ELEKTRO • DTU AUTOMATION

Linear Systems Control

Solutions to Problems

Problem 2.9

a) Volume conservation law:

$$\left.\begin{aligned} A_1\dot{x}_1 &= ku_1 - q_1 \\ A_2\dot{x}_2 &= q_1 - q_0 + q_2 \\ A_3\dot{x}_3 &= -q_2 + ku_2 \end{aligned}\right\} \qquad (1)$$

Flow equations

$$\left.\begin{aligned} q_1 &= c_1\sqrt{x_1 - x_2} \\ q_2 &= c_2\sqrt{x_3 - x_2} \\ q_0 &= c_0\sqrt{x_2} \end{aligned}\right\} \qquad (2)$$

(2) is inserted into (1):

$$\dot{x}_1 = \frac{1}{A_1}(-c_1\sqrt{x_1 - x_2} + ku_1) = f_1(x, u)$$

$$\dot{x}_2 = \frac{1}{A_2}(c_1\sqrt{x_1 - x_2} + c_2\sqrt{x_3 - x_2} - c_0\sqrt{x_2}) = f_2(x, u)$$

$$\dot{x}_3 = \frac{1}{A_3}(ku_2 - c_2\sqrt{x_3 - x_2}) = f_3(x, u)$$

$$y = q_0 = c_0\sqrt{x_2} = g(x, u)$$

b) Stationary states:

$$\dot{x}_1 = 0 \quad ; \quad \dot{x}_2 = 0 \quad ; \quad \dot{x}_3 = 0$$

$$ku_{10} = c_1\sqrt{x_{10} - x_{20}} \qquad 3)$$

$$c_1\sqrt{x_{10} - x_{20}} + c_2\sqrt{x_{30} - x_{20}} = c_0\sqrt{x_{20}} \qquad 4)$$

$$ku_{20} = c_2\sqrt{x_{30} - x_{20}} \qquad 5)$$

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3) and 5) are inserted into 4):

$$ku_{10} + ku_{20} = c_0\sqrt{x_{20}}$$
$$\Rightarrow x_{20} = \left(\frac{k}{c_0}(u_{10} + u_{20})\right)^2$$

From 3) and 5) is obtained:

$$x_{10} = \left(\frac{k}{c_1}u_{10}\right)^2 + x_{20}$$
$$x_{30} = \left(\frac{k}{c_2}u_{20}\right)^2 + x_{20}$$

Output:

$$y_0 = c_0\sqrt{x_{20}}$$

We define the deviation variables:

$$x_1 = x_{10} + \Delta x_1 \qquad u_1 = u_{10} + \Delta u_1$$
$$x_2 = x_{20} + \Delta x_2 \qquad u_2 = u_{20} + \Delta u_2$$
$$x_3 = x_{30} + \Delta x_3 \qquad y = y_0 + \Delta y$$

$$A = \left\{\frac{\partial f_i}{\partial x_j}\right\}_0$$

$$= \begin{bmatrix} \frac{-c_1}{2A_1\sqrt{x_{10}-x_{20}}} & \frac{c_1}{2A_1\sqrt{x_{10}-x_{20}}} & 0 \\ \frac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} & -\frac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} - \frac{c_0}{2A_2\sqrt{x_{20}}} - \frac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} & \frac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} \\ 0 & \frac{c_2}{2A_3\sqrt{x_{30}-x_{20}}} & \frac{c_2}{2A_3\sqrt{x_{30}-x_{20}}} \end{bmatrix}$$

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$$B = \left\{ \frac{\partial f_i}{\partial u_j} \right\}_0 =$$

$$\begin{bmatrix} \frac{k}{A_1} & 0 \\ 0 & 0 \\ 0 & \frac{k}{A_3} \end{bmatrix}$$

$$C = \left\{ \frac{\partial g_i}{\partial x_j} \right\}_0 = \left\{ 0 \quad \frac{c_0}{2\sqrt{x_{20}}} \quad 0 \right\}$$

$$D = \left\{ \frac{\partial g_i}{\partial u_j} \right\}_0 = 0$$