LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.5

The differential equations desired are found with direct differentiation of the definitions using the Leibniz's integral differentiation formula:

$$\frac{dv_{avg}}{dt} = \frac{d}{dt}\left(\frac{1}{t}\int_0^t v\ d\tau\right) = -\frac{1}{t^2}\int_0^t v\ d\tau + \frac{1}{t}\ v$$

$$= \frac{1}{t}\left(v - \frac{1}{t}\int_0^t v\ d\tau\right)$$

$$\Rightarrow \frac{dv_{avg}}{dt} = \frac{1}{t}(v - v_{avg}), \qquad t > 0$$

$$\frac{dv_{rms}}{dt} = \frac{d}{dt}\left(\sqrt{\frac{1}{t}\int_0^t v^2\ d\tau}\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}\int_0^t v^2\ d\tau\right)^{-\frac{1}{2}}\left[-\frac{1}{t^2}\int_0^t v^2\ d\tau + \frac{1}{t}\ v^2\right]$$

$$= \frac{1}{2t}\frac{\left[v^2 - \frac{1}{t}\int_0^t v^2\ d\tau\right]}{\left(\frac{1}{t}\int_0^t v^2\ d\tau\right)^{\frac{1}{2}}} = \frac{1}{2t}\frac{(v^2 - v_{rms}^2)}{v_{rms}}$$

$$\frac{dv_{rms}}{dt} = \frac{1}{2t}\left(\frac{v^2}{v_{rms}} - v_{rms}\right), \quad t > 0,\ v_{rms} > 0$$

a. If one uses these equations with a stationary input signal, one will for large times find an approximate (iterative) answer for its average and root mean square values over long integration times.

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b. The equations can be used for a practical determination of the mean and RMS values of the input signal if the averaging time is very large compared to period of signal. Practically one could use a constant integration time instead of an (effectively) infinite time.

c. The answer to b. is yes.

The advantage of using the differential equations is that they give an answer which is as exact as possible given the definition of average and RMS value.

The main disadvantage is that the answers found are dependent on the averaging times used and that the signals measured must be stationary. This is the natural result of the form of the definitions themselves.

d. The differential equations above may be used on a random or noise signal with proper choice of the integration algorithm and the sample time used. The integration time should be 20 - 100 times the noise generator sample time.

e. The differential equations can be linearized immediately with differentiation:

$$\mathbf{A} = \nabla_x f\Big|_{\substack{x = x_n \\ u = u_n}} = \begin{bmatrix} -\frac{1}{t} & 0 \\ 0 & -\frac{1}{2t}\left(1 + \frac{v_0^2}{v_{orms}^2}\right) \end{bmatrix}$$

$$\mathbf{B} = \nabla_u f\Big|_{\substack{x = x_n \\ u = u_n}} = \begin{bmatrix} \frac{1}{t} \\ \frac{v_0}{t\, v_{orms}} \end{bmatrix}$$

e. The equations above are decoupled because the definition equations for v_{avg} and v_{rms} are. The eigen frequencies for the linearized differential equations are negative and thus they represent stable filters for $t > 0$. As the eigenfrequencies are proportional with $1/t$ the filters' bandwidths become smaller and smaller as time increases. This filters out high fre-

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quency changes in the input signals. For large times the filters are very insensitive to their inputs.

f. The differential equations thus have the same limitiation as analog instruments: an increasingly accurate exact answer can be found for longer integration times assuming a stationary input signal.

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