Flight Dynamics & Control

Tutorial 1 - Solutions

1. (a) The eigenvalues of the system $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 3) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

which has roots $\lambda_1 = -1$ and $\lambda_2 = 5$. The eigenvectors can be found by solving

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $\lambda_1 = -1$

$$2\xi_{11} + 2\xi_{12} = 0$$

$$4\xi_{11} + 4\xi_{12} = 0$$

hence $\boldsymbol{\xi}_1 = [1, -1]$. For $\lambda_2 = -1$

$$-4\xi_{21} + 2\xi_{22} = 0$$

$$4\xi_{21} - 2\xi_{22} = 0$$

hence $\boldsymbol{\xi}_2 = [1, 2]$.

(b) The eigenvalues of the system $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}$ are

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda + 1) + 10 = 0$$

$$\lambda^2 + 9 = 0$$

which has roots $\lambda = \pm\sqrt{-9} = \pm 3i$. The eigenvectors can be found by solving

$$\begin{bmatrix} 1-\lambda & -5 \\ 2 & -1-\lambda \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $\lambda = \pm 3i$

$$(1 \mp 3i)\xi_{11} - 5\xi_{12} = 0$$

$$2\xi_{11} + (-1 \mp 3i)\xi_{12} = 0$$

if $\xi_{11} = 1$, $\xi_{12} = 0.2 \mp 0.6i$.

(c) The eigenvalues of the system $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ are

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2(1-\lambda) = 0$$

whose roots are $\lambda_{1,2} = 0$ and $\lambda_3 = 1$. The eigenvectors can be found by solving

$$\begin{bmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_1 = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

from the second and third equations we see that irrespective of the value of ξ_{12}, $\xi_{11} = \xi_{13} = 0$. Hence for $\xi_{12} = \pm 1$, $\boldsymbol{\xi}_1 = [0, 1, 0]$.
In the case of repeated eigenvalues ($\lambda_1 = \lambda_2$), we find the second eigenvector by solving

$$(\mathbf{A} - \mathbf{I}\lambda)\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

yielding $\boldsymbol{\xi}_2 = [1, 0, -1]$.
Finally, for $\lambda_3 = 1$ the system of equations is

$$-\xi_{31} = 0$$

$$\xi_{31} - \xi_{32} = 0$$

$$\xi_{31} = 0$$

Therefore $\xi_{31} = \xi_{32} = 0$ for all values of ξ_{33}, giving $\boldsymbol{\xi}_3 = [0, 0, 1]$.

(d) The eigenvalues of the system $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - 1)^2 = 0$$

which has roots $\lambda_{1,2} = 1$ and $\lambda_3 = 0$. The eigenvectors can be found by solving

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_1 = 1$

$$2\xi_{13} = 0$$
$$\xi_{12} = 0$$
$$\xi_{13} = 0$$

which hold true for any value of ξ_{11}. Therefore the one eigenvector for this root is $\boldsymbol{\xi}_1 = [1, 0, 0]$. The second eigenvector for $\lambda = 1$ can be found using

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

giving $\boldsymbol{\xi}_2 = [0, 0, 0.5]$.
For $\lambda_3 = 0$ we similarly find that $\boldsymbol{\xi}_3 = [0, 1, 0]$.

2. (a) Given the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + k_2 \boldsymbol{\xi}_2 e^{\lambda_2 t}$$

where $\lambda_{1,2}$ are the roots of

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 5\lambda + 6 = 0,$$

giving $\lambda_1 = -2$ and $\lambda_2 = -3$.
For $\lambda_1 = -2$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -6 & -5-\lambda_1 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving an eigenvector $\boldsymbol{\xi}_1 = [1, -2]$ and for $\lambda_1 = -3$

$$\begin{bmatrix} -\lambda_2 & 1 \\ -6 & -5-\lambda_2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving an eigenvector $\boldsymbol{\xi}_2 = [1, -3]$.
The general solution is therefore

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + k_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-3t}$$

and the weightings can be calculated based on the initial conditions

$$x_1(0) = k_1 + k_2$$
$$x_2(0) = -2k_1 - 3k_2$$

(b) Given the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whose eigenvalues are given by

$$\begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

The system therefore has a pair of equal real roots $\lambda = 3$.
The system's first eigenvector is

$$\begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda_1 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving $\boldsymbol{\xi}_1 = [1, -1]$. As there is multiplicity the second eigenvector $\boldsymbol{\xi}_2$ is the solution of

$$\begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \boldsymbol{\xi}_2 = \boldsymbol{\xi}_1$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

giving $\boldsymbol{\xi}_2 = [1, 0]$.
When we have multiplicity the general solution is also different, becoming

$$\mathbf{x}(t) = k_1 \boldsymbol{\xi}_1 e^{\lambda t} + k_2 \left(\boldsymbol{\xi}_1 t e^{\lambda t} + \boldsymbol{\xi}_2 e^{\lambda t} \right)$$

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + k_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} \right)$$

(c) Given the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whose eigenvalues are given by

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0.$$

The system therefore has a pair of complex conjugate eigenvalues $\lambda_{1,2} = \pm i$. The eigenvector for $\lambda_1 = i$ is

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving $\boldsymbol{\xi}_1 = [1, i]$ and similarly $\boldsymbol{\xi}_2 = [1, -i]$. The general solution for the system therefore is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} + k_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it}$$

(d) Given the system

$$\ddot{x} + \dot{x} - 6x = 0$$

let us first define $x_2 = \dot{x}$ and $x_1 = x$. This allows us to put our differential equation in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

The eigenvalues of the system are

$$\begin{bmatrix} -\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

which has roots $\lambda_1 = -3$ and $\lambda_2 = 2$. Since $x_2 = \dot{x}_1$ the relation between the two states is already given, meaning that we do not need to calculate the eigenvectors of the system to estimate it. The general solution is thus

$$x_1(t) = x(t) = k_1 e^{-3t} + k_2 e^{2t}$$

meaning that

$$x_2(t) = \dot{x}(t) = -3k_1 e^{-3t} + 2k_2 e^{2t}$$

You may calculate the eigenvectors of the system to check that this indeed the case.

(e) Given the system

$$\dddot{x} - 3\ddot{x} + 3\dot{x} - x = 0,$$

let us define $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = \ddot{x}$. The system then becomes

$$\dot{x_3} - 3x_3 + 3x_2 - x_1 = 0$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The eigenvalues of the system are

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0$$

which gives a triple root $\lambda = 1$. The three independent solutions for a system with a multiplicity of 3 and deficit of 2 are

$$x_1(t) = \boldsymbol{\xi}_1 e^{\lambda t}$$

$$x_2(t) = (\boldsymbol{\xi}_2 + \boldsymbol{\xi}_1 t) e^{\lambda t}$$

$$x_3(t) = \left(\boldsymbol{\xi}_3 + \boldsymbol{\xi}_2 t + \boldsymbol{\xi}_1 \frac{t^2}{2}\right) e^{\lambda t}$$

where

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 2 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 2 \end{bmatrix} \boldsymbol{\xi}_2 = \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 2 \end{bmatrix} \boldsymbol{\xi}_3 = \boldsymbol{\xi}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

giving $\boldsymbol{\xi}_1 = [1, 1, 1]$, $\boldsymbol{\xi}_2 = [-1, 0, 1]$ and $\boldsymbol{\xi}_3 = [2, 1, 1]$.

Combining the three modes,

$$x(t) = \left[k_1\boldsymbol{\xi}_1 + k_2\boldsymbol{\xi}_2 + k_3\boldsymbol{\xi}_3 + (k_2\boldsymbol{\xi}_1 + k_3\boldsymbol{\xi}_2)\, t + k_3\boldsymbol{\xi}_1 \frac{t^2}{2}\right] e^t$$

or

$$x(t) = \begin{bmatrix} k_1 - k_2 + 2k_3 \\ k_1 + k_3 \\ k_1 + k_2 + k_3 \end{bmatrix} e^t + \begin{bmatrix} k_2 - k_3 \\ k_2 \\ k_2 + k_3 \end{bmatrix} te^t + \begin{bmatrix} k_3 \\ k_3 \\ k_3 \end{bmatrix} \frac{t^2}{2} e^t.$$

3. Let us first calculate the equilibria of the system

$$\dot{x}_1 = x_1 x_2 + x_1^3$$

$$\dot{x}_2 = x_1 + x_2^2$$

which will occur when $\dot{x}_1 = \dot{x}_2 = 0$ and therefore

$$x_1(x_1^2 + x_2) = 0$$

$$x_1 + x_2^2 = 0.$$

One equilibrium will be at $[x_1, x_2]_A = [0, 0]$, while further equilibria can be found by solving the second equation for $x_1 = -x2^2$ and substituting into the first

$$-x_2^2(x_2 + x_2^4) = 0$$

giving $x_2 = -1$ and therefore one more equilibrium $[x_1, x_2]_B = [-1, -1]$.

To linearise the system about these equilibria, let us calculate the Jacobian such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \dot{x}_1}{\partial x_1} & \dfrac{\partial \dot{x}_1}{\partial x_2} \\ \dfrac{\partial \dot{x}_2}{\partial x_1} & \dfrac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$\frac{\partial \dot{x}_1}{\partial x_1} = x_2 + 3x_1^2 \qquad \frac{\partial \dot{x}_1}{\partial x_2} = x_1$$

$$\frac{\partial \dot{x}_2}{\partial x_1} = 1 \qquad\qquad \frac{\partial \dot{x}_2}{\partial x_2} = 2x_2$$

and therefore

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + 3x_1^2 & x_1 \\ 1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the equilibrium condition A, where $[x_1, x_2] = [0, 0]$, the Jacobian becomes

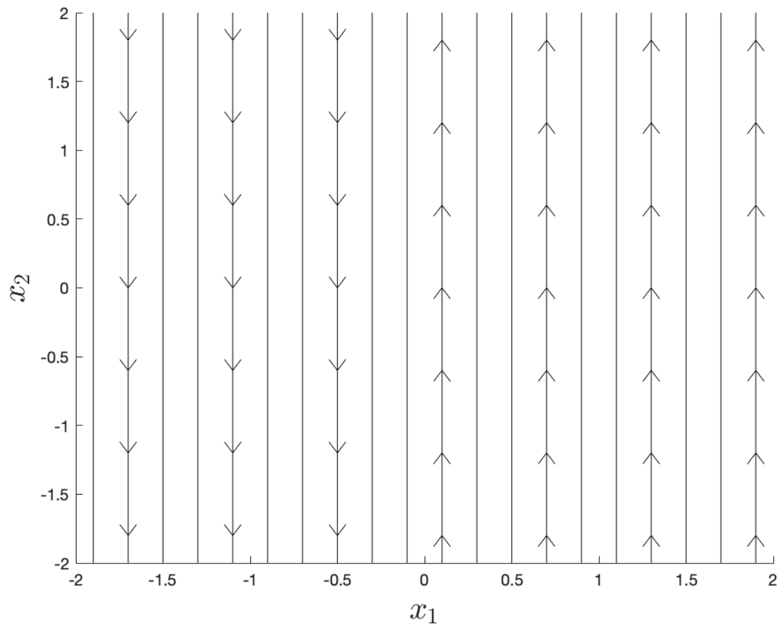
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

the dynamics of the system about which are given by the eigenvalues

$$\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix}$$

$$\lambda^2 = 0$$

indicating a neutrally stable point. The eigenvector for this mode is $\boldsymbol{\xi} = [0, 1]$. note that despite a multiplicity of 2, there is only one eigenvector for this equilibrium due to the resulting matrix being non-diagonalisable. The state space structure about this equilibrium is therefore shown below.



For the equilibrium condition B, where $[x_1, x_2] = [-1, -1]$, the Jacobian becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of the system about this equilibrium are

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3 = 0$$

yielding two real roots $\lambda = \pm\sqrt{3}$. The system therefore has one stable and one unstable trajectory through this equilibrium. To determine the trajectory of the system we must again find the eigenvector for $\lambda = \sqrt{3}$

$$\begin{bmatrix} 2-\sqrt{3} & -1 \\ 1 & -2-\sqrt{3} \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} 0$$

therefore $\boldsymbol{\xi}_1 = [1, 0.268]$. Similarly $\boldsymbol{\xi}_2 = [1, 3.73]$.

We can use our eigenvectors to determine the gradient of the trajectories passing through the equilibrium. Since the general solution about this equilibrium is

$$\mathbf{x}(t) = \mathbf{x}_{0B} + k_1 \begin{bmatrix} 1 \\ 0.268 \end{bmatrix} e^{\sqrt{3}t} + k_2 \begin{bmatrix} 1 \\ 3.73 \end{bmatrix} e^{-\sqrt{3}t}$$

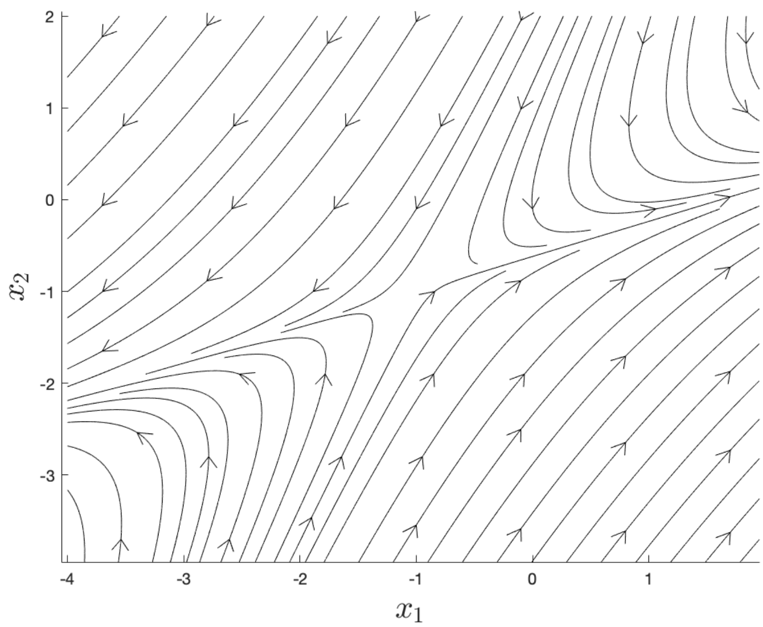
we can find that for the attractor trajectory given by $\lambda = -\sqrt{3}$

$$\frac{\partial x_2}{\partial x_1} = \frac{\partial x_2/\partial t}{\partial x_1/\partial t} = 3.73$$

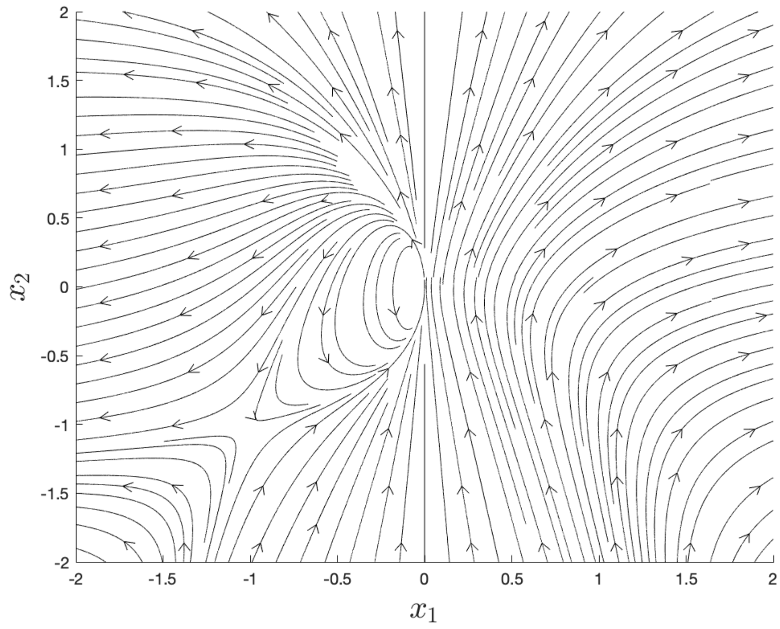
while for the diverging mode given by $\lambda = \sqrt{3}$

$$\frac{\partial x_2}{\partial x_1} = \frac{\partial x_2/\partial t}{\partial x_1/\partial t} = 0.268.$$

The resulting state space structure is seen below:



The entire state space structure can be roughly found by overlaying the structures about each equilibrium point. For illustrative purposes, a plot of the state-space trajectories of the full non-linear system is shown below



4. (a) For the system

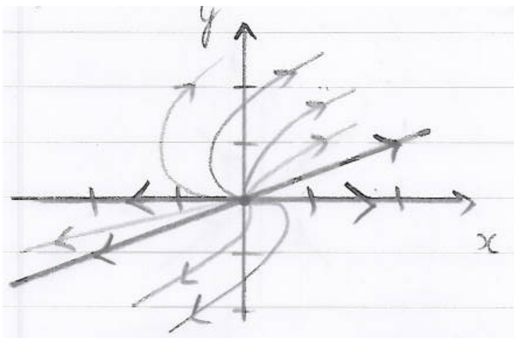
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 2$$

with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

About the equilibrium point $[0, 0]$ the structure of the state space will be



with a general solution

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + k_2 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} e^{2t}.$$

(b) For

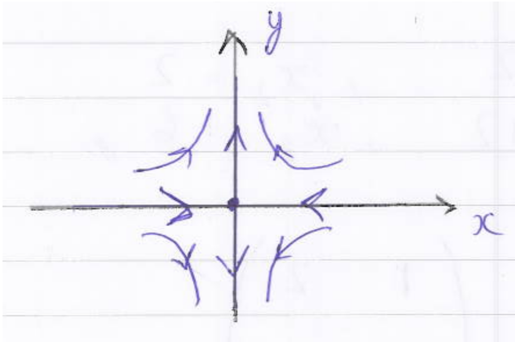
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = -1$$

with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

About the equilibrium point $[0, 0]$ the structure of the state space will be



with a general solution

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t} + k_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}.$$

(c) For

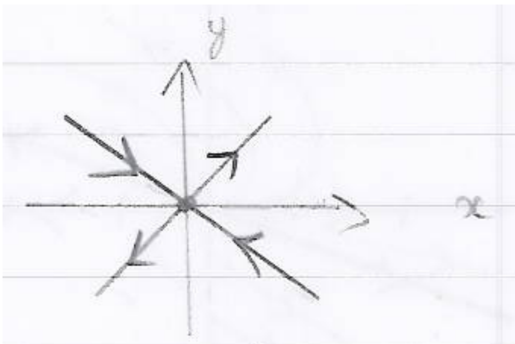
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = 1$$

with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

About the equilibrium point $[0, 0]$ the structure of the state space will be



with a general solution

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

(d) For

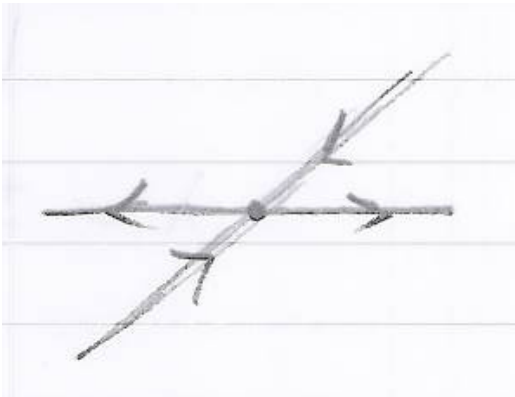
$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = -1$$

with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

About the equilibrium point $[0, 0]$ the structure of the state space will be



with a general solution

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

(e) For

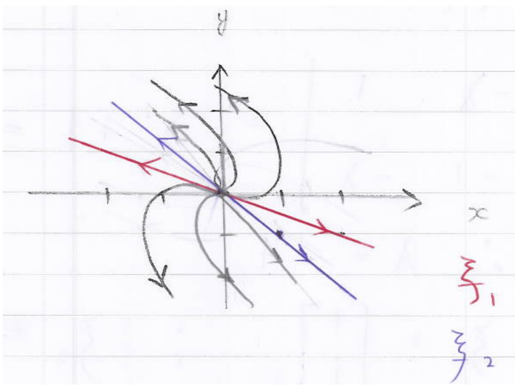
$$\mathbf{A} = \begin{bmatrix} -3 & -8 \\ 4 & 9 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 5$$

with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

About the equilibrium point $[0, 0]$ the structure of the state space will be



with a general solution

$$\mathbf{x}(t) = k_1 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} e^{t} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t}.$$

Note the fact that the larger eigenvalue is dominant and therefore "pulls" the system's trajectories towards it.

5. Consider the system

$$\dot{x} = y$$
$$\dot{y} = \cos x$$

To draw its state-space structure we must first find its equilibria.

The system is at equilibrium when $[\dot{x}, \dot{y}] = [0, 0]$, and therefore when

$$y = 0$$
$$\cos x = 0$$

giving two periodic equilibria

$$\mathbf{x}_{01} = [(2n + 0.5)\pi, 0]$$
$$\mathbf{x}_{02} = [(2n + 1.5)\pi, 0].$$

Next we can linearise the system about these equilibria by expressing it in the form

$$\dot{\mathbf{x}} = \mathbf{J}_{\mathbf{x}}\mathbf{x}$$

where

$$\mathbf{J}_{\mathbf{x}} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sin x & 0 \end{bmatrix}.$$

When evaluated at $\mathbf{x}_{01}$

$$\sin(2n\pi + \pi/2) = 1,$$

giving a Jacobian about that equilibrium

$$\mathbf{J}_{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which results in eigenvalues $\lambda_{1,2} = \pm i$, indicating a neutrally stable oscillation about the equilibrium point, and

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Similarly, when evaluated at $\mathbf{x}_{02}$

$$\sin(2n\pi + 3\pi/2) = -1,$$

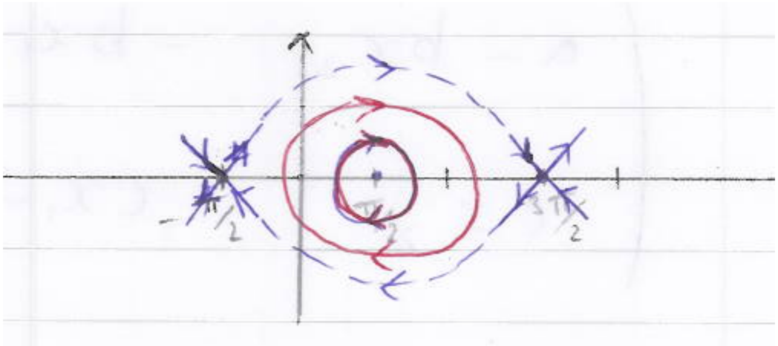
giving a Jacobian about that equilibrium

$$\mathbf{J}_{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which results in eigenvalues $\lambda_{1,2} = \pm 1$, indicating one stable and one unstable non-oscillatory trajectory through the equilibrium point, whose direction is given by

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The structure of the state-space can therefore be plotted as

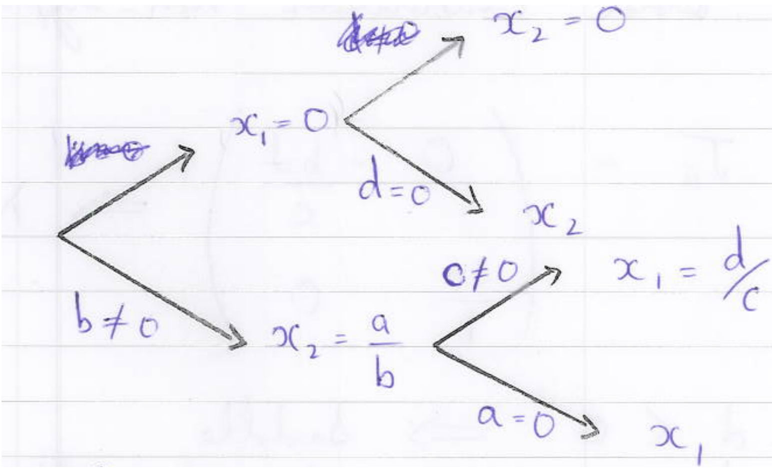


Note the cylindrical topology of the state-space in the x direction, due to its periodic nature.

6. Let us consider the system

$$\dot{x}_1 = (a - bx_2)x_1$$
$$\dot{x}_2 = (cx_1 - d)x_2.$$

The system is at equilibrium when $\dot{x}_1 = \dot{x}_2 = 0$. The equilibrium $\mathbf{x}_{01} = [x_1, x_2]_e = [0, 0]$ is evident by inspection. Further equilibria are also possible depending on the value of the constants a, b, c, d and are summarised in the tree below



The complete list of equilibria is therefore

(i) $\mathbf{x}_{01} = [0, 0]$

(ii) $\mathbf{x}_{02} = [d/c, a/b]$, if $b \neq 0$ and $c \neq 0$.

(iii) $\mathbf{x}_{03} = [x_1, 0]$, if $a = 0$, and

(iv) $\mathbf{x}_{04} = [0, x_2]$, if $d = 0$.

Expressing the system in state-space form, the Jacobian of the system will be

$$\mathbf{J}_\mathbf{x} = \begin{bmatrix} a - bx_2 & -bx_1 \\ cx_2 & cx_1 - d \end{bmatrix}$$

Let us consider each of the above equilibria

(i) If the equilibrium is at $x_1 = x_2 = 0$, the Jacobian becomes

$$\mathbf{J_x} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$$

which results in a characteristic equation

$$(\lambda - a)(\lambda + d) = 0$$

and therefore eigenvalues $\lambda_1 = a$ and $\lambda_2 = -d$. The equilibrium is therefore non-oscillatory and

- stable if $a < 0$ and $d > 0$,
- unstable if $a > 0$ and $d < 0$,
- a saddle point if $ad > 0$.

(ii) If the equilibrium is at $x_1 = d/c$ and $x_2 = a/b$, with $c \neq 0$, $b \neq 0$, the Jacobian becomes

$$\mathbf{J_x} = \begin{bmatrix} 0 & -bd/c \\ ca/b & 0 \end{bmatrix}$$

which results in a characteristic equation

$$\lambda^2 + ad = 0$$

and therefore eigenvalues $\lambda_{1,2} = \pm\sqrt{-ad}$. The equilibrium is therefore a saddle point if $ad < 0$, or neutrally stable and non-hyperbolic if $ad \geq 0$.

(iii) If $a = 0$ and the equilibrium is at $x_2 = 0$ (x_1 can take any value), the Jacobian becomes

$$\mathbf{J_x} = \begin{bmatrix} 0 & -bx_1 \\ 0 & cx_1 - d \end{bmatrix}$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = cx_1 - d$. The equilibrium is therefore non-hyperbolic and its stability is governed by λ_2.

(iv) If $d = 0$ and the equilibrium is at $x_1 = 0$ (x_2 can take any value), the Jacobian becomes

$$\mathbf{J_x} = \begin{bmatrix} a - bx_2 & 0 \\ cx_2 & 0 \end{bmatrix}$$

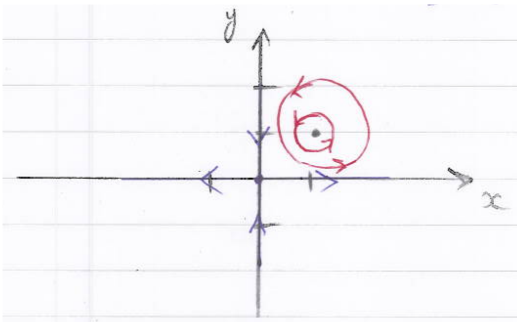
with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a - bx_2 - d$. The equilibrium is therefore non-hyperbolic and its stability is governed by λ_2.

Let us now consider the case where $a = b = c = d = 1$. Based on our prior analysis the system will have two possible equilibria at $[0, 0]$ and $[1, 1]$. The Jacobian for the above constants becomes

$$\mathbf{J_x} = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix},$$

yielding eigenvalues $\lambda_{1,2} = \pm 1$, with $\boldsymbol{\xi}_1 = [0, 1]$ and $\boldsymbol{\xi}_1 = [1, 0]$ for $\mathbf{x} = [0, 0]$. For $\mathbf{x} = [1, 1]$ the eigenvalues $\lambda_{1,2} = \pm i$, with $\boldsymbol{\xi}_1 = [1, i]$ and $\boldsymbol{\xi}_1 = [1, -i]$.

The state-space structure of the system can be approximately plotted as



and the computer generated plot of the state-space structure is