LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.4

a. The resolvent of the discrete time system is:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k) = \mathbf{F}\ \mathbf{x}(k) + \mathbf{G}\ u(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}u(k), \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad u(k) = 1$$

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -\frac{1}{8}x_1(k) + \frac{3}{4}x_2(k) + u(k)$$

$$y(k) = -\frac{1}{8}x_1(k) - \frac{1}{4}x_2(k)$$

$$\underline{y(0) = 0}$$

$$x_1(1) = 0, \qquad x_2(1) = 1, \qquad \underline{y(1) = -\frac{1}{4}}$$

$$x_1(2) = 1, \qquad x_2(2) = 0 + \frac{3}{4} + 1 = \frac{7}{4},$$

$$y(2) = -\frac{1}{8}\cdot 1 - \frac{1}{4}\cdot\frac{7}{4} = \underline{-\frac{9}{16}}$$

b. The eigenvalues of the system can be computed as:

$$\begin{vmatrix} \lambda & -1 \\ \frac{1}{8} & \lambda - \frac{3}{4} \end{vmatrix} = \lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = 0 \Rightarrow \lambda = \begin{cases} \frac{1}{2} \\ \frac{1}{4} \end{cases}$$

The natural modes are then:

$$m_i = \begin{cases} \left(\frac{1}{2}\right)^k \\ \left(\frac{1}{4}\right)^k \end{cases}$$

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The discrete resolvent matrix is:

$$\Psi(z) = (zI - F)^{-1}z = \begin{bmatrix} z & -1 \\ \frac{1}{8} & z - \frac{3}{4} \end{bmatrix}^{-1} z$$

$$\Rightarrow \Psi(z) = \begin{bmatrix} z\dfrac{z - \frac{3}{4}}{z^2 - \frac{3}{4}z + \frac{1}{8}} & z\dfrac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ z\dfrac{-\frac{1}{8}}{z^2 - \frac{3}{4}z + \frac{1}{8}} & z\dfrac{z}{z^2 - \frac{3}{4}z + \frac{1}{8}} \end{bmatrix}$$

c.

$$F^k = Z^{-1}\{z(zI - F)^{-1}\} = Z^{-1}\begin{bmatrix} z\left(-\dfrac{1}{z-\frac{1}{2}} + \dfrac{2}{z-\frac{1}{4}}\right) & z\left(\dfrac{4}{z-\frac{1}{2}} - \dfrac{4}{z-\frac{1}{4}}\right) \\ z\left(-\dfrac{\frac{1}{2}}{z-\frac{1}{2}} + \dfrac{\frac{1}{2}}{z-\frac{1}{4}}\right) & z\left(\dfrac{2}{z-\frac{1}{2}} - \dfrac{1}{z-\frac{1}{4}}\right) \end{bmatrix}$$

Note that: $\quad Z\{a^k\} = \dfrac{z}{z-a}$

$$\Rightarrow F^k = \begin{bmatrix} 2 \cdot \frac{1}{4}^k - \frac{1}{2}^k & 4 \cdot \frac{1}{2}^k - 4 \cdot \frac{1}{4}^k \\ \frac{1}{2} \cdot \frac{1}{4}^k - \frac{1}{2} \cdot \frac{1}{2}^k & 2 \cdot \frac{1}{2}^k - \frac{1}{4}^k \end{bmatrix}$$

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d. The transfer fuction of the system can be determined from:

$$H(z) = C(zI - F)^{-1}G = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \Psi(z)\frac{1}{z}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ \dfrac{z}{z^2 - \frac{3}{4}z + \frac{1}{8}} \end{bmatrix} = \frac{-\frac{1}{4}z - \frac{1}{8}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

e. The impulse response is: $y(k) = h(k)$ where

$$h(k) = \begin{cases} 0 & \text{for} \quad k = 0 \\ \mathbf{C}\mathbf{F}^{k-1}\mathbf{G} & \text{for} \quad k \geq 1 \end{cases} \quad \text{and}$$

$$\mathbf{C}\mathbf{F}^{k-1}\mathbf{G} = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \mathbf{F}^{k-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \cdot {\frac{1}{2}}^{k-1} - 4 \cdot {\frac{1}{4}}^{k-1} \\ 2 \cdot {\frac{1}{2}}^{k-1} - {\frac{1}{4}}^{k-1} \end{bmatrix} = -2 \cdot {\frac{1}{2}}^{k} + 3 \cdot {\frac{1}{4}}^{k}$$

$$\Rightarrow h(k) = \begin{cases} 0 & \text{for} \quad k = 0 \\ -2{\frac{1}{2}}^{k} + 3{\frac{1}{4}}^{k} & \text{for} \quad k \geq 1 \end{cases}$$

LINEAR SYSTEM CONTROL

Solutions to Problems

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f. From the given initial conditions and input the outout can be calculated as follows:

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i) = \sum_{i=0}^{k-1} h(k-i)u(i) + h(k-k)u(k)$$

$$= \sum_{i=0}^{k-1} h(k-i)u(i) + 0 \quad \text{(because } h(k-k) = h(0) = 0\text{)}$$

$$\Rightarrow y(k) = \sum_{i=0}^{k-1} \left(-2 \cdot \frac{1}{2}^{k-i} + 3 \cdot \frac{1}{4}^{k-i}\right) \quad \text{for} \quad k \geq 1$$

$$y(1) = -2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = -\frac{1}{4}$$

$$y(2) = -2\frac{1}{2}^2 + 3\frac{1}{4}^2 - 2 \cdot \frac{1}{2} + 3\frac{1}{4} = -\frac{9}{16}$$

This agrees with the answer in point a.

❒