LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.11

a. Newton's second law gives the expression

$$m\frac{d^2X}{dt^2} = F(V) + F(X, I) + mA - mg$$

This equation can be written in state space form as

$$\begin{aligned}\dot{X} &= V \\ \dot{V} &= \frac{1}{m}F(V) + \frac{1}{m}F(X, I) + A - g\end{aligned}$$

b. The system above can be linearized using differentiation

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= \frac{1}{m}\left.\frac{\partial F(X,I)}{\partial X}\right|_{\substack{\mathbf{X} = \mathbf{X}_n \\ u = u_n}} x + \frac{1}{m}\left.\frac{\partial F(V)}{\partial V}\right|_{\substack{\mathbf{X} = \mathbf{X}_n \\ u = u_n}} v\end{aligned}$$

where $\mathbf{X}_n = (x_n \quad v_n)^T$, $u_n = i_n$, $x = \Delta X$, $v = \Delta V$, $i = \Delta I$, $a = \Delta A$

The stationary bias current can be found by letting

$$F(X, I) = mg$$

remembering that

$$\begin{aligned}V_n &= 0 \Rightarrow F(V_n) = 0, A_n = 0 \\ X_n &= 2.5 \cdot 10^{-3} \Rightarrow \\ F(2.5 \cdot 10^{-3}, I) &= 0.01128 + 0.0361I - 0.01625 \\ &\quad + 0.02888I^2 + 0.3740I - 0.1742 \\ &= 0.02888I^2 + 0.4101I - 0.1792 = 0.0981 \\ \Rightarrow \quad I^2 + 14.20I - 9.601 &= 0 \Rightarrow I_n = 0.6467\text{amps}\end{aligned}$$

which is the requried bias current.

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Assuming this operating current the coefficients in the linearized equation can be found.

$$\frac{1}{m}\left.\frac{\partial F(X,I)}{\partial X}\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}} = \frac{1}{0.01}\left.(3.61\cdot 10^{3}X + 14.44I - 6.488)\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}}$$

$$= 11.94\ m/\sec^2 = a_{21}$$

$$\frac{1}{m}\left.\frac{\partial F(X,I)}{\partial X}\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}} = \frac{1}{0.01}\left.(14.44X + 0.05776I + 0.3740)\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}}$$

$$= 44.75\ \frac{m/\sec^2}{amp} = b$$

One can also find a graphic solution from the figure given. In this case one finds that $I_n = 0.650$ amps and

$$\left.\frac{\partial F}{\partial X}\right|_{\substack{V = X_n\\ I = I_n}} = 12.3\ N/m \quad and \quad \left.\frac{\partial F}{\partial I}\right|_{\substack{X = X_n\\ I = I_n}} = 0.42\ N/amp$$

$$\frac{1}{m}\left.\frac{\partial F(V)}{\partial V}\right|_{V_n = 0} = \frac{1}{m}\frac{\partial}{\partial V}\begin{Bmatrix}(-V(c_1 + c_2V), & V > 0\\ (-V(c_1 + c_2V), & V < 0\end{Bmatrix} = -\frac{c_1}{m}$$

Thus

$$a_{22} = -\frac{c_1}{m} = -1.55\cdot 10^{-4}$$

c. The numerical values of a_{21}, a_{22} and b have been found above. The equations $\dot{X} = V$ and $\dot{x} = v$ are linear and have the same form. Thus $a_{12} = 1$.

d. The transfer function for the system can be found from

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$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{pmatrix} \rho - a_{22} & 1 \\ a_{21} & \rho \end{pmatrix}}{\Delta} \begin{bmatrix} 0 \\ b \end{bmatrix} = \frac{b}{\rho^2 - a_{22}\rho - a_{2}}$$

since $\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}$.

where $a_{22} < 0$ and $a_{21} > 0$. The system is thus unstable in itself with roots in the right half plan. The roots are actually in $s = \pm 34.6$ rad/sec. Feedback is thus necessary to keep the ball in its nominal quiescent position.

e. In order to design a LQ regulator for the accelerometer the following specifications are necessary

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \rho$$

The Riccati equation for the system is

$$0 = \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B}^T\mathbf{P}$$

$$0 = \mathbf{P}\begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & a_{21} \\ 1 & a_{22} \end{pmatrix}\mathbf{P} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$-\mathbf{P}\begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{\rho} \begin{bmatrix} 0 & b \end{bmatrix}\mathbf{P}$$

where $\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$

This is to say that the following system of equations must be solved:

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$$2a_{11}p_{12} + 1 - \frac{p_{12}^2}{\rho} b^2 = 0$$

$$a_{22}p_{12} + a_{21}p_{22} + p_{11} - \frac{p_{12}p_{22}}{\rho} b^2 = 0$$

$$2(p_{12} + a_{22}p_{22}) - \frac{p_{22}^2}{\rho} b^2 = 0$$

One finds that

$$p_{12} = -a_{21} \frac{\rho}{b^2} \pm \sqrt{a_{21}^2 \frac{\rho^2}{b^4} + \frac{\rho}{b^2}}$$

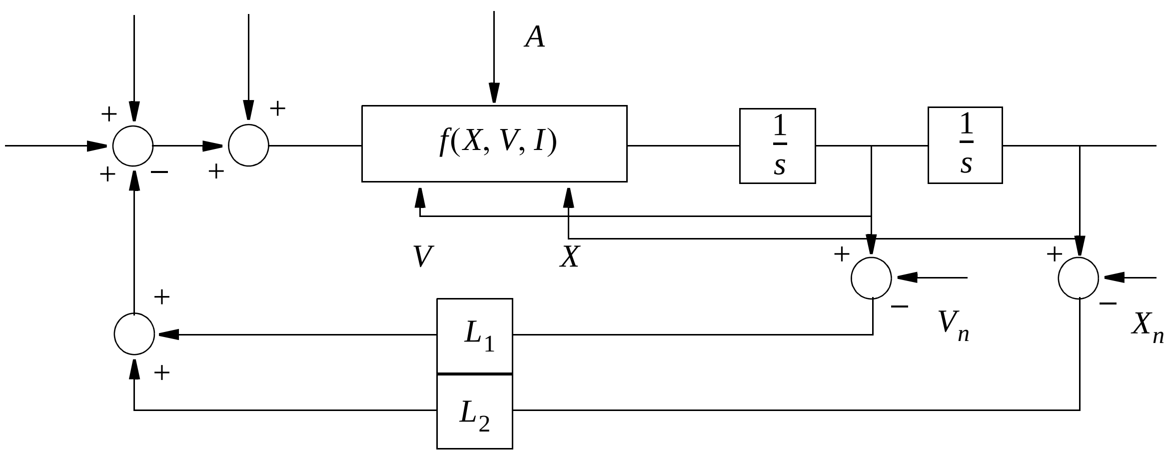
$$p_{22} = -a_{22} \frac{\rho}{b^2} \pm \sqrt{a_{22}^2 \frac{\rho^2}{b^4} + \frac{\rho}{b^2} p_{12}}$$

$$p_{11} = p_{12}p_{22} \frac{\rho}{b^2} - a_{22}p_{12} - a_{21}p_{22}$$

From this the LQR gain can be found

$$K = R_2^{-1}BP = -\frac{1}{\rho}(0 \quad b)\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = -\frac{b}{\rho}(p_{12} \quad p_{22})$$

f. The system block diagram is thus:



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g. It is possible to measure the unknown acceleration by measuring the current, *i*, which it calls forth with respect to I_n . When the system is in equilibrium one can find a from the equation

$$\frac{1}{m}F(X_n, I_n + i) = a - g$$

or from a linear approximation to this equation. The accuracy of the system is dependent on the size of the weighting factor ρ as there is no integration in the feedback loop. It is possible to make the system more accurate by including an integration in the feedback loop for the ball position.

h. In order to estimate a as a slowly varying constant the state vector must be augmented with the acceleration as an extra state.

The observer is then

$$\begin{aligned}\dot{\hat{x}} &= \hat{v} + k_1(y - \hat{x})\\ \dot{\hat{v}} &= a_{21}\hat{x} + a_{22}\hat{v} + \hat{a} + bi + k_2(y - \hat{x})\\ \dot{\hat{a}} &= k_3(y - \hat{x})\end{aligned}$$

The corresponding Riccati equation is

$$\mathbf{AQ} + \mathbf{QA}^T + \mathbf{V}_1 - \mathbf{QCV}_2^{-1}\mathbf{CQ} = 0$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and $\mathbf{V}_2^{-1} = \frac{1}{V_2}$, $\quad V_1 = diag(0, 0, V_1)$. From this the following system of equations can be found:

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$$-\frac{q_1^2}{V_2} + 2q_2 = 0$$

$$-\frac{q_1 q_2}{V_2} + a_{21}q_1 + a_{22}q_2 + q_3 + q_4 = 0$$

$$-\frac{q_1 q_3}{V_2} + q_5 = 0$$

$$-\frac{q_2^2}{V_2} + 2a_{21}q_2 + 2a_{22}q_4 + 2q_5 = 0$$

$$-\frac{q_2 q_3}{V_2} + a_{21}q_3 + a_{22}q_5 + q_6 = 0$$

$$-\frac{q_3^2}{V_2} + V_1 = 0$$

where $q_1 = q_{11}$, $q_2 = q_{12}$, $q_3 = q_{13}$, $q_4 = q_{22}$, $q_5 = q_{23}$, $q_6 = q_{33}$. No solution to these equations was requested.

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