LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.3

a. A Kalman filter is to be designed for the system which consists of the D.C. motor of problem 7.1 with the provision of a state describing the angular position of the system.

The state equation for the D.C. motor is

$$\dot{\omega}(t) = -\alpha\omega(t) + \beta V_a(t)$$

The angular position of the system can be found by integrating the angular velocity:

$$\theta(t) = \int_0^t \omega(t)dt \Rightarrow \dot{\theta}(t) = \omega(t)$$

Thus the state equation which describes the overall system is

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta V_a(t) + T_d(t) \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta V_a(t) + T_d(t) \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad T_d\text{: torque dist.}$$

b. If one attempts to design a Kalman filter based on a velocity measurement alone then one discovers that the system is not observable.

$$C = (0 \quad 1), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}: \quad \text{rank } 1$$

An observer for the system can thus not be constructed.

c. Now a Kalman filter is to be constructed which uses the position measurement. This makes the system observable.

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 7.3

$$C = (0 \quad 1), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}: \quad \text{rank} \;\; 2$$

A steady state filter is to be designed, thus the algebraic Riccati equation has to be solved.

$$0 = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} - \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{V_2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

This results in a set of quadratic equations which must be solved simultaneously for the q_{ij}‘s, $\quad i = j = 1, 2$.

$$0 = 2q_{12} - \frac{1}{V_2}\, q_{11}$$

$$0 = q_{22} - \alpha q_{12} - \frac{1}{V_2} q_{11} q_{12}$$

$$0 = -2\alpha q_{22} + V_1 - \frac{1}{V_2}\, q_{12}^2$$

The solution of these equations can be found with some difficulty to be

$$q_{11} = V_2\left(-\alpha + \sqrt{\alpha^2 + 2\sqrt{\frac{V_1}{V_2}}}\right)$$

$$q_{12} = \alpha^2 + \sqrt{\frac{V_1}{V_2}} - \alpha\sqrt{\alpha^2 + \sqrt{\frac{V_1}{V_2}}}$$

$$q_{22} = -\alpha^3 - 2\alpha\sqrt{\frac{V_1}{V_2}} + \left(\alpha^2 + \sqrt{\frac{V_1}{V_2}}\right)\sqrt{\alpha^2 + 2\sqrt{\frac{V_1}{V_2}}}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.3

The Kalman gain is thus

$$\begin{aligned}
\mathbf{L} &= \mathbf{QC}^T\mathbf{V}_2^{-1} \\
&= \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{V_2} \\
&= \frac{1}{V_2}\begin{bmatrix} q_{11} & q_{12} \end{bmatrix}
\end{aligned}$$

d. It is clear that this is the only Kalman filter which can be built for the system: this it is the “best” of the two.

❒