Electrodynamics of Particle

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e j^{\mu} A_{\mu} + \mathcal{L}_{M}$$

$$+ \underbrace{\mathcal{L}_{GF}}_{\downarrow}$$

$$\frac{\xi}{2} A_{\mu} \partial^{\mu} \partial^{\nu} A_{\nu}$$

$$Z = \int \mathcal{D}A_{\mu} \; e^{-\int d^4z \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e j^{\mu} A_{\mu} - \frac{\xi}{2} A_{\mu} \partial^{\mu} \partial^{\nu} A_{\nu} \right]}$$

$$\int d^4z \; F^{\mu\nu} F_{\mu\nu}$$

$$= \int d^4z \; (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) F_{\mu\nu}$$

$$= \int d^4z \; (\partial^{\nu} A^{\mu} - \partial^{\mu} A^{\nu}) F_{\nu\mu}$$

$$\{\mu \longleftrightarrow \nu\}$$

$$= \int d^4z \quad \partial^\nu A^\mu F_{\nu\mu} \quad - \int d^4z \quad \partial^\mu A^\nu F_{\nu\mu}$$

$$\{\mu \leftrightarrow \nu\} \qquad\qquad \{F_{\mu\nu} = -F_{\nu\mu}\}$$

$$= \int d^4z \quad \partial^\mu A^\nu F_{\mu\nu} \quad + \int d^4z \quad \partial^\mu A^\nu F_{\mu\nu}$$

$$= \int d^4z \quad 2\,\partial^\mu A^\nu F_{\mu\nu}$$

$$\int d^4z \quad 2\,\partial^\mu A^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$= 2 \int d^4z \left[(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu)\right]$$

$$= 2 \int d^4z \left[\partial^\mu\left(A^\nu \partial_\mu A_\nu\right) - A^\nu \partial^2 A_\nu - \partial^\mu\left(A^\nu \partial_\nu A_\mu\right) + A^\nu\left(\partial^\mu \partial_\nu A_\mu\right)\right]$$

$$= 2 \int d^4z \left[-A_\rho\, \eta^{\rho\nu} \partial^2 A_\nu + A_\rho\, \eta^{\rho\nu} \underbrace{\partial^\mu \partial_\nu A_\mu}_{\mu \leftrightarrow \nu} \right] \qquad \eta^{\rho\nu}\partial_\nu = \partial^\rho$$

$$= -2 \int d^4z \left[A_\rho \left(\eta^{\rho\nu} \partial^2 - \partial^\nu \partial^\rho \right) A_\nu \right]$$

$$Z = \int \mathcal{D}A_\mu \, \exp\left[\int d^4z \left[\frac{1}{2} A_\rho \left(\eta^{\rho\nu} \partial^2 - \partial^\rho \partial^\nu \right) A_\nu - e j^\mu A_\mu + \frac{\xi}{2} A_\rho \partial^\rho \partial^\nu A_\nu \right] \right]$$

$$= \int \mathcal{D}A_\mu \, \exp\left[\int d^4z \, \frac{1}{2} \left[A_\rho \mathcal{D}^{\rho\nu} A_\nu \right] - e j^\rho A_\rho \right]$$

where,

$$\mathcal{D}^{\rho\nu} = \eta^{\rho\nu} \partial^2 - (1-\xi) \partial^\rho \partial^\nu$$

Eq^n of motion :

$$\frac{\delta S}{\delta A_\rho} = 0$$

$$\Rightarrow \quad D^{\rho\nu} A_\nu = e\, j^\rho$$

$$A_\nu = e \int d^4 z' \; j^\rho(z')\, G_{\rho\nu}(z-z')$$

$$D^{\rho\nu} G_{\nu\mu} = \delta^4 (z-z')\, \delta^\rho_\mu$$

$$\int d^4 z \; \left(\eta^{\rho\nu}\partial^2 - (1-\xi)\partial^\rho \partial^\nu\right) \underbrace{G_{\nu\mu}}_{\int \frac{d^4k}{(2\pi)^4}\, \tilde{G}_{\nu\mu}(k)\, e^{ikz}} = 1$$

$$\Rightarrow \int d^4 z \, \frac{d^4 k}{(2\pi)^4} \left[-\eta^{\rho\nu} k^2 + (1-\xi) k^\rho k^\nu\right] \tilde{G}_{\nu\mu}(k)\, e^{ikz} = 1$$

$$\Rightarrow \int \frac{d^4 k}{(2\pi)^4}\, \delta(k) \left[-\eta^{\rho\nu} k^2 + (1-\xi) k^\rho k^\nu\right] \tilde{G}_{\nu\mu}(k) = 1$$

$$\left[-\eta^{\rho\nu}k^2+(1-\xi)k^\rho k^\nu\right]\tilde{G}_{\nu\mu}(k)=$$

$$\xi=1$$

$$-\eta^{\rho\nu}k^2\,\tilde{G}_{\nu\mu}(k)=\delta^\rho_\mu$$

$$\tilde{G}_{\nu\mu}=-\frac{\delta^\rho_\mu}{\eta^{\rho\nu}k^2}$$

$$\tilde{G}_{\nu\mu}=-\frac{\eta_{\nu\mu}}{k^2}$$

$$\partial\to -ik$$

$$-\partial^2\to k^2$$

$$G(z-z')=\frac{1}{\Box}$$

$$S = \int d^4z \left[\frac{1}{2} A_\rho D^{\rho\nu} A_\nu - e j^\rho A_\rho \right]$$

$$A_\rho \longrightarrow A_\rho + a_\rho$$

$\left\{ \text{Small perturbation to Gauge field } A_\mu \right\}$

$$A_\rho(z) = e \int d^4z' \; G_{\rho\nu}(z-z') \; j^\nu(z')$$

$$S = \int d^4z \left[\frac{1}{2} (A_\rho + a_\rho) D^{\rho\nu} (A_\nu + a_\nu) - e j^\rho (A_\rho + a_\rho) \right]$$

$$S_{new} = \int d^4z \Big[\underbrace{\frac{1}{2} A_\rho D^{\rho\nu} A_\nu} + \frac{1}{2} A_\rho D^{\rho\nu} a_\nu + \frac{1}{2} a_\rho \underbrace{D^{\rho\nu} A_\nu}_{e j^\rho} + \frac{1}{2} a_\rho D^{\rho\nu} a_\nu \underbrace{- e j^\rho A_\rho} - e j^\rho a_\rho \Big]$$

$$S_{new} = S_{old} + \int d^4z \, \Big[\frac{1}{2} A_\rho D^{\rho\nu} a_\nu + \frac{e}{2} a_\rho j^\rho + \frac{1}{2} a_\rho D^{\rho\nu} a_\nu - e j^\rho a_\rho \Big]$$

$$\Big\{ \text{Eq}^n \text{ of motion}: \quad D^{\rho\nu} A_\nu = e j^\rho \Big\}$$

$$S_{new} = \int d^4z \, \Big[\frac{1}{2} A_\rho D^{\rho\nu} A_\nu + \underbrace{\frac{1}{2} A_\rho D^{\rho\nu} a_\nu}_{} + \frac{1}{2} a_\rho D^{\rho\nu} a_\nu - \frac{e}{2} j^\rho a_\rho - e j^\rho A_\rho \Big]$$

$$\int d^4z \, \frac{1}{2} A_\rho D^{\rho\nu} a_\nu$$

$$= \frac{1}{2} \int d^4z \, A_\rho \left(\eta^{\rho\nu} \partial^2 - \partial^\rho \partial^\nu \right) a_\nu$$

$$= \frac{1}{2} \Big[\int d^4z \, A^\nu \partial^2 a_\nu - \int d^4x \, A_\rho \partial^\rho \partial^\nu a_\nu \Big]$$

$$= \frac{1}{2}\Big[\cancelto{0}{A^\nu (\partial a_\nu)} - \int d^4x \, (\partial A^\nu)(\partial a_\nu)$$

$$- \cancelto{0}{A_\rho (\partial^\nu a_\nu)} + \int d^4x \, (\partial^\rho A_\rho)(\partial^\nu a_\nu) \Big]$$

$\{\text{Boundary terms}\}$.

$$= \frac{1}{2}\Big[- \cancelto{0}{(\partial A^\nu) a_\nu} + \int d^4x \, a_\nu \big(\partial^2 \overset{\eta^{\rho\nu}A_\rho}{\widetilde{A^\nu}}\big)$$

$$+ \cancelto{0}{(\partial^\rho A_\rho) a_\nu} - \int d^4x \, a_\nu \left(\partial^\nu \partial^\rho A_\rho\right) \Big]$$

$$= \frac{1}{2} \int d^4z \, a_\nu \left(\eta^{\rho\nu} \partial^2 - \partial^\nu \partial^\rho \right) A_\rho$$

$$= \frac{1}{2} \int d^4z \, a_\nu D^{\nu\rho} A_\rho$$

$\{\rho \longleftrightarrow \nu\}$

$$S_{new} = \int d^4z \Big[\frac{1}{2} A_\rho D^{\rho\nu} A_\nu + \cancel{\frac{1}{2} a_\rho \underbrace{D^{\rho\nu} A_\nu}_{ej^\rho}}$$

$$+ \frac{1}{2} a_\rho D^{\rho\nu} a_\nu - \cancel{\frac{e}{2} j^\rho a_\rho}$$

$$- e j^\rho A_\rho \Big]$$

$$= \int d^4z \left[\frac{1}{2} A_\rho \underbrace{D^{\rho\nu} A_\nu}_{e j^\rho} - e j^\rho A_\rho \right]$$

$$+ \int d^4z \left[\frac{1}{2} a_\rho D^{\rho\nu} a_\nu \right]$$

$$S_{new} = \int d^4z \quad \frac{1}{2} a_\rho D^{\rho\nu} a_\nu - \frac{e}{2} j^\rho A_\rho$$

$$Z[j^\rho_{3+1}] = \int \mathcal{D}A_\rho \; e^{-\int d^4z \left[\frac{1}{2} A_\rho D^{\rho\nu} A_\nu - e j^\rho A_\rho \right]}$$

$$A_\rho \longrightarrow A_\rho + a_\rho$$

$$Z[j^\rho_{3+1}] = \int \mathcal{D}A_\rho \; e^{-\int d^4z \left[\frac{1}{2} a_\rho D^{\rho\nu} a_\nu - \frac{e}{2} j^\rho A_\rho \right]}$$

$$\frac{Z[j^\rho_{3+1}]}{Z[0]} = \exp\left[\int d^4z \; \frac{e}{2} j^\rho A_\rho \right]$$

$$\left\{ A_\rho(x) = e \int d^4z' \; G_{\rho\nu}(z-z') \, j^\nu(z') \right\}$$

$$\frac{Z\left[j^{\rho}_{3+1}\right]}{Z[0]} = \exp\left[\frac{e^2}{2}\int d^4z\, d^4z'\; j^{\rho} G_{\rho\nu}(z-z')\, j^{\nu}\right]$$

$$= \exp\left[\frac{e^2}{2}\int d^4z\, d^4z'\; j^{\rho}(z)\left[\frac{1}{\Box}\right] j^{\nu}(z')\right]$$

$$= \exp\left[S_{eff}\right]$$