LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.1

a. The time varying differential equation in the problem has two states and for an input, $u(t) = 0$, the transition matrix can be written:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi(t, t_0)x_0 = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \text{ or}$$

$$\begin{cases} x_1(t) = \Phi_{11}(t, t_0)x_{10} + \Phi_{12}(t, t_0)x_{20} \\ x_2(t) = \Phi_{21}(t, t_0)x_{10} + \Phi_{22}(t, t_0)x_{20} \end{cases} \qquad (1)$$

From section 3.1.2 one has:

$$\begin{cases} \dot{x}_1(t) = t\, x_2(t) \\ \dot{x}_2(t) = 0 \end{cases}$$

$$\begin{cases} x_2(t) = x_{20} \\ x_1(t) = \int_{t_0}^{t} \tau x_{20} d\tau + x_{10} = \frac{1}{2}\tau^2 x_{20}\Big|_{t_0}^{t} + x_{10} \end{cases}$$

$$= \frac{1}{2}t^2 x_{20} - \frac{1}{2}t_0^2 x_{20} + x_{10}$$

From (1) one obtains:

$$\Phi_{21} = 0, \quad \Phi_{22} = 1$$

$$\Phi_{11} = 1, \quad \Phi_{12} = \frac{1}{2}(t^2 - t_0^2)$$

$$\Rightarrow \Phi(t, t_0) = \begin{bmatrix} 1 & \frac{1}{2}(t^2 - t_0^2) \\ 0 & 1 \end{bmatrix}$$

b. From equation (3.21):

$$\frac{\partial \Phi(t, t_0)}{\partial t} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

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which leads to:

$$\mathbf{A} \cdot \Phi(t, t_0) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \Phi(t, t_0) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

c. The zero state solution is:

$$\mathbf{x}_{zs}(t) = \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)u(\tau)d\tau$$

$$= \int_{t_0}^{t} \begin{bmatrix} 1 & \frac{1}{2}(\tau^2 - \tau^2) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} \cdot 1 \cdot d\tau = \int_{t_0}^{t} \begin{bmatrix} \frac{1}{2}\left(\frac{t^2}{\tau} - \tau\right) \\ \frac{1}{\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \frac{t^2}{2}\ln\tau - \frac{1}{4}\tau^2 \\ \ln\tau \end{bmatrix}\Bigg|_{t_0}^{t} = \begin{bmatrix} \frac{t^2}{2}\ln\tau - \frac{1}{4}t^2 - \frac{t^2}{2}\ln\tau_0 + \frac{1}{4}t_0^2 \\ \ln\tau - \ln t_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(t_0^2 - t^2) + \frac{t^2}{2}\ln\frac{t}{t_0} \\ \ln\frac{t}{t_0} \end{bmatrix}$$

The overall solution is then:

$$\mathbf{x}(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

$$= \begin{bmatrix} x_{10} + \left(\frac{x_{20}}{2} - \frac{1}{4}\right)(t^2 - t_0^2) + \frac{t^2}{2}\ln\frac{t}{t_0} \\ x_{20} + \ln\frac{t}{t_0} \end{bmatrix} \qquad (2)$$

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Solutions to Problems

Problem 3.1

d. From (2) one finds:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \left(\frac{x_{20}}{2} - \frac{1}{4}\right)2t + \frac{t^2}{2}\frac{1}{t} + t \ \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix} = \begin{bmatrix} tx_{20} + t \ \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix}$$

$$\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\cdot\mathbf{u}(t) = \begin{bmatrix} tx_{20} + t \ \ln\frac{t}{t_0} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix} = \begin{bmatrix} tx_{20} + t \ \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix}$$

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