United
Writting Note Book

CCE 311

Numerical Methods

Name : Reem
Subject:
Institute:
Class : Section :
Roll No : 1802071 Year :

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Reference Books

1. Numerical Methods for Engineers by Chapra

United

Sub.: Lecture - 1 (MB)

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Difference between Determinants and Matrix:

1. Matrix is the set of numbers which are covered by two brakets. Determinants is also set of numbers but it is covered by two bars.

2. It is not necessary that number of rows will be equal to the number of columns in matrix. But it is necessary that number of rows will be equal to the number of columns in determinants.

3. Matrix can be used for adding, subtracting and multiplying the coefficients. Determinant can be used for calculating the value of x, y and z with Cramer's Rule.

Matrix is a shape

Determinant is number

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(Mid) **Cramer's Rule:** Best suited to small number of equations.

Example 9.3: Use Cramer's Rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01 \; (b_1)$$

$$0.5x_1 + x_2 + 1.9x_3 = 0.67 \; (b_2)$$

$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44 \; (b_3)$$

Solution: The determinant D can be written as

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

The minors are

$$A_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1 \times 0.5 - 1.9 \times 0.3 = -0.07$$

$$A_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5 \times 0.5 - 1.9 \times 0.1 = 0.06$$

$$A_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5 \times 0.3 - 1 \times 0.1 = 0.05$$

Sub.: $D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

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These minors can be used to evaluate the determinant

$$D = 0.3\times(-0.07) - 0.52\times 0.06 + 1\times(0.05) = -0.0022$$

Now applying the equation of the cramer's rule we get,

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

(Ans)

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* When Cramer's Rule becomes impractical?

⇒ For more than three equations, Cramer's rule becomes impractical because, as the number of equations increases, the determinants are time consuming to evaluate by hand (or by computer).

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(Mid) # Gauss Elimination:

Two Steps: 1) Forward Elimination

2) Back Substitution

Example 9.5: Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{------} (1)$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad \text{------} (2)$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad \text{------} (3)$$

Solution: The first part of the procedure is forward elimination. Now, to remove x_1 from (2), (3), (2) − (1) × $0.1/3$ and (3) − (1) × $0.3/3$ ⇒

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{------} (4)$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \quad \text{------} (5)$$

$$-0.190000x_2 + 10.0200x_3 = 70.6150 \quad \text{------} (6)$$

To complete the forward elimination, x_2 must be removed from (6). To remove x_2, (6) − (5) × $(-0.190000)/7.00333$ ⇒

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad \text{------} (7)$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \quad \text{------} (8)$$

$$10.0120x_3 = 70.0843 \quad \text{------} (9)$$

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We can now solve these equations by back substitution. From equ^n 9 we get,

$$x_3 = \frac{70.0843}{10.0120} = 7.0000 \text{ -------- } (10)$$

This result can be back substituted into equ^n 8

$$7.00333x_2 - (0.293333 \times 7.0000) = -19.5617$$

$$\Rightarrow x_2 = \frac{-19.5617 + 0.293333 \times 7.0000}{7.00333} = -2.50000 \text{ ---- } (11)$$

Finally equ^n 10 & 11 can be substituted in equ^n 4

$$3x_1 - 0.1 \times (-2.500) - 0.2 \times 7.0000 = 7.85$$

$$\Rightarrow x_1 = \frac{7.85 - 0.25 + 1.4}{3} = 3.00000$$

The results are identical to the exact solution of

$x_1 = 3$, $x_2 = -2.5$ and $x_3 = 7$

(Ans)

(Mid) **Gauss-Jordan:** The Gauss-Jordan method is a variation of Gauss elimination.

Example 9.12: Use Gauss-Jordan technique to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution: First, we have to express the coefficients and the right-hand side as an argumented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix} \quad r_1' = r_1/3$$

Then,

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.293333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \times 0.1 \\ r_3' = r_3 - r_1 \times 0.3 \end{matrix}$$

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Next,

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & \cancel{19}\ -0.190000 & 10.0200 & 70.6150 \end{bmatrix} r_2' = r_2/7.00333$$

Reduction of x_2 terms from first and third equation gives,

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix} \begin{matrix} r_1' = r_1 + 0.0333 r_2 \\ r_3' = r_3 + 0.1900 r_2 \end{matrix}$$

Then,

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 252356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix} r_3' = r_3/10.01200.$$

Finally reducing x_3 terms from equation one and two we get,

$$\begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix} \begin{matrix} r_1' = r_1 + 0.0680 r_3 \\ r_2' = r_2 + 0.0418 r_3 \end{matrix}$$

(Ans)

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Difference between Gauss elimination and Gauss-Jordan:

1. When an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations but in Gauss elimination, just the subsequent ones are eliminated.

2. In Gauss-Jordan method, the elimination steps result in an identity matrix whereas, in Gauss elimination, the elimination steps results in a triangular matrix.

3. It is not necessary to employ back substitution to obtain the solution ~~of~~ in Gauss-Jordan method, but in Gauss elimination, the back substitution is necessary for the solution.

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Mid ✓ Factorization Process, Triangularization Method,

Cholesky's Process:

Bala Chandra Rao Book 266 page

$AX = B \Rightarrow LUX = B \Rightarrow LY = B$ then $UX = Y$

একটি matrix কে তার lower & upper triangular matrix multiplication হিসেবে দেখানো যায়।

✓Example: Apply the Cholesky's process to locate the root of the following system -

$$x_1 + x_2 - x_3 = 2$$

$$2x_1 + 3x_2 + 5x_3 = -3$$

$$3x_1 + 2x_2 - 3x_3 = 6$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

Comparing the values,

$u_{11}=1$, $u_{12}=1$, $u_{13}=-1$, $l_{21}u_{11}=2$, so that $l_{21}=2$ $\because u_{11}=1$

Similarly we can find out another unknown,

$$AX = B \Rightarrow LUX = B$$

i.e

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} X = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Let, $UX = Y \Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

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$y_1 = 2,$

$2y_1 + y_2 = -3 \Rightarrow 4 + y_2 = -3 \Rightarrow y_2 = -7$

$3y_1 - y_2 + y_3 = 6 \Rightarrow y_3 = 6 - (3 \times 2) + (-7) = -7$

$UX = Y$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}$$

$7x_3 = -7 \Rightarrow x_3 = -1$

$x_2 + 7x_3 = -7 \Rightarrow x_2 - 7 = -7 \Rightarrow x_2 = 0$

$x_1 + x_2 - x_3 = 2 \Rightarrow x_1 + 0 - 1 = 2 \Rightarrow x_1 = \cancel{3}\,1$

$\therefore x_1 = \cancel{3}\,1, \quad x_2 = 0, \quad x_3 = -1.$

(Ans)

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* Gauss Elimination & Gauss Jordan – Code করে নিতে হবে

* Least Square Regression – brief

2 parallel lines — no solution

1 straight line on another — infinite solution

2 straight lines very close (পাশাপাশি) - ill condition

Linear Regression (Mid)

The simplest example of a least-squares approximation is fitting a straight line to a set of paired observations: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

The mathematical expression for the straight line is

$$\boxed{y = a_0 + a_1 x + e}$$

where a_0 and a_1 are coefficients representing the intercept and the slope respectively.

e is the error or residual between the model and the observations.

$$e = y - a_0 - a_1 x \quad \text{-----} \quad (1)$$

Linear regression finds a linear function that most nearly passes through the given data points - the regression (function) line best fits the data.

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* Criteria for a "best" fit:

For fitting a best line through the data যখন would be minimize হবে যখন sum of the residual errors (for all available data) minimize হবে, i.e.

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i) \quad \text{---} (1)$$ where, n = total no. of points

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i|$$

Let, $$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \quad \text{-----} (2)$$

$$\frac{\delta S_r}{\delta a_0} = -2 \sum (y_i - a_0 - a_1 x_i) \quad \text{-----} (3)$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum (y_i - a_0 - a_1 x_i) x_i \quad \text{-----} (4)$$

$$(3) \Rightarrow 0 = \sum y_i - \sum a_0 - \sum a_1 x_i \quad \text{-----} (5)$$

$$(4) \Rightarrow 0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2 \quad \text{-----} (6)$$

$$(5) \Rightarrow n a_0 + \left(\sum x_i\right) a_1 = \sum y_i \quad \text{--------} (7)$$

$$(6) \Rightarrow \left(\sum x_i\right) a_0 + \left(\sum x_i^2\right) a_1 = \sum x_i y_i$$

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~~⑦ ⇒ $na_0 + (\sum x_i)a_1 = \sum y_i$~~

From ⑥ & ⑦,

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

⑦ $\Rightarrow na_0 + (\sum x_i)a_1 = \sum y_i$

$\Rightarrow na_0 = \sum y_i - (\sum x_i)a_1$

$\Rightarrow a_0 = \frac{\sum y_i}{n} - \frac{(\sum x_i)a_1}{n}$

$\Rightarrow a_0 = \bar{y} - a_1\bar{x}$

** Find out the value of a_0 and a_1 in the case of least square regression.

$\boxed{\text{Example - 17.1}}$

Next Class → Polynomial Regression

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Example 17.1: Fit a straight line to the x and y values in the first two columns of the table.

Computations for an error analysis of the linear fit:

x_i	y_i	$(y_i - \bar{y})$	$(y_i - a_0 - a_1 x_i)^2$
1	0.5	8.5765	0.1687
2	2.5	0.8622	0.5625
3	2.0	2.0408	0.3473
4	4.0	0.3265	0.3265
5	3.5	0.0051	0.5896
6	6.0	6.6122	0.7972
7	5.5	4.2908	0.1993
Σ	24.0	22.7143	2.9911

Solution: The following quantities can be computed–

$n = 7$ $\quad \Sigma x_i y_i = 119.5$ $\quad \Sigma x_i^2 = 140$

$\Sigma x_i = 28$ $\quad \bar{x} = \frac{28}{7} = 4$

$\Sigma y_i = 24$ $\quad \bar{y} = \frac{24}{7} = 3.428571$

Now,

$$a_1 = \frac{n \Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n \Sigma x_i^2 - (\Sigma x_i)^2} = \frac{7(119.5) - 28(24)}{7(140) - (28)^2} = 0.8392857$$

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$$a_0 = \bar{y} - a_1\bar{x}$$

$$= 3.428571 - 0.8392857(4)$$

$$= 0.07142857$$

Therefore, the least- square fit is

$$y = a_0 + a_1 x$$

$$\Rightarrow y = 0.07142857 + 0.8392857x$$

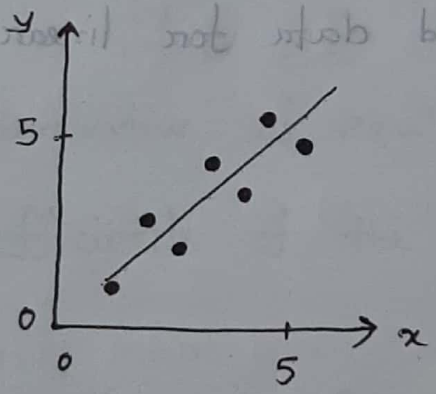

Fig 17.1(c) : More satisfactory result using the least-square fit

Polynomial Regression : (Mid)

The process of going back to an earlier or less advanced form or state.

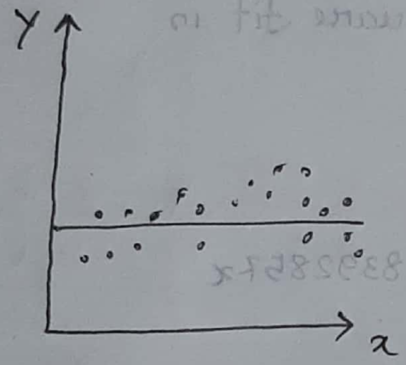

Fig: ill-suited data for linear regression

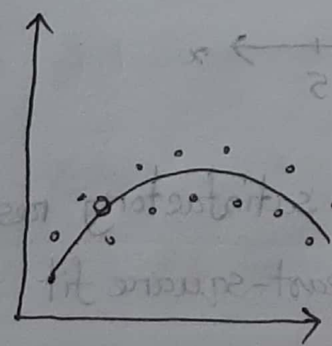

Fig: parabola is preferable

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* The least-squares procedure can be readily extended to fit the data to a higher-order polynomial. For example, Lets suppose that we fit a second-order polynomial or ~~quardic~~ quadratic:

$$y = a_0 + a_1 x + a_2 x^2 + e$$

For this case the sum of the squares of the residuals is

$$S_r = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_1 x_i^2\right)^2 \text{ - - - - - - } (1)$$

Taking the derivative of eqn (1) w.r.t each of the unknown coefficients of the polynomial, as in

$$\frac{\delta S_r}{\delta a_0} = -2 \sum \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum x_i \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)$$

$$\frac{\delta S_r}{\delta a_2} = -2 \sum x_i^2 \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)$$

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These equations can be set equal to zero and rearranged to developed the following set of normal equations:

$$(n) a_0 + \left(\sum x_i\right) a_1 + \left(\sum x_i^2\right) a_2 = \sum y_i$$

$$\left(\sum x_i\right) a_0 + \left(\sum x_i^2\right) a_1 + \left(\sum x_i^3\right) a_2 = \sum x_i y_i$$

$$\left(\sum x_i^2\right) a_0 + \left(\sum x_i^3\right) a_1 + \left(\sum x_i^4\right) a_2 = \sum x_i^2 y_i$$

Standard error is formulated as:

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

Example 17.5: Fit a second-order polynomial to the data in the first two colums of the table:

x_i	y_i	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08158
3	27.2	3.12	0.80491
4	40.9	239.22	0.61951
5	61.1	1272.11	0.09439
Σ	152.6	2513.39	3.74657

Solution: From the given data,

order $m = 2$ $\quad \Sigma x_i = 15$ $\quad \Sigma x_i^4 = 979$

$n = 6$ $\quad \Sigma y_i = 152.6$ $\quad \Sigma x_i y_i = 585.6$

$\bar{x} = 2.5$ $\quad \Sigma x_i^2 = 55$ $\quad \Sigma x_i^2 y_i = 2488.8$

$\bar{y} = 25.433$ $\quad \Sigma x_i^3 = 225$

Therefore, simultaneous linear equations are -

$$\begin{bmatrix} 6\, n & 15\, \Sigma x_i & 55\, \Sigma x_i^2 \\ 15\, \Sigma x_i & 55\, \Sigma x_i^2 & 225\, \Sigma x_i^3 \\ 55\, \Sigma x_i^2 & 225\, \Sigma x_i^3 & 979\, \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix} \end{array} \begin{array}{l} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{array}$$

using Gauss elimination we get,

$$a_0 = 2.47857$$

$$a_1 = 2.35929$$

$$a_2 = 1.86071$$

$\therefore$ The least-squares quadratic equation for this case is,

$$y = a_0 + a_1 x + a_2 x^2$$

$$\Rightarrow y = 2.47857 + 2.35929x + 1.86071x^2$$

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The standard error of the estimate based on the regression polynomial is

$$S_{y/x} = \sqrt{\frac{3.74657}{6-3}}$$

$$= 1.12$$

The coefficient of determination is

$$r^2 = \frac{2513.39 - 3.74657}{2513.39} = 0.99851$$

and the correlation coefficient is, $r = 0.99925$

These results indicate that 99.851 percent of the original uncertainty has been explained by the model.

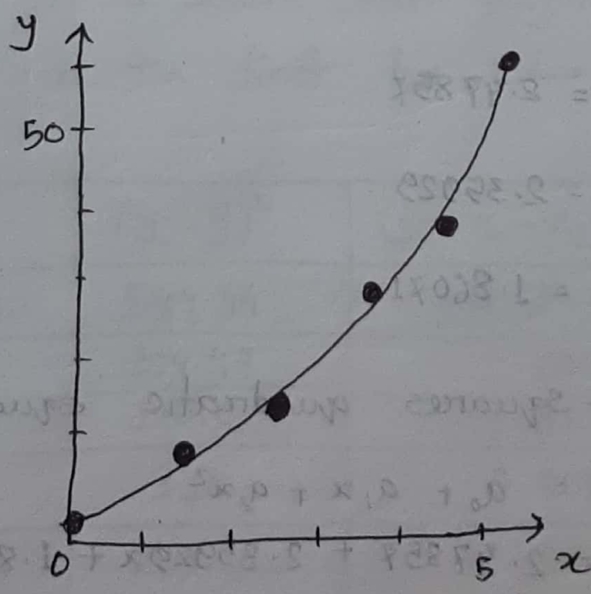


Fig: fit of a second-order polynomial

Trapezoidal Rule (Mid)

The trapezoidal rule is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in the equ^n is first order:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

A straight line can be represented as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

The area under this straight line is an estimate of the integral of $f(x)$ between the limits a and b:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right] dx$$

$$= \left[f(a) \cdot x + \frac{f(b) - f(a)}{b-a} \cdot \frac{(x-a)^2}{2} \right]_a^b$$

$$= f(a) \cdot b - f(a) \cdot a + \frac{f(b) - f(a)}{(b-a)} \cdot \frac{(b-a)^2}{2}$$

$$= f(a)\cdot(b-a) + \frac{[f(b)-f(a)]\cdot(b-a)}{2}$$

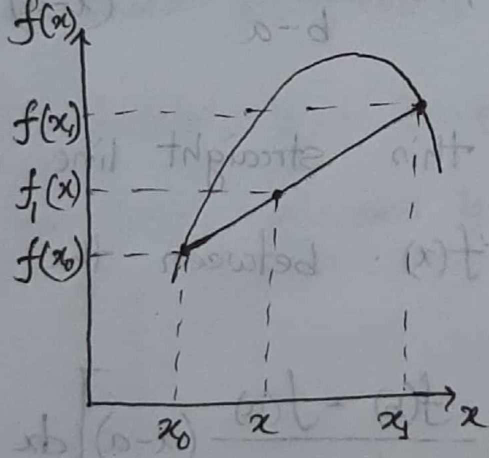
$$= \frac{2f(a)(b-a) + f(b)(b-a) - f(a)(b-a)}{2}$$

$$= \frac{f(a)(b-a) + f(b)(b-a)}{2}$$

$$\therefore I = \underbrace{(b-a)}\cdot\underbrace{\frac{f(a)+f(b)}{2}}$$

$I \simeq$ width $\times$ average height

This is the formula for the trapezoidal rule.



$$\frac{f_1(x)-f(x_0)}{x-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$\Rightarrow f_1(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$$

Example 21.1: Use Trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$; exact value $= 1.640533$

Solution: The function values

$$\varepsilon_t = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$$

V_A = actual value observed

V_E = expected value/ exact value

$$f(0) = 0.2$$

$$f(0.8) = 0.232$$

using trapezoidal rule we get,

$$I \cong \underbrace{0.8}_{(b-a)} \frac{\overset{f(a)}{0.2} + \overset{f(b)}{0.232}}{2} = 0.1728$$

which represents an error of

$$\varepsilon_t = \frac{E_t}{\text{exact value}} \times 100\%$$

$$E_t = 1.640533 - 0.1728 = 1.467733$$

which corresponds to percent relative error, $\varepsilon_t = 89.5\%$

The reason for this large error is evident from the figure as we can see that the area under the straight line neglects a significant portion of the integral

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lying above the line.

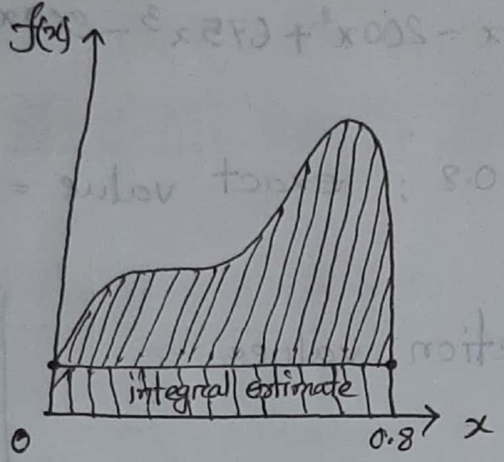

Fig: single application of Trapezoidal rule

* How to remove the error of Trapezoidal rule?

⇒ by using Simpsom's 1/3 Rule.

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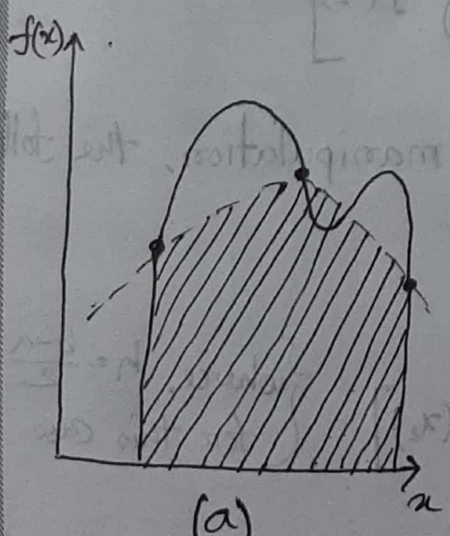
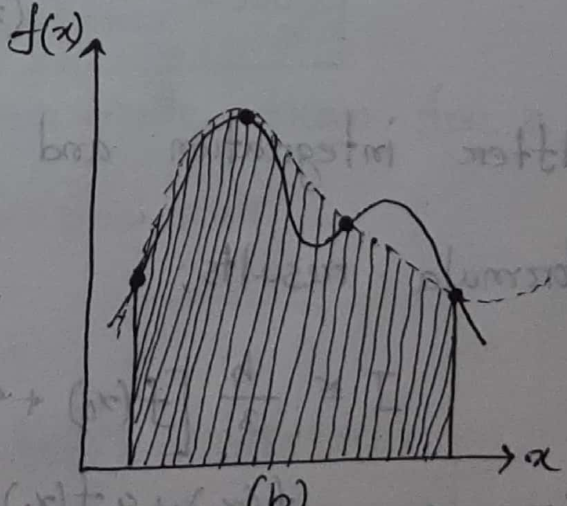
Simpson's Rules (Mid)

Another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points. For example, if there is an extra point midway between f(a) and f(b), the three points can be connected with a parabola.

If there are two points equally spaced between f(a) and f(b), the four points can be connected with a third-order polynomial. The formulas that result from taking the integrals under these polynomials are called Simpson's rule.



[1/3 rule]



[3/8 rule]

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* Simpson's 1/3 Rule (Mid)

Simpson's 1/3 rule results when a second-order interpolating polynomial is substituted into

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a 2nd-order Lagrange polynomial, the integral becomes,

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integration and algebric manipulation, the following formula results;

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

where, $h = \frac{b-a}{2}$ for this case

$$I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{average height}}$$

This is called Simpson's 1/3 rule as h is divided by 3.

Example 21.4: Use Simpson's 1/3 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$; exact integral is 1.640533

Solution: The function values

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

using Simpson's 1/3 rule,

$$I \cong 0.8 \, \frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

which represents an exact error of

$$E_t = 1.640533 - 1.367467 = 0.2730667$$

$$\varepsilon_t = 16.6\% \qquad \boxed{\varepsilon_t = \frac{E_t}{\text{exact value}} \times 100\%}$$

which is 5 times more accurate than for a single application of the trapezoidal rule.

$$\boxed{\text{The estimated error is} \quad E_a = -\frac{(0.8)^5}{2880}(-2400) = 0.2730667}$$

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* Simpson's 3/8 Rule: (Mid)

If there are two points equally spaced between $f(a)$ and $f(b)$, the four points can be connected with a third-order polynomial.

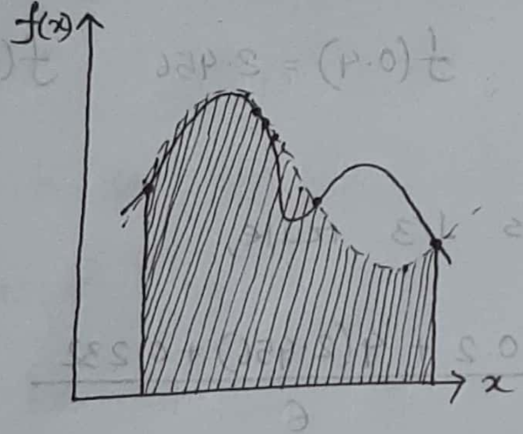

Fig: Graphical depiction of Simpson's 3/8 rule

a third-order Lagrange polynomial can be fit to four points and integrated:

$$I \cong \int_a^b f_3(x)\,dx$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] \text{ where, } h = \frac{b-a}{3}$$

This equ^n is called Simpson's 3/8 rule because h is manipulated by 3/8. It is the third Newton-Cotes closed integration formula.

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The 3/8 rule can also be ~~exposed~~ expressed as,

$$I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}}_{\text{Average height}}$$

Example 21.6:

(a) Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$

Solution: A single application of Simpson's 3/8 rule requires four equally spaced points:

$$\frac{a+b}{3} = \frac{0+0.8}{3} = 0.2667$$

$f(0) = 0.2$ $\quad f(0.2667) = 1.432724$

$0.2667 + 0.2667 = 0.533$

$f(0.5333) = 3.487177$ $\quad f(0.8) = 0.232$

$0.533 + 0.2667 = 0.8$

using Simpson's 3/8 rule,

$$I \cong 0.8 \; \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

$$E_t = 1.640533 - 1.519170 = 0.1213630$$

$$\varepsilon_t = 7.4\%$$

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(b) Use it in conjunction with Simpson's 3/8 rule to integrate the same function for five segments.

Solution: The data needed for a five-segment application ($h = 0.16$) is ($0.8/5 = 0.16$)

$f(0) = 0.2$ $f(.16) = 1.296919$

$f(0.32) = 1.743393$ $f(0.48) = 3.186015$

$f(0.64) = 3.181929$ $f(0.80) = 0.232$

The integral for the first to segments is obtained using Simpson's 1/3 rule:

$$I \cong \overset{(0.32-0)}{0.32} \frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

For the last three segments, the 3/8 rule can be used to obtain

$$I \cong \overset{(0.8-0.32)}{0.48} \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

The total integral is computed by summing the two results:

$$I = 0.3803237 + 1.264754 = 1.645077$$

$$E_t = 1.640533 - 1.645077 = -0.00454383$$

$$\varepsilon_t = -0.28\%$$

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Iterative Process / Direct Substitution Method

from youtube

Consider an equation $f(x)=0$ which can take in the form $x=\phi(x)$.

If $\phi'(x)<1$, $\phi(x)$ is convergent.

$$x_1=\phi(x_0)$$
$$x_2=\phi(x_1)$$
$$x_3=\phi(x_2)\ldots$$

Example: Find a real root of the equ^n $2x^2-4x+1=0$ using iterative process.

Sol^n: Given equ^n, $\cancel{f(x)=}\ 2x^2-4x+1=0 \Rightarrow x=\frac{1}{2}x^2+\frac{1}{4}$

$f(x)=2x^2-4x+1$

$f(0)=1$

$f(1)=-1$

$\therefore$ The root is between 0 and 1.

So, $x_0=\frac{0+1}{2}=0.5$

Now, $x=\phi(x)$

$\Rightarrow \phi(x)=\frac{1}{2}x^2+\frac{1}{4}$

$\phi'(x)=x$

$\phi'(x)_{\text{at } x_0=0.5}=0.5<1$

So, the iterative process can be applied.

We know,

$$x_{i+1}=\phi(x_i)$$

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Putting $i=0$, we get

$$x_1 = \phi(x_0) = \frac{1}{2}x_0^2 + \frac{1}{4} = \frac{1}{2}(0.5)^2 + \frac{1}{4} = 0.375$$

$$x_2 = \phi(x_1) = \frac{1}{2}x_1^2 + \frac{1}{4} = \frac{1}{2}(0.375)^2 + \frac{1}{4} = 0.320$$

$$x_3 = \phi(x_2) = \frac{1}{2}x_2^2 + \frac{1}{4} = \frac{1}{2}(0.320)^2 + \frac{1}{4} = 0.3012$$

$$x_4 = \phi(x_3) = \frac{1}{2}x_3^2 + \frac{1}{4} = \frac{1}{2}(0.3012)^2 + \frac{1}{4} = 0.295$$

$$x_5 = \phi(x_4) = \frac{1}{2}x_4^2 + \frac{1}{4} = \frac{1}{2}(0.295)^2 + \frac{1}{4} = 0.293$$

$$x_6 = \phi(x_5) = \frac{1}{2}x_5^2 + \frac{1}{4} = \frac{1}{2}(0.293)^2 + \frac{1}{4} = 0.2929$$

$$x_7 = \phi(x_6) = 0.292895205$$

$$x_8 = \phi(x_7) = 0.292865292$$

$$x_9 = \phi(x_8) = 0.29288522236$$

$$x_{10} = \phi(x_9) = 0.2928908$$

$$x_{11} = \phi(x_{10}) = 0.29289182553$$

$$x_{12} = \phi(x_{11}) = 0.2928932018$$

$$x_{13} = \phi(x_{12}) = 0.29289316$$

$$x_{14} = \phi(x_{13}) = 0.2928932016$$

$$x_{15} = \phi(x_{14}) = 0.2928932138$$

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B.C.R 28 page from class note

Example: $2x^2 - 4x + 1 = 0$

$\Rightarrow x = \left(\frac{1}{2}\right)x^2 + \frac{1}{4}$

$x = g(x)$

n	x_n
1	1.0
2	0.75
3	0.53125
⋮	⋮
13	0.292894
14	0.292893
15	0.292893

Convergence

1st step − 2nd step > 2nd step − 3rd step

n	x_n
1	2.0
2	2.25
3	2.78125
4	4.117676
5	8.727627
6	38.335736
⋮	⋮

Divergence

3rd step − 2nd step < 2nd step − 1st step

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Gauss Elimination
Gauss Jordan

$\left.\begin{array}{l}\text{Gauss Elimination}\\ \text{Gauss Jordan}\end{array}\right\}$ Lab

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Sub.: Lecture-8 (MB)

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(Mid) ## Bisection Method (Binary chopping, interval halving, Bolzano's method)

If $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have opposite signs, that is $f(x_l)f(x_u) < 0$, then there is at least one real root between x_l and x_u.

Algorithm

Step-1: Choose lower x_l and upper x_u guesses for the root such that, the function changes sign over the interval. This can be checked by ensuring that, $f(x_l)f(x_u) < 0$.

Step-2: An estimate of the root x_r, is determined by

$$x_r = \frac{x_l + x_u}{2}$$

Step-3: Make the following evaluations to determine in which sub-interval the root lies:

(a) If $f(x_l)f(x_r) < 0$, the root lies in the lower sub-interval. Therefore, set $x_u = x_r$ and return to step 2.

(b) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval. Therefore, set $x_l = x_r$ and return to step 2.

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(c) If $f(x_l)f(x_u) = 0$, the root equals to x_r, terminate the computation.

Example 5.3: Use Bisection Method to solve the following problem up to approximate percent relative error $\varepsilon_a \le 0.422$,

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$

Solution:

Let, $x_l = 12$, $x_u = 16$

Algo Step 1: $f(12)\,f(16) = 6.0669 \times -2.2687 = -13.76$

$\therefore f(x_l)f(x_u) < 0$; $x_u = x_r$

Now,

Algo Step 2: $x_r = \frac{12+16}{2} = \underline{\underline{14}}$ (iteration 1)

Step 3: $\underset{x_l}{f(12)} \cdot \underset{x_u}{f(14)} = 6.0669 \times 1.5687 = 9.52 > 0$; $x_l = x_r$

$\therefore x_l = \underline{14}$ ~~(iteration 1)~~

Algo Step 2: $x_r = \frac{14+16}{2} = 15$ (iteration 2)

Step 3: $f(14)\,f(15) = 1.5687 \times -0.4248 = -0.666 < 0$; $x_u = x_r$

$\therefore x_u = 15$ ~~(iteration 2)~~

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$x_r = \frac{14+15}{2} = 14.5$ (iteration 3)

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

Continuing this process until the appropriate error

$\varepsilon_a <= 0.422$

Iteration	x_l	x_u	x_r	ε_a (%)
1	12	16	14	
2	14	16	15	6.677
3	14	15	14.75	3.448
4	14.5	15	14.875	1.695
5	14.75	15	14.875	0.840
6	14.75	14.875	14.8125	0.422

Thus after 6 iterations, ε_a falls below 0.422%.

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False-Position Method (Linear interpolation method, regula falsi)

False-position method joins $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line and the x-axis represents an improved estimates of the root.

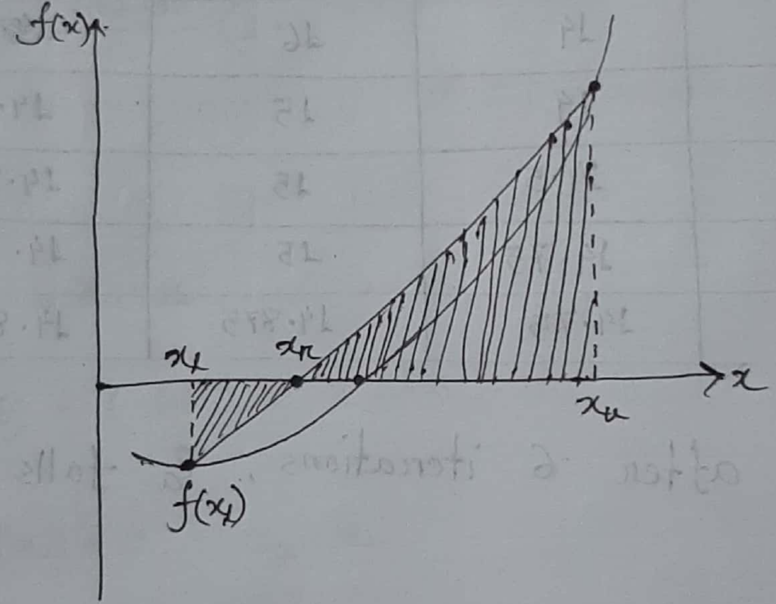

fig: Graphical depiction of the method of false position

The intersection of the straight line with the x-axis can be estimated as,

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$\Rightarrow f(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

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$$\Rightarrow x_r\left[f(x_l) - f(x_u)\right] = x_u f(x_l) - x_l f(x_u)$$

$$\Rightarrow x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$\Rightarrow x_r = \frac{x_u f(x_l)}{f(x_l)-f(x_u)} - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = x_u + \frac{x_u f(x_l)}{f(x_l)-f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = x_u + \frac{x_u f(x_u)}{f(x_l)-f(x_u)} - \frac{x_l f(x_u)}{f(x_l)-f(x_u)}$$

$$\Rightarrow x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$\boxed{***}$ Generally False position এর error আগে কমে যায়। কিন্তু সবক্ষেত্রে False position, bisection method এর থেকে better না; It depends on the type of the equation. $(x^{10}-1)$

Pitfall of False-position

* A case where Bisection is preferable to false position

Although False-position method would seem to be the bracketing method of preference, there are certain cases where it performs poorly.

Example 5.6: $f(x) = x^{10} - 1$

between $x = 0$ and 1.3

Solution: Using Bisection, the result can be summarized as

Iteration	x_l	x_u	x_r	$\varepsilon_a (\%)$
1	0	1.3	0.65	
2	0.65	1.3	0.975	33.3
3	0.975	1.3	1.1375	14.3
4	0.975	1.1375	1.05625	7.7
5	0.975	1.05625	1.015625	4.0

Using False position

Iteration	x_l	x_u	x_r	$\varepsilon_a (\%)$
1	0	1.3	0.0943	
2	0.09430	1.3	0.18176	48.1
3	0.18176	1.3	0.26287	30.9
4	0.26287	1.3	0.33811	22.3
5	0.33811	1.3	0.40788	17.1

After 5 iterations, appropriate error from bisection is 4.0 and from false-position is 17.1.

This proves that, there are cases where Bisection is preferable.

Runge-Kutta Methods

$$\frac{dy}{dx} = f(x, y)$$

New value = old value + slope × step size

$$y_{i+1} = y_i + \phi h$$

Euler's Method (Euler-Cauchy Method, Point slope method)

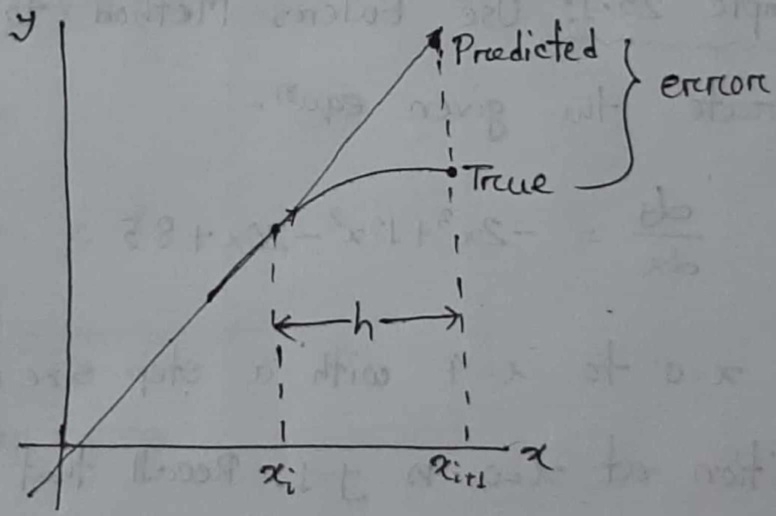

Fig: Euler's method

The first derivative provides a direct estimate of the slope at x_i

$$\phi = f(x_i, y_i)$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i & y_i

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This estimate can be substituted into the equ^n,

$$y_{i+1} = y_i + f(x_i, y_i)h$$

This formula is referred to as Euler's (or the Euler-Cauchy or the point-slope) method.

A new value of y is predicted using the slope to extrapolate linearly over the step size h.

Example 25.1: Use Euler's Method to numerically integrate the given equ^n.

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x=0$ to $x=4$ with a step size of 0.5. The initial condition at $x=0$ is $y=1$. Recall that the exact equ^n is given by

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

Solution: $y_{i+1} = y_i + f(x_i, y_i)h$ can be used to implement Euler's method.

$$y(0.5) = y(0) + f(0,1)\,0.5$$

where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0,1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5$$

$$= 8.5$$

Therefore,

$$y(0.5) = 1.0 + 8.5(0.5)$$

$$= 5.25$$

The true solution at $x = 0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1$$

$$= 3.21875$$

Thus, the error is

$$E_t = \text{true} - \text{approximate}$$

$$= 3.21875 - 5.25$$

$$= -2.03125$$

Or, expressed as percent relative error, $\varepsilon_t = -63.1\%$.

For the second step,

$$y(1) = y(0.5) + f(0.5, 5.25)\,0.5$$

$$= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]\,0.5$$

$$= 5.875$$

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The true solution at $x = 1.0$ is

$$y = -0.5(1)^4 + 4(1)^3 - 10(1)^2 + 8.5 \times 1 + 1$$

$$= 3.0$$

Therefore, the percent error is $\varepsilon_t = -95.8\%$.

The ~~resu~~ computation is repeated and the results are shown in the table and figure below:

x	y_{true}	y_{Euler}	Percent Relative Error Global	Local
0.0	1.00000	1.00000		
0.5	3.21875	5.25000	−63.1	−63.1
1.0	3.00000	5.87500	−95.8	−28.1
1.5	2.21875	5.12500	−131.0	−1.4
2.0	2.00000	4.50000	−125.0	20.3
2.5	2.71875	4.75000	−74.7	17.2
3.0	4.00000	5.87500	−46.9	3.9
3.5	4.71875	7.12500	−51.0	−11.3
4.0	3.00000	7.00000	−133.3	−53.1

Table – Comparison of true and approximate values of the integral

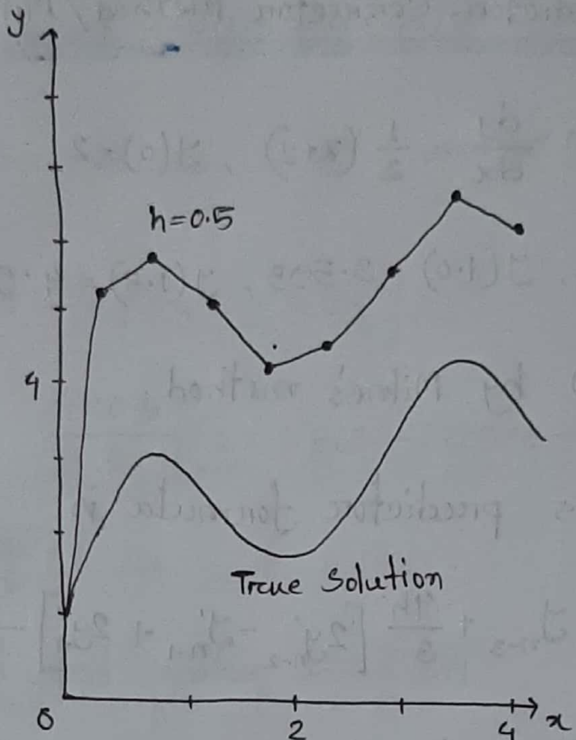

Figure: Comparison of the true solution with a numerical solution using Euler's method for the given integral.

Milne's Predictor-Corrector Method / Milne's Method

Example: Given $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0) = 2$

$y(0.5) = 2.636$, $y(1.0) = 3.595$, $y(1.5) = 4.968$

Now find $y(2)$ by Milne's method.

Solution: Milne's predictor formula is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \quad \text{—— (1)}$$

Putting $n=3$, we get,

P→Predictor

$$y_{4P} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \quad \text{—— (2)}$$

We're given that,

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$

The given differential equation is $y' = \frac{1}{2}(x+y)$

From above eqn, we'll calculate y'_1, y'_2, y'_3

$$y'_1 = \frac{1}{2}(x_1 + y_1) = \frac{1}{2}(0.5 + 2.636) = 1.568$$

$$y'_2 = \frac{1}{2}(x_2 + y_2) = \frac{1}{2}(1.0 + 3.595) = 2.2975$$

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$$y_3' = \frac{1}{2}(x_3 + y_3) = \frac{1}{2}(1.5 + 4.968) = 3.324$$

Substituting the values in ②, we get

$$y_{4P} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4 \times 0.5}{3}\left[2 \times 1.568 - 2.2975 + 2 \times 3.324\right]$$

$$= 6.871$$

So, our predicted value is 6.871

Now, we'll correct it to get actual value by Milne's Corrector formula, which is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right] \text{ ——— ③}$$

By putting $n=3$, we get

$$y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \text{ ——— ④}$$

Now,

$$y_4' = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2 + 6.871) = 4.4355$$

[∵ y(2) এর বদলে বলা হয়েছে]

Now by putting required values in ④ we get,

$$y_{4c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

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$$= 3.595 + \frac{0.5}{3}\left[2.2975 + (4 \times 3.324) + 4.4355\right]$$

$$= 6.8731$$

$\therefore y(2) = 6.8731$ (Ans)

* In Western literature, the method here called "Milne Method" is called the (explicit) ~~pid~~ midpoint rule.

* The corrector appearing in the "predictor-corrector Milne Method" is called the Milne method or Milne device.

* This method is direct generalization of the Simpson quadrature rule to differential equations.

** Why do we use the predictor-corrector method?

⇒ Predictor - Corrector methods counter this problem by using the information gained from the previous n-steps, to predict what the state of the system will be at the end of the next step.

They can then use this predicted value to evaluate the derivative at the end of the step.

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** How many basic values are required for Milne's predictor corrector method?

⇒ In Milne's method, four prior values are needed for finding the value of y at.

For finding predictor formula Milne neglect, fourth and higher order differences and for corrector formula Milne uses Simpson's 1/3 formula.

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Lagrange Interpolation

It is a way to find a polynomial, called lagrange polynomial, that takes on certain values at arbitrary points.

* Advantages of Lagrange Interpolation:

- This formula is used to find the value of the function even when the arguments are not equally spaced.
- This formula is used to find the value of independent variable x corresponding to a given value of a function.

** If the values of x are at equidistant or not at equidistant, we use Lagrange's interpolation formula.

Let, $y=f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2, \dots, y_n$ corresponding to $x=x_0, x_1, x_2, \dots, x_n$, that is $y_i = f(x_i)$; $i=0,1,2,\dots,n$.

Now there are $(n+1)$ paired values (x_i, y_i), $i=0,1,2,\dots,n$ and hence $f(x)$ can be represented by a polynomial function of degree n in x.

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Then the Lagrange formula is,

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Example: Using Lagrange's interpolation formula find $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

Solution: Here the intervals are not equal. By Lagrange's interpolation formula, we have

$x_0 = 5, \quad x_1 = 6, \quad x_2 = 9, \quad x_3 = 11$

$y_0 = 12, \quad y_1 = 13, \quad y_2 = 14, \quad y_3 = 16$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

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$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

Putting $x=10$ we get,

$$y(10) = f(10) = \frac{(10-6)(10-9)(10-11)}{(-1)\times(-4)\times(-6)} \times 12 + \frac{(10-5)(10-9)(10-11)}{1\times(-3)\times(-5)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{4\times 3\times(-2)} \times 14 + \frac{(10-5)(10-6)(10-9)}{6\times 5\times 2} \times 16$$

$$= \frac{12}{6} - \frac{13}{3} + \frac{5\times 14}{3\times 2} + \frac{4\times 16}{12}$$

$$= 14.6663$$

(Ans)

* Differentiation Based on Lagrange interpolation formula:

Example 5.10 : Using Lagrange's interpolation formula find $f'(0.12)$

Bala C. Rao

x	$f(x)$
0.05	0.05
0.10	0.0999
0.20	0.1987
0.26	0.2571

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Solution: The intervals are unequal. Using Lagrange's interpolation formula we have,

$$x_0 = 0.05, \quad x_1 = 0.10, \quad x_2 = 0.20, \quad x_3 = 0.26$$

$$y_0 = 0.05, \quad y_1 = 0.0999, \quad y_2 = 0.1987, \quad y_3 = 0.2571$$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-0.10)(x-0.20)(x-0.26)}{(0.05-0.10)(0.05-0.20)(0.05-0.26)} \times 0.05$$

$$+ \frac{(x-0.05)(x-0.20)(x-0.26)}{(0.10-0.05)(0.10-0.20)(0.10-0.26)} \times 0.0999$$

$$+ \frac{(x-0.05)(x-0.10)(x-0.26)}{(0.20-0.05)(0.20-0.10)(0.20-0.26)} \times 0.1987$$

$$+ \frac{(x-0.05)(x-0.10)(x-0.20)}{(0.26-0.05)(0.26-0.10)(0.26-0.20)} \times 0.2571$$

Putting $x = 0.12$

$$f(0.12) = \frac{(0.12-0.10)(0.12-0.20)(0.12-0.26)}{(-0.05)\times(-0.15)\times(-0.21)} \times 0.05 + \frac{(0.12-0.05)(0.12-0.20)(0.12-0.26)}{0.05 \times (-0.10) \times (-0.16)} \times 0.0999$$

$$+ \frac{(0.12-0.05)(0.12-0.10)(0.12-0.26)}{0.15 \times 0.10 \times (-0.06)} \times 0.1987 + \frac{(0.12-0.05)(0.12-0.10)(0.12-0.20)}{0.21 \times 0.16 \times 0.06} \times 0.2571$$

$$= \frac{0.02 \times (-0.08) \times (-0.14) \times 0.05}{-0.0015} + \frac{0.06 \times (-0.08) \times (-0.14) \times 0.0999}{0.0008}$$

$$+ \frac{0.07 \times 0.02 \times (-0.14) \times 0.1987}{-0.0009} + \frac{0.07 \times 0.02 \times (-0.08) \times 0.2571}{0.0020}$$

$$= \frac{0.0000112}{-0.0015} + \frac{0.0000671}{0.0008} + \frac{0.0000389}{0.0009} - \frac{0.0000287}{0.0020}$$

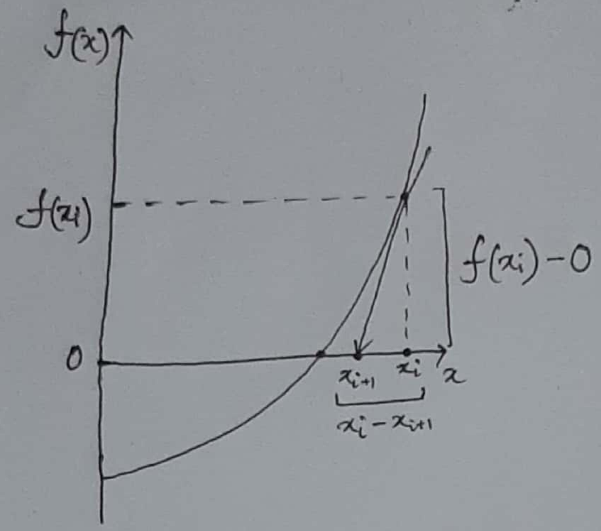
$$= -0.00746 + 0.0838 + 0.0432 - 0.0143$$

$$= 0.10524$$

$$\therefore f(0.12) = 0.10524$$

(Ans)

Newton-Raphson Method



If the initial guess at root is x_i, a tangent can be extended from the point $[x_i, f(x_i)]$. The point where this tangent crosses the x-axis usually represents an improved estimate of the root.

The first derivative ~~is~~ at x is equivalent to the slope:

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \quad \text{------} \quad (1)$$

which can be rearranged to yeild

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{-------} \quad (2)$$

This is called the ~~ne~~ Newton-Raphson formula.

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Example 6.3: Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$.

Solution: The first derivative of the function can be evaluated as

$$f'(x) = -e^{-x} - 1$$

which can be substituted along with the original function into the equation to give

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{x_i} - 1}$$

Starting with an initial guess of $x_0 = 0$, this iterative eqn can be applied to compute-

i	x_i	$\varepsilon_t (\%)$
0	0	100
1	0.500000000	11.8
2	0.566311003	0.147
3	0.567143165	0.0000220
4	0.567143290	$<10^{-8}$

Thus this approach converges rapidly on the true root. The true percent relative error at each iteration decreases much faster than it does in simple fixed-point iteration.

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**

It is a technique for solving equation of the form $f(x)=0$ by successive approximation.

It takes initial guess x_0 such that $f(x_0)$ is reasonably close to 0.

** Where is Newton-Raphson method used?

⇒ We use this method for root finding with less steps.

** Advantages

- fast convergence
- we get root in less steps
- requires only one guess

** Why this method is used?

⇒ For finding the root of given equations. As it converges on the root faster and in less steps. Besides, it requires only one guess.

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* Pitfalls of the Newton-Raphson Method

performs poorly for multiple roots, sometimes for simple roots

Example-6.5: Determine the positive root of $f(x) = x^{10} - 1$ using the Newton-Raphson method and an initial guess of $x = 0.5$.

Solution: The Newton-Raphson formula for this case is

$$x_{i+1} = x_i - \frac{x_i^{10} - 1}{10x_i^9}$$

which can be used to compute

iteration	x
0	0.5
1	51.65
2	46.485
3	41.8365
4	37.6585
5	33.887565
⋮	
∞	1.0000000

Thus after the first poor prediction, the technique is converging on the true root of 1, but at a very slow rate.

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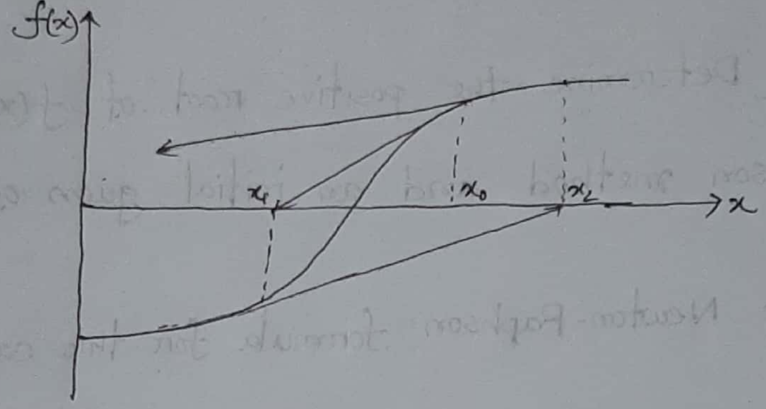
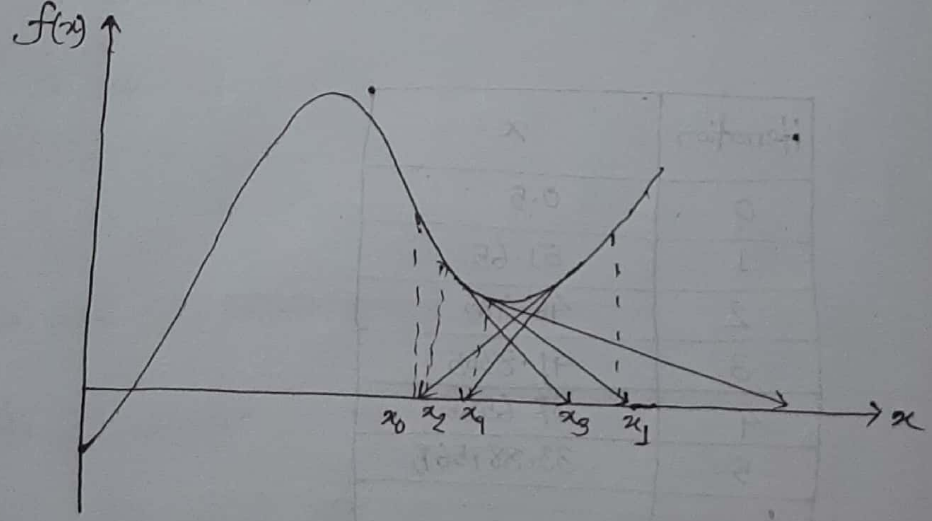
* Four cases where the Newton-Raphson method exhibits poor convergence:



(a)



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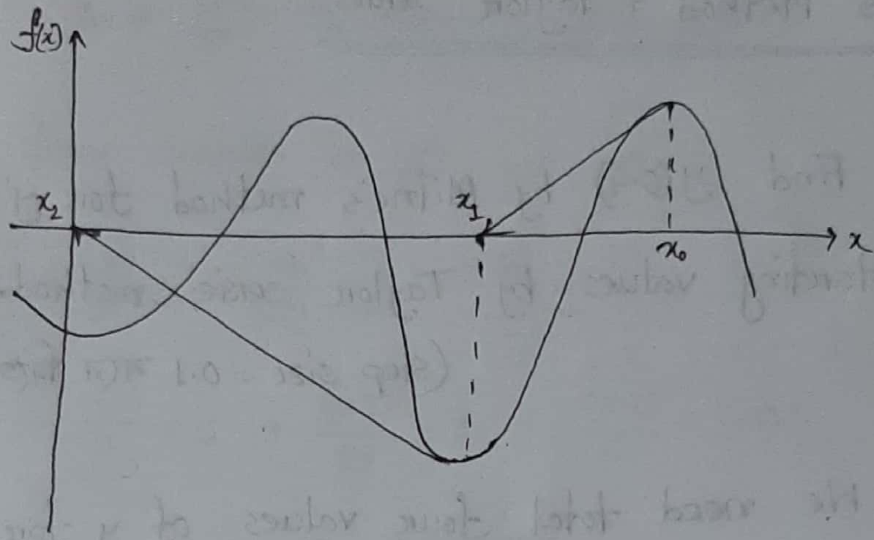

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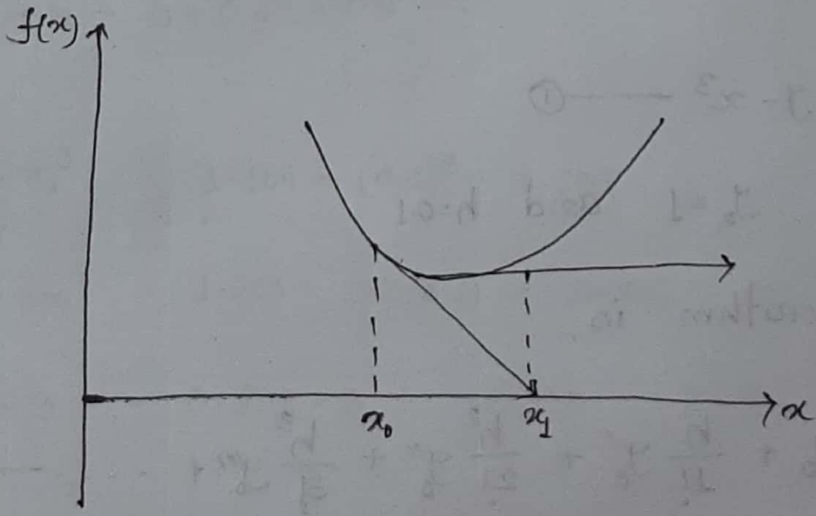

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Milne's Method + Taylor Series

Example: Find $y(0.4)$ by Milne's method for $y' = y - x^3; y(0) = 1$. obtain starting values by Taylor series method.

(step size $= 0.1$ ধরে নিতে হবে)

Solution: We need total four values of y for Predictor-Corrector method to work. Only one value of y is given, remaining values are to be calculated by Taylor series method. We need to calculate $y(0.1)$, $y(0.2)$ and $y(0.3)$ by Taylor method.

Given $y' = y - x^3$ ——— ①

and $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Taylor's Algorithm is,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{——— ②}$$

Differentiating ① w.r.t x, we get,

$$y' = y - x^3 \Rightarrow y_0' = y_0 - x_0^3 = 1 - 0^3 = 1$$

$$y'' = y' - 3x^2 \Rightarrow y_0'' = y_0' - 3x^2 = 1 - 3\times0^2 = 1$$

$y''' = y'' - \cancel{3x} \Rightarrow y_0''' = y_0'' - \cancel{3x} = 1 - \cancel{3 0} = \cancel{-2}\ 1$

Putting these values in ② we get,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 1 + \frac{0.1}{1} \times 1 + \frac{(0.1)^2}{2!} \times 1 + \frac{(0.1)^3}{6} \times (\cancel{-2})\ 1$$

$$= 1 + 0.1 + 0.005 - 0.000333$$

$\therefore y_1 = 1.104$; $y(0.1) = 1.105$

Now,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \quad ③$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1' = y_1 - x_1^3 = 1.104 - (0.1)^3 = 1.103$$

$$y_1'' = y_1' - 3x_1 = 1.103 - 3 \times (0.1) = 0.803$$

$$y_1''' = y_1'' - 3 = 0.803 - 3 = -2.197$$

Putting these values in ③, we get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$$

$$= 1.104 + \frac{0.1}{1} \times 1.103 + \frac{(0.1)^2}{2} \times 0.803 + \frac{(0.1)^3}{6} \times (-2.197)$$

$$= 1.104 + 0.1103 + 0.004015 - 0.000366 = 1.217$$

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$\therefore\ y(0.2) = 1.217$

Now,

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \cdots \quad ④$$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y_2' = y_2 - x_2^3 = 1.217 - (0.2)^3 = 1.203$

$y_2'' = y_2' - 3x_2 = 1.203 - 3 \times (0.2) = 1.185$

$y_2''' = y_2'' - 3 = 1.185 - 3 = -1.815$

Now putting these values in ④, we get,

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2'''$$

$$= 1.217 + \frac{0.2}{1} \times 1.203 + \frac{(0.2)^2}{2} \times 1.185 + \frac{(0.2)^3}{6} \times (-1.815)$$

$$= 1.343$$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$\therefore$ ~~$y_0 = 0$~~ $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3$

$y_0 = 1,\ y_1 = 1.109,\ y_2 = 1.217,\ y_3 = 1.343$

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Milne's Predictor formula is -

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

Putting $n=3$, we get

$$y_{4,P} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \quad \text{(5)}$$

$$y'_3 = y_3 - x_3^3 = 1.343 - (0.3)^3 = 1.316$$

Putting required values in (5), we get

$$y_{4,P} = 1 + \frac{4\times 0.1}{3}\left[2\times 1.103 - 1.209 + 2\times 1.316\right]$$

$$= 1.483$$

So, the predicted value is 1.483 for $y(0.4)$. Now we'll correct it using the Corrector formula to get the actual value.

Milne's Corrector formula is -

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

Putting $n=3$, we get

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$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \text{ ——— } ⑥$$

$$y_4' = y_4 - x_4^3 = 1.483 - (0.4)^3 = 1.419$$

putting required values in ⑥, we get

$$y_{4,c} = 1.217 + \frac{0.1}{3}\left[1.209 + 4 \times 1.316 + 1.419\right]$$

$$= 1.48$$

$\therefore y(0.4) = 1.48$ (Ans)

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Python basic - Javapoint

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