Rotating Cylinder on a Vibrating Plate

Sparsh Gupta, Kabir Hooda

November 2022

1 Description of the System

The above system consists of a plate performing SHM with the equation given by z=A$\sin \omega t$. The velocity of plate is given by V=A$\omega \cos \omega t$. The hollow cylinder is rotating about it's own axis and also that axis is rotating about the z axis.

Here X,Y,Z are the fixed co-ordinates and x_1, x_2, x_3 are the moving co-ordinates. θ, Φ and Ψ are the Euler angles. θ is the angle between the z-axis and x_3 axis. Φ is the angle between Y-axis and the line of node and Ψ is the angle between x_1 axis and the line of node.

It is to be noted that the angular velocities $\dot{\omega}$, $\dot{\Phi}$ and $\dot{\Psi}$ are along the line of node, z-axis and x_3 axis respectively. The centre of mass is at a distance of 'l' from the base.

2 Langrangian Equation

$$L = T - U$$

T is the kinetic energy and U is the potential energy of the given system.

$$T = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} M V_c^2$$

.

$$I_1 = I_2 = \frac{MR^2}{2} + \frac{ML^2}{12}$$

.

$$I_3 = MR^2$$

$$\omega_1 = \dot{\Phi} \sin\theta \sin\Psi + \dot{\theta} \cos\Psi$$

$$\omega_2 = \dot{\Phi} \sin\theta \cos\Psi - \dot{\theta} \sin\Psi$$

$$\omega_3 = \dot{\Phi} \cos\theta + \dot{\Psi}$$

Therefore, the kinetic energy of the system is given by,

$$T = \frac{1}{2}(\frac{MR^2}{2} + \frac{ML^2}{12})((\dot{\Phi})^2(\sin\theta)^2 + (\dot{\theta})^2) + \frac{1}{2}MR^2(\dot{\Phi}\cos\theta + \dot{\Psi})^2 + \frac{1}{2}M(\dot{z})^2$$

.
The potential energy of the system is given by,

$$U = Mg(l\cos\theta + z) + \frac{1}{2}M\omega^2 z^2$$

Therefore, Langrangian of a system is given by,

$$L = \frac{1}{2}(\frac{MR^2}{2} + \frac{ML^2}{12})((\dot{\Phi})^2(\sin\theta)^2 + (\dot{\theta})^2) + \frac{1}{2}MR^2(\dot{\Phi}\cos\theta + \dot{\Psi})^2 + \frac{1}{2}M(\dot{z})^2 - Mg(l\cos\theta + z) - \frac{1}{2}M\omega^2 z^2$$

3 Equations of motion

$$\frac{\partial L}{\partial z} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{z}})$$

$$\frac{\partial L}{\partial z} = -(Mg + M\omega^2 z)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{z}}) = M\ddot{z}$$

$$\Rightarrow \ddot{z} = -(\omega^2 z + g)$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}})$$

$$\frac{\partial L}{\partial \theta} = -\frac{MR^2(\dot{\Phi})^2 \sin 2\theta}{4} - MR^2 \sin\theta \dot{\Phi}\dot{\Psi} + \frac{ML^2 \sin 2\theta \dot{\Phi}}{24} + Mgl\sin\theta$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = (\frac{MR^2}{2} + \frac{ML^2}{12})\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = -(\frac{\frac{R^2\dot{\Phi}^2\sin 2\theta}{4} + R^2\sin\theta\dot{\Phi}\dot{\Psi} - \frac{L^2\sin 2\theta\dot{\Phi}^2}{24} - gl\sin\theta}{\frac{R^2}{2} + \frac{L^2}{12}})$$

We find that partial derivative of Langrangian wrt to Φ and Ψ is 0 i.e

$$\frac{\partial L}{\partial \Phi} = 0$$

and,

$$\frac{\partial L}{\partial \Psi} = 0$$

which implies that Φ and Ψ are cyclic co-ordinates. Therefore, corresponding generalized momenta are constant with time.