Chapter One : Complex Number

Complex number: The number having the form $a+ib$ is called a complex number where $a, b \in R$ and $i = \sqrt{-1}$

- Two complex number are said to be equal if and only if their real and imaginary parts are equal. $a+bi = c+di$

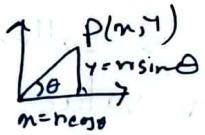
$$\Rightarrow a = c \text{ and } b = d$$

- **Absolute value on Modulus:**

Let $z = x+iy$ be a complex number, then the absolute value or modulus of z is denoted by $|z|$ and is given by

$|z| = |x+iy| = \sqrt{x^2+y^2}$ Ex: $3+4i = \sqrt{3^2+4^2} = 5$

- **Polar form:**



$y = r\sin\theta$ $x = r\cos\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\frac{y}{x}$

The form $z = x+iy = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number, where r and θ is called the polar coordinates.

- **Modulus:** Let (r,θ) be the polar coordinates corresponding to the complex number $z = x+iy = (x,y)$, then $r = \sqrt{x^2+y^2}$ is called the modulus.

- **Complex Conjugate:** The complex conjugate of the number $z = x+iy$ is denoted by $\bar{z}$ and is defined by $\bar{z} = x-iy$

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Function Limits & Continuity.

Single-valued function: A function $w=f(z)=x+iy$ is called a single valued in a domain s if and only if one value of w corresponds to each value of z in s.

Ex: If $w=f(z)=z^2+2$, then w is called a single valued function of z since to each value of z there is only one value of w.

Multiple-valued function: A function $w=f(z)=x+iy$ is called a multiple valued ~~functi~~ in a domain s if and only if more than one value of w corresponds of each value of z in s.

Ex: If $w=f(z)=z^{1/2}$, then w is called a multiple valued function of z since to each value of z there are two values of w.

Real and Imaginary parts of complex variable:

$z=x+iy$ Let $w=f(z)=u+iv$ be a single-valued function of $z=x+iy$, now replacing $x+iy$ for z, we have $u+iv=f(x+iy)$.

Then equating real and imaginary parts we have $u=u(x,y)$ and $v=v(x,y)$

Ex: If $w=e^z$ then $u+iv=e^{x+iy}=e^x.e^{iy}$

$=e^x(\cos y+i\sin y)$

$\therefore\ u=e^x\cos y=u(x,y)$ and $v=e^x\sin y=v(x,y)$

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Analytic Function

#Differential coefficient in a region or domain:

Let $f(z)$ be a single-valued function defined in a region or domain R, ~~of the~~ a then the derivative of $f(z)$ is defined as $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z)}{\Delta z}$, Then $f(z)$ is called differentiable at z.

Analytic function: **

A single-valued function $f(z)$ is said to be analytic in a region R if the derivative $f'(z)$ exists at all points z of the region R.

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Cauchy-Riemann equations: Necessary and sufficient conditions for the function $f(z) = u(x,y) + iv(x,y)$ to be analytic.

(i) Necessary condition for $f(z)$ to be analytic: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R, then it is necessary that the four partial derivatives $u_x = \frac{\partial u}{\partial x}$, $v_x = \frac{\partial v}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_y = \frac{\partial v}{\partial y}$ should exits and satisfy the Cauchy Riemann equations

$u_x = v_y \quad u_y = -v_x$

(ii) Sufficient condition for $f(z)$ to be analytic: The continuous single-valued function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R if the four partial derivatives u_x, v_x, u_y, v_y exits are continuous and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ in R.

Cauchy-Riemann equations in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

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Page: 109: Ex 95 - Verify the Cauchy-Riemann equations are satisfied for the following functions:

(i) $f(z) = e^{z^2}$ and (ii) $f(z) = \cos 2z$

Solution: (i) $f(z) = e^{z^2}$ we have $f(z) = u + iv = e^{(x+iy)^2}$

$= e^{x^2 + 2xiy - y^2} = e^{x^2-y^2} \cdot e^{2xiy} = e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$

Then $u = e^{x^2-y^2}\cos 2xy$ $\quad v = e^{x^2-y^2}\sin 2xy$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2} \cdot -2y\sin 2xy + \cos 2xy \cdot e^{x^2-y^2} \cdot 2x$$

$$= e^{x^2-y^2}(2x\cos 2xy - 2y\sin 2xy) = \frac{\partial v}{\partial y} \text{ and}$$

$$\frac{\partial u}{\partial y} = e^{x^2-y^2} \cdot -2x\sin 2xy - 2y e^{x^2-y^2}\cos 2xy$$

$$= -e^{x^2-y^2}(2y\cos 2xy + 2x\sin 2xy) = -\frac{\partial v}{\partial x}$$

Thus u and v satisfy the Cauchy-Riemann equations.

(ii) We have $f(z) = u + iv = \cos(2x + 2yi) = \cos 2x\cosh 2y - i\sin 2x\sinh 2y$ $\quad u = \cos 2x\cosh 2y$ and $v = -\sin 2x\sinh 2y$

$$\frac{\partial u}{\partial x} = -2\sin 2x\cosh 2y = \frac{\partial v}{\partial y} \text{ and}$$

$$\frac{\partial u}{\partial y} = 2\cos 2x\sinh 2y = -\frac{\partial v}{\partial x}$$

Thus u and v satisfy the Cauchy-Riemann equations.

Show that the laplace's equation in polar form is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta} = 0$$

$\cos(iy) = \cosh(y)$; $\sin(iy) = i\sinh(y)$

Laplace's equation in two dimension: (111 pg.)

In two dimensions a second order partial differential equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$ is called Laplace's equation. The vector operator ∇^2 is called Laplacian where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Harmonic function: A real-valued function $\phi(x,y)$ is called a harmonic function or simply harmonic ~~function~~ in a region R if all its second-order partial derivatives are continuous in R and at each point of the region it satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Conjugate harmonic function: If $f(z) = u + iv$ is an analytic function then u and v are called conjugate harmonic functions.

Laplace's equation in polar form: $\boxed{\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0}$

Ex:10:- Find the harmonic conjugate of the function $u = x^3 + 6x^2y - 3xy^2 - y^3$

Sol^n: We have $\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2$; $\frac{\partial^2 u}{\partial x^2} = 6x + 12y$ — (i)

$\frac{\partial u}{\partial y} = 6x^2 - 6xy - 6y^2$; $\frac{\partial^2 u}{\partial y^2} = -6x - 12y$ — (ii)

Now (i) + (ii) $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y = 0 \Rightarrow u$ is harmonic

Since u is harmonic, therefore it will be possible to find the harmonic conjugate v of u. Now by Cauchy-Riemann equations we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$\therefore \frac{\partial v}{\partial y} = 3x^2 + 12xy - 3y^2$. Integrating this equation w.r.t y,

$v = 3x^2y + 6xy^2 - y^3 + F(x)$. Differentiating this equation w.r.t x, we get $\frac{\partial v}{\partial x} = 6xy + 6y^2 + F'(x) = -6x^2 + 6xy + 6y^2$

$F'(x) = -6x^2$. Integrating w.r.t. x —

$F(x) = -2x^3 + c$ $\quad \therefore v = 3x^2y + 6xy^2 - y^3 - 2x^3 + c$

Ex: 109 $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

we have $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$; $\frac{\partial^2 u}{\partial x^2} = 6x + 6$

$\frac{\partial u}{\partial y} = -6xy - 6y$; $\frac{\partial^2 u}{\partial y^2} = -6x - 6$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0 \Rightarrow u$ is harmonic function

Again by Cauchy-Riemann equ^n, we have,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$ — ① and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 6y$

Integrating ① w.r.t. $y \Rightarrow v = 3x^2y - y^3 + 6xy + F(x)$

$\frac{\partial v}{\partial x} = 6xy + 6y + F'(x) = 6xy + 6y \Rightarrow F'(x) = 0$

$\therefore F(x) = c \quad \therefore \quad v = 3x^2y - y^3 + 6xy + c$ Ans

Ex: 110: We have $\varphi = \frac{1}{2}\log(x^2+y^2)$

The $\frac{\partial \varphi}{\partial x} = \frac{x}{x^2+y^2}$ $\quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$ —— ①

$\frac{\partial \varphi}{\partial y} = \frac{y}{x^2+y^2}$ $\quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$ —— ②

Now ① + ② $\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \Rightarrow \varphi$ is harmonic.

Now by Cauchy-Riemann equation $\Rightarrow \frac{\partial \varphi}{\partial x} = -\frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}$

and integrating this with respect to y we get.

$\phi = -x/x \tan^{-1} y/x + F(x) = -\tan^{-1} y/x + F(x)$ —— ③

Not Solve

Construction of an Analytic function:- (Milne Method)

If $f(z) = u(x,y) + iv(x,y)$ is analytic and if $\frac{\partial u}{\partial x} = \phi_1(x,y)$, $\frac{\partial u}{\partial y} = \phi_2(x,y)$ and $\frac{\partial v}{\partial x} = \psi_1(x,y)$, $\frac{\partial v}{\partial y} = \psi_2(x,y)$, then

(i) $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$; (ii) $f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$

Ex: 111 $u = 3x^2y + 2x^2 - y^3 - 2y^2$ Thus $\frac{\partial u}{\partial x} = 6xy + 4x = \phi_1(x,y)$ ---- (1)

$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y = \phi_2(x,y)$ — (3) $\frac{\partial^2 u}{\partial x^2} = 6y + 4$ — (2)

$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z)}{\Delta z}$

$\frac{\partial^2 u}{\partial y^2} = -6y - 4$ ——— (4)

Now (2) + (4) $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic

Now using Milne's Method we have,

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$ [putting $x = z$, $y = 0$ in 1 and 3]

$= 4z - 3iz^2$. Now integrating we have $f(z) = 2z^2 - iz^3 + ic$

$\Rightarrow f(z) = u + iv = 2(x+iy)^2 - i(x+iy)^3 + ic = 2(x^2 + y^2 + 2xyi) - i(x^3 + 3x^2yi - 3xy^2 - iy^3) + ic$

$v = 4xy - x^3 + 3xy^2 + c$.

Ex: 118: $u = 2x - x^3 + 3xy^2$

We have $\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \phi_1(x,y)$; $\frac{\partial^2 u}{\partial x^2} = -6x$,

$\frac{\partial u}{\partial y} = 6xy = \phi_2(x,y)$; $\frac{\partial^2 u}{\partial y^2} = 6x$

Thus $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow u$ is harmonic

We have $f'(z) = \phi_1(z,0) - i\phi_2(z,0) = 2 - 3z^2$

$f(z) = u + iv = 2z - z^3 + c = 2(x+iy) - (x+iy)^3 + ic$

$= 2(x+iy) - (x^3 + 3x^2yi - 3xy^2 - iy^3) + ic$

$v = 2y - 3x^2y + y^3 + c$ where c is a constant.

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Ex: 119: $u = \frac{y}{x^2+y^2}$

We have, $\frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} = \phi_1(x,y)$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

$\frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} = \phi_2(x,y)$

$\frac{\partial^2 u}{\partial y^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$, Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

u is harmonic.

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$f'(x) = -2y$

$g'(x) = 4x(x^2+y^2)$

$f'(x)g(x) = -2y(x^2+y^2)^2$

$g'(x)f(x) = -2xy\,4x(x^2+y^2)$

$f'(x)g(x) - f(x)g'(x)$

$= -2y(x^2+y^2)^2 + 8x^2y(x^2+y^2)$

$\frac{\partial^2 u}{\partial x^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

Now by Milne method, we have

$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = \frac{-i}{z^2}$

$f(z) = \frac{i}{z} + c = u + iv = \frac{i}{x+iy} + c_1 + ic_2$

$= \frac{i(x-iy)}{x^2+y^2} + c_1 + ic_2$

$\therefore v = \frac{x}{x^2+y^2} + c_2$

Ex: 120: $u = e^{x^2-y^2}\cos 2xy$

We have $\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) = \phi_1(x,y)$

$\frac{\partial^2 u}{\partial x^2} = 2e^{x^2-y^2}[2(x^2-y^2)\cos 2xy - 4xy\sin 2xy + \cos 2xy]$

$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy) = \phi_2(x,y)$

$\frac{\partial^2 u}{\partial y^2} = -2e^{x^2-y^2}[2(x^2-y^2)\cos 2xy + \cos 2xy - 4xy\sin 2xy]$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic. Thus the harmonic conjugate v of u exists.

Now by Milne method we have, $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$= 2ze^{z^2} = e^{z^2}d(z^2)$

$f(z) = u + iv = e^{z^2} + c$

$= e^{x^2-y^2+2ixy} + c = e^{x^2-y^2}[\cos 2xy + i\sin 2xy + c_1 + ic_2]$

$\Rightarrow v = e^{x^2-y^2}\sin 2xy + c_2$

Ex: 121 $\quad u = x^2 - y^2 - 2xy - 2x + 3y$

We have $\frac{\partial u}{\partial x} = 2x - 2y - 2 = \phi_1(x,y)$; $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y - 2x + 3 = \phi_2(x,y)$ $\quad \frac{\partial^2 u}{\partial y^2} = -2$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; u is harmonic.

By the definition of i,
$i^2 = -1$
$i = 1$

Now by Milne method, we have $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$= 2z - 2 - i(-2z + 3)$

$= 2z(1+i) - (2+3i)$

$f(z) = u + iv = z^2(1+i) - (2+3i)z + c$

$= (1+i)(x^2 - y^2 + 2xyi) - (2+3i)(x+yi) + c_1 + ic_2$

$\therefore v = x^2 - y^2 + 2xy - 3x - 2y + c_2$ where $c = c_1 + ic_2$

Formula: $e^{ix} = \cos x + i\sin x$

122: $u = e^x(x\cos y - y\sin y)$

Then we have, $\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y) = \phi_1(x,y)$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y + 2\cos y)$ —— (I)

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = \phi_2(x,y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y)$ —— (II)

Now (I) + (II) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic function.

Now using Milne's method, we have $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$= e^z(z+1)$

Now integrating, $f(z) = e^z(z+1) - e^z + c = u + iv$

$= ze^z + c = (x+iy)e^{x+iy} + c$

$= (x+iy)e^x(\cos y + i\sin y) + c_1 + ic_2$

$\therefore v = e^x(x\sin y + y\cos y) + c_2$

Ex: 123: $u = 2x(1-y)$ Then we have $\frac{\partial u}{\partial x} = 2(1-y) = \phi_1(x,y)$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = -2x = \phi_2(x,y) \quad ; \quad \frac{\partial^2 u}{\partial y^2} = 0$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.}$$

Now using Milne's method, we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = 2 + i2z$$

Now integrating this $f(z) = 2z + iz^2 + c = u + iv$

$$= 2(x+iy) + i(x^2 - y^2 + 2xyi) + c$$

$$\therefore v = x^2 + 2y - y^2 + c_2 \quad \text{where } c = c_1 + ic_2$$

Ex: 124: $u = e^{-x}(x\sin y - y\cos y)$

Then we have

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y = \phi_1(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = \phi_2(x,y)$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + e^{-x}\cos y$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic.

Now using Milne's method, we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = i(z-1)e^{-z}$$

Now integrating we have, $f(z) = i(z-1)e^{-z} + ie^{-z} + c$

$$= ize^{-z} + c = u + iv$$

$$= i(x+iy)e^{-(x-iy)} + c$$

$$= i(x+iy)e^{-x}(\cos y - i\sin y) + c$$

$$\therefore v = xe^{-x}\cos y + y\sin y e^{-x} + c_2$$

$e^{-iy} = \cos(-y) + i\sin(-y)$
$= \cos y - i\sin y$

Ex: 125: If $f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$ then find $f(z)$ interms of z.

We have $f(z) = u + iv$ — (I) and

$if(z) = iu - v$ — (II) Then $(1+i)f(z) = (u+iv)i + (u-v)$ (I) + (II)

$= P + iQ$

Where $P = u - v = e^x(\cos y - \sin y)$

and $Q = u + v$

Now $\frac{\partial P}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$ and

$\frac{\partial P}{\partial y} = -e^x(\sin y + \cos y) = \phi_2(x,y)$.

Now by Milne method, we have

$F'(z) = \phi_1(z,0) - i\phi_2(z,0) = e^z + ie^z = e^z(1+i)$

$F(z) = e^z(1+i) + (1+i)c$

$(1+i)f(z) = \cancel{e^z + c}\ e^z(1+i) + (1+i)c$ $\therefore f(z) = e^z + c$

where we have considered $F(z) = (1+i)f(z)$ and the constant term in the integration $= (1+i)c$.

$f'(z) = \phi_1(z,0) - i$

Chapter: 05

Cauchy's integral formula: Let $f(z)$ be analytic ~~and~~ inside and on a simple closed curve C. If a is any point inside C then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$, where C is traversed in the positive sence, i.e. in the counter clockwise direction.

Theorem: If $f(z)$ is analytic inside and on the boundary C of a simply-connected closed region R, then

$$\frac{1}{2\pi i}\oint_C \left\{\frac{1}{z-a} \pm \frac{1}{z-b}\right\} f(z)\,dz$$

$$= \begin{cases} f(a) \pm f(b) & \text{if the points } a \text{ and } b \text{ are inside } C. \\ f(a) & \text{if only the point } a \text{ lies inside } C. \\ \pm f(b) & \text{if only the point } b \text{ lies inside } C. \\ 0 & \text{if the points } a \text{ and } b \text{ are not inside } C. \end{cases}$$

Ex: 182:- Show that $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = \begin{cases} e^2 & \text{if } C \text{ is the circle } |z| = \\ 0 & \text{if } C \text{ is the circle } |z| = \end{cases}$

First Part: Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$. Again $z = a = 2$ lies inside the circle $|z| = 3$ and the Cauchy's integral formula can be applied there.

Then $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = f(2) = e^2$

Second part: Here $f(z) = e^z/z-2$ is analytic inside and on the circle $|z| = 1$, then by the Cauchy's integral theorem, we have. $\oint_C f(z)\,dz = 0$

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0$$

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Ex: 183: Show that $\oint \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i$, where c is the circle $|z|=5$

Sol^n: Here $f(z) = \sin 3z$ is analytic inside and on the circle $|z|=5$. Again $z=-\pi/2$ lies inside the given circle and therefore by Cauchy's integral formula, we have

$$\oint_c \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i f(-\pi/2)$$

$$= 2\pi i \sin 3\left(\frac{-\pi}{2}\right)$$

$$= 2\pi i$$

Cauchy's integral formula for the first derivative of an analytic function:

let $f(z)$ be an analytic inside and on a simple closed curve C and a is a point inside c. Then $f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}dz$; $f''(a) = \frac{2!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^3}dz$; $f^n(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$, $n=1,2,3.-$

Ex: 196: Show that $\frac{1}{2\pi i}\oint_c \frac{ze^{tz}}{(z+1)^3}dz = \left(t-\frac{t^2}{2}\right)e^{-t}$. —— $z=-1$

Here $f(z) = ze^{zt}$ is analytic inside and on c and $a=-1$ is any point inside C. Then using the Cauchy's integral formula $\frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^n}dz = \frac{1}{n!}\left[\frac{d^n}{dz^n}f(z)\right]$ at $z=a$.

for $n=2$, we have, $\frac{1}{2\pi i}\oint_c \frac{ze^{tz}}{(z+1)^3}dz = \frac{1}{2!}\left[\frac{d^2}{dz^2}(ze^{tz})\right]_{z=-1}$

$$= \frac{1}{2}\left[\frac{d}{dz}(e^{tz} + zte^{tz})\right] \text{ at } z=-1$$

$$= \frac{1}{2}\left[te^{tz} + te^{tz} + zt^2e^{tz}\right] \text{ at } z=-1$$

$$= \frac{1}{2}\left[2te^{tz} + zt^2e^{tz}\right] \text{ at } z=-1$$

$$= \left(t - \frac{t^2}{2}\right)e^{-t} \quad \text{(show)}$$

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197: Show that $\oint_c \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3}$

Solⁿ: Here $f(z) = e^{2z}$ is analytic inside and on the given circle and $z = a = -1$ is a point inside the given circle. Then Cauchy's integral formula $f^n(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$

$$\oint_c \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}(e^{2z})\right] = \frac{2\pi i}{6} \times 2\times2\times2\, e^{2z} \quad \text{at } z=-1$$

$$\therefore \oint_c \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3} \quad \text{(show)}$$

199: Show that $\oint_c \frac{\sin^6 z}{(z-\pi/6)^3}dz = \frac{21\pi i}{16}$

Here $f(z) = \sin^6 z$ is analytic inside and on the circle $z=1$ and $a = \frac{\pi}{6}$ is a point inside the given circle. Then Cauchy's integral

$$\oint_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n}f(z)\right] \quad \text{at } z=a \quad \text{for } n=2$$

$$\oint_c \frac{\sin^6 z}{(z-\pi/6)^3}dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\sin^6 z\right] \quad \text{at } z=\pi/6$$

$$= \pi i\left[\frac{d}{dz}(6\cos z\sin^5 z)\right] \quad \text{at } z=\pi/6$$

$$= \pi i\left[-6\sin^6 z + 30\cos^2 z\sin^4 z\right] \quad \text{at } z=\pi/6$$

$$= \pi i\left[-6\left(\frac{1}{2}\right)^6 + 30\times\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2\right]$$

$$= \frac{(-6+90)\pi i}{64} = \frac{21\pi i}{16}.$$

200: Show that $\oint_c \frac{5z^2-3z+2}{(z-1)^3}dz = 10\pi i$

$$\oint_c \frac{5z^2-3z+2}{(z-1)^3} = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(5z^2-3z+2)\right] \quad \text{at } z=1$$

$$= \pi i[10] = 10\pi i$$

201: Show that $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{a^n e^{az}}{n!\, z^{n+1}}dz$; let $f(z) = e^{az}$ $f^{(n)}(z) = a^n e^{az}$ —(1)

Now putting $z=0$ in (1) we get $a^n = f^{(n)}(0)$; But by Cauchy's integ

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-1)^{n+1}}dz \Rightarrow f^{(n)}(0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{z^{n+1}}dz = a^n$$

$$\Rightarrow \frac{a^n}{n!} = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z^{n+1}}dz \; ; \quad \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{f(z)}{n!\, z^{n+1}}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{a^n e^{az}}{n!\, z^{n+1}}dz$$

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2.2: $\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta$

We have $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{a^n e^{az}}{n!\,z^{n+1}}\,dz$ ——— (1)

Now taking summation over n in (1) from 0 to ∞, then we have

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_c \frac{a^n e^{az}}{n!\,z^{n+1}}\,dz = \frac{1}{2\pi i}\oint_c e^{az}\sum_{n=0}^{\infty}\frac{a^n}{n!\,z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\oint_c e^{az}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n\frac{1}{n!}\,\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\oint_c e^{az}e^{a/z}\,dz \quad \text{[Taylor series]}$$

$$= \frac{1}{2\pi i}\oint_c e^{a(z+\frac{1}{z})}\,dz$$

Now let c be a circle of unit radius having centre at the origin, then $|z|=1 \Rightarrow z = e^{i\theta}$ $\quad dz = ie^{i\theta}d\theta$

$z+\frac{1}{z} = e^{i\theta}+e^{-i\theta} = 2\cos\theta$ and $0 \le \theta \le 2\pi$

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_0^{2\pi} e^{2a\cos\theta}\,\frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta$$ (Ans)

202: Show that $\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta$

We have $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c a^n e$

Taylor's And Laurent's Series

Taylor's Theorem: If $f(z)$ is analytic for all values of z inside a circle c with centre at a, then

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

Ex: 220: Expand $f(z) = \log(1+z)$ in a Taylor series about $z=0$

$f(z) = \log(1+z)$ then $f(0) = 0$

$f'(z) = \frac{1}{1+z}$ " $f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ " $f''(0) = -1$

$f'''(z) = \frac{(-1)(-2)}{(1+z)^3}$ " $f'''(0) = 2!$

$$\therefore \log(1+z) = f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots$$

$$= z - \frac{z^2}{2!} + \frac{z^3}{3!}\times 2! - \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$ Ans

Ex: 221: Expand $\cos z$ about $z = \pi/2$

We have $f(z) = \cos z$, Then $f'(z) = -\sin z$, $f''(z) = -\cos z$, $f'''(z) = \sin$

and $f(\pi/2) = 0$, $f'(\pi/2) = -1$, $f''(\pi/2) = 0$, $f'''(\pi/2) = 1$

Then about $z = \frac{\pi}{2}$, we have,

$$f(z) = f(\tfrac{\pi}{2}) + (z-\pi/2)f'(\pi/2) + \frac{(z-\pi/2)^2}{2!}f''(\pi/2) + \frac{(z-\pi/2)^3}{3!}f'''(\pi/2) + \cdots$$

$$= -(z-\pi/2) + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$ Ans

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202: Show that $\sum_{n=0} \left(\frac{a}{n!}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{2a\cos\theta} \, d\theta$

We have $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i} \oint_c a^n e$

Taylor's And Laurent's Series

Taylor's Theorem: If $f(z)$ is analytic for all values of z inside a circle c with centre at a, then

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

Ex: 220: Expand $f(z) = \log(1+z)$ in a Taylor series about $z=0$

$f(z) = \log(1+z)$ then $f(0) = 0$

$f'(z) = \frac{1}{1+z}$ " $f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ " $f''(0) = -1$

$f'''(z) = \frac{(-1)(-2)}{(1+z)^3}$ " $f'''(0) = 2!$

$$\therefore \log(1+z) = f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots$$

$$= z - \frac{z^2}{2!} + \frac{z^3}{3!} \times 2! - \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Ans

Ex: 221: Expand $\cos z$ about $z = \pi/2$

We have $f(z) = \cos z$, Then $f'(z) = -\sin z$ $f''(z) = -\cos z$ $f'''(z) = \sin$

and $f(\pi/2) = 0$, $f'(\pi/2) = -1$, $f''(\pi/2) = 0$, $f'''(\pi/2) = 1$

Then about $z = \frac{\pi}{2}$, we have,

$$f(z) = f\left(\frac{\pi}{2}\right) + (z-\pi/2)f'(\pi/2) + \frac{(z-\pi/2)^2}{2!}f''(\pi/2) + \frac{(z-\pi/2)^3}{3!}f'''(\pi/2) + \cdots$$

$$= -(z-\pi/2) + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

Ans

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Ex: 222: Expand $\sin z$ about $z = \pi/4$ in a Taylor series.

We have $f(z) = \sin z$, $f'(z) = \cos z$, $f''(z) = \sin z$, $f'''(z) = -\cos z$

and $f(\pi/4) = \frac{1}{\sqrt{2}}$, $f'(\pi/4) = \frac{1}{\sqrt{2}}$, $f''(\pi/4) = -\frac{1}{\sqrt{2}}$, $f'''(\pi/4) = -\frac{1}{\sqrt{2}}$ ...

Then about $z = a = \pi/4$, we have

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

$$= \frac{1}{\sqrt{2}} + (z-\pi/4)\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^2}{2!}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{(z-\pi/4)^3}{3!} + \cdots$$

$$= \frac{1}{\sqrt{2}}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\}$$

Ex: 223: Expand $\log\left(\frac{1+z}{1-z}\right)$ in a Taylor series.

If $|z|<1$, then we have,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots \quad \text{— (1)}$$

And $$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} - \cdots \quad \text{— (2)}$$

Now (1) − (2) $\Rightarrow \log(1+z) - \log(1-z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$

$$\Rightarrow \log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

for $|z|<1$. It can be also proved that this series coverges for $|z|<1$.

Application of Taylor Series:

1. Applied in finite differentce methods and Newton's method for solving equations.
2. Helps in gradient-based optimization.
3. Efficient traning in deep learning.
4. Helps in robotic trajectory planning and motion prediction.
5. Helps in marginal cost & revenue analysis in economics.

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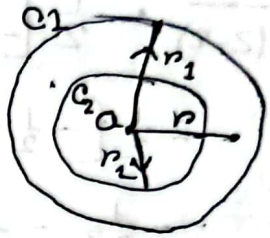
Laurent's theorem: If $f(z)$ is analytic inside and on the boundary of a ring-shaped region R bounded by two concentric circles C_1 and C_2 of centre a and radii r_1 and r_2 $(r_2 < r_1)$ respectively then for all z in R.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \text{ where}$$



$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, \quad n = 0, 1, 2, 3 \ldots$$

$$\text{and } a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, \quad n = 1, 2, 3 \ldots$$

Application of Laurent's theorem:

1. Residue theorem and complex integration.
2. Classifying singularities: (a) Removable
   (b) Poles
   (c) Essential
3. Solving partial differential equations.
4. Physics and Engineering applications.
5. Complex dynamics.

Ex: 225: Expand the function $f(z) = \frac{1}{z-3}$ in a laurent series (i) $|z| < 3$ (ii) $|z| > 3$

Sol^n: (i) We have $|z| < 3 \Rightarrow 3 > |z|$ and using this condition, we have $f(z) = \frac{1}{z-3} = -\frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} = -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \frac{z^4}{81} + \cdots\right)$

$$= -\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \frac{z^3}{81} - \cdots$$

(ii) We have $|z| > 3$ and using this we have $f(z) = \frac{1}{z-3}$

$$= \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \cdots\right)$$

$$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \cdots$$

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Ex: 226 : $f(z) = \frac{1}{z(z-2)}$ for the region (i) $0<|z|<2$ (ii) $|z|>2$

Sol^n:

Resolving into partial fractions, we have

$$f(z) = \frac{1}{2}\left[\frac{1}{z-2} - \frac{1}{z}\right] \quad \text{——— (1)}$$

We have $0<|z|<2$. Now using the condition $|z|<2$

(1) $\Rightarrow$ $\frac{1}{z(z-2)} = \frac{1}{2}\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\right]$

$$= \frac{1}{2}\left[-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}+\cdots\right) - \frac{1}{z}\right]$$

$$= -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \frac{z^4}{64} - \cdots$$

(ii) $\Rightarrow$ We have $|z|>2$ and using this condition (1) $\Rightarrow$

$$\frac{1}{z(z-2)} = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\right] = \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{2z} = \frac{1}{2z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)$$

$$= \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4} + \frac{8}{z^5} + \cdots$$

Ex: 227: $f(z) = \frac{z}{(z+1)(z+2)}$ ——— for the region $1<|z|<2$

Sol^n: We have $f(z) = \frac{z}{(z+1)(z+2)} = \frac{2}{(z+2)} - \frac{1}{(z+1)}$ ——— (1)

But we have $1<|z|<2 \Rightarrow 2>|z|$ and $|z|>1$

Then using the condition $\Rightarrow$

$$f(z) = \frac{2}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \cdots - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots$$

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228: $f(z) = \frac{1}{(z+1)(z+3)}$ given region (i) $1 < |z| < 3$ (ii) $|z| > 3$ (iii) $0 < |z+1| < 2$ (iv) $|z| < 1$

Soln: Resolving into partial fractions, we have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right) \quad \text{——— (1)}$$

(i) We have $1 < |z| < 3 \Rightarrow |z| > 1$ and $3 > |z|$

Now using this condition in equation (1) $\Rightarrow$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)$$

$$= \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots\right) - \left(\frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots\right)$$

(ii) We have $|z| > 3$. Both of the first and second term of the right hand side of (1). We will see the condition ~~(1)~~

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) -$$

$$\frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \left(\frac{1}{2z} - \frac{1}{2z}\right) + \left(\frac{3}{2} - \frac{1}{2}\right)\frac{1}{z^2} - \left(\frac{9}{2} - \frac{1}{2}\right)\frac{1}{z^3} + \cdots$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(iii) We have $0 < |z+1| < 2 \Rightarrow 2 > |z+1|$. Now using this condition

(1) $\Rightarrow \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2}\left\{\frac{1}{(z+1)+2}\right\}$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left\{1 + \frac{(z+1)}{2}\right\}^{-1}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left\{1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \cdots$$

(iv) We have $|z| < 1 \Rightarrow |z| < 1 < 3$ and using these (1)

$$\Rightarrow \frac{1}{(z+1)(z+3)} = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}(1+z/3)^{-1} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{1}{2} - \frac{1}{18}\right)z + \left(\frac{1}{2} - \frac{1}{54}\right)z^2 - \left(\frac{1}{2} - \frac{1}{162}\right)z^3 + \cdots$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

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Ex: 229: Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ — $|z|>2$.

Sol^n: We have $\frac{1}{z^2(z-1)(z+2)} = \frac{1}{z^2}\left[\frac{1}{3(z-1)} - \frac{1}{3(z+2)}\right]$

$= \frac{1}{3z^2}\left[\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}\right]$ $\quad [\because |z|>2]$

$= \frac{1}{3z^3}\left[\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right)\right]$

$= \frac{1}{3z^3}\left(\frac{3}{z} + \frac{3}{z^2} + \frac{3}{z^3} - \frac{15}{z^4} + \cdots\right)$

$= \frac{1}{z^4} - \frac{1}{z^5} + \frac{3}{z^6} - \frac{5}{z^7} + \frac{11}{z^8} - \cdots$

Ex: 230 Expand $f(z) = \frac{z^2}{(z-1)(z-2)}$ for $1<|z|<2$

We have $f(z) = \frac{z^2}{(z-1)(z-2)} = 1 - \frac{1}{z-1} + \frac{4}{z-2}$ — (1)

$1<|z|<2 \Rightarrow |z|>1$ and $2>|z|$.

Now using this condition (1) $\Rightarrow f(z) = 1 - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1-\frac{z}{2}\right)^{-1}$

$= 1 - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - 2\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$

$= \left(1-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right) - \left(2+z+\frac{z^2}{2}+\cdots\right)$

$= \cdots\cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - 1 - z - \frac{z^2}{2} - \cdots\cdots$

Ex: 232 $f(z) = \frac{1}{(z^2+1)(z+2)}$ in — $1<|z|<2$.

We have $f(z) = \frac{1}{(z^2+1)(z+2)} = \frac{1}{5}\left[\frac{1}{(z+2)} + \frac{z-2}{z^2+1}\right]$

$= \frac{1}{5}\left[\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}\right]$ $\quad [\because 2>|z|$ and $|z|>1]$

$= \frac{1}{10}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1}$

$= \frac{1}{10}\left(1-\frac{2}{z}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\cdots\right) - \frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right)$

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233: $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ (i) $2<|z|<3$ (ii) $|z|<2$ (iii) $|z|>3$

Sol^n: We have $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

(i) We have $2<|z|<3 \Rightarrow |z|>2$ and $3>|z|$ —— (2)

Now using (2), (1) $\Rightarrow$ $f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \cdots\right)$

$= \cdots - \frac{24}{z^4} + \frac{12}{z^3} - \frac{6}{z^2} + \frac{3}{z} - \frac{5}{3} + \frac{8z}{3^2} - \frac{8z^2}{3^3} + \frac{8z^3}{3^4} - \cdots$

(ii) We have $|z|<2$. Now using this (1) $\Rightarrow$

$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$

$= -\frac{1}{6} - \left(\frac{3}{2^2} - \frac{8}{3^2}\right)z + \left(\frac{3}{2^3} - \frac{8}{3^3}\right)z^2 - \left(\frac{3}{2^4} - \frac{8}{2^4}\right)z^3 + \cdots \cdots$

(iii) We have $|z|>3$ using (1) $\Rightarrow$

$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$

$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots\right)$

$= 1 + (3-8)\frac{1}{z} - (6-24)\frac{1}{z^2} + (12-72)\frac{1}{z^3} - \cdots \cdots$

$= 1 - \frac{5}{z} + \frac{18}{z^2} - \frac{60}{z^3} + \cdots$ (iii)

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23I: $f(z) = \dfrac{z^3}{(z+1)(z-2)}$ — $0 < |z+1| < 3$.

Sol^n: We have $f(z) = \dfrac{(z^3-8)+8}{(z+1)(z-2)}$

$$= \frac{(z-2)(z^2+2z+4)+8}{(z+1)(z-2)}$$

$$= \frac{z^2+2z+9}{(z+1)} + \frac{8}{(z+1)(z-2)}$$

$$= \frac{(z+1)^2+3}{z+1} - \frac{8}{3(z+1)} + \frac{8}{3(z-2)}$$

$$= (z+1) + \frac{3}{z+1} - \frac{8}{3(z+1)} + \frac{8}{3}\left\{\frac{1}{(z+1)-3}\right\}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left\{1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right\}$$

$$= \frac{1}{3(z+1)} - \frac{8}{9} + \frac{19}{27}(z+1) - \frac{8}{9}\left\{\frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\cdots\right\}$$

$=$

Chapter 07

Residues and the Residue Theorems

From Analytic function:

Singular point: A point at which a function $f(z)$ fails or create to be analytic is called a singular point.

Ex: Let $f(z) = \frac{z^2-4}{(z-1)(z-3)}$

Here $z=1$ and $z=2$ are two singular points since at these two points the function are not analytic.

Isolated singular point or singularity:

A singular point $z=z_0$ is called isolated singular point of a function $f(z)$ if we can find $\delta>0$ such that the circle $|z-z_0|=\delta$ encloses no other singularity of the function other than z_0. If there exists no such δ, in this case z_0 is called a non-isolated singularity of $f(z)$.

Ex 146: Let $f(z) = \frac{1}{(z-1)(z-3)}$, where $z=1$ and $z=3$ are the singular point of $f(z)$. Here $z=1$ is an isolated singular point of $f(z)$ since there exists at least one circle $|z-1|=\delta<2$ which does not include the singularity $z=3$. Similarly, the circle $|z-3|=\delta'<2$

Pole: A singular point $z=z_0$ is called a pole of order n of the function $f(z)$ if $\lim_{z\to z_0}(z-z_0)^n f(z) = A \neq 0$. Where n is a positive integer. If $n=1$, then $z=z_0$ = simple pole.

Ex: 147: let $f(z) = \frac{z-3}{(z-7)^5}$. Here $z=7$ is a pole of order 5 of the function $f(z)$ since $\lim_{z\to 7}(z-7)^5 f(z) = \lim_{z\to 7}(z-3) = 4 \neq 0$

Ex 148: let $f(z) = \frac{z^2+5}{(z-1)(z-3)^4(z-4)}$. Here $z=1,4$ are the simple poles of the function $f(z)$ and $z=3$ is a pole of order 4 of $f(z)$.

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Residue: If the function $f(z)$ has an isolated singularity at the point $z=a$, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laurent's expansion.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots +$$

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \cdots \quad \cdots$$

Application of Residue Theorem:

(a) It is primarily used in complex analysis to evalute complex line integrals.

(b) For calculating definite real-integrals that involve trigonometric or exponential functions.

(c) Used in single processing and control systems to solve integrals arising in Fourier Transform and Laplace Transform

(d) It is useful in finding closed-form solutions of divergent or slowly covering series.

(e) Used in solving problems related to potential flow and conformal mapping.

Residue at multiple point: If $f(z)$ is analytic inside and on a simple closed curve C ~~expect~~ except at pole $z=a$ of order m inside C. Then the residue of $f(z)$ at $z=a$ is

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

292 Ex: Evalute the residues of $f(z) = \frac{z^2}{z^2+a^2}$ at the poles.

Sol^n: The pole of $f(z)$ are given by $z^2+a^2=0 \Rightarrow z=\pm ai$ and they are simple poles.

Now $Res(ai) = \lim_{z\to ai}\{(z-ai)f(z)\} = \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{z^2+a^2}\right\}$

$$= \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{(z-ai)(z+ai)}\right\} \qquad \boxed{ai \times ai = -a^2}$$

$$= \lim_{z\to ai}\frac{z^2}{z+ai}$$

$$= \frac{(ai)^2}{2ai} = \frac{1}{2}ai$$

Similarly $Res(-ai) = -\frac{1}{2}ai$ Ans

252 Ex: $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+9)}$

Sol^n: The poles of $f(z)$ are obtained by solving $(z+1)^2(z^2+9)=0$

$f(z)$ has a double pole at $z=-1$ and simple pole at $z=\pm 3i$, Now the residue at $z=-1$ is

$$\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+9)}\right\}$$

$$= \lim_{z\to -1}\frac{d}{dz}\left\{\frac{z^2-2z}{z^2+9}\right\} = \lim_{z\to -1}\frac{z^2-2z}{z^2+9}\left\{\frac{2z-2}{z^2-2z} - \frac{2z}{z^2+9}\right\}$$

$$= \frac{3}{5}\cdot\left(\frac{-9}{3}+\frac{2}{5}\right) = -\frac{19}{15}$$

Residue at $z=3i$ is $\lim_{z\to 3i}\left\{(z-3i)\frac{z^2-2z}{(z+1)^2(z^2+9)}\right\}$

$$= \lim_{z\to 3i}\frac{z^2-2z}{(z+1)^2(z+3i)} = \frac{(3i)^2-2\cdot 3i}{(3i+1)^2\,6i} = \frac{-9-6i}{(9i^2+6i+1)6i}$$

$$= \frac{1+i}{4+3i} = \frac{7+i}{25} \quad [\text{Multiplying both side by } (4-3i)]$$

Similarly $Res(-3i)$ is $\frac{7-i}{25}$ Ans

- **Cauchy's Residue Theorem:** If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points $a_1, a_2 \ldots a_n$ inside C, then $\oint_C f(z)\,dz = 2\pi i\,[Res(a_1)+Res(a_2)+\ldots+Res(a_n)]$ times the sum of the Residues at the singularities within C.

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4 Application of Cauchy's Residue Theorem:

1. Evaluting complex integrals.
2. Computing Real definite integrals.
3. Solving improper integrals.
4. Summation of infinite series.
5. Stability analysis in control systems.

Ex: 257: $I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$

Sol^n: The poles of $\frac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving $z^2(z^2+2z+2)=0$. the integrand has a double pole at $z=0$ and two simple poles at $z = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$ and also these poles are inside C.

Now $Res(0)$ is $\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left(z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right)$

$$= \lim_{z\to 0}\left(te^{zt}(z^2+2z+2)^{-1} - e^{zt}(2z+2)(z^2+2z+2)^{-2}\right)$$

$$= \lim_{z\to 0}\frac{e^{zt}}{z^2+2z+2}\left[t - \frac{2z+2}{z^2+2z+2}\right] = \frac{t-1}{2}$$

Residue at $z=-1+i$ $= \lim_{z\to -1+i}\left\{(z+1-i)\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)} = \frac{-e^{-t}}{4}e^{ti}$$

Similarly $z=-1-i$ is $\frac{e^{-t}}{4}e^{-ti}$

Thus by the Cauchy's residue theorem

$$I = 2\pi i\left[\frac{t-1}{2} + \frac{e^{-t}}{4}(e^{ti}+e^{-ti})\right]$$

$$= 2\pi i\left[\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right]$$

$$\Rightarrow 2\pi i\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t.$$

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Ex: 258:- show that $\oint_c \frac{e^{tz}}{(z^2+1)^2}dz = \pi i(\sin t - t\cos t)$ $|z|=3$ and $t>0$

Solⁿ: Here the poles of $\frac{e^{tz}}{(z^2+1)^2} = \frac{e^{tz}}{(z-i)^2(z+i)^2}$ are at $z=\pm i$ inside c and both is of order two. Now the residue at $z=i$ is

$$\lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-i)^2e^{tz}}{(z-i)^2(z+i)^2}\right\} = \lim_{z\to i}\frac{d}{dz}\frac{e^{tz}}{(z+i)^2}$$

$$= \lim_{z\to i}\frac{t(z+i)^2e^{tz} - 2(z+i)e^{tz}}{(z+i)^4} = \lim_{z\to i}\frac{t(z+i)e^{tz} - 2e^{tz}}{(z+i)^3}$$

$$= \frac{2(it-1)e^{ti}}{(2i)^3} = \frac{(t+i)e^{ti}}{4}$$

similarly, the residue at $z=-i$ is

$$\frac{-(t-i)e^{-ti}}{4}$$

Then by the cauchy's residue theorem, $\oint_c \frac{e^{tz}}{(z^2+1)^2}dz$

$$= 2\pi i\left[\frac{(t+i)e^{ti}}{4} + \frac{-(t-i)e^{-ti}}{4}\right] = -\frac{\pi i}{2}\left\{t(e^{ti}+e^{-ti}) + i(e^{ti}-e^{-ti})\right\}$$

$$= \pi i(\sin t - t\cos t)$$

Ex: 252: show that $\oint_c \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{i}{\pi}$ where c is the circle $|z|=4$

Solⁿ: Here $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-i\pi)^2(z+i\pi)^2}$ has double pole at $z=\pi i$ and $z=-\pi i$

Now the reduce at $z=\pi i$ is $\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-\pi i)^2e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$

$$\lim_{z\to\pi i}\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\} = \lim_{z\to\pi i}\frac{(z+\pi i)^2e^z - 2e^z(z+\pi i)}{(z+\pi i)^4}$$

$$= \frac{(2\pi i-2)e^{\pi i}}{(\pi i+\pi i)^3} = \frac{\pi+i}{4\pi^3}$$

$f(z) = \frac{g}{h}$

$f'(\frac{g}{h}) = \frac{g'h - gh'}{h^2}$

similarily at $z=-\pi i$ $= \frac{\pi-i}{4\pi^3}$

then by the cauchy's residue theorem,

$$\oint_c \frac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i\left(\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$$

Ans

Ex: 260: Show that $\oint_c \frac{ze^z}{z^2-1}\,dz = 2\pi i \cosh(1)$ —— $|z|=2$

since $f(z) = \frac{ze^z}{z^2-1}$ has poles at $z = \pm 1$ and they are simple.

$$Res(1) = \lim_{z\to 1}\left\{(z-1)\frac{ze^z}{(z^2-1)}\right\} = \lim_{z\to 1}\frac{ze^z}{(z+1)} = \frac{e}{2}$$

$$Res(-1) = \frac{e^{-1}}{2}.$$

Then $\oint_c \frac{ze^z}{z^2-1}\,dz = 2\pi i\left(\frac{e}{2}+\frac{e^{-1}}{2}\right) = 2\pi i\cosh(1)$ (pm)

Ex: 261: $\oint_c \frac{e^z}{z(z-1)^2}\,dz = 2\pi i$ —— $|z|=2$

The poles of $f(z) = \frac{e^z}{z(z-1)^2}$, here $z=0$ is a simple pole and $z=1$ is a double pole.

$$Res(0) = \lim_{z\to 0}\left\{z\frac{e^z}{z(z-1)^2}\right\} = \lim_{z\to 0}\frac{e^z}{(z-1)^2} = 1$$

$$Res(1) = \lim_{z\to 1}\frac{d}{dz}\left\{(z-1)^2\frac{e^z}{z(z-1)^2}\right\} = \lim_{z\to 1}\frac{d}{dz}\frac{e^z}{z}$$

$$= \lim_{z\to 1}\frac{ze^z - e^z}{z^2} = 0$$

Now by the residue theorem, we have

$$\oint_c \frac{e^z}{z(z-1)^2}\,dz = 2\pi i\left[Res(0)+Res(1)\right]$$

$$= 2\pi i(1+0) = 2\pi i$$

Ch. 08

Contour Integration

We will evalute a varity types of real definite integrals with the help of the Cauchy's residue theorem using a suitable types of closed path or contour. For this reason, the process is called contour integration.

(FORM 01)

Integrals of the form:

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$, where ~~f(sinθ, cos~~ $f(\cos\theta, \sin\theta)$ is a national function of $\cos\theta$ and $\sin\theta$.

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ ---- ① where $f(\cos\theta, \sin\theta)$ is a national function of $\cos\theta$ and $\sin\theta$. Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

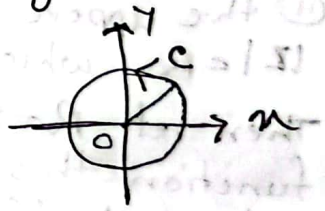
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$ and $dz = ie^{i\theta} d\theta$

$= iz\, d\theta$

or $d\theta = \frac{dz}{iz}$ Now using these

$I = \oint_C g(z)\, dz$

where C is the unit circle $|z| = 1$, whose centre is at the origin ~~and~~ and radius is



Ex: 276: Show that $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi$

Solⁿ:

Suppose $I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$ —— ①

let $z = e^{i\theta}$, then $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $dz = ie^{i\theta} d\theta$

$dz = iz\, d\theta$

Now using these (1) $\Rightarrow I = \oint_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}$ —— ②, $\therefore d\theta = \frac{dz}{iz}$

where C is the circle $|z| = 1$,

The poles of $\frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}$ are obtained by solving

$(1-2i)z^2 - 6iz - 1 - 2i \Rightarrow z = \frac{-6i \pm 4i}{2(1-2i)}$ [L. Hospital rule]

$\Rightarrow z = \frac{1}{5}(2-i)$, $2-i$ and they are simple.

of them only the pole $\frac{1}{5}(2-i)$ lies inside C. Now

$$\text{Res}\left\{\tfrac{1}{5}(2-i)\right\} = \lim_{z\to\frac{1}{5}(2-i)} \left[\left\{z-\tfrac{1}{5}(2-i)\right\}\frac{2}{(1-2i)z^2+6iz-1-2i}\right]$$

$$= \lim_{z\to\frac{1}{5}(2-i)} \frac{2}{2(1-2i)z+6i}$$

$$= \frac{5}{2-2-i-4i+15i} = \frac{1}{2i}$$

Now by the Cauchy's residue theorem (2) $\Rightarrow$

$$I = 2\pi i \,\text{Res}\left\{\tfrac{1}{5}(2-i)\right\} = 2\pi i \cdot \frac{1}{2i} = \pi$$

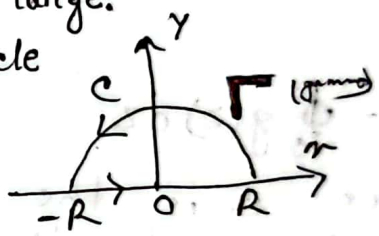
FORM 02

Integrals of the form: $\int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is a rational function of x:

Consider $\oint_C f(z)\,dz$, where C is the contour consisting of:

(i) the x-axis from $-R$ to R, where R is large.

(ii) the upper semi-circle Γ (gamma) of the circle $|z|=R$ which lies above the x-axis.



Then let $R\to\infty$ and if $f(x)$ is an even function, this can be used to evalute the integral $\int_0^{\infty} f(x)\,dx$.

Ex 290: Show that $\int_0^{\infty}\frac{dx}{x^6+1} = \frac{\pi}{3}$.

Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{1}{z^6+1}$ and C is the contour consisting of:

(i) the x-axis from $-R$ to R where R is large;

(ii) the upper semi-circle Γ of the circle $|z|=R$. Which lies above the x-axis.

Now the poles of $f(z)$ are given by $x^6+1=0 \Rightarrow z^6=-1$

$$z^6 = e^{(2n+1)\pi i}$$

$$\Rightarrow z = e^{(2n+1)\pi i/6}$$

where $n = 0, 1, 2, 3, \ldots$

$$\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{(2n+1)\pi i} = -1 \\ n = 0, 1, 2, 3 \ldots \end{array}}$$

$z = e^{\pi i/6}, e^{\pi/2}, e^{5\pi i/6}, e^{7\pi i/6} \ldots$ and they are simple. Of them only the poles $z = e^{\pi i/6}, e^{\pi i/2}$ and $e^{5\pi i/6}$ lie within C.

Now using L. Hospital's rule, the residue at the pole

$$z = a = \lim_{z \to a} (z-a) f(z) = \lim_{z \to a} \frac{z-a}{z^6+1} = \lim_{z \to a} \frac{1}{6z^5} = \frac{1}{6a^5}$$

$$\frac{1}{6a^5} = \begin{cases} \frac{1}{6} e^{-5\pi i/6} = \frac{1}{6} e^{-\pi i} e^{\pi i/6} = -\frac{1}{6} e^{\pi i/6} & \text{at } z = e^{\pi i/6} \\ \frac{1}{6} e^{-5\pi i/2} = \frac{1}{6} e^{-2\pi i} \cdot e^{-\pi i/2} = -\frac{i}{6} & \text{at } z = e^{\pi i/2} \\ \frac{1}{6} e^{-25\pi i/6} = \frac{1}{6} e^{-4\pi i} \cdot e^{-\pi i/6} = \frac{1}{6} e^{-\pi i/6} & \text{at } z = e^{5\pi i/6} \end{cases}$$

Now by the Cauchy's residue theorem, we have.

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i \left(-\frac{1}{6} e^{\pi i/6} - \frac{i}{6} + \frac{1}{6} e^{-\pi i/6}\right) = \frac{-\pi i}{3} \left(e^{\pi i/6} - e^{-\pi i/6} + i\right)$$

$$= \left(-\frac{\pi i}{3}\right)\left(2i \sin\frac{\pi}{6} + i\right) = \frac{\pi}{3}(1+1) = \frac{2\pi}{3} \quad \text{— (1)}$$

Here $\lim_{z \to \infty} z f(z) = \lim_{z \to \infty} \frac{z}{z^6+1} = 0$ then $\lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = 0$ — (2)

Now taking limit $R \to \infty$ in (1) and using (2)

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3} \Rightarrow 2\int_0^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \Rightarrow \int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

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FORM : 03

$\int_{-\infty}^{\infty} f(x)\cos mx\,dx$ or $\int_{-\infty}^{\infty} f(x)\sin mx\,dx$ where $f(x)$ is a rational function of x: Consider $\oint_C e^{imx} f(z)\,dz$, $m>0$ where C is the contour consisting of:

(i) the x-axis from $-R$ to R where R is large.

(ii) the upper semi-circle Γ of the circle $|z|=R$, which lies above the x-axis.

Then let $R\to\infty$ and if $f(x)$ is an even function, this can be used to evalute the integral $\int_{-\infty}^{\infty} e^{imx} f(x)\,dx$, $m>0$.

Jordan Lemma: $\lim_{R\to\infty}\int_{\Gamma} e^{imz} f(z)\,dz = 0$, wher Γ is the upper semi-circle of Radius R and m is a positive constant

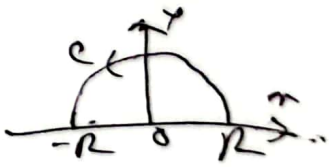
Ex: 296 show that $\int_0^{\infty}\frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-ma}$ $m\geq 0$ and $a>0$

Sol^n: Consider $\oint_C f(z)\,dz$ where $f(z)=\frac{e^{imz}}{z^2+a^2}$ and C is the contour consisting of:

(i) the x-axis from $-R$ to R where R is large.

(ii) the upper semi-circle Γ of the circle $|z|=R$ which lies above the x-axis.

The poles of $f(z)=\frac{e^{imx}}{z^2+a^2}$ are obtained by solving $z^2+a^2=0 \Rightarrow z=\pm ai$ and they are simple poles. Of which only $z=ai$ lies inside C.



$$\text{Res}[f(z).ai] = \lim_{z\to ai}\left\{\frac{(z-ai)e^{imz}}{(z-ai)(z+ai)}\right\} = \frac{e^{-ma}}{2ai}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i\,\text{Res}[f(z).ai]$$

Here $\lim_{z\to\infty}\frac{1}{z^2+a^2}=0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where

$$= 2\pi i\,\frac{e^{-ma}}{2ai} = \frac{\pi e^{-ma}}{a} \quad \text{—(1)}$$

$F(z)=\frac{1}{z^2+a^2}$ and then by the Jordan's lemma, $\lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0$

Now taking limit $R \to \infty$ in ① $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{a} \Rightarrow 2\int_0^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{a}$$

$$\therefore \int_0^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2a} \quad \text{(Ans)}$$

N.T. Practices $\Rightarrow$ 297(a)(b)(c), 301

301: show that $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx$ where $a>0$ and $m \geq 0$

Sol^n: Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ and C is the contour consisting of:

(i) the x-axis from $-R$ to R where R is large;

(ii) the upper semi-circle Γ of the circle $|z|=R$ which lies above the x-axis. The poles of $f(z) = \frac{e^{imz}}{(z^2+a^2)^2}$ are obtained by solving $(z^2+a^2)^2 = 0 \Rightarrow z = \pm ai$ which are of order 2. Only the pole $z = ai$ lies inside C. Now $\text{Res}(ai) = \lim_{z\to ai} \frac{d}{dz}\left\{(z-ai)^2 \frac{e^{imz}}{(z-ai)^2(z+ai)^2}\right\}$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{\frac{e^{imz}}{(z+ai)^2}\right\}$$

$$= \lim_{z\to ai}\left[\frac{e^{imz}}{(z+ai)^2}\left\{im - \frac{2}{z+ai}\right\}\right]$$

$$= \frac{e^{-ma}}{(2ai)^2}\left\{mi - \frac{2}{2ai}\right\} = -\frac{e^{-ma}}{4a^2}\left(\frac{-am-1}{ai}\right) = \frac{e^{-ma}(ma+1)}{4a^3 i},$$

Then by the Cauchy's residue theorem, we have $\oint_C f(z)\,dz$

$$= \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i\,\text{Res}(ai) = 2\pi i\left\{\frac{e^{-ma}(ma+1)}{4a^3 i}\right\}$$

$$= \frac{\pi(1+ma)e^{-ma}}{2a^3} \quad \text{——— ①}$$

Here $\lim_{z\to\infty} \frac{1}{(z^2+a^2)^2} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$ where $F(z) = \frac{1}{(z^2+a^2)^2}$ and then by the Jordan's lemma,

$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$ .... ② Now taking limit $R\to\infty$ in ① and using ② $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi(1+ma)e^{-ma}}{2a^3} \Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi(1+ma)e^{-ma}}{2a^3}$

$$2\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi(1+ma)e^{-ma}}{2a^3}$$

$$\therefore \int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{\pi(1+ma)e^{-ma}}{4a^3} \quad \text{(Ans)}$$

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