The Laplace Transform

Defination: Let $F(t)$ be a function of t defined for all $t>0$. Then the laplace transform of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$ or $f(s)$ is denoted by $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$

Application:
(i) It is used to analyze and design electrical circu
(ii) It is widely used to analyze and design control systems
(iii) It is used to solve partial differential equations.
(iv) It is used in probability theory to derive the moment-generating function of a probability distribution.

Some Elementary Function:

(i) $F(t) = 1$ $\quad \mathcal{L}\{1\} = f(s) = \frac{1}{s}$ — $s>0$

(ii) $F(t) = t$ $\quad \mathcal{L}\{t\} = f(s) = \frac{1}{s^2}$ — $s>0$

(iii) $F(t) = t^n$ $\quad \mathcal{L}\{t^n\} = f(s) = \frac{n!}{s^{n+1}}$ — $s>0$

(iv) $F(t) = e^{at}$ $\quad \mathcal{L}\{e^{at}\} = f(s) = \frac{1}{s-a}$ — $s>0$

(v) $F(t) = \sin at$ $\quad \mathcal{L}\{\sin at\} = f(s) = \frac{a}{s^2+a^2}$ — $s>0$

(vi) $F(t) = \cos at$ $\quad \mathcal{L}\{\cos at\} = f(s) = \frac{s}{s^2+a^2}$ — $s>0$

(vii) $F(t) = \sinh at$ $\quad \mathcal{L}\{\sinh at\} = f(s) = \frac{a}{s^2-a^2}$ — $s>|a|$

(viii) $F(t) = \cosh at$ $\quad \mathcal{L}\{\cosh at\} = f(s) = \frac{s}{s^2-a^2}$ — $s>|a|$

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Example: ① $e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t$

By applying linearity properity, we have:

$$\mathcal{L}\{e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t\}$$

$$= \mathcal{L}\{e^{4t}\} + \mathcal{L}\{4t^3\} - \mathcal{L}\{2\sin 3t\} + \mathcal{L}\{3\cos 5t\}$$

$$= \frac{1}{s-4} + 4\frac{3!}{s^4} - 2\frac{3}{s^2+3^2} + 3\frac{s}{s^2+5^2}$$

$$= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25}$$ Ans.

Find the laplace transform of the following function:

① e^{at} (ii) e^{-at}

① We know from the laplace structure,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-t(s-a)}\,dt = \left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$= \frac{e^{-\infty} - e^0}{-(s-a)} = \frac{0-1}{-(s-a)}$$

$$= \frac{1}{s-a}$$ (proved)

(ii) $$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st}e^{-at}\,dt$$

$$= \int_0^\infty e^{-t(s+a)}\,dt = \left|\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \frac{e^{-\infty} - e^0}{-(s+a)} = \frac{1}{s+a}$$

Prove: first shifting:

From Laplace transform, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s)$

Then $\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)\,dt = \int_0^\infty e^{-(s-a)t}F(t)\,dt$

$= f(s-a)$ (show)

Ex: $\boxed{\mathcal{L}\{e^{-2t}\cos 3t\}}$ $\boxed{f(s+2)}$ $\quad f(t) = \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+3^2}$

$$= \frac{s+2}{(s+2)^2+3^2} = \frac{s+2}{s^2+4s+4+9} = \frac{s+2}{s^2+4s+13}$$ Ans

Find the Laplace transform of $G(t)$ — Second shifting

Q1 If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & \text{when } t > a, \\ 0 & \text{when } t \le a. \end{cases}$ then

Theorem

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt \qquad \mathcal{L}\{G(t)\} = e^{-as} f(s).$$

$$= \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

Let $t-a = u$, $dt = u+a = du$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} F(t-a)\,dt = \int_a^\infty e^{-st} F(t-a)\,dt$$

Limit change

$t = \alpha$: $u = t-a = \alpha - \alpha$

$t = a$: $u = t-a = a-a = 0$

$$= \int_0^\infty e^{-s(u+a)} F(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} F(u)\,du = \boxed{e^{-as} f(s)}$$

Q1. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\,dt = \int_0^\infty e^{-s(u/a)} F(u)\,d(u/a)$$

let $at = u$, $t = \frac{u}{a}$, $dt = du$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} F(u)\,du$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$ Ans

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Laplace transform of derivatives:

$\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

Sol^n: $\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$ $\qquad \int uv = u\int v - \int\{\frac{d}{dx}u \int v\, dx\}dx$

$$= \left| e^{-st} F(t) \right|_0^\infty + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= 0 - e^0 F(0) + s\mathcal{L}\{F(t)\} = sf(s) - F(0)$$ proved

From this.

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

and $\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

Now n^{th} order:

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0)$$ Ans

(19) (17^m) Multiplication by powers of t

$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s)$

Ex: (20) $\mathcal{L}\{t\sin at\}$ $\qquad$ Sol^n: $\mathcal{L}\{F(t)\} = f(s)$ $\qquad F(t) = \sin at$

$f(s) = \frac{a}{s^2+a^2}$

$$\mathcal{L}\{t\sin at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$ (w)

(11) $\mathcal{L}\{t^3 e^t\}$ $\qquad$ we know $\mathcal{L}\{e^t\} = \frac{1}{s-1}$ and $\mathcal{L}\{t^3\} = \frac{3!}{t^4}$

$$\therefore \mathcal{L}\{t^3 e^t\} = \frac{6}{(s-1)^4}$$

Laplace transform of a function division by t. (262)

If $\mathcal{L}\{F(t)\} = f(s)$ then show that $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(u)\,du$

(12) Laplace transform of integrals: If $\mathcal{L}\{F(t)\} = f(s)$

(i) $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{f(s)}{s}$ (ii) $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$

2nd: The Inverse Laplace ~~Function~~ Transform

Defination: If $\mathcal{L}\{F(t)\} = f(s)$ then $F(t)$ is called an inverse Laplace transform of $f(s)$ and it is denoted by $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ (ii) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ (iii) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ (iv) $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!}$

(v) $\mathcal{L}^{-1}\left\{\frac{4}{s^2+2^2}\right\} = 2\sin 2t$ (vi) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$

Ex: 267 page:

$$\mathcal{L}^{-1}\left\{\frac{5}{s-3} - \frac{2s}{s^2+25} + \frac{3}{s^2+4} + \frac{1}{s^4}\right\}$$

$$= 5e^{3t} - 2\cos 5t + \frac{3}{2}\sin 2t + \frac{1}{3!}$$

(i) $$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+4^2}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

$$= 6 \cdot e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t + \sin 4t)$$ Ans

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(ii) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+3)^2+4^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+4^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4^2}\right\} = e^{-3t}\cos 4t - \frac{1}{2}e^{-3t}\sin 4t$

$= e^{-3t}(\cos 4t - \frac{1}{2}\sin 4t)$

(iii) $\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$

$= 4e^{-4t} - 4te^{-4t}$

$= 4e^{-4t}(1-t)$ Ans:

Ex: 07

$Y''(x) - Y'(x) = x$; when $Y(0)=2$ and $Y'(0)=-3$

Now taking Laplace transform on (1), we get

$\mathcal{L}\{Y''(x)\} - \mathcal{L}\{Y'(x)\} = \mathcal{L}\{x\}$

$\Rightarrow s^2Y(s) - sY(0) - Y'(0) - \{sY(s) - Y(0)\} = \frac{1}{s^2}$

$\Rightarrow s^2Y(s) - 2s + 3 - sY(s) + 2 = \frac{1}{s^2}$

$\Rightarrow Y(s)(s^2-s) = \frac{1}{s^2} + 2s - 5$

$\Rightarrow Y(s) = \frac{1}{s^3(s-1)} + \frac{2}{s-1} - \frac{5}{s^2-s}$

Let, $\frac{1}{s^3(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}$

$1 = As^2(s-1) + Bs(s-1) + C(s-1) + Ds^3$ —— (4)

Putting $s=0$ and $s=1$ successively in (4) then we get,

$1 = C(0-1) = -C$ and $1 = D.1 = D$

Now equating the coefficients of s^3 and s^2 from both sides of (4) then we get,

$0 = A + D \Rightarrow A = -1$

$0 = -A + B \Rightarrow B = -1$

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$$\Rightarrow \frac{1}{s^3(s-1)} = \frac{-1}{s} + \frac{-1}{s^2} + \frac{-1}{s^3} + \frac{1}{s-1} \quad \text{—— (5)}$$

And $$\frac{5}{s(s-1)} = -\frac{5}{s} + \frac{5}{s-1} \quad \text{—— (6)}$$

So, $$y(s) = -\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s-1} + \frac{2}{s-1} + \frac{5}{s} - \frac{5}{s-1}$$

$$= \frac{4}{s} - \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{s-1}$$

Taking inverse Laplace transform on both sides then we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{s-1}\right\}$$

$$Y(x) = 4.1 - \frac{x}{1} - \frac{x^2}{2!} - 2e^x = 4 - x - \frac{1}{2}x^2 - 2e^x \quad \text{(11)}$$

Ex: 08 : $Y''(t) + 9Y(t) = \cos 2t$ —— (1) If $Y(0) = 1$ and $Y(\pi/2) = -1$

Now taking Laplace transform in 1 then we get

$$\mathcal{L}\{Y''(t) + 9Y(t)\} = \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow s^2 y(s) - sY(0) - Y'(0) + 9y(s) = \frac{s}{s^2+2^2} \qquad \text{Let } Y'(0) = a$$

$$\Rightarrow y(s)(s^2+9) - s - a = \frac{s}{s^2+2^2}$$

$$\Rightarrow y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$$\Rightarrow y(s) = \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) + \frac{s}{s^2+9} + \frac{a}{s^2+9} = \frac{1}{5(s^2+4)} + \frac{4}{5}\frac{1}{(s^2+9)} + \frac{a}{s^2+}$$

Taking inverse Laplace transform on both sides then we ge

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{s}{s^2+4} + \frac{4s}{5(s^2+9)} + \frac{a}{s^2+9}\right\}$$

$$Y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{a}{3}\sin 3t \quad \text{—— (11)}$$

Now putting $t = \pi/2$ in (11) then we get

$$Y\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos 2\frac{\pi}{2} + \frac{4}{5}\cos\frac{3\pi}{2} + \frac{a}{3}\sin\frac{3\pi}{2}$$

$$-1 = \frac{-1}{5} + 0 + \frac{a}{3}(-1) \qquad \Rightarrow 1 = \frac{a}{3} + \frac{1}{5} \Rightarrow \frac{4}{5} = \frac{a}{3}$$

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$\therefore\ Y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

Ex: 09 $\quad Y''(t) + a^2 Y(t) = F(t)$ — ① when $Y(0) = 1$ $Y'(0) = -2$

Now taking Laplace transform in ① then we get

$$\mathcal{L}\{Y''(t) + a^2 Y(t)\} = \mathcal{L}\{F(t)\}$$

$$\Rightarrow s^2 y(s) - sY(0) - Y'(0) + a^2 y(s) = f(s)$$

$$\Rightarrow y(s)(s^2+a^2) - s.1 - (-2) = f(s)$$

$$\Rightarrow y(s) = \frac{s}{s^2+a^2} - \frac{2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

Taking Inverse Laplace transform on both sides then we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2} - \frac{2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}\right\}$$

$$Y(t) = \cos at - \frac{2}{a}\sin at + \frac{1}{a}\mathcal{L}^{-1}\left\{f(s)\frac{a}{s^2+a^2}\right\}$$

$$Y(t) = \cos at - \frac{2}{a}\sin at + \frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du$$

By convolution theorem.

Ex: 10 $\quad Y''(t) - a^2 Y(t) = F(t)$ — ①

Let $Y(0) = b$ and $Y'(0) = c$

Now taking Laplace transform in ① then we get

$$\mathcal{L}\{Y''(t) - a^2 Y(t)\} = \mathcal{L}\{F(t)\}$$

$$\Rightarrow s^2 y(s) - sY(0) - Y'(0) - a^2 y(s) = f(s)$$

$$\Rightarrow y(s)(s^2 - a^2) = f(s) + bs + c$$

$$\Rightarrow y(s) = \frac{c}{s^2-a^2} + \frac{bs}{s^2-a^2} + \frac{f(s)}{s^2-a^2}$$

Taking inverse Laplace transform on both sides then

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{c}{s^2-a^2} + \frac{bs}{s^2-a^2} + \frac{f(s)}{s^2-a^2}\right\}$$

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$$\Rightarrow Y(t) = b\cosh at + \frac{c}{a}\sinh at + \frac{1}{a}\int_0^t F(u)\sinh a(t-u)\,du.$$

Fourier Series

Define: Let $f(x)$ be defined in the interval $(-\pi, \pi)$ with period 2π which can be expanded in a trigonometric series.

i.e $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{-----} (1)$$

then (1) is called a Fourier series where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

Here a_0, a_n and b_n are Fourier cofficients.

Important integrals:

(i) $\int_{-\pi}^{\pi}\sin nx\,dx = 0$ (ii) $\int_{-\pi}^{\pi}\cos nx\,dx = 0$

(iii) $\int_{-\pi}^{\pi}\sin nx\cos nx\,dx = 0$ (iv) $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = 0$ If $m \neq n$

(v) $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = 0$ (vi) $\int_{-\pi}^{\pi}\cos^2 nx\,dx = \pi$

(vii) $\int_{-\pi}^{\pi}\sin^2 nx\,dx = \pi$

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If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ then show that

(i) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$ (ii) $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$ (iii) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Solⁿ:

(i) Given that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —— (1)

Now integrating both sides of (1) w.r.t x between the limits $-\pi$ and π then we get

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, dx$$

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2} [x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n \cdot 0 + \sum_{n=1}^{\infty} b_n \cdot 0 = \frac{a_0}{2} \cdot 2\pi = a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad \text{(showed)}$$

(ii) Again Multiplying both sides of (1) by $\cos nx$ and then integrating w.r.t x between the limits $-\pi$ and π, we get,

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx . \cos nx\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos nx \sin nx\, dx$$

$$= \frac{a_0}{2} \cdot 0 + a_n \pi + 0$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad \text{(showed)}$$

(iii) Multiplying both sides of (1) by $\sin nx$ and then integrating w.r.t x between the limits $-\pi$ and π, we get,

$$\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \sin nx \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin nx\, dx$$

$$= \frac{a_0}{2} \cdot 0 + 0 + b_n \cdot \pi$$

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad \text{(showed)}$$

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* **Even function:** If $f(-x) = f(x)$ then $f(x)$ is called an even function.

Lets $f(x) = \cos x$; $f(-x) = \cos(-x) = \cos x = f(x)$;

so, $\cos x$ is even function.

* **Odd function:** If $f(-x) = -f(x)$, then $f(x)$ is called an odd function.

Let $f(x) = \sin x$; $f(-x) = \sin(-x) = -\sin x = -f(x)$

so, $\sin x$ is odd function.

* If $f(x)$ is an even function then show that,

(i) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ (ii) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$ (iii) $b_n = 0$.

Solⁿ:

(i) We know,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right\} \quad \text{--- (1)}$$

If $x = -\pi$ then $-\pi = -y \Rightarrow y = \pi$

If $x = 0$ then $y = 0$

Let $x = -y$

$dx = -dy$

$$\therefore \int_{-\pi}^{0} f(x)\,dx = -\int_{\pi}^{0} f(-y)\,dy = \int_0^{\pi} f(-y)\,dy$$

$$= \int_0^{\pi} f(-x)\,dx = \int_0^{\pi} f(x)\,dx \quad [\because f(x) \text{ is}$$

Now from (1) we get

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right\} = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad \text{(showed)}$$

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(ii) We know, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

let $x=-y$ then $dx=-dy$

$$\therefore \int_{-\pi}^{0} f(x)\cos nx\,dx = -\int_{\pi}^{0} f(-y)\cos(-ny)\,dy = \int_{0}^{\pi} f(-y)\cos(-ny)\,dy$$

$$= \int_{0}^{\pi} f(-y)\cos ny\,dy = \int_{0}^{\pi} f(-x)\cos nx\,dx$$

$$= \int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\therefore a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \quad \text{(showed)}$$

(iii) We know, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

let $x=-y$ $dx=-dy$

$$\therefore \int_{-\pi}^{0} f(x)\sin nx\,dx = -\int_{\pi}^{0} f(-y)\sin(-ny)\,dy$$

$$= \int_{0}^{\pi} f(-y)\sin(-ny)\,dy$$

$$= -\int_{0}^{\pi} f(-x)\sin nx\,dx$$

$$= -\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\therefore b_n = \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx - \int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= 0 \quad \text{(showed)}$$

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Periodic Function: If $f(x+T)=f(x)$ then $f(x)$ is called periodic function and T is called its period.

Example: Since $\tan(\pi+x)=\tan(x)$, so $\tan x$ is periodic function and its period is π.

Example: 01 If $f(x)=(x+x^2)$ in $(-\pi,\pi)$, then expand in sine and cosine of the function and show that $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

At the end points $x=\pm\pi$ if $f(x)$ is discontinuous then

$$f(\pi)=\lim_{\delta\to 0}\frac{1}{2}\left[f(-\pi+\delta)+f(\pi-\delta)\right]$$

$$\Rightarrow f(x)=\frac{1}{2}\left[f(-\pi+0)+f(\pi-0)\right]$$

Sol^n: We have $f(x)=x+x^2$, $-\pi<x<\pi$

In this interval $-\pi<x<\pi$ the fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

When, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^2}{2}+\frac{\pi^3}{3}+\frac{\pi^3}{3}\right]$$

$$\cancel{=\frac{1}{\pi}\left\{\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right]}$$

$$=\frac{1}{\pi}\,\frac{2\pi^3}{3}=\boxed{\frac{2\pi^2}{3}}$$

Now $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-(1+2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{1}{n}(x+x^2)\sin nx+\frac{(1+2x)}{n^2}\cos nx-\frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\left\{0+\frac{1+2\pi}{n^2}(-1)^n-0\right\}-\left\{0+\frac{1-2\pi}{n^2}(-1)^n-0\right\}\right]$$

$$=\frac{1}{\pi}\cdot\frac{(-1)^n}{n^2}(1+2\pi-1+2\pi)=\boxed{\frac{4(-1)^n}{n^2}}$$

[since $\sin n\pi=0$, $\cos nx=(-1)^n$]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - \int_{-\pi}^{\pi}(1+2x)\left(\frac{-\cos nx}{n}\right)dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) + (1+2x)\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{1}{n}(\pi+\pi^2)(-1)^n + 0 + \frac{2}{n^3}(-1)^n\right\} - \left\{-\frac{1}{n}(-\pi+\pi^2)(-1)^n + 0 + \frac{2}{n^3}(-1)^n\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(\pi + \cancel{\pi^2} + \pi - \cancel{\pi^2})(-1)^n\right] = -\frac{1}{n\pi}\cdot 2\pi\cdot(-1)^n = \boxed{\frac{-2(-1)^n}{n}}$$

Now putting the values of a_0, a_n, b_n in (2) then we get,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right) \quad \text{--- (3)}$$

At the end points $x = \pm\pi$

$$f(\pi) = \frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right] = \frac{1}{2}\left[-\pi+\pi^2+\pi+\pi^2\right] = \pi^2$$

since $f(x) = x+x^2$

Now putting put $x = \pi$, in (3) then we get,

$$f(\pi) = \frac{\pi^2}{3} - 4\left(\frac{-1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) + 2(0)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{(proved)}$$

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Ex: 3 Given that

$$f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases} \quad \text{——— (1)}$$

We know that in the interval $-\pi < x < \pi$, the fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (2)}$$

when $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right\}$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\pi [x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi} \right\} = \frac{1}{\pi} \left\{ -\pi(0+\pi) + \frac{1}{2}(\pi^2 - 0) \right\}$$

$$= \frac{1}{\pi} \left(-\pi^2 + \frac{\pi^2}{2} \right) = -\frac{\pi}{2}$$

And $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\int_{-\pi}^{0} \pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\pi \left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-\pi}{n}(0) + 0 + \frac{1}{n^2}(\cos n\pi - \cos 0) \right\}$$

$$= \frac{1}{\pi n^2} \left\{ (-1)^n - 1 \right\}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_{0}^{\pi} f(x)\sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} \pi\sin nx + \int_{0}^{\pi} x\sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x\frac{-\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}[\cos nx]_{-\pi}^{0} + \left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi\cos n\pi}{n} + 0\right.$$

$$= \frac{1}{n}(1 - 2\cos n\pi) = \frac{1-2(-1)^n}{n}$$

Now putting the values of a_0, a_n, b_n in ② then we get,

$$f(x) = -\frac{\pi}{2}\cdot\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\{(-1)^n-1\}\cos nx}{\pi n^2} + \sum_{n=1}^{\infty}\frac{\{1-2(-1)^n\}\sin nx}{n}$$

$$= -\frac{\pi}{4} + \frac{1}{\pi}\left(\frac{-2\cos x}{1^2} + 0 - \frac{2\cos 3x}{3^2} + 0 - \frac{2\cos 5x}{5^2} + \cdots\right)$$

$$+ \left(\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) +$$

$$\left(\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \quad (3)$$

Since $f(x)$ is discontinuous at $x=0$, so

$$f(0) = \frac{1}{2}\{f(0-0) + f(0+0)\} = \frac{1}{2}(-\pi + 0) = -\frac{\pi}{2} \quad (4)$$

Now in (3) putting $x=0$ then we get

$$f(0) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + 0$$

or $$-\frac{\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

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$$-\frac{\pi}{2}+\frac{\pi}{9} = -\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$-\frac{\pi}{9} = -\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

or $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ (proved)

Fourier Integral

Dirichlet's conditions:

Let $f(x)$ function satisfy the following condition:

(i) $f(x)$ is defined in the interval $-l < x < l$

(ii) $f(x)$ and $f'(x)$ are sectionally continuous in the interval $-l < x < l$.

(iii) $f(x+2l) = f(x)$ i.e $f(x)$ is periodic and its period is $2l$.

The above conditions are called Dirichlet's conditions.

State Fourier Integral theorem.

If a function $f(x)$ satisfies Dirichlet's conditions in the interval $(-l, l)$ and $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent then at each point continuity of $f(x)$.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(x-t)\,du\,dt.$$

- If the $f(x)$ is even then,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f(t)\,dt \int_0^{\infty} \cos ut \cos ux\,du$$

This is called fourier cosine integral formula.

- If $f(x)$ is odd then,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f(t)\,dt \int_0^{\infty} \sin ut \sin ux\,du$$

This is called fourier sine integral formula.

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Fourier Transform

The finite fourier sine transform:
Let a function $F(x)$ be defined in the interval $0 < x < l$, then the finite fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$, or $f_s(n)$ and is defined as follows:

$$F_s\{F(x)\} = f_s(n)$$
$$= \int_0^l F(x) \sin \frac{n\pi x}{l} \, dx, \text{ where } n \text{ is an integer}$$

Fourier transform: Let a function $F(x)$ be defined in the interval $-\infty < x < \infty$ then the fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(n)$ and is defined as follow:

$$F\{F(x)\} = f(n) = \int_{-\infty}^{\infty} e^{inx} F(x) \, dx.$$

Inverse fourier transform: If $f(n) = \int_{-\infty}^{\infty} e^{inx} F(x) \, dx$ then $F(x)$ is called inverse fourier transform of $f(n)$ and is defined as follow: $F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{inx} f(n) \, dn$.

The finite fourier cosine transform: Let a function $F(x)$ be defined in the interval $0 < x < l$, then the finite fourier cosine transform of $F(x)$ is denoted by $F_c\{F(x)\}$ or $f_c(n)$ and is defined as follows:

$$F_c\{F(x)\} = f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} \, dx; \text{ where } n \text{ is an inte}$$

Inverse finite fourier sine transform:
If $f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} \, dx$ then $F(x)$ is called inverse finit fourier sine transform of $f_s(n)$ and is defined as follo

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

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Inverse finite fourier cosine transform: If $f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} \, dx$ then $F(x)$ is called inverse finite fourier cosine transform of $f_c(n)$ and is defined as follows:

$$F(x) = \frac{1}{l} \int_0^l F(x) \, dx + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}$$

Less Important:

ⓐ **Fourier sine transform:** Let a function $f(x)$ in the interval $0 < x < \infty$, then the fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$, or $f_s(n)$ and is defined as follow:

$$F_s\{F(x)\} = f_s(n) = \int_0^\infty F(x) \sin nx \, dx.$$

ⓑ If $f_s(n) = \int_0^\infty F(x) \sin nx \, dx$ then $F(x)$ is called inverse fourier sine transform of $f_s(n)$ and is defined as follows:

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx \, dn$$

ⓒ **Fourier cosine transform:** Let a function $F(x)$ is defined in the interval $0 < x < \infty$, then the fourier cosine transform of $F(x)$ is denoted by $F_c\{F(x)\}$ or $f_c(n)$ and is defined as follows:

$$F_c\{F(x)\} = f_c(n) = \int_0^\infty F(x) \cos nx \, dx.$$

ⓓ **Inverse:** If $f_c(n) = \int_0^\infty F(x) \cos nx \, dx$ then $F(x)$ is called inverse fourier cosine transform of $f_c(n)$ and is defined as follows:

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx \, dn$$

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4 State the Convolution: The convolution of two functions $F(x)$ and $G(x)$ in the interval $-\infty < x < \infty$ is denoted by $F*G$ and is defined by

$$F*G = \int_{-\infty}^{\infty} F(u)\, G(x-u)\, du.$$

$\boxed{***}$ Relation between Fourier and Laplace transforms.

Sol^n: We consider a function: $F(t) = \begin{cases} e^{-xt} G(t), & t > 0 \\ 0, & t < 0 \end{cases}$ — (1)

Then $F\{F(t)\} = \int_{-\infty}^{\infty} e^{int} F(t)\, dt$

$$= \int_{-\infty}^{0} e^{int} F(t)\, dt + \int_{0}^{\infty} e^{int} F(t)\, dt$$

$$= \int_{-\infty}^{0} e^{int} \cdot 0\, dt + \int_{0}^{\infty} e^{int} \cdot e^{-xt} G(t)\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-t(x-in)} \cdot G(t)\, dt$$

$$= \int_{0}^{\infty} e^{-st} G(t)\, dt \quad ; \text{ where } x - in = s$$

$$= \mathcal{L}\{G(t)\}$$

Which is the required relation between Fourier and Laplace transforms.

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482 page: Given equation $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$

From the defination of finite fourier cosine transform we get,

$$F_c\{f(x)\} = \int_0^{\pi} f(x)\cos nx\, dx$$

$$= \int_0^{\pi/2} f(x)\cos nx\, dx + \int_{\pi/2}^{\pi} f(x)\cos nx\, dx$$

$$= \int_0^{\pi/2} \cos nx\, dx - \int_{\pi/2}^{\pi} \cos nx\, dx$$

$$= \left|\frac{\sin nx}{n}\right|_0^{\pi/2} - \left|\frac{\sin nx}{n}\right|_{\pi/2}^{\pi}$$

$$= \frac{1}{n}\left[\sin\frac{n\pi}{2} - 0\right] - \frac{1}{n}\left[0 - \sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{n}\sin\frac{n\pi}{2}, \quad n = 1, 2, 3 \ldots$$

483 page: $f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$

Find the fourier sine transform.

$$F_s\{f(x)\} = f_s(n) = \int_0^{\pi} f(x)\sin nx\, dx$$

$$= \int_0^{\pi/2} f(x)\sin nx\, dx + \int_{\pi/2}^{\pi} f(x)\sin nx\, dx$$

$$= \int_0^{\pi/2} x\sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx$$

$$= \left[x\frac{(-\cos nx)}{n} - 1\frac{(-\sin nx)}{n^2}\right]_0^{\pi/2} + \left[(\pi - x)\frac{(-\cos nx)}{n} - (-1)\frac{(-\sin nx)}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[\frac{(x - \pi)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{n}\left(\frac{\pi}{2}\cos\frac{n\pi}{2} + 0\right) + \frac{1}{n^2}\left(\sin\frac{n\pi}{2} - 0\right) - \frac{1}{n}\left(0 - \frac{\pi}{2}\cos\frac{n\pi}{2}\right)$$

$$- \frac{1}{n^2}\left(0 - \sin\frac{n\pi}{2}\right)$$

$$= \frac{2}{n^2}\sin\frac{n\pi}{2}$$

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) $f(x) = e^{-|x|}$; $x \geq 0$; from the definition of fourier transform, we get

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx = 2\int_0^{\infty} f(x)\, e^{inx}\, dx$$

$$= 2\int_0^{\infty} e^{-x} e^{inx}\, dx$$

$$= 2\int_0^{\infty} e^{-(1-in)x}\, dx$$

$$= 2\left| \frac{e^{-(1-in)x}}{-(1-in)} \right|_0^{\infty}$$

$$= \frac{-2}{(1-in)}(0-1) = \frac{2}{1-in}$$

38 Page: $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > x \end{cases}$ mean $x < a$ or $x > -a$ mean $x > a$ or $x < -a$

From the definition of fourier transform we get,

$$F\{f(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{inx}\, dx$$

$$= \int_{-\infty}^{-a} F(x)\, e^{inx}\, dx + \int_{-a}^{a} F(x)\, e^{inx}\, dx + \int_a^{\infty} F(x)\, e^{inx}\, dx$$

$$= \int_{-\infty}^{-a} 0\, e^{inx}\, dx + \int_{-a}^{a} x^2 e^{inx}\, dx + \int_a^{\infty} 0\, e^{inx}\, dx$$

$$= 0 + \left[\frac{x^2 e^{inx}}{in} + \frac{2x e^{inx}}{n^2} - \frac{2 e^{inx}}{in^3} \right]_{-a}^{a}$$

$$= \frac{1}{in}\left(a^2 e^{ina} - a^2 e^{-ina}\right) + \frac{2}{n^2}\left(a e^{ina} + a e^{-ina}\right) - \frac{2}{in^3}\left(e^{ina} - e^{-ina}\right)$$

$$= \frac{a^2}{in}\left(e^{ina} - e^{-ina}\right) + \frac{2a}{n^2}\left(e^{ina} + e^{ina}\right) - \frac{2 \cdot 2i \sin na}{in^3}$$

$$= \frac{2a^2 i \sin na}{in} + \frac{2a \cdot 2\cos na}{n^2} - \frac{4 \sin na}{n^3}$$

$$= \frac{2a^2 \sin na}{n} + \frac{4a \cos na}{n^2} - \frac{4 \sin na}{n^3}$$

$$= \frac{4a^2}{n}\left(\frac{n^2}{a^2} \sin na + \frac{n}{a} \cos na - \frac{\sin n}{a^2}\right.$$

Chapter : 09 536 page:

Ex: 01 Given equations are $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ ---- ①.

$U(0,t) = 1$ ---- ② $U(\pi,t) = 3$ --- ③ $U(x,0) = 1$ --- ④

Taking finite fourier sine transform on both sides of ① then we get

$$F_s\left\{\frac{\partial U}{\partial t}\right\} = F_s\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

$$\Rightarrow \int_0^\pi \frac{\partial U}{\partial t} \sin nx\, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \quad \text{—— ⑤}$$

Consider $u = u(n,t) = \int_0^\pi U(x,t) \sin nx\, dx$ —— ⑥

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx\, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\, dx$ [using 5]

$$\Rightarrow \frac{du}{dt} = \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - \cancel{n\left\{\left[\cos nx\right.\right.} n\int_0^\pi \cos nx \frac{\partial U}{\partial x} dx$$

$$= 0 - 0 - n\left\{\left[\cos nx\, U(x,t)\right]_0^\pi - n\int_0^\pi (-\sin nx)\, U(x,t)\, dx\right\}$$

$$= -n(3\cos n\pi - 1) - n^2\int_0^\pi U(x,t) \sin nx\, dx \quad \text{by ② and ③}$$

$$= n(1 - 3\cos n\pi) - n^2 u \quad \text{by ⑥}$$

$$\frac{du}{dt} + n^2 U = n(1 - 3\cos n\pi) \quad \text{—— ⑦}$$

Which is a linear differential equation of order one.

$\therefore IF = e^{\int n^2 dt} = e^{n^2 t}$ $\qquad \frac{dy}{dx} + Py = Q$

$\therefore ue^{n^2 t} = \int n(1 - 3\cos n\pi) e^{n^2 t}$ $\qquad IF = e^{\int P dx}$

$= \frac{n(1 - 3\cos n\pi)}{n^2} e^{n^2 t} + A$ $\qquad y \cdot I.F. = \int Q\, IF\, dx$

$$u = \frac{1 - 3\cos n\pi}{n} + Ae^{-n^2 t}$$

$$u(n,t) = \frac{1 - 3\cos n\pi}{n} + Ae^{-n^2 t} \quad \text{—— ⑧}$$

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$\Rightarrow u(n,0) = \frac{1}{n}(1-3\cos n\pi) + A$ ——— (9)

Now putting $t=0$ in (6) then we get,

$$u(n,0) = \int_0^\pi U(x,0)\sin nx\,dx$$

$$u(n,0) = \int_0^\pi 1.\sin nx\,dx = \left[\frac{\cos nx}{n}\right]_0^\pi = -\frac{1}{n}(\cos n\pi - 1)$$

$$-\frac{1}{n}(\cos n\pi - 1) = \frac{1}{n}(1-3\cos n\pi) + A \quad [\text{using } 9]$$

$$A = \frac{1}{n}(1-\cos n\pi - 1 + 3\cos n\pi) = \frac{2\cos n\pi}{n}$$

From 8 $\Rightarrow$ $u(n,t) = \frac{1}{n}(1-3\cos n\pi) + \frac{2\cos n\pi}{n}e^{-n^2t}$

Taking inverse finite fourier sine transform on both sides then we get,

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n}(1-3\cos n\pi) + \frac{2\cos n\pi}{n}e^{-n^2t}\right\}\sin\frac{n\pi x}{\pi}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1-3\cos n\pi)\sin nx}{n} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}.e^{-n^2t}.\sin\frac{n\pi x}{\pi} \quad \text{—— (10)}$$

Now in the interval $0 \le x \le \pi$, the finite fourier sine transform of 1 is

$$F_s(1) = \int_0^\pi 1.\sin nx\,dx = -\left[\frac{\cos nx}{n}\right]_0^\pi = -\frac{1}{n}(\cos n\pi - 1)$$

$$F_s(1) = \frac{1}{n}(1-\cos n\pi)$$

$\therefore$ Inverse finite sine transform

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos n\pi)\sin nx \quad \text{—— (11)}$$

Again $F_s(x) = \int_0^\pi x\sin nx\,dx = \left[x\frac{-\cos nx}{n} - \frac{-\sin nx}{n^2}\right]_0^\pi$

$$F_s(x) = \left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^\pi = -\frac{1}{n}\pi\cos n\pi + 0$$

$$F_s(x) = \frac{-\pi\cos n\pi}{n}$$

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Inverse finite sine transform

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-\pi\cos n\pi}{n} \sin nx \quad \text{——} \quad (12)$$

w, $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-3\cos n\pi)\sin nx = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos n\pi-2\cos n\pi)\sin$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos n\pi)\sin nx - \frac{2}{\pi}\cdot 2\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos n\pi)\sin nx + \frac{2}{\pi}\left(\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-\pi\cos n\pi}{n}\sin nx\right)$$

$$= 1 + \frac{2}{\pi}x \quad \text{——} \quad (13)$$

rom 10 and 13 we get

$$U(x,t) = 1 + \frac{2x}{\pi} + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{\pi n}\right\} e^{-n^2 t}\sin nx$$