear function that most nearly passes through the given data points - the regression (function) line best fits the data.

It is a way of finding a stright line that best fits the data, called the line best fit.

The simplest example of a least square approximation is fitting a straight line to a set of paired observations:

$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$

The mathematical expression for the straight line is.

$$\boxed{y = a_0 + a_1 x + e}$$

where a_0 and a_1 are coefficients representing the intercept and the slope respectively.

e is the error or residual between the model and observation.

$$e = y - a_0 - a_1 x \quad \text{—— (1)}$$

* Criteria for best fit: imp

For fitting a best line through the data would be to minimize the sum of the residual errors for all available data.

i.e.

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i) \quad \text{—— (1)}$$

where, n = total no. of points.

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i| \quad \text{—— (2)}$$

Let, $$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \quad \text{—— (3)}$$

$$\frac{\delta S_r}{\delta a_0} = -2\sum (y_i - a_0 - a_1 x_i) \quad \text{—— (4)}$$

$$\frac{\delta S_r}{\delta a_1} = -2\sum (y_i - a_0 - a_1 x_i) x_i \quad \text{—— (5)}$$

(3) $\Rightarrow$ $$0 = \sum y_i - \sum a_0 - \sum a_1 x_i \quad \text{—— (6)}$$

(4) $\Rightarrow$ $$0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2$$

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(5) $\Rightarrow na_0 + (\Sigma x_i)a_1 = \Sigma y_i$ —— (7)

(6) $\Rightarrow (\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 = \Sigma x_i y_i$

From (6) & (7),

$$\boxed{a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}}$$ → math এর জন্য imp.

(7) $\Rightarrow na_0 + (\Sigma x_i)a_1 = \Sigma y_i$

$\Rightarrow na_0 = \Sigma y_i - (\Sigma x_i)a_1$

$\Rightarrow a_0 = \frac{\Sigma y_i}{n} - \frac{(\Sigma x_i)a_1}{n}$

$\Rightarrow \boxed{a_0 = \bar{y} - a_1\bar{x}}$ → math এর জন্য imp

Example 17.1 Fit a straight line to the x and y values in the first two colums of the table. computations for an error analysis of the linear fit.

x_i	y_i	$(y_i - \bar{y})$	$(y_i - a_0 - a_1 x_i)^2$
1	0.5	8.5765	0.1687
2	2.5	0.8622	0.5625
3	2.0	2.0408	0.3473
4	4.0	0.3265	0.3265
5	3.5	0.0051	0.5896
6	6.0	6.6122	0.7972
7	5.5	4.2908	0.1993
Σ	24.0	22.7143	2.9911

Solution:

The following quantities can be computed.

$n = 7$, $\Sigma x_i y_i = 119.5$ $\Sigma x_i^2 = 140$

$\Sigma x_i = 28$ $\bar{x} = \frac{28}{7} = 4$

$\Sigma y_i = 24.$ $\bar{y} = \frac{24}{7} = 3.428571$

Now,

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{7(119.5) - 28(24)}{7(140) - (28)^2}$$

$$= 0.8392857$$

$$a_0 = \bar{y} - a_1\bar{x} = 3.428571 - 0.8392857 \times 4$$

$$= 0.07142857$$

Therefore, the least square fit is,

$$y = a_0 + a_1 x$$

$$y = 0.07142857 + 0.8392857x$$

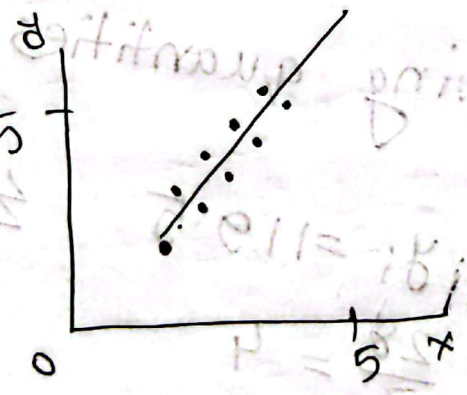

Fig - More satisfactory result using the least square fit.

Polynomial Regression

The process of going back to an earlier or less advanced form or state.

fig: Ill suited data for linear regression

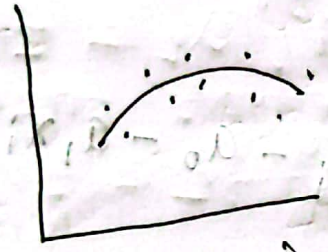

fig: Parabola is preferable

The least square procedure can be readily extended to fit the data to a higher order polynomial. for example, Let's suppose that we fit a second order polynomial or quadratic:

$$y = a_0 + a_1 x + a_2 x^2 + e$$

For this case the sum of the squares of the residuals. is,

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 \quad \text{——— (1)}$$

Taking the derivative of eqn (1) w.r.t. each of the unknown coefficients of the polynomial, as in,

$$\frac{\delta S_r}{\delta a_0} = -2\sum(y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

$$\frac{\delta S_r}{\delta a_1} = -2\sum x_i(y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{\delta S_r}{\delta a_2} = -2\sum x_i^2(y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

These equations can be set equal to zero and rearranged to develop the following set of normal equations:

$$(n)a_0 + (\sum x_i)a_1 + (\sum x_i^2)a_2 = \sum y_i$$

$$(\sum x_i)a_0 + (\sum x_i^2)a_1 + (\sum x_i^3)a_2 = \sum x_i y_i$$

$$(\sum x_i^2)a_0 + (\sum x_i^3)a_1 + (\sum x_i^4)a_2 = \sum x_i^2 y_i$$

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Standard error is formulated as:

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

Example 17.5 Fit a second-order polynomial to the data in the first two columns of table

x_i	y_i	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08158
3	27.2	3.12	0.80491
4	40.9	239.22	0.61951
5	61.1	1272.11	0.09439
Σ	152.6	2513.39	3.74657

Solution:

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	2.1	0	0	0	0	0
1	7.7	1	1	1	7.7	7.7
2	13.6	4	8	16	27.2	54.4
3	27.2	9	27	81	81.6	244.8
4	40.9	16	64	256	163.6	654.4
5	61.1	25	125	625	305.5	1527.5
$\Sigma =$ 15	152.6	55	225	979	585.6	2488.8

Order, $m = 2$, $\sum x_i = 15$, $\sum x_i^4 = 979$

$n = 6$, $\sum y_i = 152.6$, $\sum x_i y_i = 585.6$

$\bar{x} = 2.5$, $\sum x_i^2 = 55$, $\sum x_i^2 y_i = 2488.8$

$\bar{y} = 25.433$ $\sum x_i^3 = 225$

We know that,

$$n a_0 + (\sum x_i) a_1 + \{\sum (x_i)^2\} a_2 = \sum y_i$$

$$(\sum x_i) a_0 + (\sum x_i^2) a_1 + (\sum x_i^3) a_2 = \sum x_i y_i$$

$$(\sum x_i^2) a_0 + (\sum x_i^3) a_1 + (\sum x_i^4) a_2 = \sum x_i^2 y_i$$

Therefore simultaneous linear equations are,

$$6a_0 + 15a_1 + 55a_2 = 152.6 \quad -(1)$$

$$15a_0 + 55a_1 + 225a_2 = 585.6 \quad -(2)$$

$$55a_0 + 225a_1 + 979a_2 = 2488.8 \quad -(3)$$

$$\begin{bmatrix} 6 & 15 & 55 & 152 \\ 15 & 55 & 225 & 116.5 \\ 55 & 225 & 979 & 306.9 \end{bmatrix}$$

$$r_1' = \frac{r_1}{6}$$

$$\begin{array}{c} r_1 \to \\ r_2 \to \\ r_3 \to \end{array} \begin{bmatrix} 1 & 2.5 & 9.1667 & 25.4333 \\ 15 & 55 & 225 & 116.5 \\ 55 & 225 & 979 & 306.9 \end{bmatrix}$$

$$r_2' = r_2 - r_1 \times 15$$

$$r_3' = r_3 - r_1 \times 55$$

$$\begin{bmatrix} 1 & 2.5 & 9.1667 & 25.4333 \\ 0 & 17.5 & 87.4995 & -264.9997 \\ 0 & 87.5 & 474.8315 & -1091.9315 \end{bmatrix}$$

Next,

$$r_2' = \frac{r_2}{17.5}$$

$$\begin{bmatrix} 1 & 2.5 & 9.1667 & 25.4333 \\ 0 & 1 & 4.99997 & -15.1428 \\ 0 & 87.5 & 474.8315 & -1091.9315 \end{bmatrix}$$

Reduction of a_1 terms from first and third equation.

$$\begin{bmatrix} 1 & 0 & -3.3832 & 63.2903 \\ 0 & 1 & 4.99997 & -15.1428 \\ 0 & 0 & 37.3341 & 233.0635 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 - 2.5 \times r_2 \\ r_3' = r_3 - 87.5 \times r_2 \end{matrix}$$

Then,

$$\begin{bmatrix} 1 & 0 & -3.3832 & 63.2903 \\ 0 & 1 & 4.99997 & -15.1428 \\ 0 & 0 & 1 & 6.2426 \end{bmatrix} \quad r_3' = \frac{r_3}{37.3341}$$

Finally reducing a_2 terms from equation.

we get,

$$\begin{bmatrix} 1 & 0 & 0 & 84.4103 \\ 0 & 1 & 0 & -46.3539 \\ 0 & 0 & 1 & 6.2426 \end{bmatrix} \quad \begin{matrix} r_1' = r_1 + 3.3832 \times r_3 \\ r_2' = r_2 - 4.99997 \times r_3 \end{matrix}$$

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using Gauss elimination

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix}$$

$\cancel{6a_0 + 15}$

(2) $- (1) \times \frac{15}{6}$ and (3) $- (1) \times \frac{55}{6} \Rightarrow$

$$6a_0 + 15a_1 + 55a_2 = 152.6 \quad \text{— (4)}$$

$$\left(15 - 6 \times \frac{15}{6}\right) a_0 + \left(55 - 15 \times \frac{15}{6}\right) a_1 + \left(225 - 55 \times \frac{15}{6}\right) a_2 = 585.6 - 152.6$$

$$\Rightarrow 17.5 a_1 + 87.5 a_2 = 204.1 \quad \text{— (5)}$$

$$\left(55 - 6 \times \frac{55}{6}\right) a_0 + \left(225 - 15 \times \frac{55}{6}\right) a_1 + \left(979 - 55 \times \frac{55}{6}\right) a_2 = 2488.8 - 152.6 \times \frac{55}{6}$$

$$\Rightarrow 87.5 a_1 - 474.8333 a_2 = 1089.98667 \quad \text{— (6)}$$

To remove a_1 from (6) $\Rightarrow$

(6) $- (5) \times \frac{87.5}{17.5}$

$$6a_0 + 15a_1 + 55a_2 = 152.6 \quad \text{— (7)}$$

$$17.5 a_1 + 87.5 a_2 = 204.1 \quad \text{— (8)}$$

$$\left(-474.8333 - 87.5 \times \frac{87.5}{17.5}\right) a_2 = 1089.96667 - 204.1 \times \frac{87.5}{17.5}$$

$$\Rightarrow \quad -912.3333 a_2 = 69.46667$$

$$\Rightarrow a_2 = 0.07614$$

Gauss

math মিলে নাই দেখতে হবে।

(16-17)
(15-16)
(12-13)

False-Position Method

False Position method joins $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line and x axis ~~request~~ represents an improved estimates of the root.

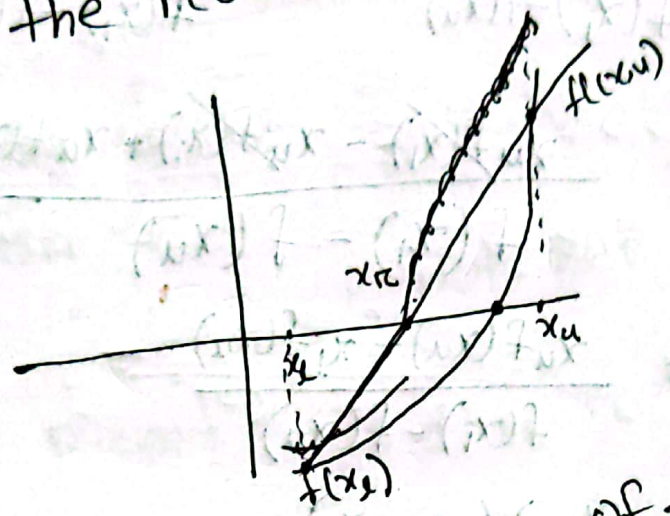

fig: Graphical depiction of the method of false position.

The intersection of the straight line with the x-axis can be estimated as,

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$\Rightarrow f(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

$$\Rightarrow f(x_l)x_r - f(x_l)x_u = f(x_u)x_r - f(x_u)x_l$$

$$\Rightarrow f(x_l)x_r - f(x_u)x_r = f(x_l)x_u - f(x_u)x_l$$

$$\Rightarrow x_r[f(x_l) - f(x_u)] = x_u f(x_l) - x_l f(x_u)$$

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$$\Rightarrow x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$= \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$= x_u + \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$= x_u + \frac{\cancel{x_u f(x_l)} - \cancel{x_u f(x_l)} + x_u f(x_u) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$= x_u + \frac{x_u f(x_u) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$= x_u + \frac{f(x_u)(x_u - x_l)}{f(x_l) - f(x_u)}$$

$$= \boxed{x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}}$$ equation

*** Generally false position এর error আগে কমে যায়। কিন্তু সবক্ষেত্রে False position, bisection method এর থেকে better না। It depends on the type of equation

*

* A case where bisection is preferable to False position: / Pitfall of false position:

Although False-position method would seem to be the bracketing method of preference, there are certain cases it performs poorly

Example 5.6 A case where Bisection is Preferable to False Position.

Use bisection and false position to locate the root of $f(x) = x^{10} - 1$ between $x = 0$ and $x = 1.3$

Solution: Using Bisection, the result can be summerized as,

$$x_r = \frac{x_l + x_u}{2}$$

Iteration	x_l	x_u	$x_r = \frac{x_l + x_u}{2}$	$\epsilon_a = \left[\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right] \times 100\%$
1	0	1.3	0.65	100.0
2	0.65	1.3	0.975	33.3
3	0.975	1.3	1.1375	14.3
4	0.975	1.1375	1.05625	7.7
5	0.975	1.05625	1.015625	4.0

Using False method

Iteration	x_l	x_u	$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$	$\epsilon_a = \left[\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right] \times$
1	0	1.3	0.0943	
2	0.09430	1.3	0.18176	48.1
3	0.18176	1.3	0.26287	30.9
4	0.26287	1.3	0.33811	22.3
5	0.33811	1.3	0.40788	17.1

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After 5 iterations, appropriate error from bisection is 4.0 and from false position is 17.1

This proves that, there are cases where Bisection is preferable than false method.

Example 5.5 Use the false position method to determine the root of the ~~same~~ equation, $f(c) = \frac{9.81(68.1)}{c}(1 - e^{-(c/68.1)10}) - 40$

Solution:

Initiate the computation with guesses of $x_l = 12$, and $x_u = 16$

First Iteration:

$x_l = 12$, $f(x_l) = 6.1139$

$x_u = 16$, $f(x_u) = -2.2303$

we know,

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$$= 16 - \frac{(-2.2303) \times (12 - 16)}{6.1139 + 2.2303}$$

$$= \cancel{+4.3} \; 14.930$$

approximate error, $\epsilon_a = \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}}$ ~~×100%~~ ×100%. [True value was $14.8011 = x_r^{old}$.]

$$= \frac{14.930 - 14.8011}{14.930} \times 100\%$$

$$= 0.88\%$$

Second iteration:

$f(x_r) = -0.2498$

$f(x_1)\, f(x_r) = 6.1139 \times -0.2498 = -1.5273$

We know that, if $f(x_1)\, f(x_r) < 0$ then the root lies in lower subinterval. Therefore x_r becomes upper limit and x_1 is lower limit.

$f(x_1) = 6.1139$

$x_1 = 12$ $\quad f(x_u) = -0.2498$

$x_u = 14.9309$

$$x_r = x_u - \frac{f(x_u)(x_1 - x_u)}{f(x_1) - f(x_u)} = 14.9309 - \frac{(-0.2498)(12-14.9309)}{6.1139 + 0.2498}$$

$$= 14.8151$$

$$\epsilon_a = \left[\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right] \times 100\% = \left[\frac{14.8151 - 14.930}{14.8151}\right] 100\%$$

$$= 0.78$$

Milne's Predictor - Corrector Method

Milne's Method

Suppose, $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0) = 2$

$y(0.5) = 2.636$, $y(1.0) = 3.595$, $y(1.5) = 4.968$

Now, find $y(2)$ by Milne's method.

Solution: Milne's predictor formula is,

$$y_{(n+1)p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \quad \text{--- (1)}$$

Predictor

Putting $n=3$, we get,

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \quad \text{--- (2)}$$

We're given that,

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$

$h = 0.5 - 0 = 0.5$

The given differential equation, $y' = \frac{1}{2}(x+y)$

From above eqn, we'll calculate y'_1, y'_2, y'_3

$y'_1 = \frac{1}{2}(x_1 + y_1) = \frac{1}{2}(0.5 + 2.636) = 1.568$

$y_2' = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1.0 + 3.595) = 2.2975$

$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5 + 4.968) = 3.324$

Substituting the values in ②, we get,

$$y_{4P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 2 + \frac{4\times 0.5}{3}[2\times 1.568 - 2.2975 + 2\times 3.324]$$

$$= 6.871$$

So, our predicted value is 6.871

Now, we will connect it to get actual value by Milne's corrector formula, which is,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}'] \quad \text{—— ③}$$

By putting $n=3$ we get,

$$y_{4c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4'] \quad \text{—— ④}$$

→ corrector

[প্রশ্নে y(2) বের করতে বলা হয়েছে]

Now,

$$y_4' = \frac{1}{2}(x_4+y_4) = \frac{1}{2}(2 + 6.871) = 4.4355$$

Now putting required value in ④ we get,

$$y_{4c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 3.595 + \frac{0.5}{3}[2.2975 + 4\times 3.324 + 4.4355]$$

$$= 6.8731 \qquad y(2) = 6.8731$$

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*Why do we use predictor-corrector method?

Ans: Predictor-corrector methods counter this problem by using the information gained from previous n-steps to predict what the state of the system will be at the end of the next step. They can use this predict values to evaluate the derivative at the end of the step.

*How many basic values are required for Milne's predictor-corrector method?

Ans: In Milne's method, four prior values are needed for finding value of y at. For finding predictor formula Milne neglect, fourth and higher order differences and for corrector formula Milne uses Simpson's 1/3 formula.

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Lagrange Interpolation Formula:

It is a way to find a polynomial called Lagrange's polynomial. It is a reformulation of the Newton polynomial that avoids the computation of divided differences. It can be represented by,

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where, $L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$

Lagrange interpolation formula for nth order

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) + \cdots + \frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})} f(x_n)$$

*Advantage of Lagrange Interpolation:

⇒ This formula is used to find the value of the function even when the argument are not equally spaced.

⇒ This formula is used to find the value of independent variable x corresponding to a given

value of a function.

*** If the values of x are at equidistant or not at equidistant, we use Lagrange interpolation formula.

Let $y=f(x)$ be a function such that $f(x)$ takes the values y_0, y_1, y_2, y_n corresponding to, $x = x_0, x_1, x_2, x_3 \ldots x_n$ that is,

$$y_i = f(x_i) \; ; i = 0, 1, 2, 3 \ldots n.$$

Now, there are $n+1$ paired values (x_i, y_i) $i=0, 1, 2, 3, \ldots n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x. The Lagrange formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x-x_n)} y_1$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

For first order:

$$f_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

second order:

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example: 18.6 Use lagrange interpolation polynomial of the first and second order to evaluate ln 2 ~~on~~.

x	1	4	6
$f(x)$	0	1.386294	1.791760

$y =$

Solution:

The first order polynomial:

$$f(x) = \frac{(x-4)}{(1-4)} \times 0 + \frac{(x-1)}{(4-1)} \times 1.386294$$

→ 1st order

→ y এর 1st value

(x- এর সাথে 1st যেই মা থাকবে x এর সা বাদে বাকি গুলো বসবে)

→ 1st number

$$f_1(2) = \frac{(2-1)}{3} \times 1.386294$$

$$= \frac{1 \times -4}{-6} \times 1.386294$$

$$= 0.462098$$

Second order,

$$f_2(x) = \frac{(x-4)(x-6)}{(1-4)(1-6)} \times 0 + \frac{(x-1)(x-6)}{(4-1)(4-6)} \times 1.386294 + \frac{(x-1)(x-4)}{(6-1)(6-4)} \times 1.791760$$

$$f_2(2) = \frac{(2-1)(2-6)}{3 \times -2} \times 1.386294 + \frac{(2-1)(2-4)}{5 \times 2} \times 1.791760$$

$$= 0.565844$$

Example: Using Lagrange interpolation formula find $y(10)$ from following:

x	5	6	9	11
y	12	13	14	16

Solution: Here intervals are not equal.

By Lagrange interpolation formula, we have.

$$x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11$$

$$y_0 = 12,\ y_1 = 13,\ y_2 = 14.\ y_3 = 16$$

we know,

$$f(x) = y = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{-24} \times 12 + \frac{(10-5)(10-9)(10-11)}{15} \times 13 +$$

$$\frac{(10-5)(10-6)(10-11)}{-24} \times 14 + \frac{(10-5)(10-6)(10-9)}{60} \times 16$$

$$= 14.6683 \text{ (Ans)}$$

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*Differentiation Based on Lagrange interpolation formula:

Example 5.10 Using Lagrange interpolation formula find $f'(0.12)$

x	$f(x)$
0.05	0.05
0.10	0.0999
0.20	0.1987
0.26	0.2571

Solution: The intervals are unequal, Using Lagrange interpolation formula, we have,

$x_0 = 0.05$, $x_1 = 0.10$, $x_2 = 0.20$, $x_3 = 0.26$

$y_0 = 0.05$, $y_2 = 0.0999$, $y_3 = 0.1987$, $y_4 = 0.2571$

$$y = f(x) = \frac{(x-0.10)(x-0.20)(x-0.26)}{(0.05-0.10)(0.05-0.20)(0.05-0.26)} \times 0.05 +$$

$$\frac{(x-0.05)(x-0.20)(x-0.26)}{(0.10-0.05)(0.10-0.20)(0.10-0.26)} \times 0.0999 +$$

$$\frac{(x-0.05)(x-0.10)(x-0.26)}{(0.20-0.05)(0.20-0.10)(0.20-0.26)} \times 0.1987 +$$

$$\frac{(x-0.05)(x-0.10)(x-0.20)}{(0.26-0.05)(0.26-0.10)(0.26-0.20)} \times 0.2571$$

Putting $x = 0.12$,

$$f(0.12) = \frac{(0.12-0.10)(0.12-0.20)(0.12-0.26)}{-1.575\times10^{-3}} \times 0.05 +$$

$$\frac{(0.12-0.05)(0.12-0.20)(0.12-0.26)}{8\times10^{-4}} \times 0.0999 +$$

$$\frac{(0.12-0.05)(0.12-0.10)(0.12-0.26)}{-9\times10^{-4}} \times 0.1987 +$$

$$\frac{(0.12-0.05)(0.12-0.10)(0.12-0.20)}{2.016\times10^{-3}} \times 0.2571$$

$$= -7.111\times10^{-3} - 0.02937 + 0.04327 + 4.285\times10^{-3}$$

$$= 0.112 \text{ (Ans)}$$

$\therefore f(0.12) =$

Trapezoidal ~~Simpson's $\frac{1}{3}$ rule~~

Trapezoidal Rule: Trapezoidal rule is the first of the Newton Cotes closed integration formulas. It corresponds to the case where the polynomial is first order.

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

A straight line can be represented as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

The area under this straight line is an estimate of the integral of $f(x)$ between the limits a and b

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right] dx$$

The result of integration

$$I = \underset{\text{lower limit}}{(b-a)} \frac{f(a) + f(b)}{2}$$

upper limit

This is called trapezoidal rule.

*Error of the trapezoidal rule:

$$E_r = \frac{-1}{12} f''(\xi)(b-a)^3$$

*Derivation of trapezoidal rule:

The equation,

$$I = \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right] dx$$

$$= \left[f(a)x + \frac{f(b)-f(a)}{b-a} \frac{(x-a)^2}{2} \right]_a^b$$

$$= \left\{ f(a).b + \frac{f(b)-f(a)}{b-a} \cdot \frac{(b-a)^2}{2} \right\} - \left\{ f(a).a + \frac{f(b)-f(a)}{b-a} \frac{(a-a)^2}{2} \right\}$$

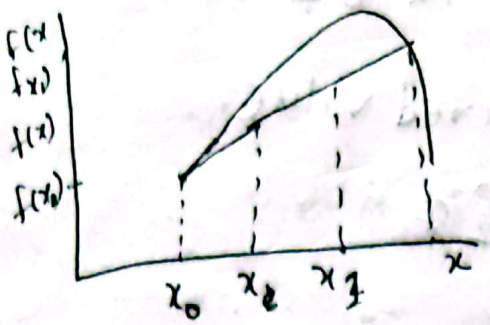
$$\Rightarrow f(a).b + \frac{f(b)-f(a)}{b-a} \cdot \frac{(b-a)^2}{2} - f(a).a$$

$$\Rightarrow f(a)(b-a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cdot \frac{(b-a)}{2}$$

$$= \frac{2f(a)(b-a) + f(b)(b-a) - f(a)(b-a)}{2}$$

$$= \frac{f(a)(b-a) + f(b)(b-a)}{2}$$

$$= \underbrace{(b-a)}_{\text{width}} \times \underbrace{\frac{f(a)+f(b)}{2}}_{\text{average height}}$$



$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0)$$

Example 21.1 Use trapezoidal rule to numerically integrate,

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$; exact value $= 1.640533$

Solution: The function values.

$$f(a) = f(0) = 0.2$$

$$f(b) = f(0.8) = 0.232$$

using trapezoidal we get,

$$I \cong (b-a)\frac{f(a) + f(b)}{2}$$

$$= (0.8 - 0)\frac{0.2 + 0.232}{2}$$

$$= 0.1728$$

$$\varepsilon_t = \left|\frac{V_A - V_E}{V_E}\right| \times 10$$

V_A - actual value

V_E - expected " / exact value

$$\varepsilon_t = \frac{\varepsilon_t}{\text{exact value}} \times 100\%$$

which represents an error of

E_t = exact value − observed value

$= 1.640533 - 0.1728$

$= 1.467733$

$\varepsilon_t = 89.5\%$ $\left[\frac{E_t}{\text{exact value}} \times 100\%\right]$

which corresponds to a percent relative error, $\varepsilon_t = 89.5\%$. The reason for this large error is evident from the figure as we can see that the area under the straight line neglects a significant portion of the integral lying above the line.

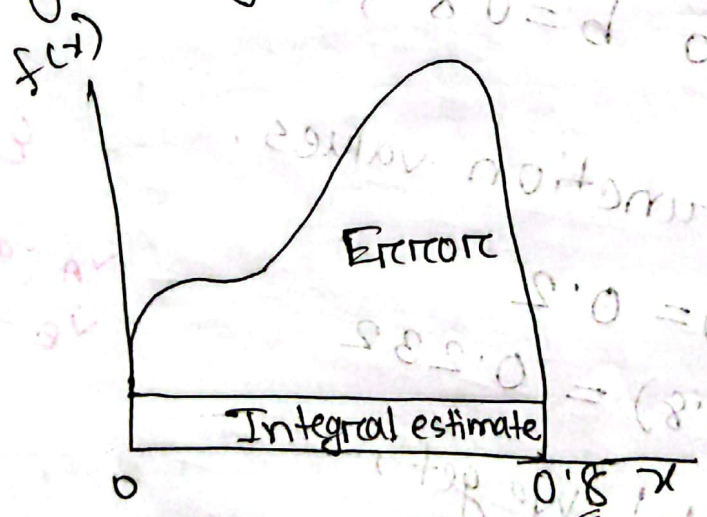


Fig: single application of trapezoidal rule.

To obtain this estimate, the functions second derivative over the interval can be computed by differentiating the original function twice.

$$f''(x) = -400 + 4050x - 10{,}800x^2 + 8000x^3$$

The average value of second derivative

$$f''(x) = \frac{\int_0^{0.8} (-400 + 4050x - 10{,}800x^2 + 8000x^3)\,dx}{0.8 - 0}$$

$$= \frac{\left[-400x + \frac{4050x^2}{2} - \frac{10{,}800x^3}{3} + \frac{8000x^4}{4}\right]_0^{0.8}}{0.8}$$

$$= -60$$

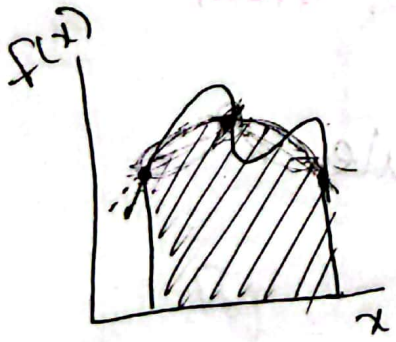
$$\therefore E_a = -\frac{1}{12}(-60)(0.8)^3 = 2.56$$

* How to remove the error of trapezoidal rule?

Ans: By using Simpson's $\frac{1}{3}$ rule.

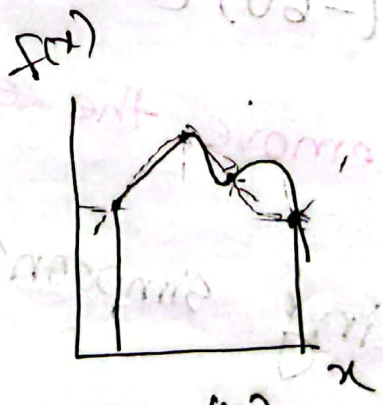
Simpson's Rule

~~*Simpson's $\frac{1}{3}$~~ Another way to obtain a more accurate estimate of an integral is to use higher order polynomials to connect the points. for example, if there is an extra point midway between f(a) and f(b), the three point can be connected with a parabola. If there are two points equally spaced between f(a) and f(b) the four points can be connected with a third order polynomial. The formulas that result from taking the integrals under these polynomials are called simpson's rule



(a)

[$\frac{1}{3}$ rule]



(b)

[$\frac{3}{8}$ rule]

*Simpson's $\frac{1}{3}$ rule.

Simpson's $\frac{1}{3}$ rule results when a second order interpolating polynomial is substituted into

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designed as x_0 and x_2 and $f_2(x)$ is represented by a second order Lagrange polynomial the integral becomes,

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integrating and algebric manipulation the following formula results:

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

[where, $h = \frac{b-a}{2}$ for this case]

$$I \cong (b-a)\frac{\left[f(x_0) + 4f(x_1) + f(x_2) \right]}{6}$$

This is called Simpson $\frac{1}{3}$ rule as h is divided by 3

Error estimation,

$$E_T = -\frac{(b-a)^5}{2880} f^4(\xi)$$

Example 21.4 Use simpson's $\frac{1}{3}$ rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$; exact integral 1.640533..

Solution: The function value,

$$f(a) = f(0) = 0.2$$

$$f(b) = f(0.8) = 0.232$$

$$f(x_1) = f\left(\frac{b-a}{2}\right) = f(0.4) = 2.456$$

Using Simpson's $\frac{1}{3}$ rule,

$$I \cong (b-a)\frac{f(a) + 4(f(x_1)) + f(b)}{6}$$

$$= 0.8 \times \frac{0.2 + 4 \times 2.456 + 0.232}{6}$$

$$= 1.367467$$

which represents an exact error of

$$E_t = 1.640533 - 1.367467$$

$$= 0.2730667$$

$$\varepsilon_t = 16.6\% \qquad \left[\frac{E_t}{\text{exact value}} \times 100\% = \varepsilon_t\right]$$

which is 5 times more accurate than for a single application of the trapezoidal rule

The estimated error

$$E_a = -\frac{0.8}{2880}(-2400) = 0.2730667$$

* Simpson's 3/8 rule:

If there are two points equally spaced between f(a) and f(b), the four points can be connected with a third order polynomial

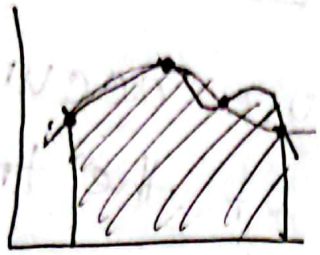


Fig: Graphical depiction of Simpson's 3/8 rule.

a third order Lagrange polynomial can be fit to four points and integrated.

$$I = \int_a^b f_3(x)\,dx$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$= \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]}{8}}_{\text{average height}}$$

This eqn is called Simpson's 3/8 rule because h is manipulated by 3/8. It is the third Newton-Cotes closed integration formula.

Example: 21.6 a) Use Simpson's 3/8 rule to integrate

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

from $a = 0$ to $b = 0.8$, exact value $= 1.640533$

b) Use it in conjunction with Simpson's 1/3 rule to integrate same function for five segment.

Solution:

a) A single application of Simpson's 3/8 rule requires four equally spaced points.

$f(x_0) = f(0) = 0.2$, $\frac{a+b}{3} = \frac{0.0.8}{3} = 0.2667$

$f(x_1) = f(0.2667) = 1.432724$

$0.2667 + 0.2667 = 0.5333$

$f(x_2) = f(0.5333) = 3.487177$ $f(x_3) = f(0.8) = 0.232$

using Simpson's 3/8 rule:

$$I \cong (b-a)\frac{[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]}{8}$$

$$= (0.8-0)\frac{0.2 + 3 \times 1.432724 + 3 \times 3.487177 + 0.232}{8}$$

$$= \cancel{0.8 \times} 1.519170$$

$E_t = 1.640533 - 1.519170 = 0.1213630$

$$\varepsilon_t = \frac{0.1213630}{1.640533} \times 100\% = 7.4\%$$

$$E_a = -\frac{(0.8)^5}{6480}(-24000) = 0.1213630$$

b) The data needed for five segment application, $h = 0.16 \rightarrow \frac{0.8}{5} = 0.16$

$f(x_0) = f(0) = 0.2$ $\quad f(x_1) = f(0.16) = 1.296919$

$f(x_2) = f(\overset{0.16+0.16}{0.32}) = 1.743393$ $\quad f(x_3) = f(\overset{0.32+0.16}{0.48}) = 3.186015$

$f(x_4) = f(0.64) = 3.181929$, $\quad f(x_5) = f(0.8) = 0.232$

The integral for first two segment ~~op~~ obtained by simpson's $\frac{1}{3}$ rule:

$$I \cong (x_2 - x_0)\frac{f(x_0) + 4(x_1) + f(x_2)}{6}$$

$$= (0.32) \times \frac{0.2 + 4 \times 1.296919 + 1.743393}{6}$$

$$= 0.3803237$$

Last three segment the 3/8 rule can be used,

$$I \cong (x_5 - x_2)\frac{f(x_2) + 3\{f(x_3) + f(x_4)\} + f(x_5)}{8}$$

$$= 0.48 \times \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8}$$

$= 1.264754$

The total integral is computed by summing the two result:

$$I = 0.3803237 + 1.264754$$

$$E_{rt} = 1.640533 - 1.645077 = -0.00454383$$

$$\varepsilon_t = \left| \frac{-0.00454383}{1.640533} \right| \times 100\%$$

$$= -0.28\%$$

Iterative Process / Direct Substitution / Simple fixed point method

Consider an equation $f(x)=0$ which can take in the form $x=\phi(x)$. If $\phi'(x)<1$, $\phi(x)$ is convergent,

$$x_1 = \phi(x_0)$$
$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2)$$

Example: Find a real root of the eqn $2x^2-4x+1=0$ using iterative process.

Sol^n: Given eqn, $2x^2-4x+1=0$

$$\Rightarrow 4x = 2x^2+1$$
$$\Rightarrow x = \frac{2x^2+1}{4}$$

$$f(x) = 2x^2-4x+1$$
$$f(0) = 1$$
$$f(1) = -1$$

The root is between 0 and 1.

So, $x_0 = \frac{0+1}{2} = 0.5$

Now, $x = \phi(x)$

$$\phi(x) = \frac{2x^2+1}{4}$$

$$\phi'(x) = \frac{x^2}{2} + \frac{1}{4}$$

$$= x$$

$$\phi(x)_{at\ x_0=0.5} = 0.5 < 1$$

So, iterative process can be applied.

We know,

$$x_{i+1} = \phi(x_i)$$

Putting $i=0$ we get:

$$x_1 = \phi(x_0) = \frac{1}{2}x_0^2 + \frac{1}{4} = \frac{1}{2}(0.5)^2 + \frac{1}{4} = 0.375$$

$$x_2 = \phi(x_1) = \frac{1}{2}x_1^2 + \frac{1}{4} = \frac{1}{2}(0.375)^2 + \frac{1}{4} = 0.320$$

$$x_3 = \phi(x_2) = \frac{1}{2}x_2^2 + \frac{1}{4} = \frac{1}{2}(0.320)^2 + \frac{1}{4} = 0.3012$$

$$x_4 = \phi(x_3) = \frac{1}{2}x_3^2 + \frac{1}{4} = \frac{1}{2}(0.3012)^2 + \frac{1}{4} = 0.295$$

$$x_5 = \phi(x_4) = \frac{1}{2}x_4^2 + \frac{1}{4} = \frac{1}{2}(0.295)^2 + \frac{1}{4} = 0.293$$

$$x_6 = \phi(x_5) = \frac{1}{2}x_5^2 + \frac{1}{4} = \frac{1}{2}(0.293)^2 + \frac{1}{4} = 0.2929$$

1st step – 2nd step 2nd step – 3rd step

convergence → $0.375 - 0.320 > 0.320 - 0.3012$

$= 0.055$ 0.0188

value কমতে থাকলে convergence. value বাড়লে divergence.

Example 6.1 Use simple fixed point iteration to locate the root of $f(x) = e^{-x} - x$, root=0.57

Solⁿ:

Let $f(x)=0$

the function can be expressed as,

$$e^{-x} - x = 0$$

$$x = e^{-x}$$

$$\Rightarrow x_{i+1} = e^{-x_i} \quad [\because x_{i+1} = g(x_i)]$$

Starting with initial guess of $x_0 = 0$

$$\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100\%$$

i	x_i	x_{i+1}	$\varepsilon_a\%$
0	0	1.000000	100%
1	1.000000	0.36787944	171.82818%
2	0.36787944	0.69220062	46.85%
3	0.69220062	0.50047354	38.3
4	0.50047354	0.60624351	17.44
5	0.60624351	0.5453957991	11.2
6	0.5453957991	0.57961245	5.90
7	0.57961245	0.5601153972	3.4808
8	0.5601153972	0.571143	1.93
9	0.571143	0.56487932	1.11
10	0.56487932	0.5684287	0.39

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at 0.564879, the error is 0

Here x true value are given 0.56714329

Here from iteration 10, we can see that.

value are close to true value. this case

s called convergence.

Example; Exm ↳ suitable example দিয়ে explain করো

এটা দিব।

$$2x^2 - 4x + 1 = 0$$

Now, the function,

$$2x^2 - 4x + 1 = 0$$

$$4x = 2x^2 + 1$$

$$x = \frac{x^2}{2} + \frac{1}{4}$$

when,

$x = 0$, $x = \frac{1}{4} = 0.25$

$x = 1$, $x = \frac{1}{2} + \frac{1}{4} = 0.75$

$\therefore \frac{0.25 + 0.75}{2} = 0.5$

Starting with initial guess of $x_0 = 1$.

i	x_i	x_{i+1}	$\varepsilon_a\%$ $\varepsilon_a = \left\lvert \frac{x_{i+1} - x_i}{x_{i+1}} \right\rvert \times 100$
0	1	0.75	33.33%
1	0.75	0.53125	0.41764
2	0.53125	0.3911132	0.358695
3	0.3911132	0.326484	0.1976
4	0.326484	0.3032996	0.1976
5	0.3032996	0.295994	
6	0.295994	0.293806	
7	0.293806	0.293161	
8	0.293161	0.292971	
9	0.292971	0.2929162	
10	0.2929162	0.29289995	
11	0.29289995	0.292895	
12	0.292895	0.2928937	
13	0.2928937	0.2928933	
14	0.2928933	0.292893	
15	0.292893	0.292893	

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As in 14, 15 iteration the difference between x_{i+1} is equal to 0. So, we reach the goal & locate the root correctly.

Convergence;

As, $1 - 0.75 = 0.25$

$0.75 - 0.531 = 0.22$

$0.531 - 0.3911 = 0.1399$

the value is decreasing and is close to root

So, it is called convergence.

Divergence;

Difference যত বেশি হবে divergenc

হবে। $x_0 = 2$ ধরে

n	x_n
1	2.0
2	2.25
3	2.78125
4	4.1176 76
5	8.727627

As, $|2 - 2.25| = 0.25$

$|2.25 - 2.78125| = 0.53$

$|2.78125 - 4.1176 76| = 1.33$

Difference increasing so it is divergence.

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এখানে করলে 5 digit পর্যন্ত করলে হবে।

*show two scenerios in the case of iteration process which *are Convergence & divergence explain with suitable example. → 17-18 → 16-17 → 18-19 → 2020

Ans: আগের পেজের $2x^2-4x+1=0$ example করলে হবে।

15-16

* Use simple fixed point iteration to locate the root of $f(x) = -0.9x^2+1.7x+2.5$. Use initial guess of $x_0=5$ and iterate until $\varepsilon_a <= 0.01\%$

Ans: Here, Given,

$$f(x) = -0.9x^2+1.7x+2.5$$

Let, $f(x)=0$ then,

$$-0.9x^2+1.7x+2.5=0$$

$$\Rightarrow \quad 1.7x = 0.9x^2-2.5$$

$$\Rightarrow \quad x = 0.5294x^2 - 1.4706$$

$$x_{i+1} = 0.5294x^2 - 1.4706$$

~~$f(x_0)=$~~ $x_0=5$, $x = 11.7644$ $= -11.5$

$x=6$, $x = 17.5878$ -19.7

$$avg = \frac{11.7644+17.5878}{2} = 14.6761$$

Given, $-0.9x^2 + 1.7x + 2.5 = 0$

$$\Rightarrow \quad -0.9x^2 = -1.7x - 2.5$$

$$\Rightarrow \quad x^2 = \frac{1.7x}{0.9} + \frac{2.5}{0.9}$$

$$x = \sqrt{\frac{1.7x}{0.9} + \frac{2.5}{0.9}}$$

$$= \sqrt{1.88889x + 2.77778}$$

$$x_{i+1} = \sqrt{1.88889x_i + 2.777778}$$

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Starcting with $x_0 = 5$

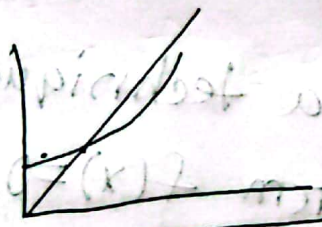
i	x_i	x_{i+1}	ε_a %
0	5	3.49603	0.43
1	3.49603	3.062907	0.1414
2	3.062907	2.9263	0.04668
3	2.9263	2.8819	0.0154
4	2.88188	2.867287	0.00509
5	2.867287	2.8624750	0.0016
6	2.8624750	2.8608871	0.00055

* Two cases of iteration process:

convergence and divergence

Difference between convergence and divergence.

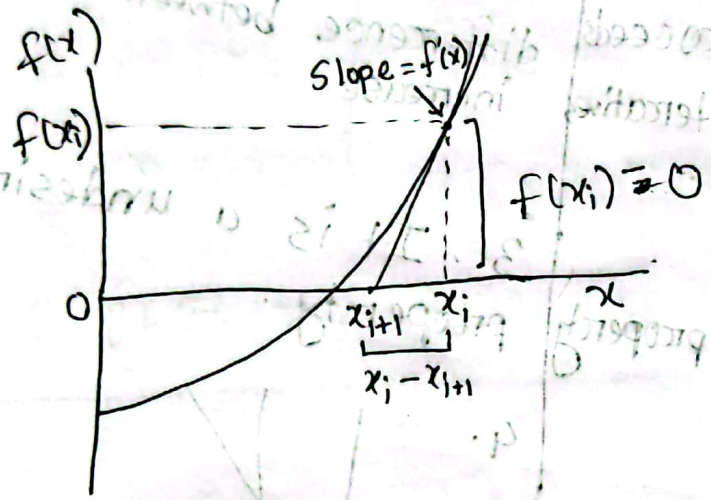
Ans:

Convergence	Divergence
1. It refers to the property of an iterative method where the sequence approaches a stable solution.	1. It refers to the property of an iterative method where the sequence fails to reach the stable solution
2. As the iteration proceeds difference between iteratives decreases.	2. As the iteration proceeds difference between iteration increase.
3. It is a desirable property	3. It is a undesirable property.
4. 	4.

Newton Rapson formula

*~~Explain~~ Simple fixed formula তে iterative বেশি হয় loop মান বেশি Newton Rapshon করে তা কমানো যায়

*what is Newton Rapson / what is NR



Newton rapson method is a technique for solving equation of the form $f(x)=0$ by succesive approximation.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

it is called Newton Rapson formula.

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* Advantage of NR :

⇒ 1) Fast convergence. It converges fast if it converges, which means in most cases we get root in less number of step.

2) It requires only one guess. formula of this method is simple

*why Newton Rapson method is used?

⇒ 1) For finding the roots of given equati

2) It can be efficiently generalized to find solution to a system of equation

3) when we approach the root, the method is quadratically convergent.

11-12

Example 6.3 Use newton-rapson method to estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$, ~~root~~ true value ≈ 0.567 until $\varepsilon_t\% < 10^{-8}$

Ans: The given function,

$$f(x_i) = e^{-x_i} - x_i \quad \text{—— (1)}$$

First derivative of eqn (1), we get,

$$f'(x_i) = -e^{-x_i} - 1$$

Now,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

$$= x_i + \frac{e^{-x_i} - x_i}{e^{-x_i} + 1}$$

Starting with initial guess $x_0 = 0$

i	x_i	x_{i+1}	ε_t % $\frac{x_{i+1}-x_i}{x_{i+1}}$
0	0	0.500000	100
1	0.500000	0.56631	11.8
2	0.56631	0.56714	0.147
3	0.567143165	0.567143290	0.00006220
4	0.567143290	0.567143847	$<10^{-8}$

Thus this approach converges on the true root

15-16

*How do you use Taylor series in the derivation of Newton Rapson method? Explain the pitfalls of newton rapson method with example.

Ans: Using Taylor series:

The taylor series is

$$f(x) \approx f(x_n) + f'(x_n)(x-x_n) + \frac{f''(x_n)}{2!}(x-x_n)^2 + \frac{f'''(x_n)}{3!}(x-x_n)^3 + \cdots$$

Now consider first two terms of taylor series,

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n)$$

Let, $f(x) = 0$

$$f(x_n) + f'(x_n)(x - x_n) = 0$$

Now,

$$\cancel{x_{n+1} = x_n - \frac{f(x_n)}{}}$$

$$f'(x_n)(x - x_n) = f(x_n)$$

$$\Rightarrow \quad x - x_n = \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad -x_n = -x + \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_n = x - \frac{f(x_n)}{f'(x_n)}$$

it can be written as,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

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itfalls of NR.

lthough NR often very efficient, there are tuation where it performs poorly.

special case – multiple roots. even dealing vith simple roots can also arise problem

For example, $f(x) = x^{10} - 1$

$$f(x_i) = x_i^{10} - 1$$

$$f'(x_i) = 10x_i^9$$

NR formual for this one's

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{10x_i^{10} - 1}{10x_i^9}$$

$x = 0.5$

i	x_i	x_{i+1}	$\epsilon_a(\%)$
0	0.5	51.65	
1	51.65	46.485	
2	46.485	37.65285	
⋮			
∞	1		

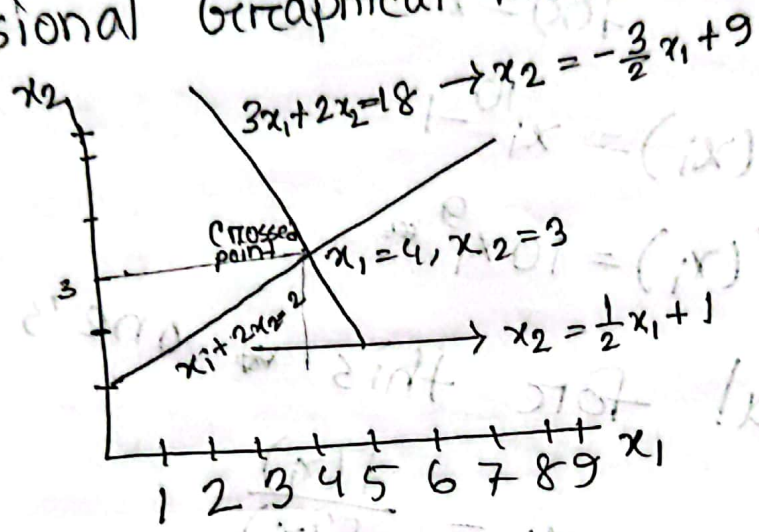
This is poor prediction.

This converging on the true root of 1 but a very slow rate

11-12

*sketch and state graphical detection of singular & ill condition Graphical Method

Two dimensional Graphical method:



Two eqn cross

For 3 or more than 3 dimensial eqn:

1) 3 simultaneously eqn should be represented by a plane.

2) The 3 planes intersect would represent the solution.

3) Beyond 3 eqn graphical method breakdown & have little practical value.

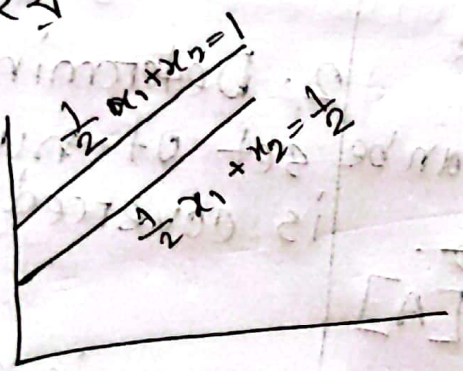
3 এর বেশি eqn হয় তখন graphical method দিয়ে solve করা যায় না, Gauss elimination, jordan দিয়ে solve করি।

4.) Sometimes Graphical method becomes useful in visualization properties of solution when the equation is 3 or less than 3.

Singular

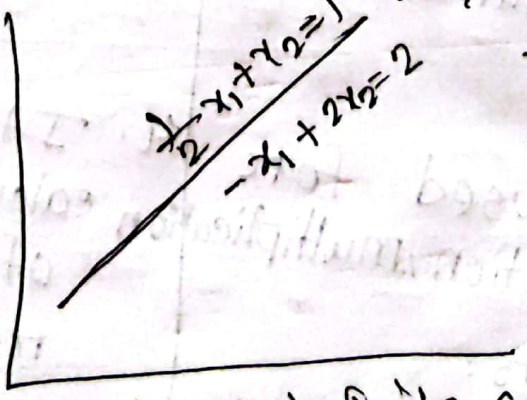
⇒ যখন দুটো straight line সমান্তরাল চলে তখন slope same হয়।

রেখাচিত্র



as slope same, so no solution will be found in parallel

⇒ দুটো straight line ২টা আরেকটার উপর পড়লে তা coincident.

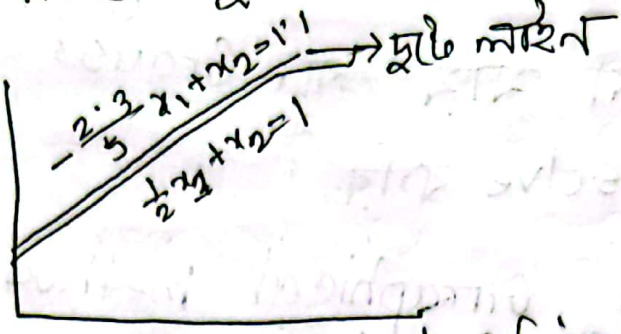


→ two line overlap each other

coincident; infinite solution

⇒ যখন দুটো line খুব কাছাকাছি থাকে

ill condition



slope very close; point of intersection is difficult to find.

imp

*Difference between determinant & matrix:

Matrix	Determinant
1. Matrix is a shape.	1. Determinant is a number.
2. Matrix is a set of numbers which ~~be~~ can be covered by ~~two bars~~ square brackets [A]	2. Determinant is a set of numbers but it is covered by |A|
3. It is not necessary the number of rows will be equal to number of column.	3. It is necessary the number of rows will be equal to number of column
4. It can be used for adding, subtraction, multiplication the coefficients.	4. It can be used for calculating the values of x, y, z in cramers rule.
5. Matrix can't be computed from a determinant.	5. Determinant value of a matrix can be computed.

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Matrix

$$\{A\} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 9.1 Use graphical method to solve

$$3x_1 + 2x_2 = 18 \quad \text{— (I)}$$

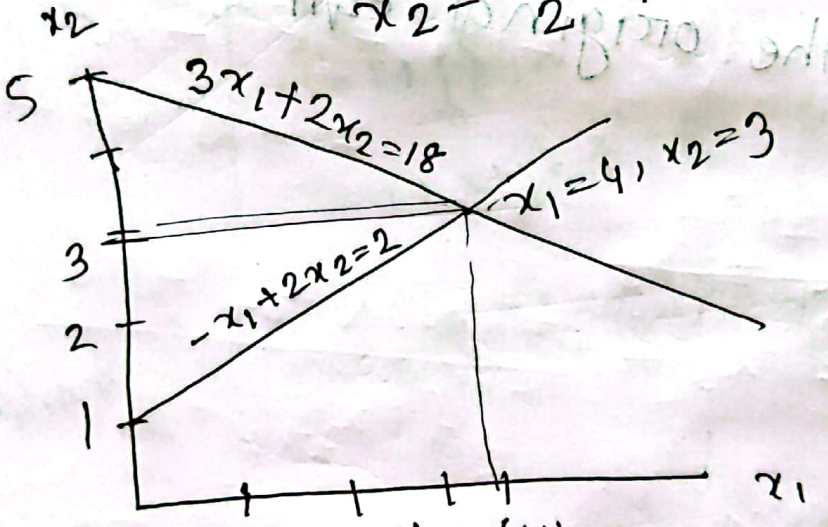
$$-x_1 + 2x_2 = 2 \quad \text{— (II)}$$

Here for eq (I),

$$x_2 = -\frac{3x_1}{2} + 9$$

for eqn (II),

$$x_2 = \frac{1}{2}x_1 + 1$$



The intersection of line

$3 \times$ (II) and + (I),

$$3x_1 + 2x_2 = 18$$
$$-3x_1 + 6x_2 = 6$$

$$8x_2 = 24$$
$$x_2 = 3$$

putting value of x_2 in (I)

$$3x_1 + 2 \times 3 = 18$$
$$3x_1 = 18 - 6$$
$$x_1 = 4$$

$x_1 = 4$

$x_2 = 3$

Now, the result can be checked by substituting these values into the original equation,

$$3.4 + 2.3$$
$$= 18$$

again $- 4 + 2 \cdot 3$
$$= 2$$

This result is equivalent to right hand side of the original eqn.

Picard's Method

15-16 | 16-17 | 17-18

Picard's method, $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$, calculate ~~y(0)~~, $y(0.1)$, $y(0.2)$ using Picard's method.

$f(x,y) = 1 + xy$ $\quad y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

First Approx

$$y_{01} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_1 = 1 + \int_0^x (1 + xy_0)\,dx$$

$$y_1 = 1 + \int_0^x 1 + x\,dx$$

$$= 1 + x + \frac{x^2}{2}$$

2nd Approx

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 1 + \int_0^x (1 + xy_1)\,dx$$

$$y_2 = 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2}\right)\right\}dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

3rd Approx

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)$$

$$y_3 = 1 + \int_0^x (1 + xy_2)$$

$$y_3 = 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\}$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

$$= 1.10534 \text{ (Ans)}$$

$$y(0.2) = 1 + (0.2) + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{8} + \frac{(0.2)^5}{15} + \frac{(0.2)^6}{48}$$

$$= 1.22289 \text{ (Ans)}$$

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Secant method

Example 6.6 Use the secant method to estim the root of $f(x) = e^{-x} - x$, start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$. true root is 0.56714329

Ans:

First iteration:

$$x_{-1} = 0; \quad f(x_{-1}) = e^{-x_{-1}} - x_{-1}$$
$$= e^{-0} - 0$$
$$= 1$$

$$x_0 = 1, \quad f(x_0) = e^{-x_0} - x_0$$
$$= -0.63212$$

$$\cancel{x_1 = x_0}$$

$$x_1 = x_{0+1} = x_0 - \frac{f(x_0)(x_{0-1} - x_0)}{f(x_{0-1}) - f(x_0)}$$

$$= 1 - \frac{-0.63212(0-1)}{1 + 0.63212}$$

$$= 0.6127$$

$$\varepsilon_a = \frac{x_{i+1} - x_{true}}{x_{i+1}} \times 100$$

$$= \cancel{0.6321} \; 7.44\%$$

Second iteration:

$x_0 = 1 \quad f(x_0) = -0.63212$

$x_1 = 0.61270 \quad f(x_1) = -0.07081$

(both estimate are now on the same side)

$$\cancel{x_2 =} \quad x_2 = x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

$$\cancel{= 0.61270 - \frac{-0.07081 \times (-0.63212 -}}$$

$$\cancel{= 0.61270 -}$$

$$= 0.61270 - \frac{-0.07081 \times (1 - 0.61270)}{-0.63212 + 0.07081}$$

$$= 0.56384$$

$$\varepsilon_t = \frac{x_{i+1} - x_{\text{true value}}}{x_{i+1}} \times 100$$

$$= 0.58\%$$

Third iteration:

$x_1 = 0.61270 \quad f(x_1) = -0.07081$

$x_2 = 0.56384 \quad f(x_2) = 0.00518$

$$x_3 = x_{2+1} = x_2 - \frac{f(x_2)(x_{2-1} - x_2)}{f(x_{2-1}) - f(x_2)}$$

$$= x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - 0.00518}$$

$$= 0.56717$$

$$\varepsilon_t = \frac{x_{i+1} - x_{trueroot}}{x_{i+1}} \times 100\%$$

$$= \frac{0.56717 - 0.56714}{0.56717} \times 100\%$$

$$= 0.0048\%$$

$\boxed{11-12}$

* Secant method can be classified as a bracketing method" – is it true or false?

Ans: False.

Bracketing method is also known as interval halving method or bisection method and work by iteratively narrowing down an interval in which root lies. It requires changes sign between estimate. On the other hand secant method requires two initial estimates of x. However, f(x) is not required to change signs between the estimates. So, it can't be classified as bracketing method.

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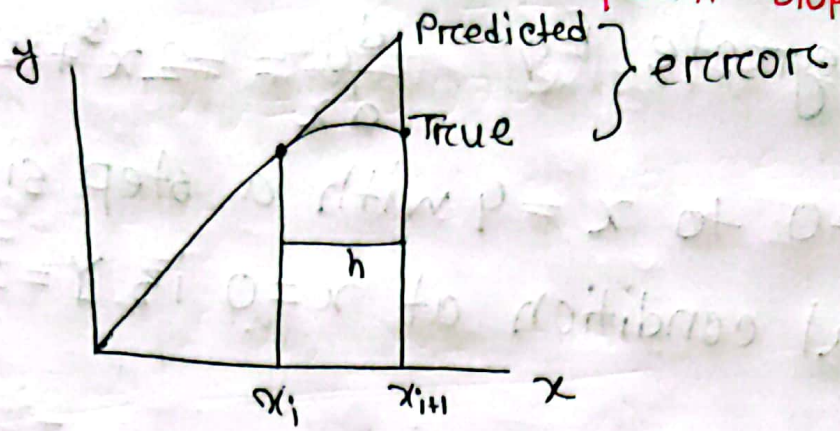
Euler's method / Euler-Cauchy or the point slope



The first derivative provide direct estimate of the slope at x_i

$$\varphi = f(x_i, y_i)$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i. This estimated can be substituted in ~~eq~~ eqn $y_{i+1} = y_i + \varphi h$

$$\therefore y_{i+1} = y_i + f(x_i, y_i)h$$

This formula is called Euler's method.

$$h = \frac{x}{5}$$

15-16

Example: 25.1) Use Euler's method to numerically integrate Eq. $\frac{dy}{dx} = -2x^3+12x^2-20x+8.$ from $x=0$ to $x=4$ with a step size of 0.5. The initial condition at $x=0$ is $Y=1$.

$h = \frac{b-a}{n}$

$n = \frac{b-a}{h}$

$= \frac{1-0}{0.5}$

$= 2$

Solution:

Given that,

$$\frac{dy}{dx} = -2x^3+12x^2-20x+8.5$$

After integrating it,

$$y = -2\cdot\frac{x^4}{4} + 12\cdot\frac{x^3}{3} - 20\cdot\frac{x^2}{2} + 8.5x + 1$$

$$= -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1 \quad \text{--- (1)}$$

we know,

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \text{--- (1)}$$

$$= y_i + h(-2x^3+12x^2-20x+8.5) \quad \text{--- (1)}$$

$$\cancel{= y_i + h(-0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1)}$$

$x_0 = 0$

$x_1 = x_0 + h = 0 + 0.5 = 0.5$

$x_2 = x_1 + h = 0.5 + 0.5 = 1$

$x_3 = x_2 + h = 1 + 0.5 = 1.5$

$x_4 = x_3 + h = 1.5 + 0.5 = 2$

for $n=0$ in eqn (ii)

$$y_{0+1} = y_0 + h\left(\cancel{-0.5x_0^4+4x_0^3-10x_0^2+8.5x_0+1}\right) -2x_0^3+12x_0^2-20x_0+8.5$$

$$y_1 = \cancel{1+0.5\times 0}\ 1+0.5\times 8.5$$

$$= \cancel{1} = 5.25$$

True solution at $x_1=0.5$ in eqn (i)

$$y = -0.5(0.5)^4+4(0.5)^3-10(0.5)^2+8.5\times 0.5+1$$

$$= 3.21875$$

Thus error,

$$E_t = \text{true} - \text{approximate}$$

$$= 3.21875 - 5.25 = -2.03125$$

relative error $\epsilon_t = \dfrac{y_{true} - y_{approx}}{y_{true}} \times 100\% = 63.1$

for $n=1$ in eqn (ii)

$$y_{1+1} = y_1 + h\left(-2x_1^3+12x_1^2-20x_1+8.5\right)$$

$$\Rightarrow y_2 = 5.25 + 0.5\left\{-2(\cancel{5.25}\ 0.5)^3+12(\cancel{5.25}\ 0.5)^2-20(\cancel{5}\ 0.5)+8.5\right\}$$

$$= 5.875$$

true solution, at $x_2=1$ in eqn (i)

$$y = -0.5+4-10+8.5+1$$

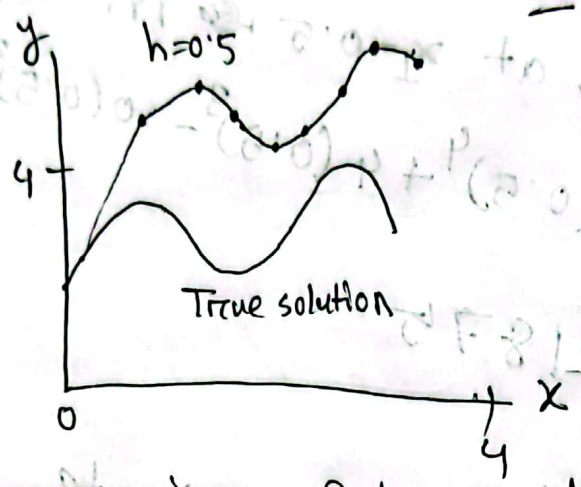
$$= 3.0$$

~~the error = 3 =~~

$$\text{percent relative error} = \frac{y_{true} - y_{approx}}{y_{true}} \times 100$$

$$= \frac{3 - 5.875}{3} \times 100$$

$$= -95.8\%$$



comparion of true solution with numerical solution using Euler's method.

x	y_{true}	y	y_{euler}	Global
0	1.00000	1.0	1.0000	
0.5	3.21875	5.25000	5.25000	−63.1
1.0	3.00000	5.87500	5.87500	−95.8
1.5	2.21875	5.12500	5.12500	−131.0
2.0	2.00000	4.500000	4.500000	−125.0
2.5	2.71875	4.75000	4.75000	−74.7
3.0	4.0000	5.8750	5.87500	−46.9
3.5	4.71875	7.12500	7.12500	−51.0
4.0	3.00000	7.00000	7.00000	−133.3

*Show effect of reduced step size on Eulers method.

Ans: The step size on Euler method determine how small the intervals between these points are. Reducing step size leading to a more accurate approximation of the solution, but it comes at the cost of increased computational effort (or requires more computations.

11-12 Weddle's Rule

Putting $n=6$ in eqn of Newton cotes formula, $I = \int_a^b f(x)\,dx \cong \int_a^b f_6(x)\,dx$ & neglecting the difference of order higher than six, we obtain weddle's rule,

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

the error in which is given by

$$E = \frac{h^6}{140} y_6$$

Heun's method

One method to improve the estimate of the slope involves the determination of two derivatives for the interval — one at initial point and another at end point. The two derivatives are then averaged to obtain an improved estimate of the slope for the entire ~~derivatives~~ interval. This is called Heun's method.

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Runge-Kutta methods

$$\frac{dy}{dx} = f(x, y)$$

New value = old value + slope × step size

$$y_{i+1} = y_i + \phi h.$$

Heun method is predictor corrector approach.

Predictor, $y^0_{i+1} = y_i + f(x_i, y_i)h.$

Corrector: $y_{i+1} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})h}{2}$

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