COS1501 Notes

December 6, 2022

Contents

Unit 1

Number Systems

1.1 Number Properties

1.1.1 Commutativity

For all integers m and n, *addition* and *multiplication* are **commutative**.

$$\begin{aligned} m + n &= n + m && \text{addition} \\ mn &= nm && \text{multiplication} \end{aligned}$$

1.1.2 Associativity

For all integers m, n and k, *addition* and *multiplication* are **associative**.

$$\begin{aligned} m + (n + k) &= (m + n) + k && \text{addition} \\ (m)(nk) &= (mn)k && \text{multiplication} \end{aligned}$$

1.1.3 Distributivity

For all integers m, n and k, *multiplication* is **distributive** over *addition*.

$$\begin{aligned} m(n + k) &= mn + mk \\ (n + k)m &= m(n + k) \\ &= mn + mk \\ &= nm + km \end{aligned}$$

1.1.4 Multiplicative Identity

There exists an integer (1) that has the property that for every integer m, $m \cdot 1 = m$.

1.1.5 Additive Identity

There exists an integer (0) that has the property that for every integer m, $m + 0 = m$.

1.1.6 Linearity

For all integers m and n, exactly one of the following is true:

$$m < n$$
$$m = n$$
$$m > n$$

1.1.7 Monotonicity

For all integers m, n and k,

If $m = n$, then $m + k = n + k$ and $mk = nk$.
If $m < n$, then $m + k < n + k$.

If $k > 0$, then $mk < nk$.
If $k < 0$, then $mk < nk$.

1.1.8 Transitivity of =, < and >

For all integers m, n and k,

If $m = n$ and $n = k$, then $m = k$.
If $m < n$ and $n < k$, then $m < k$.
If $m > n$ and $n > k$, then $m > k$.

1.1.9 Absence of Zero Divisors

For all integers m and n,

$mn = 0$ if and only if $m = 0$ or $n = 0$.

1.1.10 Additive Inverses

For every integer m there exists an integer n such that

$$m + n = 0$$

Self Assessment Exercise (Activity 1.11)

1. **Factorise the following expressions:**

   (a) $\mathbf{x^2 + 6x + 9} = (x+3)^2$

   (b) $\mathbf{x^2 - x - 2} = (x-2)(x+1)$

   (c) $\mathbf{x^2 - 5x + 6} = (x-3)(x-2)$

   (d) $\mathbf{x^2 + 4x - 12} = (x+6)(x-2)$

2. **Solve $x^2 - 4x + 4 = 0$ by factorising:**

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ \Rightarrow \quad (x-2)(x-2) &= 0 \\ \Rightarrow \quad x &= 2 \end{aligned}$$

3. **Complete the square to solve $x^2 - 4x = 12$**

$$\begin{aligned} x^2 - 4x &= 12 \\ \Rightarrow \quad x^2 - 4x + 4 &= 12 + 4 \\ \Rightarrow \quad (x-2)^2 &= 16 \\ \Rightarrow \quad x - 2 &= \pm 4 \end{aligned}$$

$$\begin{aligned} x - 2 &= 4 & & & x - 2 &= -4 \\ \Rightarrow \quad x &= 6 & &\text{or} & x &= -2 \end{aligned}$$

4. **Is 21 a prime number?**
   No, as 3 and 7 are both factors of 21.

5. **What is the value of 5! (5 factorial)?**
   $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Unit 2

Rational and Real Numbers

2.1 Rational Numbers

Rational Numbers

Denoted $\mathbb{Q}$, the set of all numbers in the form $\frac{p}{q}$ where p and q are integers are q is not zero.

2.1.1 Multiplicative Inverses

Multiplicative Inverse

For every non-zero rational number x there exists a rational number called the **multiplicative inverse**, denoted $\frac{1}{x}$ such that $(x)\left(\frac{1}{x}\right) = 1$.
This can also be written:

For every non-zero rational number x there exists a rational number y such that $xy = 1$.

2.2 Real Numbers

Real Numbers

Denoted $\mathbb{R}$, the combination of the rational and irrational numbers.

2.3 Number Systems Hierarchy

$$\mathbb{C} > \mathbb{R} > \mathbb{Q}' > \mathbb{Q} > \mathbb{Z} > \mathbb{Z}^{\geq} > \mathbb{Z}^{+}$$

Self-Assessment Exercise (Activity 2.8)

1. **Define the words "even" and "odd" for positive integers**
   An integer is **even** if it is a multiple of 2. An integer is **odd** if it is not even.

2. **Is it the case that $m + (n \cdot k) = (m + n)(m + k)$ for all positive integers m, n and k?**
   No. In order to show this, use a **counterexample**.

   Counterexample. Let $m = 1$, $n = 2$, $k = 3$. Then

$$
\begin{aligned}
m + (n \cdot k) &= 1 + \big((2)(3)\big) \\
&= 1 + 6 \\
&= 7 \\
(m + n)(m + k) &= (1 + 2)(1 + 3) \\
&= (3)(4) \\
&= 12 \\
7 &\neq 12 \\
\therefore m + (n \cdot k) &\neq (m + n)(m + k)
\end{aligned}
$$

   ■

3. **Are there any even prime numbers besides 2?**
   No.

   Proof. Let m be an even number that is not 2.
   Then $m = 2k$ where k is some real number.
   Therefore, 2 and k are factors of m.
   Therefore, m is not a prime number. ■

4. **If m and n are even, is $m + n$ even?**
   Yes.

   Proof. Let m and n be even numbers.
   Then $m = 2j$, $n = 2k$, where j and k are some real numbers.
   Then

$$
\begin{aligned}
m + n &= 2j + 2k \\
&= 2(j + k)
\end{aligned}
$$

   As the sum of the two numbers is a multiple of 2, $m + n$ is even. ■

5. **If m and n are odd, is $m \cdot n$ odd?**
   Yes.

   Proof. Let m and n be odd numbers.
   Then $m = 2j + 1$, $n = 2k + 1$, where j and k are some real numbers. Then

$$\begin{aligned} m \cdot n &= (2j + 1)(2k + 1) \\ &= 4jk + 2k + 2j + 1 \\ &= 2(2jk + k + j) + 1 \end{aligned}$$

   $\therefore m \cdot n$ is odd. ∎

Unit 3

Sets

3.1 Subset

Subset

If A and B are sets from a universal set U, then A is a **subset** of B if and only if every element of A is also an element of B.
Can be abbreviated $A \subseteq B$

3.1.1 Proper Subset

Proper Subset

If C and D are sets from a universal set U, and every element of C is an element of D, but D has some elements that are not in C, then C is a **proper subset** of D.
Can be abbreviated $C \subset D$.

Confusion Between Element and Subset

Note that $\in$ and $\subset$ are not the same. This becomes significant when dealing with power sets.

3.2 Creating Sets From Other Sets

For examples, the following sets will be used:

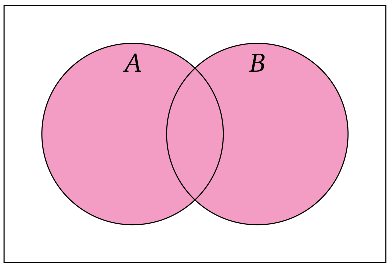
$$U = \{1,2,3,4,5\} \qquad A = \{1,2,3\} \qquad B = \{2,3,4\} \qquad C = \{4,5\}$$

3.2.1 Set Union (OR)

Set Union

The **union** of sets A and B is written $A \cup B$, and is the set of all elements that belong to A or B (or both).

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



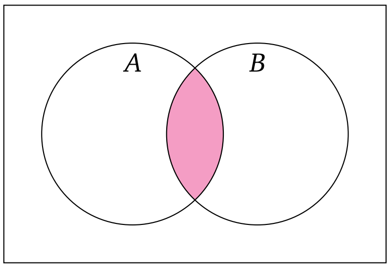
Example $A \cup B = \{1,2,3\} \cup \{2,3,4\}$
$= \{1,2,3,4\}$

3.2.2 Set Intersection (AND)

Set Intersection

The **intersection** of sets A and B is written $A \cap B$, and is the set of all elements that belong to A and B at the same time.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



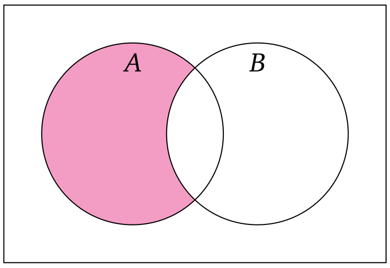
Example $A \cap B = \{1,2,3\} \cap \{2,3,4\}$
$= \{2,3\}$

3.2.3 Set Difference (MINUS)

Set Difference

The **difference** between sets A and B, also called the **complement of B relative to A**, is written $A - B$, and is the set of elements that are in A that are not in B.

$$A - B = \left\{x \mid x \in A \text{ and } x \notin B\right\}$$



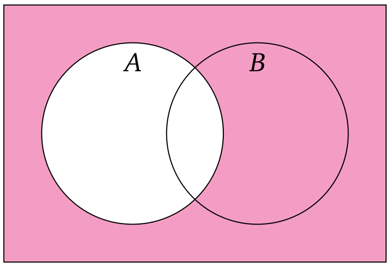
Example $\begin{aligned} A - B &= \{1, 2, 3\} - \{2, 3, 4\} \\ &= \{1\} \end{aligned}$

3.2.4 Set Complement (NOT)

Set Complement

Let A be a subset of a universal set U. Then the **complement** of A, written A' is the set of all elements that belong to U but do not belong to A.

$$A' = \left\{x \mid x \notin A\right\}$$



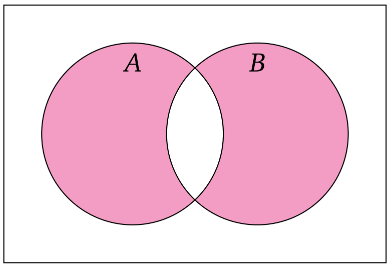
Example $\begin{aligned} A' &= U - A \\ &= \{1, 2, 3, 4, 5\} - \{1, 2, 3\} \\ &= \{4, 5\} \end{aligned}$

3.2.5 Symmetric Set Difference (XOR)

Symmetric Set Difference

The **symmetric difference** between two sets A and B, written $A + B$, is the set of elements that belong to A or to B, but not to both.

$$A + B = \{x \mid x \in A \text{ or } x \in B, \text{ but not both}\}$$



Example $\begin{aligned} A + B &= \{1, 2, 3\} + \{2, 3, 4\} \\ &= \{1, 4\} \end{aligned}$

3.3 Other Terms Significant For Sets

3.3.1 The Empty Set

Empty Set

The set that contains no elements is called the **empty set**, and is written $\emptyset$.

3.3.2 Set Disjointness

Disjointness

Two sets A and B are called **disjoint** if they have no elements in common. In other words,

$$A \cap B = \emptyset$$

Example $\begin{aligned} A \cap C &= \{1, 2, 3\} \cap \{4, 5\} \\ &= \emptyset \end{aligned}$

3.3.3 Set Cardinality

Cardinality

Let A be a set with k distinct elements that can be counted. The *number of elements* k in A is called the **cardinality** of the set. It can be written as $n(A)$ or $|A|$.

Example
$$\begin{aligned}|A| &= \big|\{1,2,3\}\big| \\ &= 3\end{aligned}$$

3.3.4 Power Sets

Power Set

Given a set A with n distinct elements, the **power set** of A, written $\mathcal{P}(A)$, is the set that has as its members *all* subsets of A.

Every Element of a Power Set is a Set

It is important to note that every element of a power set is a *set*!
That means if B is a subset of A, then B is an element of $\mathcal{P}(A)$, i.e. $B \in \mathcal{P}(A)$.
However, B is *not* a subset of $\mathcal{P}(A)$, i.e. $B \not\subseteq \mathcal{P}(A)$! A set containing B, i.e. $\{B\}$ would be a subset of $\mathcal{P}(A)$.

Example
$$\begin{aligned}\mathcal{P}(C) &= \mathcal{P}\big(\{4,5\}\big) \\ &= \big\{\emptyset, \{4\}, \{5\}, \{4,5\}\big\}\end{aligned}$$

The cardinality of a power set $\mathcal{P}(A)$ is 2^n where n is the number of elements in the set A.

Example
$$\begin{aligned}\big|\mathcal{P}(A)\big| &= \Big|\mathcal{P}\big(\{1,2,3\}\big)\Big| \\ &= 2^3 \\ &= 8\end{aligned}$$

Self Assessment Exercise 3.6

1. $$\boldsymbol{U = \{1,2,3,4,5\} \qquad A = \{1,2,3\} \qquad B = \{3,4,5\}}$$

(a) $$\begin{aligned} A \cup B &= \{1,2,3\} \cup \{3,4,5\} \\ &= \{1,2,3,4,5\} \\ B \cup A &= \{3,4,5\} \cup \{1,2,3\} \\ &= \{1,2,3,4,5\} \end{aligned}$$

(b) $$\begin{aligned} A \cap B &= \{1,2,3\} \cap \{3,4,5\} \\ &= \{3\} \\ B \cap A &= \{3,4,5\} \cap \{1,2,3\} \\ &= \{3\} \end{aligned}$$

(c) $$\begin{aligned} A - B &= \{1,2,3\} - \{3,4,5\} \\ &= \{1,2\} \\ B - A &= \{3,4,5\} - \{1,2,3\} \\ &= \{4,5\} \end{aligned}$$

(d) $$\begin{aligned} A + B &= \{1,2,3\} + \{3,4,5\} \\ &= \{1,2,4,5\} \\ B + A &= \{3,4,5\} + \{1,2,3\} \\ &= \{1,2,4,5\} \end{aligned}$$

2. $$\boldsymbol{U = \{a,e,i,o,u\} \qquad A = \{i,o,u\} \qquad B = \{a,e,o,u\}}$$

(a) $$\begin{aligned} A' &= \{i,o,u\}' \\ &= \{a,e,i,o,u\} - \{i,o,u\} \\ &= \{a,e\} \\ (A')' &= \{a,e,i,o,u\} - \{a,e\} \\ &= \{i,o,u\} \\ &= A \end{aligned}$$

(b) $$\begin{aligned} B' &= \{a,e,o,u\}' \\ &= \{a,e,i,o,u\} - \{a,e,o,u\} \\ &= \{i\} \\ (B')' &= \{a,e,i,o,u\} - \{i\} \\ &= \{a,e,o,u\} \\ &= B \end{aligned}$$

(c) $$\begin{aligned} A \cup B &= \{i,o,u\} \cup \{a,e,o,u\} \\ &= \{a,e,i,o,u\} \\ (A \cup B)' &= \{a,e,i,o,u\} - \{a,e,i,o,u\} \\ &= \emptyset \end{aligned}$$

(d) $$\begin{aligned} A' \cap B' &= \{a,e\} \cap \{i\} \\ &= \emptyset \end{aligned}$$

(e) $$\begin{aligned} A \cap B &= \{i,o,u\} \cap \{a,e,o,u\} \\ &= \{o,u\} \\ (A \cap B)' &= \{a,e,i,o,u\} - \{o,u\} \\ &= \{a,e,i\} \end{aligned}$$

(f) $$\begin{aligned} A' \cup B' &= \{a,e\} \cup \{i\} \\ &= \{a,e,i\} \end{aligned}$$

(g) $$\begin{aligned} A - B &= \{i,o,u\} - \{a,e,o,u\} \\ &= \{i\} \\ B - A &= \{a,e,o,u\} - \{i,o,u\} \\ &= \{a,e\} \end{aligned}$$

(h) $$\begin{aligned} A \cap B' &= \{i,o,u\} \cap \{i\} \\ &= \{i\} \\ B \cap A' &= \{a,e,o,u\} \cap \{a,e\} \\ &= \{a,e\} \end{aligned}$$

(i) $$\begin{aligned} A + B &= \{i,o,u\} + \{a,e,o,u\} \\ &= \{a,e,i\} \\ B + A &= \{a,e,o,u\} + \{i,o,u\} \\ &= \{a,e,i\} \end{aligned}$$

3. $$\boldsymbol{U = \{1,2,3,4,5\} \qquad A = \{3\} \qquad B = \{\{3\},4,5\}}$$

$$\begin{aligned} \mathcal{P}(A) &= \mathcal{P}(\{3\}) \\ &= \{\emptyset, \{3\}\} \\ \mathcal{P}(B) &= \mathcal{P}\Big(\{\{3\},4,5\}\Big) \\ &= \Big\{\emptyset, \{\{3\}\}, \{4\}, \{5\}, \{\{3\},4\}, \{\{3\},5\}, \{4,5\}, \{\{3\},4,5\}\Big\} \end{aligned}$$

4. $\boldsymbol{U = \{a, e, i, o, u\}} \qquad \boldsymbol{A = \{i, o, u\}} \qquad \boldsymbol{B = \{a, e, o, u\}}$

(a) $\mathcal{P}(A) = \big\{\emptyset, \{i\}, \{o\}, \{u\}, \{i, o\}, \{i, u\}, \{o, u\}, \{i, o, u\}\big\}$

$$\begin{aligned}\mathcal{P}(B) = \big\{&\emptyset, \{a\}, \{e\}, \{o\}, \{u\}, \{a, e\}, \{a, o\}, \{a, u\}, \{e, o\}, \{e, u\}, \{o, u\}, \{a, e, o\},\\ &\{a, e, u\}, \{a, o, u\}, \{e, o, u\}, \{a, e, o, u\}\big\}\end{aligned}$$

(b)
$$\begin{aligned}\mathcal{P}(A \cap B) &= \mathcal{P}\big(\{o, u\}\big)\\ &= \big\{\emptyset, \{o\}, \{u\}, \{o, u\}\big\}\end{aligned}$$

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \big\{\emptyset, \{o\}, \{u\}, \{o, u\}\big\}$$

(c)
$$\begin{aligned}\mathcal{P}\,(A') &= \mathcal{P}\big(\{a, e\}\big)\\ &= \big\{\emptyset, \{a\}, \{e\}, \{a, e\}\big\}\end{aligned}$$

$$\begin{aligned}\mathcal{P}(U) = \big\{&\emptyset, \{a\}, \{e\}, \{i\}, \{o\}, \{u\}, \{a, e\}, \{a, i\}, \{a, o\}, \{a, u\},\\ &\{e, i\}, \{e, o\}, \{e, u\}, \{i, o\}, \{i, u\}, \{o, u\},\\ &\{a, e, i\}, \{a, e, o\}, \{a, e, u\}, \{a, i, o\}, \{a, i, u\}, \{a, o, u\},\\ &\{e, i, o\}, \{e, i, u\}\{e, o, u\}, \{i, o, u\},\\ &\{a, e, i, o\}, \{a, e, i, u\}, \{a, e, o, u\}\{a, i, o, u\}, \{e, i, o, u\}, \{a, e, i, o, u\}\big\}\end{aligned}$$

$$\begin{aligned}\big(\mathcal{P}(A)\big)' = \big\{&\{a\}, \{e\}, \{a, e\}, \{a, i\}, \{a, o\}, \{a, u\}, \{e, i\}, \{e, o\}, \{e, u\},\\ &\{a, e, i\}, \{a, e, o\}, \{a, e, u\}, \{a, i, o\}, \{a, i, u\}, \{a, o, u\},\\ &\{e, i, o\}, \{e, i, u\}, \{e, o, u\},\\ &\{a, e, i, o\}, \{a, e, i, u\}, \{a, e, o, u\}\{a, i, o, u\}, \{e, i, o, u\}, \{a, e, i, o, u\}\big\}\end{aligned}$$

(d)
$$\begin{aligned}\mathcal{P}(A) \cup \mathcal{P}(B) = \big\{&\emptyset, \{a\}, \{e\}, \{i\}, \{o\}, \{u\},\\ &\{a, e\}, \{a, o\}, \{a, u\}, \{e, o\}, \{e, u\}, \{i, o\}, \{i, u\}, \{o, u\},\\ &\{a, e, o\}, \{a, e, u\}, \{a, o, u\}, \{e, o, u\}, \{i, o, u\}, \{a, e, o, u\}\big\}\end{aligned}$$

$$\begin{aligned}\mathcal{P}(A \cup B) &= \mathcal{P}\big(\{a, e, i, o, u\}\big)\\ &= \mathcal{P}(U)\\ &= \big\{\emptyset, \{a\}, \{e\}, \{i\}, \{o\}, \{u\}, \{a, e\}, \{a, i\}, \{a, o\}, \{a, u\},\\ &\qquad \{e, i\}, \{e, o\}, \{e, u\}, \{i, o\}, \{i, u\}, \{o, u\},\\ &\qquad \{a, e, i\}, \{a, e, o\}, \{a, e, u\}, \{a, i, o\}, \{a, i, u\}, \{a, o, u\},\\ &\qquad \{e, i, o\}, \{e, i, u\}\{e, o, u\}, \{i, o, u\},\\ &\qquad \{a, e, i, o\}, \{a, e, i, u\}, \{a, e, o, u\}\{a, i, o, u\}, \{e, i, o, u\}, \{a, e, i, o, u\}\big\}\end{aligned}$$

Unit 4

Proofs Involving Sets

4.1 Venn Diagrams

4.1.1 Set Equality

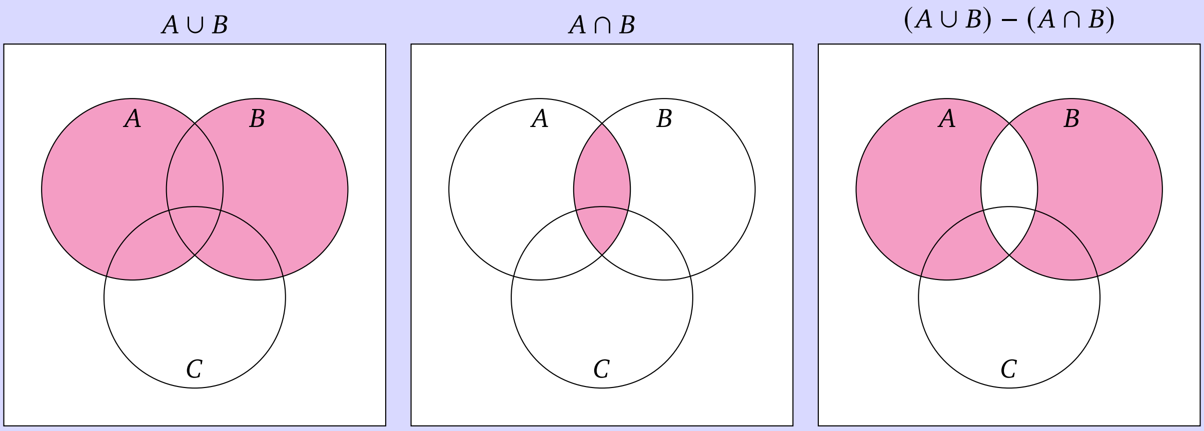
Set Equality

For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then every element of A is also an element of B, and every element of B is also an element of A, so $A = B$.

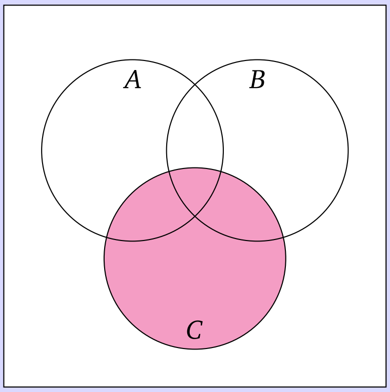
4.1.2 Drawing Complex Venn Diagrams

Draw the diagram in stages, including U, A, B and C in each diagram.

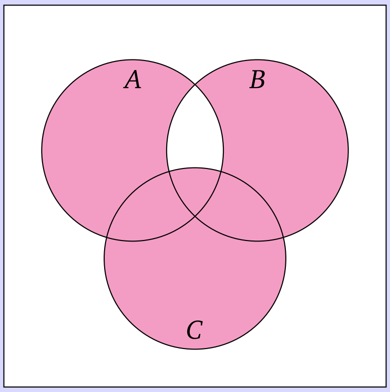
Example Let $A, B, C \subseteq U$. Draw the Venn diagram for $\left[(A \cup B) - (A \cap B)\right] \cup C$



C



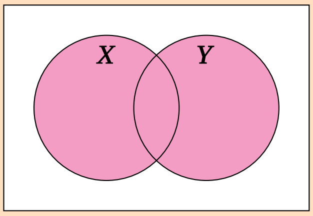
$\left[(A \cup B) - (A \cap B)\right] \cup C$



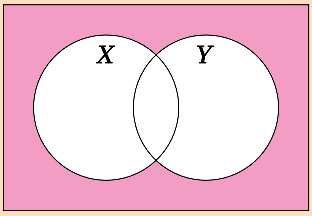
Self-Assessment Exercise 4.4

1. **(a)** $(X \cup Y)'$

$X \cup Y$

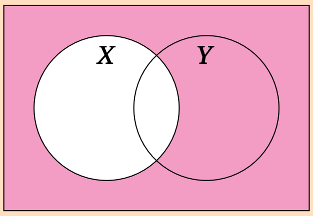


$(X \cup Y)'$

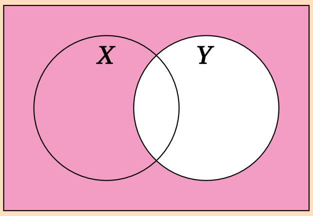


(b) $X' \cap Y'$

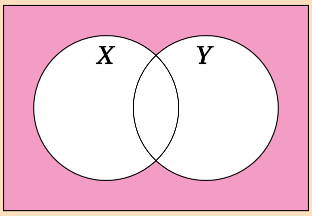
X'



Y'

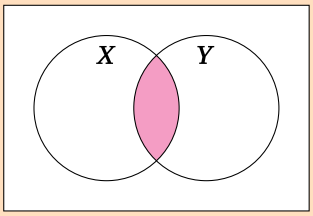


$X' \cap Y'$

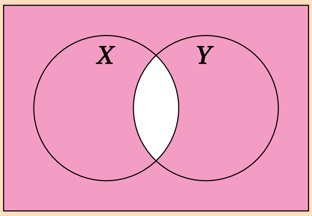


(c) $(X \cap Y)'$

$X \cap Y$

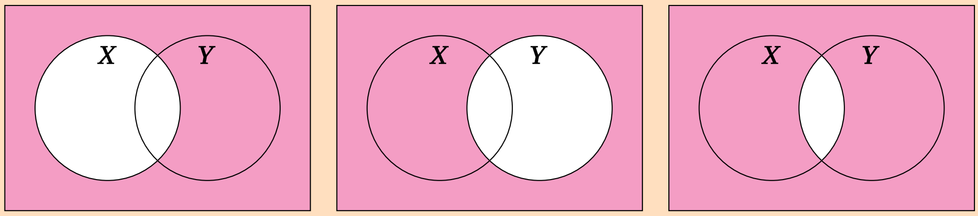


$(X \cap Y)'$



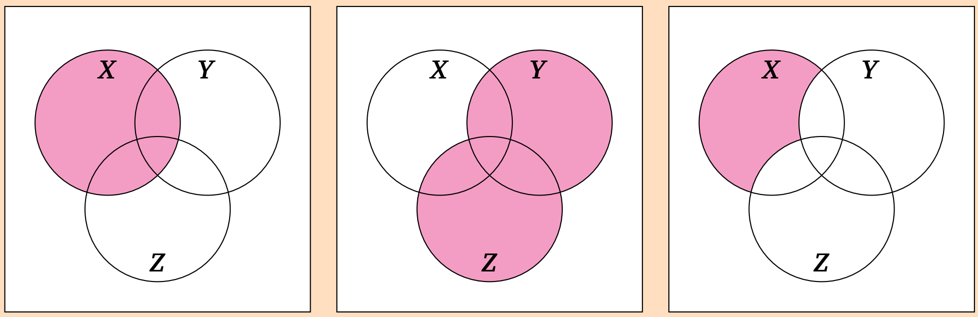
(d) $X' \cap Y'$

X' | Y' | $X' \cup Y'$



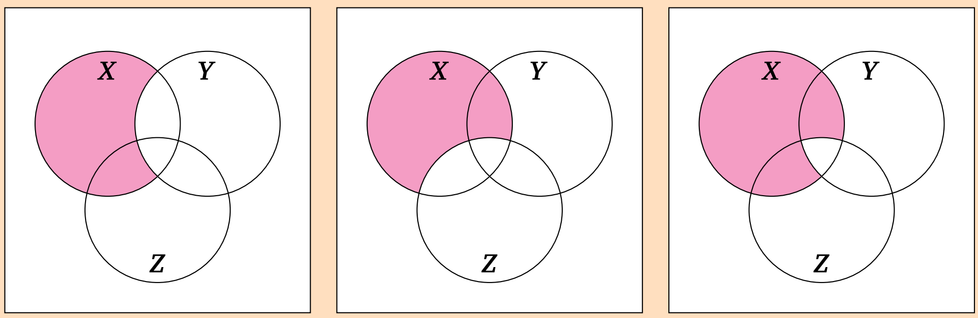
2. **(a)** $X - (Y \cup Z)$

X | $Y \cup Z$ | $X - (Y \cup Z)$



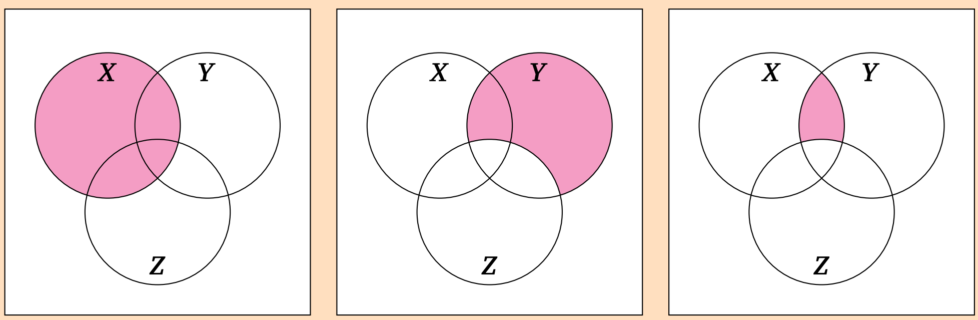
(b) $(X - Y) \cup (X - Z)$

$X - Y$ | $X - Z$ | $(X - Y) \cup (X - Z)$



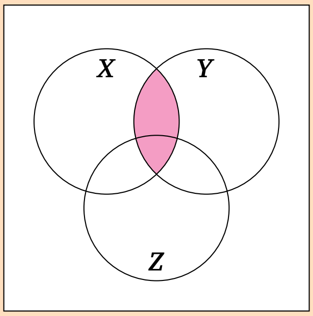
(c) $X \cap (Y - Z)$

X | $Y - Z$ | $X \cap (Y - Z)$

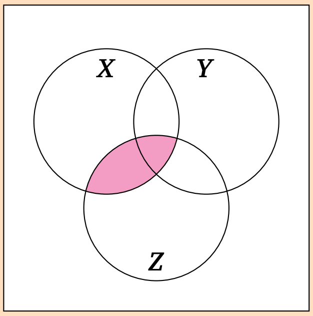


(d) $(X \cap Y) - (X \cap Z)$

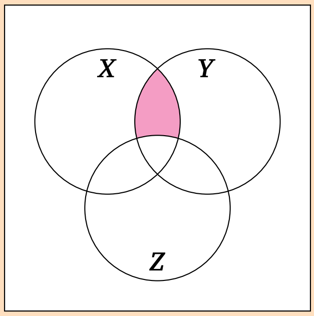
$X \cap Y$



$X \cap Z$

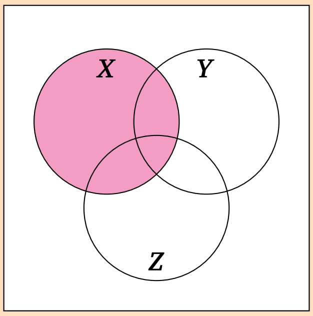


$(X \cap Y) - (X \cap Z)$

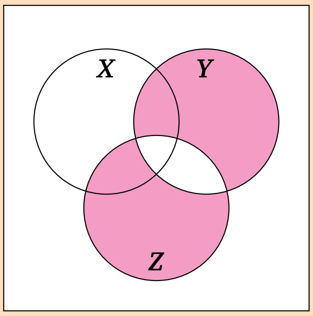


(e) $X \cap (Y + Z)$

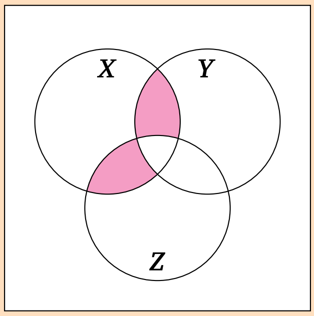
X



$Y + Z$

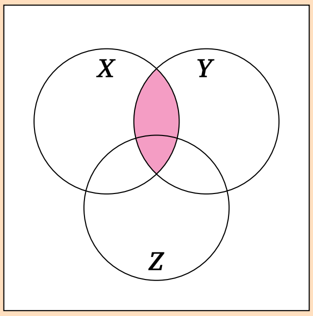


$X \cap (Y + Z)$

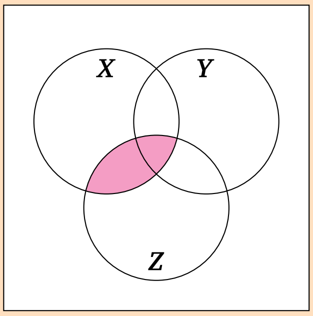


(f) $(X \cap Y) + (X \cap Z)$

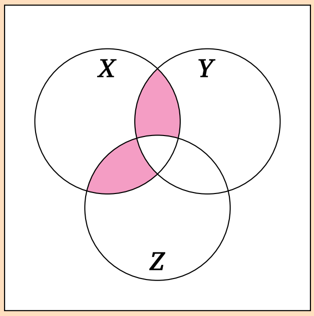
$X \cap Y$



$X \cap Z$



$(X \cap Y) + (X \cap Z)$



4.2 Proofs

For proofs with sets, one needs to prove that the sets have exactly the same elements. For this, one needs to show that each half of the equation is equal to the other half: one needs to show both forwards and backwards. However, this can be abbreviated using iff.

Example (Long Way:) Prove that for all subsets A and B of U, $A \cup B = B \cup A$

Proof. Show (i) $(A \cup B) \subseteq (B \cup A)$ and (ii) $(B \cup A) \subseteq (A \cup B)$

(i) Show $(A \cup B) \subseteq (B \cup A)$

Subproof.

Let	$x \in (A \cup B)$
If	$x \in (A \cup B)$
then	$x \in A$ or $x \in B$
i.e.	$x \in B$ or $x \in A$
i.e.	$x \in (B \cup A)$

$\therefore$ if $x \in (A \cup B)$, then $x \in (B \cup A)$,
$\therefore (A \cup B) \subseteq (B \cup A)$. □

(ii) Show $(B \cup A) \subseteq (A \cup B)$

Subproof.

Let	$x \in (B \cup A)$
If	$x \in (B \cup A)$
then	$x \in B$ or $x \in A$
i.e.	$x \in A$ or $x \in B$
i.e.	$x \in (A \cup B)$

$\therefore$ if $x \in (B \cup A)$, then $x \in (A \cup B)$,
$\therefore (B \cup A) \subseteq (A \cup B)$. □

$\therefore A \cup B = B \cup A$ ■

Using iff can shorten this, but be careful!

Example

Proof.
$x \in (X \cup Y)'$
iff $x \notin (X \cup Y)$
iff $x \notin X$ and $x \notin Y$
iff $x \in X'$ and $x \in Y'$
iff $x \in X' \cap Y'$ ■

Proof.
$x \in (X \cap Y)'$
iff $x \notin (X \cap Y)$
iff $x \notin X$ or $x \notin Y$
iff $x \in X'$ or $x \in Y'$
iff $x \in X' \cup Y'$ ■

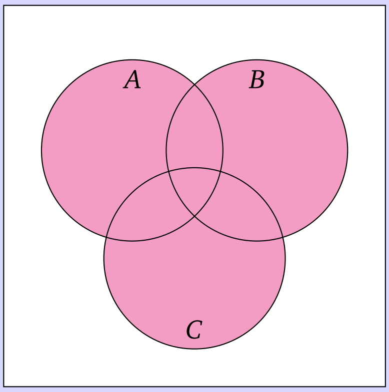
4.2.1 If and Only If Proofs

The purpose of an iff proof is to shorten a proof where you need to show that it works both forwards and backwards. Remember the symbol for iff is $\leftrightarrow$.

To do this, you convert the statement into words.

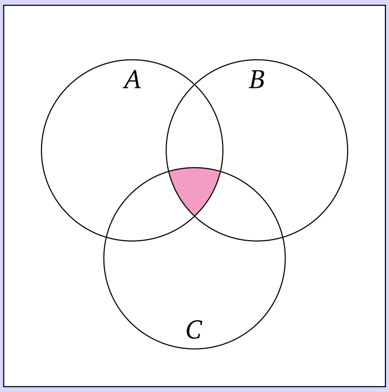
Example

1. **Prove that $A \cup (B \cup C) = (A \cup B) \cup C$ for all sets $A, B, C \subseteq U$.**
   To start off, assume that x is an element of the statement on the left:
   Let $x \in A \cup (B \cup C)$.
   Then start the proof. Convert all the $\cup$ and $\cap$ symbols to words.

	$x \in A \cup (B \cup C)$
iff	$x \in A$ or $x \in (B \cup C)$
iff	$x \in A$ or $x \in B$ or $x \in C$
iff	$(x \in A$ or $x \in B)$ or $x \in C$
iff	$(x \in (A \cup B))$ or $x \in C$
iff	$x \in (A \cup B) \cup C$
$\therefore$	$A \cup (B \cup C) = (A \cup B) \cup C$

2. **Prove that $A \cap (B \cap C) = (A \cap B) \cap C$ for all sets $A, B, C \subseteq U$.**

   Let $x \in A \cap (B \cap C)$

	$x \in A \cap (B \cap C)$
iff	$x \in A$ and $x \in (B \cap C)$
iff	$x \in A$ and $x \in B$ and $x \in C$
iff	$(x \in A$ and $x \in B)$ and $x \in C$
iff	$(x \in (A \cap B))$ and $x \in C$
iff	$x \in (A \cap B) \cap C$
$\therefore$	$A \cap (B \cap C) = (A \cap B) \cap C$

Using Nots

A basic application of the not symbol swaps $\in$ to $\notin$. If the statement has a $-$ in, this is the equivalent of and $\notin$. For example,

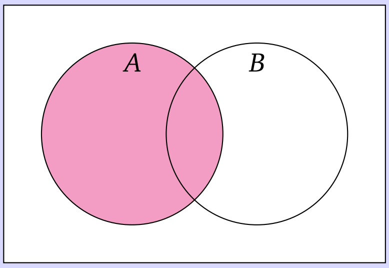
$$x \in A - B = x \in A \text{ and } x \notin B$$

Example

1. **Prove that $(A')' = A$ for all sets $A \subseteq U$.**

   Let $x \in (A')'$

	$x \in (A')'$
iff	$x \notin A'$
iff	x is not $\notin A$
iff	$x \in A$
$\therefore$	$(A')' = A$



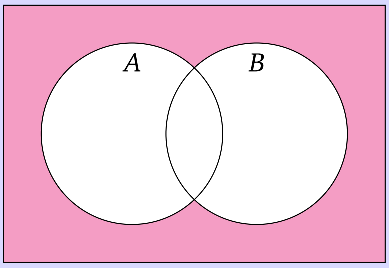
The words swap!

When you apply a $\notin$ sign in words, then $\cup$ means *and* instead of or, and $\cap$ means *or* instead of and.

Example

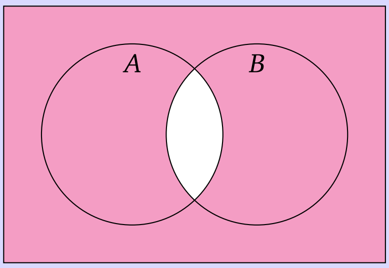
1. **Prove that $(A \cup B)' = A' \cap B'$.**
   Let $x \in (A \cup B)'$.

	$x \in (A \cup B)'$	
iff	$x \notin (A \cup B)$	
iff	$x \notin A$ *and* $x \notin B$	(and instead of or for $\cup$)
iff	$x \in A'$ and $x \in B'$	
iff	$x \in A' \cap B'$	
$\therefore$	$(A \cup B)' = A' \cap B'$	



2. **Prove that $(A \cap B)' = A' \cup B'$.**
   Let $x \in (A \cap B)'$.

	$x \in (A \cap B)'$	
iff	$x \notin (A \cap B)$	
iff	$x \notin A$ *or* $x \notin B$	(or instead of and for $\cap$)
iff	$x \in A'$ or $x \in B'$	
iff	$x \in A' \cup B'$	
$\therefore$	$(A \cap B)' = A' \cup B'$	



Self-Assessment Exercise 4.6

(a) $(X')' = X$

Proof. Let $x \in (X')'$.

$x \in (X')'$
iff $x \notin X'$
iff $x \in X$
$\therefore\ (X')' = X$ ∎

(b) $X - (Y \cap W) = (X - Y) \cup (X - W)$

Proof. Let $x \in X - (Y \cap W)$.

$x \in X - (Y \cap W)$
iff $x \in X$ and $x \notin (Y \cap W)$
iff $x \in X$ and $x \in Y'$ or $x \in W'$
iff $\big(x \in X$ and $x \in Y'\big)$ or $\big(x \in X$ and $x \in W'\big)$
iff $\big(x \in (X - Y)\big)$ or $\big(x \in (X - W)\big)$
iff $x \in (X - Y) \cup (X - W)$
$\therefore\ X - (Y \cap W) = (X - Y) \cup (X - W)$ ∎

(c) $X \cap (Y \cap W) = (X \cap Y) \cap W$

Proof. Let $x \in X \cap (Y \cap W)$.

$x \in X \cap (Y \cap W)$
iff $x \in X$ and $x \in (Y \cap W)$
iff $x \in X$ and $x \in Y$ and $x \in W$
iff $(x \in X$ and $x \in Y)$ and $x \in W$
iff $x \in (X \cap Y)$ and $x \in W$
iff $x \in (X \cap Y) \cap W$
$\therefore\ X \cap (Y \cap W) = (X \cap Y) \cap W$ ∎

(d) $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$

Proof. Let $x \in X \cap (Y \cup W)$

$x \in X \cap (Y \cup W)$
iff $x \in X$ and $x \in (Y \cup W)$
iff $x \in X$ and $(x \in Y$ or $x \in W)$
iff $(x \in X$ and $x \in Y)$ or $(x \in X$ and $x \in W)$
iff $(x \in (X \cap Y))$ or $(x \in (X \cap W))$
iff $x \in (X \cap Y) \cup (X \cap W)$
$\therefore\ X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$ ∎

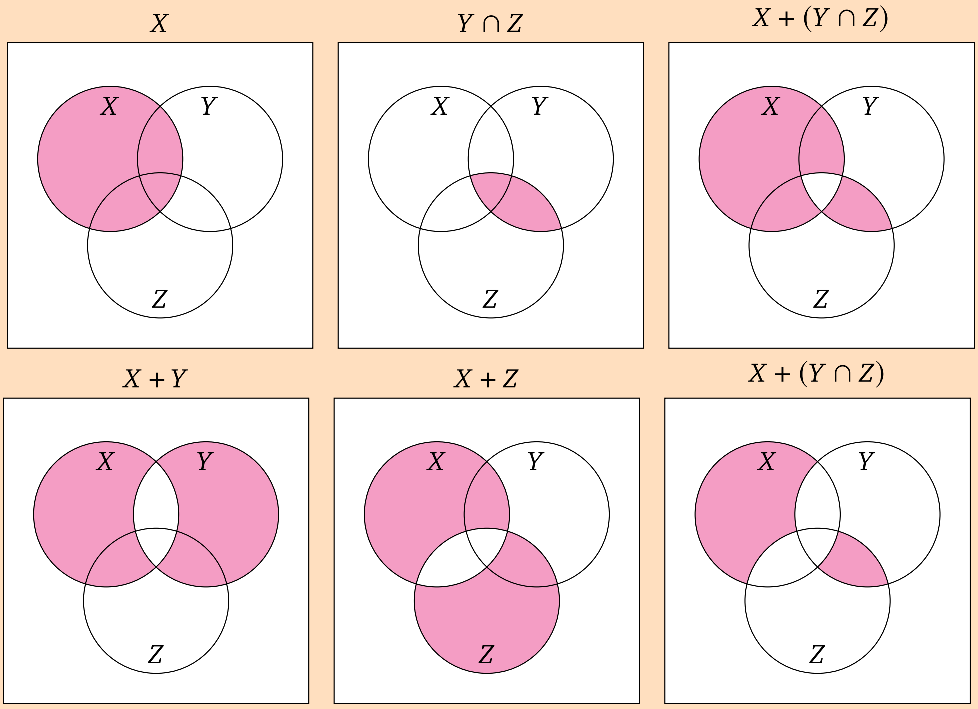
In order to prove that two sets are not equal, one needs to just provide a **counterexample** - an element that is in one set that is not in the other.

Identity

An equation which is satisfied by every possible value of the unknown(s) is called an **identity**

Self Assessment Exercise 4.8

1. **Is it the case for all $X, Y, Z \subseteq U$, $X + (Y \cap Z) = (X + Y) \cap (X + Z)$?**



   As the venn diagrams are not the same, it is not the case.
   Counterexample: Find an element that is in X and in Y, but is not in Z.

2. **Find examples of sets A and B such that $\mathcal{P}(A \cup B)$ is not a subset of $\mathcal{P}(A) \cup \mathcal{P}(B)$.**
   A and B just need to contain different elements. For example, let $A = \{1\}$ and $B = \{2\}$.

$$\begin{aligned}
\mathcal{P}(A \cup B) &= \mathcal{P}\big(\{1\} \cup \{2\}\big) \\
&= \mathcal{P}\big(\{1, 2\}\big) \\
&= \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\} \\
\mathcal{P}(A) \cup \mathcal{P}(B) &= \mathcal{P}\big(\{1\}\big) \cup \mathcal{P}\big(\{2\}\big) \\
&= \big\{\emptyset, \{1\}\big\} \cup \big\{\emptyset, \{2\}\big\} \\
&= \big\{\emptyset, \{1\}, \{2\}\big\}
\end{aligned}$$

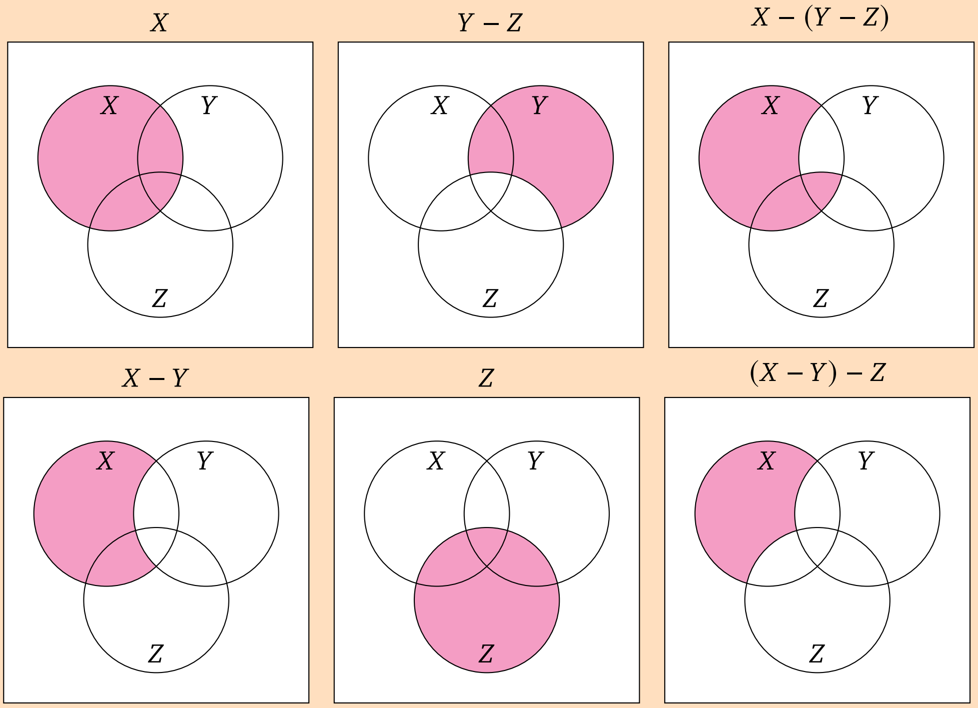
3. Is it the case that, for all $X, Y \subseteq U$, $\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y)$?

Yes.

Proof. Let $S \in P(X) \cap P(Y)$.

	$S \in P(X) \cap P(Y)$
iff	$S \in P(X)$ and $S \in P(Y)$
iff	$S \subseteq X$ and $S \subseteq Y$
iff	(The elements of S are all in X) and (The elements of S are all in Y)
iff	The elements of S are all in $X \cap Y$
iff	$S \subseteq X \cap Y$
iff	$S \in \mathcal{P}(X \cap Y)$ ∎

4. Use Venn diagrams to investigate whether, for all sets $X, Y, Z \subseteq U$ $X - (Y - Z) = (X - Y) - Z$. If it is true, provide a proof. Else, provide a counterexample.



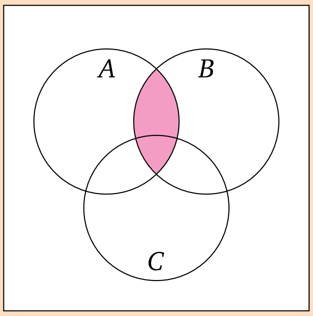
Counterexample. Let $X = \{1, 2\}$, $Y = \{4\}$, $Z = \{1, 3\}$.

$$X - (Y - Z) = \{1, 2\}$$
$$(X - Y) - Z = \{2\}$$
$$\{1, 2\} \neq \{2\}$$
$$X - (Y - Z) \neq (X - Y) - Z$$

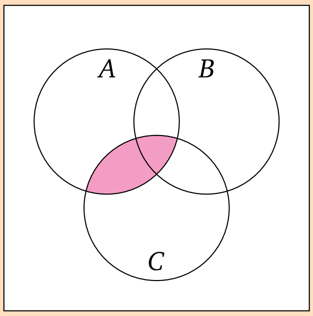
∎

5. **Use Venn diagrams to investigate whether, for all sets $A, B, C \subseteq U$ $(A \cap B) + (C \cap A) = (A \cap B') \cup (B - C)$. If it is true, provide a proof. Else, provide a counterexample.**

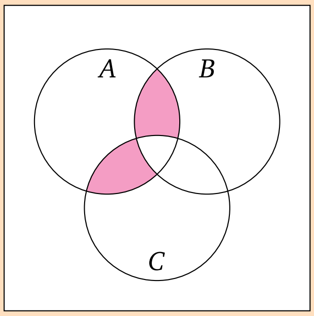
$A \cap B$



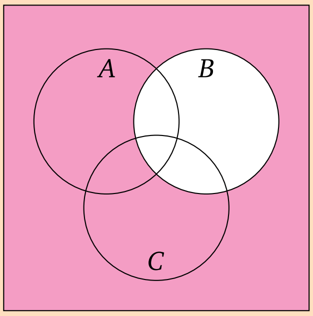
$C \cap A$



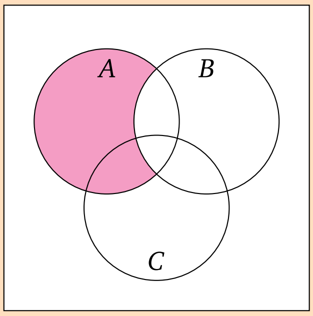
$(A \cap B) + (C \cap A)$



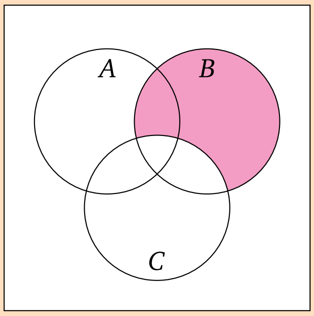
B'



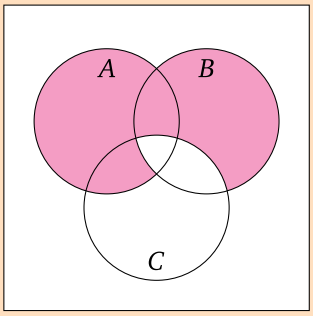
$A \cap B'$



$B - C$



$(A \cap B') \cup (B - C)$



Counterexample. Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{4\}$

$$
\begin{aligned}
(A \cap B) + (C \cap A) &= \{2\} \\
(A \cap B') \cup (B - C) &= \{1, 2, 3\} \\
\{2\} &\neq \{1, 2, 3\} \\
(A \cap B) + (C \cap A) &\neq (A \cap B') \cup (B - C)
\end{aligned}
$$

■

4.3 The Inclusion Exclusion Principle

Inclusion Exclusion Principle

For all finite sets X and Y, $|X \cup Y| = |X| + |Y| - |X \cap Y|$

Example Let $X = \{a, b, c, 1\}$ and $Y = \{1, 2, 3\}$. Then $X \cap Y = \{1\}$ and $|X \cap Y| = 1$.
$|X| = 4$, $|Y| = 3$, so $|X \cup Y| = |X| + |Y| - |X \cap Y| = 4 + 3 - 1 = 6$

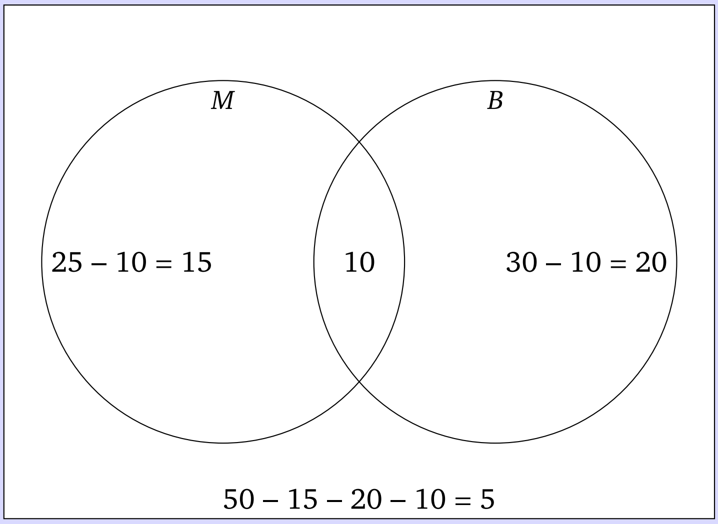
Sum Rule

If X and Y are disjoint sets ($X \cap Y = \emptyset$), and $|X| = m$ and $|Y| = n$, then $|X \cup Y| = m + n$

4.3.1 Applying the principle to Venn Diagrams

Example In a group of 50 learners, 25 play mastermind, 30 play basketball, and 10 play both. U is all the learners, M is those who play Mastermind, and B is those who play basketball.

$$|U| = 50 \qquad |M| = 25 \qquad |B| = 30 \qquad |M \cap B| = 10$$



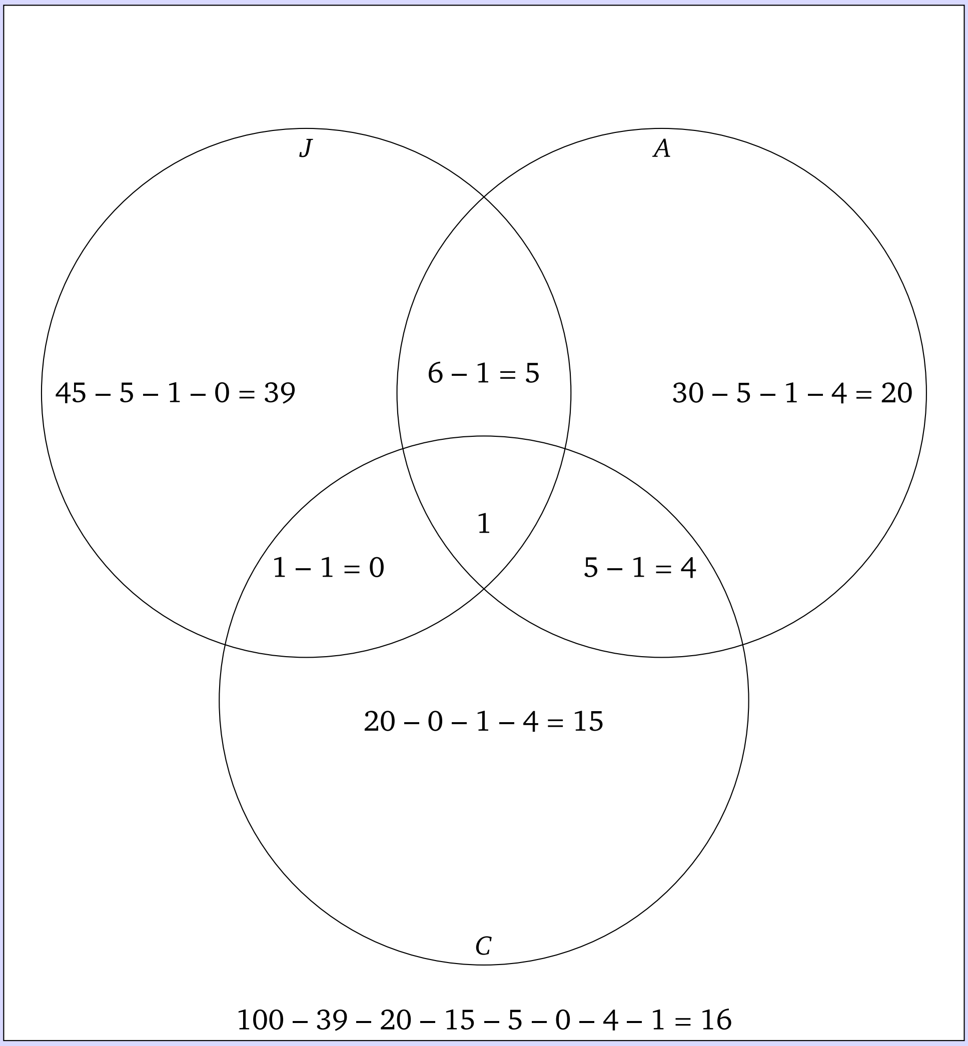
1. **How many learners play Mastermind or basketball, (or both)?**
   $|M \cup B| = 15 + 10 + 20 = 45$.
   Also, by Inclusion Exclusion,

$$\begin{aligned} |M \cup B| &= |M| + |B| - |M \cap B| \\ &= 25 + 30 - 10 \\ &= 45 \end{aligned}$$

2. **How many students do not play either Mastermind or basketball?**
   $|(M \cup B)'| = 50 - 45 = 5$

Example A questionnaire filled in by 100 subscribers to Blue Scalpel Medical Insurance who submitted no claims during 2009 reveals that 45 jog regularly, 30 do aerobics regularly, 20 cycle regularly, 6 jog and do aerobics, 1 jogs and cycles, 5 do aerobics and cycle, and 1 jogs, cycles and does aerobics.
U is the subscribers, J is those who jog, A is those who do aerobics, and C is those who cycle.

$$|U| = 100 \qquad |J| = 45 \qquad |A| = 30 \qquad |C| = 20$$
$$|J \cap A| = 6 \qquad |J \cap C| = 1 \qquad |A \cap C| = 5 \qquad |J \cap A \cap C| = 1$$

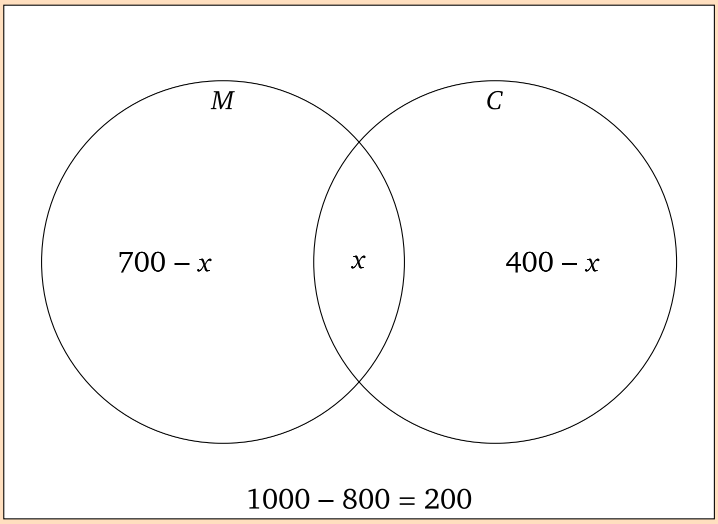


1. **How many of these healthy people do not participate regularly in any of the three activities?**
   This would be the value of the people who don't appear in any of the circles, which is 16.
2. **How many only jog?**
   This would be the value inside the circle J, which is 39.

Self Assessment Exercise 4.10

1. **Of 1000 first year students, 700 take Mathematics, 400 take Computer Science, and 800 take Mathematics or Computer Science.**
   U is the first year students, M is those who take Mathematics, and C is those who take Computer Science.

$$|U| = 1000 \qquad |M| = 700 \qquad |C| = 400 \qquad |M \cup C| = 800 \qquad |M \cap C| = x$$



$$\begin{aligned} 800 &= (700 - x) + x + (400 - x) \\ \Rightarrow \quad 800 &= 700 + 400 - x \\ \Rightarrow \quad -300 &= -x \\ \Rightarrow \quad x &= 300 \end{aligned}$$

   (a) **How many students take Mathematics and Computer Science?**
   This would be x, which is 300.

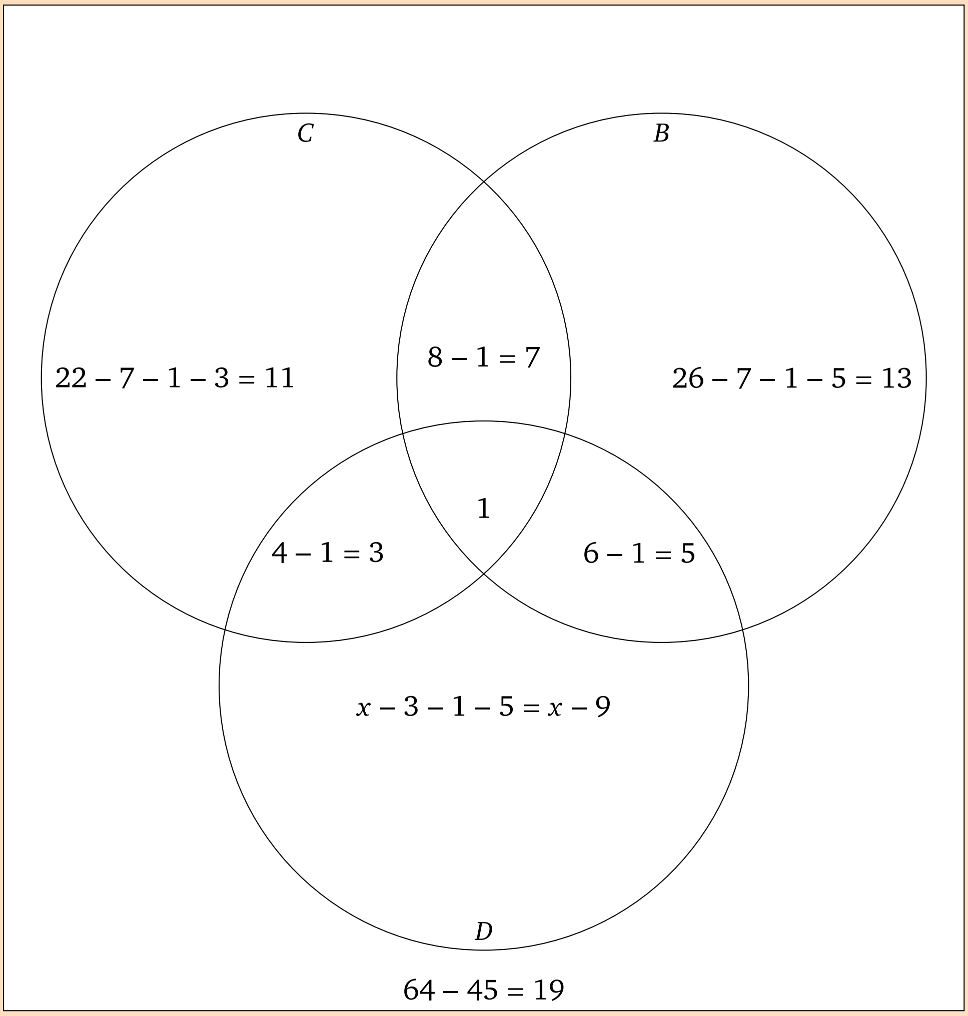
   (b) **How many students take Mathematics, but not Computer Science?**

$$\begin{aligned} 400 - x &= 400 - 300 \\ &= 100 \end{aligned}$$

   (c) **How many students do not take either of the two subjects?**
   The number occurring outside the circles, so 200.

2. **A builder has a team of 64 construction workers. 45 can use at least one of the three equipment types. 22 can operate cranes, 26 can operate backhoes, 4 can operate cranes and bulldozers, 6 can operate backhoes and bulldozers, 8 can operate cranes and backhoes, and 1 can operate all three kinds of machinery. How many can operate bulldozers?**
   U is the workers, C is the workers who can operate cranes, B is the workers who can operate backhoes, and D is the workers who can operate bulldozers.

$$|U| = 64 \qquad |C| = 22 \qquad |B| = 26 \qquad |D| = x$$
$$|C \cap D| = 4 \qquad |B \cap D| = 6 \qquad |C \cap B| = 8 \qquad |C \cap B \cap D| = 1$$
$$|C \cup B \cup D| = 45$$

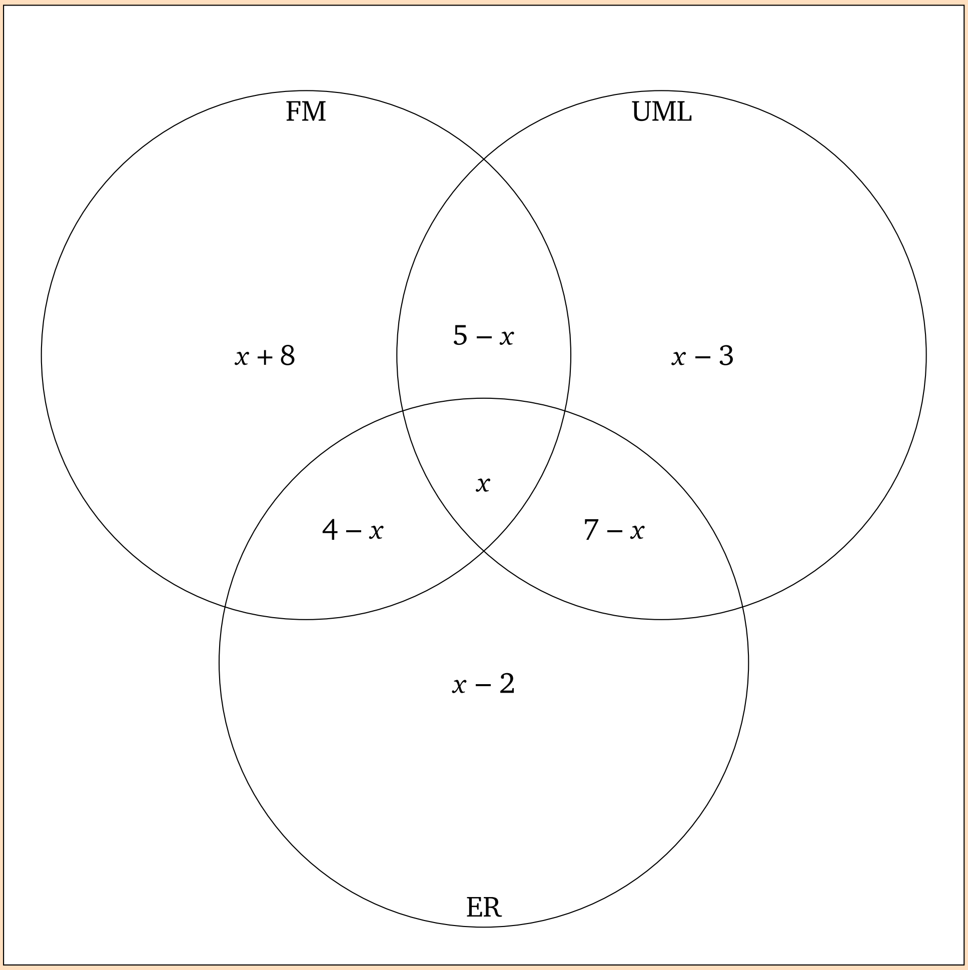


$$\begin{aligned} & 45 = 11 + 13 + (x-9) + 7 + 3 + 5 + 1 \\ \Rightarrow\ & 45 = 40 + x - 9 \\ \Rightarrow\ & x = 45 - 31 \\ \Rightarrow\ & x = 14 \end{aligned}$$

The number of people who can only operate bulldozers is $x - 9 = 14 - 9 = 5$.
The number of people who can operate bulldozers is therefore $5 + 3 + 1 + 5 = 14$.

3. **A software company employs 22 software engineers. All of them can use at least one of the three methods. 17 of them can use a formal method (FM), 9 can use Unified Modelling Language (UML), and 9 can use entity-relationship diagrams (ER). 5 engineers can use both an FM and UML, 4 both an FM and ER diagrams, and 7 both UML and ER diagrams.**

$$|U| = 22 \qquad |\mathrm{FM}| = 17 \qquad |\mathrm{UML}| = 9 \qquad |\mathrm{ER}| = 9$$
$$|\mathrm{FM} \cap \mathrm{UML}| = 5 \qquad |\mathrm{FM} \cap \mathrm{ER}| = 4 \qquad |\mathrm{UML} \cap \mathrm{ER}| = 7 \qquad |\mathrm{FM} \cap \mathrm{UML} \cap \mathrm{ER}| = x$$



For Only FM: $17 - (5 - x) - x - (4 - x) = 17 - 5 + x - x - 4 + x$
$= x + 8$

For Only UML: $9 - (5 - x) - x - (7 - x) = 9 - 5 + x - x - 7 + x$
$= x - 3$

For Only ER: $9 - (4 - x) - x - (7 - x) = 9 - 4 + x - x - 7 + x$
$= x - 2$

$$\begin{aligned}
& 22 = (x+8)+(x-3)+(x-2)+(5-x)+(4-x)+(7-x)+x \\
\Rightarrow\quad & 22 = (x+x+x-x-x-x+x)+(8-3-2+5+4+7) \\
\Rightarrow\quad & 22 = x+19 \\
\Rightarrow\quad & x = 22-19 \\
\Rightarrow\quad & x = 3
\end{aligned}$$

(a) How many engineers can use all three diagrams?
As shown above, 3 engineers.

(b) How many engineers can use UML only?

$$\begin{aligned}
x-3 &= 3-3 \\
&= 0
\end{aligned}$$

4.4 Proofs on Specific Sets

To prove that two sets are equal, prove that each member of the left-hand side belongs to the right-hand side, and vice versa.

Any variable can be used for a set description

Whether the variable is x or z does not change the members of the set.

Example Prove that $\{w \in \mathbb{R} \mid w^2 - 3w + 2 < 0\} = \{z \in \mathbb{R} \mid 1 < z < 2\}$

Proof. Let $x \in \{w \in \mathbb{R} \mid w^2 - 3w + 2 < 0\}$

$$\begin{aligned}
& x \in \{w \in \mathbb{R} \mid w^2 - 3w + 2 < 0\} \\
\text{iff}\quad & x \in \mathbb{R} \text{ and } x^2 - 3x + 2 < 0 \\
\text{iff}\quad & x \in \mathbb{R} \text{ and } (x-2)(x-1) < 0 \\
\text{iff}\quad & x \in \mathbb{R} \text{ and either } (x-2) < 0 \text{ and } (x-1) > 0 \text{ (minus times a plus is a minus) or} \\
& \qquad (x-2) > 0 \text{ and } (x-1) < 0 \text{ (plus times a minus is a minus)} \\
\text{iff}\quad & x \in \mathbb{R} \text{ and either } (x < 2 \text{ and } x > 1) \text{ or } (x > 2 \text{ and } x < 1) \\
& \qquad \text{(there are no real numbers that meet the second option)} \\
\text{iff}\quad & x \in \mathbb{R} \text{ and } (x < 2 \text{ and } x > 1) \\
\text{iff}\quad & x \in \mathbb{R} \text{ and } 1 < x < 2 \\
\text{iff}\quad & x \in \{x \in \mathbb{R} \mid 1 < x < 2\} \\
\text{iff}\quad & x \in \{z \in \mathbb{R} \mid 1 < z < 2\}
\end{aligned}$$

∎

Using Or in Proofs

Note that if there is an "or" that is connecting the statements, then the statement is true if *either* of the statements is true.

Self Assessment Exercise 4.11

1. **Prove the following:**

   (a) $\{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\} = \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$

   Proof. Let $x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\}$.

   $$\begin{array}{ll} & x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\} \\ \text{iff} & x \in \mathbb{Z}^+ \text{ and } x \text{ is an even prime number} \\ \text{iff} & x \in \mathbb{Z}^+ \text{ and } x = 2 \\ \text{iff} & x \in \mathbb{Z}^+ \text{ and } x^2 = 4 \\ \text{iff} & x \in \{x \in \mathbb{Z}^+ \mid x^2 = 4\} \\ \text{iff} & x \in \{u \in \mathbb{Z}^+ \mid u^2 = 4\} \end{array}$$

   ■

   (b) $\mathcal{P}\big(\{0, 1\}\big) = \{\emptyset\} \cup \big\{\{0\}\big\} \cup \big\{\{1\}\big\} \cup \big\{\{0, 1\}\big\}$

   Proof. Let $X \in \mathcal{P}\big(\{0, 1\}\big)$.

   $$\begin{array}{ll} & X \in \mathcal{P}\big(\{0, 1\}\big) \\ \text{iff} & X \in \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\} \\ \text{iff} & X = \emptyset \text{ or } X = \{0\} \text{ or } x = \{1\} \text{ or } X = \{0, 1\} \\ \text{iff} & X \in \{\emptyset\} \text{ or } X \in \big\{\{0\}\big\} \text{ or } X \in \big\{\{1\}\big\} \text{ or } X \in \big\{\{0, 1\}\big\} \\ \text{iff} & X \in \{\emptyset\} \cup \big\{\{0\}\big\} \cup \big\{\{1\}\big\} \cup \big\{\{0, 1\}\big\} \end{array}$$

   ■

   (c) $\{x \in \mathbb{R} \mid x^2 + 6x + 5 < 0\} = \{x \in \mathbb{R} \mid -5 < x < -1\}$

   Proof. Let $y \in \{x \in \mathbb{R} \mid x^2 + 6x + 5 < 0\}$

   $$\begin{array}{ll} & y \in \{x \in \mathbb{R} \mid x^2 + 6x + 5 < 0\} \\ \text{iff} & y \in \mathbb{R} \text{ and } y^2 + 6y + 5 < 0 \\ \text{iff} & y \in \mathbb{R} \text{ and } (y + 5)(y + 1) < 0 \\ \text{iff} & y \in \mathbb{R} \text{ and either } (y + 5) < 0 \text{ and } (y + 1) > 0 \text{ (minus times a plus is a minus) or} \\ & \qquad (y + 5) > 0 \text{ and } (y + 1) < 0 \text{ (plus times a minus is a minus)} \\ \text{iff} & y \in \mathbb{R} \text{ and either } (y < -5 \text{ and } y > -1) \text{ or } (y > -5 \text{ and } y < -1) \\ & \qquad \text{(no real numbers meet the first statement)} \\ \text{iff} & y \in \mathbb{R} \text{ and } (y > -5 \text{ and } y < -1) \\ \text{iff} & y \in \mathbb{R} \text{ and } -5 < y < -1 \\ \text{iff} & y \in \{x \in \mathbb{R} \mid -5 < x < -1\} \end{array}$$

   ■

(d) $\{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\} = \{x \in \mathbb{Z}^+ \mid x \text{ is a prime factor of } 6\}$

Proof. Let $w \in \{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\}$

$w \in \{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\}$
iff $w \in \mathbb{Z}$ and $w^2 - 5w + 4 < 0$
iff $w \in \mathbb{Z}$ and $(w-4)(w-1) < 0$
iff $w \in \mathbb{Z}$ and either $(w-4) < 0$ and $(w-1) > 0$ (minus times a plus is a minus) or
$(w-4) > 0$ and $(w-1) < 0$ (plus times a minus is a minus)
iff $w \in \mathbb{Z}$ and either $(w < 4$ and $w > 1)$ or $(w > 4$ and $w < 1)$
(no integers meet the second statement)
iff $w \in \mathbb{Z}$ and $(w < 4$ and $w > 1)$
iff $w \in \mathbb{Z}^+$ and $(1 < w < 4)$
($\mathbb{Z}^+$ as all the numbers are positive)
iff $w \in \mathbb{Z}^+$ and $w \in \{2, 3\}$
iff $w \in \{x \in \mathbb{Z}^+ \mid x \in \{2, 3\}\}$
iff $w \in \{x \in \mathbb{Z}^+ \mid x \text{ is a prime factor of } 6\}$ ■

(e) $\{x \in \mathbb{R} \mid x^2 + x - 2 > 0\} = \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$

Proof. Let $z \in \{x \in \mathbb{R} \mid x^2 + x - 2 > 0\}$

$z \in \{x \in \mathbb{R} \mid x^2 + x - 2 > 0\}$
iff $z \in \mathbb{R}$ and $z^2 + z - 2 > 0$
iff $z \in \mathbb{R}$ and $(z+2)(z-1) > 0$
iff $z \in \mathbb{R}$ and either $(z+2) < 0$ and $(z-1) < 0$ (minus times a minus is a plus) or
$(z+2) > 0$ and $(z-1) > 0$ (plus times a plus is a plus)
iff $z \in \mathbb{R}$ and either $(z < -2$ and $z < 1)$ or $(z > -2$ and $z > 1)$
iff $z \in \mathbb{R}$ and either $(z < -2)$ or $(z > 1)$
iff $z \in \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$ ■

Self Assessment Exercise 4.12

1. **Determine whether, for $V, W, Z \subseteq U$, if $V \subseteq W$, then $V \cup Z \subseteq W \cup Z$ and $V \cap Z \subseteq W \cap Z$. Provide either a proof or a counterexample.**
   Both statements are true.

   Proof. Suppose $V \subseteq W$

   Let $x \in V \cup Z$
   Then $x \in V$ or $x \in Z$
   (If $x \in V$, $x \in W$, as $V \subseteq W$)
   i.e. $x \in W$ or $x \in Z$
   i.e. $x \in W \cup Z$
   $\therefore$ $V \cup Z \subseteq W \cup Z$ ∎

   Proof. Suppose $V \subseteq W$

   Let $x \in V \cap Z$
   Then $x \in V$ and $x \in Z$
   (If $x \in V$, $x \in W$, as $V \subseteq W$)
   i.e. $x \in W$ and $x \in Z$
   i.e. $x \in W \cap Z$
   $\therefore$ $V \cap Z \subseteq W \cap Z$ ∎

2. **Is it the case that, for all subsets $X, Y, W \subseteq U$, if $X = Y$ and $Y = W$, then $X = W$, and if $X \subset Y$ and $Y \subset W$, then $X \subset W$?**
   Both statements are true.

   Proof. Suppose $X = Y$ and $Y = W$.

   Let $x \in X$
   Then $x \in Y$, as $X = Y$
   Then $x \in W$, as $Y = W$
   $\therefore$ $X = W$ ∎

   Proof. Suppose $X \subset Y$ and $Y \subset W$.

   Let $x \in X$
   Then $x \in Y$, as $X \subset Y$
   Then $x \in W$, as $Y \subset W$
   $\therefore$ $X \subseteq W$
   Y has at least one element not in X, as $X \subset Y$
   W has at least one element not in Y, as $Y \subset W$
   So W has at least two elements not in X
   i.e. $X \neq W$
   so $X \subset W$ ∎

3. Is it the case that, for all subsets X of U, $X \cup \emptyset = X$? Justify your answer.
Yes.

Proof.

Let $x \in X$
Then $x \in X$ or $x \in \emptyset$
i.e. $x \in X \cup \emptyset$
$\therefore$ $X \subseteq X \cup \emptyset$

Let $x \in X \cup \emptyset$
Then $x \in X$ or $x \in \emptyset$
i.e. $x \in X$
(x cannot be in the empty set)
$\therefore$ $X \cup \emptyset \subseteq X$

As $(X \subseteq X \cup \emptyset)$ and $(X \cup \emptyset \subseteq X)$, $X \cup \emptyset = X$ ∎

4. Is it true that for all subsets V and W of U, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$?
No.

Proof.

(i) If $V \cap W = \emptyset$ then $V = \emptyset$ or $W = \emptyset$
This claim is false.

Counterexample. Let $V = \{3, 4\}$ and $W = \{5, 6\}$.

$$\begin{aligned} V \cap W &= \{3,4\} \cap \{5,6\} \\ &= \emptyset \end{aligned}$$

$V \cap W = \emptyset$ but $V \neq \emptyset$ and $W \neq \emptyset$. □

(ii) If $V = \emptyset$ or $W = \emptyset$, then $V \cap W = \emptyset$
This claim is true.

Subproof. Let $V = \emptyset$ and W be some non-empty set.

$$\begin{aligned} V \cap W &= \emptyset \cap W \\ &= \emptyset \end{aligned}$$

$\therefore$ if $V = \emptyset$, $V \cap W = \emptyset$
Let $W = \emptyset$ and V be some non-empty set.

$$\begin{aligned} V \cap W &= V \cap \emptyset \\ &= \emptyset \end{aligned}$$

$\therefore$ if $W = \emptyset$, $V \cap W = \emptyset$
$\therefore$ $V \cap W = \emptyset$ if either $V = \emptyset$ or $W = \emptyset$ □

As the first claim is false, it is not the case that $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$. ∎

5. **Is it the case that, for every subset X of U there exists a subset Y of U such that $X \cup Y = \emptyset$? Justify your answer.**
No.

Counterexample. Let $X = \{1, \}$ and $U = \{1, 2\}$.
The possible subsets of U are $\emptyset$ or $\{1\}$ or $\{2\}$ or $\{1, 2\}$.

$$\begin{aligned} X \cup \emptyset &= \{1\} \cup \emptyset \\ &= \{1\} \\ X \cup \{1\} &= \{1\} \cup \{1\} \\ &= \{1\} \\ X \cup \{2\} &= \{1\} \cup \{2\} \\ &= \{1, 2\} \\ X \cup \{1, 2\} &= \{1\} \cup \{1, 2\} \\ &= \{1, 2\} \end{aligned}$$

From the above, there is no set Y such that $X \cup Y = \emptyset$. ■

6. **Is it the case that, for every subset X of U, there is some subset Y such that $X \cap Y = U$? Justify your answer.**
No.

Counterexample. Let $X = \{1, \}$ and $U = \{1, 2\}$.
The possible subsets of U are $\emptyset$ or $\{1\}$ or $\{2\}$ or $\{1, 2\}$.

$$\begin{aligned} X \cap \emptyset &= \{1\} \cap \emptyset \\ &= \emptyset \\ X \cap \{1\} &= \{1\} \cap \{1\} \\ &= \{1\} \\ X \cap \{2\} &= \{1\} \cap \{2\} \\ &= \emptyset \\ X \cap \{1, 2\} &= \{1\} \cap \{1, 2\} \\ &= \{1\} \end{aligned}$$

From the above, there is no set Y such that $X \cap Y = U$. ■

7. **Using "if and only if" statements, prove the following:**

   (a) **$X + Y = Y + X$ for all $X, Y \subseteq U$.**

   Proof. Let $x \in X + Y$.

	$x \in X + Y$
iff	$(x \in X \text{ or } x \in Y)$ and $x \notin X \cap Y$
iff	$(x \in Y \text{ or } x \in X)$ and $x \notin X \cap Y$
iff	$x \in Y + X$
$\therefore$	$X + Y = Y + X$

   ■

(b) $\boldsymbol{X \cap (Y + Z) = (X \cap Y) + (X \cap Z)}$ **for all** $\boldsymbol{X, Y, Z \subseteq U}$**.**

Proof. Let $x \in X \cap (Y + Z)$.

	$x \in X \cap (Y + Z)$
iff	$x \in X$ and $x \in (Y + Z)$
iff	$x \in X$ and ($x \in Y$ or $x \in Z$ and $x \notin Y \cap Z$)
iff	($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in Z$) and $x \notin (Y \cap Z)$
iff	$x \in (X \cap Y)$ or $x \in (X \cap Z)$ and $x \notin (Y \cap Z)$
iff	$x \in (X \cap Y) + (Y \cap Z)$
$\therefore$	$X \cap (Y + Z) = (X \cap Y) + (Y \cap Z)$ ■

Unit 5

Relations

5.1 Ordered Pairs

Ordered Pair

In sets the order of the elements is insignificant. If the order of the elements is significant, it is written with an **ordered pair**, which is written in round brackets ().

Example An ordered pair is written (a, b) where a and b are elements of the pair. $(a, b) \neq (b, a)$.

5.2 Cartesian Product

Cartesian Product

For any sets A and B, the **Cartesian product** of A and B is written $A \times B$, and is equal to the set

$$\{(x, y) \mid x \in A \text{ and } y \in B\}$$

In other words, the Cartesian product $A \times B$ denotes a set of ordered pairs such that all the first coordinates of the pairs are elements of set A, and all the second coordinates of the pairs are elements of set B.

Example

$$A = \{2, 3, 4\} \qquad B = \{5, 6\}$$

$$A \times B = \{(2,5), (2,6), (3,5), (3,6), (4,5), (4,6)\}$$
$$B \times A = \{(5,2), (5,3), (5,4), (6,2), (6,3), (6,4)\}$$
$$B \times B = \{(5,5), (5,6), (6,5), (6,6)\}$$
$$A \times A = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}$$

5.3 Relation

Relation

A subset of a Cartesian product from C to D is called a **relation** from C to D.

Example $A = \{2, 3, 4\}$ and $B = \{6, 7\}$. The following are some relations from A to B

$$\begin{aligned}
&\emptyset && \text{(This is a subset, even though it has no elements)} \\
&\{(3,7)\} \\
&\{(2,6),(2,7)\} \\
&\{(2,6),(3,6),(4,6)\} \\
&A \times B
\end{aligned}$$

Self-Assessment Exercise 5.4

$$A = \{1, 2, 3, 4\} \qquad B = \{2, 5\} \qquad C = \{3, 4, 7\}$$

1. **List the following Cartesian products in list notation:**
   (a) $\boldsymbol{A \times B} = \{(1,2),(1,5),(2,2),(2,5),(3,2),(3,5),(4,2),(4,5)\}$
   (b) $\boldsymbol{B \times A} = \{(2,1),(2,2),(2,3),(2,4),(5,1),(5,2),(5,3),(5,4)\}$
   (c) $\boldsymbol{(A \cup B) \times C}$

$$\begin{aligned}
A \cup B &= \{1, 2, 3, 4, 5\} \\
(A \cup B) \times C &= \{(1,3),(1,4),(1,7),(2,3),(2,4),(2,7),(3,3),(3,4),(3,7), \\
&\quad (4,3),(4,4),(4,7),(5,3),(5,4),(5,7)\}
\end{aligned}$$

   (d) $\boldsymbol{(A + B) \times B}$

$$\begin{aligned}
A + B &= \{1, 3, 4, 5\} \\
(A + B) \times B &= \{(1,2),(1,5),(3,2),(3,5),(4,2),(4,5),(5,2),(5,5)\}
\end{aligned}$$

5.3.1 Domain, Range and Codomain

Codomain

Suppose T is a relation from X to Y.
The **codomain** of T is Y.
That is, all the possible elements that could appear as second coordinates.

Domain

Suppose T is a relation from X to Y.
The **domain** of T, written $\mathrm{dom}(T)$ is:

$$\mathrm{dom}(T) = \{x \mid \text{for some } y \in Y, (x, y) \in T\}$$

That is, all the elements that actually appear as first elements in the relation T.

Range

Suppose T is a relation from X to Y.
The **range** of T, written $\mathrm{ran}(T)$ is:

$$\mathrm{ran}(T) = \{y \mid \text{for some } x \in X, (x, y) \in T\}$$

That is, all the elements that actually appear as second elements in the relation T.

Domain and Range are not equal to X and Y

$\mathrm{dom}(T) \subseteq X$. The domain of the relation is a *subset* of X, but not necessarily equal to X.
$\mathrm{ran}(T) \subseteq Y$. The range of the relation is a *subset* of Y, but not necessarily equal to Y.

Example Let $S = \big\{(a, 1), (b, 1), (a, 2)\big\}$ be a relation from $\{a, b, c\}$ to $\{1, 2, 3\}$.
Then $\mathrm{dom}(S) = \{a, b\} \subseteq \{a, b, c\}$.
And $\mathrm{ran}(S) = \{1, 2\} \subseteq \{1, 2, 3\}$.
The codomain of S is the set $\{1, 2, 3\}$.

5.3.2 Binary Relation

Binary Relation

If R is any subset of a Cartesian product $X \times Y$, then R is called a **binary relation** from X to Y, or between X and Y.
A subset R of $X \times Y$ is called the **rule** for the relation.
If $R \subseteq X \times X$, R is a binary relation on X.

5.4 Properties of Relations

5.4.1 Reflexivity

Reflexivity

A relation R on A ($R \subseteq A \times A$) is called **reflexive** on A iff for every $x \in A$, we have $(x, x) \in R$. In other words, every element needs to be related to itself (although it can also be related to other elements).

Example Let $A = \{2, 3, 5\}$. For a relation S to be reflexive on A, $\{(2,2),(3,3),(5,5)\}$ needs to be a subset of S.

$$\{(2,2),(3,3),(5,5)\} \subseteq S.$$

Therefore, the relation $\{(2,2),(3,3),(5,5)(2,3)\}$ would be a reflexive relation on A.

5.4.2 Irreflexivity

Irreflexivity

A relation R on A ($R \subseteq A \times A$) is called **irreflexive** iff there is *no* x such that $(x, x) \in R$. In other words, for any $x \in A$, $(x, x) \notin R$

Example Let $A = \{2, 3, 5\}$.

$R = \{(3,2),(2,5),(3,5)\}$.
R is *irreflexive*, as there is no element that relates to itself. i.e. None of the elements of $\{(2,2),(3,3),(5,5)\}$ are elements of R.

$S = \{(2,2),(2,5),(3,5)\}$.
S is *not reflexive*, as the elements $\{(3,3),(5,5)\}$ are not present. S is also *not irreflexive*, as the element $(2,2)$ is an element of S.

5.4.3 Symmetry

Symmetry

A relation R on A ($R \subseteq A \times A$) is called **symmetric** iff R has the property that, for all $x, y \in R$, if $(x, y) \in R$, then $(y, x) \in R$.

Example Let $B = \{1, 2, 3\}$
$R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$ is symmetric and irreflexive.
$R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$ is reflexive, but not symmetric.
$R_3 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ is symmetric and reflexive.
$R_4 = \{(1,1),(2,3)\}$ is not reflexive, irreflexive or symmetric.

5.4.4 Antisymmetry

Antisymmetry

A relation R on A ($R \subseteq A \times A$) is called **antisymmetric** iff R has the property that, for all $x, y \in R$, if $x \neq y$ and $(x, y) \in R$, then $(y, x) \notin R$.
Another definition:

A relation R on A ($R \subseteq A \times A$) is called **antisymmetric** iff R has the property that, for all $x, y \in R$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Example Let $A = \{a, b, c\}$
$P = \{(a, b), (b, b), (b, c)\}$ on A.
$a \neq b$, $(a, b) \in P$, but $(b, a) \notin P$.
$b \neq c$, $(b, c) \in P$, but $(c, b) \notin P$.
$\therefore$ P is antisymmetric on A.

Antisymmetric and Not Symmetric Are Not The Same

A relation can be both not antisymmetric and symmetric at the same time. Consider the relation: $R = \{(1, 2), (2, 1), (2, 3)\}$ on $A = \{1, 2, 3\}$.
This relation is not symmetric, as $(2, 3) \in R$, but $(3, 2) \notin R$.
This relation is also not antisymmetric, since $(1, 2)$ and $(2, 1)$ are elements of R, but $1 \neq 2$.

5.4.5 Transitivity

Transitivity

A relation R on A ($R \subseteq A \times A$) is called **transitive** iff R has the property that, for all $x, y, z \in R$, whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example If $(1, 2) \in R$ and $(2, 3) \in R$, then $(1, 3)$ must be in R.

Example Let $R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$ be a relation on $A = \{1, 2, 3\}$. This relation is transitive:
$(2, 1)$ and $(1, 2)$ mean $(2, 2)$ should be present.
Can be done with all possible combinations.

5.4.6 Trichotomy

Trichotomy

A relation R on A satisfies **trichotomy** iff, for every x and y chosen from A such that $x \neq y$, x and y are comparable.
In other words, for every $x \neq y$, every element is related to every other element. So xRy or yRx.

Example Let $S = \{(3,2),(2,1),(3,1)\}$ be a relation on $A = \{1,2,3\}$.
S satisfies the requirements for trichotomy, since:

1 is related to 2 in $(2,1)$ and related to 3 in $(3,1)$.
2 is related to 1 in $(2,1)$ and related to 3 in $(3,2)$.
3 is related to 1 in $(3,1)$ and related to 2 in $(3,2)$.

5.4.7 Inverse Relation

Inverse Relation

Given a relation R with domain A and range B, the relation R^{-1} with domain B and range A is called the **inverse of R**, and is defined such that:

$$(x,y) \in R \text{ iff } (y,x) \in R^{-1}$$

Example Let $X = a,b,c$ and $R = \{(a,b),(b,c),(a,c)\}$
Then $R^{-1} = \{(b,a),(c,b),(c,a)\}$

5.4.8 Relation Composition

Relation Composition

Given relations R from A to B and S from B to C, the **composition** of R followed by S, written $S \circ R$ or $R;S$ is the relation from A to C defined by:
$S \circ R = R;S = \{(a,c) \mid \text{there is some } b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$

Example Let $R = \{(1,a),(2,b)\}$ be a relation from $\{1,2\}$ to $\{a,b\}$
Let $S = \{(a,s),(b,s),(b,t)\}$ be a relation from $\{a,b\}$ to $\{s,t\}$.
Then $S \circ R = R;S$.

$(1,a) \longrightarrow (a,s) \longrightarrow (1,s)$
$(2,b) \longrightarrow (b,s) \longrightarrow (2,s)$
$(2,b) \longrightarrow (b,t) \longrightarrow (2,t)$

$S \circ R = R;S = \{(1,s),(2,s),(2,t)\}$

Self Assessment Activity 5.8

1. **Let P and R be relations on $A = \{1, 2, 3, \{1\}, \{2\}\}$, where $P = \Big\{(1, \{1\}), (1, 2)\Big\}$ and $R = \Big\{(1, \{1\}), (1, 3), (2, \{1\}), (2, \{2\}), (\{1\}, 3), (\{2\}, \{1\})\Big\}$**

   (a) **Is R irreflexive?**
   Yes. There are no elements that are related to themselves.

   (b) **Is R reflexive?**
   No. There are no elements that are related to themselves.

   (c) **Is R symmetric?**
   No. $(1, \{1\}) \in R$, but $(\{1\}, 1) \notin R$.

   (d) **Is R antisymmetric?**
   Yes.
   $(1, \{1\}) \in R$, and $(\{1\}, 1) \notin R$.
   $(1, 3) \in R$, and $(3, 1) \notin R$.
   $(2, \{1\}) \in R$, and $(\{1\}, 2) \notin R$.
   $(2, \{2\}) \in R$, and $(\{2\}, 2) \notin R$.
   $(\{1\}, 3) \in R$, and $(3, \{1\}) \notin R$.
   $(\{2\}, \{1\}) \in R$, and $(\{1\}, \{2\}) \notin R$.

   (e) **Is R transitive?**
   No. $(2, \{1\}) \in R$, and $(\{1\}, 3) \in R$, but $(2, 3) \notin R$

   (f) **Does R satisfy the requirement for trichotomy?**
   No. There is no pair where 1 is related to 2.

   (g) **Determine the relation $R \circ R$.**
   $R \circ R = R; R$.

   $$\begin{aligned}
   &(1, \{1\}) \to (\{1\}, 3) \to (1, 3)\\
   &(1, 3) \not\to\!\!\to\\
   &(2, \{1\}) \to (\{1\}, 3) \to (2, 3)\\
   &(2, \{2\}) \to (\{2\}, \{1\}) \to (2, \{1\})\\
   &(\{1\}, 3) \not\to\!\!\to\\
   &(\{2\}, \{1\}) \to (\{1\}, 3) \to (\{2\}, 3)
   \end{aligned}$$

   $R \circ R = R; R = \Big\{(1, 3), (2, 3), (2, \{1\}), (\{2\}, 3)\Big\}$

   (h) **Determine the relation $R \circ P$.** $R \circ P = P; R$.

   $$\begin{aligned}
   &(1, \{1\}) \to (\{1\}, 3) \to (1, 3)\\
   &(1, 2) \to (2, \{1\}) \to (1, \{1\})\\
   &(1, 2) \to (2, \{2\}) \to (1, \{2\})
   \end{aligned}$$

   $R \circ P = R; R = \Big\{(1, 3), (1, \{1\}), (1, \{2\})\Big\}$

   (i) **Give the subset T of R where $(a, B) \in T$ iff $a \in B$.**
   $T = \Big\{(1, \{1\}), (2, \{2\})\Big\}$

2. **Let $A = \{a, b\}$. For each of the specifications given below, find suitable examples of relations on $\mathcal{P}(A)$**

$$\mathcal{P}(A) = \Big\{\emptyset, \{a\}, \{b\}, \{a, b\}\Big\}$$
$$\begin{aligned}\mathcal{P}(A) \times \mathcal{P}(A) = \Big\{&(\emptyset, \emptyset), (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a, b\})\\ &(\{a\}, \emptyset), (\{a\}, \{a\}), (\{a\}, \{b\}), (\{a\}, \{a, b\})\\ &(\{b\}, \emptyset), (\{b\}, \{a\}), (\{b\}, \{b\}), (\{b\}, \{a, b\})\\ &(\{a, b\}, \emptyset), (\{a, b\}, \{a\}), (\{a, b\}, \{b\}), (\{a, b\}, \{a.b\})\Big\}\end{aligned}$$

1. **R is reflexive, symmetric and transitive on $\mathcal{P}(A)$**

Reflexivity To be reflexive, these pairs need to appear:

$$(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})$$

Symmetry Whichever pair is added, the pair that makes it symmetric needs to be added too.

If $(\emptyset, \{a\})$ is added, then $(\{a\}, \emptyset)$ needs to be added.

Examples Two relations that meet these requirements are:

$$R_1 = \Big\{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})\Big\}$$
$$R_2 = \Big\{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\}), (\emptyset, \{a\}), (\{a\}, \emptyset)\Big\}$$

2. **R is reflexive and symmetric, but not transitive on $\mathcal{P}(A)$**

Reflexivity To be reflexive, these pairs need to appear:

$$(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})$$

Symmetry Whichever pair is added, the pair that makes it symmetric needs to be added too.

If $(\emptyset, \{a\})$ is added, then $(\{a\}, \emptyset)$ needs to be added

Transitivity In order for the relation to not be transitive, two elements need to be added (for symmetry) where the first element is the second element of another pair, and the second element is the first element of a different pair.

If $(\{a\}, \{a, b\})$ and $(\{a, b\}, \{a\})$ are added.
$(\emptyset, \{a\}) \rightarrow (\{a\}, \{a, b\}) \nrightarrow (\emptyset, \{a, b\})$

Example $R_3 = \Big\{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\}),$
$(\emptyset, \{a\}), (\{a\}, \emptyset), (\{a\}, \{a, b\}), (\{a, b\}, \{a\})\Big\}$

3. **R is reflexive and transitive, but not symmetric, and not antisymmetric on $\mathcal{P}(A)$**

Reflexivity To be reflexive, these pairs need to appear:

$$(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})$$

Symmetry For the relation to not be symmetric, at least one pair cannot be flipped.

If $(\emptyset, \{a\})$ is added, then $(\{a\}, \emptyset)$ is not added.
Adding this single element would still mean R is transitive.

Antisymmetry For the relation to not be antisymmetric, at least one pair can be flipped. If $(\emptyset, \{b\})$ is added, then $(\{b\}, \emptyset)$ is added.

Example $R_4 = \Big\{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\}),$
$(\emptyset, \{a\}), (\emptyset, \{b\}), (\{b\}, \emptyset)\Big\}$

4. **R is simultaneously symmetric and antisymmetric on $\mathcal{P}(A)$**

Antisymmetry If there are no elements that are not equal to each other, then R is vacuously antisymmetric.

Symmetry If every element is equal to each other, then every element is symmetric with itself.

Example $R_5 = \Big\{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})\Big\}$

5. **R is irreflexive, antisymmetric and transitive on $\mathcal{P}(A)$**

Irreflexivity None of these pairs appear in R:

$$(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})$$

Antisymmetry No pairs (x, y) and (y, x) appear in R.

Transitivity Can go from one pair to the next.

Example $R_6 = \Big\{(\emptyset, \{a\}), (\{a\}, \{a, b\}), (\emptyset, \{a, b\})\Big\}$

3. **Prove that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.**

Proof.

(i) **If R is transitive, then $R \circ R \subseteq R$**

Subproof.

Assume	R is transitive.
Suppose	$(x, z) \in R \circ R$.
Then	there is some $y \in X$ such that $(x, y) \in R$ and $(y, z) \in R$ (By definition of composition)
And	$(x, z) \in R$ (Because R is transitive)
$\therefore$	if R is transitive, then $R \circ R \subseteq R$ ☐

(ii) If $R \circ R \subseteq R$, then R is transitive

Subproof.

Assume	$R \circ R \subseteq R$.
Suppose	$(x, y) \in R$ and $(y, z) \in R$.
Then	$(x, z) \in R \circ R$.
	(By definition of composition)
And	$(x, z) \in R$
	(Because $R \circ R \subseteq R$)
$\therefore$	if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$
$\therefore$	if $R \circ R \subseteq R$, then R is transitive. $\square$

$\therefore$ R is transitive iff $R \circ R \subseteq R$. ■

Unit 6

Special Kinds of Relation

6.1 Order Relations

6.1.1 Weak Partial Order

Weak Partial Order

A relation R on a set A is called a **weak partial order** iff R is

- reflexive on A
- antisymmetric, and
- transitive

Example Let $A = \big\{\{a\}, \{a, b\}\big\}$. A relation S on A is defined by $(B, C) \in S$ iff $B \subseteq C$. (Each first coordinate is a subset of the second coordinate.)

$$S = \Big\{\big(\{a\}, \{a\}\big), \big(\{a\}, \{a, b\}\big), \big(\{a, b\}, \{a, b\}\big)\Big\}$$

To prove this is a weak partial order, prove reflexivity, antisymmetry and transitivity.

Reflexivity Is it true that $(B, B) \in S$ for all $B \in A$? Yes.

$$\big(\{a, a\}\big) \in S \qquad \big(\{a, b\}, \{a, b\}\big) \in S$$

Antisymmetry Is it true that for all $(B, C) \in A$, if $B \neq C$, and $(B, C) \in S$, then $(C, B) \notin S$? Yes.
The elements where $B \neq C$ are $\{a\}$ and $\{a, b\}$.
$\big(\{a\}, \{a, b\}\big) \in S$, and $\big(\{a, b\}, \{a\}\big) \notin S$

Transitivity Is it true that for all $B, C, D \in A$, if $(B, C) \in S$, and $(C, D) \in S$, then $(B, D) \in S$? Yes.

$$\begin{array}{lll} \big(\{a\}, \{a\}\big) \in S & \rightarrow \big(\{a\}, \{a\}\big) \in S & \rightarrow \big(\{a\}, \{a\}\big) \in S \\ \big(\{a\}, \{a\}\big) \in S & \rightarrow \big(\{a, \}, \{a, b\}\big) \in S & \rightarrow \big(\{a\}, \{a, b\}\big) \in S \end{array}$$

The above can be done for all elements.

Weak Partial Order As S is reflexive, antisymmetric, and transitive, S is a weak partial order.

Activity 6.4

1. **Determine whether the following relations are weak partial orders.**

   (a) **Let $A = \{a, b, \{a, b\}\}$. S is the relation on A defined by $(c, B) \in S$ iff $c \in B$.**

$$S = \{(a, \{a, b\}), (b, \{a, b\})\}$$

Reflexivity Is it true that $(x, x) \in S$ for all $x \in A$?
No. Using a counterexample: $(a, a) \notin S$.
Can stop here and conclude that S is not a weak partial order, but for completeness, checking the other two conditions as well.

Antisymmetry Is it true for all $(x, y) \in A$, if $x \neq y$, and $(x, y) \in S$, then $(y, x) \notin S$?
Yes. If $(x, y) \in S$, then $x \in y$. If $x \in y$, then y cannot be an element of x, so $(y, x) \notin S$.

Transitivity Is it true for all $x, y, z \in A$, if $(x, y) \in S$, and $(y, z) \in SS$, then $(x, z) \in S$?
Yes. Vacuously true, as there is no element that has the same first coordinate as another element's second coordinate.

Weak Partial Order As S is not reflexive, S is *not* a weak partial order.

   (b) **$R \subseteq \mathbb{Z} \times \mathbb{Z}$ such that $x\,R\,y$ iff $x + y$ is even.**
If $x + y$ is even, then $x + y = 2k$ for some integer k.

Reflexivity Is it true that $(x, x) \in R$ for all $x \in \mathbb{Z}$?
Yes. $x+x = 2x$, which would be part of R if $k = x$. As $2x$ is always even, $(x, x) \in R$.

Antisymmetry Is it true that for all $(x, y) \in R$, if $x \neq y$ and $(x, y) \in R$, then $(y, x) \notin R$?
No. Let $(x, y) \in R$. Then $x + y = 2k$. But $(y + x)$ also equals $2k$. So $(y, x) \in R$.
Can stop here and conclude that R is not a weak partial order. For completeness, checking transitivity as well.

Transitivity Is it true that for all $x, y, z \in \mathbb{Z}$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?
Yes. Let $(x, y) \in R$, and $(y, z) \in R$. Then $x + y = 2k$, and $y + z = 2m$, where k and m are integers.

$$\begin{aligned}
x + y &= 2k \\
x &= 2k - y \\
y + z &= 2m \\
z &= 2m - y \\
x + z &= (2k - y) + (2m - y) \\
&= 2k + 2m - 2y \\
&= 2(k + m - y)
\end{aligned}$$

From the above, if $x + y$ is even, and $y + z$ is even, then $x + z$ is also even.

Weak Partial Order As R is not antisymmetric, R is *not* a weak partial order.

(c) R on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b)\, R\, (c, d)$ if either $a < c$ or $(a = c$ and $b \leq d)$.

Reflexivity Is it true that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b)\, R\, (a, b)$?
Yes. It is never the case that $a < a$, but $a = a$ and $b \leq b$ is true.

Antisymmetry Is it true that for all (a, b) and $(c, d) \in \mathbb{Z} \times \mathbb{Z}$, if $(a, b) \neq (c, d)$ and $\{(a, b), (c, d)\} \in R$, then $\{(c, d), (a, b)\} \notin R$?
Yes. If $(a, b)R(c, d)$, then $a < c$ or $(a = c$ and $b \leq d)$. If the first case is matched, then $a < c$, which means that $c > a$. Which means that $c \neq a$, so the second condition is not met.
If the second case is matched, then $a = c$ and $b \leq d$. If $a = c$, then $c = a$, and if $b \leq d$, then $d \geq b$. However, if $b = d$, then $(a, b) = (c, d)$, which would mean it would be excluded. Therefore, $d < b$, which means that $\{(c, d), (a, b)\} \notin R$.

Transitivity If $\{(a, b), (c, d)\} \in R$ and $\{(c, d), (e, f)\} \in R$, is $\{(a, b), (e, f)\} \in R$?
Yes.

If $\{(a, b), (c, d)\} \in R$, then either $a < c$ or $(a = c$ and $b \leq d)$.
If $\{(c, d), (e, f)\} \in R$, then either $c < e$ or $(c = e$ and $d \leq f)$.
If $a < c$ and $c < e$, then $a < e$, which means $\{(a, b), (e, f)\} \in R$.
If $a < c$ and $c = e$ and $d \leq f$, then $a < e$, which means $\{(a, b), (e, f)\} \in R$.
If $a = c$ and $b \leq d$ and $c < e$, then $a < e$, which means $\{(a, b), (e, f)\} \in R$.
If $a = c$ and $b \leq d$ and $c = e$ and $d \leq f$, then $c = e$, and $b \leq f$, which means $\{(a, b), (e, f)\} \in R$.

Weak Partial Order As R is reflexive, antisymmetric and transitive, R *is* a weak partial order.

6.1.2 Strict Partial Order

Strict Partial Order

A relation R on a set A is called a **strict partial order** iff R is

- irreflexive on A
- antisymmetric, and
- transitive

Example Let $A = \{1, 2, 3\}$ and let S on A be the relation $S = \{(1, 2), (1, 3), (2, 3)\}$. (Every first coordinate is less than the second coordinate.)
To prove this is a strict partial order, prove irreflexivity, antisymmetry and transitivity.

Irreflexivity Is it true that $(x, x) \notin S$ for any $x \in A$?
Yes, no element is related to itself, i.e. the pairs $(1, 1)$, $(2, 2)$ and $(3, 3)$ are not elements of S.

Antisymmetry Is it true that for all $x, y \in A$, if $(x, y) \in S$, then $(y, x) \notin S$?
Yes. $1 \neq 2$ and $(1, 2) \in S$ and $(2, 1) \notin S$.
$1 \neq 3$ and $(1, 3) \in S$ and $(3, 1) \notin S$.
$2 \neq 3$ and $(2, 3) \in S$ and $(3, 2) \notin S$.

Transitivity Is it true that for all $x, y, z \in A$, if $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$?
Yes. $(1, 2) \in S$ and $(2, 3) \in S$ and $(1, 3) \in S$.

Strict Partial Order As S is irreflexive, antisymmetric, and transitive, S is a strict partial order.

Activity 6.5

1. **Determine whether the following relations are strict partial orders.**

   (a) **$A = \{a, \{a\}, \{b\}\}$ and the relation S on A is $S = \{(a, \{a\}), (a, \{b\})\}$**

   Irreflexivity Is it true that $(x, x) \notin S$ for any $x \in A$?
   Yes, no element is related to itself, i.e. the pairs (a, a), $(\{a\}, \{a\})$ and $(\{b\}, \{b\})$ are not elements of S.

   Antisymmetry Is it true that for all $x, y \in A$ and $x \neq y$, if $(x, y) \in S$, then $(y, x) \notin S$?
   Yes. $a \neq \{a\}$. $(a, \{a\}) \in S$, and $(\{a\}, a) \notin S$.
   $a \neq \{b\}$. $(a, \{b\}) \in S$, and $(\{b\}, a) \notin S$.

   Transitivity Is it true that for all $x, y, z \in A$, if $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$?
   Yes. There are no ordered pairs such that $(x, y) \in R$ and $(y, z) \in R$, so R is *vacuously* transitive.

   Strict Partial Order As R is irreflexive, antisymmetric, and transitive, R *is* a strict partial order.

(b) $R \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ **such that** $(a, b) \, R \, (c, d)$ **iff** $a < c$**.**

Irreflexivity Is it true that $\big((a, b), (a, b)\big) \notin R$ for any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$?
Yes. For $\big((a, b), (a, b)\big) \in R$, it would need to satisfy the requirement $a < a$, which is never true.

Antisymmetry Is it true that for all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$, where $(a, b) \neq (c, d)$, if $((a, b), (c, d)) \in R$, then $((c, d), (a, b)) \notin R$.
Yes. If $((a, b), (c, d)) \in R$, then $a < c$. As $a < c$, $((c, d), (a, b)) \notin R$.

Transitivity Is it true for all $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, then $((a, b), (e, f)) \in R$?
Yes. If $((a, b), (c, d)) \in R$, then $a < c$. If $((c, d), (e, f)) \in R$, then $c < e$. Therefore, $a < e$, so $((a, b), (e, f)) \in R$.

Strict Partial Order As R is irreflexive, antisymmetric, and transitive, R *is* a strict partial order.

6.1.3 A Total (or Linear) Order Relation

Total Order Relation

A relation R on a set A is called a **total** or **linear order** if R is a partial order on A that also satisfies *trichotomy*.

Example The example for Strict Partial Orders:

> Let $A = \{1, 2, 3\}$ and let S on A be the relation $S = \{(1,2), (1,3), (2,3)\}$. (Every first coordinate is less than the second coordinate.)

also satisfies trichotomy.

Trichotomy Is every element of A related to every other element in the relation S?
Yes. 1 is related to 2 in $(1, 2)$, and related to 3 in $(1, 3)$.
2 is related to 1 in $(1, 2)$, and related to 3 in $(2, 3)$.
3 is related to 1 in $(1, 3)$, and related to 2 in $(2, 3)$.

Total Order Relation As this relation is a partial order relation that satisfies trichotomy, it is a **total order relation**. As the relation is a *strict* partial order, this is a **strict total order relation**.

Proof Strategies

You cannot use examples to prove a general statement, i.e, something of the form:

> For all x, or
> For all pairs (x, y)

Instead, *abstract reasoning* needs to be used to produce a *general proof*.
However, an example can be used to show that a statement is false, which is known as a **counterexample**.

Self Assessment 6.7

1. **Let $X = \{a, b, c\}$. Write down all strict partial orders on X. Which of them are linear?**
   Strict partial orders are irreflexive, antisymmetric and transitive.

   One element There are 6 relations on X that are strict partial orders that contain only one element:

$$\{(a,b)\}, \{(a,c)\}, \{(b,a)\}, \{(b,c)\}, \{(c,a)\}, \{(c,b)\}$$

   Two elements There are 6 relations on X that are strict partial orders that contain two elements:

$$\{(a,b),(a,c)\}, \{(a,b),(c,b)\}, \{(a,c),(b,c)\}, \{(b,a),(b,c)\}, \{(b,a),(c,a)\}, \{(c,a),(c,b)\}$$

Three elements There are 6 relations on X that are strict partial orders that contain three elements:

$$\{(a,b),(b,c),(a,c)\},\ \{(b,a),(a,c),(b,c)\},\ \{(c,b),(b,a),(c,a)\},\ \{(a,c),(c,b),(a,b)\},\ \{(c,a),(a,b),(c,b)\},\ \{(b,c),(c,a),(b,a)\}$$

More than three elements There are no relations on X that are strict partial orders And contain more than three elements.

Linear For a relation to be linear, it needs to satisfy *trichotomy*. As there are three elements in X the relation should contain three or more elements.
All the strict partial relations with three elements satisfy trichotomy, and so are linear.

2. In each of the following cases, determine whether R is some sort of order relation on the given set X. Justify your answer.

(a) $X = \{\emptyset, \{0\}, \{2\}\}$ **and** $R = \Big\{(\emptyset, \{0\}), (\emptyset, \{2\})\Big\}$
R is a *strict partial order*.

Proof.

Reflexivity R is not reflexive.

Counterexample. $(\emptyset, \emptyset) \notin R$ □

Irreflexivity R is irreflexive.

Proof. For all $x \in X$, $(x,x) \notin R$. □

Antisymmetry R is antisymmetric.

Proof. $(\emptyset, \{0\}) \in R$ and $(\{0\}, \emptyset) \notin R$
$(\emptyset, \{2\}) \in R$ and $(\{2\}, \emptyset) \notin R$
For all elements $(x,y) \in R$, $(y,x) \notin R$. □

Transitivity R is transitive.

Proof. There are no elements such that the second coordinate of a pair is the first coordinate of another pair. □

Trichotomy R does not satisfy trichotomy.

Counterexample. There are no pairs in the relation where $\{0\}$ and $\{2\}$ are related to each other. □

As R is irreflexive, antisymmetric and transitive, but does not satisfy trichotomy, R is a strict partial order. ■

(b) $X = \left\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\right\}$ **and** $R = \subseteq$. **(That is, each first coordinate is a subset of the second coordinate)**

$R = \left\{(\emptyset,\emptyset), (\{\emptyset\},\{\emptyset\}), \left(\{\{\emptyset\}\},\{\{\emptyset\}\}\right), (\emptyset,\{\emptyset\}), \left(\emptyset,\{\{\emptyset\}\}\right)\right\}$

R is a *weak partial order*.

Proof.

Reflexivity R is reflexive.

Proof. For all x in X, $(x,x) \in R$. □

Irreflexivity R is not irreflexive.

Counterexample. $(\emptyset,\emptyset) \in R$. □

Antisymmetry R is antisymmetric.

Proof. For all elements $(x,y) \in R$, $(y,x) \notin R$. □

Transitivity R is transitive.

Proof. Whenever $(x,y) \in R$ and $(y,z) \in R$, $(x,z) \in R$. □

Trichotomy R does not satisfy trichotomy.

Counterexample. There are no pairs in R where $\{\emptyset\}$ is related to $\{\{\emptyset\}\}$. □

As R is reflexive, antisymmetric and transitive, but does not satisfy trichotomy, R is a weak partial order. ■

3. $X = \mathbb{Z}$ **and** $R = \leq$

R is a *weak total order*.

Proof.

Reflexivity R is reflexive.

Proof. For all $x \in \mathbb{Z}$, $x = x$, so $x \leq x$, so $(x,x) \in R$. □

Irreflexivity R is not irreflexive.

Counterexample. $(1,1) \in R$ □

Antisymmetry R is antisymmetric.

Proof. If $(x,y) \in R$ and $x \neq y$, then $x < y$.
Therefore, $y \nleq x$, so $(y,x) \notin R$. □

Transitivity R is transitive.

Proof. If $(x, y) \in R$, then $x \leq y$, and if $(y, z) \in R$, then $y \leq z$.
If $x < y$ and $y < z$, then $x < z$, so $(x, z) \in R$.
If $x < y$ and $y = z$, then $x < z$, so $(x, z) \in R$.
If $x = y$ and $y < z$, then $x < z$, so $(x, z) \in R$.
If $x = y$ and $y = z$, then $x = z$, so $(x, z) \in R$.
Therefore, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. □

Trichotomy R satisfies trichotomy.

Proof. For all $x, y \in \mathbb{Z}$, either $x = y$, or $x > y$ or $x < y$. If $x > y$, then $y < x$. So x and y are always related to each other in R. □

As R is reflexive, antisymmetric and transitive, and satisfies trichotomy, R is a weak total order. ■

4. $X = \mathbb{Z}$ and R =>
R is a *strict total order*.

Proof.

Reflexivity R is not reflexive.

Counterexample. $(1, 1) \notin R$. □

Irreflexivity R is irreflexive.

Proof. For all $x \in \mathbb{Z}$, $(x, x) \notin R$. That is, $x \not> x$. □

Antisymmetry R is antisymmetric.

Proof. If $(x, y) \in R$, then $x > y$, so $y \not> x$, so $(y, x) \notin R$. □

Transitivity R is transitive.

Proof. If $(x, y) \in R$, then $x > y$. If $(y, z) \in R$, then $y > z$.
Therefore, $x > y > z$, i.e. $x > z$, so $(x, z) \in R$. □

Trichotomy R satisfies trichotomy.

Proof. For all elements $x, y \in \mathbb{Z}$, if $x \neq y$, then $x > y$ or $y > x$, so either $(x, y) \in R$, or $(y, x) \in R$ □

As R is irreflexive, antisymmetric and transitive, and satisfies trichotomy, R is a strict total order. ■

5. $X \in \mathbb{Z}^+$ and $x \, R \, y$ iff x divides into y with zero remainder. $y = kx$ for some $k \in \mathbb{Z}^+$. x is a *factor* of y and y is a *multiple* of x.
Some example elements of R are $(2, 8)$, $(7, 21)$, $(6, 36)$, $(1, 1)$.
R is a *weak partial order*.

Proof.

Reflexivity R is reflexive.

Proof. For all $x \in \mathbb{Z}^+$, $(x, x) \in R$, as

$$\begin{aligned} y &= kx && (k \in \mathbb{Z}^+) \\ &= (1)x \\ &= x \end{aligned}$$

□

Irreflexivity R is not irreflexive.

Counterexample. $(1, 1) \in R$ □

Antisymmetry R is antisymmetric.

Proof. For all $x, y \in \mathbb{Z}^+$, where $x \neq y$, let $y = kx$.
Let $x = my$. Then $y = k(my) = (km)y$. So $km = 1$. That means $x = y$, but that was assumed to be false.
So, if $(x, y) \in R$, then $(y, x) \notin R$. □

Transitive R is transitive.

Proof. Let $(x, y) \in R$ and $(y, z) \in R$. That means that $y = kx$, where $k \in \mathbb{Z}^+$, and $z = my$, where $m \in \mathbb{Z}^+$.
As $z = my$, that means $z = m(kx)$, i.e. $z = (km)x$. km is also an element of $\mathbb{Z}^+$, so $(x, z) \in R$. □

Trichotomy R does not satisfy trichotomy.

Counterexample. There are no elements of R where 2 is related to 3. □

As R is reflexive, antisymmetric and transitive, and does not satisfy trichotomy, R is a weak partial order. ■

6.2 Equivalence Relation

Equivalence Relation

A relation R on a set A is called an **equivalence relation** if R is:

- reflexive on A
- symmetric, and
- transitive

Example Let A be the set of real numbers. A relation R on A is defined as $(x, y) \in R$ iff $x = y$.

Reflexivity Is it true that $(x, x) \in R$ for all $x \in A$?
Yes. If $x = x$, then $(x, x) \in R$, and $x = x$ is always true.

Symmetry Is it true that if $(x, y) \in R$, then $(y, x) \in R$?
Yes. If $(x, y) \in R$, then $x = y$. But if $x = y$, then $y = x$, so $(y, x) \in R$.

Transitivity Is it true that is $(x, y) \in R$, and $(y, z) \in R$, then $(x, z) \in R$?
Yes. If $(x, y) \in R$, then $x = y$. And if $(y, z) \in R$, then $y = z$. So $x = y = z$, i.e. $x = z$, i.e. $(x, z) \in R$.

Equivalence Relation As R is reflexive, symmetric and transitive, R is an equivalence relation.

Equivalence relations are used to group related data together based on a specific characteristic.

Example Students get marked for an assignment using grades from A to E. All students who get an A would be in the same equivalence class, even if their individual marks are different.

6.2.1 Equivalence Class

Equivalence Class

For each $x \in A$, the **equivalence class** $[x] = \{y \mid y \in A \text{ and } x\,R\,y\}$

Example Let R be the relation on $\mathbb{Z}$ defined by $(x, y) \in R$ iff $y - x$ is even.
That is, $R = \{y \mid y - x = 2k\}$ for some $k \in \mathbb{Z}$. So,

$$\begin{aligned}[x] &= \{y \mid y - x = 2k\} \\ &= \{y \mid y = 2k + x\}\end{aligned}$$

Then substitute elements of x until there are no more equivalence classes.

$$\begin{aligned}[0] &= \{y = 2k\} \\ &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} = [2] = [4] \ldots \\ [1] &= \{y = 2k + 1\} \\ &= \{\ldots, -5, -3, -1, 1, 3, 5, 7, \ldots\} = [3] = [5] \ldots\end{aligned}$$

$[0]$ is the set of even integers, and $[1]$ is the set of odd integers.
These two equivalence classes would be the parts of the **partition** S of the set $\mathbb{Z}$ on the relation R: $S = \big\{[0], [1]\big\}$

Self-Assessment Exercise 6.10

1. **Let $X = \{a, b, c\}$. Write down all equivalence relations on X.**
   For an equivalence relation, the relation needs to be *reflexive*, *symmetric* and *transitive*.

   Reflexivity For reflexivity, {(a, a), (b, b), (c, c)} need to be part of the relation.

   Symmetry If (a, b) is added, then (b, a) must be added. This still satisfies transitivity.
   If (a, c) is added, then (c, a) must be added.
   If (b, c) is added, then (c, b) must be added.

   Transitivity If (a, b) is added, and (b, c) is added, then (a, c) must be added.

   All equivalence relations

$$
\begin{aligned}
R_1 &= \big\{(a,a),(b,b),(c,c)\big\} \\
R_2 &= \big\{(a,a),(b,b),(c,c),(a,b),(b,a)\big\} \\
R_3 &= \big\{(a,a),(b,b),(c,c),(a,c),(c,a)\big\} \\
R_4 &= \big\{(a,a),(b,b),(c,c),(b,c),(c,b)\big\} \\
R_5 &= \big\{(a,a),(b,b),(c,c),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\big\}
\end{aligned}
$$

2. **Determine whether the following relations R on X are equivalence relations. If they are, describe the equivalence classes of R.**

   (a) $X = \{a, b, c\}$ **and** $R = \big\{(c, c), (b, b), (a, a)\big\}$

   Reflexivity Yes. For all x in X, $(x, x) \in R$.

   Symmetry Yes. Each element is symmetric with itself.

   Transitivity Yes. Vacuously transitive.

   Equivalence relation R is an equivalence relation.

   Equivalence classes
$$
\begin{aligned}
[x] &= \{y \mid (x,y) \in R\} \\
[c] &= \{y \mid (c,y) \in R\} \\
&= \{c\} \\
[b] &= \{y \mid (b,y) \in R\} \\
&= \{b\} \\
[a] &= \{y \mid (a,y) \in R\} \\
&= \{a\}
\end{aligned}
$$

(b) $X = \{a, b, c\}$ **and** $R = X \times X$

$$R = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$$

Reflexivity Yes. (a,a), (b,b) and (c,c) are all in R.

Symmetry Yes. Every element in R has its mirror image.

Transitivity Yes.

Equivalence relation R is an equivalence relation.

Equivalence classes
$$\begin{aligned}[x] &= \{y \mid (x,y) \in R\} \\ [a] &= \{a \mid (a,y) \in R\} \\ &= \{a, b, c\} \\ &= [b] \\ &= [c]\end{aligned}$$

(c) $X = \mathcal{P}(Y)$ **where** $Y = \{1, 2, 3\}$ **and** R **consists of all pairs** (C, D) **such that** $C \cap \{2\} = D \cap \{2\}$

$$\begin{aligned}X &= \mathcal{P}(Y) \\ &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}\end{aligned}$$

$$S \cap \{2\} = \emptyset \text{ if } 2 \notin S. \qquad S \cap \{2\} = \{2\} \text{ if } 2 \in S.$$

Sets without 2: $\emptyset$, $\{1\}$, $\{3\}$, $\{1,3\}$

$$\begin{aligned}R_1 = \{&(\emptyset,\emptyset), (\emptyset,\{1\}), (\emptyset,\{3\}), (\emptyset,\{1,3\}), \\ &(\{1\},\emptyset), (\{1\},\{1\}), (\{1\},\{3\}), (\{1\},\{1,3\}), \\ &(\{3\},\emptyset), (\{3\},\{1\}), (\{3\},\{3\}), (\{3\},\{1,3\}) \\ &(\{1,3\},\emptyset), (\{1,3\},\{1\}), (\{1,3\},\{3\}), (\{1,3\},\{1,3\})\}\end{aligned}$$

Sets with 2: $\{2\}$, $\{1,2\}$, $\{2,3\}$, $\{1,2,3\}$

$$\begin{aligned}R_2 = \{&(\{2\},\{2\}), (\{2\},\{1,2\}), (\{2\},\{2,3\}), (\{2\},\{1,2,3\}) \\ &(\{1,2\},\{2\}), (\{1,2\},\{1,2\}), (\{1,2\},\{2,3\}), (\{1,2\},\{1,2,3\}) \\ &(\{2,3\},\{2\}), (\{2,3\},\{1,2\}), (\{2,3\},\{2,3\}), (\{2,3\},\{1,2,3\}) \\ &(\{1,2,3\},\{2\}), (\{1,2,3\},\{1,2\}), (\{1,2,3\},\{2,3\}), (\{1,2,3\},\{1,2,3\})\}\end{aligned}$$

Reflexivity Yes. All elements are related to themselves.

Symmetry Yes. All elements have their mirror image.

Transitivity Yes.

Equivalence relation R is an equivalence relation.

Equivalence classes
$$\begin{aligned}[X] &= \{Y \mid (X,Y) \in R\} \\ [\emptyset] &= \{\emptyset, \{1\}, \{3\}, \{1,3\}\} \\ [\{2\}] &= \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}\end{aligned}$$

3. **Let R be the relation on $\mathbb{Z}$ such that $(x, y) \in R$ iff $x - y$ is a multiple of 4.**
$R = (x, y) \in \mathbb{Z}$ such that $x - y = 4k$, where $k \in \mathbb{Z}$.

(a) **Do tests on R for all the following properties: reflexivity, irreflexivity, symmetry, antisymmetry, transitivity, and trichotomy.**

Reflexivity For every $x \in \mathbb{Z}$, is $(x, x) \in R$? Yes.

Proof. For all x, you would have $x - x = 4k$, that is $0 = 4k$, so $k = 0$. $(x, x) \in R$. ∎

Irreflexivity For every $x \in \mathbb{Z}$, is $(x, x) \notin R$? No.

Counterexample. $(1, 1) \in R$. ∎

Symmetry For $x, y \in \mathbb{Z}$, if $(x, y) \in R$, is $(y, x) \in R$? Yes.

Proof. Suppose $(x, y) \in R$. Then $x - y = 4k$, where $k \in \mathbb{Z}$. But $y - x = -4k$. That is, $y - x = 4(-k)$. But k can be any integer. Therefore, $(y, x) \in R$. ∎

Antisymmetry For $x, y \in \mathbb{Z}$, if $x \neq y$ and $(x, y) \in R$, is $(y, x) \notin R$? No.

Counterexample. $(8, 4) \in R$ and $(4, 8) \in R$ ∎

Transitivity For $x, y, z \in \mathbb{Z}$, if $(x, y) \in R$ and $(y, z) \in R$, is $(x, z) \in R$? Yes.

Proof. Suppose $(x, y) \in R$ and $(y, z) \in R$. Then $x - y = 4k$ for some $k \in \mathbb{Z}$, and $y - z = 4m$ for some $m \in \mathbb{Z}$.

$$\begin{aligned}
& y - z = 4m \\
\Rightarrow\quad & y = 4m + z \\
& x - y = 4k \\
\Rightarrow\quad & x - (4m + z) = 4k \\
\Rightarrow\quad & x - 4m - z = 4k \\
\Rightarrow\quad & x - z = 4k + 4m \\
\Rightarrow\quad & x - z = 4(k + m)
\end{aligned}$$

$\therefore (x, z) \in R$ ∎

Trichotomy Is every element in $\mathbb{Z}$ related to every other element in $\mathbb{Z}$? No.

Counterexample. There is no element of R where 1 is related to 2. $(1, 2) \notin R$ and $(2, 1) \notin R$. ∎

(b) **What kind of relation is R?**
R is an equivalence relation.

(c) If R is an equivalence relation, give the equivalence classes of R and show some members of each class.

$$
\begin{aligned}
[x] &= \{y \mid (x,y) \in R\} \\
&= \{y \mid x - y = 4k\} \\
&= \{y \mid y = x - 4k\} \\
[0] &= \{y = -4k\} \\
&= \{\ldots, 8, 4, 0, -4, -8, \ldots\} \\
&= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\
[1] &= \{1 - 4k\} \\
&= \{\ldots, 9, 5, 1, -3, -7, \ldots\} \\
&= \{\ldots, -7, -3, 1, 5, 9, \ldots\} \\
[2] &= \{2 - 4k\} \\
&= \{\ldots, 10, 6, 2, -2, -6, \ldots\} \\
&= \{\ldots, -6, -2, 2, 6, 10, \ldots\} \\
[3] &= \{3 - 4k\} \\
&= \{\ldots, 11, 7, 3, -1, -5, \ldots\} \\
&= \{\ldots, -5, -1, 3, 7, 11, \ldots\}
\end{aligned}
$$

The equivalence classes for [4] and up have already been covered.

4. **Suppose $\mathbb{Q}^+$ is the set of all positive quotients $\frac{m}{n}$, where $m, n \in \mathbb{Z}^+$. That is, $\mathbb{Q}^+$ is the set of positive rational numbers. Let R be the relation on $\mathbb{Q}^+$ defined by the rule $(x, y) \in R$ iff $y = \frac{a}{b}(x)$ for some $a, b \in \mathbb{Z}^+$. Prove that R is an equivalence relation, and show the equivalence classes of R.**

Some examples of elements of R: $\left(\frac{1}{2}, \frac{3}{5}\right), \left(\frac{3}{4}, \frac{5}{6}\right)$

$$\frac{3}{5} = \frac{a}{b}\left(\frac{1}{2}\right) = \left(\frac{6}{5}\right)\left(\frac{1}{2}\right) = \frac{6}{10} \qquad a = 6 \text{ and } b = 5$$

$$\frac{5}{6} = \frac{a}{b}\left(\frac{3}{4}\right) = \left(\frac{10}{9}\right)\left(\frac{3}{4}\right) = \frac{30}{36} \qquad a = 10 \text{ and } b = 9$$

Proof.

Reflexivity For every $x \in \mathbb{Q}^+$, is $(x, x) \in R$? Yes.

Subproof. For (x, x) to be in R, it needs to satisfy:

$$
\begin{aligned}
x &= \frac{a}{b}(x) \text{ for some } a, b \in \mathbb{Z}^+ \\
&= \frac{1}{1}(x) && a = 1 \text{ and } b = 1 \\
&= x
\end{aligned}
$$

$\therefore (x, x) \in R$, so R is reflexive. □

Symmetry For every $x, y \in \mathbb{Q}^+$, if $(x, y) \in R$, is $(y, x) \in R$? Yes.

Subproof. Suppose $(x, y) \in R$. Then $y = \frac{a}{b}(x)$.

$$\begin{aligned} y &= \frac{a}{b}(x) \\ \Rightarrow \quad by &= ax \\ \Rightarrow \quad \frac{b}{a}(y) &= a \\ \Rightarrow \quad x &= \frac{b}{a}(y) \end{aligned}$$

$\therefore$ $(y, x) \in R$, so R is symmetric. □

Transitivity For every $x, y, z \in \mathbb{Q}^+$, if $(x, y) \in R$, and $(y, z) \in R$, is $(x, z) \in R$? Yes.

Subproof. Suppose $(x, y) \in R$ and $(y, z) \in R$.
Then $y = \frac{a}{b}(x)$ and $z = \frac{c}{d}(y)$, where $a, b, c, d \in \mathbb{Z}^+$.

$$\begin{aligned} z &= \frac{c}{d}(y) \\ &= \frac{c}{d}\left(\frac{a}{b}\right)(x) \\ &= \frac{ab}{cd}(x) \end{aligned}$$

$\therefore$ $(x, z) \in R$, so R is transitive. □

$\therefore$ R is an equivalence relation. ■

Equivalence classes $[x] = \{y \mid (x, y) \in R\}$ for all $x \in \mathbb{Q}^+$

$$\begin{aligned} [x] &= \left\{y \mid y = \frac{a}{b}(x)\right\} \\ [1] &= \left\{y \mid y = \frac{a}{b}(1)\right\} \\ &= \left\{y \mid y = \frac{a}{b}\right\} \end{aligned}$$

This is the only equivalence class, as every equivalence class is equal to every other equivalence class.

5. Prove that if R is a relation on $\mathbb{Z}^+$, then R is symmetric iff $R = R^{-1}$.

Proof.

(i) If R is symmetric, then $R = R^{-1}$.

Proof. Assume R is symmetric on $\mathbb{Z}^+$.

Suppose $(x, y) \in R$.
Then $(y, x) \in R$ because R is symmetric.
Then $(x, y) \in R^{-1}$ by the definition of an inverse relation.
So $R \subseteq R^{-1}$.

Suppose $(x, y) \in R^{-1}$.
Then $(y, x) \in R$ by the definition of an inverse relation.
Then $(x, y) \in R$ because R is symmetric.
So $R^{-1} \subseteq R$.

As $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$, $R = R^{-1}$ □

(ii) If $R = R^{-1}$, then R is symmetric.

Proof. Assume $R = R^{-1}$.

Suppose $(x, y) \in R$.
Then $(y, x) \in R^{-1}$ by the definition of an inverse relation.
So $(y, x) \in R$ because $R = R^{-1}$
So R is symmetric. □

If R is symmetric, then $R = R^{-1}$, and if $R = R^{-1}$, then R is symmetric.
$\therefore$ R is symmetric iff $R = R^{-1}$. ■

Theorem 6.1

(i) If R is an equivalence relation to A, then $x \in [x]$ for each $x \in A$.
In other words, every member of A belongs to an equivalence class with respect to R.

(ii) If $x\,R\,y$, then $[x] = [y]$. In other words, if two elements are equivalent with respect to R, they belong to the same equivalence class.

(iii) If $[x] = [y]$, then $x\,R\,y$.

(iv) Either $[x] = [y]$ or $[x] \cap [y] = \emptyset$

6.2.2 Partitions

Partition

For a non-empty set A, a **partition** of A is a set $S = \{S_1, S_2, S_3\}$. The members of S are subsets of A (called *parts* of A) such that:

1. For all i, $S_i \neq \emptyset$. That is, every part of the partition is not empty.
2. For all i and j, if $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$. That is, different parts of the partition don't have common elements.
3. $S_1 \cup S_2 \cup S_3 \cup \ldots = A$. That is, every element of A appears in one (and only one) part of the partition.

Example Let $A = \{5, 6, 7\}$. Then A can be split into two subsets, $\{5\}$ and $\{6, 7\}$. Then $\big\{\{5\}, \{6, 7\}\big\}$ is a partition of A, as:

1. Neither of the subsets is empty.
2. There are no common elements between the subsets.
3. The union of the subsets results in A.

Going Backwards From a Partition

If one knows the original set that was partitioned, and the partitions, one can generate the original relation, using **Theorem 6.1**

Example Given an original set $A = \{a, b, c\}$ and a partition given by $\big\{\{a\}\{b, c\}\big\}$.

The subset $\{a\}$ tells us that $[a] = \{a\}$, i.e. $(a, a) \in R$.
The subset $\{b, c\}$ tells us that $[b] = \{b, c\} = [c]$, which means that b is related to b and c, and c is related to b and c, so the pairs (b, b), (b, c), (c, b) and (c, c) are in R.

$R = \big\{(a, a), (b, b), (b, c), (c, b), (c, c)\big\}$

Self Assessment Exercise 6.12

1. **Determine whether P is a partition of X in each of the following cases. If it is, describe the corresponding equivalence relation.**

   (a) $X = \{1, 2, 3\}$ **and** $P = \{\emptyset, \{1\}, \{2, 3\}\}$.
   P is *not* a partition of X, as $\emptyset$ is one of the elements of the set.

   (b) $X = \{1, 2, 3\}$ **and** $P = \{\{1\}, \{2\}, \{1, 3\}\}$.
   P is *not* a partition of X, as 1 appears in two different elements.

   (c) $X = \{1, 2, 3\}$ **and** $P = \{\{1, 3\}, \{2\}\}$.
   P *is* a partition of X.
   The part $\{2\}$ means that $(2, 2) \in R$.
   The part $\{1, 3\}$ means that $(1, 1)$, $(3, 3)$, $(1, 3)$ and $(3, 3)$ are elements of R.
   $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$

   (d) $X = \{1, 2, 3\}$ **and** $P = \{\{1\}, \{2\}\}$
   P is *not* a partition of X, as not all the members of X are included.

   (e) $X = \mathbb{Z}$ **and** $P = \{\{0\}, \mathbb{Z}^+, \text{Neg}\}$ **where** $\text{Neg} = \{x \mid x \in \mathbb{Z} \text{ and } x < 0\}$.
   P *is* a partition of X.
   The equivalence relation is:

   $$R = \{(x, y) \mid (x = 0 \text{ and } y = 0) \text{ or } (x \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}^+) \text{ or } (x \in \text{Neg and } y \in \text{Neg})\}$$

   (f) $X = \mathbb{Z}$ **and** $P = \{[0], [1], [2], [3], [4]\}$, **where** $[n] = \{x \mid x - n$ **is divisible by 5 with zero remainder**$\}$ **and** $n \in \{0, 1, 2, 3, 4\}$.
   P *is* a partition of X.
   The equivalence relation is:

   $$R = \{(x, y) \mid x - y = 5k \text{ for some } k \in \mathbb{Z}\}$$

6.3 Functions

6.3.1 Functional Relation

Functional Relation

If R is a relation from X to Y, then R is **functional** iff any element x in X only appears once as a first coordinate in an ordered pair of R.

Example Let S be a relation from $\{1,2,3\}$ to $\{a,b,c\}$, where $S = \big\{(1,a),(2,c)\big\}$. S is a functional relation as 1 and 2 only appear as first coordinates in distinct pairs.

6.3.2 Function

Function

Suppose $R \subseteq A \times B$ is a binary relation from a set A to a set B. R is a **function** from A to B if R is functional, and the domain of R is exactly the set A, i.e. $\text{dom}(R) = A$.
This is then written $R : A \longrightarrow B$.

Example Using the same relation as above:

S is a relation from $\{1,2,3\}$ to $\{a,b,c\}$, where $S = \big\{(1,a),(2,c)\big\}$

S is functional, but not a function, as $\text{dom}(S) \neq \{1,2,3\}$.

Example Prove that f defined by $(x,y) \in f$ iff $y = 5x^2 + 3$ is a function on $\mathbb{R}$.
To prove this, determine whether f is functional, and whether $\text{dom}(f) = \mathbb{R}$.

Proof.

(i) f is functional.

Proof. Suppose $(x,y) \in f$ and $(x,z) \in f$. Is it the case that $y = z$?
As $(x,y) \in f$, $y = 5x^2 + 3$. As $(x,z) \in f$, $z = 5x^2 + 3$.
Therefore, $y = 5x^2 + 3 = z$.
So f is functional. □

(ii) $\text{dom}(f) = \mathbb{R}$

Proof.
$$\begin{aligned}\text{dom}(f) &= \{x \mid \text{ for some } y \in \mathbb{R}, (x,y) \in f\} \\ &= \{x \mid \text{ for some} y \in \mathbb{R}, y = 5x^2 + 3\} \\ &= \{x \mid 5x^2 + 3 \text{ is a real number}\} \\ &= \mathbb{R}\end{aligned}$$
Therefore the domain is equal to the input set. □

As f is functional, and the domain of f is the same as the input set, f is a function. ■

Not all functional relations are functions!

Every function is a functional relation, but a relation can be functional without being a function. This just means that the domain of the relation is not the same as the input set.
If anything from the original set can be given to the relation to produce an output, it is a function.

Self Assessment Exercise 6.14

1. **Give 5 functions from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c\}$.**

$$f_1 = \{(1, a), (2, a), (3, a), (4, a)\}$$
$$f_2 = \{(1, b), (2, b), (3, b), (4, b)\}$$
$$f_3 = \{(1, c), (2, c), (3, c), (4, c)\}$$
$$f_4 = \{(1, a), (2, b), (3, c), (4, b)\}$$
$$f_5 = \{(1, b), (2, a), (3, b), (4, a)\}$$

2. **Give all the functions from $A = \{a, b\}$ to $B = \{5, 6, 7\}$.**

$$f_1 = \{(a, 5), (b, 5)\} \quad f_4 = \{(a, 6), (b, 5)\} \quad f_7 = \{(a, 7), (b, 5)\}$$
$$f_2 = \{(a, 5), (b, 6)\} \quad f_5 = \{(a, 6), (b, 6)\} \quad f_8 = \{(a, 7), (b, 6)\}$$
$$f_3 = \{(a, 5), (b, 7)\} \quad f_6 = \{(a, 6), (b, 7)\} \quad f_9 = \{(a, 7), (b, 7)\}$$

3. **Give 3 functions from $A \times A$ to B if $A = \{a, b\}$ and $B = \{5, 6, 7\}$.**

$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$f_1 = \Big\{\big((a, a), 5\big), \big((a, b), 5\big), \big((b, a), 5\big), \big((b, b), 5\big)\Big\}$$
$$f_2 = \Big\{\big((a, a), 6\big), \big((a, b), 6\big), \big((b, a), 6\big), \big((b, b), 6\big)\Big\}$$
$$f_3 = \Big\{\big((a, a), 5\big), \big((a, b), 6\big), \big((b, a), 7\big), \big((b, b), 6\big)\Big\}$$

4. **Let R be a relation on $A = \{1, 2, 3, \{1\}, \{2\}\}$ defined by $R = \Big\{\big(1, \{1\}\big), (1, 3), \big(2, \{1\}\big), \big(2, \{2\}\big), \big(\{1\}, 3\big), \big(\{2\}, \{1\}\big)\Big\}$.**

   (a) **Is R a function from A to A?**
   No. There are two elements with the same first coordinate: $\big(1, \{1\}\big)$ and $(1, 3)$, so R is not a functional relation, so R is not a function.

   (b) **Is $\operatorname{ran}(R)$ equal to the codomain of R?**
   No. $1 \in$ codomain, but $1 \notin \operatorname{ran}(R)$.

5. Consider the set $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. Show that the relations f, g and h described below are functional and have as domains $\mathcal{P}(A)$, $\mathcal{P}(A) \times \mathcal{P}(A)$, and $\mathcal{P}(A) \times \mathcal{P}(A)$ respectively.

(a) Let $f = \{(x,y) \mid x,y \in \mathcal{P}(A) \text{ and } y = x'\}$.

Functional f is functional.

Proof. Suppose $(x,y) \in f$ and $(x,z) \in f$. (f is functional iff $y = z$.)
Then $y = x'$ and $z = x'$.
So $y = x' = z$.
So f is functional. ■

Domain The domain of f is equal to the input set $\mathcal{P}(A)$.

Proof.
$$\begin{aligned} \mathrm{dom}(f) &= \{x \mid \text{for some } y \in \mathcal{P}(A), (x,y) \in f\} \\ &= \{x \mid \text{for some } y \in \mathcal{P}(A), y = x'\} \\ &= \{x \mid x' \in \mathcal{P}(A)\} \\ &= \mathcal{P}(A) \end{aligned}$$
Therefore $\mathrm{dom}(f)$ is equal to the input set. ■

(b) Let $g = \Big\{\big((u,v),y\big) \mid (u,v) \in \mathcal{P}(A) \times \mathcal{P}(A) \text{ and } y = u \cup v\Big\}$.

Functional g is functional.

Proof. Suppose $\big((u,v),y\big) \in g$ and $\big((u,v),z\big) \in g$. (g is functional iff $y = z$.)
Then $y = u \cup v$ and $z = u \cup v$.
So $y = u \cup v = z$.
So g is functional. ■

Domain The domain of g is equal to the input set $\mathcal{P}(A) \times \mathcal{P}(A)$.

Proof.
$$\begin{aligned} \mathrm{dom}(g) &= \Big\{(u,v) \mid \text{for some } y \in \mathcal{P}(A), \big((u,v),y\big) \in g\Big\} \\ &= \Big\{(u,v) \mid \text{for some } y \in \mathcal{P}(A), y = u \cup v \in g\Big\} \\ &= \Big\{(u,v) \mid u \cup v \in \mathcal{P}(A)\Big\} \\ &= \Big\{u \in \mathcal{P}(A) \text{ and } v \in \mathcal{P}(A)\Big\} \\ &= \mathcal{P}(A) \times \mathcal{P}(A) \end{aligned}$$
Therefore $\mathrm{dom}(g)$ is equal to the input set. ■

(c) Let $h = \Big\{\big((u,v),y\big) \mid (u,v) \in \mathcal{P}(A) \times \mathcal{P}(A) \text{ and } y = u \cap v\Big\}$.

Functional h is functional.

Proof. Suppose $\big((u,v),y\big) \in h$ and $\big((u,v),z\big) \in h$. (h is functional iff $y = z$.)
Then $y = u \cap v$ and $z = u \cap v$.
So $y = u \cap v = z$.
So h is functional. ■

Domain The domain of h is equal to the input set $\mathcal{P}(A) \times \mathcal{P}(A)$.

Proof.
$$\begin{aligned}\text{dom}(h) &= \Big\{(u,v) \mid \text{for some } y \in \mathcal{P}(A), \big((u,v),y\big) \in h\Big\}\\ &= \Big\{(u,v) \mid \text{for some } y \in \mathcal{P}(A), y = u \cap v \in h\Big\}\\ &= \Big\{(u,v) \mid u \cap v \in \mathcal{P}(A)\Big\}\\ &= \Big\{u \in \mathcal{P}(A) \text{ and } v \in \mathcal{P}(A)\Big\}\\ &= \mathcal{P}(A) \times \mathcal{P}(A)\end{aligned}$$
Therefore $\text{dom}(h)$ is equal to the input set. ■

6. For each of the following relations from X to Y, determine whether the relation may be regarded as a function from X to Y.

(a) $X = Y = \mathbb{Z}$ and $R = \{(x,y) \mid y = x\}$.
R is a function.

Proof.

Functional R is functional.

Subproof. Suppose $(x,y) \in R$ and $(x,z) \in R$.
Then $y = x$ and $z = x$.
So $y = x = z$.
So $y = z$.
$\therefore$ R is functional. □

Domain The domain of R is equal to the input set: $\mathbb{Z}$.

Subproof.
$$\begin{aligned}\text{dom}(R) &= \{x \mid \text{for some } y \in \mathbb{Z}, (x,y) \in R\}\\ &= \{x \mid \text{for some } y \in \mathbb{Z}, y = x\}\\ &= \{x \mid x \in \mathbb{Z}\}\\ &= \mathbb{Z}\end{aligned}$$
Therefore $\text{dom}(R)$ is equal to the input set. □

As R is functional, and the domain of R is equal to the input set, R is a function. ■

(b) $\boldsymbol{X = Y = \mathbb{Z}}$ **and** $\boldsymbol{R = \{(x, y) \mid y = x + 1\}}$**.**
R is a function.

Proof.

Functional R is functional.

Subproof. Suppose $(x, y) \in R$ and $(x, z) \in R$.
Then $y = x + 1$ and $z = x + 1$.
So $y = x + 1 = z$.
So $y = z$.
$\therefore R$ is functional. □

Domain The domain of R is equal to the input set: $\mathbb{Z}$.

Subproof.
$$\begin{aligned} \operatorname{dom}(R) &= \{x \mid \text{for some } y \in \mathbb{Z}, (x, y) \in R\} \\ &= \{x \mid \text{for some } y \in \mathbb{Z}, y = x + 1\} \\ &= \{x \mid x + 1 \in \mathbb{Z}\} \\ &= \mathbb{Z} \end{aligned}$$
Therefore $\operatorname{dom}(R)$ is equal to the input set. □

As R is functional, and the domain of R is equal to the input set, R is a function. ■

(c) $\boldsymbol{X = Y = \mathbb{Z}}$ **and** $\boldsymbol{R = \{(x, y) \mid y = 3 - x\}}$**.**
R is a function.

Proof.

Functional R is functional.

Subproof. Suppose $(x, y) \in R$ and $(x, z) \in R$.
Then $y = 3 - x$ and $z = 3 - x$.
So $y = 3 - x = z$.
So $y = z$.
$\therefore R$ is functional. □

Domain The domain of R is equal to the input set: $\mathbb{Z}$.

Subproof.
$$\begin{aligned} \operatorname{dom}(R) &= \{x \mid \text{for some } y \in \mathbb{Z}, (x, y) \in R\} \\ &= \{x \mid \text{for some } y \in \mathbb{Z}, y = 3 - x\} \\ &= \{x \mid 3 - x \in \mathbb{Z}\} \\ &= \mathbb{Z} \end{aligned}$$
Therefore $\operatorname{dom}(R)$ is equal to the input set. □

As R is functional, and the domain of R is equal to the input set, R is a function. ■

(d) $X = Y = \mathbb{Z}$ **and** $R = \{(x, y) \mid y = \sqrt{x}\}$**. (That is, the positive square root of** x**.)**
R is *not* a function.

Proof.

Functional R is functional.

Subproof. Suppose $(x, y) \in R$ and $(x, z) \in R$.
Then $y = \sqrt{x}$ and $z = \sqrt{x}$.
So $y = \sqrt{x} = z$
So $y = z$.
So R is functional. □

Domain The domain of R is *not* equal to the input set.

Counterexample. $2 \in X$, but there is no integer y (i.e. no $y \in Y$) where $y = \sqrt{2}$, because $\sqrt{2}$ is irrational.
$-1 \in X$, but there is no integer y where $y = \sqrt{-1}$.
$\therefore$ $\text{dom}(R) \neq X$ □

As the domain of R is not equal to the input set, R is not a function. ■

(e) $X = Y = \mathbb{Z}$ **and** $R = \{(x, y) \mid y^2 = x\}$**.**
R is not a function.

Proof.

Functional R is *not* functional.

Subproof. Suppose $(x, y) \in R$ and $(x, z) \in R$.
Then $y^2 = x$ and $z^2 = x$.
So $y = \pm\sqrt{x}$ and $z = \pm\sqrt{x}$.
As y can equal $\sqrt{x}$ and z can equal $-\sqrt{x}$, $y \neq z$
So R is not functional. □

Domain The domain of R is *not* equal to the input set.

Counterexample. $2 \in X$, but there is no integer y (i.e. no y in Y) where $y^2 = 2$, because $\sqrt{2}$ is irrational.
$-1 \in X$, but there is no integer y where $y^2 = -1$.
$\therefore$ $\text{dom}(R) \neq X$ □

As R is not functional, and the domain of R is not equal to the input set, R is not a function. ■

(f) $X = Y = \mathbb{R}$ **and** $S = \{(x, y) \mid x^2 + y^2 = 1\}$**.**
R is not a function.

Proof.

Functional R is *not* functional.

Subproof. Suppose $(x, y) \in S$ and $(x, z) \in S$.
Then $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
So $y^2 = 1 - x^2$ and $z^2 = 1 - x^2$.
So $y = \pm\sqrt{1 - x^2}$ and $z = \pm\sqrt{1 - x^2}$.
As y can be $\sqrt{1 - x^2}$ and z can be $-\sqrt{1 - x^2}$, $y \neq z$.
So R is not functional. □

Domain The domain of S is *not* equal to the input set.

Counterexample. $2 \in \mathbb{R}$, but there is no real number y where $2^2 + y^2 = 1$.
$\therefore \operatorname{dom}(R) \neq X$ □

As R is not functional, and the domain of R is not equal to the input set, R is not a function. ■

7. **Is the relation R on $\mathbb{Z}^+$, which consists of all pairs (x, y) such that $y = x - 1$, a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$?**
No.

Proof.

Functional R is functional.

Subproof. Suppose $(x, y) \in R$ and $(x, z) \in R$.
Then $y = x - 1$ and $z = x - 1$.
So $y = x - 1 = z$.
So $y = z$.
So R is functional. □

Domain The domain of R is *not* equal to the input set.

Subproof.
$$\begin{aligned}\operatorname{dom}(R) &= \{x \mid \text{for some } y \in \mathbb{Z}^+, (x, y) \in R\} \\ &= \{x \mid \text{for some } y \in \mathbb{Z}^+, y = x - 1\} \\ &= \{x \mid x - 1 \in \mathbb{Z}^+\} \\ &= \{x > 1 \mid x \in \mathbb{Z}^+\} \\ &\neq \mathbb{Z}^+\end{aligned}$$
For example, $y = 1 - 1 = 0$ cannot be an element of R if the domain is $\mathbb{Z}^+$ □

As the domain of R is not equal to the input set, R is not a function. ■

8. **Let $A = \{a, b, c\}$. Consider all the equivalence relations on A. How many relations are also functions from A to A?**

   Equivalence Relations

$$
\begin{aligned}
R_1 &= \big\{(a,a),(b,b),(c,c)\big\} \\
R_2 &= \big\{(a,a),(b,b),(c,c),(a,b),(b,a)\big\} \\
R_3 &= \big\{(a,a),(b,b),(c,c),(a,c),(c,a)\big\} \\
R_4 &= \big\{(a,a),(b,b),(c,c),(b,c),(c,b)\big\} \\
R_5 &= \big\{(a,a),(b,b),(c,c),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\big\}
\end{aligned}
$$

   Functions Only R_1 is a function.

   Abstract Reasoning If R is an equivalence relation, then R is reflexive. So $\text{dom}(R) = A$.
   But if R is an equivalence relation where a first coordinate appears more than once, R is not a function.
   So the first coordinates need to only appear once.
   In an equivalence relation, that means that the only function is the identity relation.
   So the only function is $\big\{(a,a),(b,b),(c,c)\big\}$

9. **Let $A = \{a, b, c\}$.**

   (a) **How many weak partial orders on A are also functions from A to A?**
   If S is a weak partial order on A, then S is reflexive. So $\text{dom}(S) = A$.
   That means that every element of A appears as the first coordinate in at least one pair.
   For S to be functional, each element of A must only appear as the first coordinate in one pair.
   The only case for this is the identity relation.
   So the only weak partial order on A that is a function is $\big\{(a,a),(b,b),(c,c)\big\}$.

   (b) **How many strict partial orders on A are also functions from A to A?**
   For a strict partial order T to be a function on A, the domain of T needs to be A, and T needs to be functional.
   Each element of A should appear as the first coordinate in exactly one pair. For the relation to be a strict partial order, it needs to be antisymmetric, irreflexive and transitive.
   There is no combination of pairs that satisfies all three requirements for a strict partial order that is also a function.

Unit 7

More About Functions

7.1 The Range of a Function

Range of a Function

Given a function $f : A \to B$, the **range** or **image set** of f is the subset

$$\{f(x) \mid x \in A\}$$

of B, written $\operatorname{ran}(f)$ or $f[A]$.
In other words, it is a subset of B where and element b of B can be reached by calling the function with a specific element a of A.

Example Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Let a function $f : A \to B$ be defined as

$$f(a) = 1 \qquad f(b) = 2 \qquad f(c) = 1$$

Then $\operatorname{ran}(f) = \{1, 2\}$.

7.1.1 Determining the Range of a Function

In order to find the range, you follow these steps:

1. Write the definition of the function $\big(f(x)\big)$, and the domain.
2. Substitute the definition with the value of $f(x)$
3. Calculate the first coordinate in terms of the second.
4. Substitute the second coordinate and that formula for it into the definition.
5. Simplify.

Easier to show with an example:

Example Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $y = 2x$.

$$\begin{aligned} \text{ran}(g) &= \{g(x) \mid x \in \mathbb{Z}\} && (1) \\ &= \{2x \mid x \in \mathbb{Z}\} && (2) \end{aligned}$$

Calculate x in terms of y (Step 3):

$$\begin{aligned} & y = 2x \\ \Rightarrow \quad & \frac{y}{2} = x \\ \Rightarrow \quad & x = \frac{y}{2} \end{aligned}$$

$$\begin{aligned} \text{ran}(g) &= \{y \mid \frac{y}{2} \in \mathbb{Z}\} && (4) \\ &= \{y \mid \frac{y}{2} \text{ is an integer}\} && (5) \\ &= \{y \mid y \text{ is an even integer}\} \end{aligned}$$

7.2 Surjectivity (MAPPING)

Surjectivity

Given a function $f : A \to B$, the function f would be **surjective** iff the *range* of f is equal to the codomain of f.
As B is the codomain of f above, that would mean that $\text{ran}(f)$ (also written $f[A]$) is equal to B.

Example Let $A = \{1, 2, 3\}$. Let $B = \{4, 5, 6\}$.

Surjective Function For a surjective function, every element of A needs to be present, and every element of B. So an example of a function $h : A \to B$ would be:

$$h = \Big\{(1, 6), (2, 4), (3, 5)\Big\}$$

$\text{ran}(h) = \{4, 5, 6\} = B$.

Non-Surjective Function For a function, every element of A needs to be present. For it to not be surjective, that means that at least one element of B is not in the range of the function. An example function $h : A \to B$ would be:

$$h = \Big\{(1, 4), (2, 4), (3, 5)\Big\}$$

$\text{ran}(h) = \{4, 5\} \neq B$.

Self Assessment Exercise 7.4

1. **In each of the following cases, write down the possible surjective functions from X to Y.**

   (a) **$X = \{a, b\}$ and $Y = \{c\}$.**
   For a surjective function, make sure each element of X appears as a first coordinate, and every element of Y is used.

   $$f_1 = \{(a,c),(b,c)\}$$

   (b) **$X = \{a, b\}$ and $Y = \{c, d\}$.**

   $$f_1 = \{(a,c),(b,d)\}$$
   $$f_2 = \{(a,d),(b,c)\}$$

   (c) **$X = \{a, b\}$ and $Y = \{c, d, e\}$.**
   There are no possible surjective functions, as there are more y elements than x elements. Either an x element appears twice, in which case it is not a function, or a y element doesn't appear, in which case it is not surjective.

2. **Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$.**

   (a) **Determine $f[\mathbb{Z}]$ (or $\operatorname{ran}(f)$).**

   $$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{x+1 \mid x \in \mathbb{Z}\} && (y = x+1 \Rightarrow x = y-1) \\ &= \{y \mid y-1 \in \mathbb{Z}\} \\ &= \{y \mid y-1 \text{ is an integer}\} \\ &= \mathbb{Z} \end{aligned}$$

   (b) **Is f surjective? If f is not surjective, provide a counterexample to show why it is not surjective.**
   f is surjective, as $f[\mathbb{Z}] = \mathbb{Z}$.

3. **Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 4x + 8$.**

   (a) **Determine $f[\mathbb{Z}]$ (or $\operatorname{ran}(f)$).**

   $$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{4x+8 \mid x \in \mathbb{Z}\} && (y = 4x+8 \Rightarrow x = \tfrac{y}{4} - 2) \\ &= \{y \mid \frac{y}{4} - 2 \in \mathbb{Z}\} \\ &= \{y \mid \frac{y}{4} \text{ is an integer}\} \\ &= \{y \mid y \text{ is an integer divisible by } 4\} \end{aligned}$$

   (b) **Is f surjective? If f is not surjective, provide a counterexample to show why it is not surjective.**
   f is not surjective, as the range of f is not equal to the codomain.

   Counterexample. $3 \in \mathbb{Z}$, which is the codomain, but there is no $x \in \mathbb{Z}$ such that $4x + 8 = 3$. ■

7.3 Injectivity (ONE TO ONE)

Injectivity

A function $f : A \to B$ is **injective** iff f has the property that whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$.
In other words, every unique first coordinate is related to a unique second coordinate.

Another definition (contrapositive) A function $f : A \to B$ is **injective** iff f has the property that whenever $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$.

Example Let $A = \{1, 2, 3\}$. Let $B = \{4, 5, 6, 7\}$.

Injective Function For an injective function, every element of A should be related to a different element of B. An example function $g : A \to B$ would be:

$$g = \Big\{(1,5), (2,7), (3,6)\Big\}$$

Non-Injective Function For a function to not be injective, two or more elements of A should be related to the same element of B. An example function $g : A \to B$ would be:

$$g = \Big\{(1,4), (2,5), (3,4)\Big\}$$

7.3.1 Determining Whether an Abstract Function is Injective

For functions defined on all elements of an infinite set such as $\mathbb{Z}$, use logic to prove the function is injective:

1. Assume that the function being applied to two different elements results in the same value.
2. Apply the function to the values.
3. Simplify using algebra.

Example

Prove Injectivity Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $y = 4x$.

$$\begin{aligned} &\text{Assume} & f(u) &= f(v) && (1)\\ &\text{Then} & 4u &= 4v && (2)\\ &\text{i.e.} & u &= v && (3) \end{aligned}$$

Non-Injective Function Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $y = x^2$.

$$\begin{aligned} &\text{Assume} & f(u) &= f(v)\\ &\text{Then} & u^2 &= v^2\\ & & \pm u &= \pm v\\ & & u &\neq v \end{aligned}$$

u is not necessarily equal to v. u could be 1, and v could be -1, and $f(u)$ would be equal to $f(v) = 1$. Therefore, f is not injective.

Self Assessment Exercise 7.5

1. **In each of the following cases, write down the injective functions from X to Y.**

   (a) $X = \{2, 4\}$ **and** $Y = \{1\}$
   There is no possible injective function, as Y has only one member, but X has two members. Either one of the members of X is excluded, in which case it is not a function, or the two members point to the same member of Y, in which case it is not injective.

   (b) $X = \{2, 4\}$ **and** $Y = \{1, 3\}$

$$f_1 = \big\{(2, 1), (4, 3)\big\}$$
$$f_2 = \big\{(2, 3), (4, 1)\big\}$$

   (c) $X = \{2, 4\}$ **and** $Y = \{1, 3, 5\}$

$$f_1 = \big\{(2, 1), (4, 3)\big\}$$
$$f_2 = \big\{(2, 3), (4, 1)\big\}$$
$$f_3 = \big\{(2, 1), (4, 5)\big\}$$
$$f_4 = \big\{(2, 3), (4, 5)\big\}$$
$$f_5 = \big\{(2, 5), (4, 1)\big\}$$
$$f_6 = \big\{(2, 5), (4, 3)\big\}$$

2. **Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = 2x - 5$. Determine whether h is injective.**
   h is injective.

   Proof. Assume $h(u) = h(v)$
   Then

$$\begin{aligned} 2u - 5 &= 2v - 5 \\ 2u &= 2v \\ u &= v \end{aligned}$$

   $\therefore$ h is injective, because when $h(u) = h(v)$, $u = v$. ∎

7.4 Composition of Functions

The composition of two functions is also a function

For any two functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the composition of the two functions $g \circ f : A \longrightarrow C$ is also a function.

Proof. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions.
As f is a function, for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.
As g is a function, for every $b \in B$, there is exactly one $c \in C$ such that $(b, c) \in g$.
As there is exactly one pair from a to b, and from b to c, there is exactly one pair in the composite function from a to c. ∎

Composite Function

Given the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the **composite function** $g \circ f : A \longrightarrow C$ is defined by

$$\begin{aligned} g \circ f : A \longrightarrow C &= g \circ f(x) \\ &= g\big(f(x)\big) \end{aligned}$$

Example Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $f(x) = 4x + 2$.
Let $g : \mathbb{Z} \longrightarrow \mathbb{Z}^{+}$ be defined by $f(x) = x^2 + 1$.
Then $g \circ f : \mathbb{Z} \longrightarrow \mathbb{Z}^{+}$.

$$\begin{aligned} (g \circ f)(x) &= g\big(f(x)\big) \\ &= g(4x + 2) \\ &= (4x + 2)^2 + 1 \\ &= \Big(16x^2 + 16x + 4\Big) + 1 \\ &= 16x^2 + 16x + 5 \end{aligned}$$

$(g \circ f)(x)$ is called the **image of x under $g \circ f$**.

The composition of two surjective functions is surjective

For any two surjective functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the composition of the two functions $g \circ f : A \longrightarrow C$ is also a surjective function.

Proof. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two surjective functions.
As g is surjective, every $c \in C$ appears as a second coordinate in g.
As f is surjective, every $b \in B$ appears as a second coordinate in f.
As f is a function, every a appears as a first coordinate in f.
As g is a function, every b appears as a first coordinate in g.
Therefore, every a maps to every b which maps to every c.
So every a maps to every c.
So the composite function is surjective. ∎

The composition of two injective functions is injective

For any two injective functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the composition of the two functions $g \circ f : A \longrightarrow C$ is also an injective function.

Proof. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two injective functions.
As f is injective, every $a \in A$ maps to a unique $b \in B$.
As g is injective, every $b \in B$ maps to a unique $c \in C$.
As every a maps to a unique b, and every b maps to a unique c, in the composite function, every a maps to a unique c.
So the composite function is injective. ■

Self Assessment Exercise 7.9

1. **Determine $f \circ f$, $g \circ g$, $g \circ f$ and $f \circ g$ in each of the following cases.**

   (a) **$f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(x) = x - 1$.**
   All these composite functions are defined on $\mathbb{Z} \longrightarrow \mathbb{Z}$.

$$\begin{aligned}(f \circ f)(x) &= f\big(f(x)\big) \\ &= f(x+1) \\ &= (x+1)+1 \\ &= x+2\end{aligned} \qquad \begin{aligned}(g \circ g)(x) &= g\big(g(x)\big) \\ &= g(x-1) \\ &= (x-1)-1 \\ &= x-2\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) \\ &= g(x+1) \\ &= (x+1)-1 \\ &= x\end{aligned} \qquad \begin{aligned}(f \circ g)(x) &= f\big(g(x)\big) \\ &= f(x-1) \\ &= (x-1)+1 \\ &= x\end{aligned}$$

   (b) **$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 2$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 + x$.**
   All these composite functions are defined on $\mathbb{R} \longrightarrow \mathbb{R}$.

$$\begin{aligned}(f \circ f)(x) &= f\big(f(x)\big) \\ &= f(3x-2) \\ &= 3(3x-2)-2 \\ &= 9x-6-2 \\ &= 9x-8\end{aligned} \qquad \begin{aligned}(g \circ g)(x) &= g\big(g(x)\big) \\ &= g\left(x^2+x\right) \\ &= \left(x^2+x\right)^2 + \left(x^2+x\right) \\ &= x^4+2x^3+x^2+x^2+x \\ &= x^4+2x^3+2x^2+x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) \\ &= g(3x-2) \\ &= (3x-2)^2+(3x-2) \\ &= 9x^2-12x+4+3x-2 \\ &= 9x^2-9x+2\end{aligned} \qquad \begin{aligned}(f \circ g)(x) &= f\big(g(x)\big) \\ &= f\left(x^2+x\right) \\ &= 3\left(x^2+x\right)-2 \\ &= 3x^2+3x-2\end{aligned}$$

(c) **$f : \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $f(x) = 113$ and $g : \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $g(x) = x + 1$.**
All these composite functions are defined on $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$.

$$\begin{aligned}(f \circ f)(x) &= f\big(f(x)\big)\\ &= f(113)\\ &= 113\end{aligned} \qquad \begin{aligned}(g \circ g)(x) &= g\big(g(x)\big)\\ &= g(x+1)\\ &= (x+1)+1\\ &= x+2\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big)\\ &= g(113)\\ &= 113+1\\ &= 114\end{aligned} \qquad \begin{aligned}(f \circ g)(x) &= f\big(g(x)\big)\\ &= f(x+1)\\ &= 113\end{aligned}$$

7.5 Bijective and Invertible Functions

7.5.1 Bijective Function

Bijective Function

A function $f : A \to B$ is **bijective** iff f is both surjective and injective.

Example Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $y = x + 2$.

Surjectivity Determine the range of f.

$$\begin{aligned}\operatorname{ran}(f) &= \{f(x) \mid x \in \mathbb{Z}\}\\ &= \{x+2 \mid x \in \mathbb{Z}\}\\ &= \{y \mid y-2 \in \mathbb{Z}\} && x = y-2\\ &= \{y \mid y \in \mathbb{Z}\}\\ &= \mathbb{Z}\end{aligned}$$

Therefore, f is surjective.

Injectivity

$$\begin{aligned}&\text{Assume} & f(u) &= f(v)\\ &\text{Then} & u+2 &= v+2\\ &\text{i.e.} & u &= v\end{aligned}$$

Therefore f is injective.

Bijectivity As f is both surjective and injective, f is bijective.

7.5.2 Invertible Functions

Invertible Function

A function $f : A \longrightarrow B$ is **invertible** iff the inverse relation f^{-1} is a function from B to A.
This occurs iff the function f is bijective.

A function f is invertible iff f is bijective

Proof.

Subproof. Suppose that $f : A \longrightarrow B$ is an invertible function.
Then $f^{-1} = \{(y, x) \mid (x, y) \in f\}$ is a function from B to A.

So the *domain* of f^{-1} is B. But the domain of f^{-1} is also the set of y's such that $(x, y) \in f$ for some x i.e. the domain of f^{-1} is the *range* of f. So the range of f is B.
So $f : A \longrightarrow B$ is surjective.

As f^{-1} is a function, an element $y \in B$ appears only once as the first coordinate in an ordered pair in f^{-1}. That is, if (y, x_1) and (y, x_2) are both in f^{-1}, then $x_1 = x_2$.
So $f : A \longrightarrow B$ is injective.

If f is an invertible function, then f is surjective and injective, so f is bijective. □

Subproof. Suppose that $f : A \longrightarrow B$ is bijective.

As f is surjective, every element of B appears as the second coordinate in an ordered pair of f. Therefore, every $b \in B$ appears as the first coordinate in an ordered pair of f^{-1}.
Therefore, the domain of f^{-1} is B.

As f is injective, every element of B appears *only once* as the second coordinate in an ordered pair of f. Therefore, every $b \in B$ appears only once in an ordered pair of f^{-1}.
Therefore, f^{-1} is functional.

If f is bijective, then f^{-1} is functional. $\operatorname{dom}(f^{-1})$ equals the codomain of f, so f^{-1} is a function. Therefore, f is invertible. □

If a function is invertible, it is bijective. If a function is bijective, it is invertible. ■

Example Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $y = x + 2$.
As was shown in the previous example, this function is bijective. As it is bijective, it is invertible.
The inverse function of f, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$$
\begin{aligned}
(y, x) \in f^{-1} &\text{ iff } (x, y) \in f \\
&\text{ iff } y = x + 2 \\
(x, y) \in f^{-1} &\text{ iff } x = y + 2 && \text{(swap variables)} \\
&\text{ iff } x - 2 = y && \text{(solve for } y\text{)} \\
&\text{ iff } y = x - 2 \\
(y, x) \in f^{-1} &\text{ iff } x = y - 2 && \text{(swap variables back)}
\end{aligned}
$$

$f^{-1} : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $f^{-1}(y) = y - 2$

7.6 Identity Function

Identity Function

For any set A, the function $i_A : A \longrightarrow A$ is the function such that $i_A(x) = x$ for all $x \in A$. This function is called the **identity function**.

Example Let $B = \{2, 4, 6, 8\}$. The identity function $i_B : B \longrightarrow B$ would be:

$$i_B = \{(2,2), (4,4), (6,6), (8,8)\}$$

Self Assessment Exercise 7.11

1. **In each of the following cases, write down the bijective functions from X to Y (if possible).**

   (a) $X = \{\emptyset, \{113\}\}$ **and** $Y = \{\{1\}\}$.
   There are no possible bijective functions, as there are more elements in X than in Y. That means that there cannot be an injective function, so there cannot be a bijective function.

   (b) $X = \{\emptyset, \{113\}\}$ **and** $Y = \{\{1\}, \{2\}\}$.

$$f_1 = \Big\{(\emptyset, \{1\}), (\{113\}, \{2\})\Big\}$$
$$f_2 = \Big\{(\emptyset, \{2\}), (\{113\}, \{1\})\Big\}$$

   (c) $X = \{\emptyset, \{113\}\}$ **and** $Y = \{\{1\}, \{2\}, \{7\}\}$.
   There are no possible bijective functions, as there are more elements in Y than in X. That means that there cannot be a surjective function, so there cannot be a bijective function.

2. **Check the following functions for injectivity, surjectivity and bijectivity, and give the inverse relation of each:**

   (a) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ **is defined by** $f(x) = x + 1$.

   Injectivity This function is injective.

   Proof. Assume $f(u) = f(v)$
   Then $u + 1 = v + 1$
   i.e. $u = v$
   Therefore f is injective. ∎

Surjectivity This function is surjective.

Proof.
$$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{x+1 \mid x \in \mathbb{Z}\} \\ &= \{y \mid y-1 \in \mathbb{Z}\} \\ &= \{y \mid y \in \mathbb{Z}\} \\ &= \mathbb{Z} \end{aligned}$$
Therefore f is surjective. ■

Bijectivity As f is injective and surjective, f is bijective.

Inverse Function
$$\begin{aligned} (y,x) \in f^{-1} &\text{ iff } (x,y) \in f \\ &\text{ iff } y = x+1 \\ (x,y) \in f^{-1} &\text{ iff } x = y+1 \\ &\text{ iff } x-1 = y \\ &\text{ iff } y = x-1 \\ (y,x) \in f^{-1} &\text{ iff } x = y-1 \end{aligned}$$
$f^{-1}(y) = y-1$

(b) $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x^2$.

Injectivity This function is *not* injective.

Proof. Assume $f(u) = f(v)$
Then $u^2 = v^2$
i.e. $\pm u = \pm v$
Therefore, f is not injective. ■

Surjectivity This function is *not* surjective.

Proof.
$$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{x^2 \mid x \in \mathbb{Z}\} \\ &= \{y \mid \pm\sqrt{y} \in \mathbb{Z}\} \\ &\neq \mathbb{Z} \end{aligned}$$

Counterexample. Suppose $y = -1$, as $-1 \in \mathbb{Z}$. There is no $x \in \mathbb{Z}$ such that $x^2 = -1$, so the range of f is not equal to the codomain. □

Therefore, f is not surjective. ■

Bijectivity As f is neither injective nor surjective, f is not bijective.

Inverse Function As f is not bijective, f^{-1} is not defined.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = 3 - x$.

Injectivity This function is injective.

Proof.
$$\begin{aligned} \text{Assume} \quad f(u) &= f(v) \\ \text{Then} \quad 3 - u &= 3 - v \\ \text{i.e.} \quad u &= v \end{aligned}$$
Therefore f is injective. ■

Surjectivity This function is surjective.

Proof.
$$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{3 - x \mid x \in \mathbb{Z}\} \\ &= \{y \mid 3 - y \in \mathbb{Z}\} \\ &= \mathbb{Z} \end{aligned}$$
Therefore f is surjective. ■

Bijectivity As this function is injective and surjective, this function is bijective.

Inverse Function
$$\begin{aligned} (y, x) \in f^{-1} &\text{ iff } (x, y) \in f \\ &\text{ iff } y = 3 - x \\ (x, y) \in f^{-1} &\text{ iff } x = 3 - y \\ &\text{ iff } x + y = 3 \\ &\text{ iff } y = 3 - x \\ (y, x) \in f^{-1} &\text{ iff } x = 3 - y \end{aligned}$$
$f^{-1}(y) = 3 - y$

(d) $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = 4x + 5$.

Injectivity This function is injective.

Proof.
$$\begin{aligned} \text{Assume} \quad f(u) &= f(v) \\ \text{Then} \quad 4u + 5 &= 4v + 5 \\ 4u &= 4v \\ \text{i.e.} \quad u &= v \end{aligned}$$
Therefore f is injective. ■

Surjectivity This function is *not* surjective.

Proof.
$$\begin{aligned} f[\mathbb{Z}] &= \{f(x) \mid x \in \mathbb{Z}\} \\ &= \{4x + 5 \mid x \in \mathbb{Z}\} \\ &= \{y \mid \frac{y-5}{4} \in \mathbb{Z}\} \\ &\neq \mathbb{Z} \end{aligned}$$
Therefore f is not surjective. ■

Bijectivity As f is not surjective, f is not bijective.

Inverse Function As f is not bijective, f^{-1} is not defined.

3. Consider an identity function $i_C : C \to C$.

(a) Prove that $i_C : C \to C$ is bijective.

Proof.

Injectivity. Assume $i_C(u) = i_C(v)$
Then $u = v$
Therefore i_C is injective. □

Surjectivity. $i_C[C] = \{i_c \mid c \in C\}$
$= \{c \mid c \in C\}$
$= C$
Therefore i_C is surjective. □

As i_C is both injective and surjective, i_C is bijective. ■

(b) Prove that i_C is an equivalence relation on C.

Proof. *Reflexivity.* For every $c \in C$, is $(c, c) \in i_C$? Yes.
For any $c \in C$, $c = c$.
So $(c, c) \in i_C$.
Therefore, i_C is reflexive. □

Symmetry. For every $c, d \in C$, if $(c, d) \in i_C$, is $(d, c) \in i_C$? Yes.
Suppose $(c, d) \in C$. Then $c = d$. So $d = c$.
Therefore, $(d, c) \in i_C$.
Therefore, i_C is symmetric. □

Transitivity. If $(c, d) \in i_C$ and $(d, e) \in i_C$, is $(c, e) \in i_C$? Yes.
Assume $(c, d) \in i_C$ and $(d, e) \in i_C$.
Then $c = d$ and $d = e$.
So $c = d = e$.
So $c = e$.
Therefore, $(c, e) \in i_C$. □

As i_C is reflexive, symmetric and transitive, i_C is an equivalence relation. ■

Unit 8

Operations

8.1 Binary Operation

Binary Operation

If $f : X \times X \to X$, then f is called a **binary operation** on X.
In other words, a binary operation takes in a pair, and returns a single value.

Operations Notation

An operation is just a function, which means it can be written in normal function *prefix* notation: $f(x, y)$. However, it is more conventional to write it in *infix* notation: xfy.

Example Addition of numbers is a binary operation. If $(x, y) = (3, 4)$, then it could be written $+(3, 4)$, but it is more conventional to write $3 + 4$.

By convention, the elements of a binary operation are all the same set.

8.1.1 Finite and Infinite Sets (Informal Definition)

Finite Set

A set whose cardinality is a non-negative number. Meaning one can count the number of elements in the set.

Example $A = \{1, 2, 3, 4\}$, where $|A| = 4$

Infinite Set

A set that is not finite. Meaning one *cannot* count the number of elements in the set.

Example The set of real numbers $\mathbb{R}$ is an infinite set.

8.1.2 Tables For Binary Operations

A way to describe a binary operation is to use a table, where the rows are based on the *first* element, and the columns on the *second*. The operator (the symbol used to describe the operation) is written in the top left corner.

Example Let $A = \{a, b, c, d\}$.
A binary operation called + (NB: This is *not* addition) could be written as follows:

+	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

This would be read (row, column). +(b, d) = a.

+	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Extra Notes for this operation

Applying concepts from later to the operation:

Identity This operation has an identity element, which is a.
Commutativity This operation is commutative.
Associativity This operation is associative.

Another binary operation, called • could be written as follows:

•	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Extra Notes for this operation

Applying concepts from later to the operation:

Identity This operation has an identity element, which is a.
Commutativity This operation is commutative.
Associativity This operation is associative.

8.2 Properties of Binary Operations

For examples below, the following binary operation has been used:

$* : \{1,2\} \times \{1,2\} \to \{1,2\}$ is defined by:

$$\Big\{((1,1),1),((1,2),2),((2,1),2),((2,2),1)\Big\}$$

In table form this would be:

$*$	**1**	**2**
1	1	2
2	2	1

8.2.1 Commutative Binary Operation

Commutativity

A binary operation $\diamond : X \times X \to X$ is **commutative** iff $x \diamond y = y \diamond x$ for all $x, y \in X$.

The easiest way to check this is it will be commutative if it is symmetrical across the diagonal from the top left to the bottom right.

Example $1 * 1 = 1 * 1 = 1$
$1 * 2 = 2 * 1 = 2$
$2 * 2 = 2 * 2 = 1$
Therefore $*$ is commutative.

8.2.2 Associative Binary Operation

Associativity

A binary operation $\diamond : X \times X \to X$ is **associative** iff $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ for all $x, y, z \in X$.

Unfortunately, for this one, you have to check each instance.

Example $(1 * 1) * 1 = 1 * 1 = 1$ and $1 * (1 * 1) = 1 * 1 = 1$
$(1 * 1) * 2 = 1 * 2 = 2$ and $1 * (1 * 2) = 1 * 2 = 2$
$(1 * 2) * 1 = 2 * 1 = 2$ and $1 * (2 * 1) = 2 * 1 = 2$
$(1 * 2) * 2 = 2 * 2 = 1$ and $1 * (2 * 2) = 1 * 1 = 1$
$(2 * 1) * 1 = 2 * 1 = 2$ and $2 * (1 * 1) = 2 * 1 = 2$
$(2 * 1) * 2 = 2 * 2 = 1$ and $2 * (1 * 2) = 2 * 2 = 1$
$(2 * 2) * 1 = 1 * 1 = 1$ and $2 * (2 * 1) = 2 * 2 = 1$
$(2 * 2) * 2 = 1 * 2 = 2$ and $2 * (2 * 2) = 2 * 1 = 2$
Therefore $*$ is associative.

8.2.3 Identity Element of a Binary Operation

Identity Element

An element e of X is an **identity element** in respect of the binary operation $\diamond : X \times X \to X$ iff $e \diamond x = x \diamond e = x$ for all $x \in X$.

The easiest way to check this is if there is a row and column in the table that is identical to the header. (NB: It needs to be *both* row and column, which contain the same element.)

Example $\mathbf{1} * 1 = 1$ and $1 * \mathbf{1} = 1$
$1 * \mathbf{2} = 2$ and $2 * \mathbf{1} = 2$

Self Assessment Exercise 8.3

1. **Let X be $\{2, 7\}$.**
   (a) **Provide 3 binary operations on X, both in list notation and in tabular form.**

$$\triangle = \Big\{((2,2),2), ((2,7),2), ((7,2),2), ((7,7),7)\Big\}$$

△	**2**	**7**
2	2	2
7	2	7

$$\triangledown = \Big\{((2,2),2), ((2,7),7), ((7,2),7), ((7,7),7)\Big\}$$

▽	**2**	**7**
2	2	7
7	7	7

$$\square = \Big\{((2,2),2), ((2,7),2), ((7,2),7), ((7,7),7)\Big\}$$

□	**2**	**7**
2	2	2
7	7	7

(b) Check the three operations for commutativity and associativity.

Commutativity $\triangle$ is commutative, as it is symmetric about the diagonal.
$\triangledown$ is commutative, as it is symmetric about the diagonal.
$\square$ is *not* commutative.

Associativity $\triangle$ is associative.

$x = 2, y = 2, z = 2$	$(2\triangle 2)\triangle 2 = 2$	$2\triangle(2\triangle 2) = 2$
$x = 2, y = 2, z = 7$	$(2\triangle 2)\triangle 7 = 2$	$2\triangle(2\triangle 7) = 2$
$x = 2, y = 7, z = 2$	$(2\triangle 7)\triangle 2 = 2$	$2\triangle(7\triangle 2) = 2$
$x = 2, y = 7, z = 7$	$(2\triangle 7)\triangle 7 = 2$	$2\triangle(7\triangle 7) = 2$
$x = 7, y = 2, z = 2$	$(7\triangle 2)\triangle 2 = 2$	$7\triangle(2\triangle 2) = 2$
$x = 7, y = 2, z = 7$	$(7\triangle 2)\triangle 7 = 2$	$7\triangle(2\triangle 7) = 2$
$x = 7, y = 7, z = 2$	$(7\triangle 7)\triangle 2 = 2$	$7\triangle(7\triangle 2) = 2$
$x = 7, y = 7, z = 7$	$(7\triangle 7)\triangle 7 = 7$	$7\triangle(7\triangle 7) = 7$

Doing the same for $\triangledown$ and $\square$. Both are associative as well.

(c) Provide 2 binary operations on $X = \{a, b, c\}$ and check them for commutativity and associativity.

★	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

♡	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

Commutativity ★ is not commutative.
♡ is not commutative.

Associativity ★ is associative.
♡ is associative.

2. Consider the $\bullet$ operation defined in the example above on $A = \{a, b, c, d\}$

(a) Examine $y \bullet x$ and $x \bullet y$ for each $x, y \in A$. Is $\bullet$ commutative?
Yes, as it is symmetric about the diagonal.

(b) Does A have an identity element for $\bullet$?
Yes, as the row and column for a matches the headers. So a is an identity element.

8.3 Operations on Vectors

8.3.1 Vector

Vector

In this course, a **vector** is considered to be an ordered *n-tuple* of numbers.
A **vector** is represented by an n-tuple u in the following way:

$$u = (u_1, u_2, u_3, \ldots, u_n) \text{ for some } n \geq 2$$

8.3.2 Vector Sum

Vector Sum

If u and v are vectors with the *same number of coordinates*, then their **sum**, written $u + v$ is the vector obtained by adding the corresponding coordinates of u and v.

$$\begin{aligned} u + v &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_2, u_2 + v_2, \ldots, u_n + v_n) \end{aligned}$$

Example Let $u = (1, 2, 3)$ and $v = (4, 5, 6)$.
Then

$$\begin{aligned} u + v &= (1, 2, 3) + (4, 5, 6) \\ &= (1 + 4, 2 + 5, 3 + 6) \\ &= (5, 7, 9) \end{aligned}$$

Vector addition is not defined for vectors of different sizes

If two vectors have a different number of coordinates, you cannot add those two vectors together.

8.3.3 Scalar-Vector Product

Scalar-Vector Product

If u is a vector and r is some scalar number, then the **product** of the number r and the vector u is the vector $r \cdot u$ obtained by multiplying each coordinate of u by r.

$$\begin{aligned} r \cdot u &= r(u_1, u_2, \dots, u_n) \\ &= (ru_1, ru_2, \dots, ru_n) \end{aligned}$$

Example Let $u = (7, 8, 9)$ and $r = 2$.
Then

$$\begin{aligned} r \cdot u &= 2(7, 8, 9) \\ &= (14, 16, 18) \end{aligned}$$

Self Assessment Exercise 8.6

1. **If $u = (3, 1)$, $v = (-4, -4)$ and $w = (0, -1)$, determine:**

   (a) $2u + v$

   $$\begin{aligned} 2u + v &= 2(3, 1) + (-4, -4) \\ &= (6, 2) + (-4, -4) \\ &= (2, -2) \end{aligned}$$

   (b) $u - 3v$

   $$\begin{aligned} u - 3v &= (3, 1) - 3(-4, -4) \\ &= (3, 1) + (12, 12) \\ &= (15, 13) \end{aligned}$$

   (c) $-3(v + w)$

   $$\begin{aligned} -3(v + w) &= -3\big((-4, -4) + (0, -1)\big) \\ &= -3(-4, -5) \\ &= (12, 15) \end{aligned}$$

8.3.4 Dot Product

Dot Product

The **dot product** (also called the **inner product**) of vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ is written $u \cdot v$ and defined by:

$$u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

The result of the dot product is a number

Unlike the other operations, which result in vectors, the dot product produces a single number.

Example Let $u = (2, 4, 6)$ and $v = (1, 3, 5)$. Then

$$\begin{aligned} u \cdot v &= (2, 4, 6)(1, 3, 5) \\ &= (2 \cdot 1) + (4 \cdot 3) + (6 \cdot 5) \\ &= 2 + 12 + 30 \\ &= 44 \end{aligned}$$

The dot product is not defined for vectors of different sizes

As with addition, if two vectors have a different number of coordinates, you cannot calculate the dot product.

Self Assessment Exercise 8.7

1. **If $u = (1, 2, 5)$ and $v = (2, 3, 5)$, determine:**

   (a) $u \cdot v$

   $$\begin{aligned} u \cdot v &= (1, 2, 5) \cdot (2, 3, 5) \\ &= (1 \cdot 2) + (2 \cdot 3) + (5 \cdot 5) \\ &= 2 + 6 + 25 \\ &= 33 \end{aligned}$$

   (b) $v(2u)$

   $$\begin{aligned} v(2u) &= (2, 3, 5)\big(2(1, 2, 5)\big) \\ &= (2, 3, 5) \cdot (2, 4, 10) \\ &= (2 \cdot 2) + (3 \cdot 4) + (5 \cdot 10) \\ &= 4 + 12 + 50 \\ &= 66 \end{aligned}$$

8.4 Operations on Matrices

8.4.1 Matrix

Matrix

A **matrix** is an array of numbers organised into rows and columns, and enclosed within brackets. The number of rows is written with the letter m and the number of columns with the letter n. So a matrix is said to have the size $m \times n$.

Example $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$ is a 2×2 matrix, and $\begin{bmatrix} -1 & 3 & 0 & 5 \\ 0 & 2 & 0 & 6 \\ 1 & -1 & 0 & 13 \end{bmatrix}$ is a 3×4 matrix.

Matrices (pronounced *may-trisseez*) have the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

8.4.2 Matrix Addition

Matrix Addition

Let A and B be two matrices of the same size. Then the matrix $\boldsymbol{A} + \boldsymbol{B}$ is:

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \end{aligned}$$

Example Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \end{aligned}$$

Self Assessment Exercise 8.8

1. **For each pair A and B determine $A + B$ (if possible):**

(a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 5 \\ 4 & -1 \end{bmatrix}$

$$A + B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4 & 0 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

This operation is not defined, as the matrices are different sizes.

(c) $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 6 \end{bmatrix}$

$$A + B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 7 & 7 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \end{bmatrix}$

This operation is not defined, as the matrices are different sizes.

8.4.3 Scalar-Matrix Multiplication

Scalar-Matrix Multiplication

Let A be a matrix, and r be some scalar number.
Then the product rA is defined as:

$$rA = r\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22} & \cdots & ra_{2n} \\ \vdots & \vdots & & \vdots \\ ra_{m1} & ra_{m2} & \cdots & ra_{mn} \end{bmatrix}$$

Example Let $r = 3$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$rA = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Self Assessment Exercise 8.9

1. **Perform the indicated operation:** $2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

$$2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 26 \end{bmatrix}$$

8.4.4 Matrix Multiplication

Matrix Multiplication

Let A and B both be matrices. In order for the product $\boldsymbol{AB}$ to be defined,

- The number of columns of A needs to be equal to the number of rows of B, i.e. $A_n = B_m$.

If the product is defined, then it will result in a matrix that is the size $A_m \times B_n$.

$$A_m \times A_n \quad \cdot \quad B_m \times B_n = A_m \times B_n$$

When multiplying matrics, it is row of first multiplied by column of second.

Example

$$\begin{bmatrix} -1 & 3 \\ 4 & 2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} (-1 \cdot 6) + (3 \cdot -8) & (-1 \cdot 9) + (3 \cdot 1) \\ (4 \cdot 6) + (2 \cdot -8) & (4 \cdot 9) + (2 \cdot 1) \\ (5 \cdot 6) + (-7 \cdot -8) & (5 \cdot 9) + (-7 \cdot 1) \end{bmatrix}$$

$$= \begin{bmatrix} -6 - 24 & -9 + 3 \\ 24 - 16 & 36 + 2 \\ 30 + 56 & 45 - 7 \end{bmatrix}$$

$$= \begin{bmatrix} -30 & -6 \\ 8 & 38 \\ 86 & 38 \end{bmatrix}$$

8.4.5 Identity Matrix

Identity Matrix

If A is matrix, then an **identity matrix** I with respect to A is a matrix such that $IA = AI = A$. For the identity matrix to be defined,

- A must be a square matrix, because matrix multiplication is *not* commutative.
- I must therefore also be a square matrix with the same number of rows and columns as A.

Then the identity matrix would have 1's for the main diagonal, and 0's elsewhere.

Example For a 2×2 matrix, the identity matrix would be:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Self Assessment Exercise 8.10

1. **Perform the indicated matrix operations (if possible)**

   (a) $\begin{bmatrix} \mathbf{31} & \mathbf{-3} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{5} \end{bmatrix} = \begin{bmatrix} (31 \cdot 0) + (-3 \cdot 1) + (2 \cdot 5) \\ (2 \cdot 0) + (5 \cdot 1) + (1 \cdot 5) \\ (3 \cdot 0) + (0 \cdot 1) + (0 \cdot 5) \end{bmatrix} = \begin{bmatrix} 0 - 3 + 10 \\ 0 + 5 + 5 \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \mathbf{9} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} \\ \mathbf{3} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} \\ \mathbf{5} & \mathbf{1} \end{bmatrix}$

   This operation is not defined.

   (c) $\begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{0} & \mathbf{6} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{\frac{1}{3}} & \mathbf{1} \\ \mathbf{\frac{1}{2}} & \mathbf{5} & \mathbf{0} \end{bmatrix}$

   $$= \begin{bmatrix} (1 \cdot 0) + (-3 \cdot 1) + \left(2 \cdot \frac{1}{2}\right) & (1 \cdot -1) + \left(-3 \cdot \frac{1}{3}\right) + (2 \cdot 5) & (1 \cdot 3) + (-3 \cdot 1) + (2 \cdot 0) \\ (0 \cdot 0) + (6 \cdot 1) + \left(4 \cdot \frac{1}{2}\right) & (0 \cdot -1) + \left(6 \cdot \frac{1}{3}\right) + (4 \cdot 5) & (0 \cdot 3) + (6 \cdot 1) + (4 \cdot 0) \\ (3 \cdot 0) + (0 \cdot 1) + \left(3 \cdot \frac{1}{2}\right) & (3 \cdot -1) + \left(0 \cdot \frac{1}{3}\right) + (3 \cdot 5) & (3 \cdot 3) + (0 \cdot 1) + (3 \cdot 0) \end{bmatrix}$$

   $$= \begin{bmatrix} (0 - 3 + 1) & (-1 - 1 + 10) & (3 - 3 + 0) \\ (0 + 6 + 2) & (0 + 2 + 20) & (0 + 6 + 0) \\ \left(0 + 0 + \frac{3}{2}\right) & (-3 + 0 + 15) & (9 + 0 + 0) \end{bmatrix}$$

   $$= \begin{bmatrix} -2 & 8 & 0 \\ 8 & 22 & 6 \\ \frac{3}{2} & 12 & 9 \end{bmatrix}$$

2. **Provide examples of matrices X and Y such that XY is a 3×3 matrix, but YX is a 2×2 matrix.**

   Any matrices X and Y such that X is a 2×3 matrix, and Y is a 3×2 matrix.

   Two examples: $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

   $A_2 = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 6 & 4 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 3 & 6 \\ 9 & 6 \\ 3 & 0 \end{bmatrix}$

3. **Provide examples of matrices X and Y such that both X and Y have at least some non-zero entries, but XY is the 2×2 zero matrix.**

   Any matrix A that has a zero column, where matrix B has a zero row that are at different indexes.

   Example: $A = \begin{bmatrix} 0 & 4 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 0 & 0 \end{bmatrix}$

4. Prove that addition is a commutative operation on the set of 2×2 matrices, and that there is a 2×2 matrix that acts as an identity element in respect of addition.

Proof. Let A and B be two 2×2 matrices, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

Commutativity. Then:
$$\begin{aligned} A + B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \\ B + A &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$
As $A + B = B + A$, matrix addition is commutative. □

Identity. The identity element for matrix addition on 2×2 matrices is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ □

So, matrix addition is commutative, and an identity element exists for matrix addition. ■

5. Prove that multiplication is *not* a commutative operation on the set of 2×2 matrices, and that there is a 2×2 matrix that acts as an identity element in respect of multiplication.

Proof.

Commutativity Counterexample. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (1 \cdot 4) + (2 \cdot 2) & (1 \cdot 3) + (2 \cdot 1) \\ (3 \cdot 4) + (4 \cdot 2) & (3 \cdot 3) + (4 \cdot 1) \end{bmatrix} = \begin{bmatrix} 4+4 & 3+2 \\ 12+8 & 9+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$
$$BA = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (4 \cdot 1) + (3 \cdot 3) & (4 \cdot 2) + (3 \cdot 4) \\ (2 \cdot 1) + (1 \cdot 3) & (2 \cdot 2) + (1 \cdot 4) \end{bmatrix} = \begin{bmatrix} 4+9 & 8+12 \\ 2+3 & 4+4 \end{bmatrix} = \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}$$

As $AB \neq BA$, matrix multiplication is not commutative. □

Identity. The identity element for matrix multiplication is the identity matrix. For a 2×2 matrix, that is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ □

So, matrix multiplication is not commutative, and an identity element exists for matrix multiplication. ■

Unit 9

Logic: Truth Tables

9.1 Declarative Statements

A declarative statement is a statement that conveys information.

Declarative Statements Some examples are:

- The capital of France is Paris
- 3 is an even integer
- This sentence is false

Non-Declarative Statements Some examples are:

- Is 3 an even integer? (*Question*: acquire information, not convey information)
- Add 3 to 5! (*Command*: induce behaviour, not convey information)

Not all declarative statements are usable

A declarative statement is not necessarily true or false, as there can be a contradiction in the statement.
However, when dealing with proofs, declarative statements are restricted to those that can be either *true* or *false*.

A declarative statement can either be

- **atomic**, (or simple) meaning they convey a single fact, or
- **compound**, meaning they combine multiple atomic statements.

9.2 Combining Statements

Statements can be combined using different **logical connectives**. Below is a list of the possible connectives.

and	$\wedge$	conjunction
or	$\vee$	disjunction
not	$\neg$	negation
if and only if	$\leftrightarrow$	biconditional
if ..., then ...	$\rightarrow$	conditional/implication

9.2.1 Conjunction (AND)

Conjunction

If p and q represent declarative statements, then $p \wedge q$ represents the statement "p and q", and is called the **conjunction** of p and q.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

9.2.2 Disjunction (OR)

Disjunction

If p and q represent declarative statements, then $p \vee q$ represents the statement "p or q", and is called the **disjunction** of p and q.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

9.2.3 Negation

Negation

If p is some declarative statement, then $\neg p$ represents the statement "not p". This is called the **negation** of a given statement.

p	$\neg p$
T	F
F	T

9.2.4 Biconditional

Biconditional

If p and q represent declarative statements, then $p \leftrightarrow q$ represents the statement p if and only if q, which can also be written p iff q. This is called the **biconditional**.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

9.2.5 Conditional/Implication

Implication

If p and q represent declarative statements, then $p \to q$ represents the statement "If p, then q", and is called a **conditional statement** or **implication**. p is called the **hypothesis** or the **antecedent** and q is called the **conclusion** or **consequent**.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If the hypothesis is false, then the statement is true

This can be quite confusing. The idea is that if the original statement is false, we can't say that the next statement is not true.

Example Consider a statement "If you read books, you are smart".

If someone reads books and is smart, this is true.
If someone reads books and is not smart, this is false.
If someone does not read books, we cannot say the statement is false, but we also cannot say it is true. So the statement would be vacuously true.

9.3 Constructing Truth Tables

1. List the statements at the top.
2. For the first column, fill half of the rows with T and half with F.
3. For the second column, for the rows that have T, write T for the upper half, and F for the lower half. Do the same for F.
4. Continue doing that until the base statements are filled.
5. Then apply the rules to the columns left to right.

Example Construct a truth table for $p \wedge (\neg q)$.

p	q
T	T
T	F
F	T
F	F

p	q	$\neg q$
T	T	F
T	F	T
F	T	F
F	F	T

p	q	$\neg q$	$p \wedge (\neg q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Activity 9.4

1. **Construct a truth table for $\left[\neg p \rightarrow (q \wedge r)\right] \vee r$**

p	q	r	$\neg p$	$q \wedge r$	$\neg p \rightarrow (q \wedge r)$	$[\neg p \rightarrow (q \wedge r)] \vee r$
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	T
F	F	F	T	F	F	F

Self Assessment Exercise 9.5

1. **Suppose that p represents the statement "It is sunny", and q represents the statement "It is humid". Write each of the following in abbreviated form.**
   (a) **It is sunny, and it is not humid** $p \wedge \neg q$
   (b) **It is humid, or it is sunny** $p \vee q$
   (c) **It is false that it is humid** $\neg q$
   (d) **It is false that it is sunny and humid** $\neg(p \wedge q)$
   (e) **It is neither sunny nor humid** $\neg p \wedge \neg q$
   (f) **It is not the case that if it is sunny then it is humid** $\neg(p \rightarrow q)$
   (g) **It is humid if it is sunny** $p \rightarrow q$
   (h) **It is humid only if it is sunny** $q \rightarrow p$
   (i) **It is sunny if and only if it is humid** $p \leftrightarrow q$
   (j) **If it is false that it is either sunny or humid (but not both), then it is not sunny** $\neg\big[(p \vee q) \wedge \neg(p \wedge q)\big] \rightarrow \neg p$

2. **Construct truth tables for the following compound statements:**
   (a) $[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)$

p	q	$\neg p$	$\neg q$	$(\neg q) \rightarrow (\neg p)$	$p \rightarrow q$	$\big[(\neg q) \rightarrow (\neg p)\big] \rightarrow (p \rightarrow q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

   (b) $\Big[\neg p \rightarrow \big(q \wedge (\neg q)\big)\Big] \rightarrow p$

p	q	$\neg p$	$\neg q$	$q \wedge \neg q$	$\neg p \rightarrow (q \wedge \neg q)$	$[\neg p \rightarrow (q \wedge \neg q)] \rightarrow p$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

   (c) $p \vee (\neg p)$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(d) $[p \wedge (p \to q)] \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(e) $(p \vee q) \wedge (\neg p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

(f) $\big(\neg p \to [q \wedge r]\big) \vee r$

p	q	r	$\neg p$	$q \wedge r$	$\neg p \to (q \wedge r)$	$[\neg p \to (q \wedge r)] \vee r$
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	T
F	F	F	T	F	F	F

(g) $\big(p \to [q \wedge r]\big) \leftrightarrow \big([p \to q] \vee [p \to r]\big)$

p	q	r	$q \wedge r$	$p \to q$	$p \to r$	$p \to (q \wedge r)$	$(p \to q) \vee (p \to r)$	S
T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T	F
T	F	T	F	F	T	F	T	F
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

S is the statement.

9.4 Relationships Between Statements

9.4.1 Tautology

Tautology

A compound statement that is always true is called a **tautology**.

Example The statement $p \vee \neg p$ is always true.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

9.4.2 Contradiction

Contradiction

A compound statement that is always false is called a **contradiction**.

Example The statement $p \wedge \neg p$ is always false.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

9.4.3 Logical Equivalence

Logical Equivalence

Two declarative statements a and b are **logically equivalent**, written $a \equiv b$, if and only if the statement $a \longrightarrow b$ is a tautology.

Example Take the declarative statement $(p \vee q) \leftrightarrow (q \vee p)$.

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

As $(p \vee q) \leftrightarrow (q \vee p)$ is a tautology, the statements are logically equivalent. That is:

$$p \vee q \equiv q \vee p$$

Important Logical Equivalences (Identities)

Let T_0 be a tautology, and F_0 be a contradiction.

(a) $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ (commutative laws)

(b) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ (associative laws)

(c) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive laws)

(d) $p \vee p \equiv p$
$p \wedge p \equiv p$ (idempotent laws)

(e) $\neg(\neg p) \equiv p$ (double negation laws)

(f) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ (De Morgan's laws)

(g) $p \vee \neg p \equiv T_0$
$p \wedge \neg p \equiv F_0$ (inverse laws)

(h) $\neg F_0 \equiv T_0$
$\neg T_0 \equiv F_0$ (negation laws)

(i) $p \vee F_0 \equiv p$
$p \wedge T_0 \equiv p$ (identity laws)

(j) $p \vee T_0 \equiv T_0$
$p \wedge F_0 \equiv F_0$ (domination laws/universal bound)

Other Useful Logical Eqiuivalences

(a) $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$ (contrapositive equivalence)

(b) $p \rightarrow q \equiv \neg p \vee q$ (implication equivalence)

(c) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ (biconditional equivalence)

(d) $\neg(p \rightarrow q) \equiv p \wedge \neg q$ (negation of implication)

Self Assessment Exercise 9.9

1. **Rewrite $p \leftrightarrow q$ as a statement built up using only $\neg$, $\vee$ and $\wedge$.**

$$\begin{aligned} p \leftrightarrow q &\equiv (p \rightarrow q) \wedge (q \rightarrow p) \\ &\equiv (\neg p \vee q) \wedge (\neg q \vee p) \end{aligned}$$

2. **Show that $\equiv$ is an equivalence relation on statements.**

Proof. For $\equiv$ to be an equivalence relation, $\equiv$ needs to be reflexive, symmetric and transitive.
Let p and q be two statements, where $p \equiv q$.

(i) *Reflexivity.* Is it the case that $p \equiv p$ for all statements? Yes.
If p is a statement, then $p \leftrightarrow p$ is always true. So $p \leftrightarrow p$ is a tautology.
So $p \leftrightarrow p$ is logically equivalent.
So $p \equiv p$ is part of the relation for all statements p.
So $\equiv$ is a reflexive relation. □

(ii) *Symmetry.* Is it the case that, if $p \equiv q$, then $q \equiv p$ for all statements p and q? Yes.
Suppose $p \equiv q$. Then that means that $p \leftrightarrow q$ is always true.
If $p \leftrightarrow q$ is always true, that means that $p \rightarrow q$ is always true, and $q \rightarrow p$ is always true.
If $q \rightarrow p$ is always true, and $p \rightarrow q$ is always true, then $q \leftrightarrow p$ is always true.
If $q \leftrightarrow p$ is always true, then $q \leftrightarrow p$ is a tautology.
So $q \equiv p$.
So $\equiv$ is a symmetric relation. □

(iii) *Transitivity.* Is it the case that, if $p \equiv q$ and $q \equiv r$, then $p \equiv r$ for all statements p, q and r? Yes.
Suppose $p \equiv q$ and $q \equiv r$.
Then $p \leftrightarrow q$ is always true, and $q \leftrightarrow r$ is always true.
So $p \rightarrow q$ and $q \rightarrow r$ are both always true. So $p \rightarrow r$ is always true.
And $r \rightarrow q$ and $q \rightarrow p$ are both always true. So $r \rightarrow p$ is always true.
So $p \leftrightarrow r$ is always true.
So $p \equiv r$
So $\equiv$ is a transitive relation. □

As $\equiv$ is reflexive, symmetric, and transitive, $\equiv$ is an equivalence relation. ■

3. **Suppose we want to define a new connective, the *exclusive disjunction*, also called the *"exclusive or"*, which will be written $+$. By $p+q$, we denote "p or q, but not both". Construct a truth table for this connective.**

p	q	$p+q$
T	T	F
T	F	T
F	T	T
F	F	F

4. **Find a statement that is logically equivalent to $\neg(p \vee \neg q)$**

$$\begin{aligned} \neg(p \vee \neg q) &\equiv \neg p \wedge \neg(\neg q) && \text{De Morgan's law} \\ &\equiv \neg p \wedge q && \text{Double Negation} \end{aligned}$$

5. **Use the law of double negation and De Morgan's laws to rewrite the following statements so that the not symbol ($\neg$) does not appear outside parentheses.**

   (a) $\neg\big[(p \vee q \vee \neg q) \wedge (q \wedge \neg p)\big]$

$$\begin{aligned} \neg\big[(p \vee q \vee \neg q) \wedge (q \wedge \neg p)\big] &\equiv \neg(p \vee q \vee \neg q) \vee \neg(q \wedge \neg p) && \text{De Morgan's law} \\ &\equiv \big(\neg p \wedge \neg q \wedge \neg(\neg q)\big) \vee \big(\neg q \vee \neg(\neg p)\big) && \text{De Morgan's law} \\ &\equiv (\neg p \wedge \neg q \wedge q) \vee (\neg q \vee p) && \text{Double Negation} \end{aligned}$$

   (b) $\neg\Big[\big(p \vee (p \to q)\big) \vee (p \wedge q)\Big]$

$$\begin{aligned} \neg\Big[\big(p \vee (p \to q)\big) \vee (p \wedge q)\Big] &\equiv \neg\Big[\big(p \vee (\neg p \vee q)\big) \vee (p \wedge q)\Big] && \text{Implication} \\ &\equiv \neg\big(p \vee (\neg p \vee q)\big) \wedge \neg(p \wedge q) && \text{De Morgan's law} \\ &\equiv \big(\neg p \wedge \neg(\neg p \vee q)\big) \wedge (\neg p \vee \neg q) && \text{De Morgan's law} \\ &\equiv \big(\neg p \wedge (\neg\neg p \wedge \neg q)\big) \wedge (\neg p \vee \neg q) && \text{De Morgan's law} \\ &\equiv \big(\neg p \wedge (p \wedge \neg q)\big) \wedge (\neg p \vee \neg q) && \text{Double Negation} \end{aligned}$$

6. **Determine whether the following statements are equivalent:**
   $\neg p \wedge (p \wedge \neg q)$ and $\neg\big(p \vee (p \to q)\big)$

$$\begin{aligned} \neg\big(p \vee (p \to q)\big) &\equiv \neg\big(p \vee (\neg p \vee q)\big) && \text{Implication} \\ &\equiv \neg p \wedge \neg(\neg p \vee q) && \text{De Morgan's law} \\ &\equiv \neg p \wedge (\neg\neg p \wedge \neg q) && \text{De Morgan's law} \\ &\equiv \neg p \wedge (p \wedge \neg q) && \text{Double Negation} \end{aligned}$$

   Therefore the two expressions are equivalent.

Unit 10

Logic: Quantifiers, predicates and proof strategies

10.1 Quantifiers and Predicates

10.1.1 Universal Quantifier (FOR ALL)

Universal Quantifier

A **universal quantifier** is written with the symbol $\forall$, meaning "for all".

Example Examples would be:

"For all $x \in \mathbb{R} \ldots$" (written $\forall\, x \in \mathbb{R}$)
"For every $x \in \mathbb{Z} \ldots$" (written $\forall\, x \in \mathbb{Z}$)

The variable x above is called a **quantified variable**.

This can be considered a *generalisation of conjunction (AND)*.

Example Let $A = \{1, 2, 3\}$. A declarative statement can then be made for the set:

$$\forall\, x \in A, x > 0$$

This means the same thing as

$$(1 > 0) \wedge (2 > 0) \wedge (3 > 0)$$

This statement is true.

10.1.2 Existential Quantifier (THERE EXISTS)

Existential Quantifier

An **existential quantifier** is written with the symbol $\exists$, meaning "there exists".

Example Examples would be:

"There exists an $x \in \mathbb{R}$ such that..." (written $\exists\, x \in \mathbb{R}$)
"For some $x \in \mathbb{Z}$ ..." (written $\exists\, x \in \mathbb{Z}$)

A quantified variable is a dummy variable

Any quantified variable can be replaced (everywhere it occurs) with another variable without changing the meaning.

Example $\forall\, x \in \mathbb{Z}^+, (x > 0) \equiv \forall\, y \in \mathbb{Z}^+, (y > 0)$

This can be considered a *generalisation of disjunction (OR)*.

Example Let $A = \{1, 2, 3\}$. A declarative statement can then be made for the set:

$$\exists\, x \in A, x > 4$$

This means the same thing as

$$(1 > 4) \vee (2 > 4) \vee (3 > 4)$$

This statement is false.

Self Assessment Exercise 10.3

1. **Write down the English equivalent of each of the following statements, and give an opinion on whether the statement is true.**
   (a) $\exists y \in \mathbb{Q}, y = \sqrt{2}$
   There exists some rational number that is the square root of 2. This statement is false, as $\sqrt{2}$ is an irrational number.
   (b) $\forall x \in \mathbb{R}, 2x < x^2$
   For all real numbers x, $2x$ is less than x^2. This statement is false, as $2(0) \nless 0^2$
   (c) $\forall x \in \mathbb{Z}, x > 0$
   Every integer is greater than 0. This statement is false, as -1 and 0 are both integers.
   (d) $\exists x \in \mathbb{Z}^+, x = 0$
   There exists a positive integer that is equal to 0. This statement is false, as 0 is not a positive integer.

Self Assessment Exercise 10.4

1. **Prove by means of truth tables that $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$**

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p) \vee (\neg q)$	$\neg(p \wedge q) \leftrightarrow (\neg p) \vee (\neg q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Self Assessment Exercise 10.5

1. **Prove by means of truth tables that $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$**

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p) \wedge (\neg q)$	$\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

10.1.3 Predicate

Predicate

A statement $P(x)$ is called a **predicate** if it expresses some property of a variable $x \in A$, and returns either true or false depending on the value of x. $P(x)$ is true for any variable $x \in A$ that has the property, and $P(x)$ is false if x does not have that property.

A predicate is a boolean function

A predicate takes in a value, and either returns true or false.

10.1.4 Negation of Quantified Statements

If $P(x)$ is a predicate containing some variable x, then:

1. $\neg\big(\forall\, x \in A, P(x)\big) \equiv \exists\, x \in A, \neg P(x)$
2. $\neg\big(\exists\, x \in A, P(x)\big) \equiv \forall\, x \in A, \neg P(x)$

Example Determine the negation of the quantified statement "$\forall x \in A, P(x) \vee Q(X)$".

$$\begin{aligned}\neg\big(\forall x \in A, P(x) \vee Q(X)\big) &\equiv \exists x \in A, \neg\big(P(x) \vee Q(x)\big)\\ &\equiv \exists \in A, \neg P(x) \wedge \neg Q(x)\end{aligned}$$

Self Assessment Exercise 10.6

1. **Determine the negations of the following quantified statements: (Show all steps.)**

   (a) $\forall x \in \mathbb{Z}^+, x > 3$

$$\begin{aligned} \neg(\forall x \in \mathbb{Z}^+, x > 3) &\equiv \exists x \in \mathbb{Z}^+, \neg(x > 3) \\ &\equiv \exists x \in \mathbb{Z}^+, x \leq 3 \end{aligned}$$

   (b) $\exists x \in \mathbb{R}, 2x = x^2$

$$\begin{aligned} \neg(\exists x \in \mathbb{R}, 2x = x^2) &\equiv \forall x \in \mathbb{R}, \neg(2x = x^2) \\ &\equiv \forall x \in \mathbb{R}, 2x \neq x^2 \end{aligned}$$

   (c) $\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)$

$$\begin{aligned} \neg\big(\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)\big) &\equiv \exists x \in \mathbb{Z}, \neg\big((x > 0) \vee (x^2 > 0)\big) \\ &\equiv \exists x \in \mathbb{Z}, \neg(x > 0) \wedge \neg(x^2 > 0) \\ &\equiv \exists x \in \mathbb{Z}, (x \leq 0) \wedge (x^2 \leq 0) \end{aligned}$$

   (d) $\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)$

$$\begin{aligned} \neg\big(\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)\big) &\equiv \forall y \in \mathbb{Z}^+, \neg\big((y \leq 10) \wedge (y \neq 0)\big) \\ &\equiv \forall y \in \mathbb{Z}^+, \neg(y \leq 10) \vee \neg(y \neq 0) \\ &\equiv \forall y \in \mathbb{Z}^+, (y > 10) \vee (y = 0) \end{aligned}$$

   (e) $\exists x \in A, P(x) \wedge Q(x)$

$$\begin{aligned} \neg\big(\exists x \in A, P(x) \vee Q(x)\big) &\equiv \forall x \in A, \neg\big(P(x) \vee Q(x)\big) \\ &\equiv \forall x \in A, \neg P(x) \wedge \neg Q(x) \end{aligned}$$

   (f) $\forall x \in \mathbb{Z}^+, (x \leq 3) \to (x^3 \geq 1)$

$$\begin{aligned} \neg\big(\forall x \in \mathbb{Z}^+ (x \leq 3) \to (x^3 \geq 1)\big) &\equiv \neg\big(\forall x \in \mathbb{Z}^+, \neg(x \leq 3) \vee (x^3 \geq 1)\big) \\ &\equiv \exists x \in \mathbb{Z}^+, \neg\big(\neg(x \leq 3) \vee (x^3 \geq 1)\big) \\ &\equiv \exists x \in \mathbb{Z}^+, \neg\neg(x \leq 3) \wedge \neg(x^3 \geq 1) \\ &\equiv \exists x \in \mathbb{Z}^+, (x \leq 3) \wedge \neg(x^3 \geq 1) \\ &\equiv \exists x \in \mathbb{Z}^+, (x \leq 3) \wedge (x^3 < 1) \end{aligned}$$

Self Assessment Exercise 10.7

1. **For each of (a) to (d) in the previous exercise, determine whether the original statement is true, whether the negation is true, or if neither of the two is true.**
   - **(a)** The original statement is false, as 1, 2 and 3 are positive integers. The negation is true.
   - **(b)** The original statement is true, as when $x = 2$, $2(2) = (2)^2$. The negation is false, as there is an element.
   - **(c)** The original statement is false, as $0 \not> 0$ and $0^2 \not> 0$. The negation is true, if $x = 0$.
   - **(d)** The original statement is true for any positive integer less than 10. The negation is false, as not all positive integers are greater than 10.

10.2 Proof Strategies

Given some statement "if p, then q", there are different ways to prove it.

10.2.1 Direct Proof

Assume that p is true, and then reason step-by-step to show that q is true.

Example Prove that the following statement is true for all $x \in \mathbb{R}$:

$$\text{If } x^2 - 4x + 3 < 0, \text{ then } x > 0$$

Start by assuming that $x^2 - 4x + 3 < 0$ is true.

Proof. Assume $x^2 - 4x + 3 < 0$. That is:

$$\begin{aligned} x^2 - 4x + 3 &< 0 \\ (x-3)(x-1) &< 0 && \text{(by factorisation)} \end{aligned}$$

That means either

(i) $(x-3) > 0$ and $(x-1) < 0$ (plus times minus gives minus), or

(ii) $(x-3) < 0$ and $(x-1) > 0$ (minus times plus gives minus)

For (i): $(x-3) > 0$ and $(x-1) < 0$

$\Rightarrow$ $x > 3$ and $x < 1$

There is no x that this can be true for.

For (ii): $(x-3) < 0$ and $(x-1) > 0$

$\Rightarrow$ $x < 3$ and $x > 1$

This shows $1 < x < 3$, or $x \in (1, 3)$.
Therefore, $x < 0$ ■

10.2.2 Proof By Contradiction (*Reductio Ad Absurdum*)

Assume that p is true. Then assume that q is false, and use step-by-step reasoning until there is a contradiction. If there is a contradiction, that means that q must be true.

Example Prove that the following statement is true for all $x \in \mathbb{R}$:

$$\text{If } x^2 - 4x + 3 < 0, \text{ then } x > 0$$

Start by assuming that $x^2 - 4x + 3 < 0$ is true.

Proof. Assume $x^2 - 4x + 3 < 0$.
If the antecedent is true, then either the consequent is true or the consequent is false.
Assume that the consequent is false, i.e. assume that $x \not> 0$, that is $x \leq 0$.

If $x \leq 0$,
Then $-4x \geq 0$ (minus times minus gives plus)
And $x^2 + 3 > 0$
So $x^2 - 4x + 3 > 0$

However, this contradicts the original assumption. Therefore, $x \leq 0$ cannot be true.
Therefore, $x > 0$. ∎

10.2.3 Proof By Contrapositive

Contrapositive

The **contrapositive** of $p \to q$ is $\neg q \to \neg p$. These two statements are logically equivalent to each other.

Example Prove that the following statement is true for all $x \in \mathbb{R}$:

$$\text{If } x^2 - 4x + 3 < 0, \text{ then } x > 0$$

To use the contrapositive, swap the two statements around, and negate them:

Proof. To use the contrapositive, prove:

$$\text{If } \neg(x > 0), \text{ then } \neg(x^2 - 4x + 3 < 0).$$

Assume $\neg(x > 0)$ is true, i.e. $x \leq 0$.
Factorise the consequent:

$$x^2 - 4x + 3 = (x - 3)(x - 1)$$

As $x \leq 0$, $(x - 3) \leq 0$ and $(x - 1) \leq 0$.

If $(x - 3) \leq 0$ and $(x - 1) \leq 0$,
Then $(x - 3)(x - 1) \geq 0$ (minus times minus gives plus)
i.e. $x^2 - 4x + 3 \geq 0$
i.e. $\neg(x^2 - 4x + 3 < 0)$ ∎

The contrapositive is not the same as the converse

The **converse** of a statement just swaps them around. This is not the same as the contrapositive.

Example Given a statement, $p \to q$.

Converse $q \to p$

Contrapositive $\neg q \to \neg p$

10.2.4 Proofs Involving Quantifiers

In order to apply a proof to a quantified statement over an infinite set A, for example $\forall\, x \in A, P(x)$, think of the statement as $x \in A \to P(x)$.

Example Prove the statement

$$\forall\, x \in \mathbb{R}, x^2 + 1 > 0.$$

Proof.

If	$x \in \mathbb{R}$,
Then	$x^2 \geq 0$,
So	$x^2 + 1 \geq 1$
i.e.	$x^2 + 1 > 0$ ∎

To disprove a statement, prove that its *negation* is true. If this is a statement such as $\forall\, x \in A, P(x)$, the negation is $\exists\, x \in A, \neg P(x)$. This shows that in order to disprove the statement, one needs to simply find a counterexample.

Example Show using a *counterexample* that this statement is not true:

$$\forall\, x \in \mathbb{R}, x^2 - 4x > 0$$

Proof. To disprove $\forall\, x \in \mathbb{R}, x^2 - 4x > 0$, one needs to prove that:

$$\exists\, x \in \mathbb{R}, x^2 - 4x \geq 0$$

One could choose $x = 0$.
Then

$$\begin{aligned} x^2 - 4x &= (0)^2 - 4(0) \\ &= 0 \\ &\not> 0 \end{aligned}$$

∎

10.2.5 Vacuous Proofs

The truth table for an implication shows that if the antecedent is false, then the statement is always true.
Using the above, if you can show that the conditional statement is false, then the statement is **vacuously true**.

Example Prove that:

$$\emptyset \subseteq X$$

To prove the above statement, we need to show that:

$$\text{If } x \in \emptyset, \text{ then } x \in X$$

Proof.
$\emptyset$ is an empty set,
so "$x \in \emptyset$" is false,
therefore "if $x \in \emptyset$, then $x \in X$" is **vacuously true**. ■

Example Let S be a relation on $\{a, b, c, d\}$, where $S = \{(a, b), (a, d)\}$. Prove that S is transitive.

Proof. For a set S to be transitive, whenever $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$.
There are no two pairs of the form (x, y) and (y, z) in S,
so it is **vacuously true** that S is transitive. ■

Self Assessment Exercise 10.10

1. **Prove each of the following statements by direct proof, contrapositive and contradiction respectively.**
   (a) **If $x^2 - 3x + 2 < 0$, then $x > 0$.**

   Direct Proof. Assume that $x^2 - 3x + 2 < 0$ is true. That is $(x - 1)(x - 2) < 0$, by factorisation.
   So either

   $$\begin{array}{llll} \text{(i)} & (x-1) < 0 & \text{and} & (x-2) > 0 \\ \Rightarrow & x < 1 & \text{and} & x > 2 \end{array}$$

   There is no x that this can be true for.

   $$\begin{array}{llll} \text{(ii)} & (x-1) > 0 & \text{and} & (x-2) < 0 \\ \Rightarrow & x > 1 & \text{and} & x < 2 \end{array}$$

   So $1 < x < 2$.

   As $1 < x < 2$, $x > 0$. ■

Contrapositive. To show: If $x \leq 0$, then $x^2 - 3x + 2 \geq 0$.
Suppose $x \leq 0$. Then $-3x \geq 0$. And $x^2 + 2 > 0$.
So $x^2 - 3x + 2 \geq 0$ ■

Contradiction. Assume that $x^2 - 3x + 2 < 0$. Suppose that $x \leq 0$.
If $x \leq 0$, then $-3x \geq 0$. And $x^2 + 2 \geq 0$.
So $x^2 - 3x + 2 \geq 0$.
But that contradicts the original assumption.
So it must be the case that $x > 0$ ■

(b) If $x^2 - x - 6 > 0$, then $x \neq 1$.

Direct Proof. Assume that $x^2 - x - 6 > 0$. That is, $(x-3)(x+2) > 0$.
So either

(i)
$$\begin{array}{rrcrr} & (x-3) > 0 & \text{and} & (x+2) & > 0 \\ \Rightarrow & x > 3 & \text{and} & x & > -2 \end{array}$$

So $x > 3$.

(ii)
$$\begin{array}{rrcrr} & (x-3) < 0 & \text{and} & (x+2) & < 0 \\ \Rightarrow & x < 3 & \text{and} & x & < -2 \end{array}$$

So $x < -2$.

So either $x > 3$ or $x < -2$.
So $x \neq 1$. ■

Contrapositive. To show: If $x = 1$, then $x^2 - x - 6 \leq 0$.
Suppose $x = 1$. Then
$$\begin{aligned} x^2 - x - 6 &= (1)^2 - 1 - 6 \\ &= 1 - 1 - 6 \\ &= -6 \end{aligned}$$
As $-6 < 0$, $x^2 - x - 6 \leq 0$. ■

Contradiction. Assume that $x^2 - x - 6 > 0$.
Suppose that $x = 1$. Then
$$\begin{aligned} x^2 - x - 6 &= (1)^2 - 1 - 6 \\ &= 1 - 1 - 6 \\ &= -6 \end{aligned}$$
So $x^2 - x - 6 < 0$.
But that contradicts the original assumption.
So it must be the case that $x \neq 1$. ■

(c) For all $a, b \in \mathbb{Z}$, if $a + b$ is odd, then exactly one of a or b is odd.

Direct Proof. Assume that $a + b$ is odd (where $a, b \in \mathbb{Z}$).
Then $a + b = 2n + 1$ for some integer n. Then either

(i) a is even.
Suppose a is even. Then $a = 2k$ for some integer k.

$$\begin{aligned} a + b &= 2k + b \\ 2k + b &= 2n + 1 \\ \Rightarrow \qquad b &= 2n + 1 - 2k \\ \Rightarrow \qquad b &= 2(n - k) + 1 \end{aligned}$$

So b is odd.

(ii) a is odd.
Suppose a is odd. Then $a = 2k + 1$ for some integer k.

$$\begin{aligned} a + b &= 2k + 1 + b \\ 2k + 1 + b &= 2n + 1 \\ \Rightarrow \qquad b &= 2n + 1 - 2k - 1 \\ \Rightarrow \qquad b &= 2(n - k) \end{aligned}$$

So b is even.

So if a is even, then b is odd. If a is odd, then b is even.
So exactly one of a or b is odd. ■

Contrapositive. To show: If both a and b are odd, or both a and b are even, then $a + b$ is not odd.
There are two cases:

(i) Suppose a and b are both odd. Then $a = 2n + 1$ for some integer n, and $b = 2k + 1$ for some integer k.
So $a + b = (2n + 1) + (2k + 1)$
$= 2n + 2k + 2$
$= 2(n + k + 1)$
So $a + b$ is even, i.e. $a + b$ is not odd.

(ii) Suppose a and b are both even. Then $a = 2n$ for some integer n and $b = 2k$ for some integer k.
So $a + b = 2n + 2k$
$= 2(n + k)$
So $a + b$ is even, i.e. $a + b$ is not odd.

So, if it is not the case that exactly one of a or b is odd, then $a + b$ is not odd. ■

Contradiction. Assume that $a+b$ is odd (where $a, b \in \mathbb{Z}$).
Then either exactly one of a and b is odd, or that is not the case.
If it is not the case, then either a and b are both odd, or a and b are both even.

(i) Suppose a and b are both odd. Then $a = 2n+1$ for some integer n, and $b = 2k+1$ for some integer k.
So
$$\begin{aligned} a+b &= (2n+1)+(2k+1) \\ &= 2n+2k+2 \\ &= 2(n+k+1) \end{aligned}$$
So $a+b$ is even, which contradicts the original assumption.

(ii) Suppose a and b are both even. Then $a = 2n$ for some integer n and $b = 2k$ for some integer k.
So
$$\begin{aligned} a+b &= 2n+2k \\ &= 2(n+k) \end{aligned}$$
So $a+b$ is even, which contradicts the original assumption.

Therefore, it must be the case that exactly one of a and b is odd. ■

(d) For all $x \in \mathbb{Z}$, if x is even, then x^2+4x+2 is even.

Direct Proof. Assume that x is even, where $x \in \mathbb{Z}$. If x is even, then $x = 2k$ for some integer k.
So
$$\begin{aligned} x^2+4x+2 &= (2k)^2+4(2k)+2 \\ &= 4k^2+8k+2 \\ &= 2(2k^2+4k+1) \end{aligned}$$
So x^2+4x+2 is even. ■

Contrapositive. To show: If x^2+4x+2 is odd, then x is odd.
Suppose x^2+4x+2 is odd. Then $x^2+4x+2 = 2k+1$ for some integer k
So
$$\begin{aligned} & x^2+4x+2 = 2k+1 \\ \Rightarrow\quad & x^2+4x+4 = 2k+1+2 \qquad \text{(Complete the square)} \\ \Rightarrow\quad & (x+2)(x+2) = 2k+2+1 \\ \Rightarrow\quad & (x+2)(x+2) = 2(k+1)+1 \end{aligned}$$
So, as $(x+2)(x+2)$ is odd, that means that $x+2$ must be odd, as the product of the integers is odd iff both integers are odd.
So $x+2 = 2n+1$ for some integer n.
So $x = 2n+1-2$, so $x = 2n-1$.
So x is odd.
So, if x^2+4x+2 is odd, then x is odd.
So, if x is even, then x^2+4x+2 is even. ■

Contradiction. Assume x is even. Then either $x^2 + 4x + 2$ is even, or odd.
Suppose that $x^2 + 4x + 2$ is odd. Then $x^2 + 4x + 2 = 2k + 1$ for some integer k

$$\begin{aligned}
\text{So} \quad & x^2 + 4x + 2 = 2k + 1 \\
\Rightarrow \quad & x^2 + 4x + 4 = 2k + 1 + 2 \qquad \text{(Complete the square)} \\
\Rightarrow \quad & (x+2)(x+2) = 2k + 2 + 1 \\
\Rightarrow \quad & (x+2)(x+2) = 2(k+1) + 1
\end{aligned}$$

So, as $(x+2)(x+2)$ is odd, that means that $x + 2$ must be odd, as the product of the integers is odd iff both integers are odd.
So $x + 2 = 2n + 1$ for some integer n.
So $x = 2n + 1 - 2$, so $x = 2n - 1$.
So x is odd.
But this contradicts the original assumption, so it must be the case that $x^2 + 4x + 2$ is even. ■

(e) If n is a multiple of 3, then $n^3 + n^2$ is a multiple of 3

Direct Proof. Assume that n is a multiple of 3. Then $n = 3k$ for some integer k.

$$\begin{aligned}
\text{So } n^3 + n^2 &= (3k)^3 + (3k)^2 \\
&= 27k^3 + 9k^2 \\
&= 3(9k^3 + 3k^2)
\end{aligned}$$

Therefore $n^3 + n^2$ is a multiple of 3. ■

Contrapositive. To show: If $n^3 + n^2$ is not a multiple of 3, then n is not a multiple of 3.
Suppose that $n^3 + n^2$ is not a multiple of 3. Then $n^3 + n^2 = 3k + 1$ for some integer k.
If $n^3 + n^2 = 3k + 1$, then $n(n^2 + n) = 3k + 1$.
So neither n, nor $n^2 + n$ are multiples of 3.
So n is not a multiple of 3.
Therefore, if n is a multiple of 3, then $n^3 + n^2$ is a multiple of 3. ■

Proof. Suppose n is a multiple of 3. Then either $n^3 + n^2$ is a multiple of 3, or not.
If $n^3 + n^2$ is not a multiple of 3, then $n^3 + n^2 = 3k + 1$ for some integer k.
If $n^3 + n^2 = 3k + 1$, then $n(n^2 + n) = 3k + 1$.
So neither n, nor $n^2 + n$ are multiples of 3.
So n is not a multiple of 3.
But this contradicts the original assumption.
So, if n is a multiple of 3, then $n^3 + n^2$ is a multiple of 3. ■

2. Provide a counterexample to show that the statement "If $x > 0$, then $x^2 - 3x + 1 < 0$" is not true for all integers $x > 0$.

$$\begin{aligned}
\text{Let } x = 4. \text{ Then } x^2 - 3x + 1 &= (4)^2 - 3(4) + 1 \\
&= 16 - 12 + 1 \\
&= 5 \not< 0
\end{aligned}$$