LINEAR SYSTEMS CONTROL

Solutions to Problems

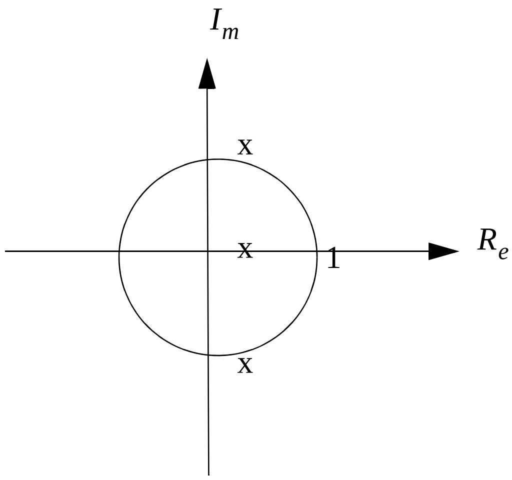
Problem 4.4

a. The characteristic equation for the system dynamic matrix is:

$$\det\begin{bmatrix}\lambda+2 & 3 & -5\\ 0.875 & \lambda-0.5 & -1\\ 1.875 & -0.5 & \lambda-3\end{bmatrix} = \lambda^3 - 0.5\lambda^2 + 1.75\lambda - 0.625 = 0$$

$$\Rightarrow \lambda = \begin{cases}0.5\\ 0.5 \pm j\end{cases}$$

The pole placement is thus as shown below.



There are two eigenvalues outside the unit circle $\Rightarrow$ the system is unstable. The controllability matrix is:

$$\mathbf{M}_c = [\mathbf{G} \quad \mathbf{F}\mathbf{G} \quad \mathbf{F}^2\mathbf{G}] = \begin{bmatrix}0 & 2 & 4\\ 1 & 0.5 & 1.5\\ 1 & 2.5 & 3\end{bmatrix}$$

The determinent of the controllability matrix is:

$$det(\mathbf{M}_c) = 1 \quad \Rightarrow \quad rank(\mathbf{M}_c) = 3 \quad \text{and the system is controllable}$$

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b. To convert the system into the controller canonical form first find:

$$P_{ch,\mathbf{F}} = \lambda^3 - 1.5\lambda^2 + 1.75\lambda - 0.625$$
$$\Rightarrow a_2 = -1.5, \qquad a_1 = 1.75, \qquad a_0 = -0.625$$

Using the method from sec. 3.9.1 in the book:

$$p_1 = \mathbf{G} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = \mathbf{F}p_1 + a_{n-1}p_1 = \begin{bmatrix} 2 \\ 1.5 \\ 2.5 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$p_3 = \mathbf{F}p_2 + a_{n-2}p_1 = \begin{bmatrix} 1 \\ -0.75 \\ -0.75 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.75 \\ 1.75 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{P} = [p_3 \;\; p_2 \;\; p_1] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad z = \mathbf{P}^{-1}x$$

$$\mathbf{F}_t = \mathbf{F}_{cc} = \mathbf{P}^{-1}\mathbf{F}\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.625 & -1.75 & 1.5 \end{bmatrix}$$

$$\mathbf{G}_t = \mathbf{G}_{cc} = \mathbf{P}^{-1}\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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c. The desired continuous eigenvalues:

$$\lambda_{cont} = \begin{cases} -2 \\ -2 \pm j \end{cases} \qquad \lambda_F = e^{T \cdot \lambda_A} \Leftrightarrow \lambda_A = \frac{1}{T}\ln\lambda_F$$

$$\lambda_{F_K} = \begin{cases} e^{0.2(-2 \pm j)} = e^{-0.4}(\cos 0.2 \pm j\sin 0.2 = 0.657 \pm j0.133 \\ e^{0.2(-2)} = 0.67 \end{cases}$$

$$\begin{aligned} P_{ch,\mathbf{F_K}}(\lambda) &= (\lambda - 0.67)(\lambda - 0.657 + j0.133)(\lambda - (0.657 + j0.133) \\ &= \lambda^3 - 1.984\lambda^2 + 1.33\lambda - 0.301 \end{aligned}$$

The closed loop system matrix (in controller canonical form):

$$\mathbf{F}_{Kt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.301 & -1.33 & 1.984 \end{bmatrix}$$

$$\mathbf{F}_{Kt} = \mathbf{F}_{cc} - \mathbf{G}_{cc}\mathbf{K}_t, \qquad \mathbf{K}_t = \begin{bmatrix} k_{1t} & k_{2t} & k_{3t} \end{bmatrix}$$

$$\mathbf{F}_{Kt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.625 - k_{1t} & -1.75 - k_{2t} & 1.5 - k_{3t} \end{bmatrix}$$

$$\Rightarrow \begin{cases} 0.301 = 0.625k_{1t} \\ -1.33 = -1.75 - k_{2t} \\ 1.984 = 1.5 - k_{3t} \end{cases} \Rightarrow \begin{cases} k_{1t} = 0.324 \\ k_{2t} = -0.42 \\ k_{3t} = -0.484 \end{cases}$$

$$\mathbf{K} = \mathbf{K}_t\mathbf{P}^{-1} = [0.808 \quad 1.552 \quad -2.036]$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.4

d.

$$P_{ch,\,\mathbf{F_K}}(\mathbf{F}) = \mathbf{F}^3 - 1.984\mathbf{F}^2 + 1.33\mathbf{F} - 0.301\mathbf{I}$$
$$= \begin{bmatrix} 2.495 & 0.292 & -3.068 \\ 0.6398 & -1.036 & 0.0035 \\ 1.483 & -1.666 & -0.5125 \end{bmatrix}$$

$$\mathbf{M}_c^{-1} = \begin{bmatrix} 0.75 & 4 & -3 \\ -1.5 & -4 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{M}_c^{-1} P_{ch,\,\mathbf{F_K}}(\mathbf{F}) = [0.808 \quad 1.552 \quad -2.036]$$

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