LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.3

a. The system to be investigated is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Its stability can be determined from its eigenvalues:

$$\det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda - 2 \end{bmatrix} = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm j$$

Both eigenvalues in right half plane and this implies that the system is unstable.

The controllability matrix for the system is:

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \text{ thus } det(\mathbf{M}_c) = -1 .$$

The controllabilitly matrix has rank 2 and hence the system is controllable. Therefore it can be stabilized using linear state feedback:

$$u = -\mathbf{K}x + r$$

b. As the system has only one output and one input, the feedback matrix for it can be calculated using Ackermann's formula. For this one needs the inverse of the controllability matrix:

$$\mathbf{M}_c^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

The desired closed loop eigenvalues are: $\lambda_{A_K} = -1 \pm j$

$$P_{Ch,\, \mathbf{A}_K}(\lambda) = (\lambda + 1 - j)(\lambda + 1 + j) = \lambda^2 + 2\lambda + 2$$

$$P_{Ch,\, \mathbf{A}_K}(\mathbf{A}) = \mathbf{A}^2 + 2\mathbf{A} + 2 \cdot \mathbf{I} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -8 & 8 \end{bmatrix}$$

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The gain is then easily calculated as:

$$\mathbf{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

To test this gain matrix one can calculate the closed loop eigenfrequencies:

$$\mathbf{A_K} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

which has the closed loop eigenfrequencies:

$$det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{bmatrix} = \lambda^2 + 2\lambda + 2 = 0 \quad \text{for} \quad \lambda = -1 \pm j$$

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