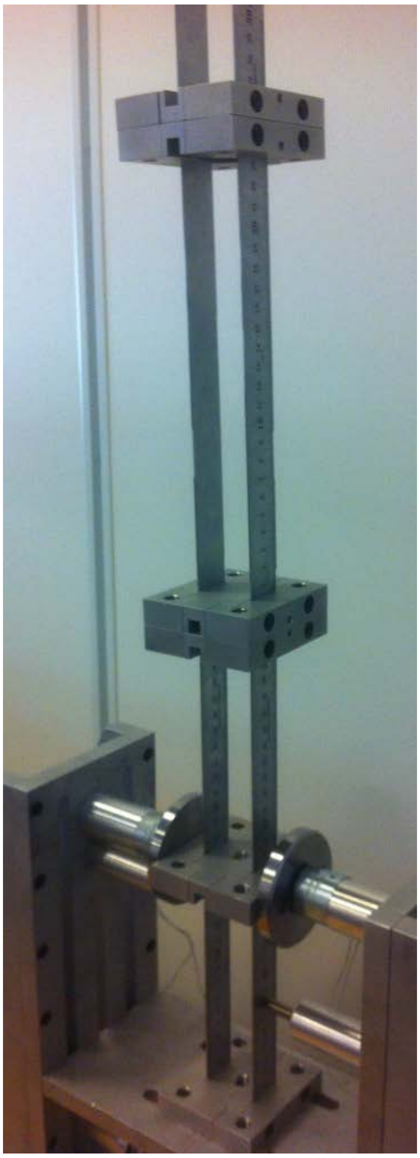
Module 4 – Similarities Transformation & Modal Analysis

In the modules 2 and 3 you have obtained a mathematical model based on principles and laws of Mechanics and Electromagnetism. Such a mathematical model uses 7 state variables which form the state vector **x**(t). With the matrices **A**, **B**, **C**, and **D** we are able to describe the dynamic behavior of the electro-mechanical system presented in the figure below.



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x} \in \mathbb{R}^7,\ u \in \mathbb{R}$$

$$y = \mathbf{C}\mathbf{x} + Du \qquad y \in \mathbb{R}$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 & x_2 \end{bmatrix}^T$$

$$= \begin{bmatrix} i & q_1 & \dot{q}_1 & q_2 & \dot{q}_2 & q_3 & \dot{q}_3 \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_i}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{K_i}{m_1} & -\frac{k_1+k_2+K_s}{m_1} & 0 & \frac{k_2}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 & \frac{k_3}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{k_3}{m_3} & 0 & -\frac{k_3}{m_3} & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{C}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ D = 0$$

The mathematical models are not unique, which you will show by building an equivalent mathematical model based on the transformation to the modal coordinates **z**(t):

$$\boxed{\mathbf{z} = \mathbf{P}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{P}^{-1}\mathbf{z}}$$

where

$$\dot{\mathbf{z}}(t) = \mathbf{PAP}^{-1}\mathbf{z}(t) + \mathbf{PBu}(t) \qquad \dot{\mathbf{z}}(t) = \mathbf{\Lambda z}(t) + \mathbf{B}_t\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{CP}^{-1}\mathbf{z}(t) + \mathbf{Du}(t) \qquad \mathbf{y}(t) = \mathbf{C}_t\mathbf{z}(t) + \mathbf{D}_t\mathbf{u}(t)$$

and **P** and its inverse **M**, and **Λ** are the matrices composed of eigenvectors and eigenvalues of the state matrix **A**. Using two different linearization points ($\mathbf{x_0}$ and $\mathbf{i_0}$) and the parameters provided in the modules 2 and 3.

1) Calculate **Λ**.
2) Calculate **M**.
3) Calculate **P**, i.e. the inverse of **M**.
4) Calculated **C**t.
5) Calculate the time constant of the electro-mechanical system.
6) Calculate the damped natural frequencies of the electro-mechanical system
7) Calculate the damping ratios associated to the damped natural frequencies of the system.
8) Explain the physical meaning of the natural modes of the electro-mechanical system and their link to eigenvectors and eigenvalues (modal superposition).

Using the matrices **A**, **B**, **C**, and **D=0** from module 2, one gets:

A =

```
1.0e+04 *

  -0.0185         0   -0.0014         0         0         0         0
        0         0    0.0001         0         0         0         0
   0.0016   -1.0334         0    0.5837         0         0         0
        0         0         0         0    0.0001         0         0
        0    0.9711         0   -1.1656         0    0.1946         0
        0         0         0         0         0         0    0.0001
        0         0         0    0.1946         0   -0.1946         0
```

B =

```
   1.2500
        0
        0
        0
        0
        0
        0
```

C =

```
     0     1     0     0     0     0     0
```

The following matrices can be used for building the equivalent electro-mechanical system using the modal coordinates **z**(t):

P =

```
  1.0056 + 0.0000i   2.9716 - 0.0000i   0.0603 + 0.0000i  -1.4242 - 0.0000i   0.0077 + 0.0000i  -0.0772 + 0.0000i   0.0004 + 0.0000i
 -0.0216 + 0.0159i  -0.1727 -53.3912i  -0.3919 - 0.0004i   0.2107 +45.8238i   0.3350 - 0.0020i  -0.0394 - 5.3176i  -0.0389 + 0.0003i
 -0.0216 - 0.0159i  -0.1727 +53.3912i  -0.3919 + 0.0004i   0.2107 -45.8238i   0.3350 + 0.0020i  -0.0394 + 5.3176i  -0.0389 - 0.0003i
 -0.0301 + 0.0102i  -0.0943 -25.4392i  -0.3958 - 0.0036i   0.3061 -16.2780i  -0.2518 - 0.0038i  -0.0671 +14.1846i   0.2194 + 0.0002i
 -0.0301 - 0.0102i  -0.0943 +25.4392i  -0.3958 + 0.0036i   0.3061 +16.2780i  -0.2518 + 0.0038i  -0.0671 -14.1846i   0.2194 - 0.0002i
  0.0162 - 0.0029i   0.2394 + 6.4619i   0.1968 - 0.0007i  -0.0109 + 6.1940i   0.1874 + 0.0000i  -0.0857 +14.1201i   0.4273 + 0.0019i
  0.0162 + 0.0029i   0.2394 - 6.4619i   0.1968 + 0.0007i  -0.0109 - 6.1940i   0.1874 - 0.0000i  -0.0857 -14.1201i   0.4273 - 0.0019i
```

Λ =

```
  1.0e+02 *

 -1.8431 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
  0.0000 - 0.0000i  -0.0017 + 1.3679i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i
  0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0017 - 1.3679i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i
  0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0024 + 0.6464i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i
  0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0024 - 0.6464i   0.0000 + 0.0000i   0.0000 - 0.0000i
 -0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0006 + 0.3305i  -0.0000 + 0.0000i
 -0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0006 - 0.3305i
```

Bt =

```
  1.2570 + 0.0000i
 -0.0270 + 0.0199i
 -0.0270 - 0.0199i
 -0.0376 + 0.0128i
 -0.0376 - 0.0128i
  0.0202 - 0.0037i
  0.0202 + 0.0037i
```

Ct =

```
  0.0004 - 0.0000i  -0.0000 + 0.0042i  -0.0000 - 0.0042i   0.0001 + 0.0090i   0.0001 - 0.0090i  -0.0001 - 0.0074i  -0.0001 + 0.0074i
```

The definition of time constant τ of the electro-mechanical system, the damped natural frequency β , the undamped natural frequency ωn , and damping ratio ζ are given by the expressions:

$$\boxed{p_{i,i+1} \in \mathbb{C}, \quad p_{i,i+1} = \alpha \pm j\beta}$$

$$\boxed{p_i \in \mathbb{R}} \longrightarrow \boxed{\tau_i = \frac{1}{|p_i|}} \longrightarrow g_i(t) = R_i e^{t/\tau_i}$$

$$\boxed{p_{i,i+1} \in \mathbb{C}, \quad p_{i,i+1} = \alpha \pm j\beta} \longrightarrow \boxed{\begin{aligned} \omega_n &= \sqrt{\alpha^2 + \beta^2} \\ \zeta &= -\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \end{aligned}}$$

$$\longrightarrow g_i(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{\alpha t} \sin(\beta t)$$

For calculating the time constant τ_i of the electro-mechanical system, the damped natural frequency β_i , the undamped natural frequency ω_{ni} , and damping ratio ζ_i one uses the values p_i in the diagonal of matrix **Λ**. The natural mode associated with the eigenvalue p_i is linked to the eigenvector $\mathbf{m}_i$ given in the matrix **M**. The eigenvectors $\mathbf{m}_i$ can be drawn and interpreted physically recalling the definition of **x**(t).

(physical interpretation of the results – Λ, P, M)

P =

```
Ps =

   1.0056 + 0.0000i   2.9716 - 0.0000i   0.0603 + 0.0000i  -1.4242 - 0.0000i   0.0077 + 0.0000i  -0.0772 + 0.0000i   0.0004 + 0.0000i
  -0.0216 + 0.0159i  -0.1727 -53.3912i  -0.3919 - 0.0004i   0.2107 +45.8238i   0.3350 - 0.0020i  -0.0394 - 5.3176i  -0.0389 + 0.0003i
  -0.0216 - 0.0159i  -0.1727 +53.3912i  -0.3919 + 0.0004i   0.2107 -45.8238i   0.3350 + 0.0020i  -0.0394 + 5.3176i  -0.0389 - 0.0003i
  -0.0301 + 0.0102i  -0.0943 -25.4392i  -0.3958 - 0.0036i   0.3061 -16.2780i  -0.2518 - 0.0038i  -0.0671 +14.1846i   0.2194 + 0.0002i
  -0.0301 - 0.0102i  -0.0943 +25.4392i  -0.3958 + 0.0036i   0.3061 +16.2780i  -0.2518 + 0.0038i  -0.0671 -14.1846i   0.2194 - 0.0002i
   0.0162 - 0.0029i   0.2394 + 6.4619i   0.1968 - 0.0007i  -0.0109 + 6.1940i   0.1874 + 0.0000i  -0.0857 +14.1201i   0.4273 + 0.0019i
   0.0162 + 0.0029i   0.2394 - 6.4619i   0.1968 + 0.0007i  -0.0109 - 6.1940i   0.1874 - 0.0000i  -0.0857 -14.1201i   0.4273 - 0.0019i
```

M =

```
VV =

   0.9976             0.0281 - 0.0205i   0.0281 + 0.0205i   0.0390 - 0.0139i   0.0390 + 0.0139i  -0.0180 + 0.0034i  -0.0180 - 0.0034i
   0.0004            -0.0000 + 0.0042i  -0.0000 - 0.0042i   0.0001 + 0.0090i   0.0001 - 0.0090i  -0.0001 - 0.0074i  -0.0001 + 0.0074i
  -0.0672            -0.5723 - 0.0040i  -0.5723 + 0.0040i  -0.5795 + 0.0035i  -0.5795 - 0.0035i   0.2456 - 0.0020i   0.2456 + 0.0020i
   0.0001            -0.0000 - 0.0059i  -0.0000 + 0.0059i   0.0000 + 0.0095i   0.0000 - 0.0095i  -0.0001 - 0.0118i  -0.0001 + 0.0118i
  -0.0143             0.8138             0.8138            -0.6135            -0.6135             0.3891 - 0.0017i   0.3891 + 0.0017i
   0.0000             0.0000 + 0.0007i   0.0000 - 0.0007i  -0.0001 - 0.0083i  -0.0001 + 0.0083i  -0.0000 - 0.0268i  -0.0000 + 0.0268i
  -0.0008            -0.0944 + 0.0003i  -0.0944 - 0.0003i   0.5345 - 0.0075i   0.5345 + 0.0075i   0.8871             0.8871
```

Λ =

```
LAM =

   1.0e+02 *

  -1.8431 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
   0.0000 - 0.0000i  -0.0017 + 1.3679i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i
   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0017 - 1.3679i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i
   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0024 + 0.6464i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i
   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0024 - 0.6464i   0.0000 + 0.0000i   0.0000 - 0.0000i
  -0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0006 + 0.3305i  -0.0000 + 0.0000i
  -0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0006 - 0.3305i
```

The drawings are not included! They will be made on the backboard at the beginning of the class 5 (September the 18th) followed by the experiments to illustrate Experimental Modal Analysis (EMA).

Ilmar