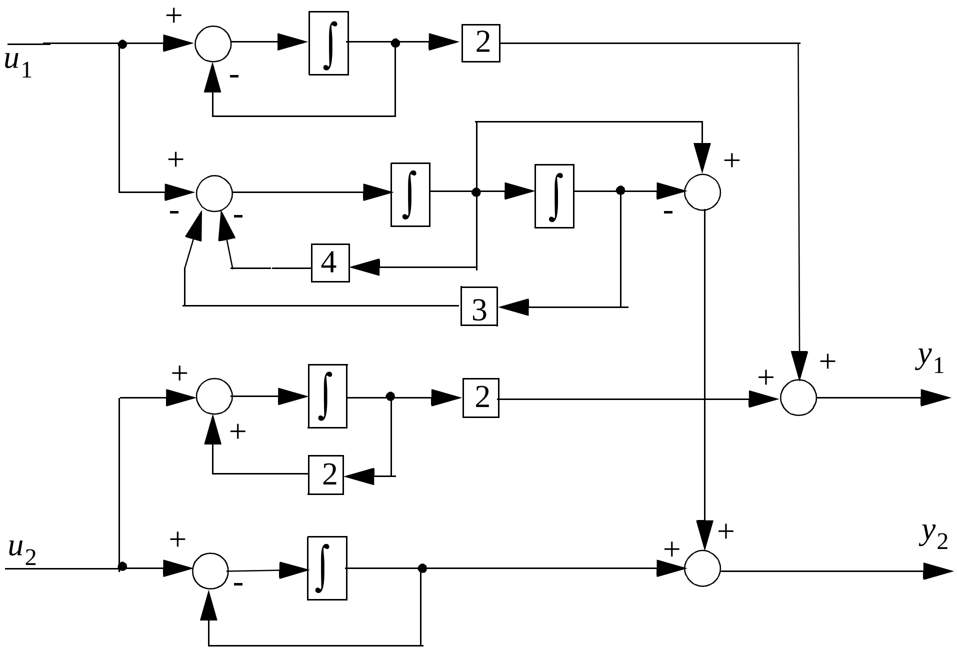
ØRSTED • DTU AUTOMATION

Linear Systems Control

Solutions to problems

Problem 3.16

a)



From this block diagram we see that:

$$\frac{q_1}{u_1} = \frac{2}{s+1} \quad , \quad \frac{q_2}{u_1} = \frac{s-1}{s^2+4s+3}$$

$$\frac{q_3}{u_2} = \frac{2}{s-2} \quad , \quad \frac{q_4}{u_2} = \frac{1}{s+1}$$

$$y_1 = q_1 + q_3 = \frac{2}{s+1}u_1 + \frac{2}{s-2}u_2$$

$$y_2 = q_2 + q_4 = \frac{s-1}{(s+1)(s+3)}u_1 + \frac{1}{s+1}u_2$$

$$\Rightarrow \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} & \frac{2}{s-2} \\ \frac{s-1}{(s+1)(s+3)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

ØRSTED • DTU AUTOMATION

Linear Systems Control

Solutions to problems

Problem 3.16

b)

$$\begin{aligned}\dot{x}_1 &= -x_1 + u_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= 3x_2 - 4x_3 + u_1 \\ \dot{x}_4 &= 2x_4 + u_2 \\ \dot{x}_5 &= -x_5 + u_2\end{aligned} \qquad \begin{aligned}y_1 &= 2x_1 + 2x_4 \\ y_2 &= -x_2 + x_3 + x_5\end{aligned}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} x$$

or:
$$\begin{aligned}\dot{x} &= Ax + Bu \\ b &= Cx\end{aligned}$$

The state space model above is minimal if it is controllable as well as observable.

$$M_c = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 \\ 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 & 121 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 16 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Problem 3.16

$$M_o = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 4 & 0 \\ 0 & -3 & -5 & 0 & -1 \\ 2 & 0 & 0 & 8 & 6 \\ 0 & 15 & 17 & 0 & 1 \\ -2 & 0 & 0 & 16 & 0 \\ 0 & -51 & -53 & 0 & -1 \\ 2 & 0 & 0 & 32 & 0 \\ 0 & 159 & 161 & 0 & 1 \end{bmatrix}$$

Using Matlab it is easy to see that both matrices have rank 4. So the system is neither controllable nor observable. Therefore it is not minimal either.

d) Denominator polynomium containing all poles of the transfer function matrix:

$$d(s) = (s+1)(s+3)(s-2) = s^3 + 2s^2 - 5s - 6$$

Then the system matrix (3.384) can be found directly

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 5 & 0 & -2 & 0 \\ 0 & 6 & 0 & 5 & 0 & -2 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 3.16

$$d(s)G(s) = \begin{bmatrix} 2(s+3)(s-2) & 2(s+1)(s+3) \\ (s-1)(s-2) & (s+3)(s-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2s^2+2s-12 & 2s^2+8s+6 \\ s^2-3s+2 & s^2+s-6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 2 & 8 \\ -3 & 1 \end{bmatrix} s + \begin{bmatrix} -12 & 6 \\ 2 & -6 \end{bmatrix}$$

$$\Rightarrow C_1 = \begin{bmatrix} -12 & 6 & 2 & 8 & 2 & 2 \\ 2 & -6 & -3 & 1 & 1 & 1 \end{bmatrix}$$

e) The fifth order system $[A, \quad B, \quad C]$ is not minimal so the sixth order system $[A_1, \quad B_1, \quad C_1]$ can obviously not be minimal either.

f) First we must find 4 linearly independent columns in M_c. We choose the first 4 columns. The transformation matrix Q is composed of these 4 columns and a fifth one chosen such that Q becomes regular. We can choose for instance:

$$Q = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \qquad \det(Q) = -3$$

and

$$Q^{-1} = \begin{bmatrix} 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & -3 & -1 & 0 & 0 \end{bmatrix}$$

Now Matlab provide us with the result:

Problem 3.16

$$A_t = Q^{-1}AQ = \left[\begin{array}{cccc|c} 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right] \quad ; \quad B_t = Q^{-1}B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$C_t = CQ = \left[\begin{array}{cccc|c} 2 & 2 & -2 & 4 & 2 \\ 1 & 1 & -5 & -1 & 0 \end{array}\right]$$

Matrix partition:

$$A_t = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline 0 & A_{nc} \end{array}\right] \quad , \quad B_t = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right] \quad , \quad C_t[C_c | C_{nc}]$$

The controllable subsystem becomes

$$A_c = \begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad , \quad B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 2 & 2 & -2 & 4 \\ 1 & 1 & -5 & -1 \end{bmatrix}$$

This system is of course of fourth order because the original system [A, B, C] has a controllability matrix of rank 4. The system $[A_c, B_c, C_c]$ is not only controllable but also observable, and consequently it is minimal.

A has the eigenvalues:
$$\lambda_A = \begin{cases} -3 \\ -1 \\ -1 \\ -1 \\ 2 \end{cases}$$

$$A_c \text{ has } \lambda_{A_c} = \begin{cases} -3 \\ -1 \\ -1 \\ 2 \end{cases}$$

Problem 3.16

The system natrix A_{nc} of the non-controllable part of the system has the eigenvalue $\lambda = -1$. Since this eigenvalue is asymptotically stable, the system [A, B, C] is stabilizable.

The system $[A_c, B_c, C_c]$ is observable, so the fifth (stable) eigenvalue must belong to the non-observable subspace, and the system [A, B, C] is detectable.