LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.2

a. Given is the continuous system below:

$$\dot{x} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ b \end{bmatrix} u, \qquad y = [1 \;\; 0]x$$

with the stable eigenvalues:

$$\begin{vmatrix} \lambda+3 & -2 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda+3)(\lambda+2)-2 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = \begin{cases} -1 \\ -4 \end{cases}$$

The resolvent matrix is:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 3+s & -2 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+5s+4}\begin{bmatrix} s+2 & 2 \\ 1 & s+3 \end{bmatrix}$$

$$\Phi(s) = (s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+4)} & \dfrac{2}{(s+1)(s+4)} \\ \dfrac{1}{(s+1)(s+4)} & \dfrac{s+3}{(s+1)(s+4)} \end{bmatrix}$$

The transfer function is then given by:

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = [1 \;\; 0](s\mathbf{I}-\mathbf{A})^{-1}\begin{bmatrix} 1 \\ b \end{bmatrix} = \frac{s+2+2b}{(s+1)(s+4)}$$

The state transition matrix can be calculated via the Laplace transform:

$$\Phi(t) = L^{-1}\{\Phi(s)\} = L^{-1}\begin{bmatrix} \dfrac{\frac{1}{3}}{s+1} + \dfrac{\frac{2}{3}}{s+4} & \dfrac{\frac{2}{3}}{s+1} - \dfrac{\frac{2}{3}}{s+4} \\ \dfrac{\frac{1}{3}}{s+1} - \dfrac{\frac{1}{3}}{s+4} & \dfrac{\frac{2}{3}}{s+1} + \dfrac{\frac{1}{3}}{s+4} \end{bmatrix}$$

Problem 3.2 (continued)

Inverse Laplace transforming the expression above gives the transition matrix:

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} \end{bmatrix}$$

The impulse response can be found as:

First method:

$$y(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{2b+1}{3}\ \frac{1}{s+1} + \frac{2b-2}{-3}\ \frac{1}{s+4}\right\}$$

$$y(t) = \frac{2b+1}{3}e^{-t} + \frac{2-2b}{3}e^{-4t}$$

Second method:

$$y(t) = \mathbf{C}\Phi(t)\mathbf{B} = [1\ \ 0]\Phi(t)\begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$= \frac{1+2b}{3}e^{-t} + \frac{2-2b}{3}e^{-4t}$$

b. The eigenvalue farthest away from the origin is: $\lambda = -4$. For $b = -\frac{1}{2}$ the response will only contain the corresponding natural mode, e^{-4t}:

$$y(t) = e^{-4t}$$

c. Note that the system is asymptotically internally stable because both eigenvalues are in the open left half plane.

$$For \quad u(t) = \begin{cases} 0 & \text{for} \quad t < 0) \\ 2e^{t} & \text{for} \quad t \geq 0) \end{cases}$$ the response is:

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Problem 3.2 (continued)

$$y(t) = \mathbf{C}\, e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{A(t-\tau)}\mathbf{B}\;\mathbf{u}(\tau)d\tau = y_1 + y_2$$

x_0 does not influence $\lim_{t\to\infty} y(t)$:

$$\mathbf{C}\, e^{\mathbf{A}(t-\tau)}\mathbf{B}\;\mathbf{u}(\tau) = [1\ \ 0]e^{A(t-\tau)}\begin{bmatrix} 2e^{\tau} \\ 2be^{\tau} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}e^{-(t-\tau)} + \frac{2}{3}e^{-4(t-\tau)} & \frac{2}{3}e^{-(t-\tau)} - \frac{2}{3}e^{-4(t-\tau)} \end{bmatrix}\begin{bmatrix} 2e^{\tau} \\ 2be^{t} \end{bmatrix}$$

$$= \frac{2+4b}{3}e^{-t+2\tau} + \frac{4-4b}{3}e^{-4t+5\tau}$$

$$\Rightarrow y_2 = \int_0^t(\ldots\ldots)d\tau = -\frac{2+4b}{6}e^{-t} - \frac{4-4b}{15}e^{-4t} + \frac{2+4b}{6}e^{t} + \frac{4-4b}{15}e^{t}$$

$$\lim_{t\to\infty}(y(t)) = 0 \quad for$$

$$\frac{2+4b}{6} + \frac{4-4b}{15} = 0 \Rightarrow b = -\frac{3}{2}$$

For $b = -\frac{3}{2}$ one obtains:

$$y_2 = \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

$$y_1 = \mathbf{C}\, e^{\mathbf{A}t}x_0 = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}$$

$$y = y_1 + y_2 = \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t}$$

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