# Frustration Driven Topological Phases and Emergent Majorana Excitations in 1D and 2D Quantum Spin Systems

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# Introduction

- Frustration arises when competing interactions prevent a system from simultaneously minimizing all local energy terms.
  - **Frustration due to geometry**: antiferromagnets on triangular plaquettes
  - **Competing exchange interactions**: if NN interaction is ferromagnetic ($J_1 > 0$) and NNN interaction is antiferromagnetic ($J_2 < 0$)

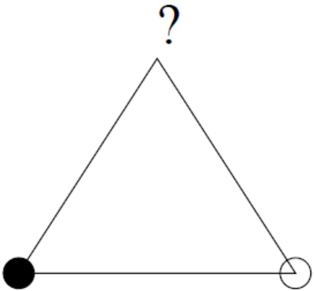



Figure: Frustrated anti-ferromagnetic triangular lattice
*Source: Hung T. Diep, Frustrated Spin Systems: History of the Emergence of a Modern Physics*

# Frustration in 1D systems and the transverse compass model

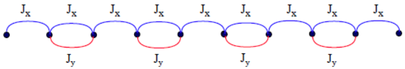



Figure: One dimensional Quantum Compass Model

- Extensive ground state degeneracy
- Suppressed magnetic order
- Enhanced quantum fluctuations
- Emergence of Majorana Zero Modes
- **Quantum Compass Model**: analytically tractable platform for studying the generation, destruction, and reemergence of EMZMs under different field configuration across sites.

- **Model Hamiltonian**:

$$\hat{H}_{OBC} = -\sum_{j=1}^{N-1} J_j^x \sigma_j^x \sigma_{j+1}^x - \sum_{j\%2=0} J_{j/2}^y \sigma_j^y \sigma_{j+1}^y - \sum_{j=1}^{N} h_j \sigma_j^z \quad (1)$$

- **Jordan-Wigner transformation**:

$$\sigma_j^x - i\sigma_j^y = 2c_j e^{i\pi \sum_{k=1}^{j-1} c_k^\dagger c_k} \quad , \quad \sigma_j^z = 2c_j^\dagger c_j - 1 \qquad (2)$$

- **Hamiltonian after JW transformation**:

$$H_{OBC}^{(JW)} = -\sum_{j=1}^{N-1} j_j^x \left[ -c_j c_{j+1} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j^\dagger \right]$$

$$- \sum_{j\%2=0} J_{j/2}^y \left[ c_j c_{j+1} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j^\dagger \right]$$

$$- \sum_{j=1}^{N} h_j \left[ 2c_j^\dagger c_j - 1 \right] \qquad (3)$$

- **Transform complex fermions into Majorana fermion operators**:

$$c_j = \frac{1}{2}\Big(\gamma_{2j-1} + i\gamma_{2j}\Big) \qquad c_j^\dagger = \frac{1}{2}\Big(\gamma_{2j-1} - i\gamma_{2j}\Big) \tag{4}$$

- **Modified Hamiltonian**:

$$H_{OBC}^{(MF)} = i\sum_{j=1}^{N-1}\left[J_j^x\gamma_{2j}\gamma_{2j+1} - \frac{1+(-1)^j}{2}J_{j/2}^y\gamma_{2j-1}\gamma_{2j+2}\right] - i\sum_{j=1}^{N}h_j\gamma_{2j-1}\gamma_{2j} \tag{5}$$

$$H_{OBC}^{(MF)} = \frac{i}{2}\Psi^T H_{OBC}\Psi \tag{6}$$

where $\Psi = (\gamma_1, \gamma_2, ....., \gamma_{12})^T$ and $H_{OBC}$ is a $12 \times 12(2N \times 2N)$ real anti-symmetric matrix of the form (considering $N = 6$):

$$
H_{OBC} = \begin{pmatrix}
0 & -h_1 & & & & & & & & & & \\
h_1 & 0 & J_1^x & & & & & & & & & \\
0 & -J_1^x & 0 & -h_2 & 0 & -J_1^y & & & & & & \\
 & & h_2 & 0 & J_2^x & 0 & & & & & & \\
 & & 0 & -J_2^x & 0 & -h_3 & & & & & & \\
 & & J_1^y & 0 & h_3 & 0 & J_3^x & & & & & \\
 & & & & & -J_3^x & 0 & -h_4 & 0 & -J_2^y & & \\
 & & & & & & h_4 & 0 & J_4^x & 0 & & \\
 & & & & & & 0 & -J_4^x & 0 & -h_5 & & \\
 & & & & & & J_2^y & 0 & h_5 & 0 & J_5^x & \\
 & & & & & & & & & -J_5^x & 0 & -h_6 \\
 & & & & & & & & & & h_6 & 0
\end{pmatrix} \tag{7}
$$

$$
\tilde{H}_{OBC} = W H_{OBC} W^T \tag{8}
$$

where W is a $2N \times 2N$ real orthogonal matrix.

$$H_{OBC}^{(MF)} = \frac{i}{2} B^T \tilde{H}_{OBC} B \quad (9)$$

where $B = (b'_1, b''_1, ......, b'_N, b''_N) = W\Psi$ is another set of Majorana fermions due to orthogonality of W.

- wavefunction of $b_j$ in the real space representation of snake chain $\Longrightarrow$ corresponding row of the W matrix
- each component will be the probability amplitude of Majorana fermion $d_j$.

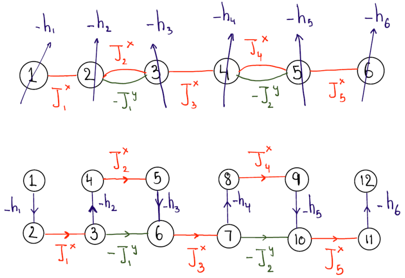



Figure: Real space (above) and the corresponding snake chain (below) representation for $H_{OBC}^{(MF)}$ in the Majorana space.

# Eigenvalue Problem

$$H_{OBC}V = \lambda V \tag{10}$$

where V is the eigenvector of the form
$V = (A_1, B_1, C_1, D_1, ....., A_{N/2}, B_{N/2}, C_{N/2}, D_{N/2})$.

- eigenvalue equations:

$$-J^x_{2j-1}B_j - h_{2j}D_j - J^y_j B_{j+1} = \lambda C_j \tag{11}$$

$$h_{2j}C_j + J^x_{2j}A_{j+1} = \lambda D_j \tag{12}$$

$$-J^x_{2j}D_j - h_{2j+1}B_{j+1} = \lambda A_{j+1} \tag{13}$$

$$J^y_j C_j + h_{2j+1}A_{j+1} + J^x_{2j+1}C_{j+1} = \lambda B_{j+1} \tag{14}$$

where $j = 1, 2, 3, ...., \frac{N}{2} - 1$.

- boundary equations:

$$-h_1 B_1 = \lambda A_1 \tag{15}$$

$$h_1 A_1 + J_1^x C_1 = \lambda B_1 \tag{16}$$

$$-J_{N/2+2}^x B_{N/2} - h_N D_{N/2} = \lambda C_{N/2} \tag{17}$$

$$h_N C_{N/2} = \lambda D_{N/2} \tag{18}$$

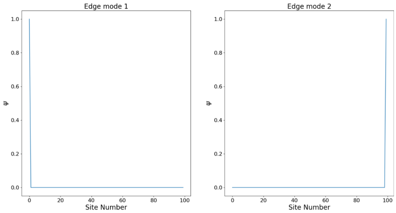
To explore the conditions under which EMZMs can emerge, we put $\lambda = 0$ in the above equation (11-18).

# Case 1: Considering zero magnetic field at all sites

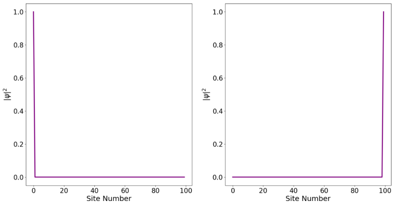
- 2 localised EMZMs at boundary

$$V_1 = (1, 0, 0, 0, ......, 0)^T$$

$$V_2 = (0, ......., 0, 0, 0, 1)^T$$




(a) Probability amplitude of Majorana Zero Modes for $h = 0$ and $J^x = J^y = 1$




(b) Probability density of Majorana Zero Modes for $h = 0$ and $J^x = J^y = 1$

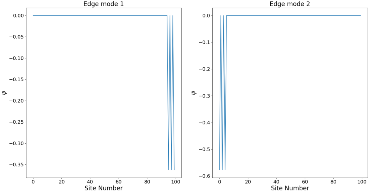
Figure: Probability density and probability amplitude of Majorana Zero Modes considering number of sites in Snake-Chain representation is 100

# Case 2: Considering uniform magnetic field across all sites

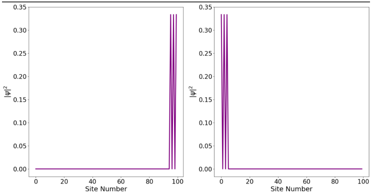
- for $h^2 = J^x_{2j} J^y_j$:

$$V_1 = \left( -\frac{J^x_1}{h}, 0, 1, 0, -\frac{h}{J^x_2}, 0, 0, 0, \ldots\ldots, 0, 0, 0, 0 \right)^T$$

$$V_2 = \left( 0, 0, 0, 0, \ldots\ldots, 0, 0, 0, -\frac{h}{J^x_{N-2}}, 0, 1, 0, -\frac{J^x_{N/2+2}}{h} \right)^T$$




(a) Probability amplitude of Majorana Zero Modes for $h = 1$ and $J^x = J^y = 1$




(b) Probability density of Majorana Zero Modes for $h = 1$ and $J^x = J^y = 1$

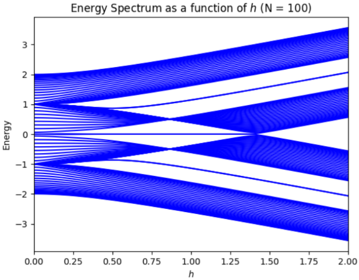
Figure: Probability density and probability amplitude of Majorana Zero Modes considering number of sites in Snake-Chain representation is 100

- for $h^2 \neq J_{2j}^x J_j^y$: No EMZMs
- **Comments**:
  - Eigenvector can also be considered of the form:

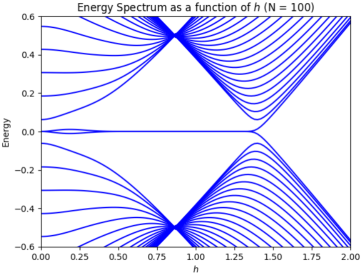
$$V = \left(-\frac{J_1^x}{h}, 0, 1, 0, -\frac{h}{J_{2}^x,}, 0, 0, 0, ....., 0, 0, 0, -\frac{h}{J_{N-2}^x}, 0, 1, 0, -\frac{J_{N/2+2}^x}{h}\right)^T$$

    $\implies$ we have a single EMZM which is non-local. This single EMZM is mentioned in a different eigenbasis.
  - $V_1 \implies$ EMZM which is in a superposition state of $d_1, d_3$ and $d_5$ Majorana fermions.
    $V_2 \implies$ EMZM which is in a superposition state of $d_{2N-4}, d_{2N-2}$ and $d_{2N}$ Majorana fermions.

- Energy Spectrum:



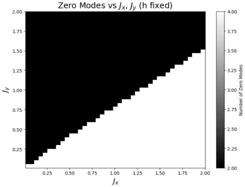

(a) full energy spectrum



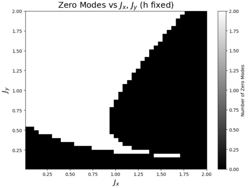

(b) zoomed-in spectrum

Figure: Energy spectrum of Hamiltonian matrix

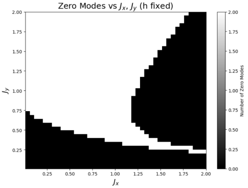
- how $J^x$ and $J^y$ interactions affects the number of Majorana zero modes, keeping magnetic field value fixed



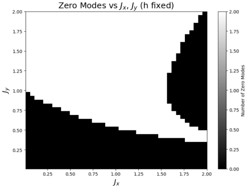

(a) $h = 0$



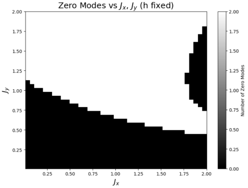

(b) $h = 0.5$



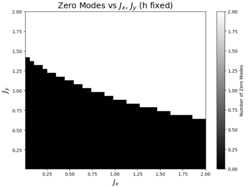

(c) $h = 0.65$




(d) $h = 0.87$




(e) $h = 1$




(f) $h = 1.26$

Figure: Phase diagram of number of zero modes. The color indicates the number of Majorana zero modes.

# Case 3: Considering non-uniform magnetic field across all sites

Magnetic field on the subset of the lattice sites, say, $S = \{l_1, l_2, ...., l_m\},\ 1 \leq l_1, l_2, ....., l_m \leq N$ vanish. The set S can always be written as the union of n ordered consecutive sequences $T_j = (l_{u_j}, l_{u_j} + 1, ....., l_{u_j} + v_j - 1)$ with length $v_j$, i.e., $S = \{T_1, T_2, ...., T_n\}$.

- **Case 1**: Considering S consist of a single consecutive sequence S consist of a single consecutive sequence T of length $\nu$, i.e., $S = T = (l_u, l_u + 1, ....., l_u + \nu - 1)$
  - $l_u =$ even, $\nu =$even

$$V_1 = \left(0, 0, 0, 0, ...., 0, 0, 0, D_{\frac{l_u+v}{2}}, 0, B_{\frac{l_u+v}{2}+1}, 0, 0, ....., 0, 0, 0, 0\right)^T$$

$$V_2 = \left(0, 0, 0, 0, ....., 0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, 0, 0, ...., 0, 0, 0, 0\right)^T$$

$$V_3 = \Bigg(0, ..., 0, 0, 0, -\frac{h_{l_u-1}}{J^x_{l_u-2}}, 0, 1^{(\frac{l_u}{2})}, 0, 0, 0, (-1)^{j+1} \prod_{j=0}^{j=\left[0, \frac{v}{2}-2\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}}, 0, 0,$$

$$..., 0, B_{\frac{l_u+v}{2}}, 0, D_{\frac{l_u+v}{2}}, 0, B_{\frac{l_u+v}{2}+1}, 0, 0, ..., 0\Bigg)^T$$

$$V_4 = \Bigg(0, ....0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, C_{\frac{l_u}{2}}, 0, 0, 0, (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[2, \frac{v}{2}\right]}^{j=\frac{v}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}}, 0,$$

$$..., 0, 0, 1^{\left(\frac{l_u+v}{2}\right)}, 0, A_{\frac{l_u+v}{2}+1}, 0, 0, 0...., 0\Bigg)^T$$

When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j \implies$ 2 more EMZMs

$$V_5 = \left(-\frac{J^x_1}{h_1}, 0, 1, 0, -\frac{h_2}{J^x_2}, 0, 0, 0, ...., 0, 0, 0, 0\right)^T$$

$$V_6 = \left(0, 0, 0, 0, ......, 0, 0, 0, -\frac{h_{N-1}}{J^x_{N-2}}, 0, 1, 0, -\frac{J^x_{N/2+2}}{h_N}\right)^T$$

- $l_u =$ even, $\nu =$odd ($\nu = 1$)

$$V_1 = \left(0, ...., 0, 0, \frac{-J^x_{l_u-1}J^x_{l_u-2}}{J^y_{\frac{l_u}{2}-1}J^x_{l_u-2} - h_{l_u-1}h_{l_u-2}}, 0, \frac{J^x_{l_u-1}h_{l_u-2}}{J^y_{\frac{l_u}{2}-1}J^x_{l_u-2} - h_{l_u-1}h_{l_u-2}},\right.$$

$$\left. 0, 1^{\left(\frac{l_u}{2}\right)}, 0, 0, 0, -\frac{J^y_{\frac{l_u}{2}}}{J^x_{l_u+1}}, 0, \frac{h_{l_u+2}J^y_{\frac{l_u}{2}}}{J^x_{l_u+2}J^x_{l_u+1}}, ..., 0\right)^T$$

$$V_2 = \left(0, ...., 0, 0, 0, -\frac{h_{l_u-1}}{J^x_{l_u-2}}, 0, 1^{\left(\frac{l_u}{2}\right)}, 0, \frac{h_{l_u+1}J^x_{l_u-1}}{J^x_{l_u}J^y_{\frac{l_u}{2}}}, 0, -\frac{J^x_{l_u-1}}{J^y_{\frac{l_u}{2}}}, 0, 0, ..., 0\right)^T$$

When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j \implies$ 2 more EMZMs

- $l_u =$ even, $\nu =$odd $(\nu \geq 3)$

$$V_1 = \Bigg(0, \ldots C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, C_{\frac{l_u}{2}}, 0, 0, 0, (-1)^{\frac{v+3}{2}-j} \prod_{j\in\left[2, \frac{v+1}{2}\right]}^{j=\frac{v+1}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}} C_{\frac{l_u}{2}+\frac{v+1}{2}}, 0,$$

$$\ldots, 0, 0, 1^{\left(\frac{l_u}{2}+\frac{v+1}{2}\right)}, 0, -\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}}, .., 0\Bigg)^T$$

$$V_2 = \Bigg(0, \ldots, -\frac{h_{l_u-1}}{J^x_{l_u-2}}, 0, 1^{\left(\frac{l_u}{2}\right)}, 0, 0, 0, (-1)^{j+1} \prod_{j=0}^{j\in\left[0, \frac{v-5}{2}\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}}, 0, 0,$$

$$\ldots, 0, B_{\frac{l_u}{2}+\frac{v-1}{2}}, 0, D_{\frac{l_u}{2}+\frac{v-1}{2}}, 0, B_{\frac{l_u+v+1}{2}}, 0, 0, .., 0\Bigg)^T$$

$$V_3 = \Bigg(0, 0, 0, 0, \ldots., 0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, 0, 0, \ldots.., 0, 0, 0, 0\Bigg)^T$$

When $h_{2j} h_{2j+1} = J^x_{2j} J^y_j \implies$ 2 more EMZMs

- $l_u =$ odd, $\nu =$even

$$V_1 = \Bigg(0, .., 0, 0, 1^{\left(\frac{l_u-1}{2}\right)}, 0, -\frac{h_{l_u-1}}{J^x_{l_u-1}}, 0, -\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u}}, 0, 0, 0,$$

$$(-1)^{j+1} \prod_{j=1}^{j\in\left[1,\frac{v}{2}\right]} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}}, 0, .., -\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}} C_{\frac{l_u+v+1}{2}}, 0, 0, 0, .., 0\Bigg)^T$$

$$V_2 = \Bigg(0, ...., -\frac{h_{l_u-2}}{J^x_{l_u-3}} B_{\frac{l_u-1}{2}}, 0, (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[0,\frac{v}{2}-1\right]}^{\frac{v}{2}} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j-2}}, 0, 0, ..., 0,$$

$$-\frac{J^y_{\frac{l_u+v-1}{2}}}{J^x_{l_u+v-2}}, 0, -\frac{h_{l_u+v}}{J^x_{l_u+v-1}}, 0, 1^{\left(\frac{l_u+v+1}{2}\right)}, 0, 0, .., 0\Bigg)^T$$

When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j \implies$ 2 more EMZMs

- $l_u = \text{odd}$, $\nu = \text{odd}$ $(\nu = 1)$

$$V_1 = \left(0, ..., 0, 0, \left(-\frac{J^x_{l_u}}{J^y_{\frac{l_u-1}{2}}}\right)^{\frac{l_u-1}{2}}, 0, \left(\frac{h_{l_u-1}J^x_{l_u}}{J^x_{l_u-1}J^y_{\frac{l_u-1}{2}}}\right)^{\frac{l_u+1}{2}}, 0, 1^{\left(\frac{l_u+1}{2}\right)}, 0, \left(-\frac{h_{l_u+1}}{J^x_{l_u+1}}\right)^{\frac{l_u+3}{2}}, 0, 0, 0, ..., 0\right)^T$$

$$V_2 = \left(0, ...., 0, 0, 0, \left(\frac{h_{l_u-2}J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-3}J^x_{l_u-2}}\right)^{\frac{l_u-3}{2}}, 0, \left(-\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-2}}\right)^{\frac{l_u-1}{2}}, 0, 0, 0, 1^{\left(\frac{l_u+1}{2}\right)}, 0, 0, ..., 0\right)^T$$

When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j \implies$ 2 more EMZMs

- $l_u = \text{odd}$, $\nu = \text{odd}$ $(\nu \geq 3)$

$$V_1 = \Bigg(0, ..., 0, 0, 0, \left(-\frac{h_{l_u-2}}{J^x_{l_u-3}}\right)^{\frac{l_u-3}{2}}, 0, 1^{\left(\frac{l_u-1}{2}\right)}, 0, 0,$$

$$0, (-1)^{j+1} \prod_{j=0}^{j=\left[0, \frac{\nu-3}{2}\right]} \frac{J^x_{l_u+2j-2}}{J^y_{\frac{l_u-1}{2}+j}}, 0, 0, .., 0, B_{\frac{l_u+\nu}{2}}, 0, D_{\frac{l_u+\nu}{2}}, .., 0\Bigg)^T$$

$$V_2 = \Bigg(0, .., 1^{\left(\frac{l_u-1}{2}\right)}, 0, \left(-\frac{h_{l_u-1}}{J^x_{l_u-1}}\right)^{\frac{l_u+1}{2}}, 0, \left(-\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u}}\right)^{\frac{l_u+1}{2}}, 0, 0, 0,$$

$$(-1)^{j+1} \prod_{j=1}^{j\in\left[1, \frac{\nu-1}{2}\right]} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}}, 0, .., \left(-\frac{h_{l_u-1}}{J^x_{l_u-1}}\right)^{\frac{l_u+\nu}{2}+1}, .., 0\Bigg)^T$$

$$V_3 = \Big(0, ....., 0, 0, 0, D_{\frac{l_u+\nu}{2}}, 0, B_{\frac{l_u+\nu}{2}+1}, 0, 0, ......, 0\Big)^T$$

When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j \implies$ 2 more EMZMs

# Future Plan

- **Floquet-driven dynamics of Majorana matter and $Z_2$ fluxes**
  - The dependence of prethermal plateaus on the frequency and strength of the drive.
  - Heating thresholds and dynamically protected regimes.
  - The role of anisotropic couplings $K_x \neq K_y \neq K_z$.
  - The onset and breakdown of prethermalization.
- **Theoretically explaining the formation of Majorana Zero Modes for different field configurations**
- **Case 2 of the Quantum Compass Model** Numerical calculations shows EMZMs for $h^2 \neq J^x_{2j} J^y_j$. This ambiguity is something that we will have to look further.