UNIT : 4 QUANTUM MECHANICS

(1)

Sol^n => We have,

the wave function

$$\psi(x) = A\cos^2(x)$$

for $-\frac{\pi}{2}$ to $\pi/2$

$$\int_{-\pi/2}^{\pi/2} |\psi(x)|^2 dx = 1$$

$$\boxed{\int_{-\pi/2}^{\pi/2} A^2\cos^4 x\, dx = 1} \Rightarrow A^2\int_{-\pi/2}^{\pi/2} \cos^4 x\, dx = 1 \Rightarrow A^2\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)^2 dx = 1$$

$$A^2\int_{-\pi/2}^{\pi/2} \frac{1+2\cos 2x+\cos^2(2x)}{4}\, dx = 1$$

$$A^2\int_{-\pi/2}^{\pi/2} \frac{1+2\cos 2x+\frac{1+\cos 4x}{2}}{4} = A^2\int_{-\pi/2}^{\pi/2} \frac{3+4\cos(2x)+\cos(4x)}{8}\, dx$$

$$A^2\int_{-\pi/2}^{\pi/2} \frac{3}{8}\, dx = 1 \qquad \left[\because \text{Integration of } \begin{matrix}\cos 2x \\ \cos 4x\end{matrix} \text{ in } -\pi/2 \text{ to } \pi/2 = 0\right]$$

$$A^2\left[\frac{3}{8}x\right]_{-\pi/2}^{\pi/2} = 1 \Rightarrow A^2\cdot\frac{3\pi}{8} = 1 \Rightarrow \boxed{A = \sqrt{\frac{8}{3\pi}}}$$

NOW, lets calculate probability between 0 to $\pi/4$

$$P = \int_0^{\pi/4} |\psi(x)|^2 dx = \int_0^{\pi/4} \frac{8}{3\pi} \cos^4(x)\, dx$$

$$P = \frac{8}{3\pi} \int_0^{\pi/4} \cos^4(x)\, dx \cancel{= \frac{8}{3\pi}\left[\frac{3}{8}x\right]_0^{\pi/4} = \frac{8}{3\pi} \cdot \frac{3}{8} \cdot \left(\frac{\pi}{4}\right)}$$

$$\boxed{\cancel{P = \frac{1}{4}}}$$

$$P = \frac{8}{3\pi} \int_0^{\pi/4} \frac{3}{8} + \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8} \cdot dx$$

$$= \frac{8}{3\pi} \left[\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}\right]_0^{\pi/4}$$

$$= \frac{8}{3\pi} \left[\frac{3}{8} \times \frac{\pi}{4} + \frac{\sin(\pi/2)}{4} + \frac{\sin(\pi)}{32}\right]$$

$$= \frac{3\pi}{32} \times \frac{8}{3\pi} + \frac{8}{3\pi} \cdot \frac{1}{4}$$

$$= \frac{1}{4} + \frac{8}{12\pi} = \frac{3\pi}{12\pi} + \frac{8}{12\pi} = \frac{8+3\pi}{12\pi}$$

$$\boxed{P = \frac{3\pi + 8}{12\pi}}$$

(2)

Sol^n :) we have,

$$\psi(x) = A\exp(-\alpha x^2)$$

To normalize the given function

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$A^2 \int_{-\infty}^{\infty} e^{-2\alpha x^2} \cdot dx = 1$$

using standard Gaussian integral

$$A^2 \int_{-\infty}^{\infty} e^{-2\alpha x^2} dx = A^2 \cdot \sqrt{\frac{\pi}{2\alpha}} = 1$$

$$\boxed{A = \left(\frac{2\alpha}{\pi}\right)^{1/4}}$$

then

$$\psi = A e^{-\alpha x^2}$$

$$\boxed{\psi = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}}$$

(3)

Sol^n => The given wave function

$$\psi(x) = c e^{-\alpha^2 x^2} \quad \text{where } -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$c^2 \int e^{-2\alpha^2 x^2} dx = 1 \implies c^2 \cdot \frac{\sqrt{\pi}}{\alpha\sqrt{2}} = 1$$ [using standard gaussian integral]

$$c = \left(\frac{\alpha\sqrt{2}}{\sqrt{\pi}}\right)^{1/2} \implies c = \left(\alpha\sqrt{\frac{2}{\pi}}\right)^{1/2}$$

$$\boxed{c = \left(\alpha\sqrt{\frac{2}{\pi}}\right)^{1/2}}$$

calculate probability

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$ [using standard result gaussian integral]

then

$$c^2 \int_0^{\infty} e^{-2\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{2\alpha}} \cdot c^2$$

$$= \sqrt{\alpha}\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^{1/2 \times 2} \times \frac{\sqrt{\pi}}{\sqrt{2\alpha}} \cdot \frac{1}{2}$$

$$= \sqrt{\alpha} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2\alpha}} \cdot \frac{1}{2} = \frac{1}{2}$$

$$\boxed{\text{Probability} = 1/2}$$

(4)

Sol^n: we have,

$$\psi(x,t) = \sqrt{\frac{2}{a}}\sin(kx)e^{-i\varepsilon t/\hbar}$$

$$\psi^*(x,t) = \sqrt{\frac{2}{a}}\sin(kx)e^{i\varepsilon t/\hbar}$$

To find the expectation value of momentum

$$\langle P\rangle = \int_{-a}^{a}\psi^*\,\hat{p}\cdot\psi\,dx = \int_{-a}^{a}\psi^*\left(-i\hbar\frac{d}{dx}\right)\psi(x,t)\,dx$$

$$\langle P\rangle = \int_{-a}^{a}\sqrt{\frac{2}{a}}\sin(kx)e^{-i\varepsilon t/\hbar}\left(-i\hbar\frac{d}{dx}\right)\cdot\sqrt{\frac{2}{a}}\sin(kx)e^{i\varepsilon t/\hbar}\,dx$$

$$\langle P\rangle = \int_{-a}^{a}\left(\frac{2}{a}\right)\sin(kx)e^{-i\varepsilon t/\hbar}(-i\hbar)\,k\cos(kx)\,e^{i\varepsilon t/\hbar}\,dx$$

$$= -i\hbar k\left(\frac{2}{a}\right)\int_{-a}^{a}\sin(kx)\cos(kx)\,dx$$

$$= -\frac{i\hbar k}{a}\int_{-a}^{a}\sin(2kx)\,dx = -\frac{i\hbar k}{a}\cdot\left[\frac{\cos(2kx)}{2k}\right]_{-a}^{a}$$

$\sin(2kx)$ is an ODD function

$$= -\frac{i\hbar k}{a(2k)}\cdot\left[\cos(2ka) - \cos(2ka)\right] = 0$$

~~<P>=0~~ $\boxed{\langle P\rangle = 0}$

(5)

Solⁿ ⇒ We have,

$$\psi(x) = A e^{(-x^2/a^2)} e^{(ikx)}$$

$$\psi(x) = A e^{(-x^2/a^2 + ikx)}$$

$$\therefore \psi^*(x) = A e^{(-x^2/a^2 - ikx)}$$

Using Normalization

$$\Rightarrow \int_{-\infty}^{\infty} |\psi^* \psi \, dx| = 1$$

$$\Rightarrow A^2 \int_{-\infty}^{\infty} \left| e^{-x^2/a^2 + ikx - x^2/a^2 - ikx} \right| = 1$$

$$\Rightarrow A^2 \int_{-\infty}^{\infty} \left| e^{-2x^2/a^2} \right| = 1$$

Let $\beta = 2/a^2$

$$A^2 \int_{-\infty}^{\infty} e^{-\beta x^2} dx = 1 \Rightarrow A^2 \int_{-\infty}^{\infty} e^{-\beta x^2} dx = 1$$

$$A^2 \sqrt{\frac{\pi}{2\beta}} = 1$$

$$A^2 \sqrt{\frac{\pi (a^2)}{2}} = 1$$

$$A^2 . a \sqrt{\pi/2} = 1$$

$$A^2 = \sqrt{\frac{2}{\pi}} \times \frac{1}{a}$$

$$\boxed{A = \left(\frac{2}{\pi}\right)^{1/4} \left(1/a\right)^{1/2}}$$

=2

$$A = \left(\frac{2}{\pi a^2}\right)^{1/4}$$

(ii) Expectation value for momentum

$$\langle P \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p}\, \psi \, dx = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$$

$$\therefore \frac{d}{dx}\left[A e^{(-x^2/a^2 + ikx)}\right] = A e^{(-x^2/a^2 + ikx)} \left(\frac{-2x}{a^2} + ik\right)$$

$$\langle P \rangle = \int_{-\infty}^{\infty} A e^{(-x^2/a^2 - ikx)} \cdot (-i\hbar)\, A e^{(-x^2/a^2 + ikx)} \left(\frac{-2x}{a^2} + ik\right) dx$$

$$\langle P \rangle = A^2 \int_{-\infty}^{\infty} e^{(-2x^2/a^2)} (-i\hbar)\left(\frac{-2x}{a^2} + ik\right) dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{(-2x^2/a^2)} \left(\frac{i2x\hbar}{a^2} + \hbar k\right) dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{(-2x^2/a^2)} \left(i\hbar \frac{2x}{a^2} + \hbar k\right) dx$$

$$= i\hbar \frac{A^2(2)}{a^2} \underbrace{\int_{-\infty}^{\infty} x\, e^{(-2x^2/a^2)} dx}_{I_1} + \hbar k A^2 \underbrace{\int_{-\infty}^{\infty} e^{(-2x^2/a^2)} dx}_{I_2}$$

$$= \left(i\hbar \frac{A^2 2}{a^2}\right) I_1 + \left(\hbar k A^2\right)(I_2) \quad \text{— (A)}$$

Let the Integral

$$I_1 = \int_{-\infty}^{\infty} x\, e^{(-2x^2/a^2)}\, dx$$

Using by parts

$$I_1 = x \cdot \int_{-\infty}^{\infty} e^{-2x^2/a^2}\, dx - \int \frac{dx}{dx} \int e^{-2x^2/a^2}\, dx$$

$$= x \cdot \sqrt{\frac{\pi a^2}{2}} - \int \sqrt{\frac{\pi a^2}{2}}\, dx$$

[Using standard Gaussian Integral]

$$= x\sqrt{\frac{\pi a^2}{2}} - x\sqrt{\frac{\pi a^2}{2}}$$

$$= 0$$

$$\boxed{I_1 = 0} \quad \text{— (iii)}$$

GAUSSIAN INTEGRAL

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

Let the Integral

$$I_2 = \int_{-\infty}^{\infty} e^{-2x^2/a^2}\, dx = \sqrt{\frac{\pi a^2}{2}} \quad \text{— (iv)}$$

putting equ (iii) and (iv) in equ (i)

$$= \left(i\hbar \frac{2A^2}{a^2}\right) \times 0 + (\hbar k A^2)\sqrt{\frac{\pi a^2}{2}} = \hbar k A^2 a \sqrt{\frac{\pi}{2}} \quad \text{— (v)}$$

put $A^2 = \left(\frac{2}{\pi a^2}\right)^{1/2}$

$$= \hbar k \sqrt{\frac{2}{\pi}} \times \frac{1}{a} \times \sqrt{\frac{\pi}{2}} \times a = \hbar k$$

$$\boxed{\langle p \rangle = \hbar k}$$

(6)

Solⁿ ⇒ We have,

$$\psi = A\exp[ik(x-a)]$$

If it is eigenfunction of the momentum operator

then $\hat{p}\psi = a\psi$ — (1)

where a = eigen value

let us check

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}Ae^{ik(x-a)}$$

$$= -i\hbar Ae^{ik(x-a)}[ik]$$

$$\hat{p}\psi(x) = \hbar k Ae^{ik(x-a)}$$

$$\boxed{\hat{p}\psi(x) = \hbar k\,\psi(x)}$$

↳ It satisfies the equation (1)

where a is eigenvalue which is a constant

Here eigen value is represent momentum

$$\boxed{a = \hbar k = \frac{hk}{2\pi}}$$

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Solⁿ:- We have,

$$\psi(x,t) = e^{i(kx-\omega t)}$$

we have the Energy operator

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \quad -(i)$$

also If $\psi(x,t)$ is an eigen function

$$\hat{E}\psi(x,t) = E\,\psi(x,t) \quad -(ii)$$

$\therefore$ E is eigen value

Lets check

$$\hat{E}\psi(x,t) = i\hbar\frac{\partial}{\partial t}e^{i(kx-\omega t)}$$

$$= i\hbar\, e^{i(kx-\omega t)}\cdot(-i\omega)$$

$$\hat{E}\psi(x,t) = \hbar\omega\, e^{i(kx-\omega t)}$$

$$\boxed{\hat{E}\psi(x,t) = E\,\psi(x,t)}$$

It satisfies eqn (ii)

so, $\psi(x,t)$ is an eigenfunction with $\boxed{\text{eigen value} = \hbar\omega}$ corresponds to Energy

$$\boxed{E = \hbar\omega}$$

(8)

Solⁿ: we have the wave function for infinite square well

For a particle in a 1D infinite square well

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{where } n = 1, 2, 3, \ldots$$

For second excited state

we have $n = 3$

$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)$$

To find the extrema, we have to find

$$\cancel{\frac{\partial \psi_3(x)}{\partial x} = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)}$$

$$\frac{d\psi_3(x)}{dx} = \frac{d}{dx}\left[\sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)\right]$$

$$= \sqrt{\frac{2}{a}} \times \left(\frac{3\pi}{a}\right) \cos\left(\frac{3\pi x}{a}\right) = 0$$

$$\sqrt{\frac{2}{a}} \times \left(\frac{3\pi}{a}\right) \cos\left(\frac{3\pi x}{a}\right) = 0$$

$$\cos\left(\frac{3\pi x}{a}\right) = 0$$

$$\frac{3\pi x}{a} = \frac{(2m+1)\pi}{2}$$

$$x = \frac{(2m+1)a}{6} \quad \text{where } m = 0, 1, 2$$

∴ we have condition

$$0 < x < a$$

$$0 < \frac{(2m+1)a}{6} < a \implies 0 < \frac{(2m+1)a}{6} < 1$$

For

$m = 0 \rightarrow a/6$ | $m = 1 \rightarrow a/2$ | $m = 2 \rightarrow \frac{5a}{6}$

We have, three extrema

$$x = \frac{a}{6}, \frac{a}{2}, \frac{5a}{6}$$

$$\frac{d^2\psi_3(x)}{dx^2} = -\sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2 \sin\left(\frac{3\pi x}{a}\right)$$

At $x = a/6$

$$\frac{d^2\psi_3(x)}{dx^2} = -\sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2 \sin\left(\frac{3\pi \times a}{6 \times a}\right)^{1}$$

$$\therefore \frac{d^2\psi_3}{dx^2} < 0 \quad (\text{maxima})$$

At $x = a/2$

$$\frac{d^2\psi_3(x)}{dx^2} = -\sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2 \sin\left(\frac{3\pi \times \cancel{a}}{2 \times a}\right)^{-1} = \sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2$$

$$\frac{d^2\psi_3}{dx^2} > 0 \quad (\text{minima})$$

At $x = \frac{5a}{6}$

$$\frac{d^2\psi_3(x)}{dx^2} = -\sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2 \sin\left(\frac{\cancel{3}\pi \times 5\cancel{a}}{\underset{2}{\cancel{6}} \times \cancel{a}}\right)^{1} = -\sqrt{\frac{2}{a}}\left(\frac{3\pi}{a}\right)^2$$

$$\frac{d^2\psi_3}{dx^2} < 0 \quad (\text{maxima})$$

Maxima is at $x = a/6$ and $\frac{5a}{6}$

Minima is at $x = a/2$

Total extreme points $= \underline{\underline{3}}$

(9)

Sol^n: For a particle trapped in Infinite potential well

we have the wave function,

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(kx)$$

$$= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

To find the probability b/w $x=0$ to $x=a/n$

$$\int_0^{a/n} \psi_n^*(x) \cdot \psi_n(x)\, dx = \int_0^{a/n} \left(\frac{2}{a}\right) \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \left(\frac{2}{a}\right) \int_0^{a/n} \frac{1 - \cos\left(\frac{2n\pi x}{a}\right)}{2}\, dx \qquad \left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= \left(\frac{2}{a}\right) \int_0^{a/n} \frac{1}{2} \cdot dx - \left(\frac{2}{a}\right) \int_0^{a/n} \frac{\cos\left(\frac{2n\pi x}{a}\right)}{2}\, dx$$

$$= \frac{1}{a} \times \frac{a}{n} - \frac{1}{a} \times [\sin(2\pi) - \sin(0)]$$

$$= \frac{1}{n} - \frac{1}{a}[0 - 0]$$

$$= \frac{1}{n}$$

$$= 2$$

$$\boxed{P = \frac{1}{n}}$$

(10)

Sol^n ⇒ we have, wave function for a particle trapped in infinite potential well

$$\psi_n(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi x}{a}\right)$$

For ground state, $n=1$

$$\psi_1(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi x}{a}\right)$$

(i) Expectation value for position (x)

$$\langle x\rangle = \int_0^a x\,|\psi_1(x)|^2\,dx$$

$$\langle x\rangle = \int_0^a x\left(\frac{2}{a}\right)\sin^2\left(\frac{\pi x}{a}\right)dx$$

$$\langle x\rangle = \left(\frac{2}{a}\right)\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)dx$$

$$\langle x\rangle = \left(\frac{2}{a}\right)\int_0^a \frac{x}{2}\,dx - \left(\frac{2}{a}\right)\int_0^a \frac{x\cos\frac{2\pi x}{a}}{2}\,dx$$

$$\langle x\rangle = \left(\frac{2}{a}\right)\frac{a^2}{4} - \left(\frac{1}{a}\right)\int_0^a x\cos\frac{2\pi x}{a}\,dx$$

$$\langle x\rangle = \frac{a}{2} - \frac{1}{a}\left[x\,\frac{\sin\frac{2\pi x}{a}}{\frac{2\pi}{a}} - \int\frac{\sin\frac{2\pi x}{a}}{\frac{2\pi}{a}}\,dx\right]$$

$$\langle x\rangle = \frac{a}{2} - \frac{1}{a}\left[x\,\frac{\sin\frac{2\pi x}{a}}{\frac{2\pi}{a}} + \frac{\cos\frac{2\pi x}{a}}{\frac{4\pi^2}{a^2}}\right]_0^a$$

$$\boxed{\langle x\rangle = a/2}$$

$$\langle P \rangle = \int_0^a \psi_1^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi_1(x)\,dx$$

$$\psi_1(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right)$$

$$\frac{\partial}{\partial x}\psi_1(x) = \sqrt{\frac{2}{a}}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)$$

$$\psi_1^*(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right)$$

$$\langle P \rangle = \int_0^a \left(\frac{2}{a}\right)(-i\hbar)\sin(\pi x/a)\cos\left(\frac{\pi x}{a}\right)\frac{\pi}{a}\cdot dx$$

$$= \int_0^a \frac{2\pi}{a^2}(-i\hbar)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)dx$$

$$= \int_0^a \frac{-\pi i\hbar}{a^2}\sin\left(\frac{2\pi x}{a}\right)dx$$

$$= -\frac{\pi i\hbar}{a^2}\int_0^a \sin\left(\frac{2\pi x}{a}\right)dx = \frac{-\pi i\hbar}{a^2}\left[\cos\left(\frac{2\pi x}{a}\right)\times\frac{a}{2\pi}\right]_0^a$$

$$= \frac{\pi i\hbar}{a(2\pi)}\left[\cos\left(\frac{2\pi x}{a}\right)\right]_0^a = \frac{\pi i\hbar}{2\pi a}\left[\cos 2\pi - \cos 0\right]$$

$$= 0$$

$$\boxed{\langle P \rangle = 0}$$

So,

$\langle x \rangle = a/2$

$\langle p \rangle = 0$

$= 2$

(12)

Solⁿ ⇒ We have the Hamiltonian operator of a system

$$H = -\frac{\partial^2}{\partial x^2} + x^2 \quad -(i)$$

$$\psi(x) = N x e^{-x^2/2} \quad -(ii)$$

Diff eqn (ii) w.r.t (x)

$$\frac{\partial \psi(x)}{\partial x} = N x(-x) e^{-x^2/2} + N e^{-x^2/2} = N e^{-x^2/2}(1-x^2)$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} = N e^{-x^2/2}(-2x) + N(1-x^2) e^{-x^2/2}(-x)$$

$$= N e^{-x^2/2}\left[-2x - x + x^3\right] = N e^{-x^2/2}(-3x + x^3)$$

$$-\frac{\partial^2 \psi(x)}{\partial x^2} = N(3x - x^3) e^{-x^2/2} \quad -(iii)$$

~~Subs~~ We have eqn (i)

$$H = -\frac{\partial^2}{\partial x^2} + x^2$$

$$\hat{H}\psi = -\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi$$

$$\hat{H}\psi = N(3x - x^3)e^{-x^2/2} + x^2 N x e^{-x^2/2}$$

$$= Ne^{-x^2/2}\left[3x - x^3 + x^3\right]$$

$$= 3Nx e^{-x^2/2}$$

$$\boxed{\hat{H}\psi = 3\psi(x)}$$

Here the eigenvalue $\boxed{E = 3}$

* Normalizing the wave function

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$\int_{-\infty}^{\infty} N^2 e^{-x^2} \cdot (x)^2 dx = 1 \Rightarrow N^2 \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2} dx = 1$$

$$\cancel{N^2 \cdot \left[x^2 \int e^{-x^2} dx - \int \frac{dx^2}{dx} \int e^{-x^2} dx \cdot dx\right]} = 1$$

$$\cancel{N^2 \left[x^2 \cdot \sqrt{\pi} - \int 2x \cdot \sqrt{\pi} \cdot dx\right] = 1 \Rightarrow N^2\left[x^2\sqrt{\pi} - x^2\sqrt{\pi}\right]}$$

Using standard Gaussian Integral $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$$N^2 \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2} dx = 1$$

$$N^2 \cdot \frac{\sqrt{\pi}}{2} = 1 \Rightarrow \boxed{N = \sqrt{\frac{2}{\sqrt{\pi}}}}$$

(13)

Sol^n: We have,

$$\psi(x,t) = A e^{\left(-ikt - \frac{km}{\hbar}x^2\right)} \quad -(A)$$

Using Schrodinger equation time dependent

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x) \quad -(i)$$

Diff equ (A) w.r.t time

$$\frac{\partial \psi}{\partial t} = A e^{\left(-ikt - \frac{km}{\hbar}x^2\right)}(-ik) = -ik\psi \quad -(ii)$$

$$i\hbar \frac{d\psi}{dt} = i\hbar(-ik)\psi = \hbar k \psi$$

Diff equ (A) w.r.t (x)

$$\frac{\partial \psi}{\partial x} = A e^{\left(-ikt - \frac{km}{\hbar}x^2\right)}\left(\frac{-2kmx}{\hbar}\right) = \frac{-2kmx}{\hbar}\psi$$

Diff eqn w.r.t x

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{-2km}{\hbar}\right)x \cdot \left(\frac{-2kmx}{\hbar}\psi\right) - \left(\frac{2km}{\hbar}\right)\psi$$

$$= -\frac{2km\psi}{\hbar} + \left(\frac{4k^2m^2x^2}{\hbar^2}\psi\right)$$

$$= \left(\frac{4k^2m^2x^2\psi}{\hbar^2} - \frac{2km\psi}{\hbar}\right) \quad -(iii)$$

putting (ii) and (iii) in equ (i)

$$\hbar k \psi = \frac{-\hbar^2}{2m}\left(\frac{4k^2m^2x^2}{\hbar^2}\psi - \frac{2km\psi}{\hbar}\right) + V(x)\psi(x)$$

$$\cancel{\hbar k\psi} = -2k^2mx^2\cancel{\psi} + \cancel{\hbar k\psi} + V(x)\cancel{\psi}$$

$$\boxed{Vx = 2k^2mx^2}$$

$$\boxed{V(x) = 2k^2mx^2}$$

(14)

Sol^n ⇒ Given :- height of the box $\longrightarrow \infty$

$a = 2.5 \times 10^{-10}$ m

$m_e = 9.1 \times 10^{-31}$ kg

$\therefore \; E_n = \dfrac{n^2 h^2}{8 m^2 a^2}$

for $E_1 = \dfrac{h^2}{8 m^2 a^2} = \dfrac{(6.626 \times 10^{-34})^2}{8 \times (9.1 \times 10^{-31})^2 \times (2.5 \times 10^{-10})^2}$

$E_1 = 9.64 \times 10^{-19}$ J

$E_1 \text{ (in eV)} = \dfrac{9.64 \times 10^{-19}}{1.6 \times 10^{-19}} = 6.03$ eV

$\boxed{E_1 = 6.03 \text{ eV}}$

~~for~~ $E_n = n^2 E_1$

for $n = 2$

$E_2 = 4 \times 6.03$

$\boxed{E_2 = 24.12 \text{ eV}}$

(15)

sol^n ⇒ We have

$\psi(x=a) = 0$ [for infinite potential well]

$A \sin(kx) = 0$

Here $x = a$ [width of the box is a]

$A \sin(ka) = 0$

$ka = n\pi \Rightarrow a = \frac{n\pi}{k} = \frac{n\pi\hbar}{p}$

$a^2 = \frac{n^2\pi^2\hbar^2}{p^2} = \frac{n^2\pi^2\hbar^2}{2mE_n}$

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

For $n=4$

$E_4 = \frac{16\pi^2\hbar^2}{2ma^2} = \frac{8\pi^2\hbar^2}{ma^2}$ — (i)

for $n=2$.

$E_2 = \frac{4\pi^2\hbar^2}{2ma^2} = \frac{2\pi^2\hbar^2}{ma^2}$ — (ii)

$\Delta E = hf$ (iii)

from (i), (ii) and (iii)

$hf = \frac{1}{ma^2}(8\pi^2\hbar^2 - 2\pi^2\hbar^2)$

$hf = \frac{1}{ma^2}(6\pi^2\hbar^2) \Rightarrow a^2 = \frac{6\pi^2\hbar^2}{mhf}$

$a = \frac{\overset{3}{\cancel{6}}\cancel{\pi^2} \cdot h \cdot \cancel{h}}{2\cancel{\pi} \times 2\cancel{\pi} \cancel{h} f \times m} = \frac{3h}{2fm} = \frac{3 \times 6.626 \times 10^{-34}}{2 \times 9.105 \times 10^{-31} \times 3.43 \times 10^{14}}$

$a = 5.64 \times 10^{-10}$ m

$\boxed{a \simeq 5.64\,Å \simeq 56.4\,nm}$ width of box