LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.11

a. The spectral density matrix is found by Fourier transforming the covariance function

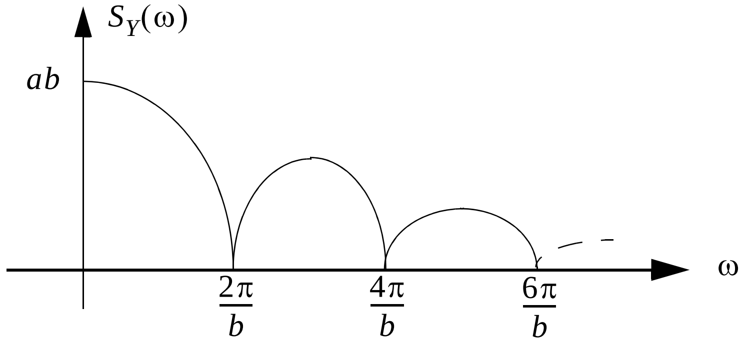
$$R_\gamma(\tau) = \begin{bmatrix} a\left(1 - b\frac{|\tau|}{b}\right) & |\tau| < b \\ 0 & \text{otherwise} \end{bmatrix}$$

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} a\left(1 - \frac{|\tau|}{b}\right) e^{-j\omega\tau} d\omega \\
&= \int_{-b}^{b} a e^{-j\omega} d\tau + \frac{a}{b}\left[\int_{-b}^{0} \tau e^{-j\omega\tau} d\tau + \int_{b}^{0} \tau e^{-j\omega\tau} d\tau\right] \\
&= \frac{a}{j\omega}(e^{j\omega b} - e^{-j\omega b}) + \frac{a}{b}\left[\frac{2}{\omega^2} - \frac{b}{j\omega}e^{j\omega b} - \frac{1}{\omega^2}e^{j\omega b} + \frac{b}{j\omega}e^{-j\omega b} - \frac{1}{\omega^2}e^{-j\omega b}\right] \\
&= \frac{2a\ \sin\omega b}{\omega} + \frac{2a}{\omega^2 b}(1 - \cos\omega b) - \frac{2a\ \sin\omega b}{\omega} = \frac{2a}{\omega^2 b}(1 - \cos\omega b)
\end{aligned}$$

As $\int \tau e^{-j\omega\tau} d\tau = -\frac{\tau}{j\omega} e^{-j\omega\tau} + \frac{1}{\omega^2}\ e^{-j\omega\tau}$, which is found by partial integration.

The D.C. value of the spectrum is found from

$$S_Y(0) = \lim_{\omega \to 0} \frac{2a}{\omega^2 b}(1 - \cos\omega b) = \lim_{\omega \to 0} \frac{2a}{\omega^2 b}\left(1 - 1 + \frac{1}{2}\omega^2 b^2\right) = ab$$



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b. No because $S_Y(\omega)$ has infinitely many zeroes.

c.

$$H(s) = \frac{p}{s+q}, \quad V = 1$$

$$S_Y(\omega) = H(j\omega)\ V\ H^T(-j\omega) = \frac{p}{q+j\omega} \cdot \frac{p}{q-j\omega} = \frac{p^2}{q^2+\omega^2}$$

For the power in Y one finds:

$$P_Y = \int_{-\infty}^{\infty} S_Y(\omega) df = \frac{p^2}{2\pi q^2} \int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{\omega}{q}\right)^2} d\omega$$

$$= \frac{p^2}{2\pi q} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{p^2}{2\pi q} [\operatorname{atan} x]\Big|_{-\infty}^{\infty} = \frac{p^2}{2q}, \quad \left(x = \frac{\omega}{q}\right)$$

The D.C. power is

$$P_\gamma = R_Y(0) = a$$

It is required that:

$$(1)\ P_\gamma = P_Y \quad \Leftrightarrow \quad \frac{p^2}{2q} = a$$

$$(2)\ S_\gamma(0) = S_Y(0) \quad \Leftrightarrow \quad \frac{p^2}{q^2} = ab$$

which must be solved for p and q.

$$\frac{p^2}{q^2} = ab = \frac{p^2}{2q} b \Leftrightarrow q = \frac{2}{b}$$

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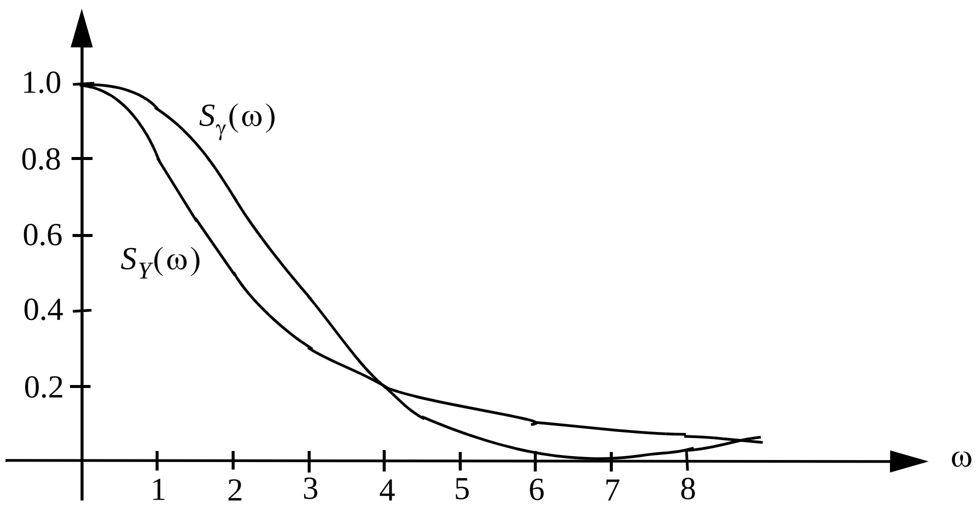
$$\frac{p^2}{q^2} = ab \quad \Leftrightarrow \quad p = q\sqrt{ab} = 2\sqrt{\frac{a}{b}}$$

$$H(s) = \frac{2\sqrt{\frac{a}{b}}}{s + \frac{2}{b}}$$

d.

$$a = 1, \quad b = 1, \quad \left\{ \begin{array}{l} S_\gamma(\omega) = \frac{2}{\omega^2}(1 - \cos\omega) \\ S_Y(\omega) = \frac{4}{\omega^2 + 4} \end{array} \right\}$$



The power bandwidth for Y can be found from:

$$P_Y^{\omega_0} = \int_{-\frac{\omega_0}{2\pi}}^{\frac{\omega_0}{2\pi}} S_Y(\omega) df = \frac{p^2}{\pi q}\left[\operatorname{atan}\frac{\omega}{q}\right]$$

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ω_0 must be found so that

$$P_Y^{\omega_0} = \frac{1}{2}P_Y \quad \Leftrightarrow \quad \operatorname{atan}\frac{\omega_0}{q} = \frac{\pi}{4}$$
$$\Leftrightarrow \omega_0 = \frac{2}{b}$$

which is the power bandwidth for *Y*.

❐