LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

a. The state equations for the ballon must be written so that they only involve the given input of the system and its state variables.

The first state equation for the ballon is

$$\dot{T} = -a(T - T_a) + cq$$

The second state equation is

$$M\frac{dV}{dt} = Vg(\rho_a - \rho) - Mg - bv$$

But it is given that $\rho T = \rho_a T_a \Rightarrow \rho_a = \frac{\rho T}{T_a}$. Thus the second state equation becomes

$$\begin{aligned} \dot{v} &= \frac{Vg}{M}\left(\frac{\rho T}{T_a} - \rho\right) - g - \frac{b}{M}v \\ &= \frac{Vg}{M}\frac{\rho}{T_a}(T - T_a) - g - \frac{b}{M}v \end{aligned}$$

b. In the stationary state

$$0 = -a(T - T_a) + cq$$

which implies that

$$T_0 = \frac{c}{a}q_0 + T_a$$

In the second state equation in the steady state $\dot{v} = 0$ and $v = 0$ thus

$$\frac{\rho_0}{T_a}(T_0 - T_a) - \frac{M}{V} = 0$$

Using the expression for T_0 above in this equation

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$$\rho_0 = \frac{T_a}{\rho_0}\frac{a}{C}\frac{M}{V}$$

also

$$\frac{\rho_0 T_0}{T_a} - \rho_0 = \frac{\rho_a T_a}{T_a} - \rho_0 = \frac{M}{V}$$

$$\rho_0 = \rho_a - \frac{M}{V}$$

Finally

$$q_0 = \frac{T_a}{\rho_0}\frac{a}{c}\frac{M}{V} = \frac{a}{c}\frac{T_a}{\rho_a - \frac{M}{V}}\frac{M}{V}$$

$$= \frac{a}{c} T_a \frac{\frac{M}{\rho_a V}}{1 - \frac{M}{\rho_a V}}$$

Notice that the term $r = \frac{M}{\rho_a V}$ is the ratio of the effective density of the air in the ballon to the ambient density.

c. Linearization can be carried out using differentiation or power series development. One finds

$$\dot{\Delta T} = -\frac{1}{\tau}\Delta T + c\Delta q + \frac{1}{\tau}\Delta T_a \quad , \quad \frac{1}{\tau} = a$$

$$\dot{\Delta v} = -\frac{b}{M}\Delta v + \frac{Vg}{M}\rho_a\frac{T_a}{T_0^2}\Delta T - \frac{Vg}{M}\frac{\rho_0 T_0}{T_a^2}\Delta T_a \quad \text{or}$$

$$\begin{pmatrix} \dot{\Delta T} \\ \dot{\Delta V} \end{pmatrix} = \begin{pmatrix} -a & 0 \\ \frac{Vg}{M}\frac{\rho_a T_a}{T_0^2} & -\frac{b}{M} \end{pmatrix}\begin{pmatrix} \Delta T \\ \Delta V \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & \frac{Vg}{M}\frac{\rho_0 T_0}{T_a} \end{pmatrix}\begin{pmatrix} \Delta q \\ \Delta T_a \end{pmatrix}$$

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d. This system is in itself stable as both of its eigenfrequencies are in the left half plan,

$$\lambda = -a \quad , \quad \lambda = -\frac{b}{m} \ .$$

In order to find out if the system is controllable, the controllability matrix has to be found.

Let $d = \dfrac{Vg\rho_0 T_0}{M}$

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} c & a & -ac & a^2 \\ 0 & -\dfrac{d}{T_a^2} & \dfrac{cd}{T_0^2} & \dfrac{cd}{T_0^2} + \dfrac{bd}{MT_0^2} \end{bmatrix}$$

The matrix has full rank so that the underlying system is controllable.

e. The state equation of the simplified system is:

$$\begin{aligned} \Delta T &= A\Delta T + B\Delta q + B_v \Delta T_a \\ &= -a\Delta T + c\Delta q + a\Delta T \quad \Rightarrow \\ A &= -a, \quad B = c, \quad B_v = a \end{aligned}$$

f. The optimization index

$$J = \int_0^a (\Delta T^2 + Q\Delta q^2) dt$$

The corresponding Riccati equation is

$$\mathbf{PA} + \mathbf{PA}^T + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B}^\mathbf{T}\mathbf{P} = 0$$

$$-2ap + 1 - \frac{c^2}{Q} p^2 = 0$$

$$p^2 + \frac{2aQ}{c^2} p - \frac{Q}{c^2} = 0$$

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$$p = -\frac{aQ}{c} + \sqrt{\frac{a^2Q^2}{c^4} + \frac{Q}{c^2}} = \frac{aQ}{c}\left(\sqrt{1 + \frac{c^2}{a^2Q}} - 1\right),$$

where the positive definite solution has been selected.

The state feedback is then

$$\Delta q = -K\Delta T \quad \text{where} \quad k = Q^{-1}B^T p$$

Thus

$$k = \frac{a}{c}\left(\sqrt{1 + \frac{c^2}{a^2Q}} - 1\right)$$

g. The system to be modelled is

$$\dot{\Delta T} = -a\Delta T + c\Delta q + a\Delta T_a$$

where $\Delta T_a \in N(0,V)$.

The measurement model is

$$y = \Delta T + w$$

where $E\{\Delta T_a^2\} \in N(0, V)$, $w \in N(0,W)$ and ΔT_a and w are independent: $E\{\Delta T \cdot w\} = 0$.

The Riccati equation for the system is:

$$-aq - aq + V + qW^{-1}q = 0$$

$$q^2 + 2aWq + VW = 0 \quad \text{or}$$

$$q = -aW + \sqrt{a^2W^2 + VW} = aW\left(\sqrt{1 + \frac{V}{a^2W}} + 1\right)$$

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The Kalman gain is then

$$L = qW^{-1} = aW\left(\sqrt{1+\frac{V}{a^2W}}+1\right)\frac{1}{W}$$

$$= a\left(\sqrt{1+\frac{V}{a^2W}}-1\right)$$

The corresponding Kalman filter is

$$\dot{\hat{\Delta T}} = -a\hat{\Delta T} + c\Delta q + L(\Delta T_{meas} - \hat{\Delta T})$$

where ΔT_{meas} is the temperature measured with respect to the linearization point.

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