A Journey from Order to Chaos: Exploring Bifurcations and Nonlinear Dynamics in a Vibrating Double Pendulum System

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Contents

1 Introduction

In a world governed by seemingly predictable laws of nature, there exists some mechanical system that defies our intuitions and reveals the remarkable beauty of nonlinear and chaotic system.

These systems often exhibit complex and intriguing behavior. One such system that has attracted considerable attention is the vibrating double pendulum. The vibrating double pendulum is a simple mechanical system consisting of two pendulums connected to each other and the suspension point is oscillating in a simple harmonic motion with certain frequency. Unlike a single pendulum that oscillates in a predictable manner, the double pendulum's motion is characterized by its sensitivity to initial conditions, resulting in rich and unpredictable dynamics. This property makes it an excellent candidate for studying nonlinear dynamics and chaos.

In this paper, we explored the dynamics and bifurcation of a vibrating double pendulum system. Bifurcation refers to the sudden qualitative changes in a system's behavior as a result of small variations in its parameters. Understanding these bifurcations is crucial for predicting and controlling the system's behavior.

Overall, the study of the dynamics and bifurcation of vibrating double pendulum systems offers a fascinating glimpse into the intricate world of nonlinear dynamics and chaos. By unraveling the complexities of this system, we hope to gain deeper insights into the behavior of nonlinear systems in general, ultimately paving the way for improved control and understanding of complex physical phenomena.

2 Mathematical Formulation

In this section, the derivation of the equation of motion of both bobs in the vibrating double pendulum system has been done with the help of Euler-Lagrangian Equation. The effects of damping has also been considered in the system.
We consider that the first bob is at the position (x_1,y_1) and it makes an angle θ_1 with the vertical from the suspension point.And the second bob will be at position (x_2,y_2) and it makes an angle θ_2 with the vertical from the first bob.The mass of both the bobs are m and length of the strings are l.The suspension point is performing SHM with amplitude a and with frequency ω.

The coordinates of both the bobs are as follow:

$$x_1 = l\sin\theta_1 + a\sin\omega t$$

$$y_1 = -l\cos\theta_1$$

$$x_2 = l\sin\theta_1 + l\sin\theta_2 + a\sin\omega t$$

$$y_2 = -l\cos\theta_1 - l\cos\theta_2$$

Differentiating above equations, we get the velocities of all the coordinates:

$$\dot{x_1} = l\dot{\theta_1}\cos\theta_1 + a\omega\cos\omega t$$
$$\dot{y_1} = l\dot{\theta_1}\sin\theta_1$$
$$\dot{x_2} = l\dot{\theta_1}\cos\theta_1 + l\dot{\theta_2}\cos\theta_2 + a\omega\cos\omega t$$
$$\dot{y_2} = l\dot{\theta_1}\cos\theta_1 + l\dot{\theta_2}\cos\theta_2$$

After writing the co-ordinates and their velocities, it's time to write the kinetic energy, potential energy and thereby finding the Lagrangian of the system.

$$T = \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2 + \dot{x_2}^2 + \dot{y_2}^2)$$

After substituting the values of velocities on T and on further simplification, we get the final result as follow:

$$T = ml^2\dot{\theta_1}^2 + \frac{ml^2\dot{\theta_2}^2}{2} + ml^2\cos(\theta_1 - \theta_2)\dot{\theta_1}\dot{\theta_2} + mla\omega\cos(\omega t)(2\cos(\theta_1)\dot{\theta_1} + \cos(\theta_2)\dot{\theta_2}) + ma^2\omega^2\cos^2\omega t$$

$$U = -mgl(2\cos\theta_1 + \cos\theta_2)$$

Lagrangian of the system is given by as follow:

$$L = T - U$$

$$L = ml^2\dot{\theta_1}^2 + \frac{ml^2\dot{\theta_2}^2}{2} + ml^2\cos(\theta_1 - \theta_2)\dot{\theta_1}\dot{\theta_2} + mla\omega\cos(\omega t)(2\cos(\theta_1)\dot{\theta_1} + \cos(\theta_2)\dot{\theta_2})$$
$$+ ma^2\omega^2\cos^2\omega t + 2mgl\cos\theta_1 + mgl\cos\theta_2$$

Now, for this system, we have also considered the effects of damping. Below, we have written the Rayleigh's Dissipation function.

$$F = \frac{1}{2}b(\dot{x_1}^2 + \dot{y_1}^2 + \dot{x_2}^2 + \dot{y_2}^2)$$

$$F = bl^2\dot{\theta_1}^2 + \frac{bl^2\dot{\theta_2}^2}{2} + bl^2\cos(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} + bla\omega\cos(\omega t)(2\cos(\theta_1)\dot{\theta_1} + \cos(\theta_2)\dot{\theta_2})$$
$$+ba^2\omega^2\cos^2\omega t$$

2.1 Euler-Lagrangian Equation

By solving the Euler-Lagrangian equation, we get the equation of motion for both bobs. Since the system also consist of damping, the Euler-Lagrangian equation for first and second bob are given by following equations respectively:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_1}}) - \frac{\partial L}{\partial \theta_1} + \frac{\partial F}{\partial \dot{\theta_1}} = 0$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_2}}) - \frac{\partial L}{\partial \theta_2} + \frac{\partial F}{\partial \dot{\theta_2}} = 0$$

Substituting the values of L and F and on solving the above equations, we get the following equation of motion for both bobs.

Equation of motion for first bob

$$2ml^2\ddot{\theta_1} + ml^2\cos(\theta_1-\theta_2)\ddot{\theta_2} + ml^2\sin(\theta_1-\theta_2)\dot{\theta_2}^2 + 2bl^2\dot{\theta_1} +$$
$$bl^2\cos(\theta_1-\theta_2)\dot{\theta_2} + 2mgl\sin(\theta_1) + 2la\omega\cos(\theta_1)(b\cos(\omega t) - m\omega\sin(\omega t)) = 0$$

Equation of motion for second bob

$$ml^2\ddot{\theta_2} + ml^2\cos(\theta_1-\theta_2)\ddot{\theta_1} - ml^2\sin(\theta_1-\theta_2)\dot{\theta_1}^2 + bl^2\dot{\theta_2} +$$
$$bl^2\cos(\theta_1-\theta_2)\dot{\theta_1} + mgl\sin(\theta_2) + la\omega\cos(\theta_2)(b\cos(\omega t) - m\omega\sin(\omega t)) = 0$$

2.2 Further Simplification to obtain the system of vibrating double pendulum

To proceed further it is helpful to express equations of motion of both bobs in dimensionless form. This reduces the number of parameters by grouping them together into dimensionless group. This reduction will simplify the analysis.

We will start with defining dimensionless time τ by:

$$\tau = \frac{t}{T}$$

where T is a characteristic time scale. Now we will express new derivatives in terms of the old ones.

$$\dot{\theta} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\theta}{d\tau}$$

and similarly

$$\ddot{\theta} = \frac{1}{T^2}\frac{d^2\theta}{d\tau^2}$$

Now on dividing equations of motion of both bobs by $\frac{ml^2}{T^2}$ and on using new derivatives, we get the following results. (Note that from now on $\dot{\theta} = \frac{d\theta}{d\tau}$ and $\ddot{\theta} = \frac{d^2\theta}{d\tau^2}$)

$$\ddot{\theta_1} + \frac{\cos(\theta_1 - \theta_2)}{2}\ddot{\theta_2} + \frac{\sin(\theta_1 - \theta_2)}{2}\dot{\theta_2}^2 + \delta\dot{\theta_1} +$$

$$\delta\frac{\cos(\theta_1 - \theta_2)}{2}\dot{\theta_2} + \epsilon\sin(\theta_1) + \gamma\cos(\theta_1)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau)) = 0$$

and

$$\ddot{\theta_2} + \cos(\theta_1 - \theta_2)\ddot{\theta_1} - \sin(\theta_1 - \theta_2)\dot{\theta_1}^2 + \delta\dot{\theta_2} +$$

$$\delta\cos(\theta_1 - \theta_2)\dot{\theta_1} + \epsilon\sin(\theta_2) + \gamma\cos(\theta_2)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau)) = 0$$

where δ, μ, ϵ and γ are parameters such that,

$$\delta = \frac{bT}{m}$$

$$\mu = \omega T$$

$$\epsilon = \frac{gT^2}{l}$$

$$\gamma = \frac{a\omega T}{l}$$

We realize that the given system is a 4-Dimensional time dependent system. The dynamical variables which are required (at a particular instant) to predict the motion of the system are θ_1, $\dot{\theta_1}$, θ_2 and $\dot{\theta_2}$. Apart from that, δ, μ, ϵ and γ are parameters which will affect the dynamics of the system at bifurcation points.

The following is the 4-Dimensional time dependent system of vibrating double pendulum:
Let $x_1 = \theta_1$, $x_2 = \dot{\theta_1}$, $x_3 = \theta_2$ and $x_4 = \dot{\theta_2}$.Then,

$$\dot{x_1} = x_2$$

$$\begin{aligned}\dot{x_2} = -\frac{\sin(x_1 - x_3)}{2 - \cos^2(x_1 - x_3)}x_4^2 - \frac{\cos(x_1 - x_3)\sin(x_1 - x_3)}{2 - \cos^2(x_1 - x_3)}x_2^2 - \delta x_2 \\ -\frac{2\epsilon\sin(x_1)}{2 - \cos^2(x_1 - x_3)} + \frac{\epsilon\cos(x_1 - x_3)\sin(x_3)}{2 - \cos^2(x_1 - x_3)} \\ -\gamma(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))\frac{2\cos(x_1) - \gamma\cos(x_1 - x_3)\cos(x_3)}{2 - \cos^2(x_1 - x_3)}\end{aligned}$$

$$\dot{x_3} = x_4$$

$$\begin{aligned}\dot{x_4} = \frac{2\sin(x_1 - x_3)}{2 - \cos^2(x_1 - x_3)}x_2^2 + \frac{\cos(x_1 - x_3)\sin(x_1 - x_3)}{2 - \cos^2(x_1 - x_3)}x_4^2 - \delta x_4 \\ -\frac{2\epsilon\sin(x_3)}{2 - \cos^2(x_1 - x_3)} + \frac{2\epsilon\cos(x_1 - x_3)\sin(x_1)}{2 - \cos^2(x_1 - x_3)} \\ -2\gamma(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))\frac{\cos(x_3) - \gamma\cos(x_1 - x_3)\cos(x_1)}{2 - \cos^2(x_1 - x_3)}\end{aligned}$$

2.3 Nonlinearity of the System

In the context of the vibrating double pendulum system, non linearity refers to the fact that the equations governing its motion are nonlinear. It exhibits complex and nonlinear motion that are sensitive to small changes in initial conditions or system parameters. Such motion are known as **Chaotic** motion.

This chaotic behavior arises due to the non linearity of the system, making it inherently unpredictable and challenging to analyze. Understanding the non linearity of the double pendulum system is essential not only for studying its inherent complexity but also for broader applications in various fields.

2.4 Phase Space of the System

Phase space is a mathematical concept that provides a comprehensive representation of the state of a dynamical system. It is a multi-dimensional space where each point corresponds to a specific configuration of the system's variables at a given time. By visualizing the system's behavior in phase space, we can gain valuable insights into its dynamics and understand how it evolves over time.

In the case of the vibrating double pendulum system, the phase space would consist of multiple dimensions, such as the angles and angular velocities of the two pendulums. The coordinates of a point in phase space specify the values of these variables at a particular instant.

Phase space allows us to study the effects of bifurcations. By analyzing the curve traced by the system in phase space, we can gain insights about the bifurcation points which are basically those values of parameters at which the attractors might born or perish at different points and as a consequence of that it completely changes the dynamics of the system at those particular points.

Additionally, phase space provides a convenient framework for studying the stability of the system. Stable states correspond to attractors that nearby trajectories tend to approach over time, while unstable states are associated with trajectories that diverge or move away from each other. By analyzing the flow of points in phase space, we can determine the stability of different regions and understand how small perturbations in initial conditions or system parameters affect the system's behavior.

In summary, phase space is a powerful tool for visualizing and understanding the dynamics of a vibrating double pendulum system. It allows us to identify attractors, analyze their stability, observe the effects of bifurcations, and gain insights into the system's long-term behavior. By examining the trajectories of points in phase space, we can unravel the complex and fascinating dynamics inherent in nonlinear systems like the double pendulum.

3 Dynamics and Phase Space of the system for different initial conditions

In this section, we will see the dynamics shown by the system for different initial conditions. Also we will take a look at the relationship between θ_1 and θ_2.

Case 1: When initial angles are very small ($\theta_1 \approx 0°$ and $\theta_2 \approx 0°$):

For small angles, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. The 4D system simplifies to the following.

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -(x_1 - x_3)x_4^2 - (x_1 - x_3)x_2^2 - \delta x_2 - 2\epsilon x_1 + \epsilon x_3 - \gamma(2-\gamma)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = 2(x_1 - x_3)x_2^2 + (x_1 - x_3)x_4^2 - \delta x_4 - 2\epsilon x_3 + 2\epsilon x_1 - 2\gamma(1-\gamma)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))$$

Let T = 1, $\delta = 0$ (b=0), l = 1, m = 1 and a = 0.1. We will see the relation between θ_1 and θ_2 for different values (order) of ω.

1) $\omega \approx 0.001$

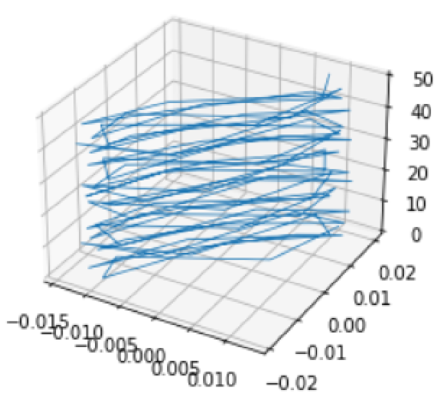

Figure 1: Plot between θ_1, θ_2 and time

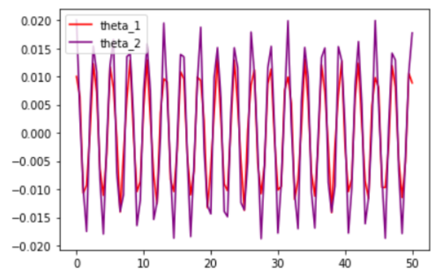

Figure 2: $\omega \approx 0.001$

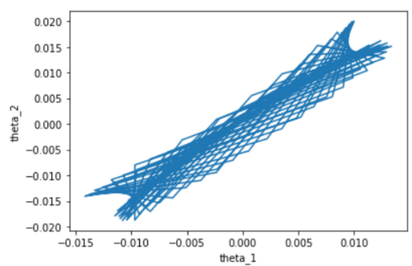

Figure 3: Plot between θ_1 and θ_2

2) $\omega \approx 0.1$

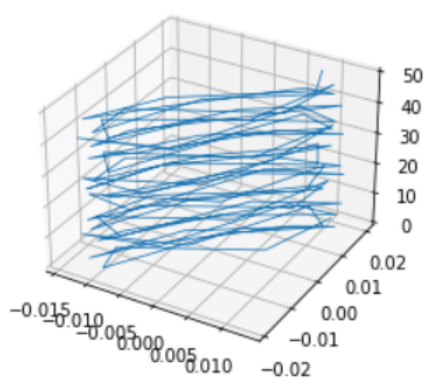

Figure 4: Plot between θ_1, θ_2 and time

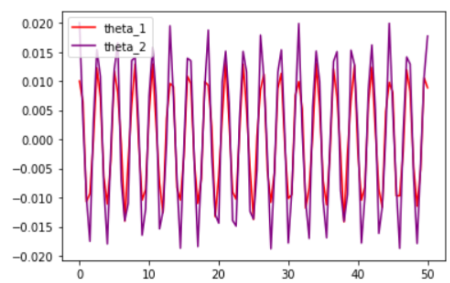

Figure 5: $\omega \approx 0.1$

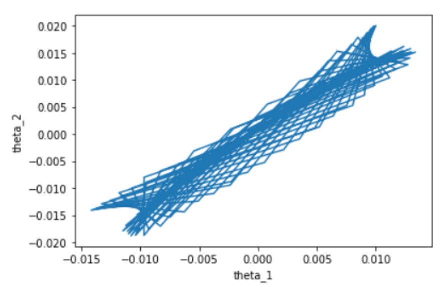

Figure 6: Plot between θ_1 and θ_2

3) $\omega \approx 1000$

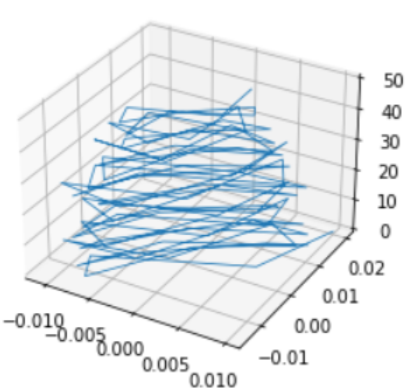

Figure 7: Plot between θ_1, θ_2 and time

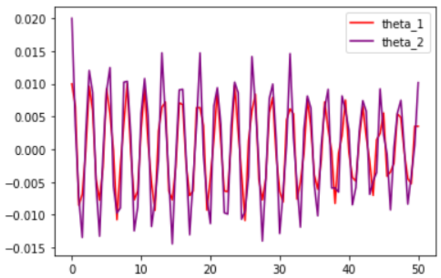

Figure 8: $\omega \approx 1000$

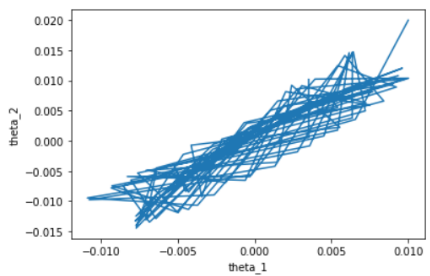

Figure 9: Plot between θ_1 and θ_2

We observe that if the initial angles are very small, then both angles are almost sinusoidal irrespective of the frequency of suspension point (ω). However it can also be seen that when omega was very large ($\omega \approx 1000$) then the amplitude of both the angles started to decrease after some time. The below link will show the simulation of the system when initial angles were very small.

Case 2: When double pendulum is inverted ($\theta_1 \approx \pi$ and $\theta_2 \approx \pi$):

For inverted double pendulum, $\sin\theta \approx 0$ and $\cos\theta \approx -1$. The 4D system simplifies to the following.

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -(x_1 - x_3)x_4^2 - (x_1 - x_3)x_2^2 - \delta x_2 + \gamma(2 - \gamma)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = 2(x_1 - x_3)x_2^2 + (x_1 - x_3)x_4^2 - \delta x_4 + 2\gamma(1 - \gamma)(\delta\cos(\mu\tau) - \mu\sin(\mu\tau))$$

1) $\omega \approx 0.001$

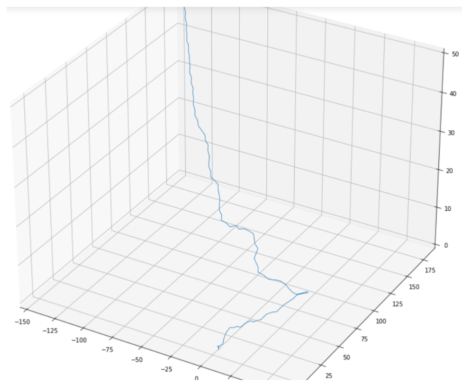

Figure 10: Plot between θ_1, θ_2 and time

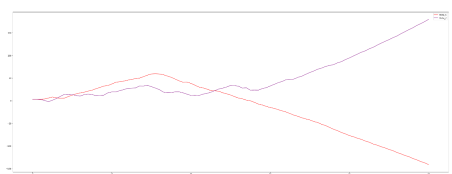

Figure 11: $\omega \approx 0.001$

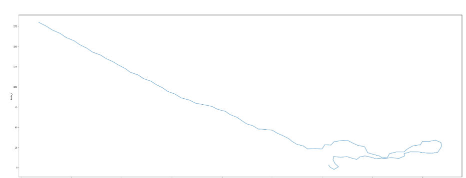

Figure 12: Plot between θ_1 and θ_2

2) $\omega \approx 0.1$

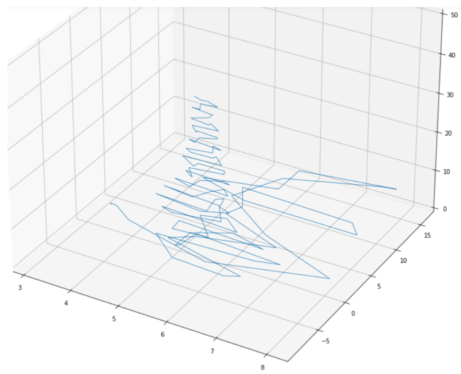

Figure 13: Plot between θ_1, θ_2 and time

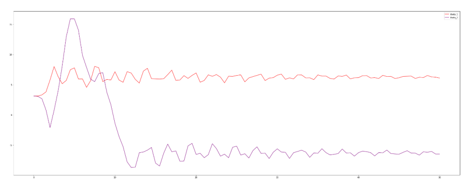

Figure 14: $\omega \approx 0.1$

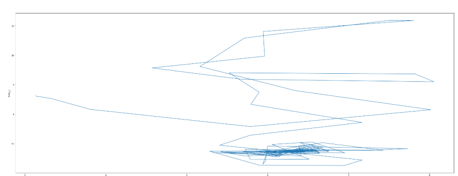

Figure 15: Plot between θ_1 and θ_2

3) $\omega \approx 1000$

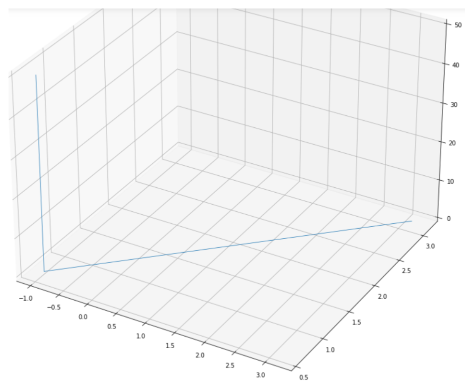

Figure 16: Plot between θ_1, θ_2 and time

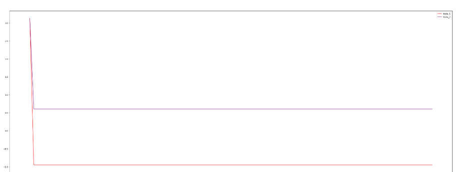

Figure 17: $\omega \approx 1000$

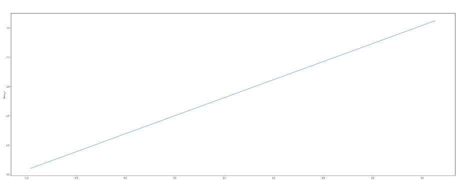

Figure 18: Plot between θ_1 and θ_2

Case 3: When $\theta_1 \approx \pi$ and $0 < \theta_2 < \pi$:

The 4D system for this particular case is as follows:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{-\sin x_3}{2-\cos^2 x_3}x_4^2 + \frac{\sin x_3 \cos x_3}{2-\cos^2 x_3}x_2^2 - \delta x_2 - \frac{\epsilon \sin x_3 \cos x_3}{2-\cos^2 x_3} - \gamma(\delta\cos\mu\tau - \mu\sin\mu\tau)\frac{\gamma\cos^2 x_3 - 2}{2-\cos^2 x_3}$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = \frac{2\sin x_3}{2-\cos^2 x_3}x_2^2 - \frac{\sin x_3\cos x_3}{2-\cos^2 x_3}x_4^2 - \delta x_4 - \frac{2\epsilon\sin x_3}{2-\cos^2 x_3} - 2\gamma(\delta\cos\mu\tau - \mu\sin\mu\tau)\frac{\cos x_3 - \gamma\cos x_3}{2-\cos^2 x_3}$$

1) $\omega \approx 0.001$

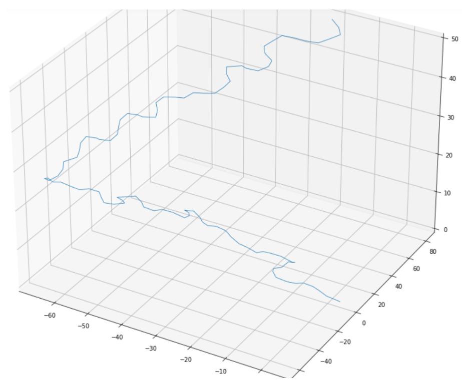

Figure 19: Plot between θ_1, θ_2 and time

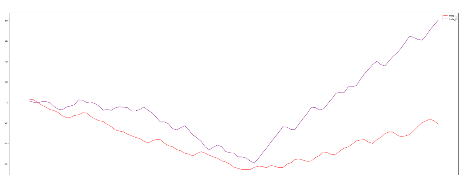

Figure 20: $\omega \approx 0.001$

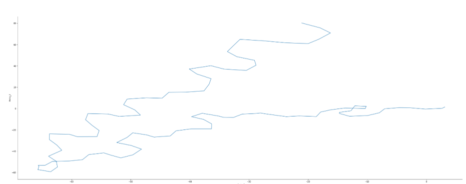

Figure 21: Plot between θ_1 and θ_2

2) $\omega \approx 0.1$

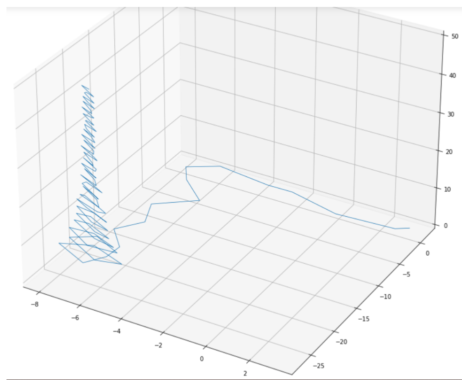

Figure 22: Plot between θ_1, θ_2 and time

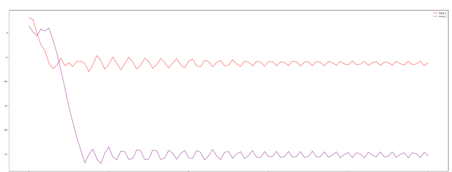

Figure 23: $\omega \approx 0.1$

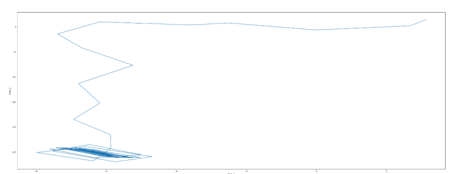

Figure 24: Plot between θ_1 and θ_2

3) $\omega \approx 1000$

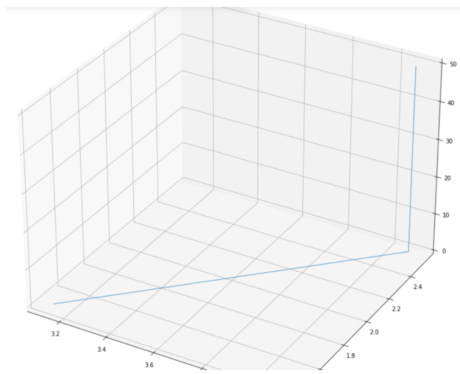

Figure 25: Plot between θ_1, θ_2 and time

Figure 26: $\omega \approx 1000$

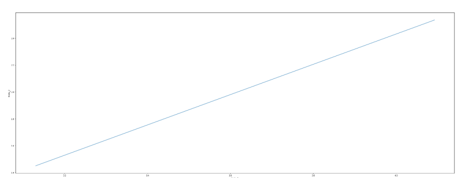

Figure 27: Plot between θ_1 and θ_2

3.1 Plot of θ_1 and θ_2 vs Time for different ranges of ω

Case1: When Initial Angles were very small

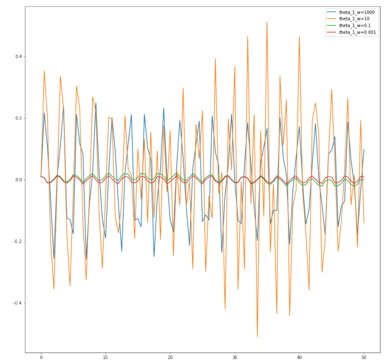

Figure 28: Plot between θ_1 and Time

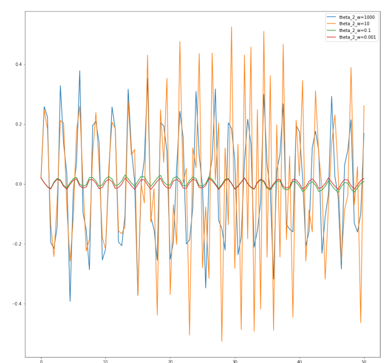

Figure 29: Plot between θ_2 and Time

Case2: When Double Pendulum is inverted

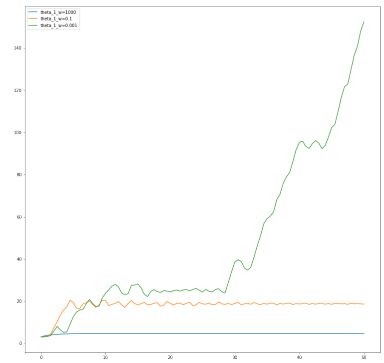

Figure 30: Plot between θ_1 and Time

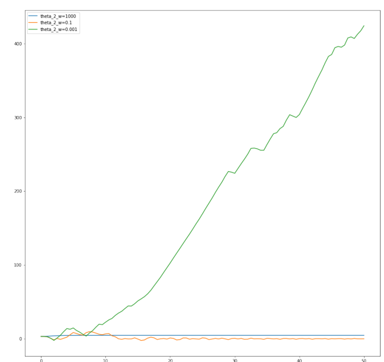

Figure 31: Plot between θ_2 and Time

Case3: When one of the angle is inverted

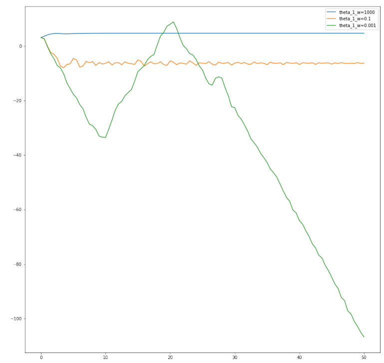

Figure 32: Plot between θ_1 and Time

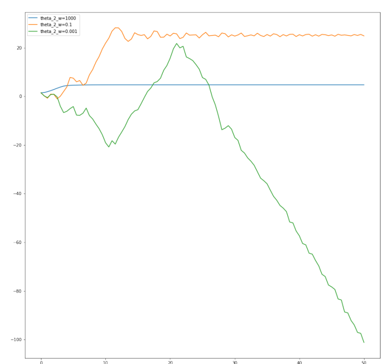

Figure 33: Plot between θ_2 and Time

3.2 Normal Modes of the System

In the context of the vibrating double pendulum system, the normal modes correspond to the eigenmodes of the system, each with its own natural frequency and mode shape.

Although, the vibrating double pendulum system is inherently nonlinear, and finding exact analytical solutions for the normal modes can be challenging. In nonlinear systems, the normal modes are more complex and can involve intricate combinations of different vibrational frequencies and mode shapes.

However, if we consider the first case, when initial angles were very small, then we can get the normal modes of the system meaning that we can find some values of ω (i.e frequency of suspension point) for which both the bobs will be either in phase or out of phase with each other. This has been done with the help of programming using numpy, scipy, sympy and matplotlib libraries. The code can be found by clicking here.

4 Numerical Simulations

To study the dynamics of the vibrating double pendulum system numerically, we can use various numerical methods for solving the ordinary differential equations (ODEs) that describe the system's motion. One common and effective method is the Runge-Kutta method (specifically, RK45) for numerical integration of the ODEs.

In the context of this system, the RK45 method is applied to integrate the system of ODEs that describe the positions and velocities of the pendulums over time. The method iteratively calculates the derivatives at different time steps and updates the state variables accordingly. The adaptive step size ensures accurate results even when the dynamics change rapidly.

Apart from that Python's Matplotlib library provides tools for creating animations and visualizations of dynamic systems. It can be used to generate 2D and 3D plots and animations to represent the evolution of the system's state variables over time.

In summary, numerical methods and simulations are powerful tools for studying complex dynamical systems, providing flexibility, accuracy, and insight into the behavior of systems that lack analytical solutions.

4.1 Simulations illustrating the motion of the angles in Phase Space

Although our system is a 4-D time dependent system or we could also say a 5-D time independent system (considering time as a dynamical variable), we present the simulation of the dynamics of both angles in 3-D phase space where θ_1 is along x-axis, θ_2 is along y-axis and time t is along the z-axis for different cases as mentioned in the previous section.

Case1: $\theta_1 \approx 0^\circ$ and $\theta_2 \approx 0^\circ$

Click to Play Animation

Case2: $\theta_1 \approx \pi$ and $\theta_2 \approx \pi$

Click to Play Animation

Case3: $\theta_1 \approx \pi$ and $0 < \theta_2 < \pi$

Click to Play Animation

Note: All Images and Animations can also be seen by clicking on the link here: `https://github.com/Sparsh1703/Journey-from-Order-to-Chaos-`

5 Chaos and Nonlinear Dynamics

Chaos is a fascinating and counterintuitive phenomenon found in certain nonlinear dynamical systems. It is characterized by complex and unpredictable behavior, often resulting from relatively simple and deterministic rules. In chaotic systems, tiny changes in initial conditions can lead to drastically different outcomes, making long-term predictions challenging or impossible.

The vibrating double pendulum system is a classic example of a chaotic system. Although the equations governing the motion of the double pendulum are deterministic (i.e., they do not involve random elements), the system can still exhibit chaotic behavior under certain conditions.

5.1 Sensitive dependence on initial conditions

The relevance of chaos to the vibrating double pendulum system lies in its sensitivity to initial conditions. When the pendulums are released from slightly different starting positions or velocities, they can follow vastly different trajectories as time progresses. This behavior is known as sensitive dependence on initial conditions, or the "butterfly effect."

Due to this sensitivity, the double pendulum system can display seemingly random and unpredictable movements over time, even though it follows deterministic laws. The complex patterns and trajectories traced by the pendulums make the system difficult to predict in the long run, resembling the behavior of chaotic systems.

The sensitivity to initial conditions also contributes to the fractal nature of many chaotic systems. Fractals are self-replicating geometrical patterns, and chaotic attractors often exhibit fractal structures in phase space due to the repeated folding and stretching of trajectories.

Despite the seemingly random behavior and unpredictability, chaotic systems are still deterministic. This means that their future states are entirely determined by their initial conditions and governing equations. There is no inherent randomness in chaotic behavior; it arises purely from the nonlinear interactions between the system's variables.

5.2 Simulations illustrating the chaotic motion of the system in Phase Space

By clicking below link, we can see the simulation of curve traced by the system in phase space. We observe that the motion will be chaotic in the phase space and it will be sensitive to initial conditions.

Click to Play Animation

The next simulation will show the vibrating double pendulum. The frequency of the suspension point has been taken around 10. The constants can be varied in the code as mentioned in the github repository.

Click to Play Animation
Github Repository : `https://github.com/Sparsh1703/Journey-from-Order-to-Chaos-` (Go to DP-Simulation.ipynb file).

6 Bifurcation Analysis

Bifurcations are critical points in the behavior of nonlinear systems where small changes in system parameters lead to qualitative changes in the system's dynamics. In other words, bifurcations are points at which the system undergoes a structural change in its behavior, such as the emergence of new stable states, periodic orbits, or chaotic behavior. They are pivotal events that mark transitions between different types of behavior in nonlinear systems.

Bifurcation Analysis can be useful in the following aspects:

Phase Transition: Bifurcations play a crucial role in phase transitions in nonlinear systems. As system parameters cross critical values, there can be sudden changes in the system's state and behavior. These phase transitions often result in the emergence or disappearance of stable equilibria or periodic orbits.

Stability Analysis: Bifurcations are closely related to stability analysis of equilibrium points in nonlinear systems. At bifurcation points, the stability of an equilibrium can change, leading to the creation or destruction of stable solutions. For instance, a stable equilibrium can become unstable or vice versa at a bifurcation.

Complexity and Chaos: Bifurcations can lead to the emergence of complex behaviors in nonlinear systems. As parameters vary, the system can transition from regular and predictable behavior to chaotic and unpredictable behavior.

Overall, bifurcations are powerful tools for understanding and characterizing the behavior of nonlinear systems. They provide insights into the emergence of different states, periodic behaviors, and chaotic dynamics, contributing to the deeper understanding of complex and unpredictable phenomena in nature.

6.1 Bifurcation Analysis in Vibrating Double Pendulum System

In this particular system, there are four parameters on which the dynamics of the system depends on. These parameters itself depends on some other quantities like mass of the bob, length of pendulum, frequency of suspension point etc. It will be very useful if we analyse the dynamics of θ_1 and θ_2 with respect to these parameters. The plot between the dynamical variable (θ_1 and θ_2) and the parameters is known as the bifurcation diagram. Below we are about to see some fascinating bifurcation diagrams which in turn depicts a larger (and might be more complicated) story.

Case1: θ vs ϵ :

Below is the bifurcation diagram of both angles vs ϵ:

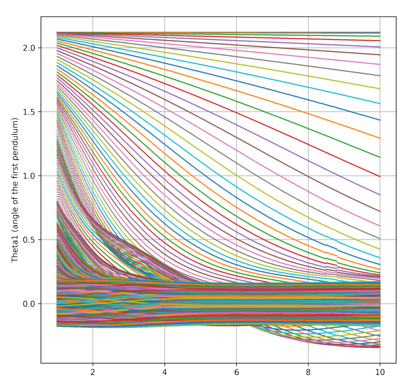

Figure 34: Plot between θ_1 and ϵ

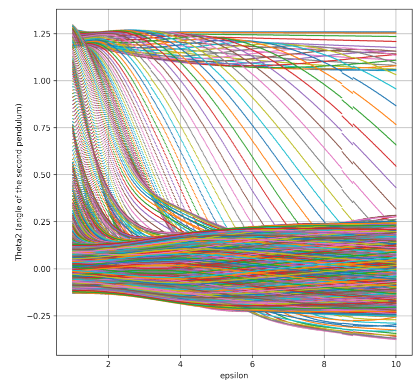

Figure 35: Plot between θ_2 and ϵ

Case2: θ vs μ :

Below is the bifurcation diagram of both angles vs μ:

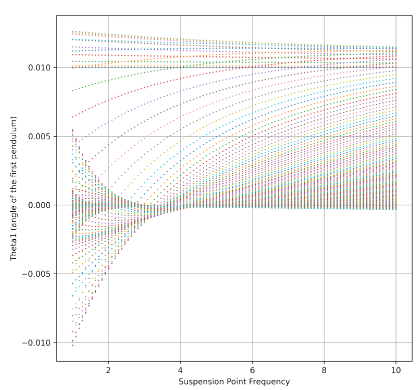

Figure 36: Plot between θ_1 and μ

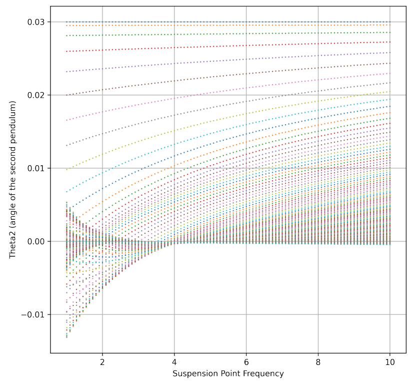

Figure 37: Plot between θ_2 and μ

Case3: θ vs δ :

Below is the bifurcation diagram of both angles vs δ:

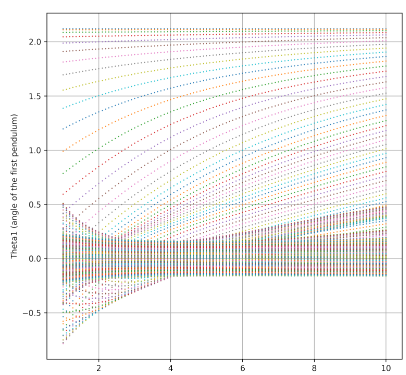

Figure 38: Plot between θ_1 and δ

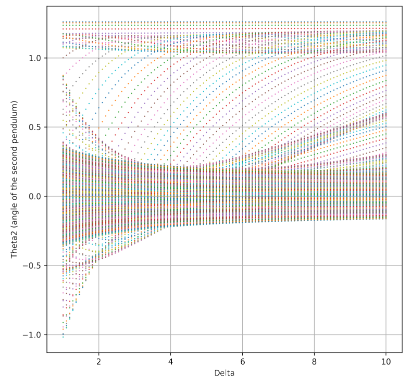

Figure 39: Plot between θ_2 and δ

Some more bifurcation plots of the system when parameters were changed:

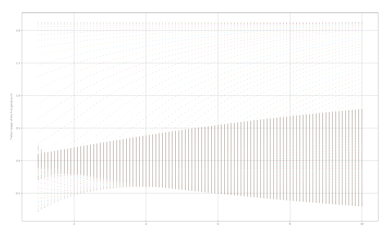

Figure 40: Plot between θ_1 and δ

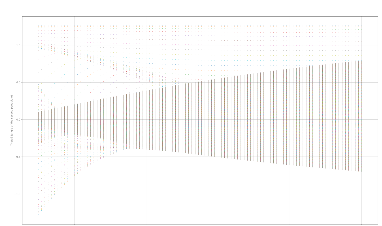

Figure 41: Plot between θ_2 and δ

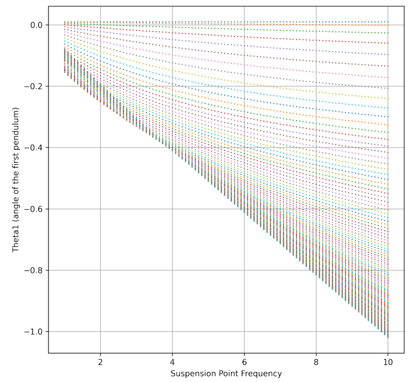

Figure 42: Plot between θ_1 and μ

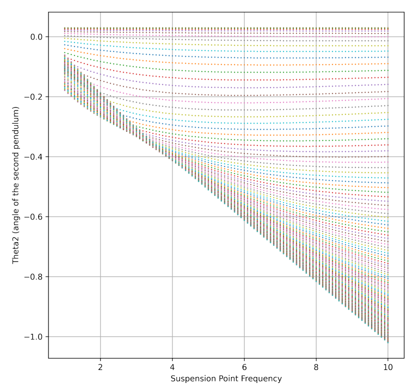

Figure 43: Plot between θ_2 and μ

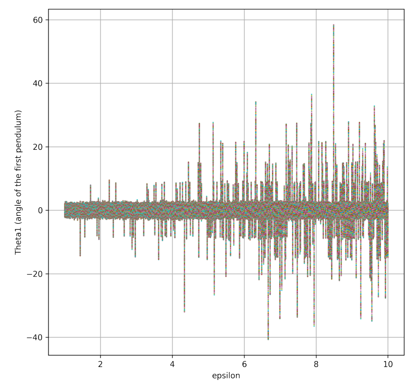

Figure 44: Plot between θ_1 and ϵ

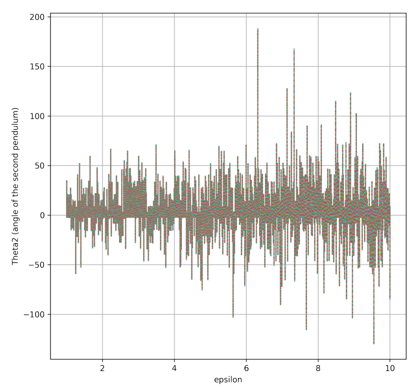

Figure 45: Plot between θ_2 and ϵ

We observe that for any range of δ, μ and ϵ, it produces chaos in the system. An attractor is the value, or set of values, that the system settles toward over time. A chaotic system has a strange attractor, around which the system oscillates forever, never repeating itself or settling into a steady state of behavior. It never hits the same point twice and its structure has a fractal form, meaning the same patterns exist at every scale no matter how much you zoom into it.

For above bifurcation diagrams, the x-axis denotes the parameters. The y-axis here depicts the angles which system tends to settle there for that particular parameter. In other words, the points above each parameter value is the parameter's attractor.

In all of the above bifurcation diagrams, we observe that for any range of the parameters, the system settles at many points (we could also say ∞ points). Or we could say that it doesn't really settle at a fixed point or a limit cycle. In reality, the system has bifurcated so many times that it now jumps, seemingly randomly, between all θ values. I only say seemingly randomly because it is definitely not random. Rather, this model follows very simple deterministic rules yet produces apparent randomness. This is chaos: deterministic and aperiodic.

7 Conclusion and Discussion

The description of our work along with the conclusion can be summarised as follow:

- We were given a double pendulum system whose suspension point is vibrating with a frequency ω. We have also considered the effects of damping. Given this system, we need to find the dynamics of the system for different initial conditions and also analyse how parameters affects the system dynamics.
- We began with deriving the equations of motion of both bobs using Euler Lagrangian Equation considering the effects of damping. We further simplified the equations by expressing them into dimensionless form. We grouped together different constants into four different parameters.
- We expressed 4-dimensional time dependent system in the form of:

$$\dot{x_1} = f(x_1, x_2, x_3, x_4)$$

$$\dot{x_2} = f(x_1, x_2, x_3, x_4)$$

$$\dot{x_3} = f(x_1, x_2, x_3, x_4)$$

$$\dot{x_4} = f(x_1, x_2, x_3, x_4)$$

- Then we saw the dynamics and Phase Space of the system for different initial conditions for different ranges of ω (frequency of suspension point). The cases were as follow:
  Case1: When initial angles were very small
  Case2: When double pendulum is completely inverted meaning both angles were approximately π radians.
  Case3: When $\theta_1 \approx \pi$ radians and θ_2 is in between 0 and π radians.

- Then with the help of programming, particularly python and its libraries, we numerically solved the set of differential equations using RK-45 method. We used matplotlib for creating animations, plots and visualizations of the evolution of state variables in Phase Space.

- We then discussed about the chaotic, nonlinear dynamics of this system and presented the simulation of the chaotic motion of the dynamical variables in Phase Space.

- At the end, we dived into the bifurcation analysis of this system where we found the relation between system's dynamical variables with the parameters.

We observe that when initial angles were very small, the curve traced were almost sinusoidal. There exists normal modes of the system which we founded with the help of programming.

For other two cases, the motion were purely chaotic for different ranges of ω (angular frequency of suspension point). However for very large values of ω ($\approx$ 1000), the angles were almost remain constant for both cases mentioned. They both might be out of phase with each other and somehow their linear combination of frequencies got cancelled and due to this, system as a whole remains constant, although the suspension point is vibrating very fast. For angles in between θ_1 and θ_2, the motion were chaotic as expected from the differential equations.

Regarding bifurcation analysis, this is a very vast topic and there is still a lot that can be done to analyse the system dependence on parameters. In this paper, the bifurcation diagrams were drawn for very limited range of certain parameters (keeping all others as constant) and it was also observed that the bifurcation diagram changes if any other constants(parameters) is changed. There is still a lot of work that is needed to be done in this.

Nonlinear dynamics and chaos play a crucial role in understanding complex physical systems, providing valuable insights into the behavior of systems that cannot be adequately described by linear equations. Some key-points are given below:

Beyond Linearity: Linear models are limited in their ability to represent nonlinear phenomena. Nonlinear dynamics explores the effects of nonlinear relationships and interactions among system variables, enabling a deeper understanding of how these interactions lead to emergent properties and phenomena.

Sensitivity to Initial Conditions: Chaos theory, a branch of nonlinear dynamics, unveils the phenomenon known as the "butterfly effect." It highlights that small changes in initial conditions can lead to drastically different outcomes over time. This sensitivity to initial conditions emphasizes the inherent unpredictability and complexity of some natural systems.

Bifurcations and Transitions: Nonlinear systems can undergo bifurcations, critical points where the system's behavior undergoes a qualitative change. Bifurcation analysis helps identify the emergence of new stable states, periodic behaviors, or chaotic regimes in response to parameter variations.