Frustration Driven Topological Phases and Emergent Majorana Excitations in 1D and 2D Quantum Spin Systems

Thesis to be submitted in partial fulfillment of the requirements for the degree

of

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by

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CERTIFICATE

This is to certify that we have examined the thesis entitled **Frustration Driven Topological Phases and Emergent Majorana Excitations in 1D and 2D Quantum Spin Systems**, submitted by **Sparsh Gupta**(Roll Number: *21PH23005*) a undergraduate student of **Department of Physics** in partial fulfillment for the award of degree of BS-MS Physics. We hereby accord our approval of it as a study carried out and presented in a manner required for its acceptance in partial fulfillment for the Under Graduate Degree for which it has been submitted. The thesis has fulfilled all the requirements as per the regulations of the Institute and has reached the standard needed for submission.

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ABSTRACT

Frustrated quantum spin systems host exotic topological phases and emergent fractionalized quasiparticles. In this thesis, we study two such settings: a 1D transverse quantum compass chain and 2D quantum spin liquid models. In the first part, we analytically and numerically investigate the emergence of exact Majorana zero modes (EMZMs) under uniform and spatially varying transverse fields. Using a Jordan Wigner mapping, we identify the conditions under which nonlocal zero modes appear, are destroyed, or reemerge at finite fields, and determine how their existence depends on bond anisotropies J^x, J^y. In the second part, we review topological order in the Kitaev honeycomb and Toric Code models, emphasizing fractionalized excitations and dynamical phenomena. Together, these results highlight how frustration and bond directional interactions enable robust topological structures across 1D and 2D spin systems.

Keywords: frustration, compass model, spin systems, majorana zero modes, quantum spin liquids, Kitaev Honeycomb model, Toric code model

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Chapter 1

Introduction to Frustrated Spin Systems

1.1 Frustration Introduction

As Leon Balents rightly said, a little frustration can make life really interesting. However, he was talking about frustration in Physics. Frustration arises when competing interactions prevent a system from simultaneously minimizing all local energy terms. It may originate from geometry, such as antiferromagnets on triangular plaquettes or from competing exchange interactions, leading to highly degenerate ground states and strong quantum fluctuations.

1.1.1 Formal definition

In Ising and Heisenberg model, the interaction energy between neighboring spins is given by $E = -J(S_i.S_j)$ where $J > 0$ for parallel spin pair and $J < 0$ for anti-parallel spin pair. If NN interaction is ferromagnetic ($J > 0$), the ground state of the system corresponds to spin configuration where all spins are parallel. This is true for any lattice structure. However if NN interaction is anti-ferromagnetic ($J < 0$), then the ground state of the system depends on the lattice structure. For square or cubic lattice, the ground state corresponds to the configuration in which spins are anti-parallel to its neighbors. But for triangular, FCC or HCP lattice, it is impossible to construct ground state where all bonds have lowest energy (figure [1.1]). In this case, we can say that the system is **frustrated**.

A more mathematical and formal definition was defined by Toulese:
We consider a unit cell of the lattice. It may be a polygon formed by faces which

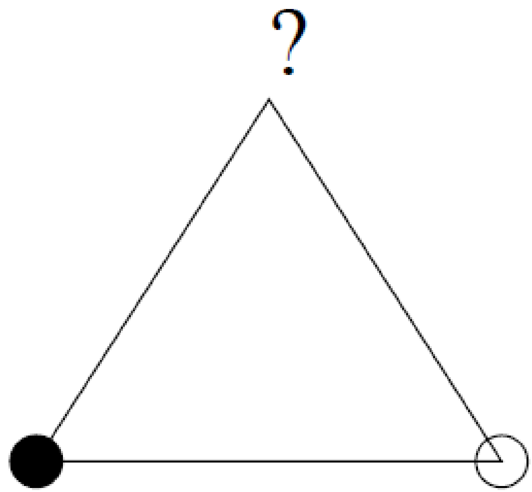


Figure 1.1: Frustrated anti-ferromagnetic triangular lattice

we call “plaquettes”. For example, the unit cell for a simple cubic lattice is a cube with 6 plaquettes. Let J_{ij} be the interaction between NN spins of plaquettes. The plaquette is said to be frustrated if the parameter P, defined below is negative.

$$P = \prod_{<ij>} sign(J_{ij}) \tag{1.1}$$

In the figure given above for triangular anti-ferromagnetic model, if we put Ising spin on the sites, then one of the bonds around plaquettes is not satisfied. Whereas if we put vector spin on the sites, each bond is only partially satisfied in lowest energy state, meaning that NN spins are not exactly anti-parallel.

We consider another kind of spin systems that can be frustrated in which each bond is only partially satisfied. This occurs when there are two or more different kind of interactions and ground state does not correspond to minimum of each kind of interaction. For example, as mentioned initially, if NN interaction is ferromagnetic ($J_1 > 0$) and NNN interaction is antiferromagnetic ($J_2 < 0$) and if $J_1 >> |J_2|$, we can say that the ground state is ferromagnetic. Every NN is satisfied but not NNN. But if $|J_2| > J_c$ (J_c is some critical value) then both NN and NNN bonds are not fully satisfied.

In conclusion, we can say that a spin system is frustrated whenever minimum of system energy does not correspond to the minimum of all local interactions, whatever be the form of interaction.

1.2 Spin liquids in frustrated magnets

Frustrated magnets are materials in which localized magnetic moments or spins interact through competing exchange interactions that cannot be simultaneously satisfied. This give rise to large degeneracy of system's ground state. Under certain conditions, frustrated magnets can lead to the formation of spin liquids in which constituent spins are highly correlated but still fluctuate strongly down to absolute zero. The fluctuations can be classical or quantum mechanical in nature. However if the fluctuations are quantum mechanical, we observe some remarkable collective phenomena like **Emergent Gauge fields** and **fractional particle excitations**.

Looking at figure [1.1], the three spins cannot be anti-parallel. Instead, the spins fluctuate or order in a less obvious manner. When we say the spins fluctuate, we basically mean quantum fluctuations, meaning particle will be in a superposition of up and down spin which we observe at very low temperature. And if we say that spin order in a less obvious manner, we are talking about thermal fluctuations. It means that even at low temperature, spins fluctuate thermally, although in a correlated manner because they are restricted to the ground state of Ising anti-ferromagnetism. By analogy to an ordinary liquid, in which molecules forms dense and correlated state, the spins in this model forms a spin liquid. So a 2D triangular lattice of Ising spins is one of the prototype of frustration. It can be showed that such model has large degeneracy of ground state. This degeneracy is a defining characteristic of frustration.

As mentioned before, fluctuations can be classical or quantum mechanical in nature. In case of quantum, spins are quantized in half integers units of $\hbar$. Classical fluctuations are driven by thermal energy and dominate for large spins. When k_bT becomes very small, classical fluctuations cease and spins get order. For small spins comparable to $\hbar/2$, uncertainty principle produces zero point motion which persist down to 0 K. These quantum fluctuations can be phase coherent and if they are strong enough, this leads to the formation of Quantum Spin Liquids(QSLs). In QSLs, spins are highly entangled in a subtle way. As a consequence, we observe exotic excitations with fractional quantum numbers and artificial gauge fields. There is a chance that we can observe QSLs in Mott Insulators in which electrons are localized to individual atomic or molecular orbitals but maintain their spin degrees of freedom.

1.2.1 Spin Ice

In spin ice materials, like $Dy_2Ti_2O_7$, $Ho_2Ti_2O_7$ and $Ho_2Sn_2O_7$, spins can be regarded as classical. Only ions of Dy and Ho are magnetic and they reside on pyrochlore lattice, which is a network of corner sharing tetrahedra (figure [1.2])

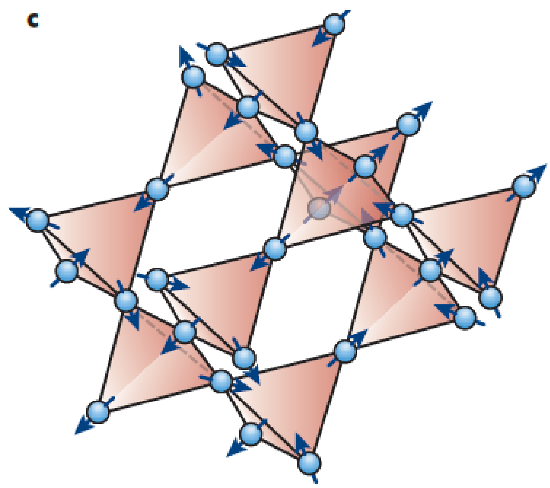


Figure 1.2: Frustrated magnetism in 3D Pyrochlore Lattice

Their f orbital electron spins are large and classical and they behave as Ising doublets, meaning they can be either up or down. But these doublets are aligned with the local axis, which connects center of 2 tetrahedra shared by that spin. The interactions can be long range and dipolar, but we consider an effective NN exchange energy J_{eff} which is ferromagnetic in nature. This ferromagnetic interaction is frustrated as we want the neighboring spins to point parallel to each other, but since their local axis are different , parallel spins are not possible. The state that minimize the energy of single tetrahedron is highly degenerate, consisting of 6 configurations ($^4C_2$) in which 2 spins point inward and 2 point outwards from center of tetrahedron. This is known as the ice rule.

If $k_bT < J_{eff}$ (thermal energy being smaller than interaction energy) , system fluctuate within 2 in and 2 out manifold of state and such states are exponentially large. Hence a low temperature entropy remains there even within this limit. Although spins remain paramagnetic in this regime, ice rule implies strong correlation. This correlated paramagnet is an example of Classical Spin liquid. The most interesting point is the fact that the local constraint will have a long range consequence. So we can distinguish Classical spin liquids from an ordinary paramagnet.

1.2.1.1 Analogies to electromagnetism

To explain long range effects due to local constraints, we can consider an analogy to electromagnetism. The spins can be thought of as an arrow pointing between centers of two tetrahedron. These arrows defines the vector field of flux lines on lattice which is divergence free due to ice rule. The vectors can be defined as an artificial magnetic field b on the lattice, and since spins are fluctuating here, so is b. The fluctuations include long loops of flux (as magnetic field lines do not start or end) that communicate spin correlations over long distances.

It can be shown mathematically that fluctuations in spin configuration at long distances is analogous to real fluctuations of magnetic field in vacuum. The spins develop power law correlations similar to that of dipole-dipole interaction. The correlation will be $\propto 1/r^3$ which is anisotropic and occurs without any symmetric breaking (meaning there is still some disorder). This is surprising as power law correlations appear near critical point or in systems with continuous symmetry breaking none of which is true for spin ice. This is the hallmark of emergent gauge fields.

1.2.1.2 Magnetic Monopoles

One of the interesting consequence of considering the picture of effective gauge field is the presence of magnetic monopole in spin ice. If we consider $k_bT << J_{eff}$, there might be violation of ice rule but it will be very rare as they are costly in energy. The simplest such defect consist of a single tetrahedron with 3 spins pointing in and 1 pointing out or vice-versa. This requires an energy of $2J_{eff}$ with respect to ground state. From magnetic viewpoint, the center of tetrahedron become source or sink for flux which we call magnetic monopole.

An important point to note is the fact that the magnetic monopole is a non-local object, as they emerge from a collective flipping of many spins. We can't just flip a single spin and create an isolated magnetic monopole without affecting the rest of the system. If we flip 1 spin, it connects 2 tetrahedron, one now has 3 in, 1 out (monopole with - charge) and the other one has 1 in, 3 out (monopole with + charge). We can separate them by flipping a chain of spins between them forming a string. This string keeps track of where the monopole came from. It does not track energy but tracks topology. The monopole carries real magnetic charges (as divergence of b or M will be non-zero), although it will be very small, but can be measurable at low temperature.

1.2.2 Quantum Spin Liquids: brief Introduction

In spin ice, if we lower the temperature, the thermal fluctuations decrease and the spins almost freeze below about 0.5 K . This is due to the large energy barrier between different ice rule configurations, which requires flipping of at least 6 spins, and the weak quantum amplitude for such large spins to tunnel through these barriers.But for materials with spin $S = 1/2$, quantum effects are strong and there will be no energy barrier. We can observe QSLs in such materials where spins continue to fluctuate even at low Temperature and avoid order. QSL strangely has a non-magnetic ground state built from local magnetic moments.

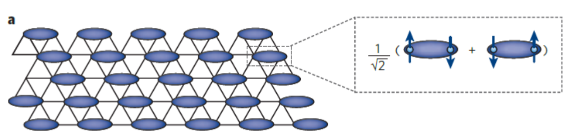


Figure 1.3: Valence-bond states of frustrated antiferromagnets

A natural building block of non-magnetic states is valence bond, a pair of spins, owing to anti-ferromagnetism, forms a spin-0 singlet state. It is a highly entangled object with 2 spins being maximally entangled and non classical. If all of the spins in a system are part of valence bonds, then the full ground state has spin 0 and is non-magnetic. One way in which this can occur is by the partitioning of all of the spins into specific valence bonds, which are static and localized. Mathematically, such a ground state is well approximated as a product of the valence bonds, so that each spin is highly entangled with only one other, its valence-bond partner. This is known as a valence-bond solid (VBS) state and is observed in many materials (figure[1.3]).

A VBS state however is not a true QSL because it breaks lattice symmetry and lack long range entanglement. To make QSL, the valence bond must undergo quantum fluctuations. The ground state is then a superposition of different partitions of spins into valence bonds. If distribution of these partitions are broad, then there is no preference for any valence bond and state can be regarded as valence bond liquid rather than solid. This type of wavefunction is known as the Resonating Valence Bond (RVB) state (figure[1.4]).

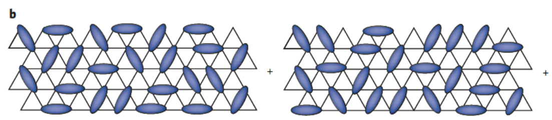


Figure 1.4: RVB state with short ranged valence bonds

Different QSL states can have different weights of each valence bond partition in their wavefunction. The valence bond can be formed from spins which are far apart (figure[1.5]) although they will be only weakly bound into singlet and valence bond can be broken to form free spins with relatively little energy. So states that have a significant weight from long-range valence bonds have more low energy spin excitations than states in which the valence bonds are mainly short range.

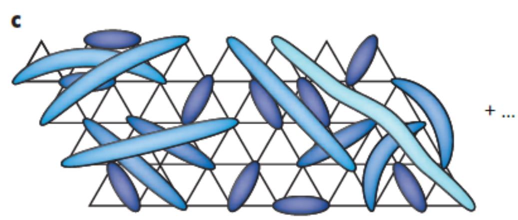


Figure 1.5: RVB state with long ranged valence bonds

One feature of QSLs is that, they support exotic excitations. In most phases of matter, we observe elementary excitations that are either electron (e) like (spin = 1/2, charge = e, -e) or magnon (m) like (spin = 1 , charge = 0). Although in FQHS, excitations differ from e and m by carrying fractional quantum number. The magnetic monopoles (or monopole- anti-monopole pair) that we observed in spin ice are also different from elementary excitations in a sense that they are classical excitations and not true coherent quasiparticle. In QSLs, the most prominent excitations are spinons (s = 1/2, charge = 0). They are established in 1D systems, in which they occur as domain walls. They can be created by flipping a semi-infinite string of spins. Strings can be associated with something known as string tension, meaning there is an energy cost, proportional to its length. String tension represents confinement of the particle. Although in 1D, string cost only a finite energy. **But in 2D/3D, string remains tensionless even at T = 0 K, owing to strong quantum fluctuations. This can be understood from the quantum superposition principle: rearranging the spins along the string simply reshuffles the various spin or valence-bond configurations that are already superposed in the ground state.

We can show that depending on the QSL type, the excitations can obey bosonic, fermionic and even anyonic statistics in higher dimensions. They may be gapped or gapless or they may even be so strongly interacting that there are no sharp excitations of any kind.

1.3 Frustration in 1D systems and the transverse compass model

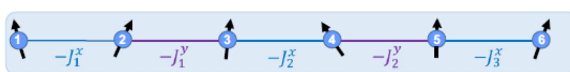


Figure 1.6: Real-space representation of an open transverse compass chain with L = 6 spins.

The 1D quantum compass model provides a canonical example of such interaction-induced frustration. In this model, neighboring bonds couple to different components of the spin operator: one bond couples through $\sigma^x\sigma^x$, while the next couples through $\sigma^y\sigma^y$ (figure[1.6]). These interaction terms do not commute, and the spin configuration that minimizes the energy on one bond cannot simultaneously minimize the energy on the next. Thus, even though the chain is geometrically unfrustrated, the competition between J^x and J^y interactions produces bond-directional frustration. This frustration has several important consequences:

1. **Extensive ground state degeneracy**: The system exhibits a large degeneracy because many spin configurations are equally favorable due to the competing constraints.

2. **Suppressed magnetic order**: The system does not develop conventional long-range magnetic order, even at zero temperature.

3. **Enhanced quantum fluctuations**: Competition between non-commuting interactions amplifies quantum fluctuations, making the ground state nontrivial.

4. **Emergence of Majorana degrees of freedom**: Upon mapping to fermions via the Jordan–Wigner transformation, the frustrated interactions translate to alternating couplings between Majorana fermions. This structure naturally supports Majorana zero modes (EMZMs) at the boundaries or at domain walls created by field inhomogeneities.

Thus, frustration in the 1D compass chain is not geometric but quantum-mechanical, originating from the directional competition between different spin interactions. In this sense, the model shares a conceptual relation with frustrated 2D models such as the Kitaev honeycomb model, which also features bond directional anisotropic interactions that lead to fractionalized Majorana excitations.

In this thesis, we explore how frustration and bond directional interactions can generate robust emergent structures, focusing on two settings: the 1D transverse quantum compass chain and 2D quantum spin liquids. The compass chain provides an analytically tractable platform for studying the generation, destruction, and reemergence of exact Majorana zero modes under spatially varying fields, while the Kitaev and Toric Code models illustrate how related mechanisms extend to two dimensions. Together, these models highlight the deep connections between frustration, fractionalization, and topological phenomena.

Chapter 2

Exact Majorana Zero Modes in a Quantum Compass Chain Model

2.1 Introduction

The emergence of EMZMs are important since they imply the existence of a degenerate ground state manifold in which quantum information can be can be stored. The manipulation of EMZMs is thus essential for the realization of a topological quantum computer. However, it is unlikely to directly observe Majorana zero modes in ordinary metals, and nonstandard systems with special properties are necessary for the emergence of EMZMs as nontrivial excitations. We study the emergence of EMZMs in a 1D quantum transverse compass model with the nearest neighbour interaction under three possible conditions:

- **Case 1**: When magnetic field h = 0
- **Case 2**: When magnetic field is uniform across all sites
- **Case 3**: When magnetic field is non-uniform across all sites

We express the spin system in terms of Majorana fermions via the Jordan Wigner transformation and then derive the possible number of EMZMs for the above mentioned cases along with their corresponding wavefunctions. The reason behind choosing 1D Quantum Compass Model is because of there exact solvability and relevance to recent experimental realization of quasi-one-dimensional materials.

2.2 Model

2.2.1 The Compass Model and Majorana Representation

The Hamiltonian of a 1D quantum compass model with both NN interactions and transverse fields varying over space can be written as:

$$\hat{H}_{OBC} = -\sum_{j=1}^{N-1} J_j^x \sigma_j^x \sigma_{j+1}^x - \sum_{j\%2=0} J_{j/2}^y \sigma_j^y \sigma_{j+1}^y - \sum_{j=1}^{N} h_j \sigma_j^z \tag{2.1}$$

for open boundary conditions (OBCs). Here J_j^x & $J_{j/2}^y$ is the interaction strength between the two nearest neighbor sites and h_j is the magnetic field experienced by the spin on site j. We assume that J_j^x & $J_{j/2}^y$ are all non-zero.

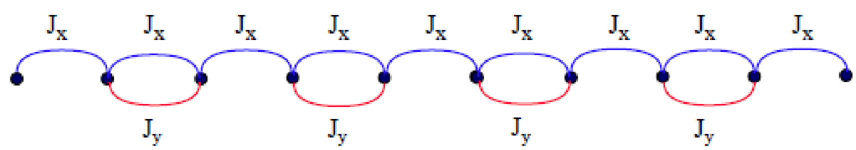


Figure 2.1: One dimensional Quantum Compass Model

Using the Jordan-Wigner transformation:

$$\sigma_j^x - i\sigma_j^y = 2c_j e^{i\pi \sum_{k=1}^{j-1} c_k^\dagger c_k} \quad , \quad \sigma_j^z = 2c_j^\dagger c_j - 1 \tag{2.2}$$

the Hamiltonian H_{OBC} can be mapped to a spinless fermion model. The final Hamiltonian $H_{OBC}^{(JW)}$ is:

$$\begin{aligned} H_{OBC}^{(JW)} = &-\sum_{j=1}^{N-1} j_j^x \Big[-c_j c_{j+1} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j^\dagger \Big] \\ &- \sum_{j\%2=0} J_{j/2}^y \Big[c_j c_{j+1} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j^\dagger \Big] \\ &- \sum_{j=1}^{N} h_j \Big[2c_j^\dagger c_j - 1 \Big] \end{aligned} \tag{2.3}$$

We transform complex fermions into Majorana fermion operators as given below and substitute that in equation (2.3):

$$c_j = \frac{1}{2}\Big(\gamma_{2j-1} + i\gamma_{2j} \Big) \qquad c_j^\dagger = \frac{1}{2}\Big(\gamma_{2j-1} - i\gamma_{2j} \Big) \tag{2.4}$$

We then obtain the modified Hamiltonian as given below:

$$H_{OBC}^{(MF)} = i\sum_{j=1}^{N-1}\left[J_j^x\gamma_{2j}\gamma_{2j+1} - \frac{1+(-1)^j}{2}J_{j/2}^y\gamma_{2j-1}\gamma_{2j+2}\right] - i\sum_{j=1}^{N}h_j\gamma_{2j-1}\gamma_{2j} \tag{2.5}$$

Now for the sake of simplicity, we consider N = 6 and write every terms of the above Hamiltonian of equation (2.5):

$$\begin{aligned} H_{OBC}^{MF} = {} & i[J_1^x\gamma_2\gamma_3 - h_1\gamma_1\gamma_2] \\ & +i[J_2^x\gamma_4\gamma_5 - J_1^y\gamma_3\gamma_6 - h_2\gamma_3\gamma_4] \\ & +i[J_3^x\gamma_6\gamma_7 - h_3\gamma_5\gamma_6] \\ & +i[J_4^x\gamma_8\gamma_9 - J_2^y\gamma_7\gamma_{10} - h_4\gamma_7\gamma_8] \\ & +i[J_5^x\gamma_{10}\gamma_{11} - h_5\gamma_9\gamma_{10}] \\ & +i[-h_6\gamma_{11}\gamma_{12}] \end{aligned}$$

$$H_{OBC}^{(MF)} = \frac{i}{2}\Psi^T H_{OBC}\Psi \tag{2.6}$$

where $\Psi = (\gamma_1, \gamma_2, ..., \gamma_{12})^T$ and H_{OBC} is a $12 \times 12(2N \times 2N)$ real anti-symmetric matrix of the form:

$$H_{OBC} = \begin{pmatrix} 0 & -h_1 & & & & & & & & & & \\ h_1 & 0 & J_1^x & & & & & & & & & \\ 0 & -J_1^x & 0 & -h_2 & 0 & -J_1^y & & & & & & \\ & & h_2 & 0 & J_2^x & 0 & & & & & & \\ & & 0 & -J_2^x & 0 & -h_3 & & & & & & \\ & & J_1^y & 0 & h_3 & 0 & J_3^x & & & & & \\ & & & & & -J_3^x & 0 & -h_4 & 0 & -J_2^y & & \\ & & & & & & h_4 & 0 & J_4^x & 0 & & \\ & & & & & & 0 & -J_4^x & 0 & -h_5 & & \\ & & & & & & J_2^y & 0 & h_5 & 0 & J_5^x & \\ & & & & & & & & & -J_5^x & 0 & -h_6 \\ & & & & & & & & & & h_6 & 0 \end{pmatrix} \tag{2.7}$$

The above Hamiltonian can be brought into diagonalized form by performing the following transformation :

$$\tilde{H}_{OBC} = W H_{OBC} W^T \tag{2.8}$$

where W is a $2N \times 2N$ real orthogonal matrix. From (2.8), we can write $H_{OBC} = W^T \tilde{H}_{OBC} W$ and substitute this in equation (2.6):

$$H_{OBC}^{(MF)} = \frac{i}{2}\Psi^T W^T \tilde{H}_{OBC} W \Psi$$

$$H_{OBC}^{(MF)} = \frac{i}{2}(W\Psi)^T \tilde{H}_{OBC} (W\Psi)$$

$$H_{OBC}^{(MF)} = \frac{i}{2} B^T \tilde{H}_{OBC} B \tag{2.9}$$

where $B = (b'_1, b''_1, ..., b'_N, b''_N) = W\Psi$ is another set of Majorana fermions due to orthogonality of W. Physically each row of W characterize the spreading of the Majorana Fermion b_j in the real space representation of snake chain, meaning each component will be the probability amplitude of Majorana fermion d_j. Whenever we talk about the wavefunction of b_j we will mean the corresponding row of the W matrix.

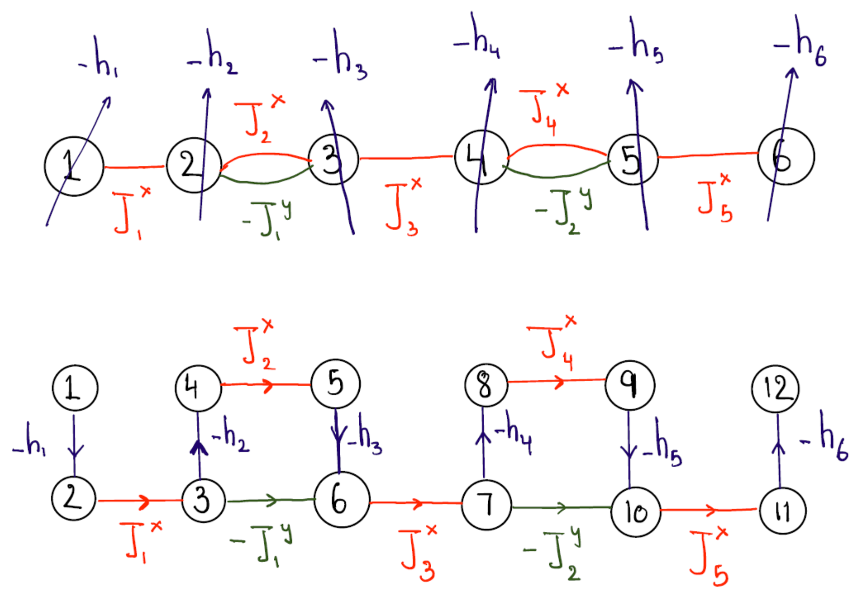


Figure 2.2: Real-space representation of an open transverse compass chain with $N = 6$ spins (above) and the corresponding snake chain representation for H_{OBC}^{MF} in the Majorana space (below).

2.3 Eigenvalue Problem

We now solve the eigenvalue problem:

$$H_{OBC}V = \lambda V \tag{2.10}$$

where V is the eigenvector of the form $V = (A_1, B_1, C_1, D_1, ..., A_{N/2}, B_{N/2}, C_{N/2}, D_{N/2})$. We have taken this form due to the quasi-periodic structure of our Hamiltonian matrix as can be seen in equation (2.7). Below we write the eigenvalue equations.

$$-h_1 B_1 = \lambda A_1 \tag{2.11}$$

$$h_1 A_1 + J_1^x C_1 = \lambda B_1 \tag{2.12}$$

$$-J_1^x B_1 - h_2 D_1 - J_1^y B_2 = \lambda C_1 \tag{2.13}$$

$$h_2 C_1 + J_2^x A_2 = \lambda D_1 \tag{2.14}$$

$$-J_2^x D_1 - h_3 B_2 = \lambda A_2 \tag{2.15}$$

$$J_1^y C_1 + h_3 A_2 + J_3^x C_2 = \lambda B_2 \tag{2.16}$$

$$-J_3^x B_2 - h_4 D_2 - J_2^y B_3 = \lambda C_2 \tag{2.17}$$

$$h_4 C_2 + J_4^x A_3 = \lambda D_2 \tag{2.18}$$

$$-J_4^x D_2 - h_5 B_3 = \lambda A_3 \tag{2.19}$$

$$J_2^y C_2 + h_5 A_3 + J_5^x C_3 = \lambda B_3 \tag{2.20}$$

$$-J_5^x B_3 - h_6 D_3 = \lambda C_3 \tag{2.21}$$

$$h_6 C_3 = \lambda D_3 \tag{2.22}$$

The first two and last two above equations are the boundary equations while the rest are the bulk equations. If we have H_{OBC} matrix of dimension $2N \times 2N$, then there will be $2N-4$ bulk equations. We now write the genral form of Bulk equations. They are mentioned below.

$$-J_{2j-1}^x B_j - h_{2j} D_j - J_j^y B_{j+1} = \lambda C_j \tag{2.23}$$

$$h_{2j} C_j + J_{2j}^x A_{j+1} = \lambda D_j \tag{2.24}$$

$$-J_{2j}^x D_j - h_{2j+1} B_{j+1} = \lambda A_{j+1} \tag{2.25}$$

$$J^y_j C_j + h_{2j+1}A_{j+1} + J^x_{2j+1}C_{j+1} = \lambda B_{j+1} \tag{2.26}$$

where $j = 1, 2, 3, ..., \frac{N}{2} - 1$. We also write the four boundary equations:

$$-h_1 B_1 = \lambda A_1 \tag{2.27}$$

$$h_1 A_1 + J^x_1 C_1 = \lambda B_1 \tag{2.28}$$

$$-J^x_{N/2+2} B_{N/2} - h_N D_{N/2} = \lambda C_{N/2} \tag{2.29}$$

$$h_N C_{N/2} = \lambda D_{N/2} \tag{2.30}$$

To explore the conditions under which EMZMs can emerge, we put $\lambda = 0$ in the above equation (2.23-2.30). We get the following equations:

$$-J^x_{2j-1}B_j - h_{2j}D_j - J^y_j B_{j+1} = 0 \tag{2.31}$$

$$h_{2j}C_j + J^x_{2j}A_{j+1} = 0 \tag{2.32}$$

$$-J^x_{2j}D_j - h_{2j+1}B_{j+1} = 0 \tag{2.33}$$

$$J^y_j C_j + h_{2j+1}A_{j+1} + J^x_{2j+1}C_{j+1} = 0 \tag{2.34}$$

$$-h_1 B_1 = 0 \tag{2.35}$$

$$h_1 A_1 + J^x_1 C_1 = 0 \tag{2.36}$$

$$-J^x_{N/2+2} B_{N/2} - h_N D_{N/2} = 0 \tag{2.37}$$

$$h_N C_{N/2} = 0 \tag{2.38}$$

We now solve these equations for different cases mentioned in the introduction to determine the EMZMs.

2.3.1 Case 1: Considering zero magnetic field at all sites

From equation (2.31),

$$-J^x_{2j-1}B_j - J^y_j B_{j+1} = 0 \implies B_{j+1} = -\frac{J^x_{2j-1}}{J^y_j}B_j$$

From equation (2.32),

$$J^x_{2j}A_{j+1} = 0 \implies A_{j+1} = 0 \quad \forall \quad j$$

Although we can't say anything about A_1 at this moment.
From equation (2.33),

$$-J^x_{2j} D_j = 0 \implies D_j = 0 \quad \forall \quad j$$

From equation (2.34),

$$J^y_j C_j + J^x_{2j+1} C_{j+1} = 0 \implies C_{j+1} = -\frac{J^y_j}{J^x_{2j+1}} C_j$$

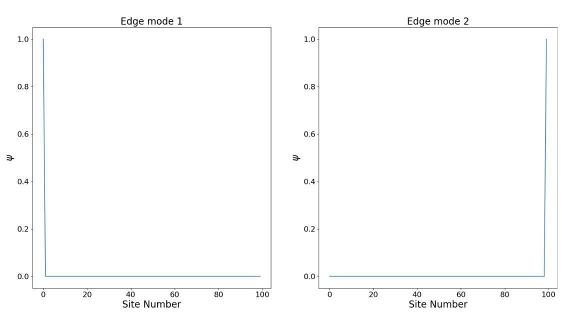
From boundary equations (2.35-2.38), we have $C_1 = 0$. Therefore, from the recursion relation of C, $C_j = 0 \quad \forall \quad j$. Similarly $B_{N/2} = 0$. Therefore, from the recursion relation of B, $B_j = 0 \quad \forall \quad j$. The only possible non-zero component is A_1. Therefore,

$$V_1 = (1, 0, 0, 0, ..., 0)^T$$

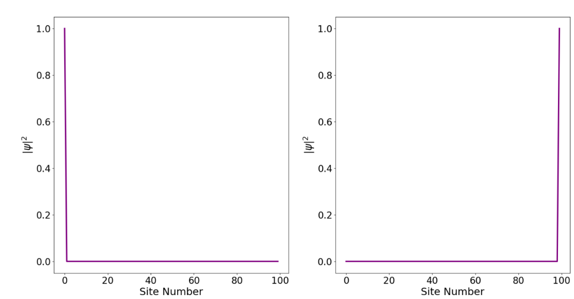
is the eigenvector ($A_1 = 1$) that we obtain. If we solve the matrix from below, then since the structure of matrix remains same, we get one more non-zero component $D_{N/2}$. Therefore,

$$V_2 = (0, ..., 0, 0, 0, 1)^T$$

As a result, we will get two localized EMZMs at the boundary. Figure[2.3a and 2.3b] depicts the probability amplitude and probability density of both edge modes respectively (considering number of sites is 50, so number of sites in snake-chain representation (figure[2.2]) will be 100).



(a) Probability amplitude of Majorana Zero Modes for $h = 0$ and $J^x = J^y = 1$



(b) Probability density of Majorana Zero Modes for $h = 0$ and $J^x = J^y = 1$

Figure 2.3: Probability density and probability amplitude of Majorana Zero Modes

2.3.2 Case 2: Considering uniform magnetic field

From equation (2.31 & 2.33),

$$-J^x_{2j-1}B_j - h\left[-\frac{h}{J^x_{2j}}B_{j+1}\right] - J^y_j B_{j+1} = 0 \implies B_{j+1} = \frac{J^x_{2j}J^x_{2j-1}}{h^2 - J^x_{2j}J^y_j}B_j$$

From equation (2.32 & 2.34),

$$J^y_j C_j + h\left[-\frac{h}{J^x_{2j}}C_j\right] + J^x_{2j+1}C_{j+1} = 0 \implies C_{j+1} = \frac{h^2 - J^x_{2j}J^y_j}{J^x_{2j}J^x_{2j+1}}C_j$$

From boundary equations (2.35-2.38), we have

$$B_1 = 0 \ , \ C_1 = -\frac{h}{J^x_1}A_1 \ , \ B_{N/2} = -\frac{h}{J^x_{N/2+2}}D_{N/2} \ , \ C_{N/2} = 0$$

Now there can be two possible scenarios:

1. $h^2 - J^x_{2j}J^y_j \neq 0$

   From $B_1 = 0$ and recursion relation of B, we conclude that $B_j = 0 \ \forall \ j$. Similarly From $C_{N/2} = 0$ and recursion relation of C, we conclude that $C_j = 0 \ \forall \ j$.

   From equation (2.33), we have $D_j = -\frac{h}{J^x_{2j}}B_{j+1}$. Since all B_j components were 0, we have $D_j = 0$ for $j = 1, 2, ..., N/2 - 1$. From Boundary equation, since $B_{N/2} = 0$, we have $D_{N/2} = 0$ and so $D_j = 0 \ \forall \ j$.

   Similarly from equation (2.32), we have $A_{j+1} = -\frac{h}{J^x_{2j}}C_j$. Since all C_j components were 0, we have $A_{j+1} = 0$ for $j = 1, 2, ..., N/2 - 1$. From Boundary equation, since $C_1 = 0$, we have $A_1 = 0$ and so $A_j = 0 \ \forall \ j$.

   As a result, there will be no EMZMs considering $h^2 - J^x_{2j}J^y_j \neq 0$. This implies that by introducing the uniform magnetic field h, the EMZMs got destroyed.

2. $h^2 - J^x_{2j}J^y_j = 0$

   Consider recursion relation of C at $j = N/2 - 1$:

$$C_{N/2} = \frac{h^2 - J^x_{N-2}J^y_{N/2-1}}{J^x_{N-2}J^x_{N-1}}C_{N/2-1}$$

Since $C_{N/2} = 0$ and $h^2 - J^x_{2j} J^y_j = 0$, we might think that $C_{N/2-1} \neq 0$. But if this is the case, then recursion relation will not be satisfied for $j = N/2 - 2$. Hence $C_{N/2-1} = 0$ and this will go on until for j = 1, we have $C_2 = \frac{h^2 - J^x_2 J^y_1}{J^x_2 J^x_3} C_1$. In this case we can have $C_1 \neq 0$. Using the similar analysis, we can obtain that $B_{N/2} \neq 0$.

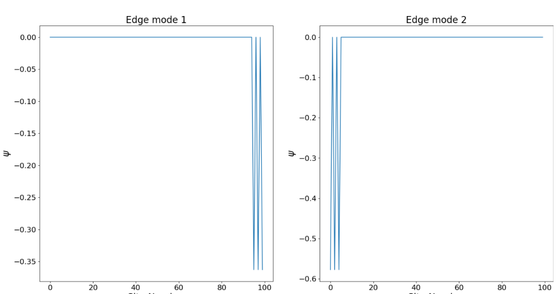
Now since $C_1 \neq 0$, from boundary equations, we can see that $A_1 \neq 0$ and from equation (2.32), $A_2 \neq 0$. Rest components of A will be 0. Similarly, since $B_{N/2} \neq 0$, from boundary equations, we can see that $D_{N/2} \neq 0$ and from equation (2.33), $D_{N/2-1} \neq 0$. Rest components of D will be 0.

As a result, we get two eigenvectors.

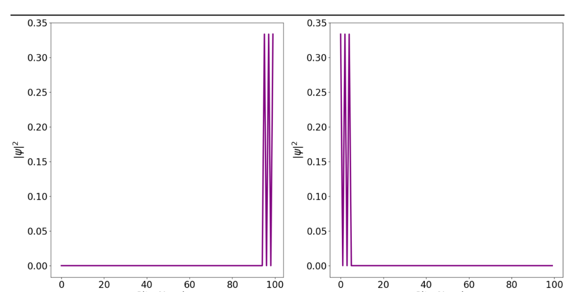
$$V_1 = \left(-\frac{J^x_1}{h}, 0, 1, 0, -\frac{h}{J^x_2,}, 0, 0, 0, ..., 0, 0, 0, 0 \right)^T$$

$$V_2 = \left(0, 0, 0, 0, ..., 0, 0, 0, -\frac{h}{J^x_{N-2}}, 0, 1, 0, -\frac{J^x_{N/2+2}}{h} \right)^T$$

Figure[2.4a and 2.4b] depicts the probability amplitude and probability density of both edge modes respectively. The reason we observe two edge modes numerically is mentioned in the comments below.



(a) Probability amplitude of Majorana Zero Modes for $h = 1$ and $J^x = J^y = 1$



(b) Probability density of Majorana Zero Modes for $h = 1$ and $J^x = J^y = 1$

Figure 2.4: Probability density and probability amplitude of Majorana Zero Modes

Comments:

1. A & C components are dependent with each other and B & D components are dependent with each other. However B and D components are independent of A and C components. We could have considered an eigenvector of the form

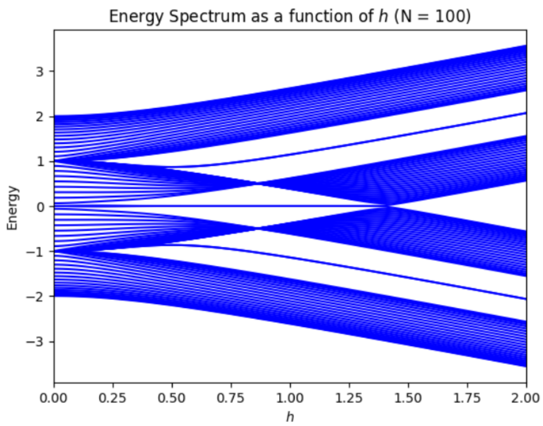
$$V = \left(-\frac{J_1^x}{h}, 0, 1, 0, -\frac{h}{J_{2}^x,}, 0, 0, 0, ..., 0, 0, 0, -\frac{h}{J_{N-2}^x}, 0, 1, 0, -\frac{J_{N/2+2}^x}{h}\right)^T$$

This implies that we have a single EMZM which is non-local. This single EMZM is mentioned in a different eigenbasis (analogous to let's say we have a 2D cartesian plane with x and y axis as basis and we simply rotate the plane by some angle so that the basis vectors will change). This eigenbasis V is written in form of $c_1V_1 + c_2V_2$.

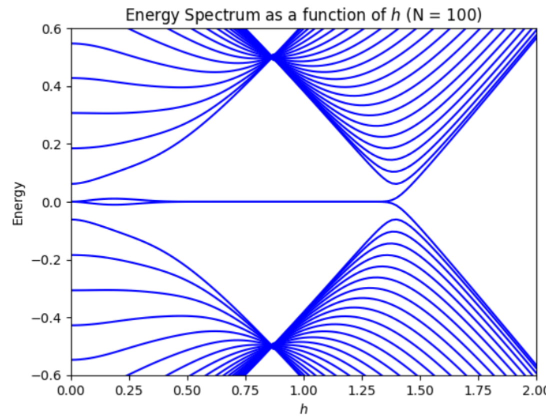
2. The eigenvectors V_1 and V_2 represent a localized EMZM at the boundary. From equation (2.9), we defined Majorana fermion as $B = W\Psi$. Each row of W can be written as eigenvector of H_{OBC}. There should be 2N eigenvectors but we only care about eigenvectors corresponding to $\lambda = 0$. Therefore, the eigenvector V_1 can be described as an EMZM which is in a superposition state of d_1, d_3 & d_5 Majorana fermions which is non-local but concentrated at boundary. Similarly V_2 can be described as an EMZM which is in a superposition state of d_{2N-4}, d_{2N-2} & d_{2N} Majorana fermions.
   It is now understandable that the eigenvector V represent an EMZM which is highly non-local (spread in the bulk but still high probability amplitude at the boundaries) and in a superposition of $d_1, d_3, d_5, d_{2N-4}, d_{2N-2}$ & d_{2N} Majorana fermion.

3. Since the matrix can be solved from below, we get two more EMZMs and so in total, we will obtain 4 EMZMs (or 2 EMZMs if we consider different basis), concentrated at the boundary (although non-local) considering $h^2 - J_{2j}^x J_j^y = 0$.

4. The figure[2.5] captures the energy spectrum of the Hamiltonian matrix (equation[2.7]), fixing the parameters $J^x = J^y = 1$ and varying the parameter h. The zoomed figure clearly captures that as soon as we introduce magnetic field, the zero energy levels will be splitted into two, meaning Majorana zero modes will be destroyed before reincarnating again at a finite magnetic value and persisting for some finite h after which they split again.



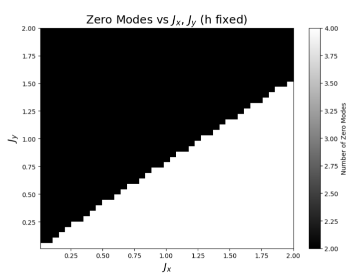
(a) full energy spectrum



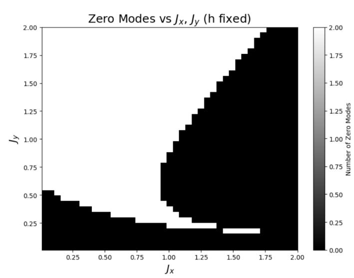
(b) zoomed-in spectrum

Figure 2.5: Energy spectrum of Hamiltonian matrix

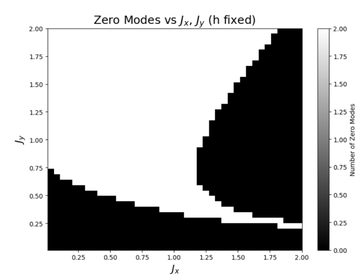
5. The phase diagram below (figure[2.6]) captures how J^x and J^y interactions affects the number of Majorana zero modes, keeping magnetic field value fixed.



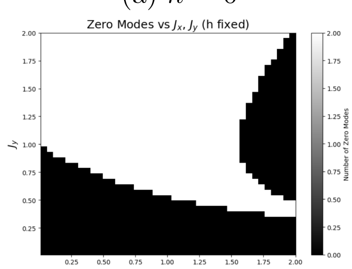
(a) $h = 0$



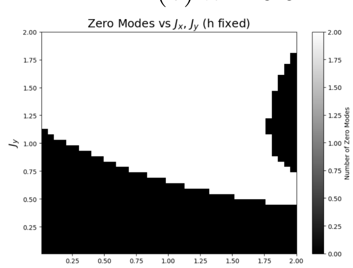
(b) $h = 0.5$



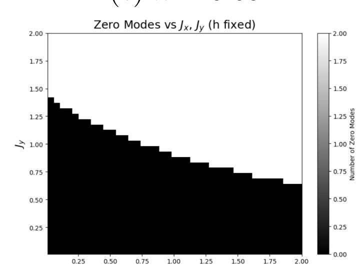
(c) $h = 0.65$



(d) $h = 0.87$



(e) $h = 1$



(f) $h = 1.26$

Figure 2.6: Phase diagram of number of zero modes. The color indicates the number of Majorana zero modes.

2.3.3 Case 3: Considering non-uniform magnetic field

We consider that the magnetic field on the subset of the lattice sites, say, $S = \{l_1, l_2, ..., l_m\}, \ 1 \leq l_1, l_2, ..., l_m \leq N$ vanish. The set S can always be written as the union of n ordered consecutive sequences $T_j = (l_{u_j}, l_{u_j}+1, ..., l_{u_j}+v_j-1)$ with length v_j, i.e., $S = \{T_1, T_2, ..., T_n\}$. We consider different cases and then try to generalize the formula to find the number of EMZMs.

2.3.3.1 Case 1: Considering S consist of a single consecutive sequence

Considering S consist of a single consecutive sequence T of length ν, i.e., $S = T = (l_u, l_u+1, ..., l_u+\nu-1)$. We first assume that l_u is even, so that $l_u \geq 2$.

- **v = even**

  This implies that $l_u + v - 1 =$ odd. We have $h_{l_u} = h_{l_u+1} = ... = h_{l_u+v-1} = 0$. From equations (2.32 and 2.33), we see that for $j \in \left[\frac{l_u}{2}, \frac{l_u}{2}+1, ..., \frac{l_u}{2}+\frac{v}{2}-1\right]$, we have

$$A_{j+1} = 0 \ , \ D_j = 0 \tag{2.39}$$

  and there is a possibility that

$$C_j \neq 0 \ , \ B_{j+1} \neq 0 \tag{2.40}$$

  For rest of j, we have from equations (2.32 and 2.33)

$$A_{j+1} = -\frac{h_{2j}}{J^x_{2j}} C_j \ , \ B_{j+1} = -\frac{J^x_{2j}}{h_{2j+1}} D_j \tag{2.41}$$

  Now we insert the above equations (2.39 & 2.40) into equations (2.31 & 2.34). We get a set of 3 equations.

$$-J^x_{l_u+2j-1} B_{\frac{l_u}{2}+j} - J^y_{\frac{l_u}{2}+j} B_{\frac{l_u}{2}+j+1} = 0 \ , \ j \in \left[1, \frac{v}{2}-1\right] \tag{2.42}$$

$$-J^x_{l_u+v-1} B_{\frac{l_u+v}{2}} - h_{l_u+v} D_{\frac{l_u+v}{2}} - J^y_{\frac{l_u+v}{2}} B_{\frac{l_u+v}{2}+1} = 0 \ , \ j = \frac{l_u+v}{2} \tag{2.43}$$

$$-J^x_{l_u-1} B_{\frac{l_u}{2}} - J^y_{\frac{l_u}{2}} B_{\frac{l_u}{2}+1} = 0 \ , \ j = \frac{l_u}{2} \tag{2.44}$$

For trivial case, we obtain:

$$B_{\frac{l_u+v}{2}+1} = -\frac{h_{l_u+v}}{J^y_{\frac{l_u+v}{2}}} D_{\frac{l_u+v}{2}} \tag{2.45}$$

Before inserting equations (2.39 & 2.40) into equation (2.34), we modify this by substituting $j \to j-1$. This will be clear in a moment when we write 3 set of equations.

$$J^y_{\frac{l_u}{2}+j-1} C_{\frac{l_u}{2}+j-1} + J^x_{l_u+2j-1} C_{\frac{l_u}{2}+j} = 0 \ , \ j \in \left[1, \frac{v}{2}-1\right] \tag{2.46}$$

$$J^x_{l_u+v-1} C_{\frac{l_u+v}{2}} + J^y_{\frac{l_u+v}{2}-1} C_{\frac{l_u+v}{2}-1} = 0 \ , \ j = \frac{l_u+v}{2} \tag{2.47}$$

$$J^y_{\frac{l_u}{2}-1} C_{\frac{l_u}{2}-1} + h_{l_u-1} A_{\frac{l_u}{2}} + J^x_{l_u-1} C_{\frac{l_u}{2}} = 0 \ , \ j = \frac{l_u}{2} \tag{2.48}$$

We substitute $j \to j-1$, so that we get a magnetic field dependent term at the boundary (just before the site at which $h = 0$, i.e. at $l_u - 1$).
For trivial case, we obtain:

$$C_{\frac{l_u}{2}-1} = -\frac{h_{l_u-1}}{J^y_{\frac{l_u}{2}-1}} A_{\frac{l_u}{2}} \tag{2.49}$$

From equation (2.44), we have $B_{\frac{l_u}{2}+1} = -\frac{J^x_{l_u-1}}{J^y_{\frac{l_u}{2}}} B_{\frac{l_u}{2}}$. From equation (2.42), we have

$$B_{\frac{l_u}{2}+j+1} = (-1)^{j+1} \prod_{j=0}^{j=\left[0,\frac{v}{2}-1\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}} B_{\frac{l_u}{2}}$$

From equation (2.43), we have (using 2.41)

$$-J^x_{l_u+v-1} B_{\frac{l_u+v}{2}} - h_{l_u+v} D_{\frac{l_u+v}{2}} - J^y_{\frac{l_u+v}{2}} \left(-\frac{J^x_{l_u+v}}{h_{l_u+v+1}} D_{\frac{l_u+v}{2}} \right) \tag{2.50}$$

$$\left(J^y_{\frac{l_u+v}{2}} J^x_{l_u+v} - h_{l_u+v} h_{l_u+v+1} \right) D_{\frac{l_u+v}{2}} = h_{l_u+v+1} J^x_{l_u+v-1} B_{\frac{l_u+v}{2}} \tag{2.51}$$

$$D_{\frac{l_u+v}{2}} = \frac{h_{l_u+v+1} J^x_{l_u+v-1}}{J^y_{\frac{l_u+v}{2}} J^x_{l_u+v} - h_{l_u+v} h_{l_u+v+1}} B_{\frac{l_u+v}{2}} \tag{2.52}$$

We also obtain:

$$B_{\frac{l_u+v}{2}+1} = -\frac{J^x_{l_u+v}}{h_{l_u+v+1}} D_{\frac{l_u+v}{2}} \ , \ D_{\frac{l_u}{2}-1} = -\frac{h_{l_u-1}}{J^x_{l_u-2}} B_{\frac{l_u}{2}} \tag{2.53}$$

From equation (2.47), we have $C_{\frac{l_u+v}{2}-1} = -\frac{J^x_{l_u+v-1}}{J^y_{\frac{l_u+v}{2}-1}} C_{\frac{l_u+v}{2}}$. From equation (2.46), we have

$$C_{\frac{l_u}{2}+j-1} = (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[1,\frac{v}{2}\right]}^{j=\frac{v}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}} C_{\frac{l_u+v}{2}} \tag{2.54}$$

From equation (2.48), we have (using 2.41)

$$J^y_{\frac{l_u}{2}-1}\left(-\frac{J^x_{l_u-2}}{h_{l_u-2}} A_{\frac{l_u}{2}}\right) + h_{l_u-1} A_{\frac{l_u}{2}} + J^x_{l_u-1} C_{\frac{l_u}{2}} = 0 \tag{2.55}$$

$$\left(-h_{l_u-2} h_{l_u-1} + J^x_{l_u-2} J^y_{\frac{l_u}{2}-1}\right) A_{\frac{l_u}{2}} = h_{l_u-2} J^x_{l_u-1} C_{\frac{l_u}{2}} \tag{2.56}$$

$$A_{\frac{l_u}{2}} = \frac{h_{l_u-2} J^x_{l_u-1}}{J^x_{l_u-2} J^y_{\frac{l_u}{2}-1} - h_{l_u-2} h_{l_u-1}} C_{\frac{l_u}{2}} \tag{2.57}$$

We also obtain:

$$C_{\frac{l_u}{2}-1} = -\frac{J^x_{l_u-2}}{h_{l_u-2}} A_{\frac{l_u}{2}} \ , \ A_{\frac{l_u+v}{2}+1} = -\frac{h_{l_u+v}}{J^x_{l_u+v}} C_{\frac{l_u+v}{2}} \tag{2.58}$$

For $j \notin \left[\frac{l_u}{2}, \frac{l_u}{2} + \frac{v}{2} - 1\right]$, the following relations remain true:

$$A_{j+1} = -\frac{h_{2j}}{J^x_{2j}} C_j \ , \ D_j = -\frac{h_{2j+1}}{J^x_{2j}} B_{j+1}$$

$$B_j = \frac{h_{2j} h_{2j+1} - J^x_{2j} J^y_j}{J^x_{2j-1} J^x_{2j}} B_{j+1} \ , \ C_{j+1} = \frac{h_{2j} h_{2j+1} - J^x_{2j} J^y_j}{J^x_{2j+1} J^x_{2j}} C_j$$

The eigenvectors obtained from all of this are mentioned below:

$$V_1 = \left(0, ..., 0, 0, 0, -\frac{h_{l_u-1}}{J^x_{l_u-2}}, 0, 1^{(\frac{l_u}{2})}, 0, 0, 0, (-1)^{j+1} \prod_{j=0}^{j=\left[0,\frac{v}{2}-2\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}}, 0, 0, ..., 0, B_{\frac{l_u+v}{2}}, 0, D_{\frac{l_u+v}{2}}, 0, B_{\frac{l_u+v}{2}+1}, 0, 0, ..., 0\right)^T$$

$$V_2 = \left(0, ...0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, C_{\frac{l_u}{2}}, 0, 0, 0, (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[2,\frac{v}{2}\right]}^{j=\frac{v}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}}, 0, ..., 0, 0, 1^{\left(\frac{l_u+v}{2}\right)}, 0, A_{\frac{l_u+v}{2}+1}, 0, 0, 0...., 0\right)^T$$

$$V_3 = \left(0, 0, 0, 0, ..., 0, 0, 0, D_{\frac{l_u+v}{2}}, 0, B_{\frac{l_u+v}{2}+1}, 0, 0, ..., 0, 0, 0, 0\right)^T$$

$$V_4 = \left(0, 0, 0, 0, ..., 0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

We get the above four eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. If we recall from Case 2, when $h^2 \neq J^x_{2j}J^y_j$, there were no eigenvectors. However when $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors in addition to above four as mentioned below:

$$V_5 = \left(-\frac{J^x_1}{h_1}, 0, 1, 0, -\frac{h_2}{J^x_2}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

$$V_6 = \left(0, 0, 0, 0, ..., 0, 0, 0, -\frac{h_{N-1}}{J^x_{N-2}}, 0, 1, 0, -\frac{J^x_{N/2+2}}{h_N}\right)^T$$

As a result, we get in total $2 \times 4 = 8$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2 \times 6 = 12$ EMZMs (multiplied by 2 because matrix is symmetric from below) for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for even l_u and even v.

- **v = odd**

  1. v = 1

     In this case, only $h_{l_u} = 0$. From equation (2.32), at $j = \frac{l_u}{2}$, we have $A_{\frac{l_u}{2}+1} = 0$ and possibly $C_{\frac{l_u}{2}} \neq 0$. Except at $j = \frac{l_u}{2}$, we have the relation $A_{j+1} = -\frac{h_{2j}}{J^x_{2j}}C_j$.From equation (2.33), we have $B_{j+1} = -\frac{J^x_{2j}}{h_{2j+1}}D_j$. From equation (2.31), we can write at $j = \frac{l_u}{2}$,

$$-J^x_{l_u-1}B_{\frac{l_u}{2}} - J^y_{\frac{l_u}{2}}B_{\frac{l_u}{2}+1} = 0 \tag{2.59}$$

$$B_{\frac{l_u}{2}+1} = -\frac{J^x_{l_u-1}}{J^y_{\frac{l_u}{2}}}B_{\frac{l_u}{2}} \tag{2.60}$$

From equation (2.41), we can show that $D_{\frac{l_u}{2}} = -\frac{h_{l_u+1}}{J^x_{l_u}}B_{\frac{l_u}{2}+1}$ and $D_{\frac{l_u}{2}-1} = -\frac{h_{l_u-1}}{J^x_{l_u-2}}B_{\frac{l_u}{2}}$.

From equation (2.34), We write two coupled equations at $j = \frac{l_u}{2}$ (where second equation is written after transforming $j \to j-1$ in equation (2.34)).

$$J^y_{\frac{l_u}{2}} C_{\frac{l_u}{2}} + J^x_{l_u+1} C_{\frac{l_u}{2}+1} = 0 \tag{2.61}$$

$$J^y_{\frac{l_u}{2}-1} C_{\frac{l_u}{2}-1} + h_{l_u-1} A_{\frac{l_u}{2}} + J^x_{l_u-1} C_{\frac{l_u}{2}} = 0 \tag{2.62}$$

Using equation (2.41), we obtain

$$C_{\frac{l_u}{2}+1} = -\frac{J^y_{\frac{l_u}{2}}}{J^x_{l_u+1}} C_{\frac{l_u}{2}} \tag{2.63}$$

From $C_{\frac{l_u}{2}+1}$, we can obtain $A_{\frac{l_u}{2}+2}$ using equation (2.41):

$$A_{\frac{l_u}{2}+2} = -\frac{h_{l_u+2}}{J^x_{l_u+2}} C_{\frac{l_u}{2}+1} \tag{2.64}$$

Similarly, we can obtain,

$$J^y_{\frac{l_u}{2}-1}\left(\frac{J^x_{l_u-2}}{h_{l_u-2}}\right) A_{\frac{l_u}{2}} - h_{l_u-1} A_{\frac{l_u}{2}} = J^x_{l_u-1} C_{\frac{l_u}{2}} \tag{2.65}$$

$$\left(J^y_{\frac{l_u}{2}-1} J^x_{l_u-2} - h_{l_u-1} h_{l_u-2}\right) A_{\frac{l_u}{2}} = J^x_{l_u-1} h_{l_u-2} C_{\frac{l_u}{2}} \tag{2.66}$$

$$A_{\frac{l_u}{2}} = \frac{J^x_{l_u-1} h_{l_u-2}}{J^y_{\frac{l_u}{2}-1} J^x_{l_u-2} - h_{l_u-1} h_{l_u-2}} C_{\frac{l_u}{2}} \tag{2.67}$$

From $A_{\frac{l_u}{2}}$, we can obtain $C_{\frac{l_u}{2}-1}$ using equation (2.41):

$$C_{\frac{l_u}{2}-1} = -\frac{J^x_{l_u-2}}{h_{l_u-2}} A_{\frac{l_u}{2}} = \frac{J^x_{l_u-1} J^x_{l_u-2}}{J^y_{\frac{l_u}{2}-1} J^x_{l_u-2} - h_{l_u-1} h_{l_u-2}} C_{\frac{l_u}{2}} \tag{2.68}$$

For other sites where magnetic field is non-zero, we have the usual relations mentioned in the first case. The eigenvectors obtained from above analysis are mentioned below:

$$V_1 = \left(0, ..., 0, 0, \frac{-J^x_{l_u-1} J^x_{l_u-2}}{J^y_{\frac{l_u}{2}-1} J^x_{l_u-2} - h_{l_u-1} h_{l_u-2}}, 0, \frac{J^x_{l_u-1} h_{l_u-2}}{J^y_{\frac{l_u}{2}-1} J^x_{l_u-2} - h_{l_u-1} h_{l_u-2}}, 0, 1^{\left(\frac{l_u}{2}\right)}, 0, 0, 0, -\frac{J^y_{\frac{l_u}{2}}}{J^x_{l_u+1}}, 0, \frac{h_{l_u+2} J^y_{\frac{l_u}{2}}}{J^x_{l_u+2} J^x_{l_u+1}}, ..., 0\right)^T$$

$$V_2 = \left(0,0,0,0,....,0,0,0,-\frac{h_{l_u-1}}{J^x_{l_u-2}},0,1^{\left(\frac{l_u}{2}\right)},0,\frac{h_{l_u+1}J^x_{l_u-1}}{J^x_{l_u}J^y_{\frac{l_u}{2}}},0,-\frac{J^x_{l_u-1}}{J^y_{\frac{l_u}{2}}},0,0,...,0,0,0,0\right)^T$$

We get the above two eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors as mentioned below:

$$V_3 = \left(-\frac{J^x_1}{h_1},0,1,0,-\frac{h_2}{J^x_2},0,0,0,....,0,0,0,0\right)^T$$

$$V_4 = \left(0,0,0,0,......,0,0,0,-\frac{h_{N-1}}{J^x_{N-2}},0,1,0,-\frac{J^x_{N/2+2}}{h_N}\right)^T$$

As a result, we get in total $2 \times 2 = 4$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2 \times 4 = 8$ EMZMs for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for even l_u and odd v ($v = 1$).

2. $v \geq 3$

We have $l_u + v - 1$ =even. From equation (2.32), we obtain $C_j \neq 0$ and $A_{j+1} = 0$ for $j \in \left[\frac{l_u}{2}, \frac{l_u}{2}+1, ..., \frac{l_u}{2}+\frac{v-1}{2}\right]$. For other values of j, we have $A_{j+1} = -\frac{h_{2j}}{J^x_{2j}}C_j$.

From equation (2.33), we obtain that $B_{j+1} \neq 0$ and $D_j = 0$ for $j \in \left[\frac{l_u}{2}, \frac{l_u}{2}+1, ..., \frac{l_u}{2}+\frac{v-3}{2}\right]$. For other values of j, we have $B_{j+1} = -\frac{J^x_{2j}}{h_{2j+1}}D_j$.

Now we insert the above assumptions in equations (2.31 and 2.34). From each, we get a set of 3 equations mentioned below:

$$-J^x_{l_u+2j-1}B_{\frac{l_u}{2}+j} - J^y_{\frac{l_u}{2}+j}B_{\frac{l_u}{2}+j+1} = 0 \;\;,\;\; j = \left[1, \frac{v-3}{2}\right] \tag{2.69}$$

$$-J^x_{l_u-1}B_{\frac{l_u}{2}} - J^y_{\frac{l_u}{2}}B_{\frac{l_u}{2}+1} = 0 \;\;,\;\; j = \frac{l_u}{2} \tag{2.70}$$

$$-J^x_{l_u+v-2}B_{\frac{l_u}{2}+\frac{v-1}{2}} - J^y_{\frac{l_u}{2}+\frac{v-1}{2}}B_{\frac{l_u}{2}+\frac{v+1}{2}} = 0 \;\;,\;\; j = \frac{l_u}{2} + \frac{v-1}{2} \tag{2.71}$$

In equation (2.34), we convert $j \to j-1$ and obtain:

$$J^y_{\frac{l_u}{2}+j-1}C_{\frac{l_u}{2}+j-1} + J^x_{l_u+2j-1}C_{\frac{l_u}{2}+j} = 0 \;\;,\;\; j = \left[1, \frac{v-1}{2}\right] \tag{2.72}$$

$$J^y_{\frac{l_u}{2}-1}C_{\frac{l_u}{2}-1} + J^x_{l_u-1}C_{\frac{l_u}{2}} + h_{l_u-1}A_{\frac{l_u}{2}} = 0, \ \ , \ \ j = \frac{l_u}{2} \tag{2.73}$$

$$J^y_{\frac{l_u}{2}+\frac{v-1}{2}}C_{\frac{l_u}{2}+\frac{v-1}{2}} + J^x_{l_u+v}C_{\frac{l_u}{2}+\frac{v+1}{2}} = 0 \tag{2.74}$$

Using equations (2.69,2.70 and 2.71), we can show that

$$B_{\frac{l_u}{2}+j+1} = (-1)^{j+1} \prod_{j=0}^{j\in\left[0,\frac{v-1}{2}\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}} B_{\frac{l_u}{2}} \tag{2.75}$$

$D_{\frac{l_u}{2}+\frac{v-1}{2}} = -\frac{h_{l_u+v}}{J^x_{l_u+v-1}} B_{\frac{l_u}{2}+\frac{v+1}{2}}$, $D_{\frac{l_u}{2}-1} = -\frac{h_{l_u-1}}{J^x_{l_u-2}} B_{\frac{l_u}{2}}$ are the non-zero component. For trivial case, all above components will vanish.

Using equations (2.72 and 2.74), we can show that

$$C_{\frac{l_u}{2}+j-1} = (-1)^{\frac{v+3}{2}-j} \prod_{j\in\left[1,\frac{v+1}{2}\right]}^{j=\frac{v+1}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}} C_{\frac{l_u}{2}+\frac{v+1}{2}} \tag{2.76}$$

Using equations (2.73 and 2.41), we can obtain $A_{\frac{l_u}{2}}$ as shown:

$$J^y_{\frac{l_u}{2}-1}\left(-\frac{J^x_{l_u-2}}{h_{l_u-2}}A_{\frac{l_u}{2}}\right) + h_{l_u-1}A_{\frac{l_u}{2}} = -J^x_{l_u-1}C_{\frac{l_u}{2}} \tag{2.77}$$

$$A_{\frac{l_u}{2}} = \frac{J^x_{l_u-1}h_{l_u-2}}{J^x_{l_u-2}J^y_{\frac{l_u}{2}-1} - h_{l_u-1}h_{l_u-2}} C_{\frac{l_u}{2}} \tag{2.78}$$

The other non-zero components are mentioned below:

$$C_{\frac{l_u}{2}-1} = -\frac{J^x_{l_u-2}}{h_{l_u-2}} A_{\frac{l_u}{2}} \ \ , \ \ A_{\frac{l_u}{2}+\frac{v+3}{2}} = -\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}} C_{\frac{l_u}{2}+\frac{v+1}{2}} \tag{2.79}$$

For trivial case, we obtain $A_{\frac{l_u}{2}} = -\frac{J^y_{\frac{l_u}{2}-1}}{h_{l_u-1}} C_{\frac{l_u}{2}-1}$. For other sites where magnetic field is non-zero, we have the usual relations mentioned previously.

The eigenvectors obtained from above analysis are mentioned below:

$$V_1 = \left(0, ...C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, C_{\frac{l_u}{2}}, 0, 0, 0, (-1)^{\frac{v+3}{2}-j} \prod_{j\in\left[2,\frac{v+1}{2}\right]}^{j=\frac{v+1}{2}} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j-1}} C_{\frac{l_u}{2}+\frac{v+1}{2}}, 0, ..., 0, 0, 1^{\left(\frac{l_u}{2}+\frac{v+1}{2}\right)}, 0, -\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}}, .., 0\right)^T$$

$$V_2 = \left(0, ..., -\frac{h_{l_u-1}}{J^x_{l_u-2}}, 0, 1^{\left(\frac{l_u}{2}\right)}, 0, 0, 0, (-1)^{j+1} \prod_{j=0}^{j\in\left[0,\frac{v-5}{2}\right]} \frac{J^x_{l_u+2j-1}}{J^y_{\frac{l_u}{2}+j}}, 0, 0, ..., 0, B_{\frac{l_u}{2}+\frac{v-1}{2}}, 0, D_{\frac{l_u}{2}+\frac{v-1}{2}}, 0, B_{\frac{l_u+v+1}{2}}, 0, 0, .., 0\right)^T$$

$$V_3 = \left(0, 0, 0, 0, ..., 0, 0, C_{\frac{l_u}{2}-1}, 0, A_{\frac{l_u}{2}}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

We get the above three eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors as mentioned below:

$$V_4 = \left(0, 0, 0, 0, ..., 0, 0, 0, -\frac{h_{N-1}}{J^x_{N-2}}, 0, 1^{\left(\frac{N}{2}\right)}, 0, -\frac{J^x_{\frac{N}{2}+2}}{h_N}\right)^T$$

$$V_5 = \left(-\frac{J^x_1}{h_1}, 0, 1, 0, -\frac{h_2}{J^x_2}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

As a result, we get in total $2 \times 3 = 6$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2 \times 5 = 10$ EMZMs for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for even l_u and odd v ($v \geq 3$).

Now we assume that l_u is odd with $l_u \geq 1$.

- **v = even**

  This implies that $l_u + v - 1 =$ even. From equation (2.32), we can say that $C_j \neq 0$ and $A_{j+1} = 0$ for $j = \left[\frac{l_u+1}{2}, \frac{l_u+3}{2}, ..., \frac{l_u+v-1}{2}\right]$. For same range of j (for which we converted $j \to j-1$ in equation (2.33)), we have from equation (2.33), $B_j \neq 0$ and $D_{j-1} = 0$.

  We imply the above relations into equations (2.31,2.34). In equation (2.31), we convert $j \to j-1$. We get a set of 3 equations which are mentioned below:

$$-J^x_{l_u+2j-2}B_{\frac{l_u-1}{2}+j} - J^y_{\frac{l_u-1}{2}+j}B_{\frac{l_u+1}{2}+j} = 0 \ , \ j = \left[1, \frac{v}{2}-1\right] \tag{2.80}$$

$$-J^x_{l_u-2} B_{\frac{l_u-1}{2}} - J^y_{\frac{l_u-1}{2}} B_{\frac{l_u+1}{2}} = 0 \ , \ j = \frac{l_u+1}{2} \tag{2.81}$$

$$-J^x_{l_u+v-2} B_{\frac{l_u}{2}+\frac{v-1}{2}} - J^y_{\frac{l_u}{2}+\frac{v-1}{2}} B_{\frac{l_u}{2}+\frac{v+1}{2}} = 0 \ , \ j = \frac{l_u}{2} + \frac{v+1}{2} \tag{2.82}$$

In equation (2.34) also, we convert $j \to j-1$ and obtain a set of 3 equations mentioned below:

$$J^y_{\frac{l_u-1}{2}+j} C_{\frac{l_u-1}{2}+j} + J^x_{l_u+2j} C_{\frac{l_u+1}{2}+j} = 0 \ , \ j = \left[1, \frac{v}{2}-1\right] \tag{2.83}$$

$$J^y_{\frac{l_u-1}{2}} C_{\frac{l_u-1}{2}} + J^x_{l_u} C_{\frac{l_u+1}{2}} = 0 \ , \ j = \frac{l_u+1}{2} \tag{2.84}$$

$$J^y_{\frac{l_u}{2}+\frac{v-1}{2}} C_{\frac{l_u}{2}+\frac{v-1}{2}} + J^x_{l_u+v} C_{\frac{l_u}{2}+\frac{v+1}{2}} = 0 \ , \ j = \frac{l_u}{2} + \frac{v+1}{2} \tag{2.85}$$

From equations (2.80-2.82) and using equation (2.41), we get the following non-zero components:

$$C_{\frac{l_u+1}{2}+j} = (-1)^{j+1} \prod_{j=0}^{j\in\left[0,\frac{v}{2}\right]} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}} C_{\frac{l_u-1}{2}} \tag{2.86}$$

$$A_{\frac{l_u+1}{2}} = -\frac{h_{l_u-1}}{J^x_{l_u-1}} C_{\frac{l_u-1}{2}} \ , \ A_{\frac{l_u+v+3}{2}} = -\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}} C_{\frac{l_u+v+1}{2}} \tag{2.87}$$

For trivial case, all above components will vanish. From equations (2.83-2.85) and using equation (2.41), we get the following non-zero components:

$$B_{\frac{l_u-1}{2}+j} = (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[0,\frac{v}{2}\right]}^{\frac{v}{2}} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j-2}} B_{\frac{l_u+v+1}{2}} \tag{2.88}$$

$$D_{\frac{l_u+v-1}{2}} = -\frac{h_{l_u+v}}{J^x_{l_u+v-1}} B_{\frac{l_u+v+1}{2}} \ , \ D_{\frac{l_u-3}{2}} = -\frac{h_{l_u-2}}{J^x_{l_u-3}} B_{\frac{l_u-1}{2}} \tag{2.89}$$

For trivial case, again we see that the above components will vanish.For other sites where magnetic field is non-zero, we have the usual relation mentioned previously. The eigenvectors obtained from above analysis are mentioned below:

$$V_1 = \left(0,..,0,0,1^{\left(\frac{l_u-1}{2}\right)},0,-\frac{h_{l_u-1}}{J^x_{l_u-1}},0,-\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u}},0,0,0,(-1)^{j+1}\prod_{j=1}^{j\in\left[1,\frac{v}{2}\right]}\frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}},0,..,-\frac{h_{l_u+v+1}}{J^x_{l_u+v+1}}C_{\frac{l_u+v+1}{2}},0,0,0,..,0\right)^T$$

$$V_2 = \left(0, ..., -\frac{h_{l_u-2}}{J^x_{l_u-3}} B_{\frac{l_u-1}{2}}, 0, (-1)^{\frac{v}{2}-j+1} \prod_{j=\left[0,\frac{v}{2}-1\right]}^{\frac{v}{2}} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j-2}}, 0, 0, ..., 0, -\frac{J^y_{\frac{l_u+v-1}{2}}}{J^x_{l_u+v-2}}, 0, -\frac{h_{l_u+v}}{J^x_{l_u+v-1}}, 0, 1\left(\frac{l_u+v+1}{2}\right), 0, 0, .., 0\right)^T$$

We get the above two eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors as mentioned below:

$$V_3 = \left(-\frac{J^x_1}{h_1}, 0, 1, 0, -\frac{h_2}{J^x_2}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

$$V_4 = \left(0, 0, 0, 0, ..., 0, 0, 0, -\frac{h_{N-1}}{J^x_{N-2}}, 0, 1\left(\frac{N}{2}\right), 0, -\frac{J^x_{\frac{N}{2}+2}}{h_N}\right)^T$$

As a result, we get in total $2\times2 = 4$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2\times4 = 8$ EMZMs for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for odd l_u and even v.

- **v = odd** We consider 2 cases here:

  1. v = 1

     In this case only $h_{l_u} \neq 0$. From equation (2.33), at $j = \frac{l_u-1}{2}$, we have $D_{\frac{l_u-1}{2}} = 0$ and possibly $B_{\frac{l_u+1}{2}} \neq 0$. From equation (2.31), at $j = \frac{l_u-1}{2}$, we obtain:

     $$-J^x_{l_u-2}B_{\frac{l_u-1}{2}} - J^y_{\frac{l_u-1}{2}}B_{\frac{l_u+1}{2}} = 0 \tag{2.90}$$

     From equation (2.34), at $j = \frac{l_u-1}{2}$, we obtain:

     $$J^y_{\frac{l_u-1}{2}}C_{\frac{l_u-1}{2}} + J^x_{l_u}C_{\frac{l_u+1}{2}} = 0 \tag{2.91}$$

     For other sites, we have our usual relation, $A_{j+1} = -\frac{h_{2j}}{J^x_{2j}}C_j$ and $B_{j+1} = -\frac{J^x_{2j}}{h_{2j+1}}$. From above analysis, we get the following non-zero components:

     $$B_{\frac{l_u-1}{2}} = -\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-2}}B_{\frac{l_u+1}{2}} \;\;,\;\; C_{\frac{l_u-1}{2}} = -\frac{J^x_{l_u}}{J^y_{\frac{l_u-1}{2}}}C_{\frac{l_u+1}{2}} \tag{2.92}$$

     $$A_{\frac{l_u+1}{2}} = -\frac{h_{l_u-1}}{J^x_{l_u-1}}C_{\frac{l_u-1}{2}} = \frac{h_{l_u-1}J^x_{l_u}}{J^x_{l_u-1}J^y_{\frac{l_u-1}{2}}}C_{\frac{l_u+1}{2}} \;\;,\;\; A_{\frac{l_u+3}{2}} = -\frac{h_{l_u+1}}{J^x_{l_u+1}}C_{\frac{l_u+1}{2}} \tag{2.93}$$

$$D_{\frac{l_u-3}{2}} = -\frac{h_{l_u-2}}{J^x_{l_u-3}} B_{\frac{l_u-1}{2}} = \frac{h_{l_u-2} J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-3} J^x_{l_u-2}} B_{\frac{l_u+1}{2}} \tag{2.94}$$

The eigenvectors obtained are mentioned below:

$$V_1 = \left(0,0,0,0,...,0,0,\left(-\frac{J^x_{l_u}}{J^y_{\frac{l_u-1}{2}}}\right)^{\frac{l_u-1}{2}},0,\left(\frac{h_{l_u-1}J^x_{l_u}}{J^x_{l_u-1}J^y_{\frac{l_u-1}{2}}}\right)^{\frac{l_u+1}{2}},0,1^{\left(\frac{l_u+1}{2}\right)},0,\left(-\frac{h_{l_u+1}}{J^x_{l_u+1}}\right)^{\frac{l_u+3}{2}},0,0,0,...,0\right)^T$$

$$V_2 = \left(0,0,0,0,....,0,0,0,\left(\frac{h_{l_u-2}J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-3}J^x_{l_u-2}}\right)^{\frac{l_u-3}{2}},0,\left(-\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u-2}}\right)^{\frac{l_u-1}{2}},0,0,0,1^{\left(\frac{l_u+1}{2}\right)},0,0,...,0\right)^T$$

We get the above two eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors as mentioned below:

$$V_3 = \left(-\frac{J^x_1}{h_1},0,1,0,-\frac{h_2}{J^x_2},0,0,0,...,0,0,0,0\right)^T$$

$$V_4 = \left(0,0,0,0,.....,0,0,0,-\frac{h_{N-1}}{J^x_{N-2}},0,1^{\left(\frac{N}{2}\right)},0,-\frac{J^x_{\frac{N}{2}+2}}{h_N}\right)^T$$

As a result, we get in total $2 \times 2 = 4$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2 \times 4 = 8$ EMZMs for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for odd l_u and odd v ($v = 1$).

2. $v \geq 3$

From equation (2.32), we have $C_j \neq 0$ and $A_{j+1} = 0$ for $j \in \left[\frac{l_u+1}{2}, \frac{l_u+3}{2}, ..., \frac{l_u+v}{2} - 1\right]$. From equation (2.33) after converting $j \to j-1$, we get $B_j \neq 0$ and $D_{j-1} = 0$ for $j \in \left[\frac{l_u+1}{2}, \frac{l_u+3}{2}, ..., \frac{l_u+v}{2}\right]$.

From equation (2.31), we get the following 3 set of equations:

$$-J^x_{l_u+2j-2} B_{\frac{l_u-1}{2}+j} - J^y_{\frac{l_u-1}{2}+j} B_{\frac{l_u+1}{2}+j} = 0 \ , \ j \in \left[1, \frac{v-1}{2}\right] \tag{2.95}$$

$$-J^x_{l_u-2} B_{\frac{l_u-1}{2}} - J^y_{\frac{l_u-1}{2}} B_{\frac{l_u+1}{2}} = 0 \ , \ j = \frac{l_u-1}{2} \tag{2.96}$$

$$-J^x_{l_u+v-1} B_{\frac{l_u+v}{2}} - h_{l_u+v} D_{\frac{l_u+v}{2}} - J^y_{\frac{l_u+v}{2}} B_{\frac{l_u+v}{2}+1} = 0 \ , \ j = \frac{l_u+v}{2} \tag{2.97}$$

From equation (2.34), we get the following 3 set of equations:

$$J^y_{\frac{l_u-1}{2}+j} C_{\frac{l_u-1}{2}+j} + J^x_{l_u+2j} C_{\frac{l_u+1}{2}+j} = 0 \ , \ j \in \left[1, \frac{v-3}{2}\right] \tag{2.98}$$

$$J^y_{\frac{l_u-1}{2}} C_{\frac{l_u-1}{2}} + J^x_{l_u} C_{\frac{l_u+1}{2}} = 0 \ , \ j = \frac{l_u-1}{2} \tag{2.99}$$

$$J^y_{\frac{l_u+v-2}{2}} C_{\frac{l_u+v-2}{2}} + J^x_{l_u+v-1} C_{\frac{l_u+v}{2}} = 0 \ , \ j = \frac{l_u+v}{2} - 1 \tag{2.100}$$

Using equations (2.95 and 2.96), we get the following:

$$B_{\frac{l_u+1}{2}+j} = (-1)^{j+1} \prod_{j=0}^{j=\left[0, \frac{v-1}{2}\right]} \frac{J^x_{l_u+2j-2}}{J^y_{\frac{l_u-1}{2}+j}} B_{\frac{l_u-1}{2}} \tag{2.101}$$

Using equations (2.97 and 2.41), we determine $D_{\frac{l_u+v}{2}}$

$$-J^x_{l_u+v-1} B_{\frac{l_u+v}{2}} - h_{l_u+v} D_{\frac{l_u+v}{2}} - J^y_{\frac{l_u+v}{2}} \left(-\frac{J^x_{l_u+v}}{h_{l_u+v+1}} D_{\frac{l_u+v}{2}} \right) = 0 \tag{2.102}$$

$$D_{\frac{l_u+v}{2}} = \frac{J^x_{l_u+v-1} h_{l_u+v+1}}{J^x_{l_u+v} J^y_{\frac{l_u+v}{2}} - h_{l_u+v} h_{l_u+v+1}} B_{\frac{l_u+v}{2}} \tag{2.103}$$

$$D_{\frac{l_u-3}{2}} = -\frac{h_{l_u-2}}{J^x_{l_u-3}} B_{\frac{l_u-1}{2}} \ , \ B_{\frac{l_u+v}{2}+1} = -\frac{J^x_{l_u+v}}{h_{l_u+v+1}} D_{\frac{l_u+v}{2}} \tag{2.104}$$

For trivial case, we obtain $B_{\frac{l_u+v}{2}+1} = -\frac{h_{l_u+v}}{J^y_{\frac{l_u+v}{2}}} D_{\frac{l_u+v}{2}}$. Using equations (2.98,2.99, 2.100 and 2.41), we determine the following non-zero components:

$$C_{\frac{l_u+1}{2}+j} = (-1)^{j+1} \prod_{j=0}^{j \in \left[0, \frac{v-1}{2}\right]} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}} C_{\frac{l_u-1}{2}} \tag{2.105}$$

$$A_{\frac{l_u+1}{2}} = -\frac{h_{l_u-1}}{J^x_{l_u-1}} C_{\frac{l_u-1}{2}} \ , \ A_{\frac{l_u+v}{2}+1} = -\frac{h_{l_u+v}}{J^x_{l_u+v}} C_{\frac{l_u+v}{2}} \tag{2.106}$$

For trivial case, the above components will vanish.

We can now obtain the eigenvectors from the above analysis:

$$V_1 = \left(0, ..., 0, 0, 0, \left(-\frac{h_{l_u-2}}{J^x_{l_u-3}} \right)^{\frac{l_u-3}{2}}, 0, 1^{\left(\frac{l_u-1}{2} \right)}, 0, 0, 0, (-1)^{j+1} \prod_{j=0}^{j=\left[0, \frac{v-3}{2}\right]} \frac{J^x_{l_u+2j-2}}{J^y_{\frac{l_u-1}{2}+j}}, 0, 0, .., 0, B_{\frac{l_u+v}{2}}, 0, D_{\frac{l_u+v}{2}}, .., 0 \right)^T$$

$$V_2 = \left(0, .., 1^{\left(\frac{l_u-1}{2}\right)}, 0, \left(-\frac{h_{l_u-1}}{J^x_{l_u-1}}\right)^{\frac{l_u+1}{2}}, 0, \left(-\frac{J^y_{\frac{l_u-1}{2}}}{J^x_{l_u}}\right)^{\frac{l_u+1}{2}}, 0, 0, 0, (-1)^{j+1} \prod_{j=1}^{j\in\left[1,\frac{v-1}{2}\right]} \frac{J^y_{\frac{l_u-1}{2}+j}}{J^x_{l_u+2j}}, 0, .., \left(-\frac{h_{l_u-1}}{J^x_{l_u-1}}\right)^{\frac{l_u+v}{2}+1}, .., 0\right)^T$$

$$V_3 = \left(0, 0, 0, 0, ..., 0, 0, 0, D_{\frac{l_u+v}{2}}, 0, B_{\frac{l_u+v}{2}+1}, 0, 0, ..., 0, 0, 0, 0\right)^T$$

We get the above three eigenvectors considering $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$. When $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we get 2 more eigenvectors as mentioned below:

$$V_4 = \left(0, 0, 0, 0, ..., 0, 0, 0, -\frac{h_{N-1}}{J^x_{N-2}}, 0, 1^{\left(\frac{N}{2}\right)}, 0, -\frac{J^x_{\frac{N}{2}+2}}{h_N}\right)^T$$

$$V_5 = \left(-\frac{J^x_1}{h_1}, 0, 1, 0, -\frac{h_2}{J^x_2}, 0, 0, 0, ..., 0, 0, 0, 0\right)^T$$

As a result, we get in total $2 \times 3 = 6$ EMZMs for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and $2 \times 5 = 10$ EMZMs for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$ for odd l_u and odd v ($v \geq 3$).

In general, the set S is a union of multiple consecutive sequences, so we still need to study the situation where more than one consecutive sequence is present. We call two neighboring consecutive sequences T_i and T_{i+1} successive if $l_{u_{i+1}} = l_{u_i} + v_i + 1$, and unsuccessive if $l_{u_{i+1}} \geq l_{u_i} + v_i + 2$.

2.3.3.2 Case 2: Considering S consist of two neighboring successive sequences

Considering S consist of two neighboring successive sequences T_i and T_{i+1} of length ν_i and ν_{i+1} respectively. The sites where magnetic field will be 0 are $l_{u_i}, l_{u_i+1}, ..., l_{u_i+v_i-1}$, $l_{u_{i+1}}, l_{u_{i+1}+1}, ..., l_{u_{i+1}+v_{i+1}-1}$. Now we consider many possible scenarios and find the number of EMZMs for each:

1. $l_{u_i} =$ even, $v_i =$ odd and so $l_{u_{i+1}} =$ even and $v_{i+1} =$ odd
   For even l_u and odd v, we recall that the following components will be zero and non-zero at those sites where magnetic field will vanish (some components at

boundary are also mentioned):

$$B_{\frac{l_{u_i}}{2}}, B_{\frac{l_{u_i}}{2}+1}, ..., B_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}} \neq 0$$

where all components of B are connected with $B_{\frac{l_{u_i}}{2}}$ using the recursion relation. Similarly,

$$C_{\frac{l_{u_i}}{2}}, C_{\frac{l_{u_i}}{2}+1}, ..., C_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}} \neq 0$$

where all components of C are connected with $C_{\frac{l_{u_i}+v_i+1}{2}}$ using the recursion relation. We also recall that

$$A_{\frac{l_{u_i}}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}-2} C_{\frac{l_{u_i}}{2}} \Big\} \ , \ D_{\frac{l_{u_i}}{2}+\frac{v_i-1}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}+v_i} B_{\frac{l_{u_i}+v_i+1}{2}} \Big\}$$

$$A_{\frac{l_{u_i}}{2}+1} = A_{\frac{l_{u_i}}{2}+2} = ... = A_{\frac{l_{u_i}+v_i+1}{2}} = 0$$
$$D_{\frac{l_{u_i}}{2}} = D_{\frac{l_{u_i}}{2}+1} = ... = D_{\frac{l_{u_i}+v_i-3}{2}} = 0$$

We get the same set of eigenvalue equations for the sequence T_{i+1} and the same zero and non-zero components as mentioned below:

$$B_{\frac{l_{u_{i+1}}}{2}}, B_{\frac{l_{u_{i+1}}}{2}+1}, ..., B_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}+1}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}}{2}}, C_{\frac{l_{u_{i+1}}}{2}+1}, ..., C_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}+1}{2}} \neq 0$$
$$A_{\frac{l_{u_{i+1}}}{2}} \neq 0 \Big\{ \sim h_{l_{u_{i+1}}-2} C_{\frac{l_{u_{i+1}}}{2}} \Big\} \ , \ D_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}-1}{2}} \neq 0 \Big\{ \sim h_{l_{u_{i+1}}+v_{i+1}} B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \Big\}$$
$$A_{\frac{l_{u_{i+1}}}{2}+1} = A_{\frac{l_{u_{i+1}}}{2}+2} = ... = A_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} = 0$$
$$D_{\frac{l_{u_{i+1}}}{2}} = D_{\frac{l_{u_{i+1}}}{2}+1} = ... = D_{\frac{l_{u_{i+1}}+v_{i+1}-3}{2}} = 0$$

The eigenequations written for T_i and T_{i+1} will couple only if

$$B_{\frac{l_{u_{i+1}}}{2}} = B_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}} \ , \ C_{\frac{l_{u_{i+1}}}{2}} = C_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}}$$

We also note that $A_{\frac{l_{u_i}+v_i+1}{2}} = A_{\frac{l_{u_{i+1}}}{2}}$ but $A_{\frac{l_{u_i}+v_i+1}{2}} = 0$ and $A_{\frac{l_{u_{i+1}}}{2}} \neq 0$. We note that $A_{\frac{l_{u_{i+1}}}{2}} \propto h_{l_{u_{i+1}}-2}$ and since $l_{u_{i+1}} = l_{u_i+v_i+1}$, we have $h_{l_{u_{i+1}}-2} = h_{l_{u_i+v_i-1}}$ but $h_{l_{u_i+v_i-1}} = 0$. Therefore, we have $A_{\frac{l_{u_{i+1}}}{2}} = 0$. There will be no common D component as can be observed from above, where $D_{\frac{l_{u_i}+v_i-3}{2}} = 0, D_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}-1}{2}} \neq 0$ and $D_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}} = D_{\frac{l_{u_{i+1}}}{2}} = 0$. We can now determine the number of EMZMs.

Considering these sequences are not present at boundary, there will be 2 EMZMs forming at successive sequences T_i and T_{i+1}. One will have non-zero B and D components where we saw $B_{\frac{l_{u_{i+1}}}{2}} = B_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}}$ and so all B components are connected. The other will have non-zero A and C components where we saw $C_{\frac{l_{u_{i+1}}}{2}} = C_{\frac{l_{u_i}}{2}+\frac{v_i+1}{2}}$ and so all C components are connected similarly.There will be one more EMZM which is trivial case with $A_{\frac{l_{u_i}}{2}}$ and $C_{\frac{l_{u_i}}{2}-1}$ as non-zero components. Therefore, for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$, we will have $2 \times 3 = 6$ EMZMs. For $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$, we will have $2 \times 5 = 10$ EMZMs (2×2 more EMZMs forming at boundaries). If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 3 = 12 \text{ EMZMs}$$
$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 3 + 2 \times 2 = 16 \text{ EMZMs}$$

2. $l_{u_i} =$ even, $v_i =$ odd and $l_{u_{i+1}} =$ even, $v_{i+1} =$ even
   For $l_{u_{i+1}} =$ even, $v_{i+1} =$ even we have the following zero and non-zero components:

$$B_{\frac{l_{u_{i+1}}}{2}}, B_{\frac{l_{u_{i+1}}}{2}+1}, \ldots.., B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}}{2}}, C_{\frac{l_{u_{i+1}}}{2}+1}, \ldots.., C_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$
$$D_{\frac{l_{u_{i+1}}}{2}} = D_{\frac{l_{u_{i+1}}}{2}+1} = \ldots.. = D_{\frac{l_{u_{i+1}}+v_{i+1}}{2}-1} = 0$$
$$A_{\frac{l_{u_{i+1}}}{2}+1} = A_{\frac{l_{u_{i+1}}}{2}+2} = \ldots.. = A_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} = 0$$
$$D_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}+1} B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \right\} \;,\; A_{\frac{l_{u_{i+1}}}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-2} C_{\frac{l_{u_{i+1}}}{2}} \right\}$$

We get the common variables connecting both eigen-equations as $B_{\frac{l_{u_{i+1}}}{2}} = B_{\frac{l_{u_i}+v_i+1}{2}}$ and $C_{\frac{l_{u_{i+1}}}{2}} = C_{\frac{l_{u_i}+v_i+1}{2}}$ and we also have $A_{\frac{l_{u_{i+1}}}{2}} = 0$ since $h_{l_{u_{i+1}}-2} = h_{l_{u_i}+v_i-1} = 0$. In total,

$$\text{Number of EMZMs} = 2 \times (1 + 1 + 1 + 1) = 8 \text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1 + 1 + 1 + 1) + 2 \times 2 = 12 \text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2\times 3+2\times 4=14 \text{ EMZMs}$$
$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2\times 3+2\times 4+2\times 2=18 \text{ EMZMs}$$

3. l_{u_i} = even, v_i = even and $l_{u_{i+1}}$ =odd, v_{i+1} = odd

   For l_{u_i} = even, v_i = even, we get the following zero and non-zero components:

$$B_{\frac{l_{u_i}}{2}}, B_{\frac{l_{u_i}}{2}+1}, ..., B_{\frac{l_{u_i}+v_i}{2}} \neq 0$$
$$C_{\frac{l_{u_i}}{2}}, C_{\frac{l_{u_i}}{2}+1}, ..., C_{\frac{l_{u_i}+v_i}{2}} \neq 0$$
$$D_{\frac{l_{u_i}}{2}} = D_{\frac{l_{u_i}}{2}+1} = ... = D_{\frac{l_{u_i}}{2}+\frac{v_i}{2}-1} = 0$$
$$A_{\frac{l_{u_i}}{2}+1} = A_{\frac{l_{u_i}}{2}+2} = ... = A_{\frac{l_{u_i}}{2}+\frac{v_i}{2}} = 0$$
$$D_{\frac{l_{u_i}}{2}+\frac{v_i}{2}} \neq 0\left\{\sim h_{l_{u_i}+v_i+1}B_{\frac{l_{u_i}+v_i}{2}}\right\} \;,\; A_{\frac{l_{u_i}}{2}} \neq 0\left\{\sim h_{l_{u_i}-2}C_{\frac{l_{u_i}}{2}}\right\}$$

For $l_{u_{i+1}}$ = odd, v_{i+1} = odd, we get the following zero and non-zero components:

$$B_{\frac{l_{u_{i+1}}-1}{2}}, B_{\frac{l_{u_{i+1}}+1}{2}}, ..., B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}-1}{2}}, C_{\frac{l_{u_{i+1}}+1}{2}}, ..., C_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$
$$D_{\frac{l_{u_{i+1}}-1}{2}} = D_{\frac{l_{u_{i+1}}+1}{2}} = ... = D_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}}{2}-1} = 0$$
$$A_{\frac{l_{u_{i+1}}+3}{2}} = A_{\frac{l_{u_{i+1}}+5}{2}} = ... = A_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}}{2}} = 0$$
$$D_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}}{2}} \neq 0\left\{\sim h_{l_{u_{i+1}}+v_{i+1}+1}B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}}\right\} \;,\; D_{\frac{l_{u_{i+1}}-3}{2}} \neq 0\left\{\sim h_{l_{u_{i+1}}-2}B_{\frac{l_{u_{i+1}}-1}{2}}\right\}$$
$$A_{\frac{l_{u_{i+1}}}{2}+\frac{v_{i+1}}{2}} \neq 0\left\{\sim h_{l_{u_{i+1}}+v_{i+1}}C_{\frac{l_{u_{i+1}}+v_{i+1}}{2}}\right\} \;,\; A_{\frac{l_{u_{i+1}}+1}{2}} \neq 0\left\{\sim h_{l_{u_{i+1}}-1}C_{\frac{l_{u_{i+1}}-1}{2}}\right\}$$

The common variables we obtain which connects both eigenequations are as follow:

$$B_{\frac{l_{u_i}+v_i}{2}} = B_{\frac{l_{u_{i+1}}-1}{2}} \;,\; C_{\frac{l_{u_i}+v_i}{2}} = C_{\frac{l_{u_{i+1}}-1}{2}}$$

We observe that $D_{\frac{l_{u_i}+v_i}{2}} = D_{\frac{l_{u_{i+1}}-1}{2}}$ but $D_{\frac{l_{u_i}+v_i}{2}}$ is non-zero. However $D_{\frac{l_{u_i}+v_i}{2}}$ is proportional to $h_{l_{u_i}+v_i+1} = h_{l_{u_{i+1}}} = 0$. Therefore, $D_{\frac{l_{u_i}+v_i}{2}} = D_{\frac{l_{u_{i+1}}-1}{2}} = 0$. Similarly $D_{\frac{l_{u_{i+1}}-3}{2}} = D_{\frac{l_{u_i}}{2}+\frac{v_i}{2}-1}$ but $D_{\frac{l_{u_{i+1}}-3}{2}} \neq 0$. But $D_{\frac{l_{u_{i+1}}-3}{2}}$ is proportional to $h_{l_{u_{i+1}-2}} = h_{l_{u_i}+v_i-1} = 0$. Therefore, $D_{\frac{l_{u_{i+1}}-3}{2}} = D_{\frac{l_{u_i}}{2}+\frac{v_i}{2}-1} = 0$. In total,

$$\text{Number of EMZMs} = 2 \times (1+1+1+1) = 8 \text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1+1+1+1) + 2 \times 2 = 12 \text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 4 = 14 \text{ EMZMs}$$
$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 4 + 2 \times 2 = 18 \text{ EMZMs}$$

4. l_{u_i} = even, v_i = even and $l_{u_{i+1}}$ = odd, v_{i+1} = even
We get the following zero and non-zero components for $l_{u_{i+1}}$ = odd, v_{i+1} = even:

$$B_{\frac{l_{u_{i+1}}-1}{2}}, B_{\frac{l_{u_{i+1}}+1}{2}}, ..., B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}-1}{2}}, C_{\frac{l_{u_{i+1}}+1}{2}}, ..., C_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$D_{\frac{l_{u_{i+1}}-1}{2}} = D_{\frac{l_{u_{i+1}}+1}{2}} = ... = D_{\frac{l_{u_{i+1}}+v_{i+1}-3}{2}} = 0$$
$$A_{\frac{l_{u_{i+1}}+3}{2}} = A_{\frac{l_{u_{i+1}}+5}{2}} = ... = A_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} = 0$$

$$D_{\frac{l_{u_{i+1}}+v_{i+1}-1}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}} B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \right\} \ , \ D_{\frac{l_{u_{i+1}}-3}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-2} B_{\frac{l_{u_{i+1}}-1}{2}} \right\}$$
$$A_{\frac{l_{u_{i+1}}+v_{i+1}+3}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}+1} C_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \right\} \ , \ A_{\frac{l_{u_{i+1}}+1}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-1} C_{\frac{l_{u_{i+1}}-1}{2}} \right\}$$

The common variables we obtain which connects both eigenequations are as follow:

$$B_{\frac{l_{u_i}+v_i}{2}} = B_{\frac{l_{u_{i+1}}-1}{2}} \ , \ C_{\frac{l_{u_i}+v_i}{2}} = C_{\frac{l_{u_{i+1}}-1}{2}}$$

Using similar analysis as mentioned in previous case, we have $D_{\frac{l_{u_{i+1}}-3}{2}}=0$ and $D_{\frac{l_{u_i}+v_i}{2}}=D_{\frac{l_{u_{i+1}}-1}{2}}=0$.In total,

$$\text{Number of EMZMs} = 2\times(1+1+1) = 6 \ \text{EMZMs}$$

for $h_{2j}h_{2j+1}\neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1}=J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2\times(1+1+1)+2\times 2 = 10 \ \text{EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1}\neq J^x_{2j}J^y_j \rightarrow 2\times 2+2\times 4 = 12 \ \text{EMZMs}$$
$$h_{2j}h_{2j+1}= J^x_{2j}J^y_j \rightarrow 2\times 2+2\times 4+2\times 2 = 16 \ \text{EMZMs}$$

5. $l_{u_i}=\text{odd}$, $v_i=\text{odd}$ and $l_{u_{i+1}}=\text{odd}$, $v_{i+1}=\text{odd}$
   For $l_{u_i}=\text{odd}$, $v_i=\text{odd}$, we have the following zero and non-zero components:

$$B_{\frac{l_{u_i}-1}{2}}, B_{\frac{l_{u_i}+1}{2}}, ..., B_{\frac{l_{u_i}+v_i}{2}} \neq 0$$
$$C_{\frac{l_{u_i}-1}{2}}, C_{\frac{l_{u_i}+1}{2}}, ..., C_{\frac{l_{u_i}+v_i}{2}} \neq 0$$
$$A_{\frac{l_{u_i}+3}{2}} = A_{\frac{l_{u_i}+5}{2}} = ... = A_{\frac{l_{u_i}+v_i}{2}} = 0$$
$$D_{\frac{l_{u_i}-1}{2}} = D_{\frac{l_{u_i}+1}{2}} = ... = D_{\frac{l_{u_i}+v_i-2}{2}} = 0$$
$$D_{\frac{l_{u_i}-3}{2}} \neq 0\Big\{\sim h_{l_{u_i}-2}B_{\frac{l_{u_i}-1}{2}}\Big\} \ , \ D_{\frac{l_{u_i}+v_i}{2}} \neq 0\Big\{\sim h_{l_{u_i}+v_i+1}B_{\frac{l_{u_i}+v_i}{2}}\Big\}$$
$$A_{\frac{l_{u_i}+1}{2}} \neq 0\Big\{\sim h_{l_{u_i}-1}C_{\frac{l_{u_i}-1}{2}}\Big\} \ , \ A_{\frac{l_{u_i}+v_i}{2}+1} \neq 0\Big\{\sim h_{l_{u_i}+v_i}C_{\frac{l_{u_i}+v_i}{2}}\Big\}$$

   We again get the same set of equations with $i\rightarrow i+1$ for the sequence T_{i+1}. The common variables that we obtain which connects both eigenequations are as follow:

$$B_{\frac{l_{u_{i+1}}-1}{2}} = B_{\frac{l_{u_i}+v_i}{2}} \ , \ C_{\frac{l_{u_{i+1}}-1}{2}} = C_{\frac{l_{u_i}+v_i}{2}}$$

   We also observe that $D_{\frac{l_{u_{i+1}}-3}{2}} = D_{\frac{l_{u_i}+v_i-2}{2}}$. But $D_{\frac{l_{u_{i+1}}-3}{2}} \neq 0$. But since $D_{\frac{l_{u_{i+1}}-3}{2}}$ is proportional to $h_{l_{u_{i+1}}-2} = h_{l_{u_i}+v_i-1} = 0$, we have $D_{\frac{l_{u_{i+1}}-3}{2}} = D_{\frac{l_{u_i}+v_i-2}{2}} = 0$.

Similarly $D_{\frac{l_{u_i}+v_i}{2}} = D_{\frac{l_{u_{i+1}}-1}{2}}$. But since $D_{\frac{l_{u_i}+v_i}{2}}$ is proportional to $h_{l_{u_i}+v_i+1} = h_{l_{u_{i+1}}} = 0$, we have $D_{\frac{l_{u_i}+v_i}{2}} = D_{\frac{l_{u_{i+1}}-1}{2}} = 0$. In total,

$$\text{Number of EMZMs} = 2 \times (1+1+1) = 6 \text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1+1+1) + 2 \times 2 = 10 \text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 3 = 12 \text{ EMZMs}$$
$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 3 + 2 \times 2 = 16 \text{ EMZMs}$$

6. $l_{u_i} = \text{odd}$, $v_i = \text{odd}$ and $l_{u_{i+1}} = \text{odd}$, $v_{i+1} = \text{even}$

For $l_{u_{i+1}} = \text{odd}$, $v_{i+1} = \text{even}$, we get the following zero and non-zero components:

$$B_{\frac{l_{u_{i+1}}-1}{2}}, B_{\frac{l_{u_{i+1}}+1}{2}}, ..., B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}-1}{2}}, C_{\frac{l_{u_{i+1}}+1}{2}}, ..., C_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$A_{\frac{l_{u_{i+1}}+3}{2}} = A_{\frac{l_{u_{i+1}}+5}{2}} = ... = A_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} = 0$$
$$D_{\frac{l_{u_{i+1}}-1}{2}} = D_{\frac{l_{u_{i+1}}+1}{2}} = ... = D_{\frac{l_{u_{i+1}}+v_{i+1}-3}{2}} = 0$$

$$D_{\frac{l_{u_{i+1}}-3}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-2} B_{\frac{l_{u_{i+1}}-1}{2}} \right\} \ , \ D_{\frac{l_{u_{i+1}}+v_{i+1}-1}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}} B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \right\}$$
$$A_{\frac{l_{u_{i+1}}+1}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-1} C_{\frac{l_{u_{i+1}}-1}{2}} \right\} \ , \ A_{\frac{l_{u_{i+1}}+v_{i+1}+3}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}+1} C_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \right\}$$

The common variables we obtain which connects both eigenequations are as follow:

$$B_{\frac{l_{u_i}+v_i}{2}} = B_{\frac{l_{u_{i+1}}-1}{2}} \ , \ C_{\frac{l_{u_i}+v_i}{2}} = C_{\frac{l_{u_{i+1}}-1}{2}}$$

Using similar analysis as mentioned in previous case, we have $D_{\frac{l_{u_{i+1}}-3}{2}} = 0$ and $D_{\frac{l_{u_i}+v_i}{2}} = D_{\frac{l_{u_{i+1}}-1}{2}} = 0$.In total,

$$\text{Number of EMZMs} = 2 \times (1+1) = 4 \text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1+1) + 2 \times 2 = 8 \text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 2 = 10 \text{ EMZMs}$$
$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 3 + 2 \times 2 + 2 \times 2 = 14 \text{ EMZMs}$$

7. $l_{u_i} = \text{odd}$, $v_i = \text{even}$ and $l_{u_{i+1}} = \text{even}$, $v_{i+1} = \text{odd}$

For $l_{u_i} = \text{odd}$, $v_i = \text{even}$, we get the following zero and non-zero components:

$$B_{\frac{l_{u_i}-1}{2}}, B_{\frac{l_{u_i}+1}{2}}, ..., B_{\frac{l_{u_i}+v_i+1}{2}} \neq 0$$
$$C_{\frac{l_{u_i}-1}{2}}, C_{\frac{l_{u_i}+1}{2}}, ..., C_{\frac{l_{u_i}+v_i+1}{2}} \neq 0$$
$$A_{\frac{l_{u_i}+3}{2}} = A_{\frac{l_{u_i}+5}{2}} = ... = A_{\frac{l_{u_i}+v_i+1}{2}} = 0$$
$$D_{\frac{l_{u_i}-1}{2}} = D_{\frac{l_{u_i}+1}{2}} = ... = D_{\frac{l_{u_i}+v_i-3}{2}} = 0$$
$$D_{\frac{l_{u_i}-3}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}-2} B_{\frac{l_{u_i}-1}{2}} \Big\} \ , \ D_{\frac{l_{u_i}+v_i-1}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}+v_i} B_{\frac{l_{u_i}+v_i+1}{2}} \Big\}$$
$$A_{\frac{l_{u_i}+1}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}-1} C_{\frac{l_{u_i}-1}{2}} \Big\} \ , \ A_{\frac{l_{u_i}+v_i+3}{2}} \neq 0 \Big\{ \sim h_{l_{u_i}+v_i+1} C_{\frac{l_{u_i}+v_i+1}{2}} \Big\}$$

For $l_{u_{i+1}} = \text{even}$, $v_{i+1} = \text{odd}$, we get the following zero and non-zero components:

$$B_{\frac{l_{u_{i+1}}}{2}}, B_{\frac{l_{u_{i+1}}}{2}+1}, ..., B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$C_{\frac{l_{u_{i+1}}}{2}}, C_{\frac{l_{u_{i+1}}}{2}+1}, ..., C_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} \neq 0$$
$$A_{\frac{l_{u_{i+1}}}{2}+1} = A_{\frac{l_{u_{i+1}}}{2}+2} = ... = A_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}} = 0$$
$$D_{\frac{l_{u_{i+1}}}{2}} = D_{\frac{l_{u_{i+1}}}{2}+1} = ... = D_{\frac{l_{u_{i+1}}+v_{i+1}-3}{2}} = 0$$

$$D_{\frac{l_{u_{i+1}}+v_{i+1}-1}{2}} \neq 0\Big\{\sim h_{l_{u_{i+1}}+v_{i+1}} B_{\frac{l_{u_{i+1}}+v_{i+1}+1}{2}}\Big\}$$

$$A_{\frac{l_{u_{i+1}}}{2}} \neq 0\Big\{\sim h_{l_{u_{i+1}}-2} C_{\frac{l_{u_{i+1}}}{2}}\Big\}$$

The common variables that we obtain which connects both eigenequations are :

$$B_{\frac{l_{u_i}+v_i+1}{2}} = B_{\frac{l_{u_{i+1}}}{2}} \;,\; C_{\frac{l_{u_i}+v_i+1}{2}} = C_{\frac{l_{u_{i+1}}}{2}}$$

We observe that $A_{\frac{l_{u_i}+v_i+1}{2}} = A_{\frac{l_{u_{i+1}}}{2}}$. But $A_{\frac{l_{u_{i+1}}}{2}} \neq 0$. However, we notice that $A_{\frac{l_{u_{i+1}}}{2}}$ is proportional to $h_{l_{u_{i+1}}-2} = h_{l_{u_i}+v_i-1} = 0$. Therefore, $A_{\frac{l_{u_i}+v_i+1}{2}} = A_{\frac{l_{u_{i+1}}}{2}} = 0$. Similarly, $A_{\frac{l_{u_i}+v_i+3}{2}} = A_{\frac{l_{u_{i+1}}}{2}+1}$. But $A_{\frac{l_{u_i}+v_i+3}{2}} \neq 0$. However, we notice that $A_{\frac{l_{u_i}+v_i+3}{2}}$ is proportional to $h_{l_{u_i}+v_i+1} = h_{l_{u_{i+1}}} = 0$. Therefore, $A_{\frac{l_{u_i}+v_i+3}{2}} = A_{\frac{l_{u_{i+1}}}{2}+1} = 0$. In total,

$$\text{Number of EMZMs} = 2 \times (1+1) = 4 \text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1+1) + 2 \times 2 = 8 \text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 2 + 2 \times 3 = 10 \text{ EMZMs}$$

$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 2 + 2 \times 3 + 2 \times 2 = 14 \text{ EMZMs}$$

8. $l_{u_i} = \text{odd}$, $v_i = \text{even}$ and $l_{u_{i+1}} = \text{even}$, $v_{i+1} = \text{even}$

For $l_{u_{i+1}} = \text{even}$, $v_{i+1} = \text{even}$, we get the following zero and non-zero components:

$$B_{\frac{l_{u_{i+1}}}{2}}, B_{\frac{l_{u_{i+1}}}{2}+1}, \ldots\ldots., B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$

$$C_{\frac{l_{u_{i+1}}}{2}}, C_{\frac{l_{u_{i+1}}}{2}+1}, \ldots\ldots., C_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0$$

$$A_{\frac{l_{u_{i+1}}}{2}+1} = A_{\frac{l_{u_{i+1}}}{2}+2} = \ldots.. = A_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} = 0$$

$$D_{\frac{l_{u_{i+1}}}{2}} = D_{\frac{l_{u_{i+1}}}{2}+1} = \ldots.. = D_{\frac{l_{u_{i+1}}+v_{i+1}}{2}-1} = 0$$

$$D_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}+v_{i+1}+1} B_{\frac{l_{u_{i+1}}+v_{i+1}}{2}} \right\}$$

$$A_{\frac{l_{u_{i+1}}}{2}} \neq 0 \left\{ \sim h_{l_{u_{i+1}}-2} C_{\frac{l_{u_{i+1}}}{2}} \right\}$$

The common variables that we obtain which connects both eigenequations are :

$$B_{\frac{l_{u_i}+v_i+1}{2}} = B_{\frac{l_{u_{i+1}}}{2}} \;\; , \;\; C_{\frac{l_{u_i}+v_i+1}{2}} = C_{\frac{l_{u_{i+1}}}{2}}$$

Using similar analysis as mentioned in previous case, we have $A_{\frac{l_{u_i}+v_i+1}{2}} = A_{\frac{l_{u_{i+1}}}{2}} = 0$ and $A_{\frac{l_{u_i}+v_i+3}{2}} = A_{\frac{l_{u_{i+1}}}{2}+1} = 0$.In total,

$$\text{Number of EMZMs} = 2 \times (1+1+1) = 6 \;\text{ EMZMs}$$

for $h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j$ and for $h_{2j}h_{2j+1} = J^x_{2j}J^y_j$,

$$\text{Number of EMZMs} = 2 \times (1+1+1) + 2 \times 2 = 10 \;\text{ EMZMs}$$

If T_i and T_{i+1} are not successive, we have the following total number of EMZMs for both cases:

$$h_{2j}h_{2j+1} \neq J^x_{2j}J^y_j \rightarrow 2 \times 2 + 2 \times 4 = 12 \;\text{ EMZMs}$$

$$h_{2j}h_{2j+1} = J^x_{2j}J^y_j \rightarrow 2 \times 2 + 2 \times 4 + 2 \times 2 = 16 \;\text{ EMZMs}$$

2.3.3.3 Case 3: Considering S consist of α consecutive sequences

Considering S consist of α consecutive sequences, that is, T_i, T_{i+1},....., $T_{i+\alpha-1}$ with $\alpha \geq 2$. There can be several possible cases and we need to do similar analysis as done above to determine total number of EMZMs.

Chapter 3

An Introduction to Quantum Spin Liquids

Quantum Spin Liquids(QSLs) are recognized as the phases of matter that fall beyond the paradigm of Landau's symmetry based classification. They are characterized by **long range many body entanglement**. QSLs are expected to be observed in **frustrated quantum magnets**. In this chapter, we focus on two examples, 2D Z_2 QSL associated with Kitaev's toric code model and Kitaev's Honeycomb Model. Using these examples, we can understand the inherent properties of QSLs.

3.1 Introduction

In much of Condensed matter physics, we study about different conventional phases, like metals, semiconductors, superconductors, magnets etc. Understanding these conventional phases relies on 2 major organizing principles: 1) phases can be classified on the basis of spontaneous symmetry breaking and 2) adiabatic continuity which connects interacting systems to non-interacting or mean field descriptions. However it was later discovered that there are some unconventional phases that cannot be explained using the above 2 principles. The first unconventional phase discovered was Quantum Hall effect that are characterized by long range entanglement. Phases that are characterized by long range entanglement are known as Quantum ordered systems. We will focus on one more quantum ordered system, i.e. QSL . It consist of a bunch of interacting quantum spins, each sitting on lattice sites in 2D/3D.

Such spin systems in which electrons are localized to lattice sites by repulsive electron-electron interaction describes the insulating phase of Mott Insulators. While the charge degree of freedom(dof) is localized, the spins interact with each other. This interaction dictate the magnetic properties of system. At low temperature, many of the Mott insulators order magnetically, but there exist some which remains disordered paramagnetic phase. They do not break any continuous symmetry near absolute zero and such systems are what we call QSLs. They have some interesting properties which are mentioned below:

1. Presence of long range quantum entanglement.
2. There is a ground state degeneracy which depends on the topology of the lattice.
3. We observe emergent photon like gauge bosonic excitations.
4. Fractionalization of quantum numbers of the excitations.

3.2 Quantum Ordered Phases in Magnets: QSL

We focus on the central features of QSL by studying Toric Code Model and Kitaev's Honeycomb model.

3.2.1 Toric Code Model

The toric code model consist of spin 1/2 particles sitting at the bonds between lattice sites (figure[3.1]). The Hamiltonian is given by:

$$H_{TC} = -\sum_{s} J_s \hat{A}_s - \sum_{p} J_p \hat{B}_p \tag{3.1}$$

with $J_s, J_p > 0 \ \forall \ s, p$ where s, p denotes the site and plaquette of a square lattice and $\hat{A}_s = \prod_{i\in s} \sigma_i^x, \hat{B}_p = \prod_{i\in p} \sigma_i^z$. If coupling constant differs from site to site, then there is no symmetry in general. The time reversal symmetry can be broken by adding a small perturbation but this does not affect the state of the system. We can show that the following relation will be satisfied:

$$A_s^2 = B_p^2 = 1 \tag{3.2}$$

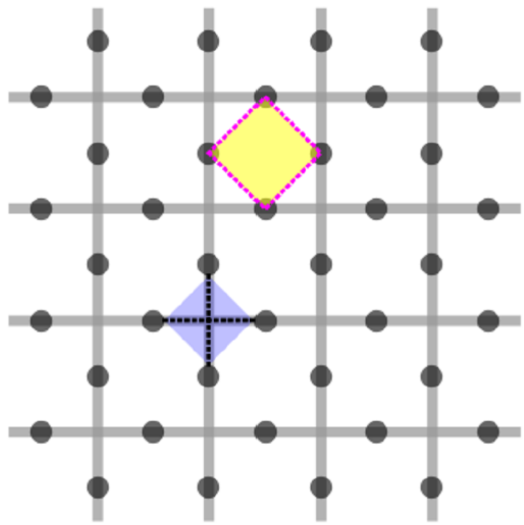


Figure 3.1: The Toric code model

so the eigenvalues of the operators can be ± 1. The below commutation relations will be satisfied as well.

$$[A_s, A_{s'}] = [B_p, B_{p'}] = 0 \quad , \quad [A_s, B_p] = 0 \tag{3.3}$$

3.2.1.1 The Ground State

To obtain ground state with minimum energy, the ground state should satisfy the below equation simultaneously,

$$A_s|\Psi_g> = |\Psi_g> \quad , \quad B_p|\Psi_g> = |\Psi_g> \tag{3.4}$$

To obtain ground state, we begin with state where all $\sigma_i^z = +1$ and call this state $|\{\sigma^z = +1\}>$. Clearly the second equation just given above can be satisfied but not the first equation as (see figure[3.2]),

$$\hat{A}_s|\{\sigma_z = +1\}> = |\{\sigma_z = 1\}'\{\sigma_z = -1, \ \forall i \in s\}> \tag{3.5}$$

Similarly, if we act with a string of $\hat{A}_s$ operators, we get a configuration where all spins are up except for those sitting on loop described by string of $\hat{A}_s$. We can define a state of the form,

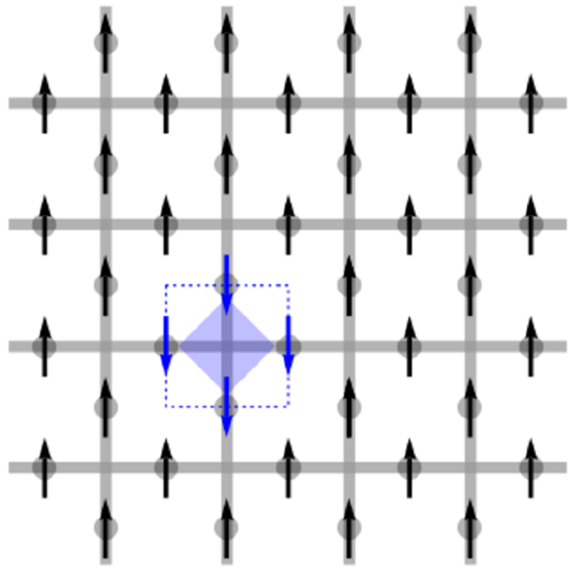


Figure 3.2: The action of $\hat{A}_s$ (the site encircled by the shaded simplex) on an all $\sigma^z = +1$ state, gives a state where all spins are up except for those on the closed dotted loop.

$$|\Psi_{gs}> = \prod_s \left(\frac{1 + \hat{A}_s}{2} \right) |\{\sigma^z = 1\}> \tag{3.6}$$

which describes an equal superposition of states composed of all closed loops of down spins in a sea of up spins. This state will satisfy both of the above equations. Now a question arises if the above state is same as a random product state of the form $|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow$ .... $\downarrow\uparrow\uparrow>$? The answer is No and we will explore further how exactly.

3.2.1.2 The excitations

The excitations can be obtained by flipping the spins $A_s = -1, B_p = -1$. This can be done by acting with σ_i^z on a bond i for the first class of excitation or σ_i^x for the second class of excitation. Since the bond i is connected with 2 plaquettes and 2 sites, we always get excitations in pair created by local spin operators (figure[3.3]).

The cost of excitations wrt ground state is given by

$$E_s = 2(J_s + J_{s'}) \quad , \quad E_p = 2(J_p + J_{p'}) \tag{3.7}$$

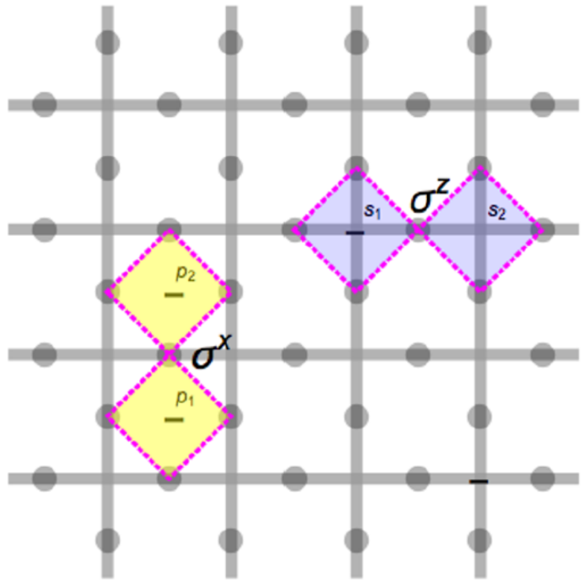


Figure 3.3: The elementary excitations in a Toric code model

$A_s = -1$ excitations are called the electric (e) charges and $B_p = -1$ excitations are magnetic (m) charges. If we use a string of σ^z operators,

$$W_e(s_1, s_2) = \prod_{i \in s_1 \rightarrow s_2} \sigma_i^z \tag{3.8}$$

where the product is taken over any path in between sites starting from s_1 and ending at s_2 then it creates 2 e charge excitations at sites s_1 and s_2. The excited state is given by

$$|e_1, e_2 > = W_e(s_1, s_2)|\Psi_g > \tag{3.9}$$

Here the energy cost is independent of the path of the string and only depends on the position of end points. Since the path of strings does not matter, interchanging the position of e charges does not affect the state of system and so the e charges are bosonic in nature. Thus they follow bosonic statistics. We can now similarly define,

$$W_m(p_1, p_2) \ = \prod_{i \in p_1 \rightarrow p_2} \sigma_i^x \tag{3.10}$$

which creates 2 magnetic charges at the end of string. The excited state is given by

$$|m_1, m_2> = W_m(p_1, p_2)|\Psi_g> \tag{3.11}$$

Here also the actual path does not matter and the m charges are also bosonic in nature (figure[3.4]).

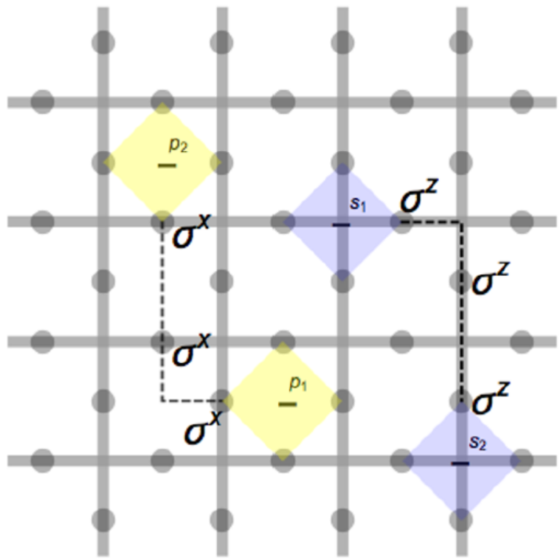


Figure 3.4: Figure 3.5

3.2.1.3 Semionic mutual statistics of e and m charges and the bound state fermion

Consider there is an e and m charge (such that the other charges of e and m are very far apart). The m charge can be moved around the e charge through a closed loop using W_m operator which we call here $W_{m,o}$ (figure[3.5]). We can observe that W_m operator has to cross W_e string an odd number of times, and 2 strings anti-commute at bonds wherever they cross. Hence,

$$W_{m,o}|e, m> = -|e, m> \tag{3.12}$$

This is just an Ising version of Aharonov-Bohm phase. The e and m together is known as mutual semion. It will follow fermionic statistics.

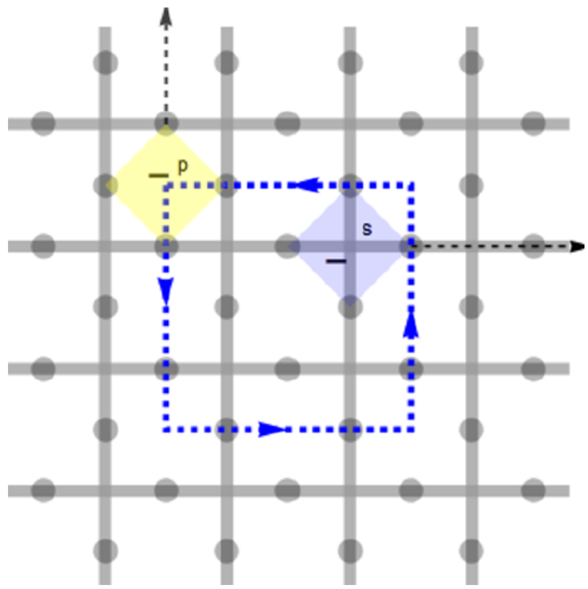


Figure 3.5: m charge moving around e charge through a closed loop

3.2.1.4 Long range Quantum Entanglement in Ground State

Now is the time to answer the question if the QSL ground state is different from random product state. Firstly, it should be noted that the excitations created in the Toric code has a finite energy gap. Now we consider a new Hamiltonian:

$$H' = -\sum_i \Gamma_i \sigma_i^z \tag{3.13}$$

where Γ_i is a coupling constant. The random product state is the ground state of H'. The excitation in this system are also gapped with minimum energy required is $min(\Gamma_i)$. Now the equivalence or non-equivalence of 2 ground states can be stated as follow:

We consider a continuous one parameter family of Hamiltonians of the form,

$$H_\lambda = (1-\lambda)H_{TC} + \lambda H' \tag{3.14}$$

where $\lambda \in [0,1]$ with $\lambda = 0$ corresponds to Toric code Hamiltonian and $\lambda = 1$ corresponds to H'. Let the ground state and excited state of H_λ be given by $|\Psi_{gs}^\lambda>$ and $|\Psi_1^\lambda>$ respectively with the energy gap given by Δ_λ. It is clear that for $\lambda = 0$ and $\lambda = 1$, $\Delta_\lambda \neq 0$. If $\Delta_\lambda \neq 0 \ \forall \ \lambda \in [0,1]$ then the ground state of Toric code and random product state are connected to each other adiabatically without closing Δ. If $\Delta_\lambda = 0$

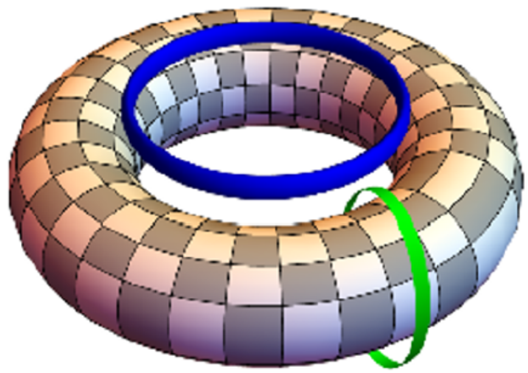


Figure 3.6: The toric code defined on a 2-torus

for at least one intermediate value of λ, then the 2 states cannot be adiabatically connected with each other. It it is true for all possible paths in parameter space, we say that the 2 ground states are necessarily separated by at least 1 phase transition. It should be emphasized here that the distinction of phases does not depend on the symmetry based classification of condensed matter physics.

Talking about QSLs, it is the second situation which occurs, i.e. QSLs ground state cannot be adiabatically deformed to random product state. The reason is the fact that 2 ground states has different signatures of many body quantum entanglement. We can measure this many body entanglement using what we call as entanglement entropy (EE). The random product state has 0 EE but the Toric code has long range entanglement which is a characteristic feature of QSL or any other quantum ordered system.

3.2.1.5 Topological degeneracy and Topological Quantum Number

Topological degeneracy and topological quantum numbers are the outcome of long range many body entanglement and has nothing to do with the symmetry of the system but only on the topology of manifold in which the system is embedded. To understand this further we consider the example of Toric code on the Torus (figure[3.6]).

We note that on a torus, there are 2 non-contractible loops. For each loop, we can define 2 operators. We observe that the 2 lines which run in the $\hat{x}$ direction, the thicker blue line is denoted as L_x^e and the thinner blue dotted line as L_x^m. Similarly, for $\hat{y}$ direction, the thicker red line is denoted as L_y^e and the thinner red dotted line as L_y^m (figure[3.7]). Now we define 4 operators for our 4 loops:

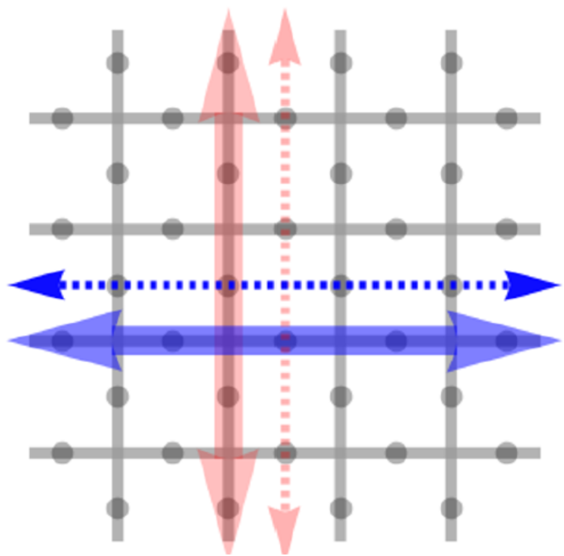


Figure 3.7: The two operators defined on the two non-contractible loops of the 2-torus

$$\hat{Z}x = \prod_{i \in L_x^e} \sigma_i^z \quad \hat{X}x = \prod_{i \in L_x^m} \sigma_i^x \quad \hat{Z}y = \prod_{i \in L_y^e} \sigma_i^z \quad \hat{X}y = \prod_{i \in L_y^m} \sigma_i^x \tag{3.15}$$

$$\hat{Z_\alpha}^2 = \hat{X_\alpha}^2 = 1. \tag{3.16}$$

Thus their eigenvalues are ± 1. The algebra of operators are given by,

$$[X_\alpha, X_\beta] = [Z_\alpha, Z_\beta] = 0 \quad [X_x, Z_x] = [X_y, Z_y] = 0 \quad \{X_x, Z_y\} = \{X_y, Z_x\} = 0 \tag{3.17}$$

We can find the eigenstates of commuting operators Z_x and Z_y which we can denote as $|Z_x = \pm 1, Z_y = \pm 1 >$. Also each of the above operators commute with H_{TC}. Thus ground state can be simultaneous eigenstate of Z_x and Z_y. Thus the ground state is 4-fold degenerate corresponding to 4 states $|Z_x = \pm 1, Z_y = \pm 1 >$. It should be noted that degeneracy here has nothing to do with the symmetry of the system. This is a topological degeneracy. These topological degeneracies and topological quantum numbers (i.e. eigenvalues of operators defined above) are manifestation of long range quantum entanglement present in the ground state. Further, symmetry related quantum numbers can be easily destroyed by adding in small local, but, symmetry violating terms to the Hamiltonian. However, this is not true for topological quantum numbers. The topological degeneracy is preserved and hence the topological quantum numbers are robust to such perturbations in the thermodynamic limit.

The ideas of topological degeneracy, quantum number and excitations together describe the topological order of Toric code model. These features are however much more general and form characteristic features of a large class of gapped quantum spin liquids called Z_2 quantum spin liquids.

3.2.2 Kitaev Honeycomb Model

3.2.2.1 Model Hamiltonian

The Kitaev model is an $S = \frac{1}{2}$ quantum spin model on a honeycomb lattice such that the interactions are bond dependent and the Hamiltonian is given by

$$H_K = -\sum_{\gamma=x,y,z} K_\gamma \sum_{<jj'>_\gamma} S_j^\gamma S_{j'}^\gamma \tag{3.18}$$

where S_j is an $S = \frac{1}{2}$ spin operator at site j on the honeycomb lattice. In the model, spins on the NN bond $< jj' >$ couple with each other by the Ising type interaction for the γ component with the exchange constant K_γ. This bond dependent interaction leads to a frustration, distinct from the geometrical frustration. For example, let us consider that the energy gain of a z bond is maximized by aligning spins on the bond along the S_z direction. In this situation, the S_x and S_y components vanish in the classical picture; therefore, the energy gains on the x and y bonds cannot be maximized simultaneously. Hence there is a macroscopic degeneracy in the classical ground state in a sense that the z components of spins at the z bond can be flipped. Consider there are N number of sites. Each site has 3 inequivalent nearest neighbor (NN), so there should be $3N$ bonds, but since 2 sites share same bond, we will have $\frac{3N}{2}$ bonds. But since the spin components S_x and S_y vanish, we won't get any energy from x and y y bonds. So for each site, x and y bonds have 0 energy. For N sites, we have $\frac{2N}{2}$ bonds giving 0 energy. So the remaining number of z bonds are $\frac{3N}{2} - N = \frac{N}{2}$. For each z bond, flipping the spins give same energy, therefore the classical ground state is $2^{\frac{N}{2}}$ fold degenerate. These classical degeneracies should be lifted by introducing quantum fluctuations.

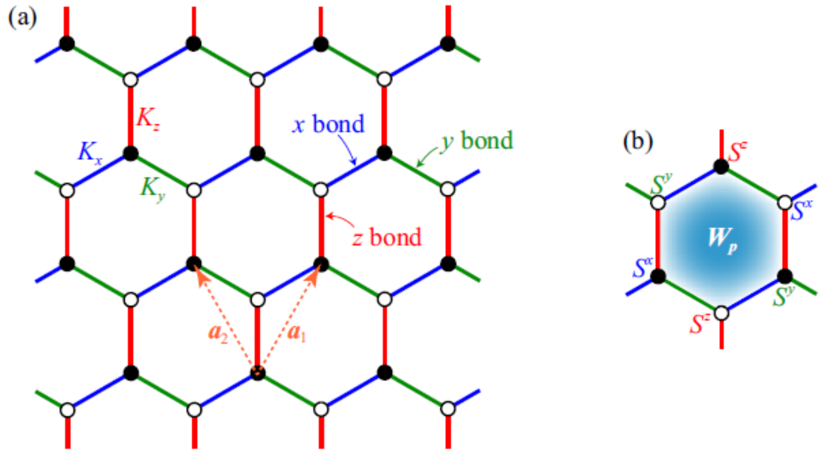


Figure 3.8: (a) Honeycomb lattice on which the Kitaev model is defined. (b) Flux operator W_p

We also define a flux operator by, $W_p = 2^6 \prod_{j \in p} S_j^{\gamma_j}$, where γ_j is the spin component corresponding to the type of the bond not belonging to the edges of the hexagon plaquette p among three bonds connected with site j.

Using the commutation of Pauli operators at different site, $[\sigma_i^\alpha, \sigma_j^\beta] = 0$ and anti-commutation relation at same site $\{\sigma_i^\alpha, \sigma_i^\beta\} = 2\delta^{\alpha\beta}$, we can show that $W_P^2 = 1$. This means that its eigenvalues $w_p = \pm 1$. We can obtain the relation $[W_p, H] = 0$. This implies that the operator W_p is a local Z_2 conserved quantity. We can also conclude that all eigenstates of the Kitaev model are assigned by the set of the eigenvalues of W_p , namely $w_p = \{\pm 1\}$.

3.2.2.2 Majorana fermion representation

From the previous section, we concluded that the eigenstates of Kitaev Hamiltonian are characterised by configurations of $\{w_p = \pm 1\}$. The number of such configurations is $2^{N/2}$ but the dimension of Hilbert space is 2^N. What about the remaining degrees of freedom ? To answer that, we introduce the Majorana fermion representation of $S = \frac{1}{2}$ quantum spin:

$$S_j^\gamma = \frac{1}{2} i b_j^\gamma c_j \tag{3.19}$$

where c_j, b_j^x, b_j^y, b_j^z are majorana fermions defined at site j. This Majorana fermion representation is schematically presented in figure[3.9(a)]. The four fermionic operators satisfy the following relations:

$$c_j^\dagger = c_j \ , \ c_j^2 = 1 \ , \ (b_j^\gamma)^\dagger = b_j^\gamma \ , \ (b_j^\gamma)^2 = 1 \tag{3.20}$$

$$\{c_j, b_j^\gamma\} = 0 \ , \ \{b_j^\gamma, b_j^{\gamma'}\} = 0 \ (\gamma \neq \gamma') \tag{3.21}$$

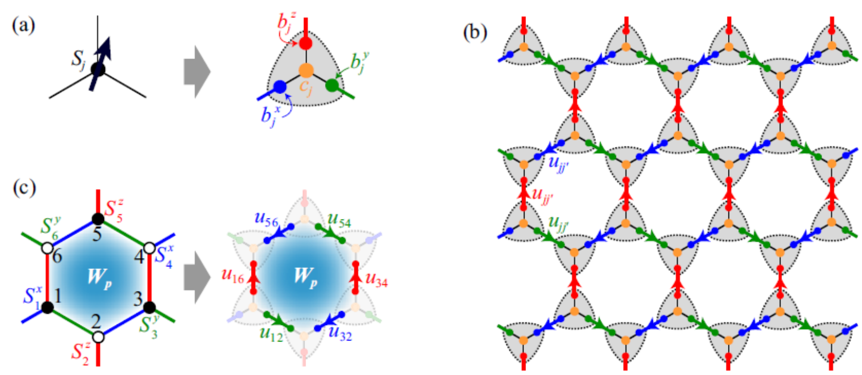


Figure 3.9: (a) Mapping of a quantum spin to four Majorana fermions, where the yellow, blue, green, and red circles stand for the Majorana fermions c_j , b_x^j , b_y^j , and b_z^j , respectively. (b) Schematic picture for the four-Majorana representation. (c) Correspondence between the original spins and gauge fields for the flux operator W_p

Furthermore, these operators at different sites also anticommute with each other.We substitute the majorana fermion representation (3.19) into the Kitaev Hamiltonian (3.18) to obtain the following Hamiltonian:

$$H_K = \frac{i}{4} \sum_{[jj']_\gamma} K_\gamma u_{jj'} c_j c_{j'} \tag{3.22}$$

where $u_{jj'} = i b_j^\gamma b_{j'}^\gamma$ on the γ bond with $[jj']_\gamma$. The direction from j to j' in $u_{jj'}$ is shown in figure[3.9(b)]. If we calculate the commutation of H_K and $u_{jj'}$, we obtain $[H_K, u_{jj'}] = 0$. This operator therefore is a local Z_2 conserved quantity. We can also show that $u_{jj'}^2 = 1$ using the relations (3.20) and (3.21).

We know that the number of hexagonal plaquettes in the honeycomb lattice is $\frac{N}{2}$ and that of the NN bond is $\frac{3N}{2}$. So the number of combinations in $\{w_p = \pm 1\}$ and $\{u_{jj'} = \pm 1\}$ is $2^{N/2}$ and $2^{3N/2}$ respectively. The difference indicates the existence of redundancy in the operators $u_{jj'}$. This redundancy comes from the Majorana representation shown in equation (3.19). Since two Majorana fermions represent a spinless fermion, the four-Majorana representation doubles the local Hilbert space. The total dimension is 2^{2N}, which is extended from the physical degrees of freedom in the N spin system, 2^N. From the algebra of an$S = 1/2$ spin, $2^3 S^j_x S^j_y S^j_z = i$ is satisfied at each site. On the other hand, this is rewritten as $ib^x_j b^y_j b^z_j c_j = iD$ in the Majorana representation, where we introduce $D = b^x_j b^y_j b^z_j c_j$ with $D^2 = 1$. Thus, the original spin space enforces $D = +1$, but introducing the four Majorana fermions extends the Hilbert space with $D = \pm 1$. This consideration indicates that one needs the projection onto the subspace with $D = +1$ in the Majorana fermion representation. The projection imposes a local constraint for the Majorana fermion system.

Using the constraint mentioned above, we can determine the relation between W_p and $u_{jj'}$:

$$W_p = \prod_{[jj'] \in p} u_{jj'} \tag{3.23}$$

To see difference in degrees of freedom, we introduce flux Φ_p at the hexagon plaquette p such that $W_p = \exp(i\Phi_p)$. $w_p = +1(-1)$ corresponds to $\Phi_p = 0(\pi)$ and hence this local state is called a zero (π) flux. The local state with π flux is also called a vison. We also introduce the gauge field $A_{jj'}$ such that $u_{jj'} = exp(iA_{jj'})$. From equation (3.23), we can show that

$$\Phi_p = \sum_{<jj'> \in p} A_{jj'} \tag{3.24}$$

The above relation is similar to the magnetic flux Φ_p crossing the surface surrounded by a closed contour C and vector potential A as $\Phi = \oint_C A.ds$, where ds is a differential element along the C in electromagnetism.

We consider a closed loop connecting the centers of hexagon plaquettes in the honeycomb lattice, as shown in figure[3.10(a) and (b)]. Even if the signs of $u_{jj'}$ on the bonds crossing the loop are flipped, the configuration of W_p does not change. Thus, the transformation with $A_{jj'} \to A_{jj'} + \pi$ for $A_{jj'}$ on an arbitrary loop is an example of local gauge transformations. On the other hand, the transformation for $A_{jj'}$ on an open string causes the change of two fluxes at the endpoints of the string (see figure[3.10 (c) and (d)]), and hence it is not a gauge transformation.

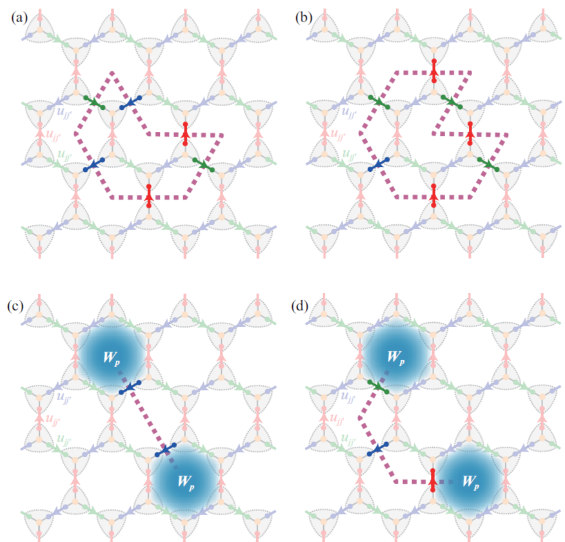


Figure 3.10: (a),(b) Examples of gauge transformations for the highlighted $u_{jj'}$ on the closed loops.(c),(d) Examples of transformations with $u_{jj'} \to -u_{jj'}$ on the highlighted bonds crossing distinct open strings, which cause the flip of the same two fluxes at their endpoints

When the gauge configuration is fixed, equation (3.22) becomes a bilinear form for the Majorana fermions c_j, which is easily solved by diagonalization or Fourier transformation. Therefore, the Kitaev model is mapped onto a free itinerant Majorana fermion system coupled with classical degrees of freedom. This is understood as the spin fractionalization into two elementary excitations, itinerant Majorana fermions and local Z_2 gauge fields (or flux W_p).

3.2.2.3 Jordan-Wigner Transformation

In this transformation, we regard the honeycomb lattice as 1D chains consisting of x and y bonds. The $S = \frac{\sigma}{2}$ spin operators are written by spinless fermion operators $(a_j, a_j^\dagger)$ as:

$$\sigma_j^x - i\sigma_j^y = 2a_j e^{i\pi \sum_{k=1}^{j-1} a_k^\dagger a_k} \;,\; \sigma_j^z = 2a_j^\dagger a_j - 1 \tag{3.25}$$

Using the representation given above, the Kitaev Hamiltonian can be rewritten as:

$$H_K = -\frac{K_x}{4}\sum_{[jj']_x}(a_j+a_j^\dagger)(a_{j'}-a_{j'}^\dagger)-\frac{K_y}{4}\sum_{[jj']_y}(a_j+a_j^\dagger)(a_{j'}-a_{j'}^\dagger)-\frac{K_z}{4}\sum_{[jj']_z}(2a_j^\dagger a_j-1)(2a_{j'}^\dagger a_{j'}-1) \tag{3.26}$$

The spinless fermion operators are represented by two Majorana fermions $(c_j, \bar{c_j})$ as

$$\begin{cases} c_j = a_j + a_j^\dagger \ , \ \bar{c_j} = \frac{a_j - a_j^\dagger}{i} & j \in A \text{ sublattice} \\ c_{j'} = \frac{a_{j'} - a_{j'}^\dagger}{i} \ , \ \bar{c_{j'}} = a_{j'} + a_{j'}^\dagger & j \in B \text{ sublattice} \end{cases} \tag{3.27}$$

A $S = \frac{1}{2}$ spin S_j is represented by two Majorana fermions $(c_j, \bar{c_j})$, as shown in figure[3.11(a) and (b)]. Using this Majorana fermion representation of spin, the Hamiltonian is rewritten as:

$$H_K = -\frac{iK_x}{4} \sum_{[jj']_x} c_j c_{j'} - \frac{iK_y}{4} \sum_{[jj']_y} c_j c_{j'} - \frac{iK_z}{4} \sum_{[jj']_z} \eta_r c_j c_{j'} \tag{3.28}$$

where the operator $\eta_r = i\bar{c_j}\bar{c_{j'}}$ is defined on each z bond r. Moreover, the flux operator W_p is given by the Z_2 conserved quantities as

$$W_p = \prod_{r \in p} \eta_r \tag{3.29}$$

where the product is taken for two z bonds on the hexagon plaquette p , as shown in figure[3.11(b)].

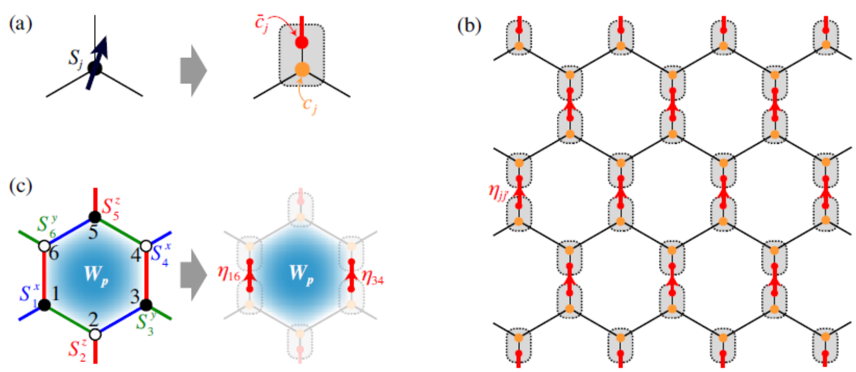


Figure 3.11: (a) Mapping of a quantum spin to two Majorana fermions, where the yellow and red circles stand for the Majorana fermions c_j and $\bar{c_j}$, respectively. (b) Schematic picture for the two Majorana representation using the Jordan–Wigner transformation.(c) Correspondence between the original spins and the Z_2 variables for the flux operator W_p

Unlike the four-Majorana representation, the present representation does not extend the model space, and hence no local constraints are needed.

Here, we introduce the vacuum in the Majorana representation using the Jordan–Wigner transformation. We introduce two complex fermions, d_r and $d_r^\dagger$, from four Majorana fermions, $(c_j, c_{j'})$ and $(\bar{c_j}, \bar{c_{j'}})$, at each z bond $r =< jj' >_z$ as

$$\begin{cases} c_j = d_r + d_r^\dagger \ , \ \bar{c}_j = \frac{\bar{d}_r - \bar{d}_r^\dagger}{i} & \text{for } j \in A \text{ sublattice} \\ c_{j'} = \frac{d_r - d_r^\dagger}{i} \ , \ \bar{c}_{j'} = \bar{d}_r + \bar{d}_r^\dagger & \text{for } j \in B \text{ sublattice} \end{cases} \tag{3.30}$$

Using the above representation, we define the vacuum $|0>$ so as to satisfy $d_r|0> = \ \bar{d}_r|0> = \ 0$ for all z bonds. Then, any states are represented by linear combinations of the Fock state, which is given by

$$|\Psi> = \ \bar{d}_r^\dagger \bar{d}_{r'}^\dagger ....\bar{d}_r^\dagger \bar{d}_{r'}^\dagger ......|0> \tag{3.31}$$

The operator η_r can be written as:

$$\eta_r = (\bar{d}_r - \bar{d}_r^\dagger)(\bar{d}_r + \bar{d}_r^\dagger) = 1 - 2\bar{d}_r^\dagger \bar{d}_r \tag{3.32}$$

This relation indicates that $\eta_r = +1(-1)$ corresponds to the absence (presence) of the fermion $\{\bar{d}_r\}$. From equation(3.29), the fermionic state of $\{\bar{d}_r\}$ determines the flux configuration $\{w_p = \pm 1\}$.

Chapter 4

Discussion and Conclusion

4.1 Discussion

In this thesis, we mainly analyzed the emergence of exact Majorana zero modes in interacting spin chains. We focus on a 1D quantum compass model with different possible configurations such as i) when magnetic field is zero, ii) when magnetic field is uniform across all sites and iii) when magnetic field is non-uniform across all sites. To diagonalize the model, we first perform a Jordan-Wigner transformation to map the spin model into a fermionic one. We then define a set of Majorana operators by linearly combining the fermion creation/annihilation operators on individual sites. The finally obtained quadratic Majorana-fermion model provides us with a convenient representation to study the existence of EMZMs.

Working in the Majorana-fermion representation, we are able to derive an analytical expression for the number of the emergent EMZMs for different possible cases, as mentioned above. We also derive explicit forms of the EMZMs. We find non-local probability amplitude of the EMZMs although they are concentrated at the boundary for all the above cases.

Finally, we discuss possible experimental platforms that can realize the model and the control of the transverse fields. Considering that the quantum Ising chain in the presence of both a transverse and a longitudinal magnetic field has recently been simulated using the nuclear magnetic resonance quantum simulator, such a setup may also serve as a suitable simulator to realize the compass model. Another possible setup to realize the current model is the superconducting quantum circuits. Recently, the spin-1/2 XXZ Heisenberg chain in a transverse field has been realized using the nanoscale arrangement technique of magnetic atoms, which provides another candidate for simulation of the transverse compass model. In practice, the local fields exerted on the quasi-one-dimensional magnetic materials can be induced by the proximity effect of a ferromagnet, and the strength of the field can be tuned by varying the distance between the spin chain and the ferromagnet. We emphasize that the advantage of the system considered here lies in its exact solvability and mathematical rigor.

A unifying theme emerging from all Chapters is the central role of bond-directional frustration in stabilizing unconventional quantum phases. In the one-dimensional transverse quantum compass model, frustration arises from competing interactions on alternating bonds, suppressing classical order and giving rise to symmetry-protected Majorana zero modes with maximum probability amplitude at boundaries. Although lacking geometric frustration, the 1D model shows how anisotropic interactions alone can generate robust topological features. In contrast, the 2D Kitaev honeycomb model demonstrates how frustration leads to intrinsic topological order and true fractionalization, where spins decompose into itinerant Majorana fermions and static Z_2 fluxes, while the Toric Code provides a paradigmatic example of a Z_2 topologically ordered phase with deconfined anyonic excitations. Together, these results highlight that whether in 1D or 2D, frustration fundamentally restructures spin degrees of freedom, enabling the emergence of topological phases and fractionalized quasiparticles.

4.2 Future work

We will now focus on a rich frontier: the non-equilibrium dynamics of fractionalized excitations in the Kitaev honeycomb model. The next steps of this project will focus primarily on:

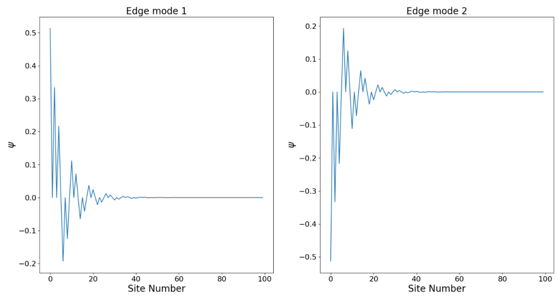
1. **Floquet-driven dynamics of Majorana matter and Z_2 fluxes**
   Under periodic driving, the Kitaev spin liquid enters a fractionalized prethermal regime, where the itenerant majoranas heat rapidly, but the Z_2 fluxes remain frozen. The two sectors acquire drastically different effective temperatures. Future work can systematically analyze the following.

   - The dependence of prethermal plateaus on the frequency and strength of the drive.
   - Heating thresholds and dynamically protected regimes.
   - The role of anisotropic couplings $K_x \neq K_y \neq K_z$
   - The onset and breakdown of prethermalization.

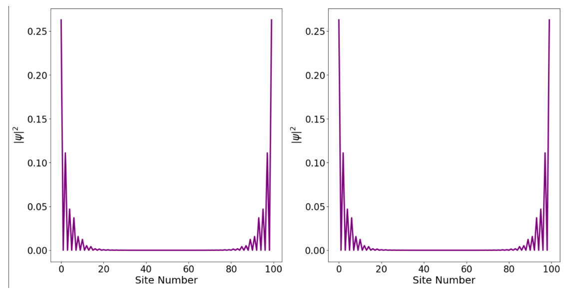
   This will help to clarify how universal fractionalized prethermalization is in topological quantum matter.

2. **Theoretically explaining the formation of Majorana Zero Modes for different field configurations**
   We realized that whatever we have done so far in the thesis is just some analytical and numerical calculations and simply presented the results and observation. However we do need to develop theoretical framework that could explain what exactly is going on microscopically (or macroscopically) that leads to the formation of EMZMs in different field configurations.

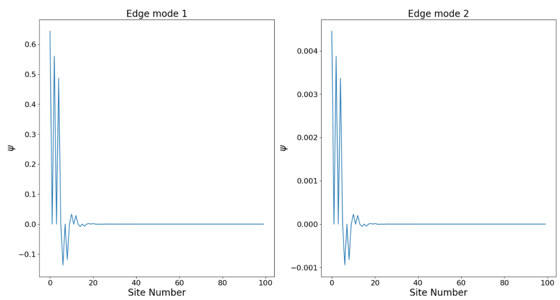
3. **Quantum Compass Model**
   In chapter 2 of the thesis, we mentioned in Case 2 that if the magnetic field (h) is uniform across all sites and if $h^2 \neq J^x_{2j} J^y_j$, then we should not observe any EMZM. However when we performed numerical calculations, we do observe Majorana zero modes with probability density concentrated at boundary. In the figure[4.1], we can see those probability amplitudes and density for different values of h, considering $J^x = J^y = 1$. This ambiguity is something that we will have to look further.



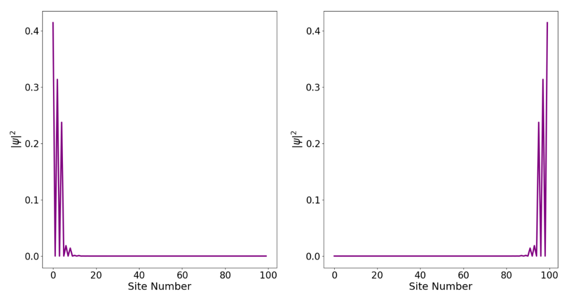
(a) Probability amplitude of Majorana Zero Modes for $h = 0.65$



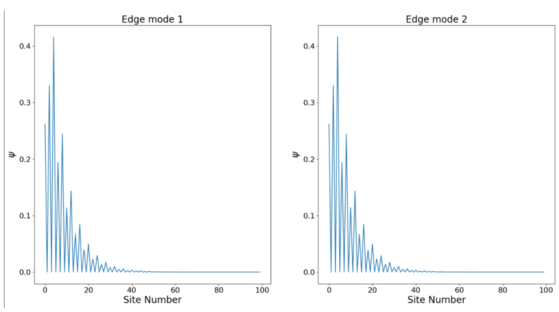
(b) Probability density of Majorana Zero Modes for $h = 0.65$



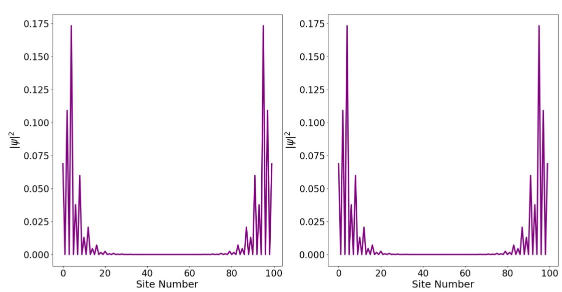
(c) Probability amplitude of Majorana Zero Modes for $h = 0.87$



(d) Probability density of Majorana Zero Modes for $h = 0.87$



(e) Probability amplitude of Majorana Zero Modes for $h = 1.26$



(f) Probability density of Majorana Zero Modes for $h = 1.26$

Figure 4.1: Probability density and probability amplitude of Majorana Zero Modes

4.3 Conclusion

This thesis explored how frustration and bond-directional anisotropy in quantum spin systems give rise to topological structures and emergent fractionalized excitations across both 1D and 2D settings. In the 1D transverse quantum compass chain, we demonstrated analytically and numerically how Exact Majorana Zero Modes (EMZMs) emerge, disappear, and reappear under different magnetic-field configurations. The dependence of these zero modes on spatial inhomogeneities J^x, J^y highlights a rich field of boundary and defect localized Majorana physics. Complementing this, the review of Kitaev and Toric Code models illustrated how similar local constraints, emergent gauge fields, and fractionalization manifest in 2D quantum spin liquids. Taken together, these results show that frustration not only suppresses conventional order but actively structures the low-energy Hilbert space into topologically nontrivial sectors supporting robust quasiparticles.

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