LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.5

a. The eigenfrequencies of the matrix in problem 3.3 can be found from.

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \Rightarrow \lambda = \begin{cases} -1 \\ -3 \\ -4 \end{cases}$$

and the eigenvectors by solving the equations: $\mathbf{A}v_i = \lambda_i v_i$.

1. $\lambda_1 = -1$:

$$\begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = -1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}$$

$$\begin{aligned} -4v_{11} + \frac{1}{2}v_{12} &= -v_{11} &\Rightarrow\quad 3v_{11} &= \frac{1}{2}v_{12} \\ -v_{12} + 8v_{13} &= -v_{12} &\Rightarrow\quad v_{13} &= 0 \\ -3v_{13} &= v_{13} \end{aligned}$$

One can choose for example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

and similarly:

2.

$$\lambda_2 = -3 \qquad \Rightarrow \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.5 (continued)

3.

$$\lambda_3 = -4 \quad \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b. The modal matrix is then given by:

$$\mathbf{M} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 6 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 6 \\ 6 & -1 & 8 \end{bmatrix}$$

The diagonal transformation is: $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$

$$\mathbf{A}_t = \Lambda = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

c. The transformed state vector then becomes:

$$\mathbf{z}(t) = \Phi_z(t)\mathbf{z}_0 = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-4t} \end{bmatrix} z_0$$

Problem 3.5 (continued)

$$\mathbf{z}_0 = \mathbf{M}^{-1}\mathbf{x}_0 = \begin{bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{bmatrix}$$

$$\Rightarrow \mathbf{z}(t) = \begin{bmatrix} \frac{1}{6}e^{-t} \\ 0 \\ -\frac{1}{6}e^{-4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{M}\mathbf{z}(t) = \begin{bmatrix} \frac{1}{6}(e^{-t} - e^{-4t}) \\ e^{-t} \\ 0 \end{bmatrix}$$

See for example problem 3.3.

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