Revisiting Underapproximate Reachability for Multipushdown Systems

Sparsa Roychowdhury, Prof. S Akshay, Prof. S Krishna, and Prof. Paul Gastin

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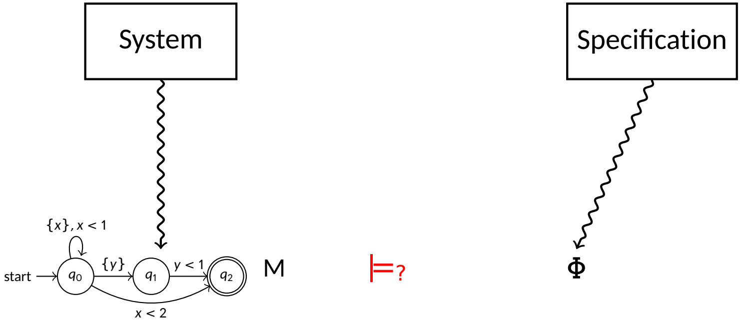
Outline

- Introduction
- Multistack Pushdown Automata (MPDA)
- Hole bounded runs of MPDA
- Algorithm
- Extension to time
- Implementation
- Experiments
- Conclusion

Introduction

- Checking the correctness of a system is essential
- Testing can guarantee the presence of bugs but can not guarantee the absence of it
- Formal verification is a celebrated area of computer science that proves the correctness of a system mathematically
- Model checking is one of the most promising approaches to formal verification

Model checking



$$L(M) \cap \overline{L(\Phi)} = \emptyset$$

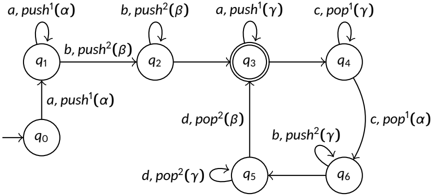
Figure: Model checking as a reachability/emptiness problem

Model checking contd.

- Finding a proper model is essential to capture the important behaviors of systems
- If the model is too powerful, model checking becomes difficult, sometimes undecidable!
- Thus, finding a good model that strikes a good balance between expressivity and complexity is very crucial in model checking

Multi-stack Pushdown Automata

- Verification of recursive concurrent programs is an active area of interest
- Behaviours of recursive concurrent programs can be modeled using multi-stack pushdown automata(MPDA)



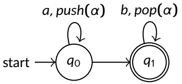
A Multistack Pushdown Automata accepting the language:
$L^{bh} = \{a^n b^n (a^{q_1} c^{q_1+1} b^{q'_1} d^{q'_1+1} \cdots a^{q_n} c^{q_n+1} b^{q'_n} d^{q'_n+1}) \mid n, q_i, q'_i \in \mathbb{N}, \forall i \in [n]\}$

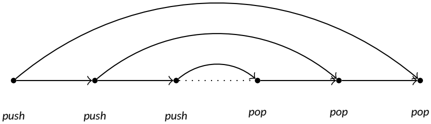
Multi-stack Pushdown Automata

- Multi-stack pushdown automata is Turing complete
- Reachability is Undecidable!!
- Under-approximation!
  1. Bounded Round La Torre et al. [2010], Qadeer and Wu [2004]
  2. Bounded Scope Torre et al. [2016]
  3. Bounded Phase La Torre et al. [2007] etc.

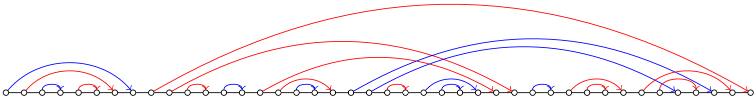
Capturing Behavior as Graphs

Graphs can be used to represent set of runs





Behavior Graph of an MPDA



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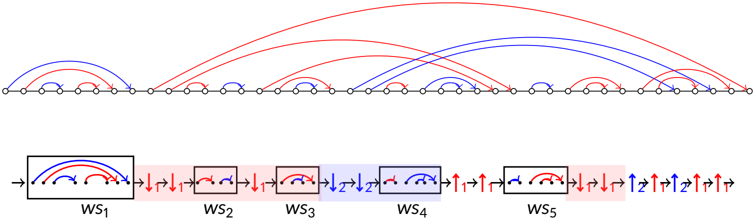

- Well-nested sequence (ws_j) can be ε

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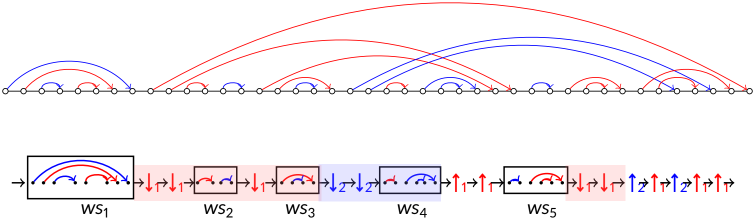

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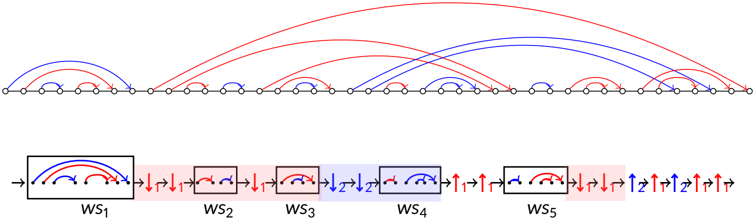

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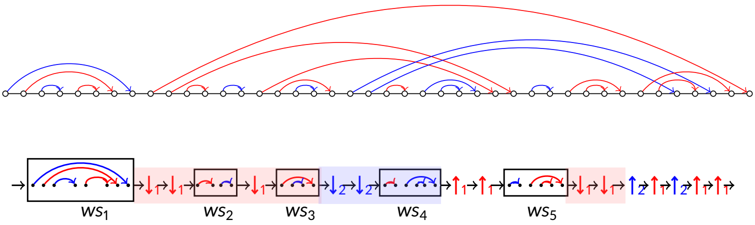

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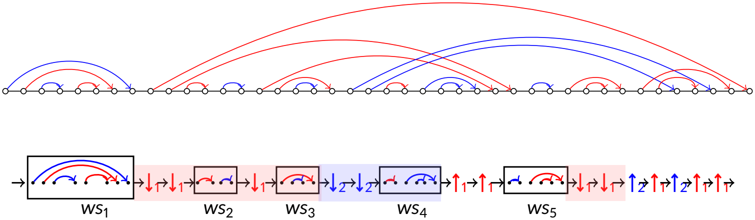

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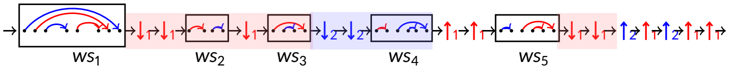
- An i-segment is the concatenation of multiple atomic-i-segments

Segments and holes



- **Pending-push($\downarrow_i$)** of stack i (w.r.t a position n)

  `No matching pop prior` n

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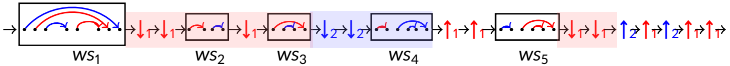

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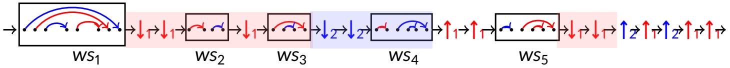

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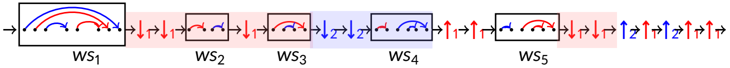
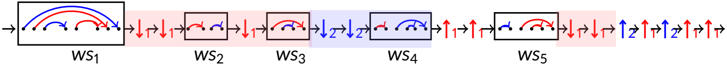


- **Pending-push($\downarrow_i$)** of stack i (w.r.t a position n)

  No matching pop prior n

- **An i-segment** is open if its first push transitions are pending
- **An i-hole** in a run is the maximal factor that is an open i-segment
- We denote by nbHoles(σ) the number of holes in a run σ

Hole Bounded runs of MPDA

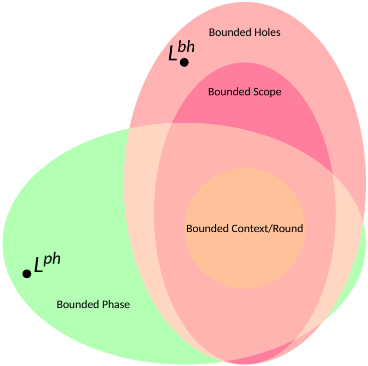


Definition

A run σ is said to be *K*-**hole** bounded if $\text{nbHoles}(\sigma) \leq K$, for $K \in \mathbb{N}$.

In **hole** bounded reachability of MPDA we only search for a **K-hole** bounded accepting run of MPDA where, **K**$\in \mathbb{N}$.

Hierarchy of Under-approximations



$$L^{ph} = \{(ab)^n c^n d^n | n \in \mathbb{N}\}$$

$$L^{bh} = \{a^n b^n (a^{q_1} c^{q_1+1} b^{q'_1} d^{q'_1+1} \cdots a^{q_n} c^{q_n+1} b^{q'_n} d^{q'_n+1}) \mid n, q_i, q'_i \in \mathbb{N}$$

Algorithm to Solve Hole bounded reachability

Question: Given a MPDA and a bound $K \in \mathbb{N}$ does there exists a K-hole bounded run that reaches a target state from the initial state?

- Algorithm
  - Uses fixed-point computation
  - Guided BFS exploration
  - Witness generation

Steps of the Algorithm

We divide the algorithm in two parts,

1. Fixed-point Computation and Memoization

   `Precomputes well nested runs (`ws`), atomic open-hole `$(\downarrow_i ws)$`, and open-hole `$(\downarrow_i ws)^+$`.`

Steps of the Algorithm

We divide the algorithm in two parts,

1. Fixed-point Computation and Memoization
2. On-The-Fly guided Breadth First Search (BFS) Exploration `Which solves hole bounded reachability for MPDA`

Computing Well-Nested Sequences

$$s_i \xrightarrow{\downarrow^n (\alpha)} s_j$$

Computing Well-Nested Sequences

$$\begin{tikzcd} s_i \arrow[r, "\downarrow^n (\alpha)"] & s_j \arrow[r] & s_k \end{tikzcd}$$

Computing Well-Nested Sequences

$$s_i \xrightarrow{\downarrow^n (\alpha)} s_j \longrightarrow s_k \xrightarrow{\uparrow^n (\alpha)} s_l$$

Computing Well-Nested Sequences

$$\begin{tikzcd} s_i \arrow[rr] \arrow[rr, bend left] & & s_l \end{tikzcd}$$

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For all state s_i, create the following entries,

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\left.
\begin{array}{c|c|c|c|c}
\ddots & s_j & \ldots & s_k & \ldots \\
\hline
\vdots & \vdots & \ldots & \vdots & \ldots \\
\hline
s_i & 1 & \ldots & 0 & \ldots \\
\hline
\vdots & \vdots & \ldots & \vdots & \ddots
\end{array}
\right\} \mid \mathscr{S} \mid
$$

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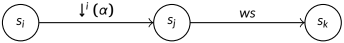
Transitive Closure will give us the **well-nested** sequences

Computing i-segments and i-holes

- An **i-segment ($\downarrow^i (\alpha) ws$)** for a stack i is a pending push($\downarrow^i$) followed by a maximal **well-nested** sequence(ws)



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$$s_i \xrightarrow{\downarrow^i(\alpha)} s_j \xrightarrow{ws} s_k$$

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- Assume we store this information in M^i

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$$

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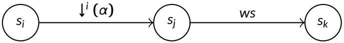
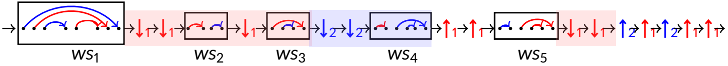
- We can pre-compute all possible **i-segments** by concatenating well-nested sequences after a push transition of stack i.
- Assume we store this information in M^i
- Dropping stack alphabet and doing transitive closure on M^i gives us information of all **i-Holes**, (M_i^h).



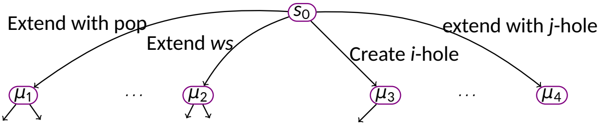
Blocks of a run of MPDA



Every run of a MPDA is made of three basic blocks

- Well-nested sequence (ws_j)
- Hole (sequence of $\downarrow_i$ ws_j)
- Matching-pops ($\uparrow_i$)

Breadth First Search Exploration



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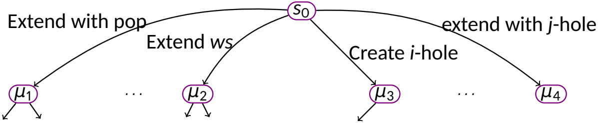

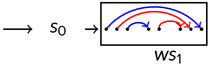

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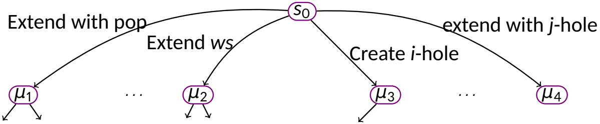

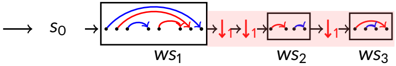

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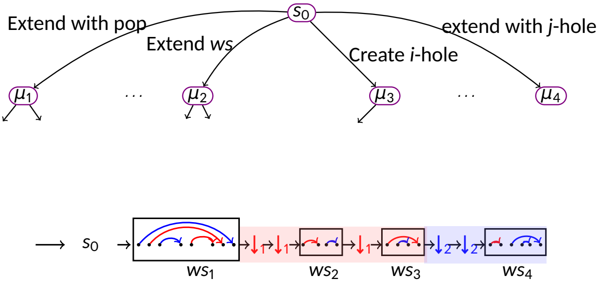

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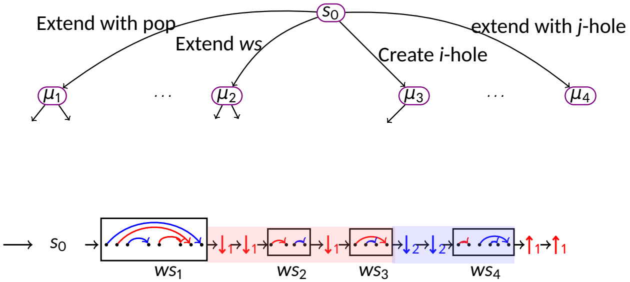

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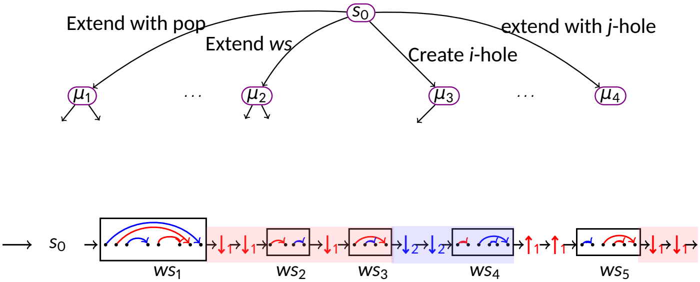

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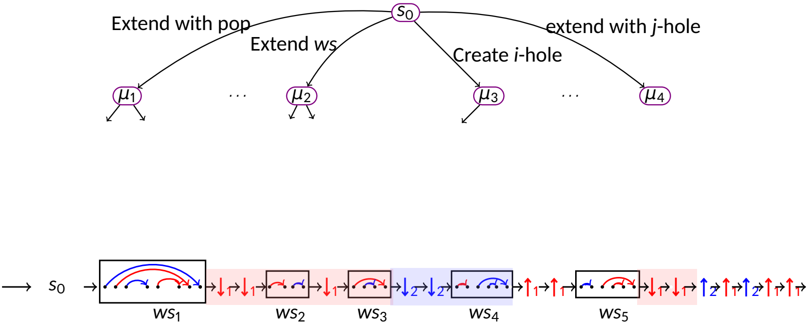

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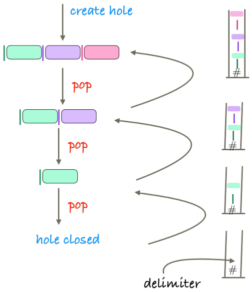

Timed Automata Alur and Dill [1994]

Timed automaton is a well-studied model that capture the behaviors of real-time systems

- Finite States + Clocks

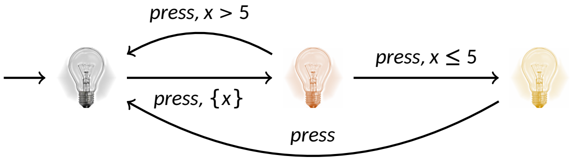

Figure: A Timed Automaton.

- Infinite state space
- Region construction
- Reachability is PSPACE-C

Extension to Time

We also extend the algorithm to handle Timed Multistack Pushdown Automata

- Clocks + Age (closed guards)
- Data Structures
- Timed witness generation

BHIM

- We have built a tool BHIM[1] based on the algorithm
- Works on both timed and untimed MPDA
- Generates a run as a witness of reachability
- We tested our tool on a set of benchmarks

[1] https://sparsa.github.io/bhim

Experiments

Name	Locations	Transitions	Stacks	Holes	Time Empty (mili sec)	Time Witness (mili sec)	Memory(KB)
Bluetooth	45	89	2	0	149.3	0.241	6934
MultiProdCons-1(2)	11	18	2	2	11.1	0.1	1796
MultiProdCons-2(3,2)	7	11	2	2	126.529	0.281	5632
MultiProdCons-2(24,7)	32	34	2	2	1879.33	10.63	21836
dm-target	22	27	2	2	26.483	0.279	6624
Binary Search Tree	29	78	2	2	60.8	5.1	5143
untimed-L^{crit}	6	10	2	2	14.9	0.7	4692
Untimed-Maze	9	12	2	0	8.25	0.07	5558
L^{bh}	7	13	2	2	22.2	0.6	4404

Table: Experimental results: Time Empty and Time Witness column represents no. of milliseconds needed for emptiness checking and to generate witness, respectively.

Name	Locations	Transitions	Stacks	Clocks	c_{max}	Aged(Y/N)	Holes	Time Empty(mili sec)	Time Witness (mili sec)	Memory(KB)
Bluetooth	45	89	2	0	2	Y	0	152.8	0.119	5568
L^{crit}	6	10	2	2	8	Y	2	9965.2	3.7	203396
Maze	9	12	2	2	5	Y	2	349.3	0.31	11604

Table: Experimental results of timed examples. The column c_{max} is defined as the maximum constant in the automaton, and Aged denotes if the stack is timed or not

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Thank You

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Questions?

Bibliography I

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Salvatore La Torre, Parthasarathy Madhusudan, and Gennaro Parlato. The language theory of bounded context-switching. In *Latin American Symposium on Theoretical Informatics*, pages 96–107. Springer, 2010.

P Madhusudan and Gennaro Parlato. The tree width of auxiliary storage. In *ACM SIGPLAN Notices*, volume 46, pages 283–294. ACM, 2011.

Shaz Qadeer and Dinghao Wu. Kiss: keep it simple and sequential. *ACM sigplan notices*, 39(6):14–24, 2004.

Salvatore La Torre, Margherita Napoli, and Gennaro Parlato. Scope-bounded pushdown languages. *International Journal of Foundations of Computer Science*, 27(02):215–233, 2016.

Decidability & Some Possible Algorithms

Theorem

The K-hole bounded reachability of MPDA *is decidable*

- Hole bounded ⇒ Tree-width bounded

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- Tree-width bounded ⇒ Decidable reachability Madhusudan and Parlato [2011]

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Difficult to design!

- Naive BFS exploration

Decidability & Some Possible Algorithms

Theorem

The K-hole bounded reachability of MPDA *is decidable*

- Hole bounded ⇒ Tree-width bounded
- Tree-width bounded ⇒ Decidable reachability Madhusudan and Parlato [2011]
- Tree-width based Algorithm Akshay et al. [2017],
  1. Capture the behaviour of the model as a graph
  2. Show that these graphs has bounded tree-width
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Difficult to design!

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Unnecessary branches!