Problem 3.3

Given the system dynamical matrix

$$\mathbf{A} = \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix}$$

the resolvent matrix $\Phi(s)$ is given by

$$\begin{aligned}
\Phi(s) &= (s\mathbf{I} - \mathbf{A})^{-1} \\
&= \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{2(s+4)(s+1)} & \frac{4}{(s+4)(s+3)(s+1)} \\ 0 & \frac{1}{s+1} & \frac{8}{(s+3)(s+1)} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}
\end{aligned}$$

The state transition matrix $\phi(t)$ can be computed as the inverse Laplace transform of the resolvent matrix $\Phi(s)$ or directly as the exponential matrix of the system dynamical matrix $\mathbf{A}$. Applying the inverse Laplace transform we obtain

$$\begin{aligned}
\phi(t) &= \mathcal{L}^{-1}\{\Phi(s)\} \\
&= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}
\end{aligned}$$

Calculating the exponential matrix we obtain

$$\begin{aligned}
\phi(t) &= e^{\mathbf{A}t} \\
&= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}
\end{aligned}$$

As expected the two approaches produce the same result.

The zero input solution of the associated 3^{rd} order LTI system with initial condition $\mathbf{x}_0 = [0, 1, 0]^{\mathrm{T}}$ is given by

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 \\
&= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} \\ e^{-t} \\ 0 \end{bmatrix}
\end{aligned}$$

The solution shows that the effect of the non-zero initial condition x_{20} naturally produces a dynamic response in the state variable $x_2(t)$ but also in the state variable $x_1(t)$. However the state variable $x_3(t)$ is not affected at all. This can be explained by looking at the structure of the system dynamical matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix}$$

which shows that the dynamics of $x_1(t)$ is coupled with the dynamics of $x_2(t)$ through the coefficient $a_{12} = 1/2$, but the dynamics of $x_3(t)$ is totally uncoupled from the dynamics of the other two state variables since $a_{31} = a_{32} = 0$.