PQED Green's function in Position Space

$$G^{\mu\nu}(\omega, k) = \frac{P^{\mu\nu}}{2\left[-\omega^2 + k^2\right]^{1/2}}$$

$$P^{\mu\nu} = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

$$G_{\mu\nu}(x-y) = \int \frac{d^2k}{(2\pi)^2} \, e^{ik.(x-y)} \, \frac{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{2\sqrt{-\omega^2 + k^2}}$$

$$= \underbrace{\int \frac{d^2k}{(2\pi)^2} \, e^{ik.(x-y)} \, \frac{\delta_{\mu\nu}}{2|\vec{k}|}}_{1}$$

$$- \underbrace{\int \frac{d^2k}{(2\pi)^2} \, e^{ik.(x-y)} \, \frac{k_\mu k_\nu}{2k^3}}_{2}$$

Evaluating (1)

$$\int \frac{d^2k}{(2\pi)^2} e^{ik\cdot r} \frac{\delta_{\mu\nu}}{2|k|}$$

$$= \int \frac{\cancel{k}\, dk\, d\theta}{4\pi^2} \frac{e^{ikr\cos\theta}}{2\cancel{k}} \delta_{\mu\nu}$$

$$= \int_0^\infty \frac{dk}{8\pi^2} \delta_{\mu\nu} \int_0^{2\pi} d\theta\, e^{ikr\cos\theta}$$

$$= \int_0^\infty \frac{dk}{8\pi^2} \delta_{\mu\nu}\, 2\pi\, J_0(kr)$$

$$= \frac{\delta_{\mu\nu}}{4\pi} \int_0^\infty dk\, J_0(kr)$$

$$= \frac{\delta_{\mu\nu}}{4\pi r}$$

Evaluating 2,

$$\int \frac{d^2k}{(2\pi)^2}\, e^{ik\cdot r}\, \underbrace{\frac{k_\mu k_\nu}{2k^3}}_{f(k)} = f(r)$$

$$f(r) = A(r)\,\delta_{\mu\nu} + B(r)\,\frac{x_\mu x_\nu}{r^2}$$

$$= I_{\mu\nu}$$

$$\left\{ r = |\vec{x} - \vec{y}| \right.$$

$$x_\mu = (x_1, x_2)$$

$$\begin{cases} x_1 = r \\ x_2 = 0 \end{cases}$$

$$\vec{k} = (k\cos\theta,\; k\sin\theta)$$

$$I_{11} = \int \frac{\cancel{k}\,dk\,d\theta}{8\pi^2}\, e^{ikr\cos\theta}\, \frac{\cancel{k^2}\cos^2\theta}{\cancel{k^3}}$$

$$= \frac{1}{8\pi^2} \int_0^\infty dk \int_0^{2\pi} d\theta\, \cos^2\theta\, e^{ikr\cos\theta}$$

$$= \frac{1}{8\pi^2} \int_0^\infty dk \quad \pi \left[J_0(kr) + J_2(kr) \right]$$

$$= \frac{1}{8\pi} \int_0^\infty dk \left[J_0(kr) + J_2(kr) \right]$$

$$= \frac{1}{8\pi} \times \frac{2}{r}$$

$$= \frac{1}{4\pi r}$$

$$\left\{ \begin{array}{l} \displaystyle\int_0^\infty J_0(kr)\, dk = \frac{1}{r} \\ \displaystyle\int J_2(kr)\, dk = \frac{1}{r} \end{array} \right.$$

$$I_{22} = \int \frac{k\, dk\, d\theta}{(2\pi)^2} \; e^{ikr} \; \frac{k^2 \sin^2\theta}{2k^3}$$

$$= \int_0^\infty \frac{dk}{8\pi^2} \int_0^{2\pi} d\theta \; \underbrace{\sin^2\theta}_{1-\cos^2\theta} \; e^{ikr\cos\theta}$$

$$= \int_0^\infty \frac{dk}{8\pi^2} \left[2\pi J_0(kr) - \left(\pi J_0(kr) + \pi J_2(kr) \right) \right]$$

$$= \int_0^\infty \frac{dk}{8\pi^2} \, \pi \left(J_0(kr) - J_2(kr) \right)$$

$$= 0$$

$$I_{22} = A(r) = 0$$

$$I_{11} = A(r) + B(r) = B(r)$$

$$= \frac{1}{4\pi r}$$

$$\therefore \quad f(r) = \frac{1}{4\pi r} \frac{x_\mu x_\nu}{r^2}$$

$$\therefore \quad G_{\mu\nu}(x-y) = \frac{\delta_{\mu\nu}}{4\pi |x-y|} - \frac{1}{4\pi |x-y|} \frac{x_\mu x_\nu}{|x-y|^2}$$

$$\boxed{G_{\mu\nu}(x-y) = \left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{|x-y|^2} \right) \frac{1}{4\pi |x-y|}}$$

$$A_\mu(x) = e \int d^2y \; G_{\mu\nu}(x-y) \; j^\nu(y)$$

$$= e \int d^2y \left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{(x-y)^2} \right) \frac{j^\nu}{4\pi |x-y|}$$

$$= e \int d^2y \; \frac{j_\mu}{4\pi |x-y|}$$

$$- e \int d^2y \; \frac{x_\mu (j \cdot x)}{4\pi (x-y)^2 |x-y|}$$

$(x-y)^2 = (x-y)^\mu (x-y)_\mu$

$\mu, \nu \longrightarrow$ space indices

$$\boxed{A_\mu = e \int d^2y \left[j_\mu - \frac{x_\mu (j \cdot x)}{(x-y)^2} \right] \frac{1}{4\pi |x-y|}}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$A_\mu = (\phi, \vec{A})$$