Meissner Effect using PQED

$$F = \int d^2r \left[a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{\psi^*}{2m^*}\left(\frac{\hbar\nabla}{i} + \frac{2e}{c}A\right)^2\psi \right]$$

$$+ \int d^2r\, d^2r' \, (\nabla_r \times A(r)) \frac{1}{\sqrt{\nabla^2_{r-r'}}} (\nabla_{r'} \times A(r'))$$

$$- \int d^2r \; j \cdot A$$

$$Z = \int DA\, D\psi \; e^{-\beta F[j, \psi, A]}$$

$$F[j, \psi, A] = \int d^2r \left[a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla + \frac{2e}{c}A\right)\psi\right|^2 \right]$$

$$+ \int d^2r\, d^2r' \, (\nabla_r \times A(r)) \frac{1}{\sqrt{\nabla^2_{r-r'}}} (\nabla_{r'} \times A(r'))$$

$$- \int d^2r \; j^\mu A_\mu$$

Let $\psi(r) = \sqrt{n_s}\, e^{i\theta(r)}$

$$f[\psi, A] = \frac{2e^2A^2}{m^*c^2} n_s + \frac{\hbar \psi^* eA}{m^*c} \nabla\theta \; \psi$$

$$+ \frac{1}{2m^*} \left| \left(\frac{\hbar}{i}\nabla + \frac{2e}{c} A \right) \psi \right|^2$$

$$\left| \left(\hbar \nabla\theta \; \psi + \frac{2e}{c} A \psi \right) \right|^2$$

$$\frac{2e^2 n_s}{m^*c^2} \left(\frac{\hbar c}{2e} \nabla\theta + A \right)^2$$

$$F[j, A, \theta] = \int d^2r \left[a n_s + \frac{b}{2} n_s^2 + \frac{\rho_s}{2} \left(A + \frac{\hbar c}{2e} \nabla\theta \right)^2 - j \cdot A \right] \qquad \left\{ \rho_s = \frac{4e^2 n_s}{m^* c^2} \right\}$$

$$+ \int d^2r \, d^2r' \; \nabla_r \times A(r) \frac{1}{\sqrt{\nabla^2_{r-r'}}} \nabla_{r'} \times A(r')$$

$$F = \int d^2r\, d^2r'\, \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2k}{(2\pi)^2}$$

$$e^{ip.r} \left[i\vec{p} \times \vec{A}(p)\right] \frac{e^{iq.(r-r')}}{|q|} \left[i\vec{k} \times \vec{A}(k)\right] e^{ik.r'}$$

$$= -\int d^2r\, d^2r'\, \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2k}{(2\pi)^2}$$

$$\frac{\left[\vec{p} \times \vec{A}(p)\right] . \left[\vec{k} \times \vec{A}(k)\right]}{|q|} e^{i(p+q)r} e^{i(k-q)r'}$$

$$= - \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \delta(p+q)\, \delta(k-q)$$

$$\frac{\left[\vec{p} \times \vec{A}(p)\right] . \left[\vec{k} \times \vec{A}(k)\right]}{|q|}$$

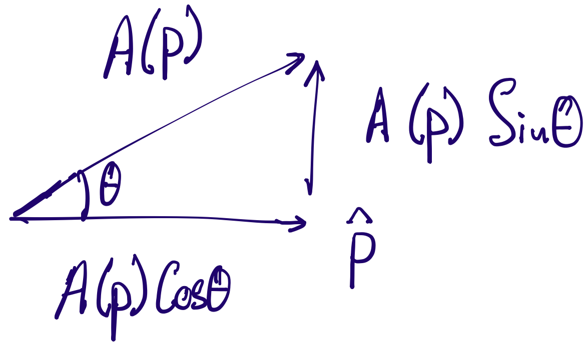
$$= -\int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \, \delta(p+q) \frac{[\vec{p} \times \vec{A}(p)] \cdot [\vec{q} \times \vec{A}(q)]}{|q|}$$

$$= \int \frac{d^2p}{(2\pi)^2} \frac{[\vec{p} \times \vec{A}(\vec{p})] \cdot [\vec{p} \times \vec{A}(-\vec{p})]}{|p|}$$

$$\left\{ \begin{aligned} & (A \times B) \cdot \underbrace{(A \times C)}_{D} \\ &= A \cdot (B \times D) \\ &= A \cdot (B \times A \times C) \\ &= A \cdot \big((B \cdot C) A - (A \cdot C) B\big) \\ &= (A \cdot A)(B \cdot C) - (A \cdot B)(A \cdot C) \end{aligned} \right.$$

$$\Rightarrow \int \frac{d^2 p}{(2\pi)^2} \frac{p^2}{|p|} \left[\vec{A}(p) \cdot \vec{A}(-p) - (\hat{p} \cdot \vec{A}(p)) (\hat{p} \cdot \vec{A}(-p)) \right]$$

$$\vec{A_T}(p) = \vec{A(p)} - (\hat{p} \cdot \vec{A}(p))\, \hat{p}$$



$$\underbrace{\vec{A_T}(-p)}_{\text{transverse components}} = \vec{A}(-p) - (\hat{p} \cdot \vec{A}(-p))\, \hat{p}$$

$$\vec{A_T}(p) \cdot \vec{A_T}(-p) = \vec{A}(p) \cdot \vec{A}(-\vec{p}) - (\hat{p} \cdot \vec{A}(p)) (\hat{p} \cdot \vec{A}(-p))$$

$$\Rightarrow \int \frac{d^2 p}{(2\pi)^2} |p| \, |A_T(p)|^2$$

$$p \leftrightarrow k \quad \Rightarrow \int \frac{d^2 k}{(2\pi)^2} |k| \, |A_T(k)|^2$$

$$\int d^2r \, \frac{\rho_s}{2} \left(A + \frac{\hbar c}{2e} \nabla \theta \right)^2$$

$$= \int d^2r \, \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{\rho_s}{2} \left(e^{ik.r} A(k) + \frac{\hbar c}{2e} (ik) e^{ikr} \theta(k) \right)$$

$$\left(e^{iq.r} A(q) + \frac{\hbar c}{2e} (iq) e^{iq.r} \theta(q) \right)$$

$$= \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{\rho_s}{2} \delta(k+q) \left[A(k) + \frac{i\hbar c}{2e} k \theta(k) \right]$$

$$\left[A(q) + \frac{i\hbar c}{2e} q \, \theta(q) \right]$$

$$= \int \frac{d^2k}{(2\pi)^2} \frac{\rho_s}{2} \left[A(k) + \frac{i\hbar c}{2e} k \, \theta(k) \right]$$

$$\left[A(-k) - \frac{i\hbar c}{2e} k \, \theta(-k) \right]$$

$\hat{k}\,|k|$

$$= \frac{\rho_s}{2} \int \frac{d^2k}{(2\pi)^2} \Big[A(k)\,A(-k) + \frac{i\hbar c}{2e}\, k \big(A(-k)\,\theta(k) - A(k)\,\theta(-k) \big) + \frac{\hbar^2 c^2}{4e^2}\, k^2\, \theta(k)\,\theta(-k) \Big]$$

$$= \frac{\rho_s}{2} \int \frac{d^2k}{(2\pi)^2} \Big[|A_T(k)|^2 + |A_L(k)|^2 - \left| i\frac{\hbar c}{2e}\, k\, \theta(k) \right|^2 + \frac{i\hbar c\, k}{2e} \big(A_L(-k)\,\theta(k) - A_L(k)\,\theta(-k) \big) \Big]$$

$$= \frac{\rho_s}{2} \int \frac{d^2k}{(2\pi)^2} \Big[|A_T(k)|^2 + \Big(A_L(k) - \frac{i\hbar c k}{2e}\,\theta(k) \Big)^2 \Big]$$

$$\int d^2r \; j \cdot A = \int d^2r \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} e^{i(k+q)r} j(k)\, A(q)$$

$$\int \frac{d^2k}{(2\pi)^2} \left[j(k)\; A(-k) \right]$$

$$= \int \frac{d^2k}{(2\pi)^2} \left[j(-k) \cdot A(k) \right]$$

$$= \int \frac{d^2k}{(2\pi)^2} \left[j_L(-k)\, A_L(k) + j_T(-k)\, A_T(k) \right]$$

$$Z = \int D\psi \; e^{-\left[a n_s + \frac{b}{2} n_s^2\right]}$$

$$\underbrace{\int DA}_{DA_T\, DA_L} \; e^{-\left[\int \frac{d^2k}{(2\pi)^2} \left(-\frac{i\,|k|}{8\pi} |A_T(k)|^2 + \frac{\rho_s}{2} |A_T(k)|^2 + \frac{\rho_s}{2} \left| A_L(k) - \frac{i\hbar c k}{2e} \theta(k) \right|^2 - J_T(-k)\, A_T(k) - J_L(-k)\, A_L(k) \right.\right]}$$

$$\underbrace{\int \mathcal{D}A_T(k)\; e^{-\int \frac{d^2k}{(2\pi)^2}\left[\left(\frac{\rho_s}{2}+\frac{|k|}{8\pi}\right)|A_T(k)|^2 - J_T(-k)\,A_T(k)\right]}}_{1}$$

$$\underbrace{\int \mathcal{D}A_L(k)\; e^{-\int \frac{d^2k}{(2\pi)^2}\left[\frac{\rho_s}{2}\left|A_L(k)-\frac{i\hbar c}{2e}k\,\theta(k)\right|^2 - J_L(-k)\,A_L(k)\right]}}_{2}$$

Evaluating 1 ,

$$\Rightarrow \quad b = J_T(-k) \qquad a = \rho_s + \frac{|k|}{4\pi}$$

$$\left\{\begin{aligned} &\int dx\; e^{-\left(\frac{1}{2}ax^2 - bx\right)} \\ &= \int dx\; e^{-\frac{1}{2}a\left(x-\frac{b}{a}\right)^2 + \frac{b^2}{2a}} \\ &= e^{b^2/2a}\sqrt{\frac{2\pi}{a}} \end{aligned}\right.$$

$$F_{eff}^{(T)} = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\,\frac{J_T(-k)\;J_T(k)}{\left[\rho_s + \frac{|k|}{4\pi}\right]}$$

$$G_T(k) = \frac{1}{\rho_s + \frac{|k|}{4\pi}}$$

$G_T \downarrow$ transverse propagator

Evaluating 2,

$$\underbrace{\int \mathcal{D}A_L(k)\; e^{-\int \frac{d^2k}{(2\pi)^2}\left[\frac{\rho_s}{2}\left|A_L(k) - \frac{i\hbar c}{2e}k\,\theta(k)\right|^2 - J_L(-k)\,A_L(k)\right]}}_{2}$$

$$\int \mathcal{D}A_L(k)\; e^{-\int \frac{d^2k}{(2\pi)^2}\left[\frac{\rho_s}{2}\left|A_L(k) - a_L(k)\right|^2 - J_L(-k)\,A_L(k)\right]}$$

$$-\frac{1}{2}p(x-a)^2$$

$$+\,b(x-a)$$

$$+\,ab$$

$$\int dx\; e^{-\left[\frac{1}{2}p(x-a)^2 - bx\right]}$$

$$= \int dx\; e^{-\frac{p}{2}x^2 + (ap+b)x - \frac{pa^2}{2}}$$

$$= \int dx\; e^{-\frac{p}{2}\left[x - \frac{(ap+b)}{p}\right]^2}\; e^{\frac{(ap+b)^2}{2p} - \frac{pa^2}{2}}$$

$$= e^{\frac{b^2}{2p} + ab} \sqrt{\frac{2\pi}{p}}$$

2

$$\Gamma_{\mathrm{eff}}^{(L)} = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\, \frac{J_L(-k)\, J_L(k)}{\rho_s}$$

$$- \int \frac{d^2k}{(2\pi)^2} \left[\frac{i\hbar c k}{2e}\,\theta(k)\right] J_L(-k)$$

$$F_{eff}[\theta, J] = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[\frac{J_T(-k)\;J_T(k)}{\left[\rho_s + \frac{|k|}{4\pi}\right]} + \frac{J_L(-k)\,J_L(k)}{\rho_s} + \frac{i\hbar c}{e}\,k\,\theta(k)\;J_L(-k)\right]$$

$$- a n_s - \frac{b}{2} n_s^2$$

$$\frac{\delta F_{eff}}{\delta \theta} = 0 \implies \frac{i\hbar c}{e}\,k\;J_L(-k) = 0$$

$$\boxed{J_L(k) = 0}$$

$$\underbrace{\int DA_T(k)\; e^{-\int \frac{d^2k}{(2\pi)^2}\left[\left(\frac{\rho_s}{2} + \frac{|k|}{8\pi}\right)|A_T(k)|^2 - J_T(-k)\,A_T(k)\right]}}_{F[A_T,\, J_T]}$$

$$\frac{\delta F}{\delta A_T(-k)} = \left(\rho_s + \frac{|k|}{4\pi}\right) A_T(k) - J_T(k) = 0$$

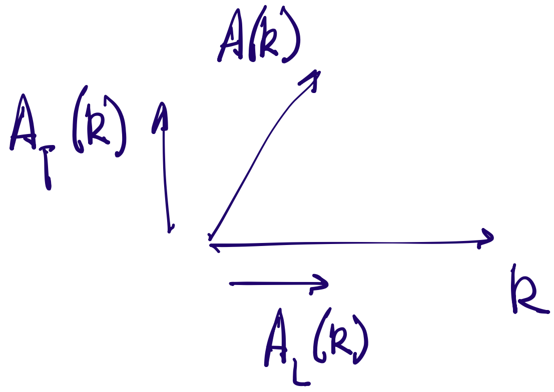
$$A_T(k) = \frac{J_T(k)}{\rho_s + \frac{|k|}{4\pi}}$$

$$B(k) = k\, A_T(k)$$

$$B(k) = \frac{k}{\rho_s + \frac{|k|}{4\pi}}\, J_T(k)$$

$$B(x) = \int \frac{dk}{(2\pi)}\, e^{ikr}\, B(k)$$

$$B = k \times A(k)$$



$$B = k \times \left(A_L(k) + A_T(k)\right)$$

$$= k\, A_T(k)$$

$$\Rightarrow \quad B(x) = \int \frac{dk}{2\pi}\, e^{ikx}\, B(k)$$

$$= \int \frac{dk}{2\pi}\, e^{ikx}\, \frac{k\, J_T(k)}{\rho_s + \frac{|k|}{4\pi}}$$

$$= \int dk\, e^{ikx}\, \frac{2k}{|k| + M}\, \underbrace{J_T(k)}_{\substack{\text{constant} \\ \text{(assumption)}}}$$

$$= 2J \int dk\, e^{ikx}\, \frac{k}{k+M} \qquad \left\{ M = 4\pi \rho_s \right\}$$

$$\sim \frac{1}{x^3}$$

Meissner effect here should have algebric screening.