A thesis on Fractional Quantum Hall Effect

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by

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Abstract

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In this work, I have presented a brief overview on Fractional Quantum Hall Effect and the different approaches to understand the fractional quantum hall states at different filling fractions, specifically, the Hierarchy approach and Composite fermion. I am currently understanding the system by focusing on the microscopic wavefunction of electrons. We can also look at the system in the field theory perspective where both Conformal field theory and Topological field theory (Chern Simons theory) plays a very important role. An overview on the Non-Abelian Quantum Hall States is also given at the end.

Contents

Chapter 1

Introduction to Laughlin States

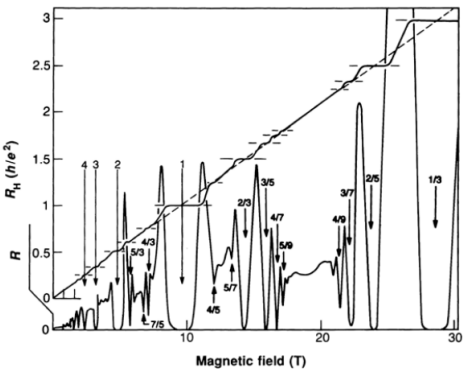


FIGURE 1.1: FQHE

It was discovered experimentally in 1982 that the plateaux in the hall resistivity can be observed at fractional values. Although the fractional values at which the plateaux is observed are some specific rational numbers. For instance, some prominient plateaux were observed at $\nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ etc. in the lowest landau level and $\nu = \frac{4}{3}, \frac{5}{3}, \frac{7}{5}, \frac{5}{2}, \frac{12}{5}, \frac{13}{5}$ etc.

How do we explain the formation of plateaux ? To explain this, we have to consider interactions between electrons into account. Let us assume that $\nu < 1$ so that the

lowest landau level is partially filled. We know that each landau level can accommodate a macroscopic number of states. Now we consider Coulomb interaction between electrons:

$$V_{coulomb} = \frac{e^2}{4\pi\epsilon_0|r_i - r_j|}$$

Now due to the effect of interactions between electrons, we see that there will be a split in the energy levels of degenerate ground states. To get correct wavefunction and energy levels, we may use degenerate perturbation theory but this would require to diagonalize a macroscopically large matrix. It will be also tough to perform this calculation numerically due to computational costs.

So how should we approach this problem ? The first person who approached this problem was **Laughlin** who described FQHE at filling fractions $\nu = \frac{1}{m}$ where m is an odd integer. The ground state wavefunction identified by Laughlin at $\nu = \frac{1}{m}$ is:

$$\Psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

Clearly this is anti-symmetric when m is an odd integer. For m an even integer, it can be thought of as a quantum Hall state for bosons. The pre-factor vanishes with a zero of order m whenever two electrons come together.

It can be shown numerically that, for small number of particles, Laughlin's wavefunction is almost correct. Although if the number of particles are very high (of the order 10^{10}), then Laughlin's wavefunction does not overlap with true ground state. Although, interestingly Laughlin states have the same fractional excitations and the same topological order as the true ground states.

1.1 Wigner Crystal

The Laughlin state should be thought of as a liquid phase of electrons. In fact, strictly speaking, it should be thought of as an entirely new phase of matter, distinguished by a property called **topological order**.

There is a competing solid phase in which the electrons form a two-dimensional triangular lattice, known as a **Wigner crystal**. It has lower energy than Laughlin state only when densities of electrons are low. It is observed at $\nu \leq \frac{1}{7}$.

Chapter 2

Quasiholes and Quasiparticles

We observe two types of Charge excitations: **Quasi-holes** and **Quasi-particles**:

2.1 Quasi-Holes

The wavefunction describing a quasi-hole at position $\eta \in C$ is:

$$\Psi_{hole}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)\prod_{k<l}(z_k - z_l)^m e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

It can be observed that the electron density vanishes at position η. In other words, a **hole** is created in the electron fluid. We can write the wavefunction for M quasi-holes at positions η_j with j = 1,2,...,M:

$$\Psi_{M-hole}(z;\eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

The quasi-hole carries a fraction of the electric charge of the electron. It has charge $e^* = \frac{e}{m}$.

2.2 Quasi-Particles

There are also excitations of the quantum hall fluid which carry charge $e^* = -\frac{e}{m}$. These are Quasi-particles. The wavefunction is given by:

$$\Psi_{particle}(z;\eta) = \left[\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i} - \bar{\eta}\right)\prod_{k<l}(z_k - z_l)^m\right]e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

Here the derivatives act only on the polynomial pre-factor; not on the exponential. It seems to be somewhat harder to write down quasi-particle eigenstates compared to quasi-hole eigenstates. To see the problem, note that we want to increase the density of electrons inside the Hall fluid and, hence, decrease the relative angular momentum of some pair of electrons. In the case of the quasi-hole, it was simple enough to increase the angular momentum: for example, for a hole at the origin we simply need to multiply the Laughlin wavefunction by the factor $\prod_i z_i$. But now that we want to decrease the angular momentum, we're not allowed divide by $\prod_i z_i$ as the resulting wavefunction is badly singular. Nor can we multiply by $\prod_i \bar{z}_i$. because, although this will decrease the angular momentum, the resulting wavefunction no longer sits in the lowest Landau level. Instead, a simple way to reduce the degree of a polynomial is to differentiate that is what is done on the wavefunction.

2.3 Anyon's Introduction

We know that there are two kinds of particles: Bosons and fermions. However, if particles are restricted to move in 2-dimensional system, then we observe another kind of particles which are neither bosons nor fermions. They are known as Anyons. They carry fractional charges and they follow fractional statistics.

We take 2 identical particles, described by the wavefunction $\Psi(r_1, r_2)$. Since the

particles are identical, all probabilities must be the same if the particles are exchanged. This tells us that $|\Psi(r_1, r_2)|^2 = |\Psi(r_2, r_1)|^2$ so that, upon exchange, the wave-functions differ by at most a phase:

$$\Psi(r_1, r_2) = e^{i\pi\alpha}\Psi(r_2, r_1)$$

Now suppose that we exchange again. Performing two exchanges is equivalent to a rotation, so should take us back to where we started. This gives the condition:

$$\Psi(r_1, r_2) = e^{i2\pi\alpha}\Psi(r_1, r_2)$$

There can be two possible values of α, which is 0 (bosons) and 1 (fermions). So where is the problem? The weak point is the statement that when we rotate two particles by we should get back to where we came from. Why should this be true? The answer lies in thinking about the topology of the worldlines particles make in spacetime.

In d = 3 spatial dimensions, the path that the pair of particles take in spacetime can always be **continuously connected** to the situation where the particles don't move at all. This is the reason the resulting state should be the same as the one before the exchange. But in d = 2 spatial dimensions, this is not the case: the worldlines of particles now wind around each other. Mathematically, what's going on is that in dimensions $d \geq 3$, the exchange of particles must be described by a representation of the permutation group. But, in $d = 2$ dimensions, exchanges are described a representation of the braid group.

Chapter 3

Ground State Degeneracy

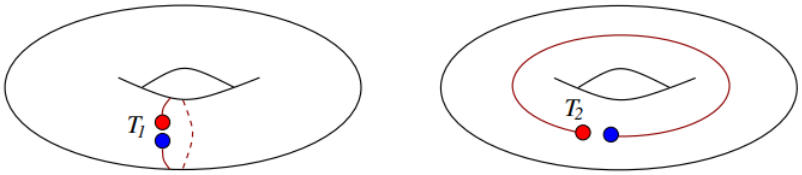


FIGURE 3.1: Taking a quasi-hole (red) and quasi-particle (blue) around one cycle of the torus or around the other

Consider the following process on a torus. We create from the vacuum a quasiparticle – quasi-hole pair. We then separate this pair, taking them around one of the two different cycles of the torus as shown in the figure, before letting them annihilate again. We'll call the operator that implements this process T_1 for the first cycle and T_2 for the second.

Now suppose we take the particles around one cycle and then around the other. Because the particles are anyons, the order in which we do this matters: there is a topological difference between the paths taken. Indeed, we can convince ourself that $T_1 T_2 T_1^{-1} T_2^{-1}$ is equivalent to taking one anyon around another: the worldlines have

linking number one. This means that the T_1 must obey the algebra

$$T_1 T_2 = e^{\frac{2\pi i}{m}} T_2 T_1$$

But such an algebra of operators can't be realised on a single vacuum state. This immediately tells us that the ground state must be degenerate. The smallest representation has dimension m, with the action

$$T_1|n>= e^{\frac{2\pi i n}{m}}|n>$$

$$T_2|n>= |n+1>$$

It should be noted that we don't have to say anything about the shape or sizes of these manifolds. The number of ground states depends only on the topology.

Chapter 4

Other Filling Fractions

We know that Laughlin states describes fractional quantum hall states at filling fraction $\nu = \frac{1}{m}$. Although there are many more states that are not governed by Laughlin's wavefunction. There are different approaches to understand the fractional quantum hall states at different filling fractions. Two most prominent ones are the **Hierarchy** and **Composite Fermion** approach.

4.1 The Hierarchy

Laughlin proposed the first model wavefunction for fractional quantum hall states at $\nu = \frac{1}{m}$.These states describe systems where strong electron-electron interactions create a highly correlated ground state that minimizes the Coulomb repulsion. After the Laughlin states, new fractional quantum Hall states can emerge from quasiparticles (excitations) of the parent Laughlin states. These quasiparticles themselves can form correlated states, giving rise to new filling factors. This **recursive generation of states** is known as the **hierarchy of states**.

Quasi-particle (or Quasi-hole) of the $\nu = \frac{1}{m}$ state form a new effective system. These quasi-particles condense into a Laughlin-like state, creating new filling factors at

$$\nu_{quasi} = \mp \frac{1}{2pm^2 \pm m}$$

where the overall sign is negative for holes and positive for particles. Adding this to the filling fraction of the original $\nu = \frac{1}{m}$ state, we have

$$\nu = \frac{1}{m} \mp \frac{1}{2pm^2 \pm m} = \frac{1}{m \pm \frac{1}{2p}}$$

Note that the filling fraction is decreased by quasi-holes and increased by quasi-particles.

Now we can go further. The quasi-objects in this new state can also form quantum Hall states. And so on. The resulting fillings are given by the continuous fractions

$$\nu = \frac{1}{m \pm \frac{1}{2p_1 \pm \frac{1}{2p_2 \pm \ldots}}}$$

For example, building on the Hall state $\nu = \frac{1}{3}$, the set of continuous fractions for quasi-particles with $p_i = 1$ leads to the sequence $\nu = \frac{2}{5}$ followed by $\nu = \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$ and $\frac{6}{13}$.

4.2 Composite Fermion

We start by introducing the idea of a **vortex**. Here, a vortex will mean a winding in the wavefunction itself. Ultimately we will be interested in vortices in the Laughlin wavefunction. We once again focus on the Laughlin's wavefunction:

$$\Psi(z_i) = \prod_{i<j} (z_i - z_j)^m$$

where we have omitted the exponential part as of now.

We observe that the wavefunction has a zero of order m as two electrons approach. This means that each particle can be thought of as m vortices. Of course, one of these zeros was needed by the Pauli exclusion principle. Moreover, we needed m zeros per particle to get the filling fraction right. But nothing forced us to have the other $m - 1$ zeros sitting at exactly the same place.

Motivated by this observation, we define **a composite fermion to be an electron bound to** $m - 1$ **further vortices**. The whole thing is a fermion when m is odd. Using the Chern-Simons theory, we can explain that there is a flux attached to the Composite fermion.It should be noted that the composite fermions don't carry real magnetic flux with them. They carry a different, emergent flux.

They experience a reduced effective magnetic field because the flux attached to the electrons partially cancels the external field. This reduction in the magnetic field allows composite fermions to form their own integer quantum Hall states, leading to fractional quantum Hall states for the original electrons. Because there is one electron per composite fermion, the density is the same. But because the magnetic fields experienced by electrons and composite fermions differ, the filling fractions must also differ: we must have

$$n = \frac{\nu^* B^*}{\Phi_0} = \frac{\nu B}{\Phi_0}$$

$$\nu = \frac{\nu^*}{1 + (m-1)\nu^*}$$

If we consider composite fermion to completely fill their lowest landau level so that $\nu^* = 1$. Then we get $\nu = \frac{1}{m}$. In other words, **the fractional quantum Hall effect can be thought of as an integer quantum Hall effect for composite fermions**.

Suppose that we fill the first ν^* Landau levels to get an integer quantum Hall effect for composite fermions with $\nu^* > 1$. Then we find filling fractions that are different from the Laughlin states. For example, if we pick $m = 3$, then the sequence of states arising is $\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ etc. These are the same sequence that we saw in the hierarchy construction.

If we again focus on Laughlin wavefunction, we observe:

$$\Psi(z_i) = \prod_{i<j}(z_i - z_j)^{m-1} \prod_{k<l}(z_k - z_l)$$

The second term in this decomposition is simply the wavefunction for the fully-filled lowest Landau level. We can think of the first term as attaching $m - 1$ vortices to each position z_i to form the composite fermion.

Inspired from the above Laughlin wavefunction, we again write a guess for the wavefunction, referred to as the **Jain Wavefunction** given by:

$$\Psi_\nu(z) = P_{LLL}\left[\prod_{i<j}(z_i - z_j)^{m-1}\Psi_{\nu^*}(z, \bar{z})\right]$$

where we have again neglected the exponential part.

Here $\prod_{i<j}(z_i - z_j)^{m-1}$ factor attaches the $(m-1)$ vortices to each electron. Ψ_{ν^*} is the wavefunction for $\nu^* \in Z$ for fully filled landau level although it should be noted that it is not for the **lowest** landau level since Ψ_{ν^*} is not the holomorphic

function. Therefore P_{LLL} project to the lowest Landau level. Operationally, P_{LLL} is defined by moving all factors of $\bar{z}_i$ in [...] to the left. We then make the substitution

$$\bar{z}_i \implies 2l_B^2 \frac{\partial}{\partial z_i}$$

Chapter 5

Non-Abelian Quantum Hall States

We recall that the quantum Hall effect arises when a two-dimensional electron gas is subjected to a perpendicular magnetic field. The resulting highly quantized conductance plateaus correspond to distinct quantum Hall states. These states can be classified as **Abelian** or **Non-Abelian**, depending on the mathematical properties of the anyons that emerge in these systems.

When two identical Abelian anyons are exchanged, the wavefunction acquires a phase factor. The braiding (interchange) of particles leads to a commutative operation. While in the case of Non-Abelian anyons, the wavefunction transforms into a different state in a manner dependent on the order of exchanges. The braiding operations do not commute and are described by unitary matrices. Exchanging quasi-particles creates a transformation in the multi-particle quantum state, which encodes information non-locally. This property is key to fault-tolerant quantum computing.

5.1 Theory of Non Abelian Anyons

We consider a list of the different types of anyons that we have in our model. We'll call them a, b, c, etc. We include in this list a special state which has no particles. This is called the vacuum and is denoted as 1.

5.1.1 Fusion

When we bring two anyons together, the object that we're left with must, when viewed from as far, also be one of the anyons on our list. The subtlety is that we need not be left with a unique type of anyon when we do this. We denote the possible types of anyon that can arise as a and b are brought together as

$$a \star b = \sum_c N_{ab}^c \ c$$

where N_{ab}^c is an integer that tells us how many different ways are there to get an anyon of type c. It should also be noted that $a \star b = b \star a$ and $N_{ab}^c = N_{ba}^c$. We can also interpret the equation the other way round: if a specific anyon c appears on the right of this equation, then there is a way for it to split into anyons of type a and b. The vacuum 1 is the trivial state in the sense that $a \star 1 = a$ for all a.

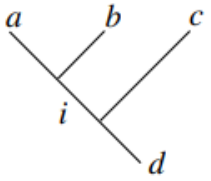
One of the surprising fact about Non-Abelian anyons is the fact that we don't think of a single anyon as having any internal degrees of freedom and, correspondingly, it has no associated Hilbert space beyond its position degree of freedom. Yet a pair of anyons do carry extra information. The Hilbert space H_{ab} describing the "internal" state of a pair of anyons has dimension:

$$dim(H_{ab}) = \sum_c N_{ab}^c$$

The anyons are called Non-Abelian anyons whenever $N_{ab}^c \geq 2$ for some a, b and c. The information contained in this Hilbert space is not carried by any local degree of freedom. Indeed, when the two anyons a and b are well separated, the wavefunctions describing different states in Hab will typically look more or less identical in any local region. The information is carried by more global properties of the wavefunction. For this reason, the Hilbert space H_{ab} is sometimes called the topological Hilbert space.

5.1.2 The Fusion Matrix

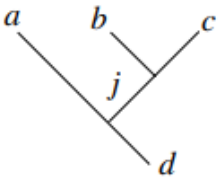
Suppose that we have three anyons, a, b and c. We first fuse a and b together and, of all the possibilities allowed by the fusion rules, we get some specific anyon i. We subsequently fuse i with c and end up with a specific anyon d. All of this is captured by a fusion tree which looks like this:



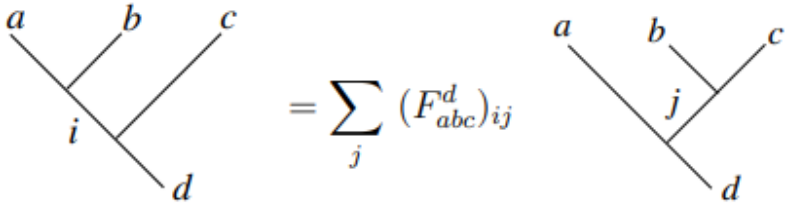
We list the anyons that we start with at the top and then read the tree by working downwards to see which anyons fuse to which. Alternatively, we can read the tree by starting at the bottom and thinking of anyons as splitting. Importantly, there

can be several different anyons i that appear in the intermediate channel.

Now suppose that we do the fusing in a different order: we first fuse b with c and subsequently fuse the product with a. We ask that the end product will again be the anyon d. But what will the intermediate state be? There could be several different possibilities j.



The question we want to ask is: if we definitely got state i in the first route, which of the states j appear in the second route. In general, there won't be a specific state j, but rather a linear combination of them. This is described graphically by the equation



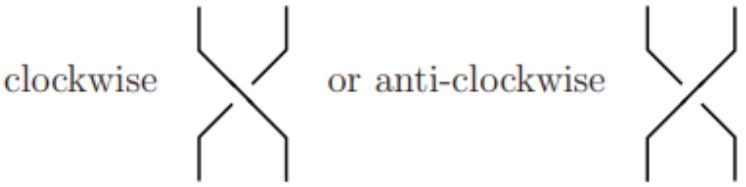
where the coefficients $(F^d_{abc})_{ij}$ are thought of as the coefficients of a unitary matrix, (F^d_{abc}), specified by the four anyons a, b, c and d. This is called the **fusion matrix**.

If there are more than one ways in which the anyons j can appear in intermediate states then we should sum over all of them and, correspondingly, the fusion matrix

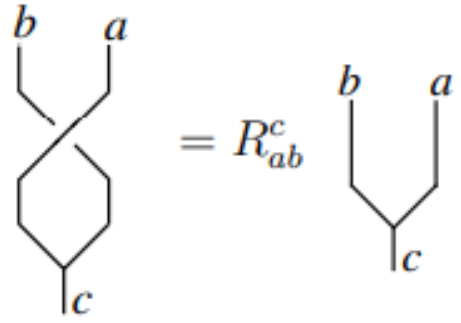
should have more indices. More crucially, sometimes there will be multiple ways in which the final state d can appear. This will happen whenever $N_{aj}^{d} \geq 2$ for some j. In this case,the process on the left will typically give a linear combination of the different d states on the right. The fusion matrix should also include indices which sum over these possibilities.

5.1.3 Braiding

The braiding of two anyons can be done either in clockwise or anti-clockwise direction:



Suppose that we fuse two anyons a and b together to get c. We then do this again, but this time braiding the two anyons in an anti-clockwise direction before fusing. The resulting states are related by the R-matrix, defined by:



If $N_{ab}^c = 1$, so that there only a single option for the final anyon, then R_{ab}^c is simply a complex phase. However, if $N_{ab}^c \geq 2$, so that there are several different ways of getting the final anyon c, then there's no reason we should get the same state after the exchange. In this case, the R-matrix is a genuine matrix of size N_{ab}^c x N_{ab}^c and we should be summing over all possible final states on the right-hand side.

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