LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.1

a. Here the positions and velocities are selected as the states:

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The state space model is then:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \dfrac{\frac{3g}{4L}(M+m)\sin x_1 - \frac{3m}{4}\sin x_1 \cos x_1 \cdot x_2^2 - \frac{3}{4L}\cos x_1 \cdot u}{M + m - \frac{3}{4}m\cos^2 x_1} \\ x_4 \\ \dfrac{mL\sin x_1 \cdot x_2^2 - \frac{3}{4}mg\sin x_1 \cos x_1 + u}{M + m - \frac{3}{4}m\cos^2 x_1} \end{bmatrix}.$$

$$y = x_3$$

b. The stationary state is selected as $\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ and it is assumed that:

$$\sin\theta \cong \theta,\ \cos\theta \cong 1 \text{ and } \dot{\theta}^2 \cong 0.$$

This leads to

$$\dot{x}_2 = \ddot{\theta} \cong \frac{\frac{3g}{4L}(M+m)\theta - \frac{3}{4L}u}{M + \frac{m}{4}},$$

$$\dot{x}_4 = \ddot{p} \cong \frac{-\frac{3}{4}mg\theta + u}{M + \frac{m}{4}}.$$

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The linearized state equation then becomes:

$$\dot{\Delta \mathbf{x}} = 0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{3g(M+m)}{\left(M+\dfrac{m}{4}\right)4L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-\dfrac{3}{4}mg}{M+\dfrac{m}{4}} & 0 & 0 & 0 \end{bmatrix} \Delta \mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4L}}{M+\dfrac{m}{4}} \\ 0 \\ \dfrac{1}{M+\dfrac{m}{4}} \end{bmatrix} \Delta u$$

$$\Delta y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \Delta \mathbf{x}$$

Note that the incremental variables (the Δ-variables) are the same as the absolute (large signal) variables because $\mathbf{x}_0 = \mathbf{0}$.

c. Defining

$$\Delta \mathbf{x} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \text{where } \mathbf{z}_1 = \begin{bmatrix} \Delta\theta \\ \dot{\Delta\theta} \end{bmatrix} \text{ and } \mathbf{z}_2 = \begin{bmatrix} \Delta p \\ \dot{\Delta p} \end{bmatrix},$$

one obtains:

$$\mathbf{z}_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{3g(M+m)}{\left(M+\dfrac{m}{4}\right)4L} & 0 \end{bmatrix} \mathbf{z}_1 + \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4L}}{M+\dfrac{m}{4}} \end{bmatrix} \Delta u .$$

Note that the angle system with state vector $\mathbf{z}_1$ is independent of the position system with state vector $\mathbf{z}_2$. The opposite is not true.

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Therefore the model for $\mathbf{z}_2$ can only be considered a valid state model if θ is assumed to be a disturbance:

$$\dot{\mathbf{z}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_2 + \begin{bmatrix} 0 \\ \dfrac{1}{M + \dfrac{m}{4}} \end{bmatrix} \Delta u + \begin{bmatrix} 0 & 0 \\ \dfrac{-\dfrac{3}{4} mg}{M + \dfrac{m}{4}} & 0 \end{bmatrix} \mathbf{z}_1$$

or

$$\dot{\mathbf{z}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_2 + \begin{bmatrix} 0 \\ \dfrac{1}{M + \dfrac{m}{4}} \end{bmatrix} \Delta u + \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4} mg}{M + \dfrac{m}{4}} \end{bmatrix} \Delta\theta ,$$

where the disturbance is $v = \Delta\theta$ and

$$\mathbf{B}_v = \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4} mg}{M + \dfrac{m}{4}} \end{bmatrix}$$

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