3 October 2002

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Linear Systems Control

Solutions to problems

Problem 3.13

a)

$$\dot{x} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x$$

$$\begin{vmatrix} \lambda+1 & -1 & -1 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{vmatrix} = \lambda(\lambda+1)(\lambda+3) + 2(\lambda+1) = (\lambda+1)(\lambda^2+3\lambda+2) = 0$$

Eigenvalues: $$\lambda = \begin{cases} -1 \\ -1 \\ -2 \end{cases}$$

$$Av_i = \lambda_i v_i \quad , \quad \lambda = -1$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \quad = -1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \begin{cases} -v_{11} + v_{12} + v_{13} = -v_{11} \\ v_{13} = -v_{12} \\ -2v_{12} - 3v_{13} = -v_{13} \end{cases}$$

For $v_{11} = 1$ we can choose $v_{12} = 0$, $v_{13} = 0$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly: $$v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad , \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

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b)

$$M = [v_1 \ \ v_2 \ \ v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad , M^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\Lambda = M^{-1}AM = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

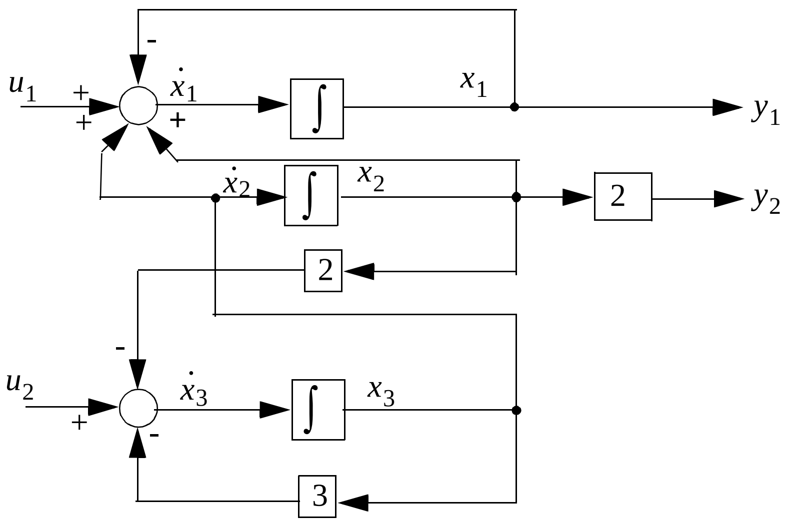
$$\underline{B_\Lambda = M^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}} \quad , \quad \underline{C_\Lambda = CM = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}}$$

c)

$$\dot{x}_1 = -x_1 + x_2 + x_3 + u_1$$

$$\dot{x}_2 = x_3 \qquad\qquad y_1 = x_1$$

$$\dot{x}_3 = -2x_2 - 3x_3 + u_2 \qquad\qquad y_2 = 2x_2$$



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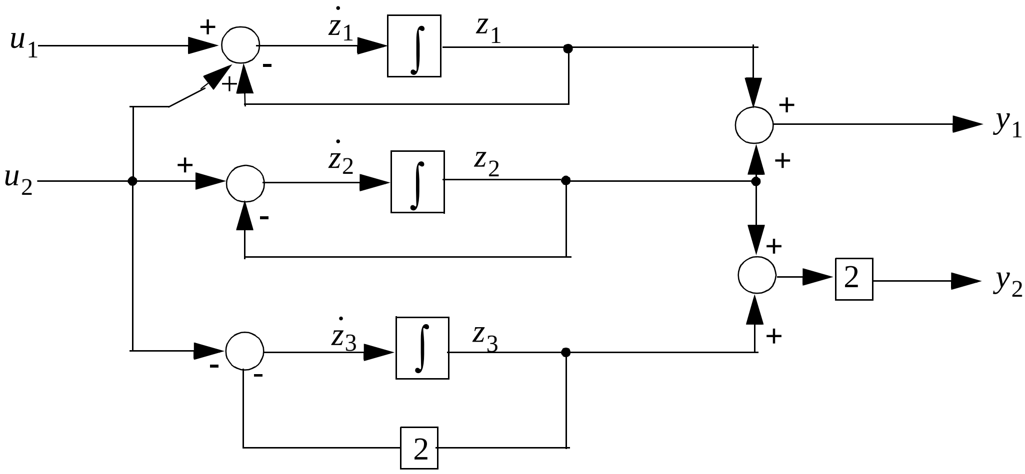
The diagonal form:

$$\dot{z}_1 – z_1 + u_1 + u_2$$
$$\dot{z}_2 = -z_2 + u_2 \qquad y_1 = z_1 + z_2$$
$$\dot{z}_3 = -2z_3 - u_2 \qquad y_2 = 2z_2 + 2z_3$$



d)

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$$G(s) = C(sI - A)^{-1}B$$

$$(sI-A)^{-1} = \begin{bmatrix} s+1 & -1 & -1 \\ 0 & s & -1 \\ 0 & z & s+3 \end{bmatrix}^{-1} = \frac{1}{|sI-A|}\begin{bmatrix} s(s+3)+2 & 0 & 0 \\ s+1 & (s+1)(s+3) & -2(s+1) \\ s+1 & s+1 & s(s+1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{s+1}{(s+1)(s^2+3s+2)} & \frac{1}{s^2+3s+2} \\ 0 & \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ 0 & \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

Problem 3.13

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s^2 3s + 2} \\ 0 & \dfrac{2}{s^2 + 3s + 2} \end{bmatrix}$$

Diagonal form:

$$(sI - \Lambda)^{-1} = \begin{bmatrix} (s+1)^{-1} & 0 & 0 \\ 0 & (s+1)^{-1} & 0 \\ 0 & 0 & (s+2)^{-1} \end{bmatrix}$$

$$G(s) = C_\Lambda (sI - \Lambda)^{-1} B_\Lambda$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} (sI - \Lambda)^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s^2 + 3s + 2} \\ 0 & \dfrac{2}{s^2 + 3s + 2} \end{bmatrix}$$

e)

Left eigenvectors are easiest found by taking the w_i^T vectors as the rows of M^{-1} (see equation (3.245):

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$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_1 = -1$$

$$w_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_2 = -1$$

$$w_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \quad \text{for} \quad \lambda_3 = -2$$

If we find w_i from the definition (3.342)

$$w_i^T A = \lambda_i w_i^T \quad \text{or} \quad A^T w_i = \lambda_i w_i$$

it is important to norm the w_i's such that the condition

$$\begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} [v_1 \;\; v_2 \;\; v_3] = I \tag{1}$$

(see (3.245))

is fulfilled. If we let

$$w_1^T = [p \;\; p \;\; p] \quad , \quad w_2^T = [0 \;\; 2q \;\; q] \quad \text{and} \quad w_3^T = [0 \;\; r \;\; r]$$

we can find from 1 that

$$p = 1 \quad , q = 1 \quad , r = -1$$

Problem 3.13

f)

$$M_C = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 & 0 & 7 \end{bmatrix}$$

$$\Rightarrow rank(M_C) = 3 \Rightarrow \text{The system is controllable}$$

$$M_0 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -3 & -3 \\ 0 & -4 & -6 \end{bmatrix} \quad \text{rank}(M_0) = 3$$

$$\Rightarrow \text{The system is observable}$$

g) PHS-test:

The system is controllable if $w_i^T B \neq 0$ for all i

$$w_1^T B = [1 \;\; 1 \;\; 1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [1 \;\; 1] \neq 0$$

$$w_2^T B = [0 \;\; 1] \neq 0, \qquad w_3^T B = [0 \;\; -1] \neq 0 =$$

$$\Rightarrow \text{The system is controllable}$$

Similarly, the system is observable if $w_i^T B \neq 0$ for all i

$$Cv_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Cv_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \qquad Cv_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \text{The system is observable}$$