LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.10

This problem concerns a D.C. motor which is to be used as an electronic throttle control for a spark ignition engine. The control object is basically a D.C. motor which has an increased moment of inertia due to the throttle plate (see example 2.3).

a. The system matrices are (given that position control is desired):

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{R}_2 = r_2$$

The Riccati equation for this system is

$$0 = \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B}^T\mathbf{P}$$

$$0 = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 \\ \beta \end{pmatrix}\frac{1}{r_2}\begin{pmatrix} 0 & \beta \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$0 = 1 - \frac{\beta^2}{r_2} p_{12}^2$$

$$\Rightarrow \qquad 0 = p_{11} - \alpha\, p_{12} - \frac{\beta^2}{r_2} p_{12} p_{22}$$

$$0 = 2(p_{12} - \alpha p_{22}) - \frac{\beta^2}{r_2} p_{22}^2$$

These equations can be solved simultaneously to find that:

$$p_{12} = \frac{\sqrt{r_2}}{\beta}$$

$$p_{22} = \frac{r_2}{\beta^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}\right)$$

$$p_{11} = \frac{\sqrt{r_2}}{\beta}\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}$$

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The LQR gain is then given by

$$\mathbf{K} = \mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P} = -\frac{1}{r_2}\begin{pmatrix} 0 & \beta \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$= -\frac{\beta}{r_2}\begin{pmatrix} p_{12} & p_{22} \end{pmatrix}$$

$$= -\left[\frac{1}{\sqrt{r_2}} \quad \frac{1}{\beta}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}\right)\right]$$

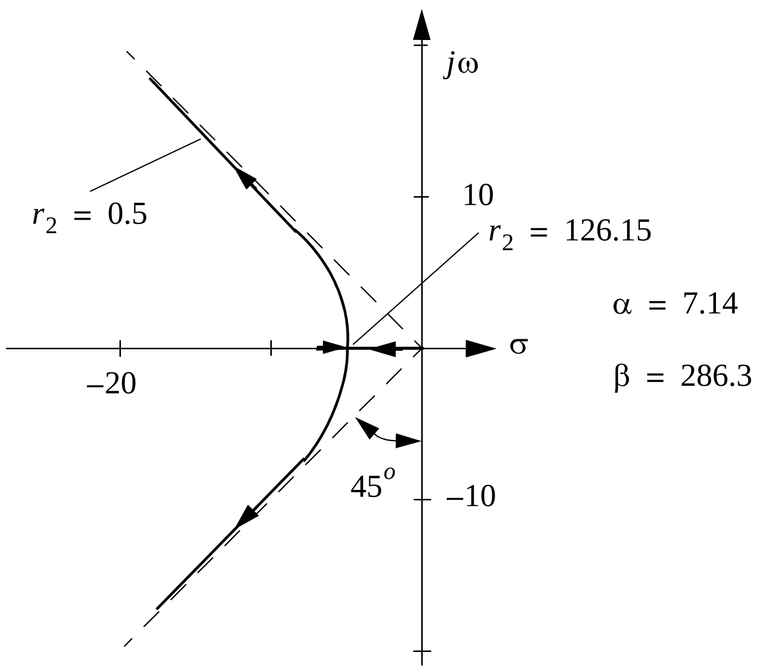
b. The characteristic equation for the system is

$$\det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] = s^2 + s\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}} + \frac{\beta}{\sqrt{r_2}} = 0$$

which has the solutions:

$$s_0 = -\frac{1}{2}\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}} \pm \frac{1}{2}\sqrt{\alpha^2 - \frac{2\beta}{\sqrt{r_2}}}.$$

The plot below shows the root curve as a function of the weight paramter r_2.



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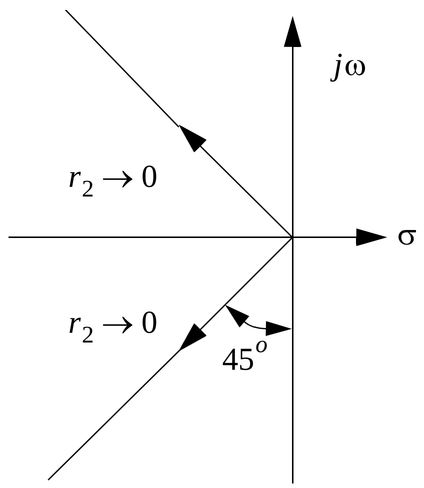
The asymptotes for the root curve are shown dashed on the figure: they correspond to the order of the system which is 2. Using the parameter values, $\alpha = 7.14$, $\beta = 286.3$, and the input weighting, $r_2 = 0.5$, the specified system response time (time constant) of 50 msec can be achieved with the feedback matrix:

$$\mathbf{K} = [1.414 \quad 0.0775].$$

The time constant is reasonable for a real automotive throttle control.

The branch point is at $s_{bp} = -5.049$. Notice that as $r_2 \to 0$ more and more power is used to control the system as it becomes faster and faster.

c. If the motor damping, α, is zero then the root curve look like the figure below



d. The value of r_2 which gives a response time (time constant) of 50 msec is 0.5. The system response to a given set of initial conditions can be calculated by finding the systems transition matrix.

$$\bar{x}(t) = \bar{\Phi}(t, 0)\,\bar{x}(0) \qquad \text{since} \quad u(t) = 0$$

Problem 5.10

$$\mathbf{A_K} = \mathbf{A} - \mathbf{BK} = \begin{pmatrix} 0 & 1 \\ -404.8 & -29.33 \end{pmatrix}$$

$$\Rightarrow [s\mathbf{I} - \mathbf{A_K}]^{-1} = \begin{pmatrix} \dfrac{s+29.33}{\Delta} & \dfrac{1}{\Delta} \\ \dfrac{-404.8}{\Delta} & \dfrac{s}{\Delta} \end{pmatrix},$$

$$\Delta = s^2 + 29.33s + 404.8$$

The roots of the characteristic equation are $s_0 = -14.67 \pm j13.77$.

Note now that:

$$L^{-1}\left\{\frac{1}{(s+a)^2 + b^2}\right\} = \frac{e^{-at}\sin bt}{b}$$

$$L^{-1}\left\{\frac{s+a}{(s+a)^2 + b^2}\right\} = e^{-at}\cos bt$$

$$s^2 + 2sa + a^2 + b^2 : = s^2 + 29.33s + 404.8$$

$$\begin{cases} \Rightarrow 2a = 29.33 \Rightarrow a = 14.665 \\ \Rightarrow a^2 + b^2 = 404.8 \quad \Rightarrow b = 13.775 \end{cases}$$

$$\Phi(t,0) = L^{-1}[s\mathbf{I} - \mathbf{A_K}]^{-1} = \begin{pmatrix} \Phi_{11}(t,0) & \Phi_{12}(t,0) \\ \Phi_{21}(t,0) & \Phi_{22}(t,0) \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x(t) = \Phi(t,0)x(0) = 0.3\begin{bmatrix} \Phi_{11}(t,0) \\ \Phi_{21}(t,0) \end{bmatrix}$$

Thus

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$$x(t) = 0.3L^{-1}\begin{bmatrix} \dfrac{s+29.33}{(s+14.665)^2+13.775^2} \\ \dfrac{-404.8}{(s+14.665)^2+13.775^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.3e^{-14.665t}\cos(1.89.74t) \\ -0.64e^{-14.665t}\sin(189.74t) \end{bmatrix}$$

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